BACHELOR’S DEGREE THESIS

Degree in Mathematics

Title: Maps and Trees

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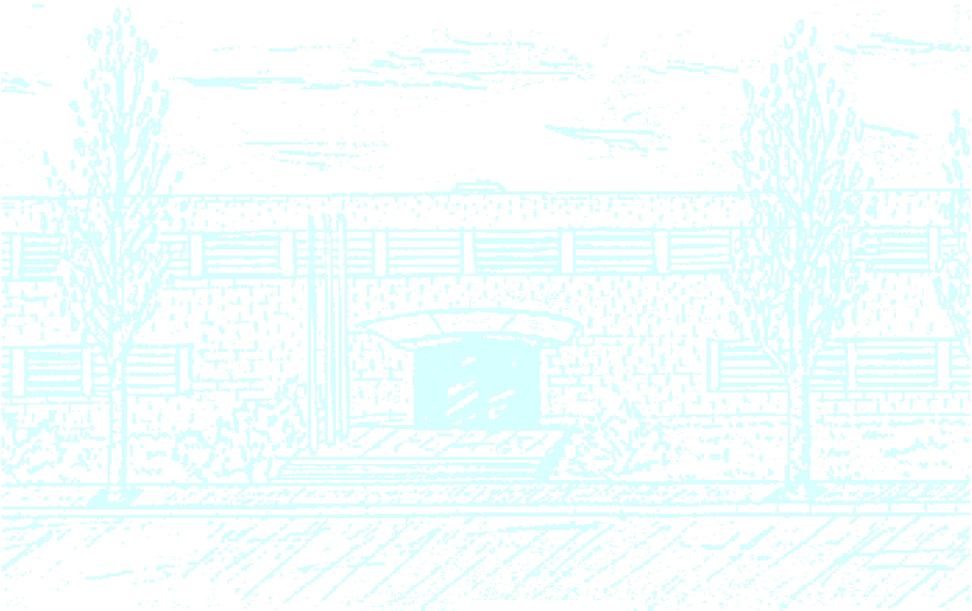
Academic year: 2016



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Abstract

MSC2010: 05A19, 05C05, 05C10, 05C30

We present bijective proofs for the enumeration of planar maps and non-separable planar maps, and apply the same method to rederive the enumeration formula for self-dual maps.

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Introduction

In the early sixties W. T. Tutte published a series of papers where he obtained formulas for the enumeration of different families of planar maps. To obtain these, he used some properties about their decomposition to write equations satisfied by their generating series. These equations are quite complicated but he still managed to solve them in most cases, obtaining in this way closed formulas for certain families of maps.

However, Tutte's approach does not give any insight about the combinatorial structure of maps, and it does not explain why the formulas obtained are so simple. Since then different proofs and generalizations have appeared, based on matrix integrals, algebraic approach, and bijective constructions. This work deals with the latter, which give elegant ways to rederive Tutte's formulas and at the same time help understanding the combinatorial structure of maps.

In particular, the enumeration formulas of different families of planar maps suggest that they can be interpreted as trees with some decorations. These are what we call blossoming trees, which roughly are spanning trees of the maps with some dangling half-edges that enable to reconstruct their faces. Additionally, in the case of arbitrary planar maps we also show a different bijective proof in which the trees involved are well-labelled.

The aim of this work is to present bijective proofs for two of the classical results in the enumeration of maps, that is, for the number of planar maps and for the number of non-separable planar maps, and then finally to offer a new proof for the enumeration formula for self-dual maps using the same method.

Chapter 1

Maps and enumeration

In this first chapter we give all the definitions we need, the basic concepts about maps which we need for the construction of the bijections [10]. We also present the classical results of enumeration by Tutte, which are the focus of this work and which we will prove later on. We assume familiarity with basic graph theory concepts, as in [12].

1.1 Planar maps

Definition 1.1. *A planar map M is a proper embedding of a connected graph G in the sphere S, considered up to orientation preserving homeomorphisms of S.*

Here, an embedding (or drawing) of a graph G on the oriented sphere S is *proper* if the vertices are represented by distinct points and the edges are represented by paths that only intersect at their endpoints and in agreement with the incidence relation of G.

Notice that in this definition loops and multiple edges are allowed. If a map does not have loops or multiple edges, it is said to be *simple.*

The basic elements of a map are natural; we have that the *edges* and *vertices* of a map are the drawings of the edges and vertices of the embedded graph. The *faces* are the connected components of $S \setminus G$.

We define a *corner* as the angular sector between two consecutive edges around a vertex. The concept of corner will be very useful when we later define rooted maps. Each corner c is incident to a vertex $v(c)$, a face $f(c)$ and two edges. We denote these two edges $cw(c)$ and $ccw(c)$, respectively preceeding and following c in counterclockwise direction around $v(c)$. Note $cw(c)$ and $ccw(c)$ are the same if $v(c)$ has degree 1. An example can be seen in Figure 1.1.

The *degree* of a vertex or face is the number of incident corners. A map is *Eulerian* if all its vertices have even degree. It is *m-valent* if all its vertices have

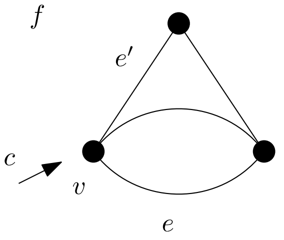

Figure 1.1: The corner c is incident to the vertex $v = v(c)$, the face $f = f(c)$, and the edges $e = ccw(c)$ and $e' = cw(c)$.

degree m, and an *m-angulation* if all its faces have degree m. In the particular cases of $m = 3, 4$, they are usually called trivalent, tetravalent, triangulation and quadrangulation.

A *plane map* is a proper embedding of a connected graph in the plane. It has one unbounded face called the *outer face*, and the other faces are called inner faces. Vertices and edges are also called outer or inner depending on whether or not they are incident with the outer face.

A plane map is actually a planar map with a marked face, which is the outer face.

A planar/plane map is (corner) *rooted* if it has a marked corner. $v(c)$, $ccw(c)$ and $f(c)$ are the *root vertex*, *root edge* and *root face*. The classical definition of rooted planar map was actually a planar map with a distinguished oriented edge. The newer definition is clearly equivalent; the oriented edge $\vec{e}$ has an associated corner c which is the one that has $v(c)$ as the origin of $\vec{e}$ and $ccw(c) = e$.

Usually each rooted planar map is associated with the corresponding rooted plane map in which the root face is the outer face.

1.2 Dual map, quadrangulation and radial map

Given a map M there are some basic constructions that we can consider associated to it.

The *dual* $M^\star$ of a map M is the map obtained by drawing one vertex for each face in M, and where two vertices are incident if the faces they come from are adjacent, that is, they have an edge in common. In practice, we can draw the dual map on top of the original map by drawing each vertex $f^\star$ in its corresponding face f of M, and one edge $e^\star$ across each edge e of M. Corners of M and $M^\star$ are in clear bijection and we consider them equal.

Given a map M with vertex set V and its dual $M^\star$ with vertex set $V^\star$, the *quadrangulation* $Q(M)$ is defined as the map that has vertex set $V \cup V^\star$ and an edge for each corner c of M connecting $v(c)$ and $f(c)^\star$ (remember that each corner c is incident to a vertex $v(c)$ and a face $f(c)$).

The *radial map* $R(M)$ of a map M with edge set E is the map that has E as vertex set and with an edge for each corner c connecting $cw(c)$ and $ccw(c)$. $R(M)$ is the dual of $Q(M)$.

Given a rooted map (M, c), its dual rooted map $(M, c)^\star$ is $(M^\star, c)$, its rooted quadrangulation is $(Q(M), c')$ where c' is the only corner incident to the root vertex of M and such that $cw(c')$ is the edge joining c and $f(c)$, and its rooted radial map is $(R(M), c)$ the dual of the rooted quadrangulation. We can see an example of these constructions in Figure 2.1.

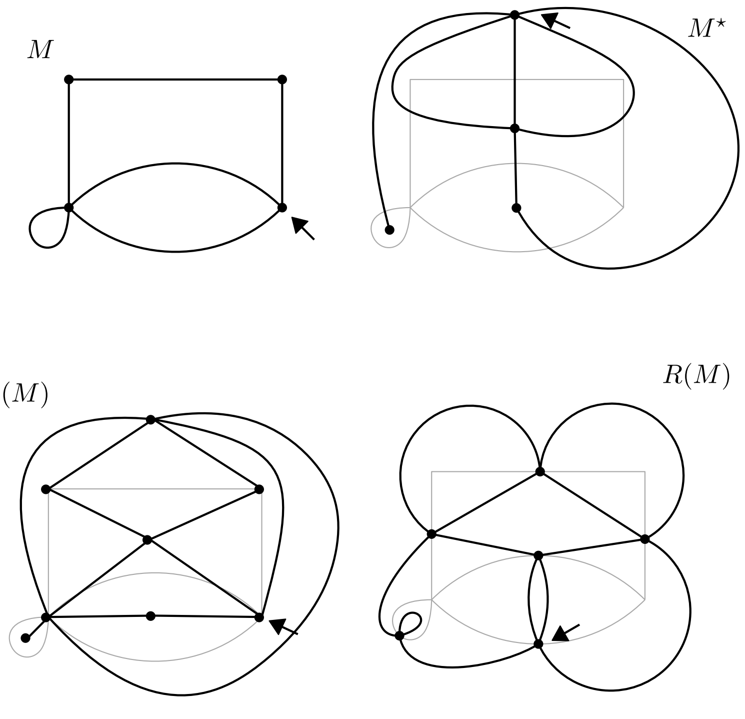

Figure 1.2: A rooted map M and the construction of its dual $M^\star$, $Q(M)$ quadrangulation and $R(M)$ radial map.

Since $Q(M)$ is a quadrangulation, it is clear then that $R(M)$ is a tetravalent map. Using Euler's formula we can count how many vertices, edges and faces the quadrangulation and radial map of a map with n edges have. $Q(M)$ has $n + 2$ vertices, $2n$ edges and n faces, while $R(M)$ has n vertices, $2n$ edges and $n + 2$ faces.

1.3 Classical results on map enumeration

In the year 1962 W. T. Tutte published the first in a series of "Census" papers, but it was in the fourth of these papers where he obtained formulae for the number of rooted maps (with n edges), the number of non-separable rooted maps, and the number of 3-connected rooted maps without multiple edges. It is the first two of these results that are considered the classical results in enumeration of maps, and the ones that we will state in this thesis.

The two classical results by Tutte are the following:

Theorem 1.1. *The number of rooted planar maps with n edges is:*

$$\frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}$$

A rooted planar map is *non-separable* (or *2-connected*) if it is either a single edge, which can be a bridge or a loop, or it is 2-connected as a graph, that is, there is no vertex v such that $G \setminus v$ is not connected.

Theorem 1.2. *The number of rooted planar non-separable maps with n+1 edges is:*

$$\frac{2}{(n+1)(2n+1)} \binom{3n}{n}$$

Tutte obtained his enumeration formulae via generating functions. The results though, were quite simple, which begged the question of whether there was not another proof via bijection with other well-known structures.

In fact, not only the formulae are simple, but the presence of the Catalan numbers in them reminds of the enumeration results for trees, and the other factors that appear in the formulae can be justified by adding certain decorations to these trees, as we will see in the next chapters.

Chapter 2

Counting planar maps

This chapter is dedicated to proving Theorem 1.1, or counting the number of planar maps. We give two different bijective proofs. The first one relates maps with a family of decorated trees called blossoming trees [1], and the second with well-labelled trees [3].

2.1 Orientations

Definition 2.1. *An orientation of a map is a choice of direction for each of its edges.*

We define the *indegree* and *outdegree* of a vertex as the number of edges oriented inward and outwards the vertex. Let M be a planar map of vertex set V, and $\alpha : V \longrightarrow N$ an application which assigns a natural number to each vertex of the map. Then an *α-orientation* of M is an orientation such that for every $v \in V$ the outdegree of v is $\alpha(v)$. If such an orientation exists, it is said to be *feasible.*

An *oriented path* is a sequence of adjacent vertices $v_0, v_1, ..., v_l$ where each edge $v_i v_{i+1}$ is oriented from v_i to v_{i+1}. It is called an *oriented cycle* if $v_l = v_0$. In a rooted map, we call *clockwise* or *counterclockwise cycle* an oriented cycle with the outer face (the root face) on its left or right respectively. We will call *minimal* any orientation without counterclockwise cycles. An orientation of a rooted map is *accessible* if from every vertex in M there is an oriented path to the root vertex.

Proposition 2.1. *[4]. Let M be a plane map and α be a feasible function on its vertices. Then, there exists a unique minimal α-orientation.*

Proof. Suppose there are two different minimal α-orientations $O_1 \neq O_2$. Then the edges where O_1 and O_2 disagree form an eulerian suborientation of O_1 (same in- and out- degree) so O_1 and O_2 contain a cycle, and it is counterclockwise in either O_1 or O_2.

□

The relevance of this result to our purpose is that to any given map with a feasible function α we can associate one specific α-orientation canonically, which is its unique minimal orientation.

For some families of maps the choice of accessible orientation is simple, either because it is unique or because we know a way to orient the edges in a canonical way so that we obtain one. Such as the following two families:

A rooted tree (which is a rooted map with only one face) always has a unique accessible orientation, which is trivially minimal since it does not have any cycles. This orientation is the one where all the edges are directed towards the root vertex, as we can see in the following figure.

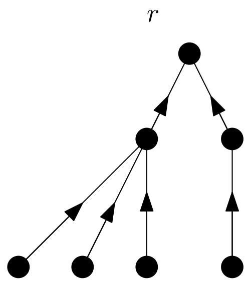

Figure 2.1: A minimal, accessible orientation on a rooted tree.

Another family that can be canonically endowed with a minimal accessible orientation are Eulerian maps. We call the *Eulerian orientation* of an Eulerian map the orientation that we obtain recursively by orienting clockwise the outer cycle and erasing it, as we can see in Figure 2.2. If the map is disconnected when the outer cycle is erased in any of the iterations then the algorithm is applied to each of the resulting connected components.

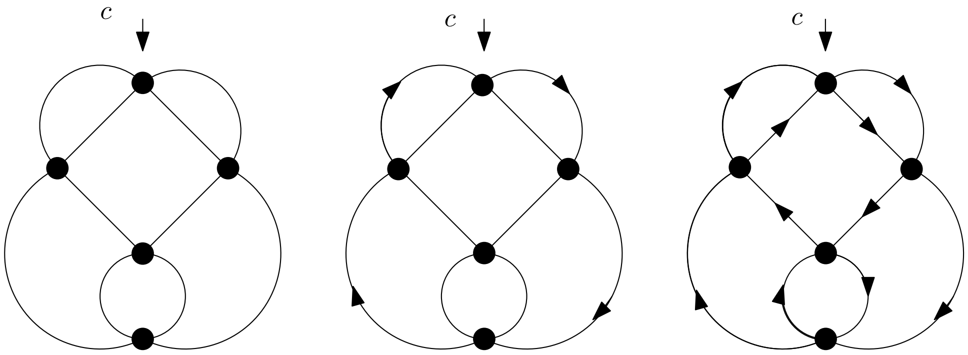

Figure 2.2: Construction of the Eulerian orientation on an Eulerian map.

2.2 The blossoming tree approach

A *blossoming map* is a plane map in which each outer corner can carry a sequence of opening and closing stems. In the case of a plane tree, we talk about blossoming trees, that is:

Definition 2.2. *A blossoming tree is a plane tree in which each corner can carry a sequence of opening and closing stems.*

We will consider the blossoming trees rooted and oriented, and represent opening stems with green arrows and closing stems with reverse green arrows (see Figure 2.3).

The basic operation that allows us to construct a bijection between blossoming trees and maps is called closure, and it is a way to match opening and closing stems in a blossoming map. The result of the closure of a blossoming map will be a (possibly) blossoming map. It will be non-blossoming if the number of opening stems is equal to the number of closing stems.

We define the *contour word* of a blossoming tree as the word on the alphabet e, b, $\overline{b}$ which encodes the clockwise order of stems along the border of the outer face starting at the root, where e encodes an edge, b an opening stem and $\overline{b}$ a closing stem (see Figure 2.3).

A *local closure* of a blossoming tree is a substitution of a factor $be^{\star}\overline{b}$ by e, where $e^{\star}$ is a possibly empty sequence of e. Graphically, this means we go around the tree clockwise and "throw" an opening stem to the next closing stem, this way construcing an oriented edge, as long as there are not any other stems between them. We will call such an edge a *closure edge*.

The *closure* of a blossoming tree is the oriented map obtained after doing all the possible local closures. Clearly the closure of a blossoming tree is unique. An example can be seen in Figure 2.3.

Notice that closure edges are canonically oriented, and since all closures are done clockwise around the map, no counterclockwise cycle can be formed during closure. Thus, if the initial orientation is minimal, so is the orientation after closure. Accessibility is also preserved during closure.

We need now the inverse operation to closure, which we call opening. This operation is given by the following theorem:

Theorem 2.1. *Let M be a plane map with root vertex r endowed with a minimal accessible orientation O. Then M admits a unique edge-partition (T,C) such that:*

- *edges in T (tree edges) form a spanning tree of M, rooted at r, on which the restriction of O is accessible;*

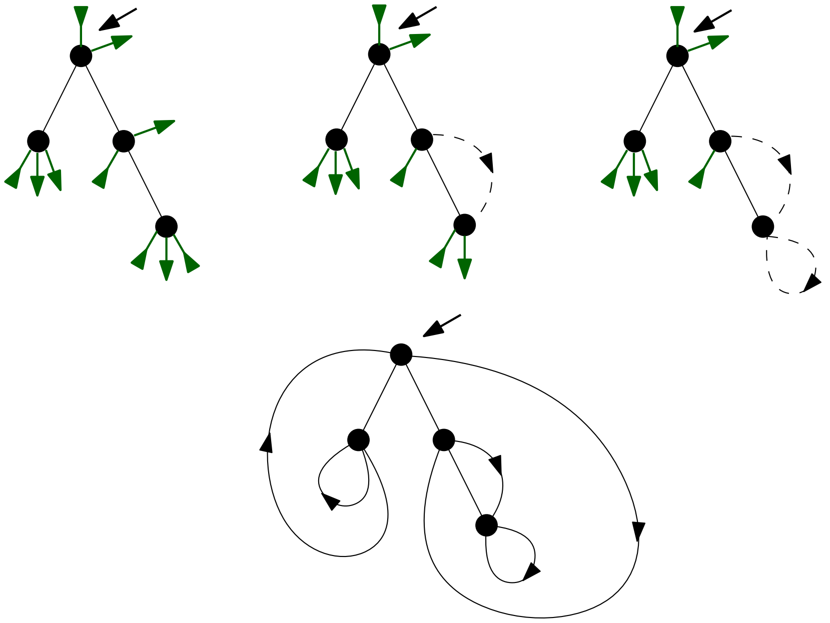

Figure 2.3: A blossoming tree; a local closure; another local closure; the total closure of the tree. The contour word of the tree is $bebe\bar{b}b\bar{b}e\bar{b}ee\bar{b}b\bar{b}e$, and after the first local closure is $beeb\bar{b}e\bar{b}ee\bar{b}b\bar{b}e$

- *any edge in C (closure edge) is a clockwise edge in the unique cycle it forms with edges in T.*

We call this partition a *tree-and-closure partition.*

Proof. By induction on the number n of faces of M. If M only has one face, then M is a tree so the property is satisfied.

Suppose $n \geq 2$, and that the property is satisfied for any plane map with a minimal orientation and less than n faces.

Let M be a rooted plane map with a minimal orientation and n faces.

We see that we can remove an outer edge e of M so that $M \setminus e$ is still endowed with a minimal accessible orientation. Since the outer edges do not form a counterclockwise cycle, there is at least one of them that has the outer face on its left and it is not a bridge. Let $\vec{uv}$ be such an edge and consider $M\setminus \vec{uv}$. If $M\setminus \vec{uv}$ is accessible, we choose $e = \vec{uv}$ as the edge to be removed. If it is not, let C be the accessible component of the root vertex r in $M\setminus \vec{uv}$, and let D be the complement of C. Since M was accessible, it is clear that v belongs to C and u does not, and that u is accessible from all vertices in D. Then, the cut between C and D, made up of edges oriented from C to D, is incident twice to the outer face of $M\setminus \vec{uv}$. This is illustrated in Figure 2.4 for clarification. Let e be the edge with the outer face on its left. Since e is not a bridge in M, $M \setminus e$ has $n-1$ faces and the orientation of $M \setminus e$ induced by the orientation of M is minimal and accessible.

Then by induction $M \setminus e$ admits a unique tree-and-closure partition (T, M) and $(T, M \cup e)$ is a tree-and-closure partition for M.

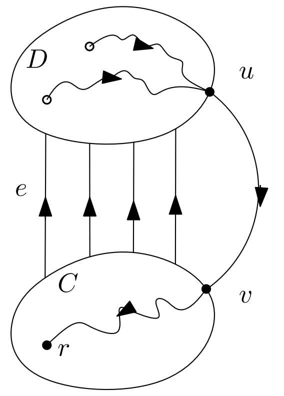

Figure 2.4: e is a closure edge of M.

We need to see now that M does not admit a different tree-and-closure partition, that is, it does not admit a tree-and-closure partition with e in the tree.

Suppose by contradiction that (T', C') is a tree-and-closure partition for M with $e = \overrightarrow{xy} \in T'$. Consider the simple path γ from x to r in T. At least one edge in γ does not belong to T', otherwise there would be a cycle in T'. Let $e_1 = \overrightarrow{x_1 y_1}$ be the first edge in γ such that it belongs to C'. Let γ_1 be the path from y_1 to r in T'. Since e has the outer face on its left, it cannot be wrapped by γ_1, hence y_1 belongs to $T'(x)$ (T' rooted at x). In particular, there exists another edge e_2 of γ that belongs to C', and does not belong to γ_1, for which the previous reasoning applies (see Figure 2.5). So there is an infinite sequence of edges in γ that do not belong to T', which is a contradiction.

□

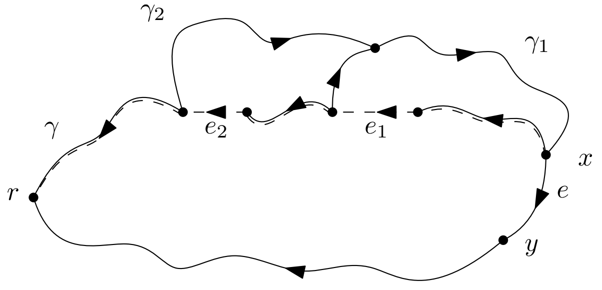

Figure 2.5: Uniqueness. Dashed lines represent γ in T and plain lines the tree T'.

Corollary 2.1. *Let M be a plane map with a root vertex r endowed with a minimal accessible orientation O. Then there exists a unique blossoming tree rooted at r, with an accessible orientation, the closure of which is M oriented with O.*

Proof. Let M be a rooted plane map with a minimal accessible orientation O. Consider the tree-and-closure partition of (T, C) of M given by Theorem 2.1. The edges of the blossoming tree are the edges of T and each edge of C is cut in two to produce a pair of opening and closing stems. ☐

An example of the blossoming tree resulting from opening can be seen in Figure 2.6.

The consequence of this is that, for any family of plane maps that can be canonically endowed with a specific minimal accessible orientation, there is a bijection between this family of maps and a family of blossoming trees with the same distribution of in- and out- degrees.

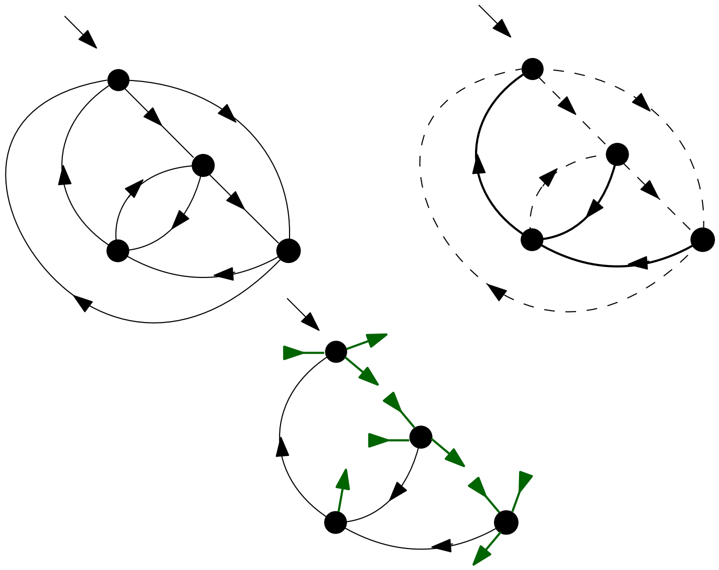

Figure 2.6: A map with a minimal accessible orientation; its tree-and-closure partition (T in thick lines, C in dashed lines); the corresponding blossoming tree.

Recall that when we talked about orientations we remarked that Eulerian maps can be canonically endowed with the Eulerian orientation, and that this orientation is minimal and accessible. Also recall that tetravalent maps, that are Eulerian, are in bijection with arbitrary maps, since each map is in one-to-one correspondence with its radial map.

Moreover, since we consider corner-rooted planar maps, which are equivalent to corner-rooted plane maps with the root corner in the outer face, no edge formed during the operation of closure will wrap around the root vertex. We call a blossoming tree *balanced* if the restriction of its contour word to $\{b, \bar{b}\}$ is a Dick word, that is, no initial segment of the string has more b's than $\bar{b}$'s. It is clear that if a blossoming tree is balanced then no closure edge of the tree will wrap around the root.

Thus, rooted planar maps with n edges are in bijection with balanced blossoming trees with n nodes, of in- and out- degrees 2. In order to count these, we consider planted binary trees with n nodes, and consider its leaves as closing stems (including the root), and then add un opening stem to each node, we get a blossoming tree with $n+2$ closing stems and n opening stems. The closure of this blossoming tree yields a blossoming map with two unmatched closing stems. Among all the blossoming trees that give this same map, a proportion $\frac{2}{n+2}$ of them are rooted on one of the unmatched closing stems. For those ones, changing their root, which is a closing stem, to an opening stem leads to a balanced blossoming tree that now can be fully closed. An example of the construction can be found in Figure 2.7.

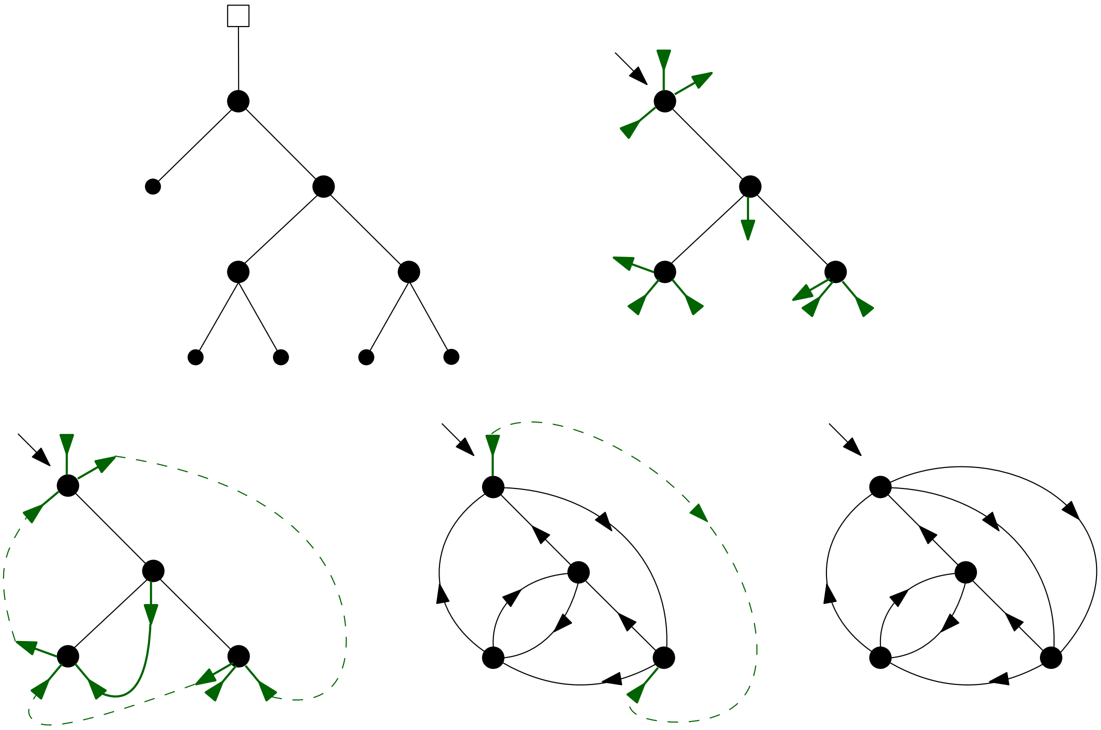

Figure 2.7: A planted binary tree; a planted binary blossoming tree; Closure of the tree, with two unmatched closing stems left; total closure of the tree; the corresponding tetravalent map. Notice that closure edges are canonically oriented (from opening to closing stem) and tree edges are oriented towards the root.

Conversely, for every map with n edges we find the tree-and-closure partition and convert the opening stem corresponding to $cw(c)$ (where c is the root corner) into the root of a binary tree. The closing stems we turn into leaves, and we get a planted binary tree with n nodes such that each node has an opening stem.

Since there are $\frac{1}{(n+1)}\begin{pmatrix} 2n \\ n \end{pmatrix}$ binary trees, a $\frac{2}{(n+2)}$ fraction of which are balanced, and for each node we can add its corresponding opening stem in either of the node's three incident corners, we have that the number of rooted planar maps with n vertices is:

$$\frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n}$$

And so Theorem 1.1 is proved.

2.3 Enumeration of Eulerian maps

Even though the classic result is the formula that counts rooted planar maps, with this bijective proof via blossoming trees we can get a stronger result, the number of Eulerian planar maps with prescribed vertex degrees. The number of planar maps is a particular case of this result, since radial maps are Eulerian maps where all vertices have in- and out- degree 2.

Consider an Eulerian planar rooted map M with n_i vertices of degree $2i$ for any $i \in [1, k]$ (which has $n = \Sigma_i i n_i$ edges) endowed with its unique minimal Eulerian orientation. The opening of this map leads to a balanced blossoming tree with the same distribution of in- and out- vertex degrees, and $l = 1 + \Sigma_i (i-1) n_i$ opening and closing stems. Each non-root vertex has exactly one outgoing edge that belongs to the blossoming tree, and since the tree is balanced, the root corner is necessarily followed by an opening stem (see Figure 2.8).

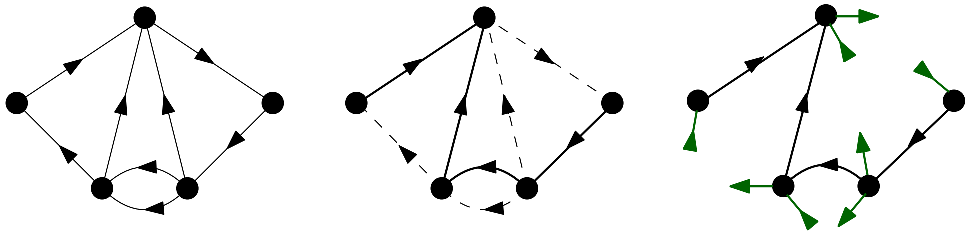

Figure 2.8: An Eulerian map; its tree-and-closure partition; its corresponding balanced blossoming tree.

Without going into too much detail, the strategy used to count the corresponding blossoming trees in this case is the same as in the previous section. We consider planted trees with n_i nodes of degree $i+1$ (arity i) for any $i > 0$, which have $l + 1 = 2 + \Sigma_i (i-1) n_i$ leaves inlcuding the root. To each node of arity i we add $i - 1$ opening stems. In this way we obtain a blossoming tree with $l - 1$ opening stems and $l + 1$ closing stems, so after all local closures there are two unmatched closing stems. A proportion $\frac{2}{l+1}$ of these trees are rooted in one of the unmatched stems, and in these last ones we change the root to an opening stem, obtaining in this way a balanced blossoming tree, whose closure gives an Eulerian map with the desired vertex degrees. Hence:

Theorem 2.2. *The number of Eulerian rooted planar maps with n_i vertices of degree $2i$ for any $i \in [1, k]$ is given by:*

$$\frac{2 \cdot (n-1)!}{(l+1)!} \Pi_{i=1}^{k} \frac{1}{n_i!}$$

2.4 The well-labelled tree approach

A *labelled tree* is a rooted plane tree in which the vertices are labelled with positive integers, such that the labels of two adjacent vertices differ at most in 1, and such that the minimum label is 1.

Definition 2.3. *A well-labelled tree is a labelled tree in which the root vertex has label 1.*

In the bijection between maps and blossoming trees we used that maps are in one-to-one correspondence with tetravalent maps, since each map has its corresponding radial map, which is tetravalent. For this second proof we will instead use that maps are in bijection with quadrangulations, since each map has a corresponding quadrangulation $Q(M)$ as we saw in the first chapter.

A *pointed quadrangulation* is a quadrangulation with a distinct marked vertex, not necessarily the same as the root vertex. To count maps via well-labelled trees we will find a bijection between labelled trees and pointed quadrangulations. We represent the pointed vertex with a double circle. For this section it is convenient to root maps in the classical way, in an oriented edge instead of in a corner, but as previously said both definitions of rooting are equivalent.

We see first how we can obtain a pointed quadrangulation from a labelled tree. In each corner of the tree we draw a "leg" or half-edge. Then, recursively, we take each leg of label $i > 1$ and "throw" it counterclockwise to the next corner of label $i - 1$ (meaning we create an edge by joining the leg with the vertex of label $i - 1$, in the first corner). Once this has been done for all legs of label $i > 1$, we create a new vertex labelled 0 in the outer face, which will be the pointed vertex of the quadrangulation, and connect all the remaining legs (the ones labelled 1) with this new vertex. Once all of this has been done, we erase the edges of the original labelled tree, and we are left with a pointed quadrangulation, since this way we can only obtain faces of degree 4. To root this quadrangulation we choose the root edge to be in the face of the root edge from the tree, and to orient this edge we follow the rules illustrated on Figure 2.9. An example of the whole construction can be seen in Figure 2.10.

We describe now the opposite construction. Let Q be a quadrangulation with a pointed vertex v, such that the root edge is oriented away from v (in terms of distance the starting point of the root edge is closer to v than the endpoint). We label each vertex u with the distance from u to v . The labels of two neighbours differ by 1. If the starting point of the root edge has label i then the enpoint has

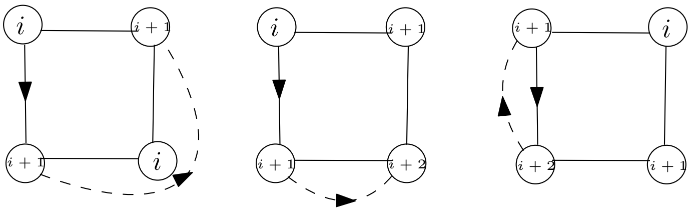
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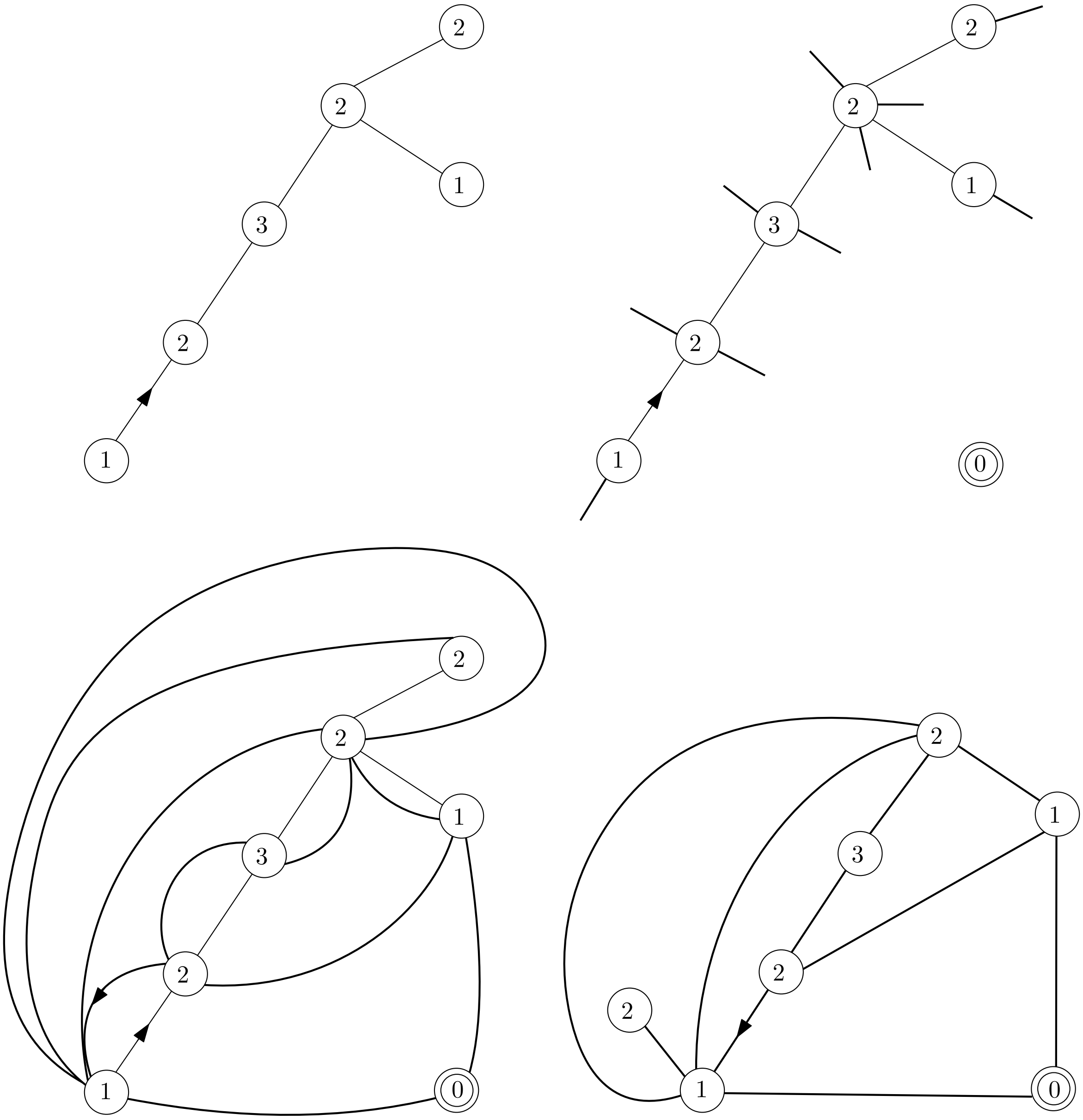
Figure 2.9: Rules for rooting the pointed quadrangulation. New edges in dashed lines.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1
0

Figure 2.10: Well-labelled tree; add a leg to each corner and vertex 0; create new edges; quadrangulation obtained after deleting tree edges.

label $i+1$. There are two possible types of faces resulting from this labelling, as shown in Figure 2.11: walking inside a face with the edges on the left, we see a cyclic sequence of the form i, $i+1$, i, $i+1$, or i, $i+1$, $i+2$, $i+1$. We draw a new edge in each face as shown in the figure, following this local rule: for the first type of face we draw an edge between the vertices of label $i+1$; for the second type of face we draw an edge from the "first" vertex labelled $i+1$ to the vertex labelled $i+2$, and by first we mean first to appear in the cyclic sequence we just described.

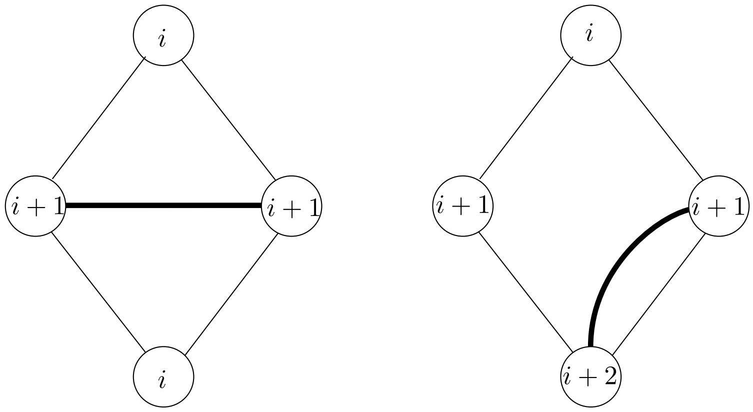


Figure 2.11: Local rule for the construction of the edges of the tree.

These new edges form a labelled tree, spanning all vertices except the pointed vertex v. We root it at the edge created in the outer face of $Q(M)$, oriented away from the enpoint of the root edge of $Q(M)$ (as in Figure 2.9 again). That it is a tree is clear by construction of the new edges, since they can not form any cycles and there are n edges for $n+1$ vertices. That it is labelled is also clear since we only allowed the endpoints in these new edges to differ in 1. Figure 2.12 shows an example of this construction.

We have a bijection between pointed rooted quadrangulations with n faces such that the root edge is oriented away from the pointed vertex and labelled trees with n edges. When specialised to rooted quadrangulations pointed canonically at their root vertex, the bijection is between rooted quadrangulations and well-labelled trees.

Now that we have the bijection, we need to count how many well-labelled trees there are. Labelled trees with n vertices are in 1-to-1 correspondence with rooted plane trees with n vertices labelled with integers, such that the labels of two adjacent vertices differ at most in 1, and such that the root is labelled with 0.

Indeed, if we have a labelled tree with root label k, substract k to all labels and we get a rooted plane tree with root label 0. And if we have a rooted plane tree with $l \leq 0$ the minimum label, then add $l+1$ to all labels and we get a labelled tree (see Figure 2.13).

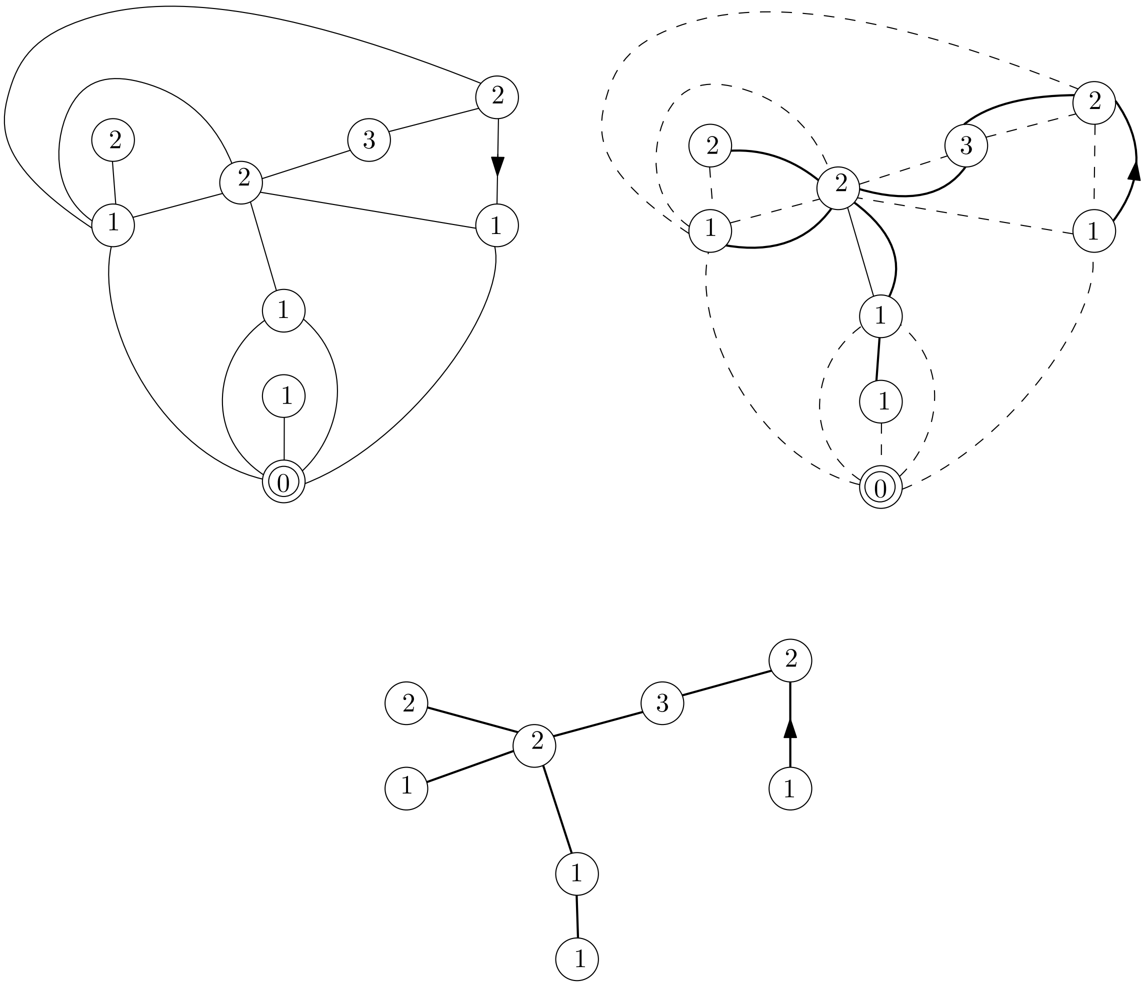

2
2
3
2
1
1
1
1
0
2
3
2
2
1
1
1
1
0
2
2
3
2
1
1
1
1

Figure 2.12: Quadrangulation; new edges in thicker lines; the corresponding well-labelled tree.

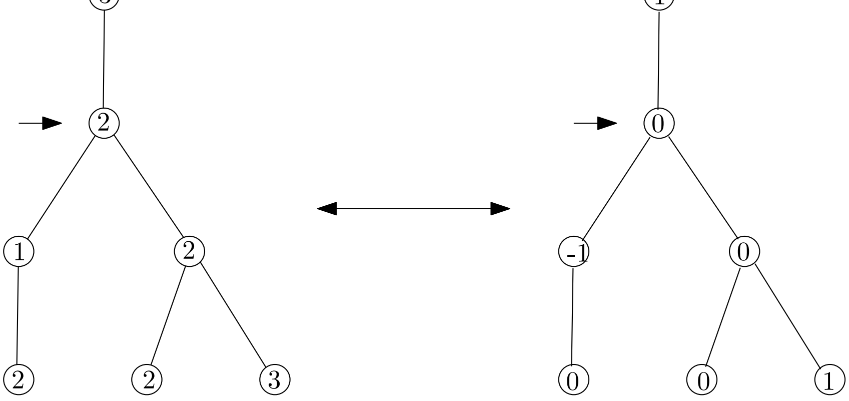

3
2
1
2
2
2
3
1
0
-1
0
0
0
1

Figure 2.13: Labelled tree and its corresponding plane rooted tree with label 0 in the root vertex.

We have $\frac{1}{n+1}\binom{2n}{n}$ rooted plane trees with n edges. If we label the root 0, then for each edge (we start at the edges incident to the root and from there we can label the rest of the vertices) we have three choices for labelling: same label at both ends, 1 higher, or 1 lower for the parent. This gives us 3^n possibilities for labelling each tree, which consequently means that the number of labelled trees is $\frac{3^n}{n+1}\binom{2n}{n}$. Now, the general bijection between pointed quadrangulations and labelled trees implies that $\frac{3^n}{n+1}\binom{2n}{n} = \frac{(n+2)q_n}{2}$, where q_n is the number of quadrangulations with n faces. Indeed, there are $n+2$ ways to point a vertex in such a quadrangulation, and half of these pointings are such that the root edge is oriented away from the pointed vertex. Which means that the number of rooted quadrangulations with n faces and so the number of rooted planar maps with n vertices is:

$$\frac{2 \cdot 3^n}{(n+2)(n+1)}\binom{2n}{n}$$

And so Theorem 1.1 is proved.

Chapter 3

Counting non-separable planar maps

This chapter is dedicated to proving Theorem 1.2, or counting the number of non-separable planar maps. We do this again via bijection with blossoming trees, as in [1] and [9].

3.1 Bipolar orientations

An *bipolar orientation* of a map (and in general of a graph) is an acyclic orientation of its edges with a single source (vertex that has only outgoing edges) and a single sink (vertex that has only intcoming edges), which are called the *poles* of the orientation. Non-separable maps (recall that these are 2-connected maps) are characterized by the following:

Proposition 3.1. *A rooted map with* $n > 1$ *vertices is non-separable if and only if it can be endowed with a bipolar orientation with the two ends of the root edge as poles.*

Proof. It is clear that in an oriented map, each 2-connected component has either a source and a sink, or a cycle. And a map endowed with a bipolar orientation has no cycles. Now suppose M has a cut-vertex x and consider two of its 2-connected components. In the best of cases the source of one component is the sink of the other, and they are both x, but even in this case the remaining source and sink are not adjacent so there is not a bipolar orientation where the poles are endpoints of the root edge.

Conversely, to each non-separable map we can assign a bipolar orientation with the following algorithm:

```
function BipolarOrientation($M$, $e$)
    Input: $M$ rooted non-separable map, $e = x_0 y_0$ root edge oriented from $x_0$ to $y_0$.
    Output: A bipolar orientation of $M$ with endpoints of $e$ as poles.
    Create stack of vertices $L = [x_0]$.
    $n(x)$ is an edge assigned to vertex $x$ the first time we see the vertex. $n(x_0)$ is the edge next to $e$ in clockwise order around $x_0$.
    while $L$ is not empty do
        Let $x$ be the first element in $L$.
        if all edges incident to $x$ are oriented then remove $x$ from $L$.
        else
            Remove $x$ from $L$
            Starting at $n(x)$, orient the edges of the face that is to the left of $n(x)$ (seen from $x$) in counterclockwise order until meeting a vertex that has already been seen.
            If the vertices visited in the previous step are $x, x_1, ..., x_k, y$, with $y$ the already seen before vertex, add $x_1, ..., x_k$ to the top of $L$.
            For $i = 1, ..., k$ assign $n(x_i)$ to be the edge next to $x_i x_{i+1}$ in clockwise order around $x_i$.
            Change $n(x)$ to be the edge next to $n(x)$ in clockwise order around $x$ and add $x$ to the top of $L$.
        end if
    end while
end function
```

No cycle can be formed in any iteration of algorithm and so the orientation obtained is bipolar. Clearly the x_0 is the source since all its edges are oriented outwards and y_0 the sink since we never add y_0 to L so no edges can be oriented outwards from the vertex. We can see an example of this algorithm in Figure 3.1.

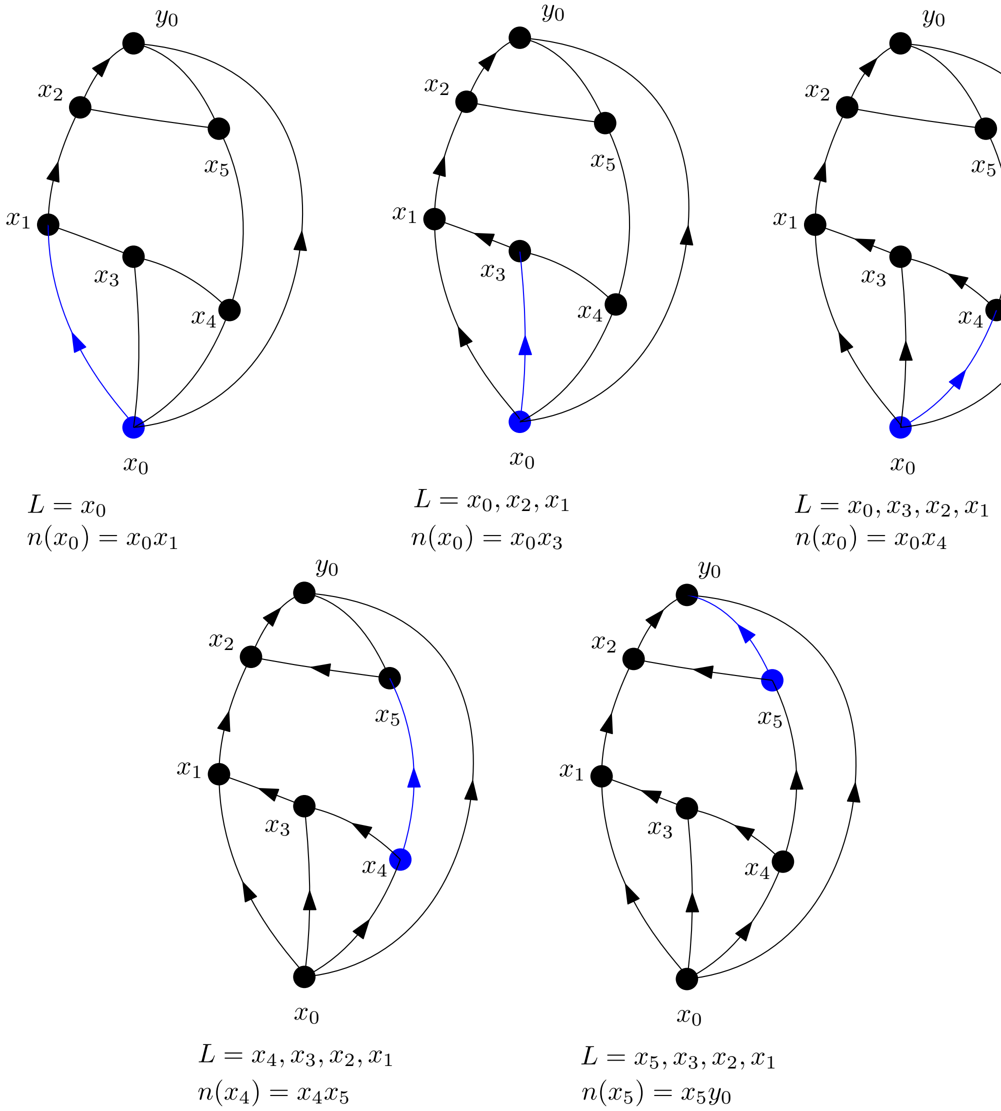
□

By convention we take the root vertex as the sink, and the other endpoint of the root edge as the source. There are some observations we can make about a rooted non-separable planar map endowed with its bipolar orientation that we list here because they will be useful for their enumeration:

1. Each face of the map is bipolar in itself, so we can classify its corners as lateral (left and right) or polar (source and sink). So each face has two special corners, the polar ones. See Figure 3.2.

2. Each vertex except the poles has a bundle of incoming edges and a bundle of outgoing edges, which allows us to classify its corners into polar (source and sink) and exactly two lateral (right and left). See Figure 3.2.

3. The quadrangulation $Q(M)$ can be endowed with an α-orientation so that the endpoints of the root edge of $Q(M)$ have indegree 0 and the rest of vertices have indegree 2. Recall that each vertex of $Q(M)$ corresponds to either a vertex or a face of M, and as said in items 1 and 2, each face and vertex of M has exactly two special corners (polar corners for a face, lateral corners for a vertex). There is one edge in $Q(M)$ for each corner of M. We obtain this α-orientation by orienting towards each vertex the two edges the correspond to the two special corners.



$L = x_0$
$n(x_0) = x_0x_1$

$L = x_0, x_2, x_1$
$n(x_0) = x_0x_3$

$L = x_0, x_3, x_2, x_1$
$n(x_0) = x_0x_4$

$L = x_4, x_3, x_2, x_1$
$n(x_4) = x_4x_5$

$L = x_5, x_3, x_2, x_1$
$n(x_5) = x_5y_0$

Figure 3.1: At each step of the algortihm the vertex and edge in blue are x and $n(x)$.

4. To each orientation of $Q(M)$ that is as described in item 3 corresponds a bipolar orientation of M.

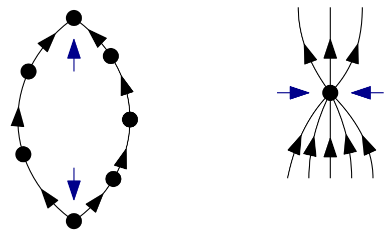


Figure 3.2: A face of M with a bipolar orientation, its polar corners marked with a blue arrow; a vertex of M with its incoming and outgoing bundle of edges, its lateral corners marked with a blue arrow.

We say that a bipolar orientation of a map M is *minimal* if the corresponding orientation of $Q(M)$ is minimal (as defined in Chapter 2). To a minimal orientation of $Q(M)$, the corresponding natural orientation of its dual $R(M)$ does not have a cocycle supported in a counterclockwise cycle such that the edges of the cocycle are oriented towards the cycle, and this is what we call a minimal orientation of $R(M)$ in this section. This forbidden configuration of a minimal orientation of $R(M)$ can be seen in Figure 3.3.

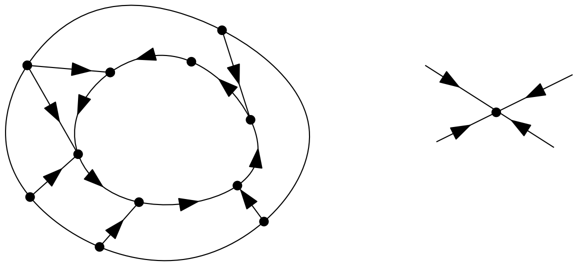


Figure 3.3: A cocycle supported in a counterclockwise cycle with its edges oriented towards it (it can be an empty cycle as seen in the second example.)

By Proposition 2.1 there is a unique minimal orientation of $Q(M)$ with the in- and out- degrees described in item 3, and so:

Proposition 3.2. *A rooted non-separable map has a unique minimal bipolar orientation with the two ends of the root edge as poles.*

In a minimal bipolar orientation the following configuration is forbidden: let u, v distinct vertices of M and f, g distinct faces, then u sink of f and on the left of g, v on the right of f and source of g. This configuration would give a counterclockwise 4-cycle in the quadrangulation. See Figure 3.4. The bipolar orientation given by the algorithm in the proof of Proposition 3.1 is actually the unique minimal orientation of a map, because it is not possible to form this configuration.

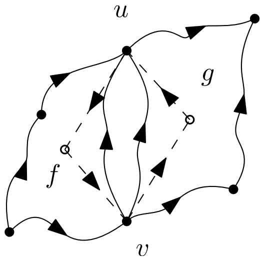


Figure 3.4: Forbidden configuration in a minimal bipolar orientation.

3.2 The combinatorial proof

The blossoming tree constructions used for the bijection with non-separable maps are based on bipolar orientations. In this section we call s and t the source and sink respectively, say that a face is *generic* if it is not incident to the root edge, and by convention draw the maps on the plane with oriented paths going upwards, with s and t being outer vertices, s at the bottom and t at the top, and the root edge the one most to the right.

As in the proof given for Theorem 1.1 using blossoming trees, the bijection we describe in this chapter is a particular case of the general bijective scheme we constructed in Chapter 2. It is not with a non-separable map M but with an extension of its radial map $R(M)$, with root vertex r corresponding to the root edge of M, and root face the same root face of M.

We consider the following orientation of $R(M)$. In a bipolar orientation of M we observed in properties 1 and 2 that each face and each vertex have two special corners. Each generic face of $R(M)$ corresponds to either a face or a vertex of M, and each edge of $R(M)$ to a corner of M. Then in each face of $R(M)$, we orient clockwise the two edges corresponding to the two special corners of the face or vertex of M associated to that face of $R(M)$, as shown in Figure 3.5. In the faces of M incident to the root edge, by convention we orient clockwise the two outer edges incident to r, and the remaining two edges outgoing from r.

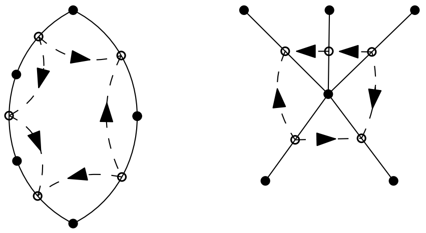


Figure 3.5: Orientation of a face of $R(M)$ associated to a face of M; orientation of a face of $R(M)$ associated to a vertex of M. In both cases M is represented with disks and black lines, $R(M)$ with circles and dashed lines, and M is represented following the convention, so that its edges are oriented upwards.

In this orientation of $R(M)$ each generic face has two special corners, corresponding to the origins of the two clockwise edges of the face. The face that corresponds to t has these two special corners as well. We define $\overline{R(M)}$ as the map resulting from adding an edge in each generic face and in the face corresonding to t between the two special corners in each face. We define $T(M)$ as the map made of these extra edges and their incident vertices. See Figure 3.6 for an example. Then:

Lemma 3.1. *$T(M)$ is a spanning tree of $\overline{R(M)}$ if and only if the underlying orientation of $R(M)$ is minimal. In this case, the edges of $R(M)$ go clockwise around $T(M)$.*

Proof. By definition, if the orientation of $R(M)$ is not minimal then there is a cocycle supported in a counterclockwise cycle and oriented towards it. The rules for the construction of the new edges that form $\overline{(R(M))}$ do not make an edge joining the component of $R(M)$ inside the cycle with the component outside the cycle, so $T(M)$ is not a spanning tree.

Reciprocally, if $T(M)$ is not a tree, the added edges define at least two connected components. Let A and B be the two non-empty subsets of edges corresponding to two connected components. Let F_A and F_B be the union of faces which contain the edges from A and B. The boundary between F_A and F_B contains only either vertices of A or vertices of B. Indeed, if $x \in A$ is a vertex in the boundary, then the preceding edge in counterclockwise order around A is oriented towards x (or else an edge of B would arrive to x). This means an edge of A arrives to the preceding vertex in the boundary, then this vertex is covered by A. This way we can go around the whole boundary finding a counterclockwise cycle, and the edges adjacent to the cycle on the outside are oriented towards it, so the orientation fo $R(M)$ is not minimal. □

So when the bipolar orientation of M is minimal, $R(M)$ is a valid set of closure edges around $T(M)$, and $(T(M), R(M))$ is the tree-and-closure partition of $\overline{R(M)}$.

The resulting balanced blossoming tree is such that each non-root vertex is incident to 4 stems, and to as many opening stems as edges. If we consider closing stems to be leaves, and remove the root vertex (with its incident edge) we get a ternary tree with an extra opening stem at each corner before an inner edge (clockwise around each vertex). This means we do not get a choice of placement for the opening stems. See Figure 3.6.

Reciprocally, the closure edges of such a blossoming tree form a 4-regular map R endowed with an orientation with two clockwise edges per face, and by Proposition 3.4 this means it is the radial map of a non-separable map, endowed with its unique minimal orientation.

A planted blossoming ternary tree with n nodes has $2n + 2$ leaves (including the root), that are the closing stems, and $2n - 2$ opening stems. A fraction $\frac{4}{2n+2}$ of them is balanced, corresponding to the ones rooted in one of the unmatched closing stems. The number of planted ternary trees with n nodes is $\frac{1}{(2n+1)}\begin{pmatrix} 3n \\ n \end{pmatrix}$ and so the number of non-separable rooted planar maps with $n + 1$ edges is:

$$\frac{2}{(n+1)(2n+1)}\begin{pmatrix} 3n \\ n \end{pmatrix}$$

And so Theorem 1.2 is proved.

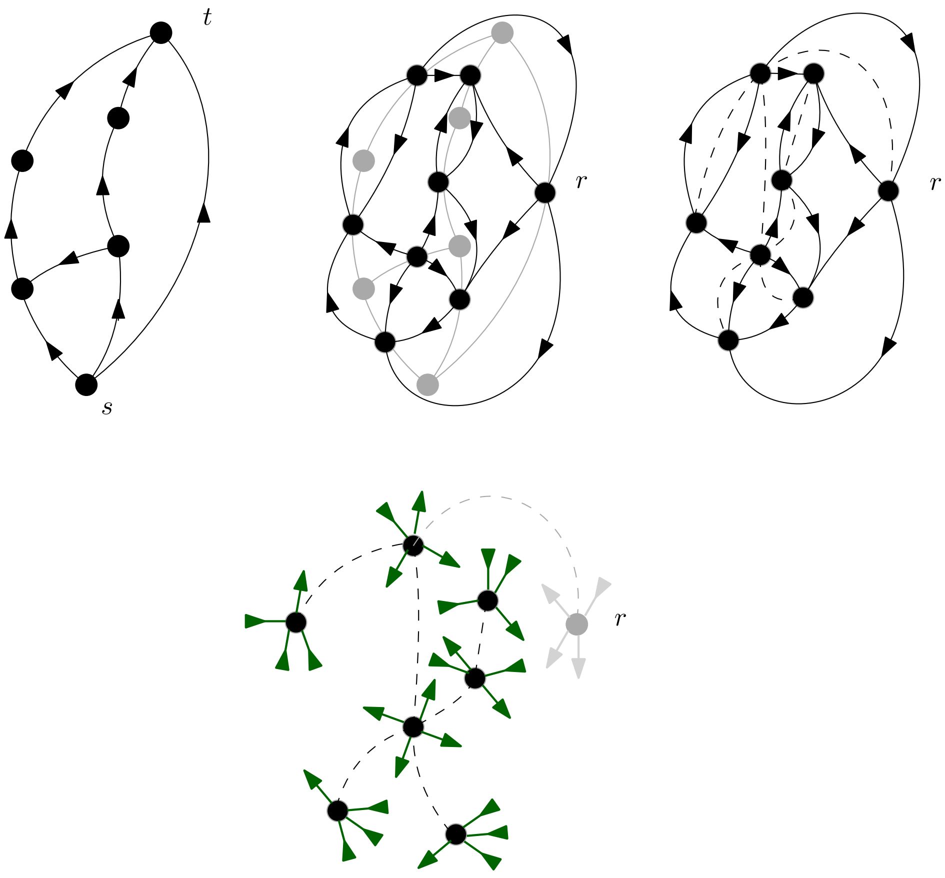


Figure 3.6: Non-separable map M with its minimal bipolar orientation; its radial map $R(M)$ oriented such that the edges corresponding to the two special corners of each vertex and face of M are clockwise; $\overline{R(M)}$; the corresponding blossoming tree.

Chapter 4

Counting self-dual maps

This chapter is dedicated to counting the number of another family of maps, self-dual maps. This number was obtained using generating functions in [7] and we offer a different and new proof using blossoming trees.

4.1 Self-dual maps

A rooted map M is *self-dual* if M and $M^\star$ are isomorphic. In Figure 4.1 we can see an example of such a map, with numbered vertices to see that the map and its dual are isomorphic.

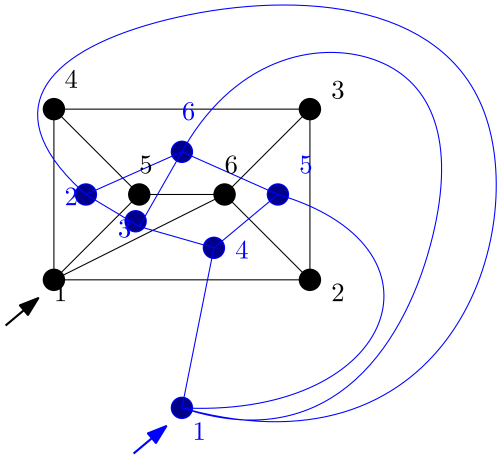


Figure 4.1: A self dual map in black and its dual in blue.

A self-dual map must have the same number of vertices and faces, and thus by Euler's formula it has an even number of edges. The aim of this chapter is to establish a bijection with blossoming trees to prove the following theorem:

Theorem 4.1. *For $n \geq 1$, the number of self-dual maps with $2n$ edges is equal to*

$$\frac{3^n}{n+1}\binom{2n}{n}.$$

4.2 The combinatorial proof

Let M be a map and consider its quadrangulation $Q(M)$ in which we colour the vertices from M in black and the vertices from $M^\star$ in white, and let bw be the root edge of $Q(M)$. Then M is self-dual if and only if the quadrangulation Q' obtained by interchanging the colours of the vertices in $Q(M)$ and changing the root edge to wb is isomorphic to $Q(M)$. We will call these quadrangulations *symmetric*. If $Q(M)$ is the same rooting it at bw or at wb, it is clear picturing the map in the sphere that M is invariant by a rotation of π degrees with the axis of symmetry going through the root edge (and maybe another edge). The quadrangulation and radial map of the self-dual map in Figure 4.1 can be seen in Figure 4.2.

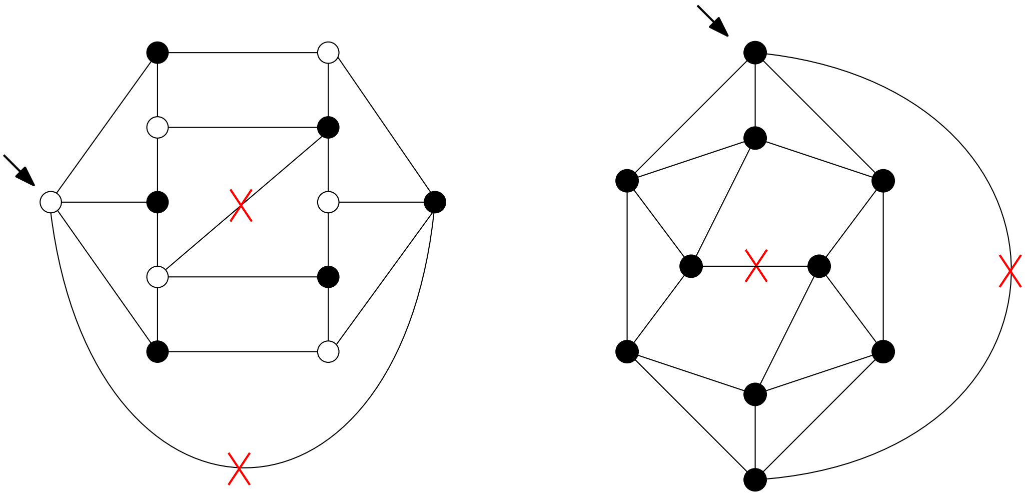


Figure 4.2: $Q(M)$ and $R(M)$, symmetric. The axis of symmetry when drawn in the sphere goes through the red X's.

We once again use that maps are in bijection with their radial maps, and so find a bijection between a family of blossoming trees and tetravalent maps corresponding to self-dual maps. In particular by the previous observation we consider only symmetric tetravalent maps.

The opening of a symmetric radial map (which we orient with the Eulerian orientation) gives a“symmetric” blossoming tree: the tree is symmetric except for the blossom corresponding to the root edge, which is an opening stem in one node of the tree and a closing stem in the symmetric node. Since the radial map had $2n$ vertices, the blossoming tree has $2n-1$ edges, so being symmetric there is an edge that we can consider “in the middle” of the tree. We split the tree in half by this middle edge, and plant both trees in this half edge that comes from the middle edge. The blossom corresponding to the root edge, as said before, is an opening stem in one of the trees; in this tree we turn it into a closing stem. Turning the closing stems into leaves we obtain two copies of a planted binary blossoming tree with n nodes, with an opening stem hanging from each node. We can see this construction in Figure 4.3.

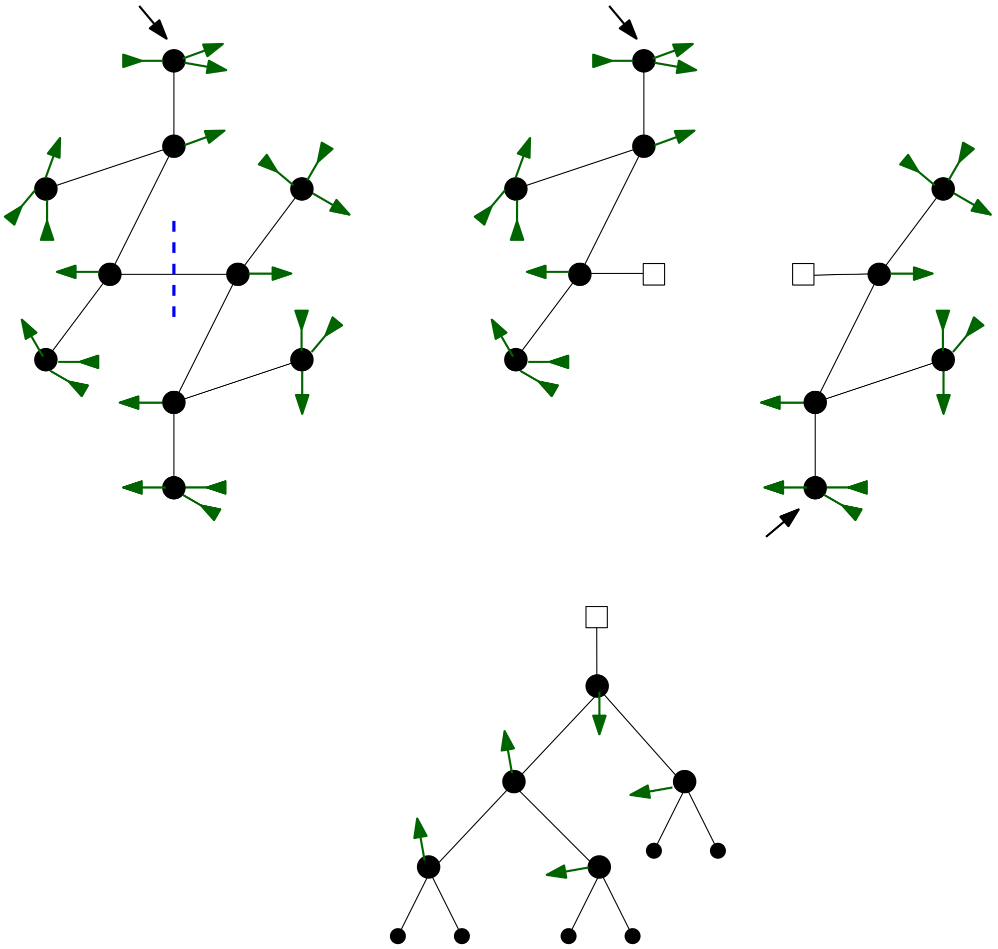


Figure 4.3: The blossoming tree of the symmetric radial map $R(M)$ (from the previous figure);the blossoming trees obtained when splitting it; the resulting planted binary blossoming tree.

For the opposite construction, we consider a planted binary tree with n nodes and an opening stem in each node. We take another copy of this tree, and for each node x of the original we call x' the node in the copy. We join both copies by the square where they are planted, and close the resulting blossoming tree.

We are left with a blossoming map that has two unmatched closing stems. We see that these stems are a x from one tree and its symetric x' from the other. The closure of each copy of the tree, as seen in Chapter 1, gives a blossoming map with two unmatched stems. There are two cases to consider. If one of the unmatched stems is the square $\square$ and the other one is x, in joining the copies of the tree all the local closures remain the same and x and x' are unmatched. Otherwise, if $\square$ is matched in the closure of the original tree and the unmatched stems are x and y where x is the first unmatched stem counterclockwise from $\square$, when we join both trees and do the closure we have that all local closures are the same except for one. The opening stem that closed $\square$ now closes y' and y. And the two unmatched stems that remain are once again x and x'. This is easy to see if we consider the contour word of the tree. This word is $A\bar{b}B$, where A is a sequence of b's and $\bar{b}$'s that represents the opening and closing stems from the root to $\square$, then $\square$ is a closing stem and as such the word has a $\bar{b}$, and finally comes another sequence B that represents the stems from $\square$ to the root. The contour word of the tree we obtain when joining both copies of the tree is then $ABAB$, so that clearly if $\square$ was an unmatched stem all local closures in both symmetric sides of the tree are the same as before, and in the other case the opening stem that was previously matched with $\square$ now it is matched with the first unmatched stem that we encounter when following the tree clockwise.

We join x and x' to make the root edge. Since the resulting radial map is symmetric, we obtain the same map orienting this edge one way or another. We can see this construction in Figure 4.4.

We have seen then that there is a bijection between symetric maps with $2n$ vertices and planted binary blossoming trees with n nodes and an opening stem hanging from each node. Since there are C_n planted binary blossoming trees and in each node we can add its opening stem in any of its 3 corners, we have that the number of self-dual maps with $2n$ edges is:

$$\frac{3^n}{n+1}\binom{2n}{n}$$

And Theorem 4.1 is proved.

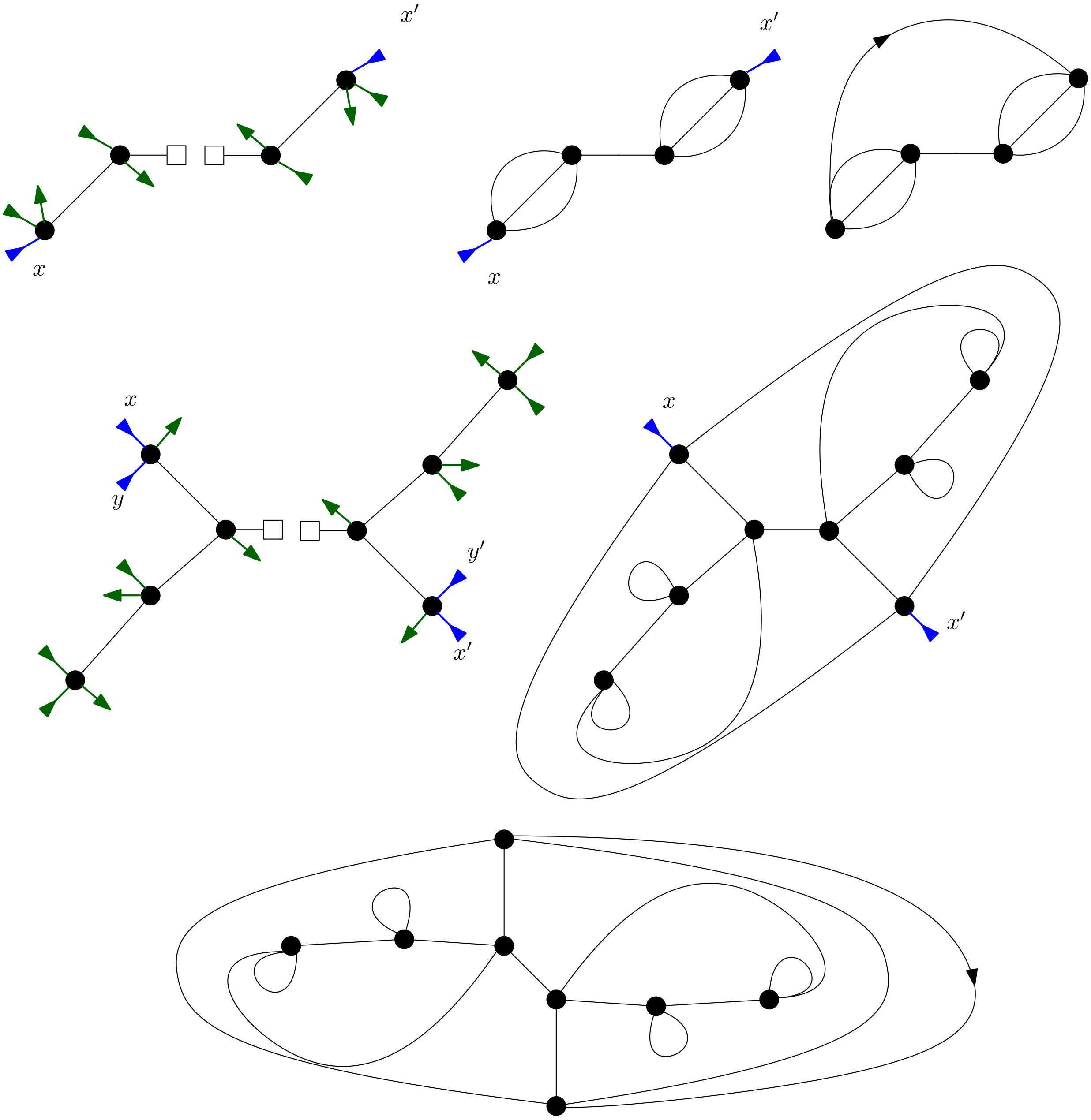


Figure 4.4: Case 1: one of the unmatched stems of the binary tree is □, the other in blue; closure of the symmetric tree; adding the root edge. Case 2: unmatched stems x and y in blue; closure of the symmetric tree; adding the root edge.

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