BACHELOR'S DEGREE THESIS

Degree in Mathematics

Title: **Transformada de Fourier: aplicacions a la resolució d'equacions en derivades parcials**

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Para mis padres. Por haberme apoyado en todo momento, por sus consejos, sus valores, por la motivación constante, por los ejemplos de perseverancia y constancia que me ha infundado siempre pero más que nada, por su amor.

Contents

Introduction

The objective of this thesis is to make a theoretical and formal study of the Fourier Transform and to introduce some of its many applications. In particular, the Fourier Transform will allow us to solve, analyze and understand more two of the most well-known and important Partial Differential Equations: the Heat equation and the Wave equation. Finally, we will introduce and study the most relevant properties of filters. In order to give the most general results and exploit the full potential of the Fourier Transform, we will introduce the distributions, their basic properties and the theory of the Fourier Transform for distributions.

We have divided this bachelor thesis in three parts. In the first part, called "Fourier Transform of functions in L^1 and L^2", we study the Fourier Transform in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, the convolution and all its properties. Moreover, we generalize the results on $\mathbb{R}^d$, $d \in \mathbb{N}$, with the aim of providing the most general results in the second part of the work, "Application of the Fourier transform to the resolution of the Heat Equation and the Wave Equation". Finally, in the last part of the thesis, named "Fourier Transform of distributions: application to filters", we have studied the filters after introducing and stating the Theory of the Fourier Transform for distributions. We have done all this in a total of 12 chapters in which we have studied:

- In chapter one, we have introduced the definition of the Fourier Transform in $L^1(\mathbb{R})$. We have also seen that the Fourier Transform is a linear continuous operator from $L^1(\mathbb{R})$ to $L^\infty(\mathbb{R})$ and we have analyzed its behavior respect some basic operations like the conjugation, the reflexion and the derivation. Finally, we have stated the inverse Fourier Transform in $L^1(\mathbb{R})$ and the principal value Fourier inversion formula.

- The second chapter introduces the Schwartz Space $\mathcal{S}(\mathbb{R})$ which is a dense subspace of $L^p(\mathbb{R})$. We will see that the restriction of the Fourier Transform to this subspace defines a bijective mapping. We will use this property and the density of the Schwartz Space in $L^2(\mathbb{R})$ to extend the Fourier Transform operator to $L^2(\mathbb{R})$ in chapter four. Hence, the Schwartz Space will play a key role in this thesis.

- In chapter three, we define the convolution of two functions and we study it in several spaces. All the results that we introduce in this chapter are also true in the multidimensional case. Convolution will be fundamental when we study the applications of the Fourier Transform to the Heat equation and the Wave equation, since the solutions will be given in terms of the convolution.

- The fourth chapter extends the Fourier Transform and the Inverse Fourier Transform to $L^2(\mathbb{R})$ using the density of the Schwartz Space $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$. Moreover, we state some analogous properties as the ones we saw for the Fourier Transform on $L^1(\mathbb{R})$. Finally, we conclude that the Fourier Transform on $L^1(\mathbb{R})$ and the one obtained by extension to $L^2(\mathbb{R})$ coincide in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$.

- In chapter five, we analyze the behavior of the Fourier Transform respect to the convolution. We will see that it has the remarkable property that it interchanges convolution and multiplication. We will formalize the results in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$.
- In chapter six, we study the multidimensional Fourier Transform in $L^1(\mathbb{R}^d)$, $L^2(\mathbb{R}^d)$ and $\mathcal{S}(\mathbb{R}^d)$. In particular, we prove in $\mathbb{R}^d$ some key results shown in the previous chapters. The aim is to apply these results to the resolution of the Heat Equation and the Wave Equation in $\mathbb{R}^d$.
- The seventh chapter studies the solution and the properties of the Heat Equation. First, we use the Fourier Transform to derive a solution of the homogeneous time-dependent Heat Equation on $\mathbb{R}^d$ and later, we introduce Duhamel's Principle to deduce the inhomogeneous solution. We also show the uniqueness of the solution in several cases using the Energy Method and the Maximum Principle. Finally, we solve the steady-state Heat Equation in the upper half-plane and we state the mean-value formulas for Laplace's equation in order to show uniqueness.
- In chapter eight, we derive the solution of the Wave Equation in terms of Fourier Transform and we see that the energy is conserved in time. However, we will see that this formula is quite indirect and involves the calculation of several Fourier Transforms. Hence, we use other methods to find a more explicit formula and to have a better understanding of the properties of the Wave Equation. First, we will apply these methods in dimension 2 and 3 to derive Poisson's and Kirchhoff's Formula respectively. Later, we will generalize these arguments in the d-dimensional case, distinguishing the case odd and even. Finally, we will be able to study some properties of the Wave Equation such that Huygen's Principle and Finite Speed Propagation.
- The ninth chapter introduces the notion of distribution and all the properties we need to do a further study in the following chapters. We define the space of distributions $\mathcal{D}^*(\mathbb{R})$ and we state the elementary operations, the derivative and the primitive of distributions and the notion of convergence of a sequence of distributions.
- In chapter ten we generalize the concept of Fourier Transform to distributions. To do that, we define the space of tempered distributions $\mathcal{S}^*(\mathbb{R})$, which is a subspace of $\mathcal{D}^*(\mathbb{R})$. We define the Fourier Transform on $\mathcal{S}^*(\mathbb{R})$ and we show that it is a linear, bijective and bicontinuous mapping. Finally, we introduce a particular case of tempered distributions, the subspace $\mathcal{K}^*(\mathbb{R})$ formed by the distributions with compact support.
- The eleventh chapter introduces the notion of convolution for distributions. We study the convolution in several cases using the spaces $\mathcal{K}^*(\mathbb{R})$, $\mathcal{S}^*(\mathbb{R})$ and $\mathcal{D}^*(\mathbb{R})$. Furthermore, we analyze the behavior of the Fourier Transform and the convolution of distributions. Finally, we define the space $\mathcal{D}^*_+(\mathbb{R})$ which is the space of distributions whose supports lie to the right of some finite point. This space will be fundamental in our later study of filters.
- In chapter twelve, we apply the Theory of the Fourier Transform of distributions to study filters and their properties. We introduce the concept of analog filter, impulse response and Transfer function. We use it to study the tempered solutions and the causal solutions of linear differential equations.

I would like to note that we have included an appendix where it can be found concepts and theorems of real analysis that are essential for the development of this thesis. Moreover, we have also included a section of notation. Finally, I would like to apologize beforehand for the possible mistakes this thesis contains.

Part I

Fourier Transform of functions in L^1 and L^2

Chapter 1

The Fourier Transform of integrable functions

1.1 Definition

Definition 1.1. *Let $f \in L^1(\mathbb{R})$. We define its Fourier transform as:*

$$\widehat{f}(\xi) := \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\, dx$$

Note that computing $\xi = 0$ we get $\hat{f}(0) = \int_{\mathbb{R}} f(x)\, dx$, that is the integral of the function f.

Definition 1.2. *Let $f \in L^1(\mathbb{R})$. We define its conjugate Fourier transform as:*

$$\check{f}(\xi) := \int_{\mathbb{R}} e^{2i\pi\xi x} f(x)\, dx$$

We will see later that $\check{f}$ is the inverse of the Fourier transform $\widehat{f}$ whenever $\widehat{f} \in L^1(\mathbb{R})$.

Remark 1.1. *The previous definitions are well-defined, i.e. the (conjugate) Fourier transform of f makes sense if and only if $f \in L^1(\mathbb{R})$. In effect:*

We have that $\hat{f}(\xi)$ and $\check{f}(\xi)$ are well-defined if and only if the integrals $\int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\, dx$ and $\int_{\mathbb{R}} e^{2i\pi\xi x} f(x)\, dx$ are well-defined. Hence, we need that $e^{\pm 2i\pi\xi x} f(x) \in L^1(\mathbb{R})$. But notice that

$$\left| e^{\pm 2i\pi\xi x} f(x) \right| = |f(x)|$$

Thus, we conclude that $e^{\pm 2i\pi\xi x} f(x) \in L^1(\mathbb{R})$ if and only if $f \in L^1(\mathbb{R})$.

1.2 Basic properties

In this section we consider f, h and g integrable functions and $\hat{f}$, $\hat{g}$ and $\hat{h}$ their respective Fourier transforms. We state the following basic properties:

Property 1.1. (Linearity) *Let $a, b \in \mathbb{C}$, if $h(x) = af(x) + bg(x)$ then $\hat{h}(\xi) = a\hat{f}(\xi) + b\widehat{g}(\xi)$.*

PROOF. $\hat{h}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} h(x)\,dx = \int_{\mathbb{R}} e^{-2i\pi\xi x}(af(x) + bg(x))\,dx = a\int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\,dx$ $+ \; b\int_{\mathbb{R}} e^{-2i\pi\xi x} g(x)\,dx = a\hat{f}(\xi) + b\widehat{g}(\xi)$.

Property 1.2. (Translation) *Let* $x_0 \in \mathbb{R}$, *if* $h(x) = f(x - x_0)$ *then* $\hat{h}(\xi) = e^{-i\,2\pi\, x_0\,\xi}\hat{f}(\xi)$.
PROOF. $\hat{h}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} h(x)\,dx = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x - x_0)\,dx$. Making the change of variables $y = x - x_0$ we get $\hat{h}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi(y+x_0)} f(y)\,dy = e^{-2i\pi\xi x_0}\int_{\mathbb{R}} e^{-2i\pi\xi y} f(y)\,dy = e^{-i\,2\pi\,x_0\,\xi}\hat{f}(\xi)$.

Property 1.3. (Modulation) *Let* $\xi_0 \in \mathbb{R}$, *if* $h(x) = e^{i\,2\pi\,x\,\xi_0} f(x)$, *then* $\hat{h}(\xi) = \hat{f}(\xi - \xi_0)$.
PROOF. $\hat{h}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} h(x)\,dx = \int_{\mathbb{R}} e^{-2i\pi\xi x} e^{i\,2\pi\,x\,\xi_0} f(x)dx = \int_{\mathbb{R}} e^{-2i\pi(\xi-\xi_0)x} f(x)dx = \hat{f}(\xi - \xi_0)$.

Property 1.4. (Time Scaling) *Let* $a \in \mathbb{R}\backslash\{0\}$, *if* $h(x) = f(ax)$, *then* $\hat{h}(\xi) = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right)$. *The particular case* $a = -1$ *leads to the time-reversal property, which states: if* $h(x) = f(-x)$, *then* $\hat{h}(\xi) = \hat{f}(-\xi)$.
PROOF. $\hat{h}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} h(x)\,dx = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(ax)dx$. Making the change of variables $y = ax$ we get $\hat{h}(\xi) = \frac{1}{|a|}\int_{\mathbb{R}} e^{-2i\pi\xi\frac{y}{a}} f(y)dy = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right)$.

Property 1.5. (Continuity and boundedness) $\hat{f}(\xi)$ *is continuous and bounded for all* $\xi \in \mathbb{R}$.
PROOF.

- The boundedness of $\hat{f}$ is consequence of $f \in L^1(\mathbb{R})$ and the following estimation:

$$\left|\hat{f}(\xi)\right| = \left|\int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\,dx\right| \leq \int_{\mathbb{R}} \left|e^{-2i\pi\xi x} f(x)\right| dx = \int_{\mathbb{R}} |f(x)|\,dx = \|f\|_1 < \infty \quad \text{for all } \xi \in \mathbb{R}.$$

- For the continuity we apply Theorem A.4.: we define $F(\xi,\, x) := e^{-2i\pi\xi x} f(x)$. We observe that:

1. The function $\xi \longmapsto F(\xi,\, x)$ is continuous for all $\xi \in \mathbb{R}$, for almost all $x \in \mathbb{R}$.
2. F is dominated by an integrable function: $|F(\xi,\, x)| = \left|e^{-2i\pi\xi x} f(x)\right| = \left|e^{-2i\pi\xi x}\right| |f(x)| = |f(x)|$ for almost all $x \in \mathbb{R}$. And by hypothesis f is integrable.

Under these conditions, Theorem A.4. holds and we get $\hat{f}(\xi) = \int_{\mathbb{R}} F(\xi,\, x)dx$ is continuous for all $\xi \in \mathbb{R}$.

Property 1.6. (Interpretation as a Operator) $\widehat{\cdot}$ *is a continuous linear operator from* $L^1(\mathbb{R})$ *to* $L^\infty(\mathbb{R})$ *and* $\left\|\hat{f}\right\|_\infty \leq \|f\|_1$.
PROOF. Let us consider the operator $\widehat{\cdot}: L^1(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$. We should show:

1. $\widehat{\cdot}$ is well-defined, i.e., if $f \in L^1(\mathbb{R})$ then $\hat{f} \in L^\infty(\mathbb{R})$: we have seen in Property 1.5. $\left|\hat{f}(\xi)\right| \leq \|f\|_1 < +\infty$ for all $\xi \in \mathbb{R}$. Thus $\hat{f} \in L^\infty(\mathbb{R})$.
2. $\widehat{\cdot}$ is linear: we have already shown it in property 1.1.
3. $\left\|\hat{f}\right\|_\infty \leq \|f\|_1$: taking supremums in $\left|\hat{f}(\xi)\right| \leq \|f\|_1$, we get immediately the inequality. Note that the inequality shows that the operator is continuous.

Property 1.7. $\lim_{|\xi|\to+\infty}\left|\hat{f}(\xi)\right| = 0.$

PROOF.

• First, we show the result for a characteristic function $f(x) = \mathbf{1}_{[a,\,b]}(x)$:

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x}\mathbf{1}_{[a,\,b]}(x)dx = \int_a^b e^{-2i\pi\xi x}dx = \begin{cases} [x]_a^b & if\ \xi = 0 \\ \left[\frac{e^{-2i\pi\xi x}}{-2i\pi\xi}\right]_a^b & if\ \xi \neq 0 \end{cases} = \begin{cases} b-a & if\ \xi = 0 \\ \frac{ie^{-2i\pi\xi b}-ie^{-2i\pi\xi a}}{2\pi\xi} & if\ \xi \neq 0 \end{cases}$$

Hence, if $\xi \neq 0$: $\left|\hat{f}(\xi)\right| = \left|\frac{ie^{-2i\pi\xi b}-ie^{-2i\pi\xi a}}{2\pi\xi}\right| \leq \frac{\left|ie^{-2i\pi\xi b}\right|+\left|ie^{-2i\pi\xi a}\right|}{2\pi|\xi|} = \frac{1}{\pi|\xi|} \overset{|\xi|\to+\infty}{\longrightarrow} 0.$

• Now, let $f \in L^1(\mathbb{R})$. Using the fact that simple functions are dense in $L^1(\mathbb{R})$, we get that there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of simple functions such that $\lim_{n\to\infty}\|f-\varphi_n\|_1 = 0$.

Moreover, fixing n and applying the result for the simple function φ_n , it holds $\lim_{|\xi|\to+\infty}|\hat{\varphi}_n(\xi)| = 0$.

By property 1.6.:

$$\left|\hat{f}(\xi)-\hat{\varphi}_n(\xi)\right| \leq \|f-\varphi_n\|_1 \implies \lim_{|\xi|\to+\infty}\left|\hat{f}(\xi)-\hat{\varphi}_n(\xi)\right| \leq \lim_{|\xi|\to+\infty}\|f-\varphi_n\|_1 \implies \left|\lim_{|\xi|\to+\infty}\hat{f}(\xi)\right|$$

$$\leq \|f-\varphi_n\|_1 \implies \lim_{n\to\infty}\left|\lim_{|\xi|\to+\infty}\hat{f}(\xi)\right| \leq \lim_{n\to\infty}\|f-\varphi_n\|_1 = 0 \implies \lim_{|\xi|\to+\infty}\hat{f}(\xi) = 0.$$

Property 1.8. *Let f and g be two functions in $L^1(\mathbb{R})$. Then $\widehat{f}g$ and $f\hat{g}$ are in $L^1(\mathbb{R})$ and*

$$\int_{\mathbb{R}} g(x)\widehat{f}(x)dx = \int_{\mathbb{R}} f(t)\hat{g}(t)dt \tag{1.2.1}$$

PROOF.

• To show that $\widehat{f}g$ is in $L^1(\mathbb{R})$ note that $\int_{\mathbb{R}}\left|\hat{f}(x)\,g(x)\right|dx = \int_{\mathbb{R}}\left|\hat{f}(x)\right||g(x)|\,dx \overset{(Prop.1.6.)}{\leq} \int_{\mathbb{R}}\|f\|_1\,|g(x)|\,dx$ $= \|f\|_1\int_{\mathbb{R}}|g(x)|\,dx = \|f\|_1\,\|g\|_1 < +\infty$ (because f and g are in $L^1(\mathbb{R})$). Analogously one shows that $f\hat{g}$ is in $L^1(\mathbb{R})$.

• To show (1.2.1), we will apply Theorem A.6. (Fubini), since $e^{-2i\pi tx}f(t)g(x)$ is integrable in $\mathbb{R}\times\mathbb{R}$:

$$\int_{\mathbb{R}} f(t)\hat{g}(t)dt = \int_{\mathbb{R}} f(t)\left(\int_{\mathbb{R}} e^{-2i\pi tx}g(x)dx\right)dt \overset{T.A.6.}{=} \int_{\mathbb{R}} g(x)\left(\int_{\mathbb{R}} e^{-2i\pi tx}f(t)dt\right)dx = \int_{\mathbb{R}} g(x)\hat{f}(x)dx.$$

1.3 Rules for computing with the Fourier transform

Lemma 1.1. *Let $f \in C^n(\mathbb{R})\cap L^1(\mathbb{R})$, $n \geq 1$, such that the derivatives $f^{(k)} \in L^1(\mathbb{R})$ for $k = 1,..,n$. Then* $\lim_{a\longrightarrow+\infty}\left|f^{(k)}(\pm a)\right| = 0$ for $k = 0,..,n-1$.

PROOF. Let us take $k \in \{0,..,n-1\}$.

• First, we show that $\lim_{a\longrightarrow+\infty}f^{(k)}(\pm a)$ exists and is finite. In effect, by hypothesis $f^{(k+1)}$ is continuous. Hence

$$f^{(k)}(a) = f^{(k)}(0) + \int_0^a f^{(k+1)}(x)dx \implies \lim_{a\longrightarrow+\infty}f^{(k)}(a) = f^{(k)}(0) + \int_0^{+\infty} f^{(k+1)}(x)dx.$$

Then, the existence of $\int_0^{+\infty} f^{(k+1)}(x)dx$ proves that $\lim_{a \longrightarrow +\infty} f^{(k)}(a)$ exists and is finite. Analogously, we see this result for $\lim_{a \longrightarrow +\infty} f^{(k)}(-a)$.

• Now, suppose that $\lim_{a \longrightarrow +\infty} f^{(k)}(a) = K \neq 0$. Then for $0 < \epsilon < |K|$ there exists $M_\epsilon \geq 0$ s.t. $-\left|f^{(k)}(x)\right| + |K| \leq \left|f^{(k)}(x) - K\right| < \epsilon$ for all $x \geq M_\epsilon$. Thus $\left|f^{(k)}(x)\right| > -\epsilon + |K|$ for all $x \geq M_\epsilon$:

$$\int_{\mathbb{R}} \left|f^{(k)}(x)\right| dx = \int_{-\infty}^{M_\epsilon} \left|f^{(k)}(x)\right| dx + \int_{M_\epsilon}^{+\infty} \left|f^{(k)}(x)\right| dx > (-\epsilon + |K|) \int_{M_\epsilon}^{+\infty} dx = +\infty$$

But $f^{(k)} \in L^1(\mathbb{R})$, so we get a contradiction. Thus, we conclude $\lim_{a \longrightarrow +\infty} f^{(k)}(a) = 0$. Analogously, we see $\lim_{a \longrightarrow +\infty} f^{(k)}(-a) = 0$.

Proposition 1.1. (Derivation) *The following statements hold:*

1. *If $x^k f(x)$ is in $L^1(\mathbb{R})$ for $k = 0, 1, .., n$, then $\hat{f}$ is n times differentiable ($\hat{f} \in C^n(\mathbb{R})$) and:*

$$\hat{f}^{(k)}(\xi) = \left[\widehat{(-2i\pi x)^k f(x)}\right](\xi) \quad \textit{for} \;\; k = 1, .., n$$

2. *If $f \in C^n(\mathbb{R}) \cap L^1(\mathbb{R})$ and if all the derivatives $f^{(k)} \in L^1(\mathbb{R})$ for $k = 1, .., n$. Then:*

$$\widehat{f^{(k)}}(\xi) = (2i\pi\xi)^k \hat{f}(\xi) \quad \textit{for} \;\; k = 1, .., n$$

3. *If $f \in L^1(\mathbb{R})$ has bounded support, then $\hat{f} \in C^\infty(\mathbb{R})$.*

PROOF.
• Proof 1: we define $G(\xi, x) := e^{-2i\pi\xi x} f(x)$ and check that its n first derivatives satisfy the assumptions of Theorem A.5.:

1. Note that $\xi \longmapsto G(\xi, x)$ is infinitely times continuously differentiable for almost all $x \in \mathbb{R}$ and for all ξ.
2. Note that $\frac{\partial^k}{\partial \xi^k} G(\xi, x) = (-2i\pi x)^k e^{-2i\pi\xi x} f(x)$. Then $\left|\frac{\partial^k}{\partial \xi^k} G(\xi, x)\right| \leq (2\pi)^k \left|x^k f(x)\right|$ which is integrable. This holds for $k = 1, .., n$.

Therefore, applying the differentiation under the integral sign Theorem

$$\hat{f}^{(k)}(\xi) = \int_{\mathbb{R}} \frac{\partial^k}{\partial \xi^k} G(\xi, x) dx = \int_{\mathbb{R}} (-2i\pi x)^k e^{-2i\pi\xi x} f(x) dx = (-2i\pi)^k \int_{\mathbb{R}} e^{-2i\pi\xi x} x^k f(x) dx$$

$$= (-2i\pi)^k \left[\widehat{x^k f(x)}\right](\xi) \quad \textit{for} \quad k = 1, .., n$$

• Proof 2: note first that $f^{(k)} \in L^1(\mathbb{R})$ implies that the respective Fourier Transforms are well-defined. We prove the result by induction. For $k = 1$:

$$\widehat{f'}(\xi) = \lim_{a \longrightarrow +\infty} \int_{-a}^{+a} e^{-2i\pi\xi x} f'(x) dx \overset{parts}{=} \left[\begin{array}{c} u = e^{-2i\pi\xi x} \longrightarrow du = -2i\pi\xi e^{-2i\pi\xi x} \\ dv = f'(x) \longrightarrow v = f(x) \end{array}\right] = \lim_{a \longrightarrow +\infty} \left[e^{-2i\pi\xi x} f(x)\right]_{-a}^{+a}$$

$$+ \lim_{a \longrightarrow +\infty} (2i\pi\xi) \int_{-a}^{+a} e^{-2i\pi\xi x} f(x) dx \overset{(lemma\,1.1)}{=} (2i\pi\xi) \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) dx = (2i\pi\xi) \hat{f}(\xi)$$

Suppose the result holds for $(k-1) \in \{1, 2, .., n-1\}$. We show that it also holds for k:

$$\widehat{f^{(k)}}(\xi) = \lim_{a \longrightarrow +\infty} \int_{-a}^{+a} e^{-2i\pi\xi x} f^{(k)}(x)dx \overset{parts}{=} \left[\begin{array}{c} u = e^{-2i\pi\xi x} \longrightarrow du = -2i\pi\xi e^{-2i\pi\xi x} \\ dv = f^{(k)}(x) \longrightarrow v = f^{(k-1)}(x) \end{array} \right] =$$

$$\lim_{a \longrightarrow +\infty} \left[e^{-2i\pi\xi x} f^{(k-1)}(x) \right]_{-a}^{+a} + \lim_{a \longrightarrow +\infty} (2i\pi\xi) \int_{-a}^{+a} e^{-2i\pi\xi x} f^{(k-1)}(x)dx \overset{(lemma\,1.1)}{=} (2i\pi\xi) \int_{\mathbb{R}} e^{-2i\pi\xi x} f^{(k-1)}(x)dx$$

$$= (2i\pi\xi)\widehat{f^{(k-1)}}(\xi) \overset{(induc.hyp.)}{=} (2i\pi\xi)^k \hat{f}(\xi)$$

• Proof 3: f has bounded support, i.e., there exists $M > 0$ large enough such that $f(x) = 0$ for $x \notin [-M, M]$. Hence:

$$\int_{\mathbb{R}} \left|x^k f(x)\right| dx = \int_{-M}^{+M} \left|x^k f(x)\right| dx \leq M^k \int_{-M}^{+M} |f(x)|\, dx = M^k \int_{\mathbb{R}} |f(x)|\, dx < +\infty \quad \text{for } k \in \mathbb{N}$$

Consequently, $x^k f(x) \in L^1(\mathbb{R})$ for all $k \in \mathbb{N}$ and we can apply statement 1 to conclude that $\hat{f} \in C^\infty(\mathbb{R})$.

Proposition 1.2. (Conjugation and parity) *Let $f \in L^1(\mathbb{R})$. Then, the following statements hold:*

1. *Conjugation:* $\overline{\hat{f}(-\xi)} = \widehat{\left[\overline{f}\right]}(\xi)$.
2. *Reflexion:* $\hat{f}(-\xi) = \widehat{f_\sigma}(\xi)$.
3. f *even (odd)* $\Longrightarrow$ $\hat{f}$ *even (odd).*
4. f *real and even* $\Longrightarrow$ $\hat{f}$ *real and even.*
5. f *real and odd* $\Longrightarrow$ $\hat{f}$ *purely imaginary and odd.*

PROOF.
• Proof 1: it holds $e^{-2i\pi\xi x} = cos(-2\pi\xi x) + i{\cdot}sin(-2\pi\xi x) = cos(2\pi\xi x) - i{\cdot}sin(2\pi\xi x) = \overline{e^{2i\pi\xi x}}$. Hence:

$$\widehat{\left[\overline{f}\right]}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x}\overline{f(x)}dx = \int_{\mathbb{R}} \overline{e^{2i\pi\xi x} f(x)}dx = \overline{\int_{\mathbb{R}} e^{2i\pi\xi x} f(x)dx} = \overline{\hat{f}(-\xi)}$$

• Proof 2: $\hat{f}(-\xi) = \int_{-\infty}^{+\infty} e^{2i\pi\xi x} f(x)dx \overset{y=-x}{=} -\int_{+\infty}^{-\infty} e^{-2i\pi\xi y} f(-y)dy = \int_{\mathbb{R}} e^{-2i\pi\xi y} f_\sigma(y)dy = \widehat{f_\sigma}(\xi)$

• Proof 3: let us suppose f is even, i.e. $f = f_\sigma$. Applying statement 2: $\hat{f}(-\xi) \overset{2.}{=} \widehat{f_\sigma}(\xi) = \hat{f}(\xi)$, that is $\hat{f}$ is even.
Let us suppose f is odd. i.e $f = -f_\sigma$. Applying statement 2: $\hat{f}(-\xi) \overset{2.}{=} \widehat{f_\sigma}(\xi) = \widehat{[-f]}(\xi) = -\hat{f}(\xi)$, that is $\hat{f}$ is even.

• Proof 4: applying statement 3, it suffices to show that $\hat{f}$ is real: $\overline{\hat{f}(\xi)} \overset{1.}{=} \hat{\overline{f}}(-\xi) \overset{(f\,real)}{=} \hat{f}(-\xi) \overset{(\hat{f}\,even)}{=} \hat{f}(\xi)$. Hence, $\hat{f}$ is real.

• Proof 5: applying statement 3, it suffices to show that $\hat{f}$ is imaginary: $\overline{\hat{f}(\xi)} \overset{1.}{=} \hat{\overline{f}}(-\xi) \overset{(f\,real)}{=} \hat{f}(-\xi) \overset{(\hat{f}\,odd)}{=} -\hat{f}(\xi)$. Hence, $\hat{f}$ is purely imaginary.

Example 1.1. As an example of the utility of the previous properties, we consider the function $f(x) = e^{-ax^2}$, $a > 0$ and we compute its Fourier transform.
If we apply the direct method, i.e. we compute $\widehat{f}(\xi) = \int_{\mathbb{R}} e^{-2\pi xi\xi} f(x)dx = \int_{\mathbb{R}} e^{-2\pi xi\xi - ax^2} dx$, we will need to evaluate a contour integral in the complex plane.

However, there is another way to proceed: we differentiate the function and we obtain the relation

$$f^{'}(x) \;=\; -2axf(x) \;\Longrightarrow\; \widehat{[f^{'}]}(\xi) \;=\; [\widehat{-2axf(x)}](\xi) \;=\; \frac{a}{i\pi}[\widehat{-2i\pi xf(x)}](\xi)$$

$$\overset{Prop.\,1.1.2.\,(left)}{\underset{Prop.\,1.1.1.\,(right)}{\Longrightarrow}} \;(2i\pi\xi)\cdot\widehat{f}(\xi) \;=\; \frac{a}{i\pi}\hat{f}^{'}(\xi)$$

Hence, we obtain an ordinary differential equation of first order: $\hat{f}^{'}(\xi) = -\frac{2\pi^2}{a}\xi\widehat{f}(\xi)$. We can solve it by separation of variables:

$$\frac{d\hat{f}}{d\xi} \;=\; -\frac{2\pi^2}{a}\xi\widehat{f} \;\Longrightarrow\; \frac{d\hat{f}}{\hat{f}} \;=-\frac{2\pi^2}{a}\xi d\xi \;\;\Longrightarrow\; \ln\left(\hat{f}\right) \;=\; -\frac{\pi^2}{a}\xi^2 \,+\, K \;\Longrightarrow\; \hat{f}(\xi) \;=\; K'e^{\frac{-\pi^2}{a}\xi^2}$$

$$K' \;=\; \hat{f}(0) \;=\; \int_{\mathbb{R}} e^{-ax^2}dx \;=\; \sqrt{\frac{\pi}{a}} \;\;\Longrightarrow\;\; \hat{f}(\xi) \;=\; \sqrt{\frac{\pi}{a}}e^{\frac{-\pi^2}{a}\xi^2}$$

1.4 The inverse Fourier Transform

Lemma 1.2. *For $n > 0$, we define $g_n(x) = e^{-\frac{2\pi}{n}|x|}$. Then its Fourier transform is $\hat{g}_n(\xi) = \frac{n}{\pi(1+n^2\xi^2)}$ and it holds that g_n and $\hat{g}_n$ are in $L^1(\mathbb{R})$. Moreover:*

$$\int_{\mathbb{R}} \hat{g}_n(\xi)d\xi \;=\; \int_{\mathbb{R}} \frac{n}{\pi\left(1+n^2\xi^2\right)}d\xi \;=\; \frac{1}{\pi}\left[\arctan(n\xi)\right]_{-\infty}^{+\infty} \;=\; 1$$

Lemma 1.3. *Let $g_n(x) := e^{-\frac{2\pi}{n}|x|}$ and $f \in L^1(\mathbb{R})$. Then $\lim\limits_{n\longrightarrow+\infty} \int_{\mathbb{R}} f(u)\hat{g}_n(u-t)du \;= f(t)$ at all the points where f is continuous.*

PROOF.

As f is continuous in t: for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ s.t if $|y-t| < \delta_\epsilon$ then $|f(y) - f(t)| < \epsilon$.

$$\left|\int_{\mathbb{R}} f(u)\hat{g}_n(u-t)du \,-\, f(t)\right| \underset{Lemma\,1.2}{\overset{u\,=\,\xi+t}{\leq}} \int_{|\xi|\leq\delta_\epsilon} |f(\xi+t)-f(t)|\,|\hat{g}_n(\xi)|\,d\xi \,+\, \int_{|\xi|>\delta_\epsilon} |f(\xi+t)-f(t)|\,|\hat{g}_n(\xi)|\,d\xi \tag{1.4.1}$$

- On the one hand, it holds that

$$\int_{|\xi|\leq\delta_\epsilon} |f(\xi+t)-f(t)|\,|\hat{g}_n(\xi)|\,d\xi \overset{continuity}{\leq} \epsilon\int_{|\xi|\leq\delta_\epsilon} |\hat{g}_n(\xi)|\,d\xi \;\leq\; \epsilon\int_{\mathbb{R}} |\hat{g}_n(\xi)|\,d\xi \overset{Lemma\,1.2}{=} \epsilon.$$

- On the other hand, we have that

$$\int_{|\xi|>\delta_\epsilon} |f(\xi+t)-f(t)|\,|\hat{g}_n(\xi)|\,d\xi \overset{triang.\,ineq.}{\leq} \int_{|\xi|>\delta_\epsilon} |f(\xi+t)\hat{g}_n(\xi)|\,d\xi \,+\, \int_{|\xi|>\delta_\epsilon} |f(t)\hat{g}_n(\xi)|\,d\xi \overset{(1)}{\leq}$$

$$|\hat{g}_n(\delta_\epsilon)|\,\|f\|_1 \,+\, |f(t)|\int_{|\xi|>\delta_\epsilon} |\hat{g}_n(\xi)|\,d\xi \;=\; \left|\frac{n}{\pi\left(1+n^2\delta_\epsilon^2\right)}\right|\|f\|_1 + |f(t)|\,(1 \,-\, \frac{2}{\pi}\arctan(n\delta_\epsilon)) \overset{n\to+\infty}{\longrightarrow} 0.$$

Where in (1) we have used that $\hat{g}_n$ is even and decreasing in $\mathbb{R}_+$.

Taking the limit in the inequality (1.4.1), we get that

$$\lim_{n\longrightarrow+\infty}\left|\int_{\mathbb{R}} f(u)\hat{g}_n(u-t)du \,-\, f(t)\right| \leq \epsilon \quad \forall\epsilon>0 \;\Longrightarrow\; \lim_{n\longrightarrow+\infty}\int_{\mathbb{R}} f(u)\hat{g}_n(u-t)du \;=\; f(t)$$

Theorem 1.1. *Let $f \in L^1(\mathbb{R})$ and let us suppose $\hat{f} \in L^1(\mathbb{R})$. Then $\check{\hat{f}}(t) = f(t)$ at all the points where f is continuous.*

PROOF.

Let $g_n(x) := e^{-\frac{2\pi}{n}|x|}$. By Lemma 1.2, g_n, $\hat{g}_n \in L^1(\mathbb{R})$, so we can apply Property 1.8.:

$$\int_{\mathbb{R}} \hat{f}(x)g_n(x)e^{2\pi itx}dx \overset{Property\,1.8}{=} \int_{\mathbb{R}} f(u)[\widehat{g_n(x)e^{2\pi itx}}](u)du \overset{Property\,1.3.}{=} \int_{\mathbb{R}} f(u)\hat{g}_n(u-t)du \tag{1.4.2}$$

Now, we apply Theorem A.2. (Convergence Dominated Theorem) to the function $\hat{f}(x)g_n(x)e^{2\pi itx}$, since for all x, it holds that $\left|\hat{f}(x)g_n(x)e^{2\pi itx}\right| \leq \left|\hat{f}(x)\right|$ (which is integrable by hypothesis). Then:

$$\lim_{n\longrightarrow+\infty}\int_{\mathbb{R}} \hat{f}(x)g_n(x)e^{2\pi itx}dx \;=\; \int_{\mathbb{R}} \lim_{n\longrightarrow+\infty}\hat{f}(x)g_n(x)e^{2\pi itx}dx \;=\; \int_{\mathbb{R}} \hat{f}(x)e^{2\pi itx}dx \;=\; \check{\hat{f}}(t)$$

Finally, taking the limit in (1.4.2), and using Lemma 1.3 in the right hand side, we conclude:

$$\check{\hat{f}}(t) \;=\; f(t)$$

Corollary 1.2. *Let $f \in L^1(\mathbb{R})$ be a continuous function except for a jump discontinuity at $x = a$. Then $\hat{f} \notin L^1(\mathbb{R})$.*

PROOF.

Let us suppose $\hat{f} \in L^1(\mathbb{R})$. Then, theorem 1.1. implies $\check{\hat{f}}(x) = f(x)$ except for $x = a$. And since $f(x)$ has a jump of discontinuity at $x = a$. It holds

$$\lim_{x\longrightarrow a^-}\check{\hat{f}}(x) \;=\; \lim_{x\longrightarrow a^-}f(x) \neq \lim_{x\longrightarrow a^+}f(x) \;=\; \lim_{x\longrightarrow a^+}\check{\hat{f}}(x) \tag{1.4.3}$$

However, property 1.5. implies $\check{\hat{f}}(x)$ is continuous everywhere. This is a contradiction with the previous fact (1.4.3).

Remark 1.2. $f \in L^1(\mathbb{R}) \nRightarrow \hat{f} \in L^1(\mathbb{R})$. *In effect, we consider the characteristic function $f(x) = \mathbf{1}_{[a,\,b]}(x) \in L^1(\mathbb{R})$. Its Fourier Transform is not in $L^1(\mathbb{R})$:*

$$\hat{f}(\xi) \;=\; \begin{cases} b-a & if\,\xi = 0 \\ \frac{ie^{-2i\pi\xi b}-ie^{-2i\pi\xi a}}{2\pi\xi} & if\,\xi \neq 0 \end{cases} \;\Longrightarrow\; \int_{\mathbb{R}}\left|\hat{f}(\xi)\right|d\xi \;=\; 2\int_0^{\infty}\left|\frac{\sin(\pi\,(b-a)\,\xi)}{\pi\xi}\right|d\xi \;=\; +\infty$$

Proposition 1.3. *If $f \in C^2(\mathbb{R})$ and f, f', $f'' \in L^1(\mathbb{R})$, then $\hat{f} \in L^1(\mathbb{R})$.*

PROOF.

Under the hypothesis of the statement, we can apply Proposition 1.1.2 to see that $\widehat{f''}(\xi) = -4\pi^2\xi^2\hat{f}(\xi)$. Moreover, as $f'' \in L^1(\mathbb{R})$, by property 1.7.: $\lim_{|\xi|\longrightarrow+\infty}\left|\widehat{f''}(\xi)\right| = 0$.

Consequently $\lim\limits_{|\xi|\longrightarrow+\infty}\left|4\pi^2\xi^2\hat{f}(\xi)\right| = 0$. Then for $\epsilon = 1$, there exists $K > 0$ such that for all $|\xi| > K$: $\left|4\pi^2\xi^2\hat{f}(\xi)\right| < 1$. Furthermore, as $\hat{f}$ is continuous in the compact $|\xi| \leq K$, there exists $A > 0$ such that $\left|\hat{f}(\xi)\right| \leq A$ for all $|\xi| \leq K$. Thus

$$\int_{\mathbb{R}}\left|\hat{f}(\xi)\right|d\xi \leq \int_{|\xi|\leq K}\left|\hat{f}(\xi)\right|d\xi + \int_{|\xi|>K}\frac{1}{4\pi^2\xi^2}d\xi \leq 2KA + \frac{1}{2\pi^2}\left[\frac{-1}{\xi}\right]_K^{\infty} = 2KA + \frac{1}{2\pi^2 K} < +\infty$$

Hence, $\hat{f} \in L^1(\mathbb{R})$.

Proposition 1.4. *If $f \in C(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\hat{f} \in L^1(\mathbb{R})$, then $\hat{\hat{f}}(x) = f_\sigma(x)$.*

PROOF.

$$\hat{\hat{f}}(x) = \int_{\mathbb{R}} e^{-2i\pi x\xi}\hat{f}(\xi)d\xi \overset{Proposition\,1.2.2}{=} \int_{\mathbb{R}} e^{2i\pi x\xi}\widehat{f_\sigma}(\xi)d\xi = \check{\widehat{f_\sigma}}(x) \overset{Theorem\,1.1.}{=} f_\sigma(x)$$

1.5 The principal value Fourier inversion formula

If $\hat{f} \notin L^1(\mathbb{R})$ then the integral $\int_{\mathbb{R}} e^{2i\pi t\xi}\hat{f}(\xi)d\xi$ is not defined. However, as we will see, this does not exclude the possible existence of:

$$\lim_{a\longrightarrow+\infty}\int_{-a}^{+a} e^{2i\pi t\xi}\hat{f}(\xi)d\xi$$

Lemma 1.4. *Let us consider the function $\frac{\sin(x)}{x}$. It has the following properties:*

1. $\int_0^{+\infty}\frac{\sin(x)}{x}dx = \frac{\pi}{2}$
2. *$s(y) := \int_y^{+\infty}\frac{\sin(x)}{x}dx$ is well-defined and differentiable on $\mathbb{R}$. Moreover, $s'(y) = -\frac{\sin(y)}{y}$ and $\lim\limits_{y\longrightarrow+\infty} s(y) = 0$.*
3. *There exists $M > 0$ such that $M = \sup\limits_{y\geq 0}|s(y)|$.*

PROOF.

• Proof 1: the application of Fubini's Theorem gives us the result:

$$\int_0^{+\infty}\frac{\sin(x)}{x}dx = \int_0^{+\infty}\left(\int_0^{+\infty} e^{-xt}\sin(x)dt\right)dx \overset{Theorem\,A.6}{=} \int_0^{+\infty}\left(\int_0^{+\infty} e^{-xt}\sin(x)dx\right)dt =$$

$$\int_0^{+\infty}\left[\frac{-e^{-xt}\cos(x)}{t^2+1} - \frac{te^{-xt}\sin(x)}{t^2+1}\right]_0^{+\infty}dt = \int_0^{+\infty}\frac{1}{t^2+1}dt = [\arctan(t)]_0^{+\infty} = \frac{\pi}{2}$$

• Proof 2: we have $s(y)$ is well-defined. In effect, as $\frac{\sin(x)}{x}$ is continuous on $[y, 0]$ then there exists $K > 0$ s.t $\frac{\sin(x)}{x} \leq K$ on $[y, 0]$. Then

$$s(y) = \int_0^{+\infty}\frac{\sin(x)}{x}dx + \int_y^0\frac{\sin(x)}{x}dx \leq \frac{\pi}{2} + \int_y^0 Kdx = \frac{\pi}{2} - Ky < +\infty$$

What is more, $\frac{\sin(x)}{x} \in C(\mathbb{R}) \implies s \in C^1(\mathbb{R})$. And

$$s^{'}(y) = \frac{d}{dy}\int_y^{+\infty} \frac{\sin(x)}{x}dx = -\frac{\sin(y)}{y} \quad \text{and} \quad \lim_{y\longrightarrow+\infty} s(y) = \lim_{y\longrightarrow+\infty}\int_y^{+\infty}\frac{\sin(x)}{x}dx = 0$$

• Proof 3: since $\lim_{y\longrightarrow+\infty} s(y) = 0$, then for $\epsilon = 1$ there exists $M_1 > 0$ s.t. $|s(y)| < 1$ for all $y > M_1$. Moreover, since s is continuous, then there exists $K > 0$ s.t $|s(y)| \leq K$ on $[0, M_1]$. Hence, we conclude $|s(y)| \leq \max\{K, 1\}$ $\forall y \in \mathbb{R}^+$ and the supremum exists.

Theorem 1.2. *Let $f \in L^1(\mathbb{R})$ such that $f^{'} \in L^1(\mathbb{R})$. Assume that there exists a finite number of real numbers $a_1, a_2, ..., a_p$ such that f is continuously differentiable on $(-\infty, a_1), ..., (a_p, +\infty)$. Let $f(t^+) := \lim_{x\longrightarrow t^+} f(x)$ and $f(t^-) := \lim_{x\longrightarrow t^-} f(x)$. Then*

$$\lim_{a\longrightarrow+\infty}\int_{-a}^{+a} e^{2i\pi t\xi}\hat{f}(\xi)d\xi = \frac{1}{2}\left(f(t^+) + f(t^-)\right)$$

PROOF.

Note first that the hypothesis imply that the limits $f(t^+)$ and $f(t^-)$ exist for all t. Let $g(\xi) := e^{2i\pi t\xi}\mathbf{1}_{[-a,\, a]}(\xi)$. The properties of the Fourier Transform imply that: $\hat{g}(x) = \frac{\sin(2\pi a(x-t))}{\pi(x-t)}$. Since $f, g \in L^1(\mathbb{R})$, it follows from Property 1.8.

$$v(a) := \int_{-a}^{+a} e^{2i\pi t\xi}\hat{f}(\xi)d\xi = \int_{\mathbb{R}} g(\xi)\hat{f}(\xi)d\xi = \int_{\mathbb{R}}\hat{g}(x)f(x)dx = \int_{\mathbb{R}}\frac{\sin(2\pi a(x-t))}{\pi(x-t)}f(x)dx =$$

$$\overset{x=t+u}{=} \int_{\mathbb{R}}\frac{\sin(2\pi au)}{\pi u}f(t+u)du = \int_0^{+\infty}\frac{\sin(2\pi au)}{\pi u}f(t+u)du + \int_0^{+\infty}\frac{\sin(2\pi au)}{\pi u}f(t-u)du =$$

$$= \int_0^{+\infty}\frac{\sin(2\pi au)}{\pi u}\left(f(t+u) + f(t-u)\right)du$$

Let us define $h_t(u) := f(t+u) + f(t-u)$. By the properties of f, we conclude that this function is integrable in $[0, +\infty)$, has at most a finite number of discontinuities $b_1, b_2, ..., b_q$, is continuously differentiable on $(0, b_1)$, (b_1, b_2), ..., $(b_q, +\infty)$ (it can happen $b_1 = 0$) and $h_t^{'}$ is integrable. Let us take $b_0 := 0$ and $b_{q+1} := +\infty$. Thus, using the notation of Lemma 1.4.

$$v(a) = \int_0^{+\infty}\frac{\sin(2\pi au)}{\pi u}\left(f(t+u) + f(t-u)\right)du = -2a\int_0^{+\infty} s^{'}(2\pi au)h_t(u)du =$$

$$= -2a\sum_{j=0}^{q}\int_{b_j}^{b_{j+1}} s^{'}(2\pi au)h_t(u)du \overset{parts}{=} \begin{bmatrix} u = h_t(u) \longrightarrow du = h_t^{'}(u) \\ dv = s^{'}(2\pi au) \longrightarrow v = \frac{s(2\pi au)}{2\pi a}\end{bmatrix} =$$

$$= -\frac{1}{\pi}\sum_{j=0}^{q}\left[s(2\pi ab_{j+1})h_t(b_{j+1}^-) - s(2\pi ab_j)h_t(b_j^+) - \int_{b_j}^{b_{j+1}} s(2\pi au)h_t^{'}(u)du\right] \tag{1.5.1}$$

Now, we apply the limit when $a \longrightarrow +\infty$ to the different terms of the previous sum:

1. For the terms $s(2\pi ab_{j+1})h_t(b_{j+1}^-) - s(2\pi ab_j)h_t(b_j^+)$ with $j = 1...q$: as $b_{j+1} \neq 0$ and $b_j \neq 0$, then we can apply the Lemma 1.4.2 and we have

$$\lim_{a\longrightarrow+\infty} s(2\pi ab_{j+1}) = \lim_{a\longrightarrow+\infty} s(2\pi ab_j) = 0 \implies \lim_{a\longrightarrow+\infty}\left[s(2\pi ab_{j+1})h_t(b_{j+1}^-) - s(2\pi ab_j)h_t(b_j^+)\right] = 0$$

2. The term $s(2\pi ab_1)h_t(b_1^-)$ has limit 0 for the same reason as before.

3. For the terms $\int_{b_j}^{b_{j+1}} s(2\pi au)h_t^{'}(u)du$ with $j = 0...q$: we apply theorem A.2. (Dominated Convergence Theorem) since $\lim_{a\longrightarrow+\infty} s(2\pi au)h_t^{'}(u) = 0$ (a.e.), $\left|s(2\pi au)h_t^{'}(u)\right| \overset{Lemma\,1.4.3}{\leq} M\left|h_t^{'}(u)\right|$ and $h_t^{'}(u) \in L^1(\mathbb{R})$. Hence

$$\lim_{a\longrightarrow+\infty}\int_{b_j}^{b_{j+1}} s(2\pi au)h_t^{'}(u)du = \int_{b_j}^{b_{j+1}} \lim_{a\longrightarrow+\infty} s(2\pi au)h_t^{'}(u)du = 0 \quad \text{for } j = 0...q$$

Hence, the remaining term in (1.5.1) is $\frac{1}{\pi}s(0)h_t(0^+) \overset{Lemma\,1.4.1}{=} \frac{1}{\pi}\frac{\pi}{2}\left(f\left(t^+\right)+f(t^-)\right)$. We conclude:

$$\lim_{a\longrightarrow+\infty}\int_{-a}^{+a} e^{2i\pi t\xi}\hat{f}(\xi)d\xi = \lim_{a\longrightarrow+\infty} v(a) = \frac{1}{2}\left(f(t^+)+f(t^-)\right)$$

Corollary 1.3. *Under the conditions of the previous theorem, if t is a point of continuity of f then*:

$$\lim_{a\longrightarrow+\infty}\int_{-a}^{+a} e^{2i\pi t\xi}\hat{f}(\xi)d\xi = f(t)$$

Example 1.2. *Let $f(x) = \pi\mathbf{1}_{\left[-\frac{1}{2\pi},\frac{1}{2\pi}\right]}(\xi)$. We saw that $\hat{f}(\xi) = \frac{\sin(\xi)}{\xi}$. We see $f,\ f^{'} \in L^1(\mathbb{R})$ and f is continuous on $\left(-\infty,-\frac{1}{2\pi}\right)$, $\left(-\frac{1}{2\pi},\frac{1}{2\pi}\right)$, $\left(\frac{1}{2\pi},+\infty\right)$. Thus, theorem 1.2. says:*

$$\lim_{a\longrightarrow+\infty}\int_{-a}^{+a} e^{2i\pi t\xi}\hat{f}(\xi)d\xi = \begin{cases}\pi & if\ |t| < \frac{1}{2\pi}\\ \frac{\pi}{2} & if\ |t| = \frac{1}{2\pi}\\ 0 & if\ |t| > \frac{1}{2\pi}\end{cases}$$

Chapter 2

The Schwartz Space

2.1 Rapidly decreasing functions

Definition 2.1. *A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to decay rapidly, or be rapidly decreasing, if*

$$\lim_{|x|\to+\infty} |x^p f(x)| = 0 \qquad \textit{for all} \;\; p \in \mathbb{N}$$

It is important to note that in spite of the name, this definition does not imply that the function is monotonic in a neighborhood of infinity ($f(x) = e^{-|x|}\sin(x)$).

Proposition 2.1. *If f is locally integrable, $f \in L^1_{loc}(\mathbb{R})$, and rapidly decreasing, then $x^p f(x) \in L^1(\mathbb{R})$ for all $p \in \mathbb{N}$.*

PROOF.

Since f decays rapidly, $\lim_{|x|\to+\infty} \left|x^{p+2} f(x)\right| = 0 \Longrightarrow$ for $\epsilon = 1$, there exists $M > 0$ such that for all $|x| > M$ we have $\left|x^{p+2} f(x)\right| < 1$. Hence:

$$\int_{\mathbb{R}} |x^p f(x)|\, dx \le \int_{|x|\le M} |x^p f(x)|\, dx + \int_{|x|>M} \frac{1}{x^2} \left|x^{p+2} f(x)\right| dx \le M^p \int_{|x|\le M} |f(x)|\, dx$$

$$+ \int_{|x|>M} \frac{1}{x^2} dx = M^p \int_{|x|\le M} |f(x)|\, dx + \frac{2}{M} < +\infty$$

Where we have used that $\int_{|x|\le M} |f(x)|\, dx < +\infty$ because $f \in L^1_{loc}(\mathbb{R})$. Thus, $x^p f(x) \in L^1(\mathbb{R})$.

Corollary 2.1. *If $f \in L^1(\mathbb{R})$ decays rapidly, then $\hat{f} \in C^\infty(\mathbb{R})$.*

PROOF.

By the previous proposition, it holds $x^p f(x) \in L^1(\mathbb{R})$ for all $p \in \mathbb{N}$. This implies by Proposition 1.1.1 that $\hat{f}$ is infinitely times differentiable.

Proposition 2.2. *Assume that f is in $C^\infty(\mathbb{R})$. If $f^{(k)} \in L^1(\mathbb{R})$ for all $k \in \mathbb{N}$, then $\hat{f}$ decays rapidly.*

PROOF.

Proposition 1.1.2 implies that $\widehat{f^{(k)}}(\xi) = (2i\pi\xi)^k \hat{f}(\xi) \quad \textit{for} \;\; k \in \mathbb{N}$. And by Property 1.7.

$$\lim_{|\xi|\longrightarrow+\infty} \left|\widehat{\left[f^{(k)}\right]}(\xi)\right| = \lim_{|\xi|\longrightarrow+\infty} 2\pi \left|\xi^k \hat{f}(\xi)\right| = 0 \; \textit{for} \;\; k \in \mathbb{N}$$

Hence, $\lim\limits_{|\xi|\longrightarrow+\infty}\left|\xi^k\hat{f}(\xi)\right| = 0$ *for* $k \in \mathbb{N}$ and we conclude $\hat{f}$ decays rapidly.

Remark 2.1. *We can conclude from the previous propositions:*

1. *The faster f decreases at infinity, the greater the regularity of $\hat{f}$.*
2. *The more regular f is, the faster $\hat{f}$ decays.*

In particular, if $f \in C^{\infty}(\mathbb{R})$ and decreases rapidly, then the same is true for $\hat{f}$.

2.2 The space $\mathcal{S}(\mathbb{R})$ (Schwartz Space)

Definition 2.2. *We define the space $\mathcal{S}(\mathbb{R})$, also known as Schwartz Space, as the set of functions $f : \mathbb{R} \longrightarrow \mathbb{C}$ such that f is infinitely times differentiable and f and all of its derivatives decay rapidly. It holds that $\mathcal{S}(\mathbb{R})$ is a vector space.*

Proposition 2.3. *The space $\mathcal{S}(\mathbb{R})$ has the following properties:*

1. *$\mathcal{S}(\mathbb{R})$ is invariant under multiplication by a polynomial.*
2. *$\mathcal{S}(\mathbb{R})$ is invariant under derivation.*
3. $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$

PROOF.

• Proof 1: let $p \in \mathbb{C}[X]$ (polynomial with complex coefficients) and $f \in \mathcal{S}(\mathbb{R})$. We want to show that $pf \in \mathcal{S}(\mathbb{R})$. It is clear that $pf \in C^{\infty}(\mathbb{R})$ since f and $p \in C^{\infty}(\mathbb{R})$. It remains to see pf and all its derivatives decay rapidly. Let us suppose $p(x) = a_n x^n + ... + a_1 x + a_0$ where $a_0, a_1, ..., a_n \in \mathbb{C}$. Then $p(x)f(x) = a_n x^n f(x) + ... + a_1 x f(x) + a_0 f(x)$ and hence

$$\lim_{|x|\to+\infty} |x^p p(x)f(x)| \leq \lim_{|x|\to+\infty} \left(|a_n|\left|x^{n+p}f(x)\right| + ... + |a_1|\left|x^{p+1}f(x)\right| + |a_0|\,|x^p f(x)|\right) = 0 \text{ for all } p \in \mathbb{N}$$

For the derivatives of pf the argument will be same. But in these cases, there will be products of polynomials and derivatives of f. However, the limit will be 0 due to $f \in \mathcal{S}(\mathbb{R})$.

• Proof 2: let $f \in \mathcal{S}(\mathbb{R})$. We want to show $f' \in \mathcal{S}(\mathbb{R})$. As $f \in C^{\infty}(\mathbb{R})$ then $f' \in C^{\infty}(\mathbb{R})$. And as all the derivatives of f decay rapidly, then all the derivatives of f' decay rapidly.

• Proof 3: let $f \in \mathcal{S}(\mathbb{R})$. In consequence, f decay rapidly and is $C^{\infty}(\mathbb{R})$ (and hence, is $L^1_{loc}(\mathbb{R})$). Therefore, we can apply Proposition 2.1 and conclude that $f \in L^1(\mathbb{R})$. Thus, $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$.

Theorem 2.1. *The space $\mathcal{S}(\mathbb{R})$ is invariant under the Fourier Transform. That is, if $f \in \mathcal{S}(\mathbb{R})$ then $\hat{f} \in \mathcal{S}(\mathbb{R})$.*

PROOF.

• First, we show that $\hat{f} \in C^{\infty}(\mathbb{R})$. In effect, as $f \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ and decays rapidly, we conclude by Corollary 2.1. that $\hat{f} \in C^{\infty}(\mathbb{R})$.

• Next, we prove that $\hat{f}$ is rapidly decreasing: as $f \in \mathcal{S}(\mathbb{R})$ then by Proposition 2.3. $f^{(k)} \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ for all $k \in \mathbb{N}$. Moreover $f \in C^{\infty}(\mathbb{R})$. Hence we can apply Proposition 2.2. and conclude that $\hat{f}$ is rapidly decreasing.

• Finally, we shall show that all the derivatives of $\hat{f}$ decays rapidly. Let $q \geq 0$ and let $g(x) := (-2i\pi)^q x^q f(x)$. As $f \in \mathcal{S}(\mathbb{R})$, we can apply Proposition 2.3.1 and conclude $g \in \mathcal{S}(\mathbb{R})$. Moreover, by Proposition 2.3.2 $g^{(p)} \in \mathcal{S}(\mathbb{R})$ for $p \geq 0$. Therefore, we can apply Proposition 1.1.2 to the function $g^{(p)}$

$$\widehat{g^{(p)}}(\xi) \overset{Prop. 1.1.2}{=} (2i\pi\xi)^p \hat{g}(\xi) \implies \left(\frac{1}{2\pi i}\right)^p \left[\widehat{((-2i\pi x)^q f(x))^{(p)}}\right](\xi) = \xi^p [\widehat{(-2i\pi x)^q f(x)}](\xi) \quad (2.2.1)$$

Moreover, as $x^q f(x) \in L^1(\mathbb{R})$ for all $q \geq 0$, we can apply Proposition 1.1.1 on the right-hand side of (2.2.1) and we get:

$$\left(\frac{1}{2\pi i}\right)^p \left[\widehat{((-2i\pi x)^q f(x))^{(p)}}\right](\xi) = \xi^p \hat{f}^{(q)}(\xi) \implies \lim_{|\xi| \to +\infty} \left|\xi^p \hat{f}^{(q)}(\xi)\right| =$$

$$= \left(\frac{1}{2\pi i}\right)^p \lim_{|\xi| \to +\infty} \left| \left[\widehat{((-2i\pi x)^q f(x))^{(p)}}\right](\xi)\right| \overset{Property\, 1.7.}{=} 0$$

Definition 2.3. *We will say that a sequence $(f_n)_{n \in \mathbb{N}}$ of elements in $\mathcal{S}(\mathbb{R})$ tends to 0 as n tends to infinity, and we will write $f_n \longrightarrow 0$ in $\mathcal{S}(\mathbb{R})$, if*

$$\lim_{n \longrightarrow +\infty} \sup_{x \in \mathbb{R}} \left| x^p f_n^{(q)}(x) \right| = 0 \qquad \textit{for all} \quad p, q \in \mathbb{N}$$

Remark 2.2. *Note that if $f_n \longrightarrow 0$ in $\mathcal{S}(\mathbb{R})$, then $\left(f_n^{(q)}\right)_{n \in \mathbb{N}}$ converges uniformly on $\mathbb{R}$ to 0 for all $q \in \mathbb{N}$. In effect, taking $p = 0$ in the definition 2.3., we get that*

$$\lim_{n \longrightarrow +\infty} \sup_{x \in \mathbb{R}} \left| f_n^{(q)}(x) \right| = 0 \qquad \text{for all} \quad q \in \mathbb{N}$$

Definition 2.4. *We will say that a sequence $(f_n)_{n \in \mathbb{N}}$ of elements in $\mathcal{S}(\mathbb{R})$ tends to f in $\mathcal{S}(\mathbb{R})$ as n tends to infinity if $f_n - f \longrightarrow 0$ in $\mathcal{S}(\mathbb{R})$.*

Proposition 2.4. *Let $(f_n)_{n \in \mathbb{N}}$ be a sequence of elements in $\mathcal{S}(\mathbb{R})$. Let us suppose $f_n \longrightarrow 0$ in $\mathcal{S}(\mathbb{R})$. Then, the following statements hold*

1. $f_n' \longrightarrow 0$ *in $\mathcal{S}(\mathbb{R})$ (continuity of derivation).*
2. $P f_n \longrightarrow 0$ *in $\mathcal{S}(\mathbb{R})$ for all polynomials P with complex coefficients.*

3. $f_n \longrightarrow 0$ *in* $L^1(\mathbb{R})$.
4. $\widehat{f_n} \longrightarrow 0$ *in* $\mathcal{S}(\mathbb{R})$ *(continuity of the Fourier transform).*

PROOF.

• Proof 1: it is consequence of the definition. In effect, for all p and q in $\mathbb{N}$:

$$\lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}} \left| x^p \left(f_n^{'}\right)^{(q)}(x)\right| = \lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}} \left| x^p f_n^{(q+1)}(x)\right| = 0 \implies f_n^{'} \longrightarrow 0 \text{ in } \mathcal{S}(\mathbb{R})$$

• Proof 2: it is sufficient to prove this for $P(x) = x^k$, $k \in \mathbb{N}$, i.e., we should prove

$$\lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}} \left| x^p \left(x^k f_n(x)\right)^{(q)}\right| = 0 \quad \text{for all} \quad p, q \in \mathbb{N}$$

But this follows immediately from the Leibniz's formula for the derivatives of a product. Hence, if we have a general polynomial $P(x) = \sum_{i=0}^{m} a_i x^i$. It holds

$$\left| x^p \left((Pf_n)(x)\right)^{(q)}\right| \leq \sum_{i=0}^{m} |a_i| \left| x^p \left(x^i f_n(x)\right)^{(q)}\right| \implies \lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}} \left| x^p \left((Pf_n)(x)\right)^{(q)}\right| \leq$$

$$\leq \sum_{i=0}^{m} \left(|a_i| \lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}} \left| x^p \left(x^i f_n(x)\right)^{(q)}\right| \right) = 0 \quad \text{for all} \quad p, q \in \mathbb{N}$$

• Proof 3: let us consider $P(x) = (1+x^2)$. It holds by 2 that $Pf_n \longrightarrow 0$ in $\mathcal{S}(\mathbb{R})$. In particular, it holds (for $p = q = 0$) that $\lim_{n\longrightarrow+\infty} \sup_{x\in\mathbb{R}} \left|(1+x^2)f_n(x)\right| = 0$. Then for $\epsilon > 0$, there exists $N > 0$ such that for all $n \geq N$ and $x \in \mathbb{R}$, $\left|(1+x^2)f_n(x)\right| < \epsilon$. Thus, for all $n \geq N$:

$$\int_{\mathbb{R}} |f_n(x)|\, dx \leq \int_{\mathbb{R}} \frac{\epsilon}{1+x^2} dx = \epsilon\pi \implies f_n \longrightarrow 0 \text{ in } L^1(\mathbb{R})$$

• Proof 4: we use (2.2.1) from Theorem 2.1.: $\left|\xi^p \hat{f_n}^{(q)}(\xi)\right| = \left(\frac{1}{2\pi}\right)^{p-q} \left| \left[\widehat{(x^q f_n(x))^{(p)}}\right](\xi)\right|$. Let us define now $g_n(x) = (x^q f_n(x))^{(p)}$. By Proposition 2.3. we get that $g_n \in \mathcal{S}(\mathbb{R})$ and by 1. and 2. we get that $g_n \longrightarrow 0$ in $\mathcal{S}(\mathbb{R})$. Now, applying 3. we get $\|g_n\|_1 \longrightarrow 0$. Finally, using Property 1.6.

$$\left|\xi^p \hat{f_n}^{(q)}(\xi)\right| = \left(\frac{1}{2\pi}\right)^{p-q} |\hat{g}_n(\xi)| \leq \left(\frac{1}{2\pi}\right)^{p-q} \|\hat{g}_n\|_\infty \overset{\text{Property 1.6.}}{\leq} \left(\frac{1}{2\pi}\right)^{p-q} \|g_n\|_1 \longrightarrow 0$$

for all $p, q \geq 0$. Hence we conclude $\widehat{f_n} \longrightarrow 0$ in $\mathcal{S}(\mathbb{R})$.

2.3 The inverse Fourier transform on $\mathcal{S}(\mathbb{R})$

Theorem 2.2. *Let us consider the mapping* $\widehat{\cdot} : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ *defined by* $\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x)\, dx$. *Then, it holds that* $\widehat{\cdot}$ *is bijective and* $\widehat{\cdot}^{-1} = \check{\cdot}$. *In other words, the following relations hold*

$$\hat{f}(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x} f(x) dx \quad \text{and} \quad f(x) = \int_{\mathbb{R}} e^{+2i\pi\xi x} \hat{f}(\xi) d\xi \qquad \text{for all } f \in \mathcal{S}(\mathbb{R}) \text{ and } x, \xi \in \mathbb{R}$$

Moreover, $\widehat{\cdot}$ and $\check{\cdot}$ are continuous in the sense of convergence on $\mathcal{S}(\mathbb{R})$.

PROOF.

• First of all, we observe that the mapping if well-defined. In effect, if $f \in \mathcal{S}(\mathbb{R})$ then by Theorem 2.1. $\hat{f} \in \mathcal{S}(\mathbb{R})$. Moreover, by Proposition 2.3.3 $\hat{f} \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ and as $f \in C^\infty(\mathbb{R})$ we can apply Theorem 1.1. and conclude that $f(x) = \check{\hat{f}}(x)$ for all $x \in \mathbb{R}$. On the other hand, note that

$$\widehat{\check{f}}(x) = \left[\widehat{\hat{f}(-\xi)}\right](x) \stackrel{Proposition\,1.2.2}{=} \left[\widehat{\widehat{f_\sigma}(\xi)}\right](x) = \left[\check{\widehat{f_\sigma}}(\xi)\right](-x) = f_\sigma(-x) = f(x)$$

Hence, we have proved $f = \check{\hat{f}}(x) = \widehat{\check{f}}(x) \ \forall x \in \mathbb{R}$. Hence we conclude that $\widehat{\cdot}$ *is bijective and* $\widehat{\cdot}^{-1} = \check{\cdot}$ on $\mathcal{S}(\mathbb{R})$.

• Finally, it remains to prove the continuity of $\widehat{\cdot}$ and $\check{\cdot}$. Let us prove that $\widehat{\cdot}$ is continuous (for $\check{\cdot}$ is analogous). We should show that given $(f_n)_{n\in\mathbb{N}}$ a sequence of elements in $\mathcal{S}(\mathbb{R})$ and $f \in \mathcal{S}(\mathbb{R})$ such that $f_n \longrightarrow f$ in $\mathcal{S}(\mathbb{R})$ then $\widehat{f_n} \longrightarrow \widehat{f}$ in $\mathcal{S}(\mathbb{R})$. In effect

$$f_n \to f \text{ in } S(\mathbb{R}) \implies f_n - f \to 0 \text{ in } \mathcal{S}(\mathbb{R}) \stackrel{Proposition\,2.4.4}{\implies} [\widehat{f_n - f}] \to 0 \text{ in } \mathcal{S}(\mathbb{R})$$
$$\implies \widehat{f_n} \to \widehat{f} \text{ in } \mathcal{S}(\mathbb{R})$$

Proposition 2.5. *Let us consider $g(x) = e^{-\pi x^2}$. Then it holds that g is in $\mathcal{S}(\mathbb{R})$ and is a fixed point of the Fourier transform, that is $\widehat{g}(\xi) = g(\xi) = e^{-\pi\xi^2}$.*

PROOF.

• We start showing that g is in $\mathcal{S}(\mathbb{R})$. It is clear that g is infinitely times differentiable. Moreover, its derivatives are of the form $g^{(n)}(x) = P(x)e^{-\pi x^2}$ where P is a polynomial. Hence, g and all its derivatives are rapidly decreasing since $\lim_{|x|\to+\infty} \left|x^p P(x)e^{-\pi x^2}\right| = 0$. We conclude $g \in \mathcal{S}(\mathbb{R})$.

• Finally we show that $\widehat{g}(\xi) = g(\xi)$. We apply example 1.1. taking $a = \pi$ and we get the result.

Chapter 3

The convolution of functions, derivation and regularization

3.1 Definitions

Definition 3.1. *We define the convolution of two functions f, $g : \mathbb{R} \longrightarrow \mathbb{C}$ as the function $f * g$, if it exists, defined by*

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)dt = \int_{\mathbb{R}} f(u)g(x-u)du$$

Definition 3.2. (support of a measurable function) *Let $f : \mathbb{R} \longrightarrow \mathbb{C}$ be a measurable function. Let θ_i, $i \in I$, be the family of open sets in $\mathbb{R}$ such that $f = 0$ a.e. on θ_i, for all $i \in I$. Let $\theta = \underset{i \in I}{\cup} \theta_i$. We define the support of f, supp(f), as the closed set $\mathbb{R} \setminus \theta$, that is*

$$\text{supp}\,(f) := \mathbb{R} \setminus \theta$$

Remark 3.1. *Let f, $g : \mathbb{R} \longrightarrow \mathbb{C}$ with supp $(f) = \mathbb{R} \setminus \theta$ and $f = g$ a.e., then it holds that supp(f) = supp(g).*

PROOF.

Suppose supp(f) $\neq$ supp(g), then there exists a set A with $\mu(A) \neq 0$ such that $f = 0$ and $g \neq 0$ a.e. on A (or viceversa). Then, $f \neq g$ on A with $\mu(A) \neq 0$. That is a contradiction with the assumption that $f = g$ almost everywhere.

Lemma 3.1. *Let f, $g : \mathbb{R} \longrightarrow \mathbb{C}$ be two functions for which $f * g$ exists. Then:*

$$\text{supp}\,(f * g) \subseteq \overline{\text{supp}\,(f) + \text{supp}\,(g)}$$

PROOF.

Let us define $S := \mathbb{R} \setminus (\text{supp}\,(f) + \text{supp}\,(g))$ and let $\mathring{S}$ denote the interior of S. Let $x \in S$, i.e. $x \notin \text{supp}\,(f) + \text{supp}\,(g)$. Then for all $t \in \text{supp}\,(f)$, we have $(x-t) \notin \text{supp}\,(g)$. Hence $g(x-t) = 0$ and $(f * g)(x) = \int_{\mathbb{R}} g(x-t)f(t)dt = 0$.

Let us denote θ_{f*g} the largest open set on which $f * g = 0$ a.e. We have seen that $x \in \mathring{S}$ implies $x \in \theta_{f*g}$. Thus,

$$\mathring{S} \subseteq \theta_{f*g} \implies \text{supp}\,(f * g) = \mathbb{R} \setminus \theta_{f*g} \subseteq \mathbb{R} \setminus \mathring{S} = \overline{\mathbb{R} \setminus S} = \overline{\text{supp}\,(f) + \text{supp}\,(g)}$$

3.2 The Convolution of functions in $L^1(\mathbb{R})$

Proposition 3.1. *Let f, g be in $L^1(\mathbb{R})$. Then, the following statements hold:*

1. *$f * g$ is defined almost everywhere and belongs to $L^1(\mathbb{R})$.*
2. *The convolution is a continuous bilinear operator from $L^1(\mathbb{R}) \times L^1(\mathbb{R})$ to $L^1(\mathbb{R})$ with:*

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1$$

PROOF.
• Proof 1: let us consider the function $F(y,t) := f(y)g(t)$. As $f,\, g \in L^1(\mathbb{R})$, Theorem A.6. (Fubini) implies that $F(y,t) \in L^1(\mathbb{R}^2)$. Moreover:

$$\int_{\mathbb{R}} |(f * g)(x)|\, dx = \int_{\mathbb{R}} \left| \int_{\mathbb{R}} f(x-t)g(t)dt \right| dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-t)g(t)|\, dtdx$$
$$\overset{x=y+t}{=} \int_{\mathbb{R}} \int_{\mathbb{R}} |f(y)g(t)|\, dtdy \overset{Fubini}{<} +\infty \tag{3.2.1}$$

Hence, we conclude $f * g$ is defined almost everywhere and belongs to $L^1(\mathbb{R})$.

• Proof 2: we have seen in (3.2.1) that:

$$\|f * g\|_1 = \int_{\mathbb{R}} |(f * g)(x)|\, dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} |f(y)g(t)|\, dtdy \overset{Fubini}{=} \left(\int_{\mathbb{R}} |f(y)|\, dy \right) \left(\int_{\mathbb{R}} |g(t)|\, dt \right) = \|f\|_1 \|g\|_1$$

Moreover, from the statement 1 we deduce that the operator $f * g : L^1(\mathbb{R}) \times L^1(\mathbb{R}) \longrightarrow L^1(\mathbb{R})$ is well-defined and the previous inequality shows that the operator is continuous with respect the two variables. Finally, it is bilinear since the linearity of the integral implies:

$$[(\lambda_1 f_1 + \lambda_2 f_2) * (\mu_1 g_1 + \mu_2 g_2)](x) = \lambda_1 \mu_1 (f_1 * g_1)(x) + \lambda_1 \mu_2 (f_1 * g_2)(x) +$$
$$+ \lambda_2 \mu_1 (f_2 * g_1)(x) + \lambda_2 \mu_2 (f_2 * g_2)(x)$$

Proposition 3.2. *Let $f \in L^1_{loc}(\mathbb{R})$ and $g \in L^1(\mathbb{R})$. Then, the following statements hold:*

1. *If $supp\,(g)$ is bounded, then $f * g$ exists almost everywhere and belongs to $L^1_{loc}(\mathbb{R})$.*
2. *If f is bounded, then $f * g$ exists everywhere and belongs to $L^\infty(\mathbb{R})$. Moreover, $\|(f * g)\|_\infty \leq \|f\|_\infty \|g\|_1$.*

PROOF.
• Proof 1: as supp(g) is bounded, there exists an interval $[-a,\, a]$ such that g is zero a.e. outside this interval. Let us take x in a finite interval $[\alpha,\, \beta]$. Then for all $t \in [-a,\, a]$ and for all $x \in [\alpha,\, \beta]$

$$f(x-t)\, g(t) = \mathbf{1}_{[\alpha - a,\, \beta + a]}(x-t) f(x-t) g(t)$$

and thus, for all $x \in [\alpha,\, \beta]$

$$(f * g)(x) = \int_{\mathbb{R}} f(x-t)g(t)dt = \int_{-a}^{+a} f(x-t)g(t)dt = \int_{\mathbb{R}} \mathbf{1}_{[\alpha - a,\, \beta + a]}(x-t) f(x-t) g(t) dt =$$

$$= \left(\mathbf{1}_{[\alpha-a,\,\beta+a]} f * g\right)(x)$$

Hence, $f * g$ coincides on $[\alpha,\,\beta]$ with the convolution of two functions in $L^1(\mathbb{R})$ since $g \in L^1(\mathbb{R})$ and $\mathbf{1}_{[\alpha-a,\,\beta+a]} f \in L^1(\mathbb{R})$ because $f \in L^1_{loc}(\mathbb{R})$. Thus, by Proposition 3.1.1 $f * g$ is defined almost everywhere and is integrable on all compacts sets.

• Proof 2: if f is bounded, then $|f| \leq \|f\|_\infty$. Thus

$$|(f * g)(x)| = \left|\int_{\mathbb{R}} f(u)g(x-u)du\right| \leq \int_{\mathbb{R}} |f(u)|\,|g(x-u)|\,du \leq \|f\|_\infty \int_{\mathbb{R}} |g(x-u)|\,du$$

$$= \|f\|_\infty \|g\|_1 < +\infty$$

Consequently, $f * g$ exists everywhere and belongs to $L^\infty(\mathbb{R})$. Moreover, taking supremums in the previous inequality: $\|(f * g)\|_\infty \leq \|f\|_\infty \|g\|_1$.

3.3 The Convolution of functions in $L^p(\mathbb{R})$

Definition 3.3. *Let p and q be two real numbers belonging to $[1,\,+\infty]$. We say that p and q are conjugates if $\frac{1}{p} + \frac{1}{q} = 1$.*

Proposition 3.3. *Assume that $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, where p and q are conjugates. Then, the following statements hold:*

1. *$f * g$ is defined everywhere and is continuous and bounded on $\mathbb{R}$.*
2. $\|f * g\|_\infty \leq \|f\|_p \|g\|_q$

PROOF.

• *Inequality, well definition and boundedness:* as $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, we can use Inequality A.1. (Hölder's inequality). Then

$$|(f * g)(x)| \leq \int_{\mathbb{R}} |f(x-t)|\,|g(t)|\,dt \overset{\text{Ineq. A.1.}}{\leq} \left(\int_{\mathbb{R}} |f(x-t)|^p\,dt\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g(t)|^q\,dt\right)^{\frac{1}{q}} = \|f\|_p \|g\|_q < \infty$$

Hence, $f * g$ is defined everywhere. Moreover, taking supremums in both sides of the inequality we get $\|f * g\|_\infty \leq \|f\|_p \|g\|_q < \infty$ and we conclude that $f * g$ is bounded.

• *Continuity*: first note that the following inequality holds

$$|(f * g)(x) - (f * g)(y)| \leq \int_{\mathbb{R}} |f(x-t) - f(y-t)|\,|g(t)|\,dt \overset{\text{Ineq. A.1.}}{\leq}$$

$$\overset{\text{Ineq. A.1.}}{\leq} \|g\|_q \left(\int_{\mathbb{R}} |f(x-t) - f(y-t)|^p\,dt\right)^{\frac{1}{p}} \tag{3.3.1}$$

- Step 1: we will prove the continuity of $f * g$ when f is continuous with compact support contained in an open interval $(-a, a)$. As the support is compact, for $\delta_1 > 0$ small enough it holds that $\text{supp}\,(f) \subset (-a+\delta_1,\, a-\delta_1)$.

Moreover, f is continuous in $[-a, a]$ and by Theorem A.8 we conclude that f is uniformly continuous in $[-a, a]$. Hence, for all $\epsilon > 0$, there exists $\delta_{2,\epsilon} > 0$ such that for all $x, y \in [-a, a]$ with $|x-y| < \delta_{2,\epsilon}$

$$|f(x) - f(y)| < \frac{\epsilon}{(2a)^{\frac{1}{p}} \|g\|_q} \implies |f(x)-f(y)|^p < \frac{\epsilon^p}{2a\|g\|_q^p}$$

In particular $\sup\limits_{|u|\le a} |f(x-y+u) - f(u)|^p < \frac{\epsilon^p}{2a\|g\|_q^p}$. Thus, taking $\delta_\epsilon = \min(\delta_1, \delta_{2,\epsilon})$ and $|x-y| < \delta_\epsilon$

$$\int_{\mathbb{R}} |f(x-t) - f(y-t)|^p \, dt \overset{t=y-u}{=} \int_{\mathbb{R}} |f(x-y+u) - f(u)|^p \, du = \int_{-a}^{+a} |f(x-y+u) - f(u)|^p \, du$$

$$\le 2a \cdot \sup_{|u|\le a} |f(x-y+u) - f(u)|^p < \left(\frac{\epsilon}{\|g\|_q}\right)^p$$

Hence, we have seen that for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for all $x, y \in \mathbb{R}$ with $|x-y| < \delta_\epsilon$, it holds using (3.3.1) that $|(f*g)(x) - (f*g)(y)| < \epsilon$. Thus, we conclude that $(f*g)$ is uniformly continuous on $\mathbb{R}$ and particularly continuous.

- Step 2: now let us suppose $f \in L^p(\mathbb{R})$. From Theorem A.9., we know that $C_c^0(\mathbb{R})$ (space of continuous functions with compact support in $\mathbb{R}$) is dense in $L^p(\mathbb{R})$. Hence, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions of $C_c^0(\mathbb{R})$ such that $\lim\limits_{n\longrightarrow\infty} \|f_n - f\|_p = 0$. Adding and subtracting $f_n * g$ at x and y

$$|(f*g)(x) - (f*g)(y)| \le |[(f-f_n)*g](x)| + |(f_n*g)(x) - (f_n*g)(y)| +$$
$$+ |[(f-f_n)*g](y)| \le 2\|g\|_q \|f-f_n\|_p + |(f_n*g)(x) - (f_n*g)(y)| \tag{3.3.2}$$

As $\lim\limits_{n\longrightarrow\infty} \|f_n - f\|_p = 0$, we have that for all $\epsilon > 0$, there exists $N_\epsilon > 0$ such that for all $n \ge N_\epsilon$

$$\|f_n - f\|_p < \frac{\epsilon}{4\|g\|_q}$$

Moreover, for each n fixed, we have seen in step 1 that $(f_n * g)$ is uniformly continuous. In particular, it is true for $n = N_\epsilon$. Hence, there exists $\delta_\epsilon > 0$ such that for all $x, y \in \mathbb{R}$ with $|x-y| < \delta_\epsilon$, it holds $|(f_{N_\epsilon}*g)(x) - (f_{N_\epsilon}*g)(y)| < \frac{\epsilon}{2}$. Thus, we conclude that for all $\epsilon > 0$ there exists $\delta_\epsilon > 0$ such that for all $x, y \in \mathbb{R}$ with $|x-y| < \delta_\epsilon$

$$|(f*g)(x) - (f*g)(y)| \overset{(3.3.2)}{\le} 2\|g\|_q \|f - f_{N_\epsilon}\|_p + |(f_{N_\epsilon}*g)(x) - (f_{N_\epsilon}*g)(y)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

In conclusion, $f*g$ is uniformly continuous in $\mathbb{R}$ and in particular is continuous.

Corollary 3.1. *Assume that $f \in L^p(\mathbb{R})$ has bounded support and $g \in L^q_{loc}(\mathbb{R})$, where p and q are conjugates. Then, the convolution $f * g$ is defined and continuous for all $x \in \mathbb{R}$.*

PROOF.

As f has bounded support, there exists an interval $[-a,\, a]$ such that the function is zero a.e. outside this interval. Let x be in an arbitrary compact interval $[\alpha,\, \beta]$. Then

$$(f * g)(x) = \int_{\mathbb{R}} f(u)g(x-u)du = \int_{-a}^{+a} f(u)g(x-u)du = \int_{\mathbb{R}} \mathbf{1}_{[\alpha-a,\, \beta+a]}(x-u)f(u)g(x-u)du$$
$$= \left(f * \mathbf{1}_{[\alpha-a,\, \beta+a]}g\right)(x)$$

Moreover, as $f \in L^p(\mathbb{R})$ and $\mathbf{1}_{[\alpha-a,\, \beta+a]}g \in L^q(\mathbb{R})$, then by Proposition 3.3. $f * \mathbf{1}_{[\alpha-a,\, \beta+a]}g$ is defined and continuous everywhere in $\mathbb{R}$. Hence, $f * g$ coincides in all compact sets with a continuous function defined everywhere. Consequently, we conclude $f * g$ is defined and continuous for all $x \in \mathbb{R}$.

Lemma 3.2. *Let $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, with p and q conjugates. Then $fg \in L^1(\mathbb{R})$.*

PROOF.
As $\frac{1}{p} + \frac{1}{q} = 1$, we can apply Inequality A.1. (Hölder's inequality) and we see that

$$\int_{\mathbb{R}} |f(x)g(x)|\, dx \le \left(\int_{\mathbb{R}} |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |g(x)|^q\, dx\right)^{\frac{1}{q}} = \|f\|_p \|g\|_q < +\infty$$

Proposition 3.4. *If $f \in L^1(\mathbb{R})$ and $g \in L^2(\mathbb{R})$. Then, the following statements hold:*

1. *$f * g$ exists almost everywhere in $\mathbb{R}$.*
2. *$f * g$ is in $L^2(\mathbb{R})$ and $\|f * g\|_2 \le \|f\|_1 \|g\|_2$*

PROOF.
• Proof 1: first of all note that we can write

$$|f(u)g(x-u)| = \left(|f(u)|\, |g(x-u)|^2\right)^{\frac{1}{2}} (|f(u)|)^{\frac{1}{2}}$$

Since $|f|,\, |g|^2 \in L^1(\mathbb{R})$, then by Proposition 3.1.1 $|f| * |g|^2$ is defined almost everywhere. Hence $\int_{\mathbb{R}} |f(u)|\, |g(x-u)|^2\, du < +\infty$ a.e. and then $|f(u)|\, |g(x-u)|^2$ (as a function of u) is in $L^1(\mathbb{R})$ for almost every x. Thus, $\left(|f(u)|\, |g(x-u)|^2\right)^{\frac{1}{2}}$ (as a function of u) is in $L^2(\mathbb{R})$.
Moreover, $|f(u)|^{\frac{1}{2}}$ is in $L^2(\mathbb{R})$. In consequence, we can apply Lemma 3.2. (with $p = q = 2$) and we conclude that $|f(u)g(x-u)| = \left(|f(u)|\, |g(x-u)|^2\right)^{\frac{1}{2}} (|f(u)|)^{\frac{1}{2}}$ is in $L^1(\mathbb{R})$ for almost all x. Then, $f * g$ exists almost everywhere.

• Proof 2: using Hölder's inequality with $p = q = 2$ to the functions $|f(u)|^{\frac{1}{2}}$ and $|f(u)|^{\frac{1}{2}} |g(x-u)|$

$$|(f * g)(x)| \le \int_{\mathbb{R}} |f(u)g(x-u)|\, du = \int_{\mathbb{R}} \left(|f(u)|\, |g(x-u)|^2\right)^{\frac{1}{2}} (|f(u)|)^{\frac{1}{2}}\, du \le$$

$$\leq \left(\int_{\mathbb{R}} |f(u)| \, |g(x-u)|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} |f(u)| \, du\right)^{\frac{1}{2}} = \left(\left[\left(|f| * |g|^2\right)(x)\right] \|f\|_1\right)^{\frac{1}{2}}$$

Consequently, $|(f*g)(x)|^2 \leq \left[\left(|f| * |g|^2\right)(x)\right] \|f\|_1$ and integrating both sides of this inequality

$$\|f*g\|_2^2 = \int_{\mathbb{R}} |(f*g)(x)|^2 \, dx \leq \|f\|_1 \int_{\mathbb{R}} \left(|f| * |g|^2\right)(x) dx \overset{Proposition\, 3.1.2}{\leq}$$

$$\leq \|f\|_1 \|f\|_1 \left\|g^2\right\|_1 = \|f\|_1^2 \|g\|_2^2 < +\infty$$

Hence, we conclude that $f*g$ is in $L^2(\mathbb{R})$ and taking the square root we see that $\|f*g\|_2 \leq \|f\|_1 \|g\|_2$.

Remark 3.2. *The previous result can be generalized as follows: if $f \in L^p(\mathbb{R})$ and $g \in L^q(\mathbb{R})$, where $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$ and $p, q, r \geq 1$, then $f*g$ is in $L^r(\mathbb{R})$. Proposition 3.4. is a particular case with $p = 1$, $q = 2$ and $r = 2$.*

3.4 Convolution and derivation

Proposition 3.5. *Let $f \in L^1(\mathbb{R})$ and let $g \in C^p(\mathbb{R})$. Assume that $g^{(k)}$ is bounded in $\mathbb{R}$ for $k = 0, 1, .., p$. Then, the following statements hold:*

1. $f*g \in C^p(\mathbb{R})$
2. $(f*g)^{(k)} = f*g^{(k)}$ *for* $k = 1, .., p$

PROOF.
Note that the assumptions imply that $g^{(k)} \in L^\infty(\mathbb{R})$ for $k = 0, 1, .., p$. As $f \in L^1(\mathbb{R})$, we can apply Proposition 3.3.1 and conclude that $f*g^{(k)}$ is continuous for $k = 0, 1, .., p$. Let us consider the function $x \longmapsto f(t)g^{(k)}(x-t)$ for $k = 0, 1, .., p-1$ and let us check that satisfies the hypothesis of Theorem A.5.:

1. $x \longmapsto f(t)g^{(k)}(x-t)$ is continuously differentiable for almost every t.
2. As $g^{(k)}$ is bounded in $\mathbb{R}$, let us define $M_k = \sup_{x\in\mathbb{R}} \left|g^{(k)}(x)\right|$. Then, $\left|f(t)g^{(k+1)}(x-t)\right| \leq M_{k+1} |f(t)|$ for all $x \in \mathbb{R}$ and for a.e. t which is integrable.

Thus, Theorem A.5. can be applied and we conclude, $(f*g)^{(k)}$ is differentiable for $k = 0, 1, .., p-1$

$$(f*g)^{(k+1)}(x) = \int_{\mathbb{R}} f(t)g^{(k+1)}(x-t)dt = \left(f*g^{(k+1)}\right)(x)$$

Remark 3.3. *In the conditions of the previous proposition, it holds that $(f*g)^{(k)}$ is bounded on $\mathbb{R}$ for $k = 0, 1, .., p$ because $(f*g)^{(k)} = f*g^{(k)}$ is a convolution of a function of $L^1(\mathbb{R})$ with a function of $L^\infty(\mathbb{R})$.*

3.5 Convolution and regularization

Definition 3.4. *We define the space $\mathcal{D}(\mathbb{R})$ as the space of functions in $C^{\infty}(\mathbb{R})$ that have bounded support.*

Definition 3.5. *A sequence of functions $(\rho_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R})$ is called a regularizing sequence if it satisfies the following conditions:*

1. $\rho_n(x) \geq 0$ *for all* $x \in \mathbb{R}$
2. $\int_{\mathbb{R}} \rho_n(x)dx = 1$
3. *The support of ρ_n is in $[-\varepsilon_n, \varepsilon_n]$, $\varepsilon_n > 0$, and* $\lim\limits_{n\longrightarrow+\infty} \varepsilon_n = 0$

To see that such a sequence exists, we take $\rho \in \mathcal{D}(\mathbb{R})$ defined by:

$$\rho(x) = \begin{cases} \frac{1}{c}e^{-\frac{1}{1-x^2}} & \text{if } |x| \leq 1 \\ 0 & \text{if } |x| > 1 \end{cases} \quad \text{with } c := \int_{-1}^{+1} e^{-\frac{1}{1-x^2}}dx > 0$$

And we define $\rho_n(x) = n\rho(nx)$. Then it holds that $(\rho_n)_{n\in\mathbb{N}}$ is a regularizing sequence. In effect, clearly ρ_n is in $\mathcal{D}(\mathbb{R})$ since is the composition of two infinitely times differentiable functions and ρ has bounded support. Moreover, ρ_n satisfies the three conditions of definition 3.5.:

1. $\rho_n(x) = \begin{cases} \frac{n}{c}e^{-\frac{1}{1-n^2x^2}} & \text{if } |x| \leq \frac{1}{n} \\ 0 & \text{if } |x| > \frac{1}{n} \end{cases}$ is positive for all $x \in \mathbb{R}$
2. $\int_{\mathbb{R}} \rho_n(x)dx = \int_{\mathbb{R}} n\rho(nx)dx \overset{y=nx}{=} \int_{\mathbb{R}} \rho(y)dy = \frac{1}{c}\int_{-1}^{+1} e^{-\frac{1}{1-x^2}}dx = 1$
3. The support of ρ_n is in $\left[-\frac{1}{n}, \frac{1}{n}\right]$ and $\lim\limits_{n\longrightarrow+\infty} \frac{1}{n} = 0$

Definition 3.6. *Let $f \in L^1(\mathbb{R})$ and let $(\rho_n)_{n\in\mathbb{N}}$ be a regularizing sequence. We will define the functions $f * \rho_n$ as regularizations of f.*

Remark 3.4. *As $f \in L^1(\mathbb{R})$, $\rho_n \in C^{\infty}(\mathbb{R})$ and all its derivatives are bounded, we conclude by Proposition 3.5. that the regularizations of f are in $C^{\infty}(\mathbb{R})$.*

Theorem 3.1. *The space $\mathcal{D}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $1 \leq p < \infty$. In other words, given $f \in L^p(\mathbb{R})$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{D}(\mathbb{R})$ such that for all $\epsilon > 0$, $\|f - g_n\|_p < \epsilon$ for n large enough.*

PROOF.
Let us take $\epsilon > 0$. Using Theorem A.9., we choose f_ϵ in $C_c^0(\mathbb{R})$ such that $\|f - f_\epsilon\|_p < \frac{\epsilon}{2}$. Now consider the regularization sequence $(\rho_n)_{n\in\mathbb{N}}$ built in definition 3.5. and the regularizations of f_ϵ : $g_n = f_\epsilon * \rho_n$. Assume that $\text{supp}(f_\epsilon) \subset [a, b]$ and $\text{supp}(\rho_n) \subset [-1, 1]$, then by Lemma 3.1.:

$$\text{supp}(g_n) = \text{supp}(f_\epsilon * \rho_n) \subseteq \overline{\text{supp}(f_\epsilon) + \text{supp}(\rho_n)} \subseteq [a-1, b+1]$$

Hence g_n has bounded support and is $C^{\infty}(\mathbb{R})$ by Remark 3.4. Consequently, $g_n \in \mathcal{D}(\mathbb{R})$.

Moreover, since $\int_{\mathbb{R}} \rho_n(x)dx = 1$, we can write

$$|f_\epsilon(x) - g_n(x)|^p = \left| f_\epsilon(x) \int_{\mathbb{R}} \rho_n(t)dt - \int_{\mathbb{R}} f_\epsilon(x-t)\rho_n(t)dt \right|^p = \left| \int_{\mathbb{R}} (f_\epsilon(x) - f_\epsilon(x-t))\, \rho_n(t)dt \right|^p \leq$$

$$\leq \left(\sup_{|t|\leq \varepsilon_n} |f_\epsilon(x) - f_\epsilon(x-t)| \int_{\mathbb{R}} \rho_n(x)dx \right)^p \leq \sup_{|t|\leq \varepsilon_n} |f_\epsilon(x) - f_\epsilon(x-t)|^p$$

Using this inequality, we get

$$\|f_\epsilon - g_n\|_p^p = \int_{\mathbb{R}} |f_\epsilon(x) - g_n(x)|^p\, dx = \int_{a-1}^{b+1} |f_\epsilon(x) - g_n(x)|^p\, dx \leq (b-a+2) \sup_{x\in[a-1,\, b+1]} |f_\epsilon(x) - g_n(x)|^p \leq$$

$$\leq (b-a+2) \sup_{x\in[a-1,\, b+1]} \sup_{|t|\leq \varepsilon_n} |f_\epsilon(x) - f_\epsilon(x-t)|^p = (b-a+2) \sup_{\substack{|t|\leq \varepsilon_n \\ x\in[a-1,\, b+1]}} |f_\epsilon(x) - f_\epsilon(x-t)|^p$$

We have seen in Proposition 3.3. (step 1) that f_ϵ is uniformly continuous in compacts. Hence, there exists $\delta_\epsilon > 0$ such that for all $x,\, y \in \mathbb{R}$ with $|x-y| < \delta_\epsilon$ hold $|f_\epsilon(x) - f_\epsilon(y)| < \frac{\epsilon}{2(b-a+2)^{\frac{1}{p}}}$ and then $|f_\epsilon(x) - f_\epsilon(y)|^p < \frac{\epsilon^p}{2^p(b-a+2)}$. Thus, taking n large enough such that $|t| \leq \varepsilon_n < \delta_\epsilon$:

$$\|f_\epsilon - g_n\|_p^p \leq (b-a+2) \sup_{\substack{|t|\leq \varepsilon_n \\ x\in[a-1,\, b+1]}} |f_\epsilon(x) - f_\epsilon(x-t)|^p < \left(\frac{\epsilon}{2}\right)^p \tag{3.5.1}$$

Consequently, for n sufficiently large, we conclude

$$\|f - g_n\|_p \leq \|f - f_\epsilon\|_p + \|f_\epsilon - g_n\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Corollary 3.2. *$\mathcal{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$ for $p \in [1, \infty)$.*
PROOF.

• First, we see that $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$. In effect, let $f \in \mathcal{D}(\mathbb{R})$, then $f \in C^\infty(\mathbb{R})$. Moreover, f and all its derivatives decay rapidly. In effect, as f has compact support, then all its derivatives will also have compact support. Hence, for $|x|$ sufficiently large, f and all its derivatives will be zero and consequently $\lim\limits_{|x|\to+\infty} \left|x^p f^{(q)}(x)\right| = 0$ for all $p,\, q \in \mathbb{N}$. Thus, $f \in \mathcal{S}(\mathbb{R})$.

• Finally, we observe that $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \subset L^p(\mathbb{R})$ and by Theorem 3.1. $\mathcal{D}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$. Consequently, we conclude $\mathcal{S}(\mathbb{R})$ is dense in $L^p(\mathbb{R})$.

Lemma 3.3. *Let $f \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R})$, $p \in [1, \infty)$. Then $f * g$ exists almost everywhere, $f * g \in L^p(\mathbb{R})$ and $\|f * g\|_p \leq \|f\|_1 \|g\|_p$*
PROOF.
Let us first define q such that $\frac{1}{p} + \frac{1}{q} = 1$. Then $q := \frac{1}{1-\frac{1}{p}}$.

• Proof 1: note that we can write:

$$|f(u)g(x-u)| = (|f(u)|\, |g(x-u)|^p)^{\frac{1}{p}} (|f(u)|)^{1-\frac{1}{p}} = (|f(u)|\, |g(x-u)|^p)^{\frac{1}{p}} (|f(u)|)^{\frac{1}{q}}$$

Since $|f|\,,\, |g|^p \in L^1(\mathbb{R})$, then by Proposition 3.1.1 $|f| * |g|^p$ is defined almost everywhere.
Hence $\int_{\mathbb{R}} |f(u)|\, |g(x-u)|^p\, du < +\infty$ a.e. and then $|f(u)|\, |g(x-u)|^p$ (as a function of u) is in $L^1(\mathbb{R})$

for almost every x. That implies that $\left(|f(u)|\,|g(x-u)|^{p}\right)^{\frac{1}{p}}$ (as a function of u) is in $L^p(\mathbb{R})$. Moreover, $|f(u)|^{\frac{1}{q}}$ is in $L^q(\mathbb{R})$. In consequence, as p and q are conjugates, we can apply Lemma 3.2. and we conclude that $|f(u)g(x-u)| = \left(|f(u)|\,|g(x-u)|^{p}\right)^{\frac{1}{p}}\left(|f(u)|\right)^{\frac{1}{q}}$ is in $L^1(\mathbb{R})$ for almost all x. Then $f * g$ exists almost everywhere.

• Proof 2: using Hölder's Inequality to the functions $|f(u)|^{\frac{1}{q}}$ and $|f(u)|^{\frac{1}{p}}\,|g(x-u)|$ (so are in $L^q(\mathbb{R})$ and $L^p(\mathbb{R})$ respectively)

$$|(f*g)(x)| \leq \int_{\mathbb{R}} |f(u)g(x-u)|\,du = \int_{\mathbb{R}} |f(u)|^{\frac{1}{p}}\,|g(x-u)|\,|f(u)|^{\frac{1}{q}}\,du \leq$$

$$\leq \left(\int_{\mathbb{R}} |f(u)|\,|g(x-u)|^{p}\right)^{\frac{1}{p}} \left(\int_{\mathbb{R}} |f(u)|\,du\right)^{\frac{1}{q}} = \left[\left(|f| * |g|^{p}\right)(x)\right]^{\frac{1}{p}} \|f\|_1^{\frac{1}{q}}$$

Hence, we get that $|(f*g)(x)|^p \leq \left[\left(|f| * |g|^{p}\right)(x)\right] \|f\|_1^{\frac{p}{q}}$. Now, integrating both sides of this inequality

$$\|f*g\|_p^p = \int_{\mathbb{R}} |(f*g)(x)|^p\,dx \leq \|f\|_1^{\frac{p}{q}} \int_{\mathbb{R}} \left[\left(|f| * |g|^{p}\right)(x)\right] dx \overset{Proposition\,3.1.2}{\leq} \|f\|_1^{\frac{p}{q}}\,\|f\|_1\,\|g^p\|_1$$

$$= \|f\|_1^{p}\,\|g\|_p^{p} < +\infty$$

And finally we conclude that $f*g$ is in $L^p(\mathbb{R})$ and $\|f*g\|_p \leq \|f\|_1\,\|g\|_p$.

Lemma 3.4. *Let $f \in \mathcal{D}(\mathbb{R})$ and let ρ_n be a regularizing sequence. Then*

$$\lim_{n\longrightarrow\infty} \|f - f*\rho_n\|_p = 0$$

PROOF. The result follows immediately from the proof of Theorem 3.1. In effect, in (3.5.1) we have shown this result when $f \in C_c^0(\mathbb{R})$. But as $\mathcal{D}(\mathbb{R}) \subset C_c^0(\mathbb{R})$, the result also holds for $f \in \mathcal{D}(\mathbb{R})$.

Proposition 3.6. *Let $f \in L^p(\mathbb{R})$, $p \in [1,\infty)$ and let ρ_n be a regularizing sequence. Then:*

$$\lim_{n\longrightarrow\infty} \|f - f*\rho_n\|_p = 0$$

PROOF.
Let $\epsilon > 0$. By Theorem 3.1., there exists $f_\epsilon \in \mathcal{D}(\mathbb{R})$ such that $\|f - f_\epsilon\|_p < \frac{\epsilon}{4}$. As $f - f_\epsilon \in L^p(\mathbb{R})$ and $\rho_n \in L^1(\mathbb{R})$, we have from Lemma 3.3.

$$\|f*\rho_n - f_\epsilon*\rho_n\|_p = \|(f-f_\epsilon)*\rho_n\|_p \overset{Lemma\,3.3}{\leq} \|f-f_\epsilon\|_p\,\|\rho_n\|_1 = \|f-f_\epsilon\|_p$$

Moreover, from Lemma 3.4. $\lim_{n\longrightarrow\infty} \|f_\epsilon - f_\epsilon*\rho_n\|_p = 0$, i.e. there exists $N > 0$ such that for all $n \geq N$ holds that $\|f_\epsilon - f_\epsilon*\rho_n\|_p < \frac{\epsilon}{2}$. Then, for $n \geq N$

$$\|f - f*\rho_n\|_p \leq \|f - f_\epsilon\|_p + \|f_\epsilon - f_\epsilon*\rho_n\|_p + \|f*\rho_n - f_\epsilon*\rho_n\|_p \leq 2\,\|f - f_\epsilon\|_p$$

$$+ \|f_\epsilon - f_\epsilon*\rho_n\|_p < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

3.6 The convolution in $\mathcal{S}(\mathbb{R})$

Proposition 3.7. *Let f, g be in $\mathcal{S}(\mathbb{R})$. Then, $f * g$ is defined almost everywhere and belongs to $\mathcal{S}(\mathbb{R})$.*

PROOF.

As $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$, then $f, g \in L^1(\mathbb{R})$ and by Proposition 3.1. $f * g$ is defined almost everywhere and is in $L^1(\mathbb{R})$. Moreover, $f, g \in C^\infty(\mathbb{R})$ and its derivatives are bounded. Then, by Proposition 3.5., we conclude that $f * g \in C^\infty(\mathbb{R})$. It remains to prove that $f * g$ and all its derivatives decay rapidly.

• Note that $|f(x-t)g(t)| \le \|f\|_\infty |g(t)|$ which is integrable. Thus, we can apply the Dominated Convergence Theorem (Theorem A.2.)

$$\lim_{|x|\to+\infty} (f*g)(x) = \lim_{|x|\to+\infty} \int_{\mathbb{R}} f(x-t)g(t)dt \overset{Dom. Conv. Th.}{=} \int_{\mathbb{R}} \lim_{|x|\to+\infty} f(x-t)g(t)dt \overset{f\, decays\, rapidly}{=} 0$$

• Now, we take $p, q \ge 0$ and we use the formula

$$x^p (f*g)^{(q)}(x) = \sum_{j=0}^{p} \beta_j (x^{p-j} f) * (x^j g^{(q)}) \quad \text{where } \beta_j \text{ are binomial coefficients}$$

Thus, $x^p (f*g)^{(q)}(x)$ is written as a sum of convolutions of elements in $\mathcal{S}(\mathbb{R})$ which is invariant under differentiation and multiplication by polynomials (Proposition 2.3.1 and 2.3.2). Thus, applying the same argument for each element as for $f * g$ we conclude

$$\lim_{|x|\to+\infty} x^p (f*g)^{(q)}(x) = 0 \qquad \text{for all } \; p, q \ge 0$$

Chapter 4

The Fourier Transform on $L^2(\mathbb{R})$

4.1 Extension of the Fourier Transform

Proposition 4.1. *$\mathcal{S}(\mathbb{R})$ is a dense linear subspace of $L^2(\mathbb{R})$.*

PROOF.
First, we show that $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$. Let us take $f \in \mathcal{S}(\mathbb{R})$ then by Proposition 2.3.1, $(1+x^2)\, f(x) \in \mathcal{S}(\mathbb{R})$. Consequently:

$$\lim_{|x| \longrightarrow +\infty} \left|(1+x^2)\, f(x)\right| = 0$$

Then for $\epsilon = 1$, there exists $M > 0$ such that $\left|(1+x^2)\, f(x)\right| < 1$ for all $|x| > M$. Moreover, as $(1+x^2)\, f(x)$ is continuous in $|x| \leq M$ then, there exists $A > 0$ such that $\left|(1+x^2)\, f(x)\right| < A$ for all $|x| \leq M$. Defining $K := \max\{1, A\}$:

$$\left|(1+x^2)\, f(x)\right| < K \implies |f(x)| < \frac{K}{1+x^2} \quad \text{for all} \quad x \in \mathbb{R}$$

Using this, we get that $f \in L^2(\mathbb{R})$, since:

$$\int_{\mathbb{R}} |f(x)|^2 \, dx \leq K^2 \int_{\mathbb{R}} \frac{1}{(1+x^2)^2} dx < +\infty$$

Consequently, as $\mathcal{S}(\mathbb{R})$ is a linear space, we conclude that $\mathcal{S}(\mathbb{R})$ is a linear subspace of $L^2(\mathbb{R})$.
The density follows from Corollary 3.2. with $p = 2$.

Proposition 4.2. (The Plancherel-Parseval equality) *Let $f,\, g \in \mathcal{S}(\mathbb{R})$. Then, it holds:*

1. $\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{g}(\xi)} d\xi = \int_{\mathbb{R}} f(x)\overline{g(x)} dx$
2. $\left\|\hat{f}\right\|_2 = \|f\|_2$

PROOF.
• Proof 1: let us define $h(\xi) := \overline{\hat{g}(\xi)}$. Note that $g \in \mathcal{S}(\mathbb{R})$, the by Theorem 2.1. $\hat{g} \in \mathcal{S}(\mathbb{R})$ and hence $h = \overline{\hat{g}} \in \mathcal{S}(\mathbb{R})$. Finally, as by Proposition 2.3.3 $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ we get $f,\, h \in L^1(\mathbb{R})$. Now, applying Property 1.8.:

$$\int_{\mathbb{R}} \hat{f}(\xi) h(\xi) d\xi = \int_{\mathbb{R}} f(x)\hat{h}(x) dx$$

But $h(\xi) = \overline{\hat{g}(\xi)} = [\check{\overline{g}}](\xi)$. Thus, by Theorem 2.2. $\hat{h} = \overline{g}$ which proves 1.

• Proof 2: now taking $f = g$ and using the previous equality:

$$\int_{\mathbb{R}} \hat{f}(\xi)\overline{\hat{f}(\xi)}d\xi = \int_{\mathbb{R}} f(x)\overline{f(x)}dx \implies \int_{\mathbb{R}} \left|\hat{f}(\xi)\right|^2 d\xi = \int_{\mathbb{R}} |f(x)|^2 dx \implies \left\|\hat{f}\right\|_2 = \|f\|_2$$

Proposition 4.3. *Let E and F be two normed vector spaces (we will denote the norm $\|\cdot\|$). Assume that F is complete and that G is a dense linear subspace of E. If $A : G \longrightarrow F$ is a continuous linear operator, then there exists a unique continuous linear extension $\widetilde{A} : E \longrightarrow F$. Furthermore, the norm of $\widetilde{A}$ is equal to the norm of A.*

PROOF.
Let $f \in E$. Due to the density of G in E, there exists a senquence $(f_n)_{n\in\mathbb{N}}$ in G s.t. $\lim_{n\longrightarrow+\infty} \|f - f_n\| = 0$. Consequently, $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence, and since A is continuous, for each n, $m \in \mathbb{N}$

$$\|A(f_n) - A(f_m)\| \leq \|A\| \|f_n - f_m\|$$

showing that $(A(f_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in F. Since F is complete, there exists $g \in F$ s.t. $A(f_n) \overset{n\longrightarrow+\infty}{\longrightarrow} g$. Let us we define $\widetilde{A} : E \longrightarrow F$ as $\widetilde{A}(f) := g = \lim_{n\longrightarrow+\infty} A(f_n)$. We will show that $\widetilde{A}$ is the extension we are looking for. We shall prove:

• $\widetilde{A}$ is well-defined: we have already seen that $\widetilde{A}(f) \in F$. It remains to show that g doesn't depend on the sequence f_n that converges to f. Let us suppose there exist two different sequences $\left(f_n^1\right)_{n\in\mathbb{N}}$ and $\left(f_n^2\right)_{n\in\mathbb{N}}$ that converge to f and such that $A(f_n^1) \overset{n\longrightarrow+\infty}{\longrightarrow} g_1$ and $A(f_n^2) \overset{n\longrightarrow+\infty}{\longrightarrow} g_2$. Then

$$\left\|A\left(f_n^1\right) - A\left(f_n^2\right)\right\| \leq \|A\| \left\|f_n^1 - f_n^2\right\| \leq \|A\| \left(\left\|f_n^1 - f\right\| + \left\|f_n^2 - f\right\|\right) \overset{n\longrightarrow+\infty}{\longrightarrow} 0$$

Hence $\lim_{n\longrightarrow+\infty} A(f_n^1) = \lim_{n\longrightarrow+\infty} A(f_n^2) \implies g_1 = g_2$. Thus, $\widetilde{A}$ is well-defined.

• $\widetilde{A}$ is linear: this is consequence of the linearity of the operator A. In effect:

$$\widetilde{A}(\lambda_1 f_1 + \lambda_2 f_2) = \lim_{n\longrightarrow+\infty} A(\lambda_1 f_n^1 + \lambda_2 f_n^2) = \lambda_1 \lim_{n\longrightarrow+\infty} A\left(f_n^1\right) + \lambda_2 \lim_{n\longrightarrow+\infty} A\left(f_n^2\right) = \lambda_1 \widetilde{A}(f_1) + \lambda_2 \widetilde{A}(f_2)$$

• $\widetilde{A}$ extends A: i.e. we should show that if $f \in G$ then $\widetilde{A}(f) = A(f)$. By taking $f_n := f$ for all $n \in \mathbb{N}$, we have $f_n \overset{n\longrightarrow+\infty}{\longrightarrow} f$. Hence, $\widetilde{A}(f) = \lim_{n\longrightarrow+\infty} A(f_n) = \lim_{n\longrightarrow+\infty} A(f) = A(f)$.

• $\widetilde{A}$ is continuous: on the one hand, we have the following inequality

$$\left\|\widetilde{A}(f)\right\| = \|g\| = \left\|\lim_{n\longrightarrow+\infty} A(f_n)\right\| = \lim_{n\longrightarrow+\infty} \|A(f_n)\| \overset{continuity}{\leq} \lim_{n\longrightarrow+\infty} \|A\| \|f_n\| =$$

$$= \|A\| \left\|\lim_{n\longrightarrow+\infty} f_n\right\| = \|A\| \|f\| \implies \left\|\widetilde{A}\right\| = \sup_{f\in E\setminus\{0\}} \frac{\left\|\widetilde{A}(f)\right\|}{\|f\|} \leq \|A\|$$

On the other hand, since $\widetilde{A}(f) = A(f)$ for all $f \in G$:

$$\left\|\widetilde{A}\right\| = \sup_{f\in E\setminus\{0\}} \frac{\left\|\widetilde{A}(f)\right\|}{\|f\|} \overset{G\subseteq E}{\geq} \sup_{f\in G\setminus\{0\}} \frac{\left\|\widetilde{A}(f)\right\|}{\|f\|} = \sup_{f\in G\setminus\{0\}} \frac{\|A(f)\|}{\|f\|} = \|A\|$$

Hence, we conclude $\left\|\widetilde{A}\right\| = \|A\|$ and $\widetilde{A}$ is continuous.

• $\widetilde{A}$ is the unique continuous linear extension of A: let us suppose there exist two extensions $\widetilde{A}_1$ and $\widetilde{A}_2$ satisfying the previous properties. Let $x \in E$. Since G is dense in E, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ of elements in G such that $x_n \overset{n\longrightarrow+\infty}{\longrightarrow} x$ in G. But, as $\widetilde{A}_1$ and $\widetilde{A}_2$ extend A, then $\widetilde{A}_1(x_n) = \widetilde{A}_2(x_n) = A(x_n)$ for all $n \in \mathbb{N}$. Hence, taking limits

$$\widetilde{A}_1(x) \overset{continuity\,of\,\widetilde{A}_1}{=} \lim_{n\longrightarrow+\infty} \widetilde{A}_1(x_n) = \lim_{n\longrightarrow+\infty} \widetilde{A}_2(x_n) \overset{continuity\,of\,\widetilde{A}_2}{=} \widetilde{A}_2(x)$$

Thus, $\widetilde{A}_1(x) = \widetilde{A}_2(x)$ for all $x \in E$. Hence, the extension is unique.

Theorem 4.1. *The Fourier transform* $\widehat{\cdot}$ *and its inverse* $\check{\cdot}$ *extend uniquely to isometries on* $L^2(\mathbb{R})$. *We will denote these extensions by* $\mathcal{F}$ *and* $\overline{\mathcal{F}}$ *respectively. Moreover, given* $f, g \in L^2(\mathbb{R})$, *the following results hold:*

1. $\mathcal{F}\overline{\mathcal{F}}(f) = \overline{\mathcal{F}}\mathcal{F}(f) = f$ *almost everywhere.*
2. $\int_{\mathbb{R}} f(x)\overline{g(x)}dx = \int_{\mathbb{R}} \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)}d\xi$
3. $\|f\|_2 = \|\mathcal{F}(f)\|_2$. *Hence,* $\mathcal{F}$ *defines an isometry in* $L^2(\mathbb{R})$.

PROOF.
Let us define $E = F = L^2(\mathbb{R})$ and let $G = \mathcal{S}(\mathbb{R})$. We know that $\widehat{\cdot}, \check{\cdot} : G \longrightarrow G \subset F$ are continuous linear operators (Theorem 2.2.) and G is dense in E (Proposition 4.1.). Hence, by Proposition 4.3., there exist unique continuous linear extensions of $\widehat{\cdot}$ and $\check{\cdot}$ which we will denote $\mathcal{F}$ and $\overline{\mathcal{F}}$ respectively.
On the other hand, since $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exist two sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{S}(\mathbb{R})$ such that converge to f and g in $L^2(\mathbb{R})$ respectively.
Moreover, as f_n and g_n are in $\mathcal{S}(\mathbb{R})$, and $\mathcal{F}, \overline{\mathcal{F}}$ are extensions of the Fourier Transform in $\mathcal{S}(\mathbb{R})$ then $\overline{\mathcal{F}}\mathcal{F}(f_n) = \mathcal{F}\overline{\mathcal{F}}(f_n) = f_n$ and $\overline{\mathcal{F}}\mathcal{F}(g_n) = \mathcal{F}\overline{\mathcal{F}}(g_n) = g_n$.

• Proof 1: we show $\mathcal{F}\overline{\mathcal{F}}(f) = f$ almost everywhere. The proof $\overline{\mathcal{F}}\mathcal{F}(f) = f$ is analogous.

$$\left\|\mathcal{F}\overline{\mathcal{F}}(f) - f\right\|_2 \leq \left\|\mathcal{F}\overline{\mathcal{F}}(f) - \mathcal{F}\overline{\mathcal{F}}(f_n)\right\|_2 + \left\|\mathcal{F}\overline{\mathcal{F}}(f_n) - f\right\|_2 \overset{\mathcal{F},\overline{\mathcal{F}}\,continuous}{\leq}$$
$$\leq \|\mathcal{F}\|_2 \left\|\overline{\mathcal{F}}\right\|_2 \|f_n - f\|_2 + \|f_n - f\|_2 \overset{n\longrightarrow+\infty}{\longrightarrow} 0$$

Hence we conclude $\mathcal{F}\overline{\mathcal{F}}(f) = f$ in $L^2(\mathbb{R})$, i.e. $\mathcal{F}\overline{\mathcal{F}}(f) = f$ almost everywhere.

• Proof 2: as $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are in $\mathcal{S}(\mathbb{R})$, we will use Proposition 4.2.1. to deduce that

$$\left|\int_{\mathbb{R}} f(x)\overline{g(x)}dx - \int_{\mathbb{R}} \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)}d\xi\right| \leq \left|\int_{\mathbb{R}} f(x)\overline{g(x)}dx - \int_{\mathbb{R}} \mathcal{F}(f_n)(\xi)\overline{\mathcal{F}(g_n)(\xi)}d\xi\right| +$$

$$+\left|\int_{\mathbb{R}} \mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)}d\xi - \int_{\mathbb{R}} \mathcal{F}(f_n)(\xi)\overline{\mathcal{F}(g_n)(\xi)}d\xi\right| \overset{Prop. 4.2.1}{\leq} \int_{\mathbb{R}} \left|f(x)\overline{g(x)} - f_n(x)\overline{g_n(x)}\right| dx +$$

$$+\int_{\mathbb{R}} \left|\mathcal{F}(f)(\xi)\overline{\mathcal{F}(g)(\xi)} - \mathcal{F}(f_n)(\xi)\overline{\mathcal{F}(g_n)(\xi)}\right| d\xi \overset{n\to+\infty}{\longrightarrow} 0$$

• Proof 3: taking $g = f$ in the equality in 2, we get

$$\int_{\mathbb{R}} f(x)\overline{f(x)}dx = \int_{\mathbb{R}} \mathcal{F}(f)(\xi)\overline{\mathcal{F}(f)(\xi)}d\xi \Longrightarrow \int_{\mathbb{R}} |f(x)|^2 dx = \int_{\mathbb{R}} |\mathcal{F}(f)(\xi)|^2 d\xi \Longrightarrow \|f\|_2 = \|\mathcal{F}(f)\|_2$$

Proposition 4.4. *Let $f,\, g \in L^2(\mathbb{R})$. Then, the following statements hold:*

1. *$\mathcal{F}(f)\cdot g$ and $f\cdot\mathcal{F}(g)$ are in $L^1(\mathbb{R})$.*
2. *$\int_{\mathbb{R}} \mathcal{F}(f)(t)\cdot g(t)dt = \int_{\mathbb{R}} f(u)\cdot\mathcal{F}(g)(u)du$*

PROOF.

• Proof 1: we have seen in Theorem 4.1. that $\mathcal{F}(f)$ is in $L^2(\mathbb{R})$. Consequently, by Hölder's Inequality with $p = q = 2$, we conclude $\mathcal{F}(f)\cdot g$ is in $L^1(\mathbb{R})$. We can apply exactly the same argument to deduce that $f\cdot\mathcal{F}(g) \in L^1(\mathbb{R})$.

• Proof 2: as $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exist two sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{S}(\mathbb{R})$ such that converge to f and g in $L^2(\mathbb{R})$ respectively. Moreover, since $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$, we have that f_n and g_n are in $L^1(\mathbb{R})$. Hence, we can apply Property 1.8.

$$\int_{\mathbb{R}} \mathcal{F}(f_n)(t)\cdot g_n(t)dt = \int_{\mathbb{R}} f_n(u)\cdot\mathcal{F}(g_n)(u)du$$

Finally, using this last equality, we deduce that

$$\left|\int_{\mathbb{R}} \mathcal{F}(f)(t)\cdot g(t)dt - \int_{\mathbb{R}} f(u)\cdot\mathcal{F}(g)(u)du\right| \leq \left|\int_{\mathbb{R}} \mathcal{F}(f)(t)\cdot g(t)dt - \int_{\mathbb{R}} \mathcal{F}(f_n)(t)\cdot g_n(t)dt\right| +$$

$$+\left|\int_{\mathbb{R}} f_n(u)\cdot\mathcal{F}(g_n)(u)du - \int_{\mathbb{R}} f(u)\cdot\mathcal{F}(g)(u)du\right| \leq \int_{\mathbb{R}} |\mathcal{F}(f)(t)\cdot g(t) - \mathcal{F}(f_n)(t)\cdot g_n(t)|\, dt +$$

$$+\int_{\mathbb{R}} |f_n(u)\cdot\mathcal{F}(g_n)(u) - f(u)\cdot\mathcal{F}(g)(u)|\, du \overset{n\to+\infty}{\longrightarrow} 0$$

Lemma 4.1. *Let $f \in L^1_{loc}(\mathbb{R})$ such that $\int_{\mathbb{R}} f(t)\phi(t)dt = 0$ for all $\phi \in \mathcal{D}(\mathbb{R})$. Then $f = 0$ a.e. on $\mathbb{R}$.*

PROOF.

We divide the proof in 3 steps:

• Step 1: let $(\rho_n(t))_{n\in\mathbb{N}}$ be the regularizing sequence built in Definition 3.5. We will show that the regularizations $\rho_n * f$ are zero. In effect, let $x \in \mathbb{R}$ and let us consider $(\rho_n(x-t))_{n\in\mathbb{N}}$. It is clear $\rho_n(x-t) \in \mathcal{D}(\mathbb{R})$ since $\rho_n(t) \in \mathcal{D}(\mathbb{R})$. Thus, using the hypothesis

$$(\rho_n * f)(x) = \int_{\mathbb{R}} f(t)\rho_n(x-t)dx = 0 \quad \text{for all } x \in \mathbb{R}$$

• Step 2: let $a > 0$ and let $b := a + 1$. We will show that for all $|x| \leq a$, $(\rho_n * f)(x) = \left(\rho_n * \mathbf{1}_{[-b,b]} f\right)(x) = 0$. Let $|x| \leq a$, we know that ρ_n has the support contained in $[-1, 1]$. Hence

$$(\rho_n * f)(x) = \int_{\mathbb{R}} f(t)\rho_n(x-t)dx = \int_{\mathbb{R}} \mathbf{1}_{[-b,b]}(t)f(t)\rho_n(x-t)dx = \left(\rho_n * \mathbf{1}_{[-b,b]} f\right)(x) \overset{step\, 1}{=} 0$$

where we have used that $|x - t| \leq 1 \implies |t| \leq 1 + |x| \leq 1 + a = b$.

• Step 3: finally we conclude $f = 0$ a.e. on $\mathbb{R}$. For this, we note the following fact:

$$\int_{-a}^{+a} |f(x)|\, dx \leq \int_{-a-1}^{+a+1} |f(x)|\, dx = \int_{\mathbb{R}} \left|f(x)\mathbf{1}_{[-b,b]}(x)\right| dx \overset{step\, 2}{=}$$

$$= \int_{\mathbb{R}} \left|f(x)\mathbf{1}_{[-b,b]}(x) - \left(\rho_n * \mathbf{1}_{[-b,b]} f\right)(x)\right| dx = \left\|f\mathbf{1}_{[-b,b]} - \left(f\mathbf{1}_{[-b,b]}\right) * \rho_n\right\|_1$$

Now, taking the limit when n goes to $+\infty$ in both sides and using Proposition 3.6. with $p = 1$, we get

$$\int_{-a}^{+a} |f(x)|\, dx \leq \lim_{n \longrightarrow +\infty} \left\|f\mathbf{1}_{[-b,b]} - \left(f\mathbf{1}_{[-b,b]}\right) * \rho_n\right\|_1 = 0 \implies \int_{-a}^{+a} |f(x)|\, dx = 0$$

Hence $f(x) = 0$ a.e. for $|x| \leq a$. As we have taken $a > 0$ arbitrary, we conclude $f = 0$ almost everywhere in $\mathbb{R}$.

Proposition 4.5. *The Fourier Transform defined on $L^1(\mathbb{R})$ and the one obtained by extension to $L^2(\mathbb{R})$ coincide on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Moreover, if $f \in L^2(\mathbb{R})$, then $\mathcal{F}(f)$ is the limit in $L^2(\mathbb{R})$ of the sequence g_n defined by*

$$g_n(\xi) := \int_{-n}^{+n} e^{-2\pi i \xi x} f(x) dx$$

PROOF.

• Let $f \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ and $\psi \in \mathcal{S}(\mathbb{R})$. As $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ and $\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$ then $\psi \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Consequently, we can apply Property 1.8. and Proposition 4.4.2. Moreover, as for all functions $\psi \in \mathcal{S}(\mathbb{R})$ hold that $\hat{\psi} = \mathcal{F}(\psi)$, we get:

$$\int_{\mathbb{R}} \psi(x)\hat{f}(x)dx \overset{Prop.1.8}{=} \int_{\mathbb{R}} \hat{\psi}(u)f(u)du = \int_{\mathbb{R}} \mathcal{F}(\psi)(u)f(u)du \overset{Prop.4.4.2}{=} \int_{\mathbb{R}} \psi(x)\mathcal{F}(f)(x)dx$$

Hence, $\int_{\mathbb{R}} \psi(x)\left(\hat{f}(x) - \mathcal{F}(f)(x)\right) dx = 0$ for all $\psi \in \mathcal{S}(\mathbb{R})$. Thus, using Lemma 4.1. and the density of $\mathcal{D}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$, we conclude $\hat{f} = \mathcal{F}(f)$ almost everywhere.

• Let us consider $f_n := f\mathbf{1}_{[-n,n]}$. We apply the Dominated Convergence Theorem to the function $(f_n - f)^2$ since we see that converges pointwise to 0 and is dominated by $4f^2 \in L^1(\mathbb{R})$. Thus:

$$\lim_{n \longrightarrow +\infty} \|f_n - f\|_2 = 0$$

Moreover, as f and $\mathbf{1}_{[-n,n]}$ are in $L^2(\mathbb{R})$, it holds $f_n = f\mathbf{1}_{[-n,n]}$ is in $L^1(\mathbb{R})$ (Hölder's inequality). Hence $f_n \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$, and consequently we get $g_n = \hat{f}_n = \mathcal{F}(f_n)$. Moreover, as we have seen that $\mathcal{F} : L^2(\mathbb{R}) \longrightarrow L^2(\mathbb{R})$ is continuous and $f_n \overset{n \longrightarrow +\infty}{\longrightarrow} f$ in $L^2(\mathbb{R})$ then: $g_n = \mathcal{F}(f_n) \overset{n \longrightarrow +\infty}{\longrightarrow} \mathcal{F}(f)$ in $L^2(\mathbb{R})$.

Remark 4.1. *The previous proposition is also true for the extension of the operator* $\check{\ }$ *on* $L^2(\mathbb{R})$. *The Inverse Fourier Transform defined on* $L^1(\mathbb{R})$ *and the one obtained by extension to* $L^2(\mathbb{R})$ *coincide on* $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. *Moreover, if* $f \in L^2(\mathbb{R})$, *then* $\overline{\mathcal{F}}(f)$ *is the limit in* $L^2(\mathbb{R})$ *of the sequence* h_n *defined by:*

$$h_n(\xi) := \int_{-n}^{+n} e^{2\pi i \xi x} f(x) dx$$

Note that the proof is completely analogous since the extension $\overline{\mathcal{F}}$ *has the same properties as the extension* $\mathcal{F}$.

Theorem 4.2. *Let* $f \in L^2(\mathbb{R})$. *Suppose that* $\mathcal{F}(f) \in L^1(\mathbb{R})$. *Then:*

$$f(x) = \int_{\mathbb{R}} \mathcal{F}(f)(\xi) e^{2\pi i \xi x} d\xi \quad a.e.$$

i.e. computing the Inverse Fourier Transform of $\mathcal{F}(f)$ *on* $L^1(\mathbb{R})$, *we recover* f *a.e.*

PROOF.
As $f \in L^2(\mathbb{R})$, we get by Theorem 4.1. that $\mathcal{F}(f) \in L^2(\mathbb{R})$. Hence, using the hypothesis of the statement, we conclude that $\mathcal{F}(f) \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Now, using the fact that the Inverse Fourier Transform defined on $L^1(\mathbb{R})$ and the one obtained by extension to $L^2(\mathbb{R})$ coincide on $L^2(\mathbb{R}) \cap L^1(\mathbb{R})$ (remark 4.1.) and that $\overline{\mathcal{F}}\mathcal{F}(f) = f$ a.e. (Theorem 4.1.1) we conclude:

$$f(x) = \int_{\mathbb{R}} \mathcal{F}(f)(\xi) e^{2\pi i \xi x} d\xi \quad \text{a.e.}$$

Remark 4.2. *If* $f \in L^1(\mathbb{R})$, *definition 1.1. defines* $\hat{f}(\xi)$ *unambiguously for every* $\xi \in \mathbb{R}$ *(since* $\hat{f}$ *is defined everywhere and is continuous). However, if* $f \in L^2(\mathbb{R})$, *then the extension* $\mathcal{F}$ *defines* $\mathcal{F}(f)$ *uniquely as an element of the Hilbert space* $L^2(\mathbb{R})$, *but as a point function is only determined almost everywhere. This is an important difference between the theory of Fourier transforms in* $L^1(\mathbb{R})$ *and* $L^2(\mathbb{R})$.

4.2 Application to the computation of certain Fourier transforms

Proposition 4.6. *The following statements hold:*

1. *If* $f \in L^2(\mathbb{R})$, *then* $\mathcal{F}\mathcal{F}(f) = f_\sigma$ *almost everywhere.*
2. *If* $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, *then* $\mathcal{F}\left(\hat{f}\right) = f_\sigma$ *almost everywhere.*

PROOF.
• Proof 1: as $\mathcal{S}(\mathbb{R})$ is dense in $L^2(\mathbb{R})$, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{S}(\mathbb{R})$ such that $\lim_{n\longrightarrow+\infty} \|f_n - f\|_2 = 0$. Now, as $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$, by Proposition 4.5. and Proposition 1.2.2:

$$\mathcal{F}(f_n)(\xi) \overset{Prop.\,4.5}{=} \widehat{f_n}(\xi) \overset{Prop.\,1.2.2}{=} \widehat{[(f_n)_\sigma]}(-\xi) = [(\check{f_n})_\sigma](\xi) \overset{Remark\,4.1.}{=} \overline{\mathcal{F}}([f_n]_\sigma)(\xi)$$

Taking the limit in both sides of this equality and using the continuity $\mathcal{F}$ and $\overline{\mathcal{F}}$ on $L^2(\mathbb{R})$:

$$\mathcal{F}(f) = \overline{\mathcal{F}}(f_\sigma) \implies \mathcal{F}\mathcal{F}(f) = \mathcal{F}\overline{\mathcal{F}}(f_\sigma) \overset{Theorem\ 4.1.1}{\implies} \mathcal{F}\mathcal{F}(f) = f_\sigma \quad \text{a.e.}$$

• Proof 2: as $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, from Proposition 4.5. it holds $\hat{f} = \mathcal{F}(f)$ almost everywhere. Using this in 1. we get: $\mathcal{F}\left(\hat{f}\right) = f_\sigma$ a.e.

Example 4.1. Let us consider the function $f(x) = e^{-ax}u(x)$ $(a > 0)$ where u is the Heaviside function defined as:

$$u(x) = \begin{cases} 1 & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

Its Fourier Transform is $\hat{f}(\xi) = \frac{1}{a+2i\pi\xi}$. This function is not in $L^1(\mathbb{R})$ and we can't compute its Fourier Transform using definition 1.1., since the integral is not defined. However, it holds that $\hat{f}$ is in $L^2(\mathbb{R})$. Consequently, we can compute its Fourier Transform using the extension defined in Theorem 4.1. Moreover, as $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$, Proposition 4.6.2 can be applied and we get

$$\mathcal{F}\left(\hat{f}\right)(x) = f_\sigma(x) = f(-x) = e^{ax}u(-x)$$

Chapter 5

Convolution and the Fourier Transform

5.1 Convolution and the Fourier transform in $L^1(\mathbb{R})$

Proposition 5.1. *If f and $\hat{f}$ are in $L^1(\mathbb{R})$, then $\check{\hat{f}} = f$ almost everywhere.*

PROOF.
From Corollary 3.2., $\mathcal{S}(\mathbb{R})$ is dense in $L^1(\mathbb{R})$. Hence, there exists a sequence f_n in $\mathcal{S}(\mathbb{R})$ such that $\lim_{n\to+\infty} \|f - f_n\|_1 = 0$. Moreover, by Theorem 2.2. $f_n = \check{\hat{f}}_n$. Hence, taking $\varphi \in \mathcal{S}(\mathbb{R})$

$$\int_{\mathbb{R}} f_n(t)\varphi(t)dt = \int_{\mathbb{R}} \check{\hat{f}}_n(t)\varphi(t)dt \overset{Property\,1.8}{=} \int_{\mathbb{R}} \hat{f}_n(u)\check{\varphi}(u)du \tag{5.1.1}$$

• Moreover, on the one hand, as $\varphi \in \mathcal{S}(\mathbb{R})$, then it is bounded, that is, $\varphi \in L^\infty(\mathbb{R})$. Thus:

$$\lim_{n\to+\infty} \left| \int_{\mathbb{R}} f_n(t)\varphi(t)dt - \int_{\mathbb{R}} f(t)\varphi(t)dt \right| \le \lim_{n\to+\infty} \int_{\mathbb{R}} |f_n(t) - f(t)|\,|\varphi(t)|\,dt \le \lim_{n\to+\infty} \|\varphi\|_\infty \int_{\mathbb{R}} |f_n(t) - f(t)|\,dt$$

$$= \|\varphi\|_\infty \lim_{n\to+\infty} \|f - f_n\|_1 = 0 \implies \lim_{n\to+\infty} \int_{\mathbb{R}} f_n(t)\varphi(t)dt = \int_{\mathbb{R}} f(t)\varphi(t)dt \tag{5.1.2}$$

• On the other hand, using Property 1.6.: $\lim_{n\to+\infty} \left\| \hat{f} - \hat{f}_n \right\|_\infty \le \lim_{n\to+\infty} \|f - f_n\|_1 = 0$. Applying this:

$$\lim_{n\to+\infty} \left| \int_{\mathbb{R}} \hat{f}_n(u)\check{\varphi}(u)du - \int_{\mathbb{R}} \hat{f}(u)\check{\varphi}(u)du \right| \le \lim_{n\to+\infty} \int_{\mathbb{R}} \left| \hat{f}_n(u) - \hat{f}(u) \right| |\check{\varphi}(u)|\,du \le$$

$$\le \|\check{\varphi}\|_1 \lim_{n\to+\infty} \left\| \hat{f} - \hat{f}_n \right\|_\infty = 0 \implies \lim_{n\to+\infty} \int_{\mathbb{R}} \hat{f}_n(u)\check{\varphi}(u)du = \int_{\mathbb{R}} \hat{f}(u)\check{\varphi}(u)du \tag{5.1.3}$$

Hence, taking the limit in both sides of (5.1.1) and using (5.1.2) and (5.1.3), we get $\int_{\mathbb{R}} f(t)\varphi(t)dt = \int_{\mathbb{R}} \hat{f}(u)\check{\varphi}(u)du$. Finally, as $\hat{f}$ is in $L^1(\mathbb{R})$ and $\varphi \in \mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ we can apply Property 1.8. to the right hand side. Thus, we get

$$\int_{\mathbb{R}} f(t)\varphi(t)dt = \int_{\mathbb{R}} \check{\hat{f}}(u)\varphi(u)du \implies \int_{\mathbb{R}} \left(f(u) - \check{\hat{f}}(u) \right) \varphi(u)du$$

Last equality is true for all $\varphi \in \mathcal{S}(\mathbb{R})$. Hence by Lemma 4.1. we conclude that $\check{\hat{f}} = f$ almost everywhere.

Remark 5.1. *The result implies that if f and $\hat{f}$ are in $L^1(\mathbb{R})$, then the equivalence class to which f belongs (integrable functions that are equal to f almost everywhere), contains a continuous representative, namely $\check{\hat{f}}$.*

Proposition 5.2. *Given f and g in $L^1(\mathbb{R})$, the following statements hold:*

1. $\widehat{f * g}\,(\xi) = \hat{f}\,(\xi)\,\hat{g}\,(\xi)$ *for all* $\xi \in \mathbb{R}$
2. *If in addition $\hat{f}$ and $\hat{g}$ are in $L^1(\mathbb{R})$ then* $\widehat{fg}\,(\xi) = \left(\hat{f} * \hat{g}\right)(\xi)$ *for all* $\xi \in \mathbb{R}$.

PROOF.
• Proof 1: by Proposition 3.1. $f * g$ is in $L^1(\mathbb{R})$. Hence the function $e^{-2i\pi\xi x}\,(f * g)\,(x)$ is in $L^1(\mathbb{R})$ for all $\xi \in \mathbb{R}$, and we can apply Theorem A.6. (Fubini):

$$\widehat{f * g}\,(\xi) = \int_{\mathbb{R}} e^{-2i\pi\xi x}\left(\int_{\mathbb{R}} f(x-t)g(t)dt\right)dx = \int_{\mathbb{R}} g(t)\left(\int_{\mathbb{R}} e^{-2i\pi\xi x} f(x-t)dx\right)dt \overset{u=x-t}{=}$$

$$\int_{\mathbb{R}} e^{-2i\pi\xi t} g(t)\left(\int_{\mathbb{R}} e^{-2i\pi\xi u} f(u)du\right)dt = \hat{f}(\xi)\int_{\mathbb{R}} e^{-2i\pi\xi t} g(t)dt = \hat{f}(\xi)\hat{g}(\xi) \quad \text{for all } \xi \in \mathbb{R}$$

• Proof 2: note that 1. is also true changing $\hat{\cdot}$ by $\check{\cdot}$, we only have to change i by $-i$ in the previous proof. Thus, since $\hat{f},\, \hat{g} \in L^1(\mathbb{R})$, we can apply statement 1. with $\check{\cdot}$ and we get:

$$\check{\left[\hat{f} * \hat{g}\right]}(x) = \check{\hat{f}}\,(x)\,\check{\hat{g}}\,(x) \overset{Proposition\, 5.1.}{=} f(x)g(x) \quad \text{a.e.}$$

Note now that fg is in $L^1(\mathbb{R}) \cap C(\mathbb{R})$ since f and g are in $L^1(\mathbb{R}) \cap L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. Hence, taking the Fourier transform in both sides of the previous equality and using Theorem 1.1., we conclude

$$\left(\hat{f} * \hat{g}\right)(\xi) = \widehat{fg}(\xi) \quad \text{for all } \xi \in \mathbb{R}$$

Proposition 5.3. *Given f and g in $\mathcal{S}(\mathbb{R})$, the following statements hold:*

1. $\widehat{f * g}\,(\xi) = \hat{f}\,(\xi)\,\hat{g}\,(\xi)$ *for all* $\xi \in \mathbb{R}$
2. $\widehat{fg}\,(\xi) = \left(\hat{f} * \hat{g}\right)(\xi)$ *for all* $\xi \in \mathbb{R}$

PROOF. This result is direct consequence of Proposition 5.2. since $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ is invariant under the Fourier transform.

5.2 Convolution and the Fourier transform in $L^2(\mathbb{R})$

In chapter 4, we extended the Fourier transform from $\mathcal{S}(\mathbb{R})$ to $L^2(\mathbb{R})$. Moreover, we saw in chapter 3, in Proposition 3.3., that the convolution is a continuous operator from $L^2(\mathbb{R}) \times L^2(\mathbb{R})$ to $L^\infty(\mathbb{R}) \cap C^0(\mathbb{R})$. We will see that a similar result to Proposition 5.3 holds in the case that $f, g \in L^2(\mathbb{R})$, understanding the Fourier transform as the extension defined in Theorem 4.1.

Proposition 5.4. *Given f and g in $L^2(\mathbb{R})$, then the following statements hold:*

1. $(f * g)(t) = [\mathcal{F}f \cdot \mathcal{F}g]\check{}\,(t)$ *for all* $t \in \mathbb{R}$
2. $\widehat{fg}\,(t) = (\mathcal{F}f * \mathcal{F}g)(t)$ *for all* $t \in \mathbb{R}$

PROOF.
• Proof 1: using the density of $\mathcal{S}(\mathbb{R})$ in $L^2(\mathbb{R})$ (Proposition 4.1.), we get that there exist two sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R})$ such that:

$$\lim_{n\longrightarrow\infty} \|f - f_n\|_2 = 0 \qquad \text{and} \qquad \lim_{n\longrightarrow\infty} \|g - g_n\|_2 = 0$$

By Proposition 5.3.1. $[\widehat{f_n * g_n}](\xi) = \hat{f}_n(\xi)\,\hat{g}_n(\xi)$, taking $\check{}$ and using Theorem 2.2. we get $(f_n * g_n)(x) = \left[\hat{f}_n\hat{g}_n\right]\check{}\,(x)$ for all x. Moreover as $\mathcal{F}f,\ \mathcal{F}g,\ \hat{f}_n,\ \hat{g}_n \in L^2(\mathbb{R})$, then $\mathcal{F}f \cdot \mathcal{F}g,\ f_n g_n \in L^1(\mathbb{R})$. Thus:

$$\left\|\mathcal{F}f \cdot \mathcal{F}g - \hat{f}_n\hat{g}_n\right\|_1 = \left\|\mathcal{F}f \cdot \mathcal{F}g - \hat{f}_n\mathcal{F}g + \hat{f}_n\mathcal{F}g - \hat{f}_n\hat{g}_n\right\|_1 \overset{\text{Hölder Ineq.}}{\leq} \left\|\mathcal{F}f - \hat{f}_n\right\|_2 \|\mathcal{F}g\|_2 +$$
$$+ \|\mathcal{F}g - \hat{g}_n\|_2 \left\|\hat{f}_n\right\|_2 \overset{\text{Theor. 4.1.3}}{=} \|f - f_n\|_2 \|g\|_2 + \|g - g_n\|_2 \|f_n\|_2 \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

Moreover, applying Property 1.6. we get:

$$\lim_{n\longrightarrow\infty} \left\|[\mathcal{F}f \cdot \mathcal{F}g]\check{} - \left[\hat{f}_n\hat{g}_n\right]\check{}\right\|_\infty \leq \lim_{n\longrightarrow\infty} \left\|\mathcal{F}f \cdot \mathcal{F}g - \hat{f}_n\hat{g}_n\right\|_1 = 0 \tag{5.2.1}$$

On the other hand, by proposition 3.3. with $p = q = 2$:

$$\|f * g - f_n * g_n\|_\infty = \|f * g - f_n * g + f_n * g - f_n * g_n\|_\infty \leq \|f - f_n\|_2 \|g\|_2 +$$
$$+ \|g - g_n\|_2 \|f_n\|_2 \overset{n\longrightarrow\infty}{\longrightarrow} 0 \tag{5.2.2}$$

Finally, using (5.2.1) and (5.2.2), we get:

$$\left\|f * g - [\mathcal{F}f \cdot \mathcal{F}g]\check{}\right\|_\infty \leq \|f * g - f_n * g_n\|_\infty + \left\|f_n * g_n - \left[\hat{f}_n\hat{g}_n\right]\check{}\right\|_\infty + \left\|\left[\hat{f}_n\hat{g}_n\right]\check{} - [\mathcal{F}f \cdot \mathcal{F}g]\check{}\right\|_\infty \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

Hence, as $f * g$ and $[\mathcal{F}f \cdot \mathcal{F}g]\check{}$ are continuous, we conclude that $(f * g)(t) = [\mathcal{F}f \cdot \mathcal{F}g]\check{}\,(t)$ for all $t \in \mathbb{R}$.

• Proof 2: let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ be the sequences defined in 1. By proposition 5.3.2 we have $\hat{f}_n * \hat{g}_n = \widehat{f_n g_n}$. Moreover, as $\mathcal{F}f,\ \mathcal{F}g,\ \hat{f}_n,\ \hat{g}_n \in L^2(\mathbb{R})$, by Proposition 3.3. $\hat{f}_n * \hat{g}_n$ and $\mathcal{F}f * \mathcal{F}g$ are in $L^\infty(\mathbb{R}) \cap C(\mathbb{R})$. Thus

$$\left\|\mathcal{F}f * \mathcal{F}g - \hat{f}_n * \hat{g}_n\right\|_\infty = \left\|\mathcal{F}f * \mathcal{F}g - \hat{f}_n * \mathcal{F}g + \hat{f}_n * \mathcal{F}g - \hat{f}_n * \hat{g}_n\right\|_\infty \overset{\text{Prop. 3.3}}{\leq} \left\|\mathcal{F}f - \hat{f}_n\right\|_2 \|\mathcal{F}g\|_2 +$$
$$+ \|\mathcal{F}g - \hat{g}_n\|_2 \left\|\hat{f}_n\right\|_2 \overset{\text{Theor. 4.1.3}}{=} \|f - f_n\|_2 \|g\|_2 + \|g - g_n\|_2 \|f_n\|_2 \overset{n\longrightarrow\infty}{\longrightarrow} 0 \tag{5.2.3}$$

On the other hand, using Property 1.6:

$$\left\|\widehat{fg} - \widehat{f_n g_n}\right\|_\infty \leq \|fg - f_n g_n\|_1 = \|fg - f_n g + f_n g - f_n g_n\|_1 \overset{\text{Hölder}}{\leq}$$
$$\leq \|f - f_n\|_2 \|g\|_2 + \|g - g_n\|_2 \|f_n\|_2 \overset{n\longrightarrow\infty}{\longrightarrow} 0 \tag{5.2.4}$$

Finally, using (5.2.3) and (5.2.4), we get:

$$\left\|\mathcal{F}f * \mathcal{F}g - \widehat{fg}\right\|_\infty \leq \left\|\mathcal{F}f * \mathcal{F}g - \hat{f}_n * \hat{g}_n\right\|_\infty + \left\|\hat{f}_n * \hat{g}_n - \widehat{f_n g_n}\right\|_\infty + \left\|\widehat{fg} - \widehat{f_n g_n}\right\|_\infty \longrightarrow 0$$

Hence, as $\mathcal{F}f * \mathcal{F}g$ and $\widehat{fg}$ are continuous, we conclude that $\widehat{fg}\,(t) = (\mathcal{F}f * \mathcal{F}g)(t)$ for all $t \in \mathbb{R}$.

Remark 5.2. *Note that with the assumptions of the previous proposition, the formula* $\widehat{(f*g)}\,(t) = (\mathcal{F}f \cdot \mathcal{F}g)\,(t)$ *does not make sense a priori, since* $f*g$ *is only in* $L^\infty(\mathbb{R}) \cap C(\mathbb{R})$*. This formula is true whenever* $f*g$ *is in* $L^1(\mathbb{R})$*.*

Proposition 5.5. *If* $f \in L^2(\mathbb{R})$ *and* $g \in L^1(\mathbb{R})$*, then* $\mathcal{F}f \cdot \hat{g} \in L^2(\mathbb{R})$ *and* $f*g = \overline{\mathcal{F}}(\mathcal{F}f \cdot \hat{g})$ *in* $L^2(\mathbb{R})$*.*

PROOF.

Let $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R})$ such that $\lim\limits_{n\longrightarrow\infty} \|f-f_n\|_2 = 0$ and $\lim\limits_{n\longrightarrow\infty} \|g-g_n\|_1 = 0$. Note that $\mathcal{F}f \in L^2(\mathbb{R})$ (Theorem 4.1.) and $\hat{g} \in L^\infty(\mathbb{R})$ (property 1.6.). Then $\mathcal{F}f \cdot \hat{g} \in L^2(\mathbb{R})$ since:

$$\int_{\mathbb{R}} (\mathcal{F}f(x)\hat{g}(x))^2\,dx \;\leq\; \|g\|_\infty^2 \int_{\mathbb{R}} \mathcal{F}f(x)^2 dx \;=\; \|g\|_\infty^2 \,\|\mathcal{F}f\|_2^2 \;=\; \|g\|_\infty^2 \,\|f\|_2^2 \;<\; +\infty \tag{5.2.5}$$

• Moreover, on the one hand we have

$$\left\|\overline{\mathcal{F}}(\mathcal{F}f\cdot\hat{g}) - \overline{\mathcal{F}}\left(\hat{f}_n\hat{g}_n\right)\right\|_2 \overset{Theor.\,4.1.}{=} \left\|\mathcal{F}f\cdot\hat{g} - \hat{f}_n\hat{g}_n\right\|_2 = \left\|\mathcal{F}f\cdot\hat{g} - \hat{f}_n\hat{g} + \hat{f}_n\hat{g} - \hat{f}_n\hat{g}_n\right\|_2 \overset{(5.2.5)}{\leq} \left\|\mathcal{F}f - \hat{f}_n\right\|_2 \|\hat{g}\|_\infty$$

$$+ \left\|\hat{f}_n\right\|_2 \|\hat{g}-\hat{g}_n\|_\infty \underset{Property\,1.6.}{\overset{Theorem\,4.1}{\leq}} \|f-f_n\|_2\,\|\hat{g}\|_\infty + \|f_n\|_2\,\|g-g_n\|_1 \overset{n\longrightarrow\infty}{\longrightarrow} 0 \tag{5.2.6}$$

• On the other hand, by Lemma 3.3., the convolution is continuous from $L^2(\mathbb{R}) * L^1(\mathbb{R})$ to $L^2(\mathbb{R})$. Hence:

$$\lim_{n\longrightarrow\infty} \|f*g - f_n*g_n\|_2 \;=\; 0 \tag{5.2.7}$$

Finally, using (5.2.6), (5.2.7) and $f_n * g_n = \overline{\mathcal{F}}\left(\hat{f}_n\hat{g}_n\right)$ (Proposition 5.3.1):

$$\left\|f*g - \overline{\mathcal{F}}(\mathcal{F}f\cdot\hat{g})\right\|_2 \leq \|f*g - f_n*g_n\|_2 + \left\|f_n*g_n - \overline{\mathcal{F}}\left(\hat{f}_n\hat{g}_n\right)\right\|_2 + \left\|\overline{\mathcal{F}}\left(\hat{f}_n\hat{g}_n\right) - \overline{\mathcal{F}}(\mathcal{F}f\cdot\hat{g})\right\|_2 \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

Hence, we conclude $f*g = \overline{\mathcal{F}}(\mathcal{F}f\cdot\hat{g})$ in $L^2(\mathbb{R})$.

Chapter 6

The multidimensional Fourier Transform

The previous chapters introduced the theory of the Fourier transform on $\mathbb{R}$. We will use it to study some applications in Partial Differential Equations such as the heat equation. However, we are also interested in the study of these Partial Differential Equations in higher dimensions. For this reason, the aim of this chapter is to extend the notion of the Fourier transform to $\mathbb{R}^d$ with $d > 1$ and make a brief study of its properties.

6.1 Preliminaries

- The setting in this chapter will be $\mathbb{R}^d$, the vector space of all d-tuples of real numbers $(x_1, x_2, ..., x_d)$ with $x_i \in \mathbb{R}$.
- Given $\mathbf{x} = (x_1, x_2, ..., x_d)$, we define its Euclidian norm as:

$$|\mathbf{x}| = \left(x_1^2 + ... + x_d^2\right)^{\frac{1}{2}}$$

- In fact, we equip $\mathbb{R}^d$ with the standard inner product:

$$\mathbf{x} \cdot \mathbf{y} = x_1 y_1 + ... + x_d y_d$$

- Given a multi-index $\alpha = (\alpha_1, ..., \alpha_n)$ of non-negative integers, we define the monomial $\mathbf{x}^\alpha$ and the operator $\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha$ by:

$$\mathbf{x}^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} ... x_d^{\alpha_d} \quad \text{and} \quad \left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \left(\frac{\partial}{\partial x_2}\right)^{\alpha_2} ... \left(\frac{\partial}{\partial x_d}\right)^{\alpha_d} = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} ... \partial x_d^{\alpha_d}}$$

Definition 6.1. *A rotation in $\mathbb{R}^d$ is a linear transformation $R : \mathbb{R}^d \longrightarrow \mathbb{R}^d$ which preserves the inner product. That is:*

1. $R(a\mathbf{x} + b\mathbf{y}) = aR(\mathbf{x}) + bR(\mathbf{y})$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ *and* $a, b \in \mathbb{R}$
2. $R(\mathbf{x}) \cdot R(\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$ *for all* $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$

Equivalently, the last condition can be replaced by $|R(\mathbf{x})| = |\mathbf{x}|$ for all $\mathbf{x} \in \mathbb{R}^d$. Or, $R^t = R^{-1}$ where R^t and R^{-1} denote the transpose and inverse of R, respectively. In particular, it holds that $\det(R) = \pm 1$. If $\det(R) = 1$ we say that R is a proper rotation. Otherwise, we say that R is an improper rotation.

Definition 6.2. *A function $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ is said to decay rapidly, or be rapidly decreasing, if for every multi-index α:*

$$\lim_{|\mathbf{x}| \longrightarrow +\infty} |\mathbf{x}^\alpha f(\mathbf{x})| = 0$$

The definition implies that for all $\epsilon > 0$ there exists $M_{\epsilon,\alpha} > 0$ such that $|\mathbf{x}^\alpha f(\mathbf{x})| < \epsilon$ for all $\mathbf{x} \in \mathbb{R}^d$ such that $|\mathbf{x}| > M_{\epsilon,\alpha}$.

Proposition 6.1. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a continuous rapidly decreasing function. Then $f \in L^1(\mathbb{R}^d)$.*

PROOF.
Let us define Q_N as the closed cube centered at the origin, with sides of length N parallel to the coordinate axis, that is,

$$Q_N := \left\{ \mathbf{x} \in \mathbb{R}^d : |x_i| \leq \frac{N}{2} \text{ for } i = 1, ..., d \right\}$$

Let us show that $I_N := \int_{Q_N} |f(\mathbf{x})| \, d\mathbf{x}$ defines a Cauchy sequence as N tends to infinity. Let us take $\epsilon > 0$ and $\alpha = (2, ..., 2)$. We have seen in definition 6.2. that there exists $M_{\epsilon,\alpha} > 0$ such that $|f(\mathbf{x})| < \frac{\epsilon}{2^{2d+1}|\mathbf{x}^\alpha|}$ for all $\mathbf{x} \in \mathbb{R}^d$ such that $|\mathbf{x}| > M_{\epsilon,\alpha}$. Hence, for all $N, \widetilde{N} > 2\max\{M_{\epsilon,\alpha}, 1\}$ hold (assuming $\widetilde{N} > N$):

$$|I_N - I_{\widetilde{N}}| = \left| \int_{Q_N} |f(\mathbf{x})| \, d\mathbf{x} - \int_{Q_{\widetilde{N}}} |f(\mathbf{x})| \, d\mathbf{x} \right| = \left| \int_{\mathbb{R}^d} |f(\mathbf{x})| \, d\mathbf{x} - \int_{\mathbb{R}^d \setminus Q_N} |f(\mathbf{x})| \, d\mathbf{x} - \int_{\mathbb{R}^d} |f(\mathbf{x})| \, d\mathbf{x} + \right.$$

$$\left. + \int_{\mathbb{R}^d \setminus Q_{\widetilde{N}}} |f(\mathbf{x})| \, d\mathbf{x} \right| = \left| - \int_{\mathbb{R}^d \setminus Q_N} |f(\mathbf{x})| \, d\mathbf{x} + \int_{\mathbb{R}^d \setminus Q_{\widetilde{N}}} |f(\mathbf{x})| \, d\mathbf{x} \right| \leq 2 \int_{\mathbb{R}^d \setminus Q_N} |f(\mathbf{x})| \, d\mathbf{x} \leq$$

$$\leq 2 \int_{\mathbb{R}^d \setminus Q_N} \frac{\epsilon}{2^{2d+1} |\mathbf{x}^\alpha|} d\mathbf{x} = \frac{\epsilon}{2^d} \int_{+\frac{N}{2}}^{+\infty} \cdots \int_{+\frac{N}{2}}^{+\infty} \left(\frac{1}{x_1^2 x_2^2 ... x_d^2} \right) dx_1 dx_2 ... dx_d = \frac{\epsilon}{2^d} \left(\int_{+\frac{N}{2}}^{+\infty} \frac{1}{x^2} dx \right)^d =$$

$$= \frac{\epsilon}{2^d} \left(\left[-\frac{1}{x} \right]_{\frac{N}{2}}^{+\infty} \right)^d = \frac{\epsilon}{2^d} \frac{2^d}{N^d} = \frac{\epsilon}{N^d} < \epsilon$$

Hence, I_N is a Cauchy sequence. As $\mathbb{R}^d$ with the Euclidean norm is a complete space, we conclude that $\lim_{N \longrightarrow +\infty} I_N$ exists. Thus:

$$\int_{\mathbb{R}^d} |f(\mathbf{x})| \, d\mathbf{x} = \lim_{N \longrightarrow +\infty} \int_{Q_N} |f(\mathbf{x})| \, d\mathbf{x} < +\infty \implies f \in L^1(\mathbb{R}^d)$$

Proposition 6.2. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ be a continuous rapidly decreasing function. Then, the following statements hold:*

1. $\int_{\mathbb{R}^d} f(\mathbf{x} + h) d\mathbf{x} = \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x}$ *for all* $h \in \mathbb{R}^d$
2. $\int_{\mathbb{R}^d} f(\delta \mathbf{x}) d\mathbf{x} = \frac{1}{\delta^d} \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x}$ *for all* $\delta > 0$
3. $\int_{\mathbb{R}^d} f(R\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^d} f(\mathbf{x}) d\mathbf{x}$ *for every rotation* R.

PROOF.
Note that all the integrals are well-defined since $f \in L^1(\mathbb{R}^d)$ by Proposition 6.1. We will prove the proposition by using Theorem A.10. (changes of variables).

1. $\int_{\mathbb{R}^d} f(\mathbf{x}+h)d\mathbf{x} = \left[\begin{array}{c} \mathbf{x} = g(\mathbf{y}) = \mathbf{y} - h \\ (Dg)(\mathbf{y}) = Id \implies |det\,(Dg)(\mathbf{y})| = 1 \end{array} \right] = \int_{\mathbb{R}^d} f(\mathbf{y})d\mathbf{y}$

2. $\int_{\mathbb{R}^d} f(\delta\mathbf{x})d\mathbf{x} = \left[\begin{array}{c} \mathbf{x} = g(\mathbf{y}) = \frac{1}{\delta}\mathbf{y} \\ (Dg)(\mathbf{y}) = \frac{1}{\delta}(Id) \implies |det\,(Dg)(\mathbf{y})| = \frac{1}{\delta^d} \end{array} \right] = \frac{1}{\delta^d}\int_{\mathbb{R}^d} f(\mathbf{y})d\mathbf{y}$

3. $\int_{\mathbb{R}^d} f(R\mathbf{x})d\mathbf{x} = \left[\begin{array}{c} \mathbf{x} = g(\mathbf{y}) = R^{-1}\mathbf{y} \\ (Dg)(\mathbf{y}) = R^{-1} \implies |det\,(Dg)(\mathbf{y})| = 1 \end{array} \right] = \int_{\mathbb{R}^d} f(\mathbf{y})d\mathbf{y}$

Definition 6.3. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{C}$. We say that f is a radial function if depends only on $|\mathbf{x}|$. In other words, f is radial if there exists a function $f_0(u)$ defined for $u \geq 0$ such that $f(\mathbf{x}) = f_0(|\mathbf{x}|)$.*

Proposition 6.3. *Let $f : \mathbb{R}^d \longrightarrow \mathbb{C}$. Then f is radial if and only if $f(R\mathbf{x}) = f(\mathbf{x})$ for all rotations.*

PROOF.

• Let R be a rotation. It holds that $|R\mathbf{x}| = |\mathbf{x}|$. Hence, as f is radial we have: $f(R\mathbf{x}) = f_0(|R\mathbf{x}|) = f_0(|\mathbf{x}|) = f(\mathbf{x})$.

• Conversely, let us define:

$$f_0(u) = \begin{cases} f(0) & \text{if } u = 0 \\ f(\mathbf{x}) & \text{if } u = |\mathbf{x}| \end{cases}$$

Note that f_0 is well defined, since if $\mathbf{x}$ and $\mathbf{x}'$ are points such that $|\mathbf{x}| = |\mathbf{x}'|$, then there exists a rotation R such that $\mathbf{x}' = R\mathbf{x}$. Hence $f(\mathbf{x}') = f(R\mathbf{x}) = f(\mathbf{x})$.

Finally, we note that $f(\mathbf{x}) = f_0(|\mathbf{x}|)$. Consequently, we conclude that f is radial.

6.2 Multidimensional Fourier transform on $\mathcal{S}\left(\mathbb{R}^d\right)$

Definition 6.4. *We define the space $\mathcal{S}\left(\mathbb{R}^d\right)$, also known as Schwartz Space, as the set of functions $f : \mathbb{R}^d \longrightarrow \mathbb{C}$ such that f is infinitely times differentiable on $\mathbb{R}^d$ and f and $\left(\frac{\partial}{\partial \mathbf{x}}\right)^\beta f$ decay rapidly for every multi-index β.*

Proposition 6.4. *The space $\mathcal{S}\left(\mathbb{R}^d\right)$ has the following properties:*

1. *$\mathcal{S}(\mathbb{R}^d)$ is invariant under multiplication by a polynomial.*
2. *$\mathcal{S}(\mathbb{R}^d)$ is invariant under derivation.*
3. $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$

PROOF.

• Proof 1: let $p \in \mathbb{C}[X_1, ..., X_d]$ (polynomial with complex coefficients) and $f \in \mathcal{S}(\mathbb{R}^d)$. We want to show that $pf \in \mathcal{S}(\mathbb{R}^d)$. It is clear that $pf \in C^\infty(\mathbb{R}^d)$ since f and $p \in C^\infty(\mathbb{R}^d)$. It remains to see that pf and all its derivatives decay rapidly. Let us suppose $p(\mathbf{x}) = \sum_{\alpha \in I} a_\alpha \mathbf{x}^\alpha$ where $a_\alpha \in \mathbb{C}$. Then

$$\lim_{|\mathbf{x}| \to +\infty} |\mathbf{x}^p p(\mathbf{x}) f(\mathbf{x})| \leq \lim_{|\mathbf{x}| \to +\infty} \sum_{\alpha \in I} |a_\alpha| \left|\mathbf{x}^{\alpha+p} f(\mathbf{x})\right| = 0 \text{ for all multi-index } p$$

For the terms $\left(\frac{\partial}{\partial \mathbf{x}}\right)^\beta pf$ the argument will be the same. But in these cases, there will be products of polynomials and partial derivatives of f. However, the limit will be 0 due to $f \in \mathcal{S}(\mathbb{R}^d)$.

• Proof 2: let $f \in \mathcal{S}(\mathbb{R}^d)$. We want to show $\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\beta} f \in \mathcal{S}(\mathbb{R}^d)$. As $f \in C^{\infty}(\mathbb{R}^d)$ then $\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\beta} f \in C^{\infty}(\mathbb{R}^d)$. And as all the partial derivatives of f decay rapidly, then all the partial derivatives of $\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\beta} f$ decay rapidly.

• Proof 3: let $f \in \mathcal{S}(\mathbb{R}^d)$. Then, f decays rapidly and is in $C^{\infty}(\mathbb{R}^d)$. Therefore, we can apply Proposition 6.1 and conclude that $f \in L^1(\mathbb{R}^d)$. Thus $\mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$.

Definition 6.5. *We define the Fourier transform of $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ by:*

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} \qquad \text{for } \boldsymbol{\xi} \in \mathbb{R}^d$$

Proposition 6.5. *Let $f \in \mathcal{S}\left(\mathbb{R}^d\right)$. Then, the following properties hold:*

1. *Let $g(\mathbf{x}) := f(\mathbf{x}+h)$. Then $\hat{g}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}) e^{2\pi i \boldsymbol{\xi}\cdot h}$ whenever $h \in \mathbb{R}^d$.*
2. *Let $g(\mathbf{x}) := f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot h}$. Then $\hat{g}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi}+h)$ whenever $h \in \mathbb{R}^d$.*
3. *Let $g(\mathbf{x}) := f(\delta \mathbf{x})$. Then $\hat{g}(\boldsymbol{\xi}) = \frac{1}{\delta^d} \hat{f}(\frac{\boldsymbol{\xi}}{\delta})$ whenever $\delta > 0$.*
4. *Let $g(\mathbf{x}) := \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} f(\mathbf{x})$. Then $\hat{g}(\boldsymbol{\xi}) = (2\pi i \boldsymbol{\xi})^{\alpha} \hat{f}(\boldsymbol{\xi})$ for all multi-index α.*
5. *Let $g(\mathbf{x}) := (-2\pi i \mathbf{x})^{\alpha} f(\mathbf{x})$. Then $\hat{g}(\boldsymbol{\xi}) = \left(\frac{\partial}{\partial \boldsymbol{\xi}}\right)^{\alpha} \hat{f}(\boldsymbol{\xi})$ for all multi-index α.*
6. *Let $g(\mathbf{x}) := f(R\mathbf{x})$. Then $\hat{g}(\boldsymbol{\xi}) = \hat{f}(R\boldsymbol{\xi})$ whenever R is a rotation.*

PROOF.
Observation: as $f \in \mathcal{S}\left(\mathbb{R}^d\right)$ then f, $\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\beta} f$ and $(-2\pi i \mathbf{x})^{\alpha} f(\mathbf{x})$ are in $C^{\infty}\left(\mathbb{R}^d\right)$ and decay rapidly for every multi-index β and α. Consequently, by proposition 6.1., we conclude that are in $C^{\infty}\left(\mathbb{R}^d\right) \cap L^1\left(\mathbb{R}^d\right)$.

• Proof 1: $\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{x}+h) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} = \left[\begin{array}{c} \mathbf{x} = g(\mathbf{y}) = \mathbf{y} - h \\ (Dg)(\mathbf{y}) = Id \implies |det\,(Dg)(\mathbf{y})| = 1 \end{array}\right] =$
$= e^{2\pi i h\cdot\boldsymbol{\xi}} \int_{\mathbb{R}^d} f(\mathbf{y}) e^{-2\pi i \mathbf{y}\cdot\boldsymbol{\xi}} d\mathbf{y} = e^{2\pi i h\cdot\boldsymbol{\xi}} \hat{f}(\boldsymbol{\xi})$

• Proof 2: $\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} e^{-2\pi i \mathbf{x}\cdot h} d\mathbf{x} = \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot(\boldsymbol{\xi}+h)} d\mathbf{x} = \hat{f}(\boldsymbol{\xi}+h)$

• Proof 3: $\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(\delta\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} = \left[\begin{array}{c} \mathbf{x} = g(\mathbf{y}) = \frac{1}{\delta}\mathbf{y} \\ (Dg)(\mathbf{y}) = \frac{1}{\delta}(Id) \implies |det\,(Dg)(\mathbf{y})| = \frac{1}{\delta^d} \end{array}\right] =$
$= \frac{1}{\delta^d} \int_{\mathbb{R}^d} f(\mathbf{y}) e^{-2\pi i \frac{\mathbf{y}}{\delta}\cdot\boldsymbol{\xi}} d\mathbf{y} = \frac{1}{\delta^d} \hat{f}(\frac{\boldsymbol{\xi}}{\delta})$

• Proof 4: let $\alpha = (\alpha_1, ..., \alpha_d)$ be an arbitrary multi-index. Then we have

$$\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} f(\mathbf{x}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} = \int_{\mathbb{R}^d} \left[\frac{\partial^{(\alpha_1+\alpha_2+..+\alpha_d)}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} ... \partial x_d^{\alpha_d}} f(\mathbf{x})\right] e^{-2\pi i (x_1\xi_1 + x_2\xi_2 + ... + x_d\xi_d)} d\mathbf{x} =$$

$$= \int_{\mathbb{R}^{d-1}} e^{-2\pi i (x_2\xi_2+...+x_d\xi_d)} \left(\int_{\mathbb{R}} e^{-2\pi i x_1 \xi_1} \left(\frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\right) \left[\frac{\partial^{(\alpha_2+..+\alpha_d)}}{\partial x_2^{\alpha_2} ... \partial x_d^{\alpha_d}} f(\mathbf{x})\right] dx_1\right) dx_2 ... dx_d$$

Let us define $h(x_1) := \frac{\partial^{(\alpha_2+..+\alpha_d)}}{\partial x_2^{\alpha_2}...\partial x_d^{\alpha_d}} f(\mathbf{x})$. We know that $\frac{\partial^{(\alpha_2+..+\alpha_d)}}{\partial x_2^{\alpha_2}...\partial x_d^{\alpha_d}} f(\mathbf{x}) \in C^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Hence $h \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and all its derivatives are in $L^1(\mathbb{R})$. Thus, we can apply Proposition 1.1.2 and we get:

$$\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^{d-1}} e^{-2\pi i(x_2\xi_2+...+x_d\xi_d)} \left[(2\pi i\xi_1)^{\alpha_1} \hat{h}(\xi_1)\right] dx_2...dx_d =$$

$$= \int_{\mathbb{R}^{d-1}} e^{-2\pi i(x_2\xi_2+...+x_d\xi_d)} \left[(2\pi i\xi_1)^{\alpha_1} \int_{\mathbb{R}} e^{-2\pi i x_1\cdot\xi_1} \frac{\partial^{(\alpha_2+..+\alpha_d)}}{\partial x_2^{\alpha_2}...\partial x_d^{\alpha_d}} f(\mathbf{x}) dx_1\right] dx_2...dx_d =$$

$$= (2\pi i\xi_1)^{\alpha_1} \int_{\mathbb{R}^d} \left[\frac{\partial^{(\alpha_2+..+\alpha_d)}}{\partial x_2^{\alpha_2}...\partial x_d^{\alpha_d}} f(\mathbf{x})\right] e^{-2\pi i(x_1\xi_1+x_2\xi_2+...+x_d\xi_d)} dx_1dx_2dx_3...dx_d =$$

$$= (2\pi i\xi_1)^{\alpha_1} \int_{\mathbb{R}^{d-1}} e^{-2\pi i(x_1\xi_1+x_3\xi_3...+x_d\xi_d)} \left(\int_{\mathbb{R}} e^{-2\pi i x_2\xi_2} \left(\frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}}\right) \left[\frac{\partial^{(\alpha_3+..+\alpha_d)}}{\partial x_3^{\alpha_3}...\partial x_d^{\alpha_d}} f(\mathbf{x})\right] dx_2\right) dx_1dx_3...dx_d$$

Let us define $h(x_2) := \frac{\partial^{(\alpha_3+..+\alpha_d)}}{\partial x_3^{\alpha_3}...\partial x_d^{\alpha_d}} f(\mathbf{x})$. We know that $\frac{\partial^{(\alpha_3+..+\alpha_d)}}{\partial x_3^{\alpha_3}...\partial x_d^{\alpha_d}} f(\mathbf{x}) \in C^\infty(\mathbb{R}^d) \cap L^1(\mathbb{R}^d)$. Hence $h \in C^\infty(\mathbb{R}) \cap L^1(\mathbb{R})$ and all its derivatives are in $L^1(\mathbb{R})$. Thus, we can apply Proposition 1.1.2

$$\hat{g}(\boldsymbol{\xi}) = (2\pi i\xi_1)^{\alpha_1} \int_{\mathbb{R}^{d-1}} e^{-2\pi i(x_1\xi_1+x_3\xi_3...+x_d\xi_d)} \left[(2\pi i\xi_2)^{\alpha_2} \hat{h}(\xi_2)\right] dx_1dx_3...dx_d =$$

$$= (2\pi i\xi_1)^{\alpha_1} \int_{\mathbb{R}^{d-1}} e^{-2\pi i(x_1\xi_1+x_3\xi_3...+x_d\xi_d)} \left[(2\pi i\xi_2)^{\alpha_2} \int_{\mathbb{R}} e^{-2\pi i x_2\cdot\xi_2} \frac{\partial^{(\alpha_3+..+\alpha_d)}}{\partial x_3^{\alpha_3}...\partial x_d^{\alpha_d}} f(\mathbf{x}) dx_2\right] dx_1dx_3...dx_d =$$

$$= (2\pi i\xi_1)^{\alpha_1} (2\pi i\xi_2)^{\alpha_2} \int_{\mathbb{R}^d} \left[\frac{\partial^{(\alpha_3+..+\alpha_d)}}{\partial x_3^{\alpha_3}...\partial x_d^{\alpha_d}} f(\mathbf{x})\right] e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} dx_1dx_2dx_3...dx_d$$

Doing the same for the coordinates $x_3, ..., x_d$ we finally get that:

$$\hat{g}(\boldsymbol{\xi}) = (2\pi i\xi_1)^{\alpha_1} (2\pi i\xi_2)^{\alpha_2} ... (2\pi i\xi_d)^{\alpha_d} \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} dx_1dx_2dx_3...dx_d = (2\pi i\boldsymbol{\xi})^\alpha \hat{f}(\boldsymbol{\xi})$$

- Proof 5: let $\alpha = (\alpha_1, ..., \alpha_d)$ be an arbitrary multi-index. Then we have:

$$\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} (-2i\pi\mathbf{x})^\alpha f(\mathbf{x}) e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} d\mathbf{x} = \int_{\mathbb{R}^d} (-2i\pi x_1)^{\alpha_1} ... (-2i\pi x_d)^{\alpha_d} f(\mathbf{x}) e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} dx_1dx_2...dx_d =$$

$$= \int_{\mathbb{R}^{d-1}} (-2i\pi x_2)^{\alpha_2} ... (-2i\pi x_d)^{\alpha_d} e^{-2\pi i(x_2\xi_2+...+x_d\xi_d)} \left[\int_{\mathbb{R}} (-2i\pi x_1)^{\alpha_1} f(\mathbf{x}) e^{-2\pi i x_1\xi_1} dx_1\right] dx_2...dx_d$$

Let us define $h(x_1) := f(\mathbf{x})$. We know that $(-2i\pi x_1)^\alpha h(x_1) \in L^1(\mathbb{R})$ for all α. Thus, we can apply Proposition 1.1.1 and we get:

$$\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^{d-1}} (-2i\pi x_2)^{\alpha_2} ... (-2i\pi x_d)^{\alpha_d} e^{-2\pi i(x_2\xi_2+...+x_d\xi_d)} \left[\frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}} \hat{h}(\xi_1)\right] dx_2...dx_d =$$

$$= \frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}} \int_{\mathbb{R}^d} (-2i\pi x_2)^{\alpha_2} ... (-2i\pi x_d)^{\alpha_d} e^{-2\pi i(x_1\xi_1+x_2\xi_2+...+x_d\xi_d)} f(\mathbf{x}) d\mathbf{x} =$$

$$\frac{\partial^{\alpha_1}}{\partial\xi_1^{\alpha_1}} \int_{\mathbb{R}^{d-1}} (-2i\pi x_3)^{\alpha_3} .. (-2i\pi x_d)^{\alpha_d} e^{-2\pi i(x_1\xi_1+x_3\xi_3..+x_d\xi_d)} \left[\int_{\mathbb{R}} (-2i\pi x_2)^{\alpha_2} f(\mathbf{x}) e^{-2\pi i x_2\xi_2} dx_2\right] dx_1dx_3..dx_d$$

Let us define $h(x_2) := f(\mathbf{x})$. We know that $(-2i\pi x_2)^{\alpha} h(x_2) \in L^1(\mathbb{R})$ for all α. Thus, we can apply Proposition 1.1.1 and we get:

$$\hat{g}(\boldsymbol{\xi}) = \frac{\partial^{\alpha_1}}{\partial \xi_1^{\alpha_1}} \int_{\mathbb{R}^{d-1}} (-2i\pi x_3)^{\alpha_3} ... (-2i\pi x_d)^{\alpha_d} e^{-2\pi i (x_1\xi_1 + x_3\xi_3 .. + x_d \xi_d)} \left[\frac{\partial^{\alpha_2}}{\partial \xi_2^{\alpha_2}} \hat{h}(\xi_2) \right] dx_1 dx_3 ... dx_d =$$

$$= \frac{\partial^{(\alpha_1+\alpha_2)}}{\partial \xi_1^{\alpha_1} \partial \xi_2^{\alpha_2}} \int_{\mathbb{R}^d} (-2i\pi x_3)^{\alpha_3} ... (-2i\pi x_d)^{\alpha_d} e^{-2\pi i (x_1\xi_1 + x_2\xi_2 + ... + x_d \xi_d)} f(\mathbf{x}) d\mathbf{x}$$

Doing the same for the coordinates $x_3, ..., x_d$ we finally get:

$$\hat{g}(\boldsymbol{\xi}) = \frac{\partial^{(\alpha_1+...+\alpha_d)}}{\partial \xi_1^{\alpha_1} ... \partial \xi_d^{\alpha_d}} \int_{\mathbb{R}^d} f(\mathbf{x}) e^{-2\pi i (x_1 \cdot \xi_1 + ... + x_d \cdot \xi_d)} d\mathbf{x} = \left(\frac{\partial}{\partial \boldsymbol{\xi}} \right)^{\alpha} \hat{f}(\boldsymbol{\xi})$$

• Proof 6: as R is a rotation, it holds that $R^{-1} = R^t$. Hence, $R^{-1}\mathbf{y} \cdot \boldsymbol{\xi} = R^t \mathbf{y} \cdot \boldsymbol{\xi} = \mathbf{y} \cdot R\boldsymbol{\xi}$:

$$\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} f(R\mathbf{x}) e^{-2\pi i \mathbf{x} \cdot \boldsymbol{\xi}} d\mathbf{x} = \left[\begin{array}{c} \mathbf{x} = g(\mathbf{y}) = R^{-1}\mathbf{y} \\ (Dg)(\mathbf{y}) = R^{-1} \implies |det\,(Dg)(\mathbf{y})| = 1 \end{array} \right] =$$

$$= \int_{\mathbb{R}^d} f(\mathbf{y}) e^{-2\pi i R^{-1}\mathbf{y} \cdot \boldsymbol{\xi}} d\mathbf{y} = \int_{\mathbb{R}^d} f(\mathbf{y}) e^{-2\pi i \mathbf{y} \cdot R\boldsymbol{\xi}} d\mathbf{y} = \hat{f}(R\boldsymbol{\xi})$$

Corollary 6.1. *Let $f \in \mathcal{S}(\mathbb{R}^d)$. If f is radial, then its Fourier transform $\hat{f}$ is also radial.*

PROOF.

From Proposition 6.3., as f is radial, it holds that $f(\mathbf{x}) = f(R\mathbf{x})$ for any rotation R. Thus, taking the Fourier transform in both sides and applying Proposition 6.5.6:

$$\hat{f}(\boldsymbol{\xi}) = [\widehat{f(R\mathbf{x})}](\boldsymbol{\xi}) = \hat{f}(R\boldsymbol{\xi}) \qquad \text{for any rotation } R$$

Hence, we conclude by Proposition 6.3. that $\hat{f}$ is radial.

Proposition 6.6. *Let $f \in L^1(\mathbb{R}^d)$. Then, its Fourier transform satisfies that* $\lim_{|\boldsymbol{\xi}| \longrightarrow +\infty} \left| \hat{f}(\boldsymbol{\xi}) \right| = 0$.

PROOF.

• Step 1: we show the result for a simple function $f(\mathbf{x}) := \mathbf{1}_{[a_1,b_1] \times ... \times [a_d,b_d]}(x_1, ..., x_d)$. Let us define $f_i(x) = \mathbf{1}_{[a_i,b_i]}(x)$ for all $i = 1...d$. Then

$$\hat{f}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-2\pi i \mathbf{x} \cdot \boldsymbol{\xi}} \mathbf{1}_{[a_1,b_1] \times ... \times [a_d,b_d]}(\mathbf{x})\, d\mathbf{x} = \prod_{i=1}^{d} \left(\int_{a_i}^{b_i} e^{-2\pi i x_i \xi_i} dx_i \right) = \prod_{i=1}^{d} \hat{f}_i(\xi)$$

Now, using property 1.7. for simple functions in $\mathbb{R}$, we get that $\lim_{|\boldsymbol{\xi}| \longrightarrow +\infty} \left| \hat{f}(\boldsymbol{\xi}) \right| = 0$.

• Step 2: we use the density of the simple functions in $L^1(\mathbb{R}^d)$ (Theorem A.7.): there exists a sequence $\{\varphi_n\}_{n \in \mathbb{N}}$ of simple functions such that $\lim_{n \to \infty} \|f - \varphi_n\|_1 = 0$.

Moreover, fixing n and applying step 1 for the simple function φ_n , it holds $\lim_{|\boldsymbol{\xi}| \to +\infty} |\hat{\varphi}_n(\boldsymbol{\xi})| = 0$. Then:

$$\left| \hat{f}(\boldsymbol{\xi}) - \hat{\varphi}_n(\boldsymbol{\xi}) \right| \leq \int_{\mathbb{R}^d} \left| e^{-2\pi i \mathbf{x} \cdot \boldsymbol{\xi}} \right| |f(\mathbf{x}) - \varphi_n(\mathbf{x})|\, d\mathbf{x} = \|f - \varphi_n\|_1 \implies \lim_{|\boldsymbol{\xi}| \to +\infty} \left| \hat{f}(\boldsymbol{\xi}) - \hat{\varphi}_n(\boldsymbol{\xi}) \right| \leq$$

$$\leq \lim_{|\boldsymbol{\xi}|\to+\infty} \|f-\varphi_n\|_1 = \|f-\varphi_n\|_1 \implies \left|\lim_{|\boldsymbol{\xi}|\to+\infty} \hat{f}(\boldsymbol{\xi})\right| \leq \|f-\varphi_n\|_1 \overset{n\longrightarrow+\infty}{\longrightarrow} 0$$

Thus, we finally conclude that $\lim_{|\boldsymbol{\xi}|\longrightarrow+\infty} \left|\hat{f}(\boldsymbol{\xi})\right| = 0.$

Theorem 6.1. *The space $\mathcal{S}(\mathbb{R}^d)$ is invariant under the Fourier transform. That is, if $f\in\mathcal{S}(\mathbb{R}^d)$ then $\hat{f}\in\mathcal{S}(\mathbb{R}^d)$.*

PROOF.

• On the one hand, we have seen in Proposition 6.5.5 that the Fourier transform of f is differentiable for all multi-index α. Hence $\hat{f}\in C^\infty(\mathbb{R}^d)$.

• On the other hand, let us consider $\left(\frac{\partial}{\partial\boldsymbol{\xi}}\right)^\beta \hat{f}(\boldsymbol{\xi})$ for any multi-index β. We want to see that it decays rapidly. Let α be another multi-index. As $f\in\mathcal{S}(\mathbb{R}^d)$, we have by proposition 6.5.5:

$$\boldsymbol{\xi}^\alpha \left(\frac{\partial}{\partial\boldsymbol{\xi}}\right)^\beta \hat{f}(\boldsymbol{\xi}) = \boldsymbol{\xi}^\alpha \left[\widehat{(-2\pi i\mathbf{x})^\beta f(\mathbf{x})}\right](\boldsymbol{\xi})$$

Now, as by Proposition 6.4.1 $(-2\pi i\mathbf{x})^\beta f(\mathbf{x})$ is in $\mathcal{S}(\mathbb{R}^d)$, we can apply Proposition 6.5.4:

$$\boldsymbol{\xi}^\alpha \left[\widehat{(-2\pi i\mathbf{x})^\beta f(\mathbf{x})}\right](\boldsymbol{\xi}) = \frac{1}{(2i\pi)^{|\alpha|}}(2i\pi\boldsymbol{\xi})^\alpha \left[\widehat{(-2\pi i\mathbf{x})^\beta f(\mathbf{x})}\right](\boldsymbol{\xi}) = \frac{1}{(2i\pi)^{|\alpha|}}\left[\widehat{\left(\frac{\partial}{\partial\mathbf{x}}\right)^\alpha (-2\pi i\mathbf{x})^\beta f(\mathbf{x})}\right](\boldsymbol{\xi})$$

As $\left(\frac{\partial}{\partial\mathbf{x}}\right)^\alpha \left[(-2\pi i\mathbf{x})^\beta f(\mathbf{x})\right]\in\mathcal{S}(\mathbb{R}^d)\subset L^1(\mathbb{R}^d)$, applying Proposition 6.6.:

$$\lim_{|\boldsymbol{\xi}|\longrightarrow+\infty}\left|\boldsymbol{\xi}^\alpha \left(\frac{\partial}{\partial\boldsymbol{\xi}}\right)^\beta \hat{f}(\boldsymbol{\xi})\right| = \frac{1}{(2i\pi)^{|\alpha|}}\lim_{|\boldsymbol{\xi}|\longrightarrow+\infty}\left|\left[\widehat{\left(\frac{\partial}{\partial\mathbf{x}}\right)^\alpha (-2\pi i\mathbf{x})^\beta f(\mathbf{x})}\right](\boldsymbol{\xi})\right| = 0$$

Proposition 6.7. *The Fourier transform of $e^{-\pi|\mathbf{x}|^2}$ is $e^{-\pi|\boldsymbol{\xi}|^2}$. In other words, $e^{-\pi|\mathbf{x}|^2}$ is a fix point of the operator $\widehat{\cdot}$.*

PROOF.

Let us define $g(\mathbf{x}) := e^{-\pi|\mathbf{x}|^2}$. Note first that g is in $\mathcal{S}(\mathbb{R}^d)$. Hence, we can compute its Fourier transform and we get:

$$\hat{g}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\xi}} e^{-\pi|\mathbf{x}|^2} d\mathbf{x} = \int_{\mathbb{R}^d}\left(e^{-2\pi i x_1\xi_1}e^{-\pi x_1^2}\right)\ldots\left(e^{-2\pi i x_d\xi_d}e^{-\pi x_d^2}\right)d\mathbf{x} =$$

$$= \left(\int_{\mathbb{R}} e^{-2\pi i x_1\xi_1}e^{-\pi x_1^2}dx_1\right)\ldots\left(\int_{\mathbb{R}} e^{-2\pi i x_d\xi_d}e^{-\pi x_d^2}dx_d\right) = \left(e^{-\pi\xi_1^2}\right)\ldots\left(e^{-\pi\xi_d^2}\right) = e^{-\pi|\boldsymbol{\xi}|^2} = g(\boldsymbol{\xi})$$

Where we have used example 1.1. with $a = \pi$.

Lemma 6.1. $\int_{\mathbb{R}} e^{-\pi x^2} dx = 1$

PROOF.

$$\left(\int_{\mathbb{R}} e^{-\pi x^2} dx\right)^2 = \left(\int_{\mathbb{R}} e^{-\pi x^2} dx\right)\left(\int_{\mathbb{R}} e^{-\pi y^2} dy\right) \overset{Theor.\ A.6.}{=} \int_{\mathbb{R}}\int_{\mathbb{R}} e^{-\pi\left(x^2+y^2\right)} dxdy = \left[\begin{array}{c} r = x^2+y^2 \\ \theta = \arctan\left(\frac{y}{x}\right) \end{array}\right] =$$

$$= \int_0^{2\pi}\int_0^{+\infty} e^{-\pi r^2} r dr d\theta = 2\pi \int_0^{+\infty} e^{-\pi r^2} r dr = 2\pi\left[-\frac{e^{-\pi r^2}}{2\pi}\right]_0^{+\infty} = 1$$

Definition 6.6. *Let us consider $f,\ g \in \mathcal{S}\left(\mathbb{R}^d\right)$, we define its convolution $f * g$ as:*

$$(f * g)(\mathbf{x}) = \int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{t}) g(\mathbf{t}) d\mathbf{t}$$

Definition 6.7. *A family of integrable functions $\{K_\delta(\mathbf{x})\}_{\delta>0}$ is said to be a family of good kernels if it satisfies the following properties:*

1. *For all $\delta > 0$, $\int_{\mathbb{R}^d} K_\delta(\mathbf{x}) d\mathbf{x} = 1$*
2. *There exists $M > 0$ such that $\int_{\mathbb{R}^d} |K_\delta(\mathbf{x})| d\mathbf{x} \leq M$ for all $\delta > 0$*
3. *For every $\eta > 0$, we have $\int_{|\mathbf{x}|>\eta} |K_\delta(\mathbf{x})| d\mathbf{x} \longrightarrow 0$ as $\delta \longrightarrow 0$*

Proposition 6.8. *Let us consider the family of functions $\{K_\delta(\mathbf{x})\}_{\delta>0}$ defined by $K_\delta(\mathbf{x}) := \delta^{-\frac{d}{2}} e^{-\frac{\pi|\mathbf{x}|^2}{\delta}}$. Then, it holds that $\{K_\delta(\mathbf{x})\}_{\delta>0}$ is a family of good kernels:*

PROOF.

We should prove the three properties of definition 6.7. Note that making the change of variables $\mathbf{y} = \frac{1}{\sqrt{\delta}}\mathbf{x}$ and using Lemma 6.1.:

$$\int_{\mathbb{R}^d} |K_\delta(\mathbf{x})| d\mathbf{x} = \int_{\mathbb{R}^d} K_\delta(\mathbf{x}) d\mathbf{x} = \int_{\mathbb{R}^d} \delta^{-\frac{d}{2}} e^{-\frac{\pi|\mathbf{x}|^2}{\delta}} d\mathbf{x} = \int_{\mathbb{R}^d} e^{-\pi|\mathbf{y}|^2} d\mathbf{y} = \left(\int_{\mathbb{R}} e^{-\pi y^2} dy\right)^d = 1$$

which proves property 1 and 2. Finally, making again the change of variables $\mathbf{y} = \frac{1}{\sqrt{\delta}}\mathbf{x}$, we prove property 3. In effect:

$$\int_{|\mathbf{x}|>\eta} |K_\delta(\mathbf{x})| dx = \int_{|y|>\frac{\eta}{\sqrt{\delta}}} e^{-\pi|\mathbf{y}|^2} d\mathbf{y} \overset{\delta\longrightarrow 0}{\longrightarrow} 0$$

Proposition 6.9. *Let $\{K_\delta(\mathbf{x})\}_{\delta>0}$ be a family of good kernels. Let $f \in \mathcal{S}(\mathbb{R}^d)$, then $(f * K_\delta)(\mathbf{x})$ tends to $f(\mathbf{x})$ uniformly in $\mathbf{x}$ as $\delta \longrightarrow 0$.*

PROOF.

• First we prove that f is uniformly continuous in $\mathbb{R}^d$. Let us take $\epsilon > 0$. We make the following observations:

1. As $f \in \mathcal{S}(\mathbb{R}^d)$, then it decays rapidly. Thus, there exists $R_\epsilon > 0$ such that $|f(\mathbf{x})| < \frac{\epsilon}{4}$ whenever $|\mathbf{x}| \geq R_\epsilon$.
2. As $f \in \mathcal{S}(\mathbb{R}^d)$, then it is continuous. Thus, f is uniformly continuous in the compact $B_{R_\epsilon}(0) = \left\{\mathbf{x} \in \mathbb{R}^d \text{ s.t. } |\mathbf{x}| \leq R_\epsilon\right\}$. Hence, there exists $\delta_\epsilon > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \frac{\epsilon}{2}$ for all $\mathbf{x}, \mathbf{y} \in B_{R_\epsilon}(0)$ satisfying $|\mathbf{x}-\mathbf{y}| < \delta_\epsilon$.

We want to find $\eta_\epsilon > 0$ such that $|f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ satisfying $|\mathbf{x} - \mathbf{y}| < \eta_\epsilon$. Let us take $\eta_\epsilon = \delta_\epsilon$. We distinguish the following three cases:

1. If $\mathbf{x}, \mathbf{y} \in B_{R_\epsilon}(0)$, by observation 2, $|f(\mathbf{x}) - f(\mathbf{y})| < \frac{\epsilon}{2} < \epsilon$.
2. If $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d \setminus B_{R_\epsilon}(0)$, by observation 1, $|f(\mathbf{x}) - f(\mathbf{y})| \leq |f(\mathbf{x})| + |f(\mathbf{y})| < \frac{\epsilon}{2} < \epsilon$
3. If $\mathbf{x} \in B_{R_\epsilon}(0)$ and $\mathbf{y} \in \mathbb{R}^d \setminus B_{R_\epsilon}(0)$. Let us take $\mathbf{z}$ such that $|\mathbf{z}| = R_\epsilon$. Then: $|f(\mathbf{x}) - f(\mathbf{y})| \leq |f(\mathbf{x}) - f(\mathbf{z})| + |f(\mathbf{y}) - f(\mathbf{z})| < \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon$

• Now let us show that $(f * K_\delta)(\mathbf{x}) \longrightarrow f(\mathbf{x})$ uniformly in $\mathbf{x}$ as $\delta \longrightarrow 0$. Let us take $\epsilon > 0$. First, we make the following observations:

1. We have seen that for $|\mathbf{x} - \mathbf{y}| < \eta_\epsilon$ it holds $|f(\mathbf{x}) - f(\mathbf{y})| < \epsilon$
2. As $f \in \mathcal{S}(\mathbb{R}) \subset L^\infty(\mathbb{R})$, then there exists $M > 0$ such that $|f(\mathbf{x})| \leq M$ for all $\mathbf{x} \in \mathbb{R}^d$.
3. By the property 3 of good kernels with η_ϵ, there exists $\delta_\epsilon > 0$ such that $\int_{|\mathbf{x}|>\eta_\epsilon} |K_\delta(\mathbf{x})|\, d\mathbf{x} < \frac{\epsilon}{M}$ for all $\delta < \delta_\epsilon$

Using this three observations, for all $\delta < \delta_\epsilon$ we have:

$$|(f * K_\delta)(\mathbf{x}) - f(\mathbf{x})| \leq \left|\int_{\mathbb{R}^d} K_\delta(\mathbf{t})\,(f(\mathbf{x}-\mathbf{t}) - f(\mathbf{x}))\,d\mathbf{t}\right| \leq \int_{|\mathbf{t}|>\eta_\epsilon} K_\delta(\mathbf{t})\,|f(\mathbf{x}-\mathbf{t}) - f(\mathbf{x})|\,d\mathbf{t} +$$

$$+ \int_{|\mathbf{t}|\leq\eta_\epsilon} K_\delta(\mathbf{t})\,|f(\mathbf{x}-\mathbf{t}) - f(\mathbf{x})|\,d\mathbf{t} \leq 2M \int_{|t|>\eta_\epsilon} K_\delta(\mathbf{t})d\mathbf{t} + \epsilon \int_{\mathbb{R}^d} K_\delta(\mathbf{t})d\mathbf{t} = 2\epsilon + \epsilon = 3\epsilon$$

Hence, taking supremums, we get $\|f * K_\delta - f\|_\infty < 3\epsilon$ for all $\delta < \delta_\epsilon$ and we conclude $(f * K_\delta)(\mathbf{x}) \overset{\delta\longrightarrow 0}{\longrightarrow} f(\mathbf{x})$ uniformly in $\mathbf{x}$.

Proposition 6.10. *Let $f, g \in \mathcal{S}(\mathbb{R}^d)$. Then, it holds:*

$$\int_{\mathbb{R}^d} f(\mathbf{x})\,\hat{g}(\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}^d} \hat{f}(\mathbf{y})\,g(\mathbf{y})\,d\mathbf{y}$$

PROOF.

First, we observe that $f, g \in \mathcal{S}(\mathbb{R}^d) \subset L^1(\mathbb{R}^d)$. Thus $\|f\|_1 < +\infty$ and $\|g\|_1 < +\infty$. We will use Theorem A.6. (Fubini). Let us prove first that $F(\mathbf{x},\mathbf{y}) := f(\mathbf{x})g(\mathbf{y})e^{-2\pi i \mathbf{x}\cdot\boldsymbol{y}}$ is integrable in $\mathbb{R}^d \times \mathbb{R}^d$:

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \left|f(\mathbf{x})\,g(\mathbf{y})e^{-2\pi i \mathbf{x}\cdot\boldsymbol{y}}\right| d\mathbf{y}d\mathbf{x} = \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} |f(\mathbf{x})\,g(\mathbf{y})|\,d\mathbf{y}d\mathbf{x} = \int_{\mathbb{R}^d} |f(\mathbf{x})| \int_{\mathbb{R}^d} |g(\mathbf{y})|\,d\mathbf{y}d\mathbf{x} =$$

$$= \int_{\mathbb{R}^d} |f(\mathbf{x})|\,\|g\|_1\,d\mathbf{x} = \|f\|_1\,\|g\|_1 < +\infty$$

Thus, applying Fubini's Theorem we conclude that the result of the statement is true. In effect,

$$\int_{\mathbb{R}^d} f(\mathbf{x})\,\hat{g}(\mathbf{x})\,d\mathbf{x} = \int_{\mathbb{R}^d} f(\mathbf{x}) \int_{\mathbb{R}^d} g(\mathbf{y})e^{-2\pi i \mathbf{x}\cdot\boldsymbol{y}} d\mathbf{y}d\mathbf{x} = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(\mathbf{x})\,g(\mathbf{y})e^{-2\pi i \mathbf{x}\cdot\boldsymbol{y}} d\mathbf{y}\right) d\mathbf{x} =$$

$$= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} f(\mathbf{x})\,g(\mathbf{y})e^{-2\pi i \mathbf{x}\cdot\boldsymbol{y}} d\mathbf{x}\right) d\mathbf{y} = \int_{\mathbb{R}^d} g(\mathbf{y})\left(\int_{\mathbb{R}^d} f(\mathbf{x})\,e^{-2\pi i \mathbf{x}\cdot\boldsymbol{y}} d\mathbf{x}\right) d\mathbf{y} = \int_{\mathbb{R}^d} g(\mathbf{y})\,\hat{f}(\mathbf{y})\,d\mathbf{y}$$

Theorem 6.2. (Inversion Theorem) *Let* $f \in \mathcal{S}\left(\mathbb{R}^d\right)$*. Then* $f(\mathbf{x}) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi}) e^{-2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}$*.*

PROOF.

• Step 1: we see that $f(\mathbf{0}) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi}$. Let us define $G_\delta(\mathbf{x}) := e^{-\pi\delta|\mathbf{x}|^2}$. By Proposition 6.7., we deduce that $\widehat{G_\delta}(\boldsymbol{\xi}) = K_\delta(\boldsymbol{\xi}) = \delta^{-\frac{d}{2}} e^{-\frac{\pi|\boldsymbol{\xi}|^2}{\delta}}$. Hence, using Proposition 6.10. with G_δ and f:

$$\int_{\mathbb{R}^d} f(\mathbf{x})\, K_\delta(\mathbf{x})\, d\mathbf{x} = \int_{\mathbb{R}^d} \hat{f}(\mathbf{y})\, G_\delta(\mathbf{y})\, d\mathbf{y}$$

We observe that $\int_{\mathbb{R}^d} f(\mathbf{x})\, K_\delta(\mathbf{x})\, d\mathbf{x} = (f * K_\delta)(\mathbf{0}) \overset{\delta\longrightarrow 0}{\longrightarrow} f(\mathbf{0})$ (by Proposition 6.9.). Moreover, as $\left|\hat{f}(\mathbf{y})\, G_\delta(\mathbf{y})\right| \leq \left|\hat{f}(\mathbf{y})\right|$ is integrable (because $\hat{f} \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^1\left(\mathbb{R}^d\right)$) we get by the Dominated Convergence Theorem that $\int_{\mathbb{R}^d} \hat{f}(\mathbf{y})\, G_\delta(\mathbf{y})\, d\mathbf{y} \overset{\delta\longrightarrow 0}{\longrightarrow} \int_{\mathbb{R}^d} \hat{f}(\mathbf{y})\, d\mathbf{y}$. Hence, $f(\mathbf{0}) = \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi})\, d\boldsymbol{\xi}$.

• Step 2: let us consider $F(\mathbf{y}) := f(\mathbf{y}+\mathbf{x})$ for some $\mathbf{x} \in \mathbb{R}^d$. It holds that $F \in \mathcal{S}\left(\mathbb{R}^d\right)$. Then by step 1 we get:

$$f(\mathbf{x}) = F(\mathbf{0}) \overset{step\,1}{=} \int_{\mathbb{R}^d} \hat{F}(\boldsymbol{\xi})\, d\boldsymbol{\xi} \overset{Prop.\,6.5.1}{=} \int_{\mathbb{R}^d} \hat{f}(\boldsymbol{\xi})\, e^{2\pi i \mathbf{x}\cdot\boldsymbol{\xi}} d\boldsymbol{\xi}$$

Proposition 6.11. (convolution) *Let* $f, g \in \mathcal{S}(\mathbb{R}^d)$*. Then, the following statements hold:*

1. $f * g$ *is in* $\mathcal{S}(\mathbb{R}^d)$ *and* $\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha (f * g) = \left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha f\right] * g$

2. $\widehat{(f * g)}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})\, \hat{g}(\boldsymbol{\xi})$

PROOF.

• Proof 1: first we will see that $f * g$ is in $C^\infty\left(\mathbb{R}^d\right)$. We will apply Differentiation under the integral sign Theorem (Theorem A.5.). Let us consider the coordinate x_i for $i \in \{1, .., d\}$ and let $F(x_i, \mathbf{y}) := \left[\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i} f(\mathbf{x}-\mathbf{y})\right] g(\mathbf{y})$ for a non-negative integer α_i. Then, $F(x_i, \mathbf{y})$ satisfies:

1. For all $\mathbf{y} \in \mathbb{R}^d$, $x_i \longmapsto F(x_i, \mathbf{y})$ is continuously differentiable in $\mathbb{R}$.

2. Note that $\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i+1} f(\mathbf{x}) \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^\infty\left(\mathbb{R}^d\right)$. Thus, $|F(x_i, \mathbf{y})| = \left|\left[\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i+1} f(\mathbf{x}-\mathbf{y})\right] g(\mathbf{y})\right| \leq$ $\leq \left\|\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i+1} f\right\|_\infty |g(\mathbf{y})|$ which is integrable for $\mathbf{y} \in \mathbb{R}^d$.

Hence, applying Theorem A.5. we get that $\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i} (f * g)(\mathbf{x}) = \int_{\mathbb{R}^d} F(x_i, \mathbf{y}) d\mathbf{y}$ is differentiable for all $x_i \in \mathbb{R}$ and moreover:

$$\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i+1} (f * g)(\mathbf{x}) = \left(\frac{\partial}{\partial x_i}\right) \int_{\mathbb{R}^d} F(x_i, \mathbf{y}) d\mathbf{y} = \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial x_i}\right) F(x_i, \mathbf{y}) d\mathbf{y} =$$
$$= \int_{\mathbb{R}^d} \left[\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i+1} f(\mathbf{x}-\mathbf{y})\right] g(\mathbf{y}) d\mathbf{y} = \left(\left[\left(\frac{\partial}{\partial x_i}\right)^{\alpha_i+1} f\right] * g\right)(\mathbf{x})$$

Thus, we have seen that $f * g \in C^\infty(\mathbb{R})$ for each coordinate x_i such that $i \in \{1, .., d\}$, consequently $f * g \in C^\infty\left(\mathbb{R}^d\right)$. Moreover, we conclude:

$$\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha (f * g)(\mathbf{x}) = \left(\left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha f\right] * g\right)(\mathbf{x})$$

• Now, let us prove that f and $\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} f$ decay rapidly for every multi-index α. Let us take a multi-index β. Then:

$$\left|\mathbf{x}^{\beta}\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha}(f*g)(\mathbf{x})\right| = \left|\mathbf{x}^{\beta}\left(\left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} f\right]*g\right)(\mathbf{x})\right| \leq \int_{\mathbb{R}^d}\left|\mathbf{x}^{\beta}\left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} f(\mathbf{x}-\mathbf{y})\right] g(\mathbf{y})\right| d\mathbf{y}$$

Now, applying the Dominated Convergence Theorem (Theorem A.2.) we get that

$$\lim_{|\mathbf{x}|\longrightarrow+\infty}\left|\mathbf{x}^{\beta}\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha}(f*g)(\mathbf{x})\right| \leq \int_{\mathbb{R}^d}\lim_{|\mathbf{x}|\longrightarrow+\infty}\left|\mathbf{x}^{\beta}\left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} f(\mathbf{x}-\mathbf{y})\right] g(\mathbf{y})\right| d\mathbf{y} = 0$$

In conclusion, we conclude that f and $\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} f$ decay rapidly for every multi-index α. Then, $f \in \mathcal{S}\left(\mathbb{R}^d\right)$.

• Proof 2: let us consider $F(\mathbf{x},\mathbf{y}) = e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})$ for $\boldsymbol{\xi} \in \mathbb{R}^d$. Let us prove that $F(\mathbf{x},\mathbf{y})$ is integrable in $\mathbb{R}^d \times \mathbb{R}^d$.

$$\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left|e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\right| d\mathbf{x}d\mathbf{y} = \int_{\mathbb{R}^d}|g(\mathbf{y})|\int_{\mathbb{R}^d}|f(\mathbf{x}-\mathbf{y})|\, d\mathbf{x}d\mathbf{y} =$$

$$= \|f\|_1 \int_{\mathbb{R}^d}|g(\mathbf{y})|\, d\mathbf{y} = \|f\|_1\|g\|_1 < +\infty$$

Consequently, F satisfies the hypothesis of Fubini's Theorem and we can apply it:

$$\widehat{(f*g)}(\boldsymbol{\xi}) = \int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}}\left(\int_{\mathbb{R}^d} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} = \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, d\mathbf{y}\right) d\mathbf{x} =$$

$$= \int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} f(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, d\mathbf{x}\right) d\mathbf{y} = \left[\begin{array}{c} \mathbf{x} = g(\mathbf{u}) = \mathbf{u}+\mathbf{y} \\ (Dg)(\mathbf{u}) = Id \Longrightarrow |det\,(Dg)(\mathbf{u})| = 1 \end{array}\right] =$$

$$\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{u}} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{y}} f(\mathbf{u})\, g(\mathbf{y})\, d\mathbf{u}\right) d\mathbf{y} = \left(\int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{y}} g(\mathbf{y})\, d\mathbf{y}\right)\left(\int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{u}} f(\mathbf{u})\, d\mathbf{u}\right) = \hat{f}(\boldsymbol{\xi})\,\hat{g}(\boldsymbol{\xi})$$

Theorem 6.3. (Plancherel in $\mathcal{S}\left(\mathbb{R}^d\right)$) *Let $f \in \mathcal{S}\left(\mathbb{R}^d\right)$. Then* $\left\|\hat{f}\right\|_2 = \|f\|_2$.

PROOF.

Note that it holds that $\widehat{\left(\overline{f}\right)_{\sigma}}(\boldsymbol{\xi}) = \overline{\hat{f}(\boldsymbol{\xi})}$. Let us define $h(\mathbf{x}) := \left(f*\left(\overline{f}\right)_{\sigma}\right)(\mathbf{x})$. By Proposition 6.11.2 we have:

$$\hat{h}(\boldsymbol{\xi}) = \left(f*\widehat{\left(\overline{f}\right)_{\sigma}}\right)(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})\,\widehat{\left(\overline{f}\right)_{\sigma}}(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})\,\overline{\hat{f}(\boldsymbol{\xi})} = \left|\hat{f}(\boldsymbol{\xi})\right|^2$$

Moreover, $h(\mathbf{0}) = \int_{\mathbb{R}^d}|f(\mathbf{x})|^2\, d\mathbf{x}$. Finally, by the Inversion Theorem (Theorem 6.2.):

$$h(\mathbf{0}) = \int_{\mathbb{R}^d}\hat{h}(\boldsymbol{\xi})\, e^{-2\pi i \mathbf{0}\cdot\mathbf{u}} d\boldsymbol{\xi} = \int_{\mathbb{R}^d}\hat{h}(\boldsymbol{\xi})\, d\boldsymbol{\xi} = \int_{\mathbb{R}^d}\left|\hat{f}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi}$$

Hence we conclude:

$$\int_{\mathbb{R}^d}|f(\mathbf{x})|^2\, d\mathbf{x} = \int_{\mathbb{R}^d}\left|\hat{f}(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \Longrightarrow \left\|\hat{f}\right\|_2 = \|f\|_2$$

6.3 Multidimensional Fourier transform on $L^2\left(\mathbb{R}^d\right)$

Let $f \in L^2\left(\mathbb{R}^d\right)$. We know that $\mathcal{S}\left(\mathbb{R}^d\right)$ is dense in $L^2\left(\mathbb{R}^d\right)$. Hence, there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of functions in $\mathcal{S}\left(\mathbb{R}^d\right)$ such that $\|f_n - f\|_2 \overset{n\longrightarrow+\infty}{\longrightarrow} 0$. As $(f_n)_{n\in\mathbb{N}}$ converge in $L^2\left(\mathbb{R}^d\right)$, in particular it is a Cauchy sequence in $L^2\left(\mathbb{R}^d\right)$. We are going to see now that $\left(\hat{f}_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2\left(\mathbb{R}^d\right)$. In effect, by Theorem 6.3. (Plancherel), it holds:

$$\left\|\hat{f}_n - \hat{f}_m\right\|_2 = \left\|\widehat{f_n - f_m}\right\|_2 = \|f_n - f_m\|_2 \overset{n,m\longrightarrow+\infty}{\longrightarrow} 0$$

Hence, $\left(\hat{f}_n\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^2\left(\mathbb{R}^d\right)$ and as $L^2\left(\mathbb{R}^d\right)$ is a Hilbert Space, in particular is complete. Thus, $\left(\hat{f}_n\right)_{n\in\mathbb{N}}$ converge to a function $g \in L^2\left(\mathbb{R}^d\right)$, i.e.

$$\lim_{n\longrightarrow+\infty} \left\|\hat{f}_n - g\right\|_2 = 0$$

Definition 6.8. Let $f \in L^2\left(\mathbb{R}^d\right)$. Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{S}\left(\mathbb{R}^d\right)$ such that $\|f_n - f\|_2 \overset{n\longrightarrow+\infty}{\longrightarrow} 0$. We define the Fourier transform of f as:

$$\mathcal{F}f\left(\boldsymbol{\xi}\right) := g\left(\boldsymbol{\xi}\right) = \lim_{n\longrightarrow+\infty} \hat{f}_n\left(\boldsymbol{\xi}\right) \quad \left(\text{in } L^2\left(\mathbb{R}^d\right)\right)$$

The Fourier transform of f in $L^2\left(\mathbb{R}^d\right)$ doesn't depend on the sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $\mathcal{S}\left(\mathbb{R}^d\right)$ that $\|f_n - f\|_2 \overset{n\longrightarrow+\infty}{\longrightarrow} 0$.

Theorem 6.4. (Plancherel in $L^2\left(\mathbb{R}^d\right)$) *Let $f \in L^2\left(\mathbb{R}^d\right)$. Then $\|\mathcal{F}f\|_2 = \|f\|_2$.*

PROOF.

Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of elements of $\mathcal{S}\left(\mathbb{R}^d\right)$ such that $\|f_n - f\|_2 \overset{n\longrightarrow+\infty}{\longrightarrow} 0$. It holds by Theorem 6.3.:

$$\int_{\mathbb{R}} f_n(x)\overline{f_n(x)}dx = \int_{\mathbb{R}} \widehat{f_n}(\xi)\overline{\widehat{f_n}(\xi)}d\xi$$

Hence, using this:

$$\left|\int_{\mathbb{R}^d} f(\mathbf{x})\overline{f(\mathbf{x})}d\mathbf{x} - \int_{\mathbb{R}^d} \mathcal{F}\left(f\right)\left(\boldsymbol{\xi}\right)\overline{\mathcal{F}\left(f\right)\left(\boldsymbol{\xi}\right)}d\boldsymbol{\xi}\right| \leq \left|\int_{\mathbb{R}^d} f(\mathbf{x})\overline{f(\mathbf{x})}dx - \int_{\mathbb{R}^d} f_n(\mathbf{x})\overline{f_n(\mathbf{x})}d\mathbf{x}\right| +$$

$$+\left|\int_{\mathbb{R}^d} f_n(\mathbf{x})\overline{f_n(\mathbf{x})}d\mathbf{x} - \int_{\mathbb{R}^d} \widehat{f_n}(\xi)\overline{\widehat{f_n}(\xi)}dd\boldsymbol{\xi}\right| + \left|\int_{\mathbb{R}^d} \mathcal{F}\left(f\right)\left(\boldsymbol{\xi}\right)\overline{\mathcal{F}\left(f\right)\left(\boldsymbol{\xi}\right)}d\boldsymbol{\xi} - \int_{\mathbb{R}^d} \widehat{f_n}(\xi)\overline{\widehat{f_n}(\xi)}dd\boldsymbol{\xi}\right| \overset{n\longrightarrow+\infty}{\longrightarrow} 0$$

Thus we conclude that

$$\int_{\mathbb{R}^d} \left|f(\mathbf{x})\right|^2 d\mathbf{x} = \int_{\mathbb{R}^d} \left|\mathcal{F}f(\boldsymbol{\xi})\right|^2 d\boldsymbol{\xi} \implies \|\mathcal{F}f\|_2 = \|f\|_2$$

Part II

Application of the Fourier transform to the resolution of the Heat Equation and the Wave Equation

• One of the most important applications of the convolution and the Fourier transform is in solving differential equations. Given a PDE for $u(x,t)$ with $x \in \mathbb{R}$, the general strategy will be:

1. Use the Fourier transform to get an ODE for the the transformed $\hat{u}$ (or a PDE of lower dimensionality) .
2. Solve the ODE.
3. Use the inverse Fourier transform (and operational formulas) to get back to a representation for u.

• We will explode the property of the Fourier transform proved in Proposition 1.1.2. It states that under certain hypothesis:

$$\widehat{f^{(k)}}(\xi) = (2i\pi\xi)^k \hat{f}(\xi)$$

In general, a differential operator can be thought of as a polynomial in $\frac{d}{dx}$, say of the form:

$$P\left(\frac{d}{dx}\right) = a_n \left(\frac{d}{dx}\right)^n + a_{n-1}\left(\frac{d}{dx}\right)^{n-1} + ... + a_1\left(\frac{d}{dx}\right) + a_0$$

and when it is applied to a function $f(x)$, the result is

$$P\left(\frac{d}{dx}\right) f = a_n f^{(n)} + a_{n-1} f^{(n-1)} + ... + a_1 f^{'} + a_0 f$$

Now, if we take the Fourier transform of this expression, we wind up with the Fourier transform of f multiplied by the corresponding n-th degree polynomial evaluated at $2\pi i\xi$. In effect,

$$\left[\widehat{P\left(\frac{d}{dx}\right) f}\right](\xi) = \Big(a_n (2\pi i\xi)^n + a_{n-1}(2\pi i\xi)^{n-1} + ... + a_1(2\pi i\xi) + a_0\Big)\hat{f}(\xi) = P(2\pi i\xi)\hat{f}(\xi)$$

• As an example, let us consider the ordinary differential equation.

$$u^{''} - u = -f$$

We assume that f is in $L^1(\mathbb{R})$. Then, taking the Fourier transform of both sides of the equation and applying the mentioned property:

$$(2\pi i\xi)^2\, \widehat{u}(\xi) - \widehat{u}(\xi) = -\widehat{f}(\xi) \implies \widehat{u}(\xi) = \frac{1}{1+4\pi^2\xi^2}\widehat{f}(\xi)$$

We recognize $\frac{1}{1+4\pi^2\xi^2}$ as the Fourier transform of $\frac{1}{2}e^{-|t|}$. Thus:

$$\hat{u} = \widehat{\left[\frac{1}{2}e^{-|t|}\right]} \cdot \widehat{[f]}$$

As $\frac{1}{2}e^{-|t|}$ and f are in $L^1(\mathbb{R})$, it holds by Proposition 5.2. that the right hand side of the expression is the Fourier transform of the convolution of this two functions. Hence, using the inversion Theorem:

$$u(t) = \frac{1}{2}e^{-|t|} * f(t) = \frac{1}{2}\int_{\mathbb{R}} e^{-|t-x|} f(x)dx$$

Chapter 7

The Heat Equation

7.1 Derivation of the heat equation

The heat equation describes how heat and particles get transported, (typically) under conduction. Consequently, we will provide two derivations, one for the diffusion of particles and the other for the heat. We will use Fourier's Law:

Fourier's Law: the amount of heat (or concentration of particles) flows from hot (or more concentrated) regions to cold (or less concentrated) regions at a rate $k > 0$ proportional to the temperature gradient (or concentration gradient).

7.1.1. Diffusion

Consider a liquid in which a dye is being diffused through the liquid. Suppose the liquid is contained in a pipe of length L. By Fourier's Law, the dye will move from higher concentration to lower concentration. Let $u(x,t)$ be the concentration (mass per unit length) of the dye at the position $x \in [0, L]$ in the pipe at time t. The total mass of dye M in the pipe in $[x_0, x_1] \subset [0, L]$ at time t is given by:

$$M(t) = \int_{x_0}^{x_1} u(x,t)dx \implies \frac{dM(t)}{dt} = \int_{x_0}^{x_1} u_t(x,t)dx$$

By Fourier's Law, it holds:

$$\frac{dM(t)}{dt} = \text{flow in at } x_1 - \text{ flow out at } x_0 = ku_x(x_1,t) - ku_x(x_0,t)$$

Hence, $\int_{x_0}^{x_1} u_t(x,t)dx = ku_x(x_1,t) - ku_x(x_0,t)$ and differentiating with respect to x_1 we get $u_t(x_1,t) = ku_{xx}(x_1,t)$. And as we have take and arbitrary $x_1 \in [0, L]$ we can conclude:

$$u_t(x,t) = ku_{xx}(x,t) \text{ for all } x \in [0, L]$$

This is known as the diffusion equation.

7.1.2. Heat Flow

Let D be a region in $\mathbb{R}^d$ and let $\mathbf{x} = (x_1, ..., x_d)^t$ be a vector in $\mathbb{R}^d$. Let $u(\mathbf{x}, t)$ be the temperature at point $\mathbf{x}$, time t, and let $H(t)$ be the total amount of heat (in calories) contained in D at time t. Let

c be the specific heat of the material and ρ its density (mass per unit volume). Then the amount of heat and the change in heat is given by:

$$H(t) = \int_D c\rho u(\mathbf{x},t)d\mathbf{x} \implies \frac{dH(t)}{dt} = \int_D c\rho u_t(\mathbf{x},t)d\mathbf{x}$$

Fourier's Law says that heat flows from hot to cold regions at a rate $k > 0$ proportional to the temperature gradient. The only way heat will leave D is through the boundary. That is:

$$\frac{dH(t)}{dt} = \int_{\partial D} k\nabla u \cdot nd\mathcal{S}$$

where ∂D is the boundary of D, n is the outward unit normal vector to ∂D and $d\mathcal{S}$ is the surface measure over ∂D. Therefore, using Theorem A.11. (Divergence Theorem):

$$\int_{\partial D} k\nabla u \cdot nd\mathcal{S} = \int_D \nabla\cdot(k\nabla u)\,d\mathbf{x} = \int_D k\triangle u d\mathbf{x}$$

where $\triangle u = \sum_{i=1}^{d} u_{x_i x_i}$. Hence, as we have taken an arbitrary volume D we get:

$$\int_D c\rho u_t(\mathbf{x},t)d\mathbf{x} = \int_D k\triangle u d\mathbf{x} \implies c\rho u_t(\mathbf{x},t) = k\triangle u(\mathbf{x},t)$$

And, as we suppose c, ρ and k are constants, we are led to the heat equation:

$$u_t(\mathbf{x},t) = \tilde{k}\triangle u(\mathbf{x},t)$$

1.3. Simplification

We will take $k = 1$. We can do this simplification by rescaling the variables of the equation. In effect, let us define $\mathbf{y} = \frac{1}{\sqrt{k}}\mathbf{x}$, and let $\widetilde{u}(\mathbf{y},t) := u(\sqrt{k}\mathbf{y},t)$. Then, it holds:

$$\frac{\partial}{\partial y_i}\widetilde{u}(\mathbf{y},t) = \frac{\partial}{\partial x_i}u(\sqrt{k}\mathbf{y},t)\sqrt{k} \implies \frac{\partial^2}{\partial y_i^2}\widetilde{u}(\mathbf{y},t) = \frac{\partial^2}{\partial x_i^2}u(\sqrt{k}\mathbf{y},t)k$$

Hence, $\triangle\widetilde{u}(\mathbf{y},t) = k\triangle u(\sqrt{k}\mathbf{y},t) = u_t\left(\sqrt{k}\mathbf{y},t\right) = \widetilde{u}_t(\mathbf{y},t)$. Thus:

$$\widetilde{u}_t(\mathbf{y},t) = \triangle\widetilde{u}(\mathbf{y},t)$$

7.2 The Heat Kernel

Definition 7.1. *Let us consider the family of functions* $\{K_\delta(\mathbf{x})\}_{\delta>0}$ *defined in Proposition 6.8. We define the Heat Kernel of* $\mathbb{R}^d$ *as the family of functions* $\{\mathcal{H}_t(\mathbf{x})\}_{t>0}$ *such that:*

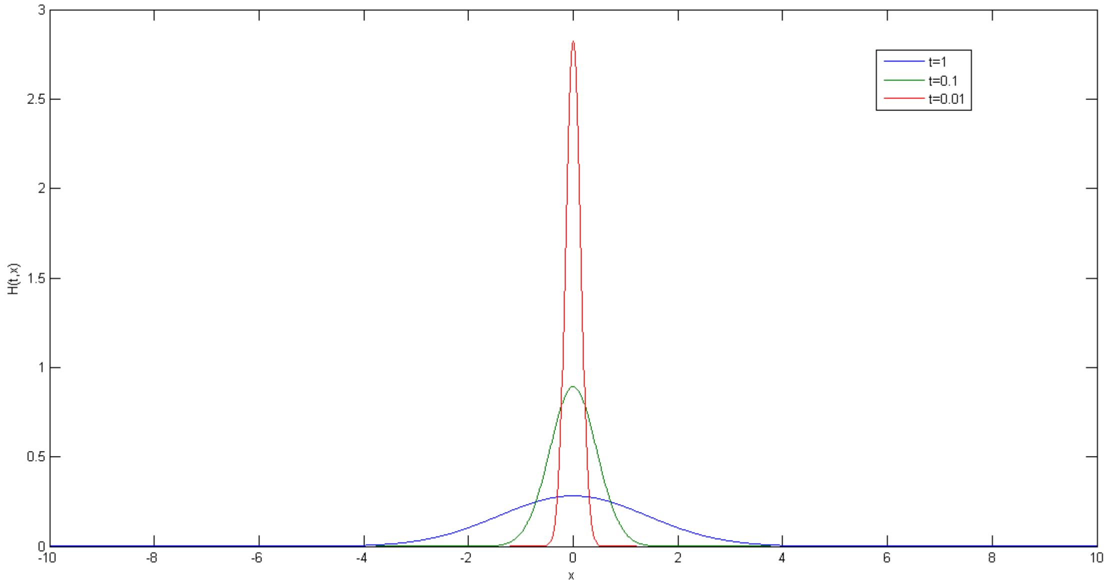
$$\mathcal{H}_t(\mathbf{x}) := K_{4\pi t}(\mathbf{x}) = \frac{1}{(4\pi t)^{\frac{d}{2}}}e^{-\frac{|\mathbf{x}|^2}{4t}}$$

Proposition 7.1. *The Fourier transform of* $\mathcal{H}_t$ *for* $t > 0$ *is* $\hat{\mathcal{H}}_t(\boldsymbol{\xi}) = e^{-4\pi^2 t|\boldsymbol{\xi}|^2}$.

PROOF. It follows immediately from Proposition 6.7. by making a change of variable.

In the following figure, we have plotted the 1-dimensional Heat Kernel respect the spacial variable x for times $t = 1$, $t = 0.1$ and $t = 0.01$.

We can see that the curves are becoming more concentrated near $x = 0$ when t becomes smaller. Nevertheless, they are doing so in a way that keeps the area under each curve equal to 1.



Lemma 7.1. *Let $\delta > 0$ be any real positive number. Then, $\mathcal{H}_t(\mathbf{x})$ and all its partial derivatives are uniformly bounded on $(\mathbf{x}, t) \in \mathbb{R}^d \times [\delta, +\infty)$.*

PROOF

• We start showing that $\mathcal{H}_t(\mathbf{x})$ is uniformly bounded on $(\mathbf{x}, t) \in \mathbb{R}^d \times [\delta, +\infty)$. In effect, note that:

$$|\mathcal{H}_t(\mathbf{x})| = \left| \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t}} \right| \leq \left| \frac{1}{(4\pi t)^{\frac{d}{2}}} \right| \leq \frac{1}{(4\pi \delta)^{\frac{d}{2}}} \quad \text{for all} \quad (\mathbf{x}, t) \in \mathbb{R}^d \times [\delta, +\infty)$$

• Now, let us take $k \in \mathbb{N}$ and a multi-index α. One proves that the partial derivatives of $\mathcal{H}_t(\mathbf{x})$ can be written as:

$$\left(\frac{\partial^k}{\partial t^k}\right) \left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \mathcal{H}_t(\mathbf{x}) = \sum_{i=1}^{l} \frac{P_i(\mathbf{x})}{t^{\frac{d}{2}+c_i}} e^{-\frac{|\mathbf{x}|^2}{4t}}$$

for some $l, c_i \in \mathbb{N} \setminus \{0\}$ and polynomials P_i. Note that as $e^{-\frac{|\mathbf{x}|^2}{4t}} \in \mathcal{S}\left(\mathbb{R}^d\right)$ then $P_i\left(\mathbf{x}\right) e^{-\frac{|\mathbf{x}|^2}{4t}} \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^\infty\left(\mathbb{R}^d\right)$. Consequently, there exists $K_i > 0$ such that $\left| P_i\left(\mathbf{x}\right) e^{-\frac{|\mathbf{x}|^2}{4t}} \right| \leq K_i$. Then,

$$\left| \left(\frac{\partial^k}{\partial t^k}\right) \left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \mathcal{H}_t(\mathbf{x}) \right| \leq \sum_{i=1}^{l} \left| \frac{P_i(\mathbf{x})}{t^{\frac{d}{2}+c_i}} e^{-\frac{|\mathbf{x}|^2}{4t}} \right| \leq \sum_{i=1}^{l} \left| \frac{K}{\delta^{\frac{d}{2}+c_i}} \right| \quad \text{for all} \quad (\mathbf{x}, t) \in \mathbb{R}^d \times [\delta, +\infty)$$

Thus, all the partial derivatives of $\mathcal{H}_t(\mathbf{x})$ are uniformly bounded on $(\mathbf{x}, t) \in \mathbb{R}^d \times [\delta, +\infty)$.

Lemma 7.2. *Let $\delta > 0$ be any real positive number. Then, $\mathcal{H}_t(\cdot) \in \mathcal{S}\left(\mathbb{R}^d\right)$ for all $t \in [\delta, +\infty)$.*

PROOF

- First of all, note that $\mathcal{H}_t(\mathbf{x}) = \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t}}$ is in $C^\infty(\mathbb{R}^d)$ respect $\mathbf{x}$ for all $t \geq \delta$.
- Moreover, the function $\mathcal{H}_t(\cdot)$ decays rapidly since $\lim\limits_{|\mathbf{x}| \longrightarrow +\infty} \left| \frac{\mathbf{x}^\alpha}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}|^2}{4t}} \right| = 0$ for all $t \geq \delta$. Finally, its derivatives also decay rapidly, since it holds that $\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \mathcal{H}_t(\mathbf{x}) = \sum\limits_{i=1}^{l} \frac{P_i(\mathbf{x})}{t^{\frac{d}{2}+c_i}} e^{-\frac{|\mathbf{x}|^2}{4t}}$ (where $P_i(\mathbf{x})$ is a polynomial) for all multi-index α. Hence, $\lim\limits_{|x| \longrightarrow +\infty} \left| \mathbf{x}^\alpha \sum\limits_{i=1}^{l} \frac{P_i(\mathbf{x})}{t^{\frac{d}{2}+c_i}} e^{-\frac{|\mathbf{x}|^2}{4t}} \right| = 0$ for all $t \geq \delta$.

7.3 Solution of the time-dependent heat equation on $\mathbb{R}^d$

Let us consider a non-bounded d-dimensional surface, which we model by $\mathbb{R}^d$, and suppose we are given an initial temperature distribution $g(\mathbf{x})$ on the surface at time $t = 0$. We wish to determine the temperature $u(\mathbf{x}, t)$ at a point $\mathbf{x} \in \mathbb{R}^d$ at time $t > 0$. Hence, we need to solve the following initial-value problem:

$$\begin{cases} u_t(\mathbf{x}, t) = \triangle u(\mathbf{x}, t) & \mathbf{x} \in \mathbb{R}^d,\ t > 0 \\ u(\mathbf{x}, 0) = g(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{7.3.1}$$

Now, let us suppose that the solution u and the function g satisfy the hypothesis of the theorems proved in the previous chapters and we can apply them freely. Note that we only assume this to find a possible solution of (7.3.1). Later, we will formalize it.

Let us take the Fourier transform respect $\mathbf{x}$ in both sides of the heat equation. On the left hand side, we get:

$$\begin{aligned} \widehat{[u_t]}(\boldsymbol{\xi}, t) &= \int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} u_t(\mathbf{x}, t) d\mathbf{x} = \int_{\mathbb{R}^d} \frac{\partial}{\partial t} \left(e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} u(\mathbf{x}, t) \right) d\mathbf{x} = \\ &\overset{Theor\ A.5.}{=} \frac{\partial}{\partial t} \left(\int_{\mathbb{R}^d} e^{-2\pi i \boldsymbol{\xi}\cdot\mathbf{x}} u(\mathbf{x}, t) d\mathbf{x} \right) = \hat{u}_t(\boldsymbol{\xi}, t) \end{aligned}$$

On the right hand side, we apply Proposition 6.5.4. with $\alpha_i = (0, .., 0, 2, 0, .., 0)$.

$$\widehat{[\triangle u]}(\boldsymbol{\xi}, t) = \left[\sum_{i=1}^{n} \widehat{\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha_i}} u \right](\boldsymbol{\xi}, t) = \sum_{i=1}^{n} (2\pi i \boldsymbol{\xi})^{\alpha_i} \hat{u}(\boldsymbol{\xi}, t) = -4\pi^2 \sum_{i=1}^{n} \xi_i^2 \hat{u}(\boldsymbol{\xi}, t) = -4\pi^2 |\boldsymbol{\xi}|^2 \hat{u}(\boldsymbol{\xi}, t)$$

Hence, fixing $\boldsymbol{\xi}$, we get an ordinary differential equation of the Fourier transform of u with respect the temporal variable. We can solve it by separation of variables:

$$\hat{u}_t(\boldsymbol{\xi}, t) = -4\pi^2 |\boldsymbol{\xi}|^2 \hat{u}(\boldsymbol{\xi}, t) \implies \frac{d\hat{u}}{\hat{u}} = -4\pi^2 |\boldsymbol{\xi}|^2 dt \implies ln(\hat{u}) = -4\pi^2 |\boldsymbol{\xi}|^2 t + C$$

$$\implies \hat{u}(\boldsymbol{\xi}, t) = A(\boldsymbol{\xi}) e^{-4\pi^2 |\boldsymbol{\xi}|^2 t}$$

Note that the constant depends on $\boldsymbol{\xi}$, since we have considered the ODE for a $\boldsymbol{\xi}$ fix. Hence, for each $\boldsymbol{\xi}$ we will obtain a different constant which we denote by $A(\boldsymbol{\xi})$. Moreover, in $t = 0$ we get:

$$\hat{u}(\boldsymbol{\xi}, 0) = A(\boldsymbol{\xi}) \implies A(\boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi})$$

Finally, using Proposition 7.1. i.e. $\hat{\mathcal{H}}_t(\boldsymbol{\xi}) = e^{-4\pi^2 t|\boldsymbol{\xi}|^2}$ and the inverse Fourier transform we get:

$$\hat{u}(\boldsymbol{\xi}, t) = \hat{g}(\boldsymbol{\xi})\hat{\mathcal{H}}_t(\boldsymbol{\xi}) = \widehat{g * \mathcal{H}_t}(\boldsymbol{\xi}) \implies u(\mathbf{x}, t) = (g * \mathcal{H}_t)(\mathbf{x})$$

Theorem 7.1. *Let us consider the initial-value problem (7.3.1). Assume that* $g \in C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$. *Then the following statements hold:*

1. *The function* $u(\mathbf{x}, t) = (g * \mathcal{H}_t)(\mathbf{x})$ *is* $C^\infty(\mathbb{R}^d \times (0, +\infty))$
2. $u(\mathbf{x}, t)$ *solves the heat equation for all* $\mathbf{x} \in \mathbb{R}^d$ *and* $t > 0$.
3. $\lim_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u(\mathbf{x}, t) = g(\mathbf{x}_0)$ *where* $\mathbf{x}$, $\mathbf{x}_0 \in \mathbb{R}^d$ *and* $t > 0$.
4. *If* $g \in \mathcal{S}(\mathbb{R}^d)$, *then* $\lim_{t\longrightarrow 0} \|u(\cdot, t) - g\|_\infty = 0$ *(uniformly convergence in* $\mathbf{x}$*). Hence, if we set* $u(\mathbf{x}, 0) = g(\mathbf{x})$, *it holds* $u \in C^\infty(\mathbb{R} \times (0, +\infty)) \cap C^0(\mathbb{R} \times [0, +\infty))$
5. *If* $g \in \mathcal{S}(\mathbb{R}^d)$, *then* $\lim_{t\longrightarrow 0} \|u(\cdot, t) - g\|_2 = 0$

PROOF.

• Proof 1: if $g \in L^1(\mathbb{R}^d)$, we can apply Differentiation under the integral sign Theorem. In effect, let $k \in \mathbb{N}$, α be a multi-index and $\delta > 0$. By Lemma 7.1., there exists $C > 0$ such that $\left|\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \mathcal{H}_t(\mathbf{x})\right| < C$ on $(\mathbf{x}, t) \in \mathbb{R}^d \times [\delta, +\infty)$. Consequently, $\left|\left[\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \mathcal{H}_t(\mathbf{x} - \mathbf{y})\right] g(\mathbf{y})\right| \leq C\,|g(\mathbf{y})|$ which is integrable. Then, we conclude that $u \in C^\infty(\mathbb{R}^d \times [\delta, +\infty))$ and

$$\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha u(\mathbf{x}, t) = \left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha (g * \mathcal{H}_t)(\mathbf{x}) = \int_{\mathbb{R}^d} \left[\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \mathcal{H}_t(\mathbf{x} - \mathbf{y})\right] g(\mathbf{y})d\mathbf{y}$$

As we have shown it for any $\delta > 0$, the result is true for $(\mathbf{x}, t) \in \mathbb{R}^d \times (0, +\infty)$. If $g \notin L^1(\mathbb{R}^d)$, a similar argument can be applied but we will have to work the term $\mathcal{H}_t(\mathbf{x} - \mathbf{y})$.

Alternatively, we can use the Theory of distributions that we introduce in Chapters 9, 10, 11 and 12 (these results can be generalized to $\mathbb{R}^d$). As $\mathcal{H}_t(\mathbf{x}) \in \mathcal{S}(\mathbb{R}^d)$ and the generalized function T_g of g is in $\mathcal{S}^*(\mathbb{R}^d)$ (Proposition 10.3.). Then by Proposition 11.2. we conclude that

$$(\mathcal{H}_t * T_g)(\mathbf{x}) = < T_g, \mathcal{H}_t(\mathbf{x} - \mathbf{y}) > = \int_{\mathbb{R}^d} \mathcal{H}_t(\mathbf{x} - \mathbf{y})g(\mathbf{y})d\mathbf{y} = u(\mathbf{x}, t)$$

is in $C^\infty(\mathbb{R}^d)$ and $\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha u(\mathbf{x}, t) = \left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \mathcal{H}_t\right] * T_g(\mathbf{x}) = \int_{\mathbb{R}^d} \left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha \mathcal{H}_t(\mathbf{x} - \mathbf{y})g(\mathbf{y})d\mathbf{y}$.

• Proof 2: we shall see that $u_t(\mathbf{x}, t) = \triangle u(\mathbf{x}, t)$. Note $u(\mathbf{x}, t) = (g * \mathcal{H}_t)(\mathbf{x}) = \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} g(\mathbf{y})d\mathbf{y}$. Then, by a straightforward calculation, we get:

$$u_t(\mathbf{x}, t) - \triangle u(\mathbf{x}, t) = \int_{\mathbb{R}^d} \left[\frac{\partial}{\partial t} - \triangle_{\mathbf{x}}\right] \mathcal{H}_t(\mathbf{x} - \mathbf{y})\, g(\mathbf{y})d\mathbf{y} =$$

$$= \int_{\mathbb{R}^d} \left(\frac{-d}{2t\,(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} + \frac{|\mathbf{x} - \mathbf{y}|^2}{4t^2\,(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} - \frac{|\mathbf{x} - \mathbf{y}|^2}{4t^2\,(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} + \frac{d}{2t\,(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}}\right) g(\mathbf{y})d\mathbf{y} = 0$$

• Proof 3: let $\mathbf{x}_0 \in \mathbb{R}^d$ and $\epsilon > 0$. We need to show that there exists $\delta > 0$ such that $|u(\mathbf{x},t) - g(\mathbf{x}_0)| < \epsilon$ for $|(\mathbf{x},t) - (\mathbf{x}_0, 0)| < \delta$. Note that by Proposition 6.8. $K_{4\pi t}(\mathbf{x}) = \mathcal{H}_t(\mathbf{x})$ is a good kernel. Thus, we can write:

$$|u(\mathbf{x},t) - g(\mathbf{x}_0)| = \left| \int_{\mathbb{R}^d} \frac{1}{(4\pi t)^{\frac{d}{2}}} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} \left(g(\mathbf{y}) - g(\mathbf{x}_0)\right) d\mathbf{y} \right|$$

Let $B(\mathbf{x}_0, \gamma)$ be the ball of radius γ centered at $\mathbf{x}_0$. Using the fact that g is continuous, if we take γ sufficiently small:

$$\left| \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{B(\mathbf{x}_0,\gamma)} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} \left(g(\mathbf{y}) - g(\mathbf{x}_0)\right) d\mathbf{y} \right| \le \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{B(\mathbf{x}_0,\gamma)} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} |g(\mathbf{y}) - g(\mathbf{x}_0)| \, d\mathbf{y} < \frac{\epsilon}{2}$$

Now, we look at the integral over the complement of $B(\mathbf{x}_0, \gamma)$. Let us take $\mathbf{y} \in \mathbb{R}^d \setminus B(\mathbf{x}_0, \gamma)$ and $\mathbf{x}$ such that $|\mathbf{x} - \mathbf{x}_0| < \frac{\gamma}{2}$. Then:

$$|\mathbf{y} - \mathbf{x}_0| \le |\mathbf{y} - \mathbf{x}| + |\mathbf{x} - \mathbf{x}_0| < |\mathbf{y} - \mathbf{x}| + \frac{\gamma}{2} < |\mathbf{y} - \mathbf{x}| + \frac{1}{2} |\mathbf{y} - \mathbf{x}_0|$$

Therefore, we have that $|\mathbf{y} - \mathbf{x}_0| < 2|\mathbf{y} - \mathbf{x}|$. Moreover $|g(\mathbf{x})| \le C$ for all $\mathbf{x} \in \mathbb{R}^d$ since $g \in L^\infty\left(\mathbb{R}^d\right)$. Thus

$$\left| \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d \setminus B(\mathbf{x}_0,\gamma)} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} \left(g(\mathbf{y}) - g(\mathbf{x}_0)\right) d\mathbf{y} \right| \le \frac{D}{t^{\frac{d}{2}}} \int_{\mathbb{R}^d \setminus B(\mathbf{x}_0,\gamma)} e^{-\frac{|\mathbf{x}_0-\mathbf{y}|^2}{16t}} d\mathbf{y} =$$

$$= \left[\begin{array}{c} \mathbf{y} = h(\mathbf{z}) = \mathbf{x}_0 - \sqrt{t}\mathbf{z} \\ (Dh)(\mathbf{z}) = -\sqrt{t}\,(Id) \implies |det\,(Dh)(\mathbf{y})| = t^{\frac{d}{2}} \end{array} \right] = D \int_{\mathbb{R}^d \setminus B(0, \frac{\gamma}{\sqrt{t}})} e^{-\frac{|\mathbf{z}|^2}{16}} d\mathbf{z} \overset{t \longrightarrow 0^+}{\longrightarrow} 0 \quad \text{where } D = \frac{2C}{(4\pi)^{\frac{d}{2}}}$$

Thus, there exists $\eta > 0$ such that for $0 < t < \eta$, it holds $\left| \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d \setminus B(\mathbf{x}_0,\gamma)} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} \left(g(\mathbf{y}) - g(\mathbf{x}_0)\right) d\mathbf{y} \right| < \frac{\epsilon}{2}$. Hence, taking $\delta = \min(\frac{\gamma}{2}, \eta)$, for all $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$ such that $|(\mathbf{x},t) - (\mathbf{x}_0, 0)| < \delta$, it holds:

$$|u(\mathbf{x},t) - g(\mathbf{x}_0)| \le \left| \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{B(\mathbf{x}_0,\gamma)} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} \left(g(\mathbf{y}) - g(\mathbf{x}_0)\right) d\mathbf{y} \right| +$$

$$+ \left| \frac{1}{(4\pi t)^{\frac{d}{2}}} \int_{\mathbb{R}^d \setminus B(\mathbf{x}_0,\gamma)} e^{-\frac{|\mathbf{x}-\mathbf{y}|^2}{4t}} \left(g(\mathbf{y}) - g(\mathbf{x}_0)\right) d\mathbf{y} \right| < \epsilon$$

• Proof 4: as $g \in \mathcal{S}\left(\mathbb{R}^d\right)$ and $\mathcal{H}_t(\mathbf{x}) = K_{4\pi t}(\mathbf{x})$, we can apply proposition 6.9. and conclude that $u(\mathbf{x},t) = (g * \mathcal{H}_t)(\mathbf{x}) \overset{t \longrightarrow 0}{\longrightarrow} g(\mathbf{x})$ uniformly in $\mathbf{x}$. That is:

$$\lim_{t \longrightarrow 0} \|u(\cdot,t) - g\|_\infty = 0$$

• Proof 5: let us fix $t \in (0, +\infty)$. Note that g, $\mathcal{H}_t \in \mathcal{S}\left(\mathbb{R}^d\right)$ (Lemma 7.2.), then by proposition 6.11.1 $u(\cdot,t) \in \mathcal{S}\left(\mathbb{R}^d\right)$ respect $\mathbf{x}$ and by Proposition 6.11.2 $\hat{u}(\boldsymbol{\xi},t) = \hat{g}(\boldsymbol{\xi})\,\hat{\mathcal{H}_t}(\boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi})\, e^{-4\pi^2 t|\boldsymbol{\xi}|^2}$. Thus using Theorem 6.3. we get:

$$\|u(\cdot,t) - g\|_2^2 = \|\hat{u}(\cdot,t) - \hat{g}\|_2^2 \implies \int_{\mathbb{R}^d} |u(\mathbf{x},t) - g(\mathbf{x})|^2 \, d\mathbf{x} = \int_{\mathbb{R}^d} |\hat{u}(\boldsymbol{\xi},t) - \hat{g}(\boldsymbol{\xi})|^2 \, d\boldsymbol{\xi} =$$

$$= \int_{\mathbb{R}^d} |\hat{g}(\boldsymbol{\xi})|^2 \left| e^{-4\pi^2 t|\boldsymbol{\xi}|^2} - 1 \right|^2 d\boldsymbol{\xi}$$

Let us take $\epsilon > 0$. We note the following facts:

- Let us define $I_n = \int_{|\boldsymbol{\xi}|\geq n} |\hat{g}(\boldsymbol{\xi})|^2 \left|e^{-4\pi^2 t|\boldsymbol{\xi}|^2} - 1\right|^2 d\boldsymbol{\xi}$ for $n \in \mathbb{N}$. Note that $I_n < +\infty$ since $\left|e^{-4\pi^2 t|\boldsymbol{\xi}|^2} - 1\right|^2 \leq 4$ and $\hat{g} \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^2\left(\mathbb{R}^d\right)$. Hence, $(I_n)_{n\in\mathbb{N}}$ defines a sequence in $\mathbb{R}$ such that $I_n \overset{n\longrightarrow+\infty}{\longrightarrow} 0$. Thus, there exists $N > 0$ such that $|I_n| < \frac{\epsilon}{2}$ for all $n \geq N$. In particular:

$$I_N = \int_{|\boldsymbol{\xi}|\geq N} |\hat{g}(\boldsymbol{\xi})|^2 \left|e^{-4\pi^2 t|\boldsymbol{\xi}|^2} - 1\right|^2 d\boldsymbol{\xi} < \frac{\epsilon}{2}$$

- Note that $\hat{g} \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^\infty\left(\mathbb{R}^d\right)$. Let us define $H(t) := \sup_{|\boldsymbol{\xi}|\leq N} \left(|\hat{g}(\boldsymbol{\xi})|^2 \left|e^{-4\pi^2 t|\boldsymbol{\xi}|^2} - 1\right|^2\right)$. It holds that $\lim_{t\longrightarrow 0} H(t) = 0$. Hence, there exists $\delta > 0$ such that $|H(t)| < \frac{\epsilon}{2(2N)^d}$ for all $|t| < \delta$. Then:

$$\int_{|\boldsymbol{\xi}|\leq N} |\hat{g}(\boldsymbol{\xi})|^2 \left|e^{-4\pi^2 t|\boldsymbol{\xi}|^2} - 1\right|^2 d\boldsymbol{\xi} \leq \int_{|\boldsymbol{\xi}|\leq N} H(t) d\boldsymbol{\xi} < \frac{\epsilon}{2\,(2N)^d}\,(2N)^d = \frac{\epsilon}{2}$$

Then, it holds that there exists $\delta > 0$ such that for all $|t| < \delta$:

$$\int_{\mathbb{R}^d} |u(\mathbf{x},t) - g(\mathbf{x})|^2 d\mathbf{x} \leq \int_{|\boldsymbol{\xi}|\geq N} |\hat{g}(\boldsymbol{\xi})|^2 \left|e^{-4\pi^2 t|\boldsymbol{\xi}|^2} - 1\right|^2 d\boldsymbol{\xi} + \int_{|\boldsymbol{\xi}|\leq N} |\hat{g}(\boldsymbol{\xi})|^2 \left|e^{-4\pi^2 t|\boldsymbol{\xi}|^2} - 1\right|^2 d\boldsymbol{\xi} < \epsilon$$

Thus, we conclude that $\lim_{t\longrightarrow 0} \|u(\cdot,t) - g\|_2 = 0$.

Lemma 7.3. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence in $C^k\left(\mathbb{R}^d\right)$ and let f be a function in $C^k\left(\mathbb{R}^d\right)$. Assume that $(u_n)_{n\in\mathbb{N}}$ converges uniformly to a function u in $\mathbb{R}^d$. If $\left(\frac{\partial}{\partial x}\right)^\alpha u_n$ converges uniformly to $\left(\frac{\partial}{\partial x}\right)^\alpha f$ in $\mathbb{R}^d$ for all multi-index α such that $1 \leq |\alpha| \leq k$, then it holds that $\left(\frac{\partial}{\partial x}\right)^\alpha u = \left(\frac{\partial}{\partial x}\right)^\alpha f$ in $\mathbb{R}^d$ for all multi-index α such that $1 \leq |\alpha| \leq k$. In particular, $u \in C^k\left(\mathbb{R}^d\right)$.*

PROOF.

• Step 1: first, we prove the lemma for a multi-index α such that $|\alpha| = 1$. We will apply Theorem A.14. Let us fix $\hat{\mathbf{x}} = (x_1,..,x_{i-1},x_{i+1},..,x_d)$ and define $\tilde{u}_n(x_i) := u_n(x_i,\hat{\mathbf{x}})$, $\tilde{u}(x_i) := u(x_i,\hat{\mathbf{x}})$ and $\tilde{f}(x_i) := f(x_i,\hat{\mathbf{x}})$. Let us take an arbitrary interval $[a,b] \subset \mathbb{R}$. Note that:

- As $(u_n)_{n\in\mathbb{N}}$ is a sequence in $C^1\left(\mathbb{R}^d\right)$, then $\tilde{u}_n(x_i)$ is $C^1(\mathbb{R})$.
- As $(u_n)_{n\in\mathbb{N}}$ converges uniformly to u in $\mathbb{R}^d$, then $(\tilde{u}_n)$ converges uniformly to $\tilde{u}$ in $[a,b]$. In effect:

$$\sup_{x_i\in[a,b]} |u_n(x_i,\hat{\mathbf{x}}) - u(x_i,\hat{\mathbf{x}})| \leq \sup_{(x_i,\hat{\mathbf{x}})\in\mathbb{R}^d} |u_n(x_i,\hat{\mathbf{x}}) - u(x_i,\hat{\mathbf{x}})| = \|u_n - u\|_\infty \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

- As $\frac{\partial}{\partial x_i} u_n$ converges uniformly to a function $\frac{\partial}{\partial x_i} f$ in $\mathbb{R}^d$, then $\frac{\partial}{\partial x_i}\tilde{u}_n$ converges uniformly to $\frac{\partial}{\partial x_i}\tilde{f}$ in $[a,b]$. In effect:

$$\sup_{x_i\in[a,b]} \left|\frac{\partial}{\partial x_i} u_n(x_i,\hat{\mathbf{x}}) - \frac{\partial}{\partial x_i} f(x_i,\hat{\mathbf{x}})\right| \leq \sup_{(x_i,\hat{\mathbf{x}})\in\mathbb{R}^d} \left|\frac{\partial}{\partial x_i} u_n(x_i,\hat{\mathbf{x}}) - \frac{\partial}{\partial x_i} f(x_i,\hat{\mathbf{x}})\right| \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

Thus, by Theorem A.14. we get that $(\tilde{u}_n)$ converges uniformly to a function $h \in C^1((a,b))$ such that $\frac{\partial}{\partial x_i} h(x_i) = \frac{\partial}{\partial x_i}\tilde{f}(x_i)$ in $[a,b]$. However, we have seen that it should be $h(x_i) = \tilde{u}(x_i)$. Hence we conclude that $\frac{\partial}{\partial x_i} u(x_i,\hat{\mathbf{x}}) = \frac{\partial}{\partial x_i} f(x_i,\hat{\mathbf{x}})$ in $[a,b]$. Since we have proved that for an arbitrary interval $[a,b]$ and $\hat{\mathbf{x}} \in \mathbb{R}^{d-1}$ we conclude that:

$$\frac{\partial}{\partial x_i} u(x_i,\hat{\mathbf{x}}) = \frac{\partial}{\partial x_i} f(x_i,\hat{\mathbf{x}}) \qquad \text{for all } (x_i,\hat{\mathbf{x}}) \in \mathbb{R}^d$$

• Step 2: if we have an arbitrary multi-index such that $1 \leq |\alpha| \leq k$, we get by induction and applying the argument of step 1 that $\left(\frac{\partial}{\partial x}\right)^{\alpha} u = \left(\frac{\partial}{\partial x}\right)^{\alpha} f$ in $\mathbb{R}^d$.

Theorem 7.2. *Let us consider the initial-value problem (7.3.1). Assume $g \in L^2\left(\mathbb{R}^d\right)$. Then the following statements hold:*

1. *The function $u(\mathbf{x}, t) = (g * \mathcal{H}_t)(\mathbf{x})$ is $C^{\infty}\left(\mathbb{R}^d \times (0, +\infty)\right)$*
2. *$u(\mathbf{x}, t)$ solves the heat equation for all $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$.*
3. $\lim\limits_{t \longrightarrow 0} \|u(\cdot, t) - g\|_2 = 0$

PROOF.
We start making the following observations:

- As $\mathcal{S}\left(\mathbb{R}^d\right)$ is dense in $L^2\left(\mathbb{R}^d\right)$, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ of elements in $\mathcal{S}\left(\mathbb{R}^d\right)$ such that $\lim\limits_{n \longrightarrow \infty} \|g_n - g\|_2 = 0$.
- As $\mathcal{S}\left(\mathbb{R}^d\right) \subset C\left(\mathbb{R}^d\right) \cap L^{\infty}\left(\mathbb{R}^d\right)$, Theorem 7.1. holds for each of the functions g_n. We define $u_n = g_n * \mathcal{H}_t$.

• Proof 1: we will apply Lemma 7.3. By Theorem 7.1. $u_n \in C^{\infty}\left(\mathbb{R}^d \times (0, +\infty)\right)$.

1. First, we show that $(u_n)_n$ converges uniformly to u on $\mathbb{R}^d \times (0, +\infty)$. In effect, as $(g_n - g)$ and $\mathcal{H}_t$ are in $L^2\left(\mathbb{R}^d\right)$ for all $t > 0$, we can apply Proposition 3.3.2 (this result can be generalized to $\mathbb{R}^d$):
$$|u_n(\mathbf{x}, t) - u(\mathbf{x}, t)| \leq \|(g_n - g) * \mathcal{H}_t\|_{\infty} \leq \|g_n - g\|_2 \|\mathcal{H}_t\|_2 \overset{n \longrightarrow +\infty}{\longrightarrow} 0$$
for all $(\mathbf{x}, t) \in \mathbb{R}^d \times (0, +\infty)$. Thus, $(u_n)_n$ converges uniformly to u on $\mathbb{R}^d \times (0, +\infty)$.
2. Finally, we prove that $\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} u_n \overset{Theor. 7.1.}{=} g_n * \left[\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} \mathcal{H}_t\right]$ converges uniformly to $g * \left[\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} \mathcal{H}_t\right]$. In effect, again as $(g_n - g)$ and $\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} \mathcal{H}_t$ are in $L^2\left(\mathbb{R}^d\right)$ for all $t > 0$, we can apply Proposition 3.3.2:
$$\left|\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} u_n(\mathbf{x}, t) - g * \left[\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} \mathcal{H}_t\right](\mathbf{x})\right| \leq \left\|(g_n - g) * \left[\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} \mathcal{H}_t\right]\right\|_{\infty}$$
$$\leq \|g_n - g\|_2 \left\|\left(\frac{\partial^k}{\partial t^k}\right)\left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} \mathcal{H}_t\right\|_2 \overset{n \longrightarrow +\infty}{\longrightarrow} 0$$

Hence, as we have proved it for an arbitrary multi-index α and $k \in \mathbb{N}$, we can apply Lemma 7.3. and we get that $u \in C^{\infty}(\mathbb{R}^d \times (0, +\infty))$ and
$$\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} u(\mathbf{x}, t) = g * \left[\left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} \mathcal{H}_t\right](\mathbf{x}) = \lim_{n \longrightarrow +\infty} \left(\frac{\partial}{\partial t}\right)^k \left(\frac{\partial}{\partial \mathbf{x}}\right)^{\alpha} u_n(\mathbf{x}, t) \qquad (7.3.2)$$

• Proof 2: we will use (7.3.2) and that u_n solves the heat equation. Thus:
$$\triangle u(\mathbf{x}, t) = \lim_{n \longrightarrow +\infty} \triangle u_n(\mathbf{x}, t) = \lim_{n \longrightarrow +\infty} \left(\frac{\partial}{\partial t}\right) u_n(\mathbf{x}, t) = u_t(\mathbf{x}, t)$$

Hence, we get that $u(\mathbf{x}, t)$ solves the heat equation for all $\mathbf{x} \in \mathbb{R}^d$ and for all $t > 0$.

• Proof 3: as $\mathcal{H}_t \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^1\left(\mathbb{R}^d\right)$ for all $t > 0$ and $g - g_n \in L^2\left(\mathbb{R}^d\right)$, we can apply Lemma 3.3. (this result can be generalized in $\mathbb{R}^d$) and we get that:

$$\|u_n(\cdot, t) - u(\cdot, t)\|_2 = \|(g - g_n) * \mathcal{H}_t\|_2 \leq \|g - g_n\|_2 \|\mathcal{H}_t\|_1 = \|g - g_n\|_2$$

Hence, using the triangular inequality:

$$\|u(\cdot, t) - g\|_2 \leq \|u(\cdot, t) - u_n(\cdot, t)\|_2 + \|u_n(\cdot, t) - g_n\|_2 + \|g_n - g\|_2 =$$

$$= \|u_n(\cdot, t) - g_n\|_2 + 2\|g_n - g\|_2$$

Now, as by Theorem 7.1. $\lim\limits_{t \longrightarrow 0} \|u_n(\cdot, t) - g_n\|_2 = 0$, we get that $\lim\limits_{t \longrightarrow 0} \|u(\cdot, t) - g\|_2 \leq 2\|g_n - g\|_2$. Finally, taking the limit as $n \longrightarrow +\infty$ in this expression, we conclude:

$$\lim_{t \longrightarrow 0} \|u(\cdot, t) - g\|_2 = 0$$

7.4 Properties of the heat equation

1. *Smoothness of solutions.* As it can be seen from the above theorem, solutions of the heat equation are infinitely times differentiable. Even if there are singularities in the initial data (for example in the case that the initial data is in $L^2\left(\mathbb{R}^d\right)$) , they are instantly "smoothed out" and the solution $u(\mathbf{x}, t) \in C^\infty\left(\mathbb{R}^d \times (0, +\infty)\right)$.

2. *Domain of dependence.* The value of the solution at the point $\mathbf{x} \in \mathbb{R}^d$, time $t > 0$ depends on the value of the initial data on the whole space. In other words, there is an infinite domain of dependence for solutions to the heat equation. This is in contrast to hyperbolic equations where solutions are known to have finite domains of dependence.

Example 7.1.

Let us consider an infinite rod with an initial temperature distribution

$$g(x) := \begin{cases} \frac{1}{1+x^2} & x \in \mathbb{R} \setminus \mathbb{N} \\ 0 & x \in \mathbb{N} \end{cases}$$

We want to determine the temperature of the rod $u(x, t)$ at a point $x \in \mathbb{R}$ and time $t > 0$. It is given by the solution of (7.3.1). Note that $g \in L^2(\mathbb{R})$ since $\int_{\mathbb{R}} g(x)^2 dx = \frac{\pi}{2}$. Hence, by Theorem 7.2. the solution is given by $u(x, t) = (g * \mathcal{H}_t)(x) = \int_{\mathbb{R}} \frac{1}{1+y^2} \frac{1}{\sqrt{4\pi t}} e^{-\frac{(x-y)^2}{4t}} dy$. In the following figures, we show the temperature of the rod at times $t = 0$, $t = 0.00001$, $t = 1$ and $t = 100$:

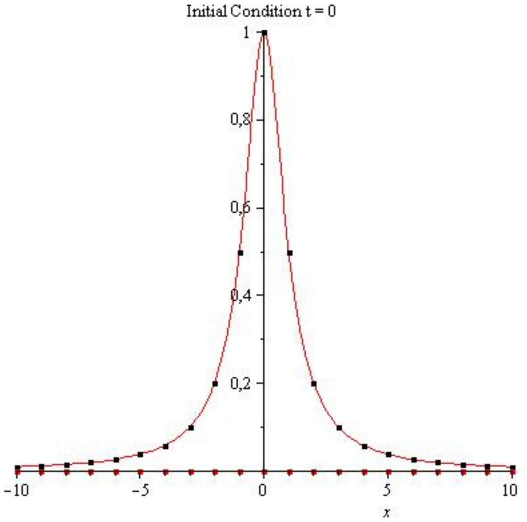


Figure 7.4.1: $t = 0$

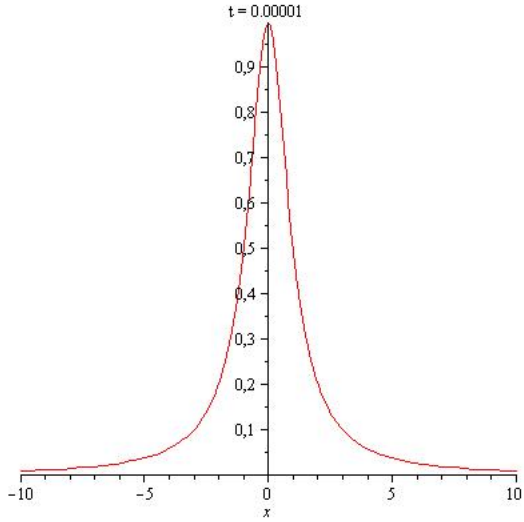


Figure 7.4.2: $t = 0.00001$

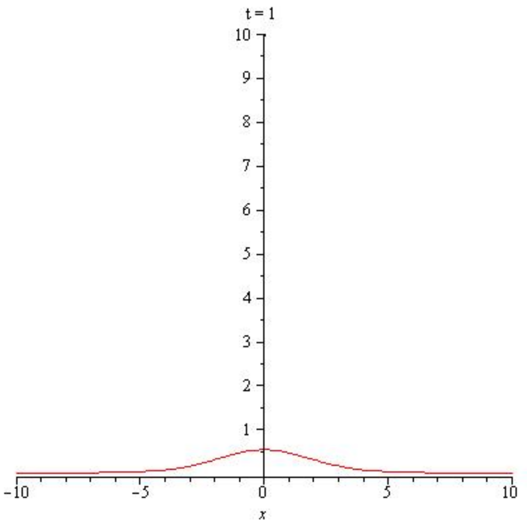


Figure 7.4.3: $t = 1$

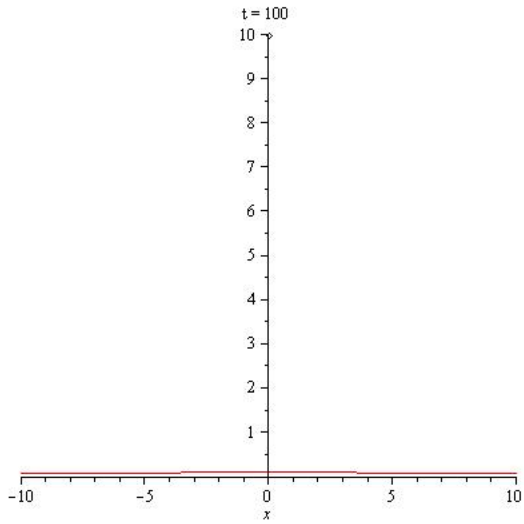


Figure 7.4.4: $t = 100$

In the figures, we can see the property of smoothness of solutions. In effect, the initial-condition f is not continuous and has an infinite number of discontinuities. However, the solution becomes immediately infinitely times differentiable, as we can see in the second figure.

We also note that at $t = 0$ the rod is very hot near the origin, while the points which are far from the origin are rather colder. However, as we can notice in figure 3 and 4, the heat flows from the origin to the distant points. In fact, the rod tends to be in thermal equilibrium when $t \longrightarrow +\infty$. This fact is due to the Fourier's Law introduced at the beginning of the chapter.

7.5 Inhomogeneous heat equation on $\mathbb{R}^d$

Let us consider a non-bounded d-dimensional surface, which we model by $\mathbb{R}^d$, and suppose we are given an initial temperature distribution $g(\mathbf{x})$ on the surface at time $t = 0$. Moreover, let us suppose there is a sink or source of heat on the surface given by $f(\mathbf{x}, t)$. We wish to determine the temperature $u(\mathbf{x}, t)$ at a point $\mathbf{x} \in \mathbb{R}^d$ at time $t > 0$. Hence, we need to solve the following initial-value problem:

$$\begin{cases} u_t(\mathbf{x}, t) \,-\, \triangle u(\mathbf{x}, t) \,=\, f(\mathbf{x}, t) & \mathbf{x} \in \mathbb{R}^d,\, t > 0 \\ u(\mathbf{x}, 0) \,=\, g(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{7.5.1}$$

We claim that we can use solutions of the homogeneous equation to construct solutions of the inhomogeneous equation.

Duhamel's Principle

Consider the following ODE:

$$\begin{cases} u_t(t) + au(t) \,=\, f(t) \\ u(0) \,=\, g \end{cases}$$

where a is a constant. The solution of this ODE is given by $u(t) = e^{-at}g + \int_0^t e^{-a(t-s)} f(s)ds$. In other words, the solution $u(t)$ is the propagator e^{-at} applied to the initial data, plus the propagator "convolved" with the nonlinear term.

Thus, if we let $S(t)$ be the operator which multiplies functions by e^{-at}, we see that the solution of the homogeneous problem ($f(t) = 0$) is given by $u_h(t) = S(t)g = e^{-at}g$ and the solution of the inhomogeneous problem is given by:

$$u(t) \,=\, S(t)g \,+\, \int_0^t S(t-s)f(s)ds$$

This same technique will allow us to find a solution of the inhomogeneous heat equation. Being able to construct solutions of the inhomogeneous problem from solutions of the homogeneous problem is known as Duhamel's principle.

Hence, let us suppose we can solve the homogeneous problem (7.3.1) and that we can express the solution as $u_h(\mathbf{x}, t) = S(t)g(\mathbf{x})$. We claim that the solution of the inhomogeneous problem is given by:

$$u(\mathbf{x}, t) \,=\, S(t)g(\mathbf{x}) \,+\, \int_0^t S(t-s)f(\mathbf{x}, s)ds$$

At least formally, we see that

$$\left(\frac{\partial}{\partial t} - \triangle\right) u(\mathbf{x}, t) \,=\, \left(\frac{\partial}{\partial t} - \triangle\right) [S(t)g(\mathbf{x})] \,+\, \left(\frac{\partial}{\partial t} - \triangle\right) \left[\int_0^t S(t-s)f(\mathbf{x}, s)ds\right] \,=$$

$$= 0 \,+\, S(t-t)f(\mathbf{x}, s) \,+\, \int_0^t \left(\frac{\partial}{\partial t} - \triangle\right) S(t-s)f(\mathbf{x}, s)ds \,=\, S(0)f(\mathbf{x}, s) \,=\, f(\mathbf{x}, s)$$

Solution of the Inhomogeneous Heat Equation

We have seen that the solution of the homogeneous heat equation is given by $u_h(\mathbf{x},t) = (g * \mathcal{H}_t)(\mathbf{x}) = \int_{\mathbb{R}^d} \mathcal{H}_t(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, d\mathbf{y}$, when g satisfies certain conditions. That is, we can think of the solution operator $S(t)$ associated with the heat equation on $\mathbb{R}^d$ as defined by $S(t)g(\mathbf{x}) := \int_{\mathbb{R}^d} \mathcal{H}_t(\mathbf{x}-\mathbf{y})\, g(\mathbf{y})\, d\mathbf{y}$. Therefore, by Duhamel's Principle, we expect that the solution of the inhomogeneous heat equation to be given by

$$u(\mathbf{x},t) = S(t)g(\mathbf{x}) + \int_0^t S(t-s)f(\mathbf{x},s)ds = \int_{\mathbb{R}^d} \mathcal{H}_t(\mathbf{x}-\mathbf{y})\, f(\mathbf{y})\, d\mathbf{y} +$$

$$+ \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{t-s}(\mathbf{x}-\mathbf{y})\, f(\mathbf{y},s) d\mathbf{y} ds$$

We will show that $u_p(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{t-s}(\mathbf{x}-\mathbf{y})\, f(\mathbf{y},s)d\mathbf{y}ds$ satisfies the inhomogeneous heat equation with zero initial data. If we can prove this, then $u(x,t) = u_h(\mathbf{x},t) + u_p(\mathbf{x},t)$ will solve (7.5.1).

Theorem 7.3. *Assume $f \in C_1^2(\mathbb{R}^d \times [0,+\infty))$ (meaning f is twice continuously differentiable in the spatial variables and once continuously differentiable in the time variable) and has compact support. Then the following statements hold:*

1. *The function $u(\mathbf{x},t) := \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{t-s}(\mathbf{x}-\mathbf{y})\, f(\mathbf{y},s)d\mathbf{y}ds$ is in $C_1^2(\mathbb{R}^d \times (0,+\infty))$.*
2. *$u(\mathbf{x},t)$ solves $u_t(\mathbf{x},t) - \triangle u(\mathbf{x},t) = f(\mathbf{x},t)$ for all $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$.*
3. *$\lim_{t \longrightarrow 0} \|u(\cdot,t)\|_\infty = 0$ (uniform convergence).*

PROOF.

• Proof 1: we make a change of variables as follows: $\widetilde{\mathbf{y}} := \mathbf{x}-\mathbf{y}$ and $\widetilde{s} := t-s$. Then, we have:

$$u(\mathbf{x},t) := \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{t-s}(\mathbf{x}-\mathbf{y})\, f(\mathbf{y},s)d\mathbf{y}ds = \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}})\, f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s})d\widetilde{\mathbf{y}}d\widetilde{s} \tag{7.5.2}$$

Applying Lemma 7.1. and differentiating under the integral sign, one gets that

- $u_t(\mathbf{x},t) = \int_{\mathbb{R}^d} \mathcal{H}_t(\widetilde{\mathbf{y}})\, f(\mathbf{x}-\widetilde{\mathbf{y}},0)d\widetilde{\mathbf{y}} + \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}})\, \frac{\partial}{\partial t} f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s})d\widetilde{\mathbf{y}}d\widetilde{s}$ which is continuous.
- $u_{x_i}(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}})\, \frac{\partial}{\partial x_i} f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s})d\widetilde{\mathbf{y}}d\widetilde{s}$ for $1 \le i \le d$, which is continuous.
- $u_{x_i x_j}(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}})\, \frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s})d\widetilde{\mathbf{y}}d\widetilde{s}$ for $1 \le i < j \le d$, which is continuous.

Therefore, we conclude that $u \in C_1^2(\mathbb{R}^d \times (0,+\infty))$.

• Proof 2: using the same change of variables of (7.5.2) and the relations deduced in 1, we get:

$$\left(\frac{\partial}{\partial t} - \triangle_x\right) u(\mathbf{x},t) = \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(\frac{\partial}{\partial t} - \triangle_x\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s})d\widetilde{\mathbf{y}}d\widetilde{s} + \int_{\mathbb{R}^d} \mathcal{H}_t(\widetilde{\mathbf{y}})\, f(\mathbf{x}-\widetilde{\mathbf{y}},0)d\widetilde{\mathbf{y}} =$$

$$= \int_0^t \int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(-\frac{\partial}{\partial \widetilde{s}} - \triangle_{\widetilde{\mathbf{y}}}\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s})d\widetilde{\mathbf{y}}d\widetilde{s} + \int_{\mathbb{R}^d} \mathcal{H}_t(\widetilde{\mathbf{y}})\, f(\mathbf{x}-\widetilde{\mathbf{y}},0)d\widetilde{\mathbf{y}} =$$

$$= \int_{\epsilon}^{t}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(-\frac{\partial}{\partial \widetilde{s}} - \triangle_{\widetilde{\mathbf{y}}}\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s} + \int_{0}^{\epsilon}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(-\frac{\partial}{\partial \widetilde{s}} - \triangle_{\widetilde{\mathbf{y}}}\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s} +$$

$$+ \int_{\mathbb{R}^d} \mathcal{H}_t(\widetilde{\mathbf{y}}) f(\mathbf{x}-\widetilde{\mathbf{y}}, 0) d\widetilde{\mathbf{y}} \ \equiv \ I_\epsilon + J_\epsilon + K$$

- First we check J_ϵ. As $f \in C_1^2\left(\mathbb{R}^d \times [0,+\infty)\right)$ and has compact support, it holds that f_t, $\triangle f$ are in $L^\infty\left(\mathbb{R}^d\right)$. Then:

$$|J_\epsilon| = \left|\int_0^{\epsilon}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(-\frac{\partial}{\partial \widetilde{s}} - \triangle_{\widetilde{\mathbf{y}}}\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s}\right| \le \left(\|f_t\|_\infty + \|\triangle f\|_\infty\right) \int_0^{\epsilon}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}})\, d\widetilde{\mathbf{y}} d\widetilde{s} =$$

$$= \left(\|f_t\|_\infty + \|\triangle f\|_\infty\right)\epsilon \overset{\epsilon \longrightarrow 0^+}{\longrightarrow} 0$$

- For I_ϵ, we use the assumption that f has compact support $M \subset \mathbb{R}^d$. And as $f \in C_1^2\left(\mathbb{R}^d \times [0,+\infty)\right)$, then f, f_t, $\triangle f$, ∇f are zero in ∂M. Integrating by parts:

- $$\int_{\epsilon}^{t}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(-\frac{\partial}{\partial \widetilde{s}}\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s} \overset{Fubini}{=} \int_{\mathbb{R}^d}\int_{\epsilon}^{t} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(-\frac{\partial}{\partial \widetilde{s}}\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{s} d\widetilde{\mathbf{y}} =$$

$$\overset{parts}{=} \int_{\mathbb{R}^d} \left[-\mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s})\right]_{\epsilon}^{t} d\widetilde{\mathbf{y}} + \int_{\mathbb{R}^d}\int_{\epsilon}^{t} \frac{\partial}{\partial \widetilde{s}}\mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{s} d\widetilde{\mathbf{y}}$$

- $$\int_{\epsilon}^{t}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(-\triangle_{\widetilde{\mathbf{y}}}\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s} \overset{Theor.A.12}{=} \int_{\epsilon}^{t}\int_{\mathbb{R}^d} \nabla_{\widetilde{\mathbf{y}}}\mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \nabla_{\widetilde{\mathbf{y}}} f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s} =$$

$$\overset{Theor.A.12}{=} \int_{\epsilon}^{t}\int_{\mathbb{R}^d} \left[-\triangle_{\widetilde{\mathbf{y}}}\mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}})\right] f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s}$$

Hence, using these two equalities, we finally get that:

$$I_\epsilon = \int_{\epsilon}^{t}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) \left(-\frac{\partial}{\partial \widetilde{s}} - \triangle_{\widetilde{\mathbf{y}}}\right) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s} = \int_{\epsilon}^{t}\int_{\mathbb{R}^d} \left[\left(\frac{\partial}{\partial \widetilde{s}} - \triangle_{\widetilde{\mathbf{y}}}\right)\mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}})\right] f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s}$$

$$-\int_{\mathbb{R}^d} \mathcal{H}_t(\widetilde{\mathbf{y}}) f(\mathbf{x}-\widetilde{\mathbf{y}}, 0) d\widetilde{\mathbf{y}} + \int_{\mathbb{R}^d} \mathcal{H}_\epsilon(\widetilde{\mathbf{y}}) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\epsilon) d\widetilde{\mathbf{y}} = -K + \int_{\mathbb{R}^d} \mathcal{H}_\epsilon(\widetilde{\mathbf{y}}) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\epsilon) d\widetilde{\mathbf{y}}$$

- Finally, we get that:

$$\left(\frac{\partial}{\partial t} - \triangle_x\right) u(\mathbf{x},t) = \lim_{\epsilon \longrightarrow 0^+} \left(I_\epsilon + J_\epsilon + K\right) = \lim_{\epsilon \longrightarrow 0^+} \int_{\mathbb{R}^d} \mathcal{H}_\epsilon(\widetilde{\mathbf{y}}) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\epsilon) d\widetilde{\mathbf{y}} = f(\mathbf{x},t)$$

where the last equality can be shown using the same technique we used to prove Theorem 7.1.3.

• Proof 3: as f has compact support and is continuous, then is bounded. Note that for all $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$ hold:

$$|u(\mathbf{x},t)| = \left|\int_0^{t}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}}) f(\mathbf{x}-\widetilde{\mathbf{y}}, t-\widetilde{s}) d\widetilde{\mathbf{y}} d\widetilde{s}\right| \le \|f\|_\infty \int_0^{t}\int_{\mathbb{R}^d} \mathcal{H}_{\widetilde{s}}(\widetilde{\mathbf{y}})\, d\widetilde{\mathbf{y}} d\widetilde{s} = \|f\|_\infty \int_0^{t} d\widetilde{s} = \|f\|_\infty t$$

Taking supremums in $\mathbf{x} \in \mathbb{R}^d$, we get that:

$$\|u(\cdot,t)\|_\infty \le \|f\|_\infty t \overset{t \longrightarrow 0^+}{\longrightarrow} 0$$

7.6 Uniqueness of solutions

Energy Method for the Heat Equation in $\mathbb{R}^d$

We will use the Energy Method to show the uniqueness of solutions for the homogeneous heat equation in the case that the initial condition is a functions in $\mathcal{S}\left(\mathbb{R}^d\right)$.

Lemma 7.4. *Let $g \in \mathcal{S}\left(\mathbb{R}^d\right)$. Then $u_h(\mathbf{x},t) := (g * \mathcal{H}_t)(\mathbf{x})$ is a solution of the homogeneous heat equation (7.3.1). Moreover $u_h \in C^\infty\left(\mathbb{R}^d \times (0,+\infty)\right) \cap C^0\left(\mathbb{R}^d \times [0,+\infty)\right)$ and $u_h(\cdot,t) \in \mathcal{S}\left(\mathbb{R}^d\right)$ uniformly in t in the sense that for any $T>0$:*

$$\sup_{(\mathbf{x},t)\in\mathbb{R}^d\times(0,T)} |\mathbf{x}|^\beta \left|\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha u_h(\mathbf{x},t)\right| < +\infty \qquad \text{for any multi-index } \alpha,\, \beta \tag{7.6.1}$$

PROOF.
As $g \in \mathcal{S}\left(\mathbb{R}^d\right) \subset C\left(\mathbb{R}^d\right) \cap L^\infty\left(\mathbb{R}^d\right)$, then by Theorem 7.1., $u_h \in C^\infty\left(\mathbb{R}^d \times (0,+\infty)\right) \cap C^0\left(\mathbb{R}^d \times [0,+\infty)\right)$ is a solution of the homogeneous heat equation. Moreover, as for each t fixed, $\mathcal{H}_t \in \mathcal{S}\left(\mathbb{R}^d\right)$, then by Proposition 6.11.1 $u_h(\mathbf{x},t) = (g * \mathcal{H}_t)(\mathbf{x})$ is in $\mathcal{S}\left(\mathbb{R}^d\right)$ for each $t \in (0,+\infty)$ fixed and $\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha u_h(\mathbf{x},t) = \left(\left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha g\right] * \mathcal{H}_t\right)(\mathbf{x})$. Finally, as $|\mathbf{x}|^\beta \left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha g$, $\mathcal{H}_t \in \mathcal{S}\left(\mathbb{R}^d\right) \subset L^1(\mathbb{R}^d)$, we can apply Proposition 3.2.2 and we get that

$$\begin{aligned} |\mathbf{x}|^\beta \left|\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha u_h(\mathbf{x},t)\right| &\leq |\mathbf{x}|^\beta \left\|\left[\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha g\right] * \mathcal{H}_t\right\|_\infty \overset{Prop. 3.2.2}{\leq} |\mathbf{x}|^\beta \left\|\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha g\right\|_\infty \|\mathcal{H}_t\|_1 \\ &\leq \left\||\mathbf{x}|^\beta \left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha g\right\|_\infty < +\infty \end{aligned}$$

where we have used that $\|\mathcal{H}_t\|_1 = 1$ and $|\mathbf{x}|^\beta \left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha g \in \mathcal{S}\left(\mathbb{R}^d\right)$. As this inequality holds for all $(\mathbf{x},t) \in \mathbb{R}^d \times (0,T)$, we conclude that for any $T>0$:

$$\sup_{(\mathbf{x},t)\in\mathbb{R}^d\times(0,T)} |\mathbf{x}|^\beta \left|\left(\frac{\partial}{\partial \mathbf{x}}\right)^\alpha u_h(\mathbf{x},t)\right| < +\infty \qquad \text{for any multi-index } \alpha,\, \beta$$

Theorem 7.4. *Let $g \in \mathcal{S}\left(\mathbb{R}^d\right)$. Consider the homogeneous initial-value problem of the heat equation given by (7.3.1). There exists an unique solution $u \in C^\infty\left(\mathbb{R}^d \times (0,+\infty)\right) \cap C^0\left(\mathbb{R}^d \times [0,+\infty)\right)$ such that $u(\cdot,t) \in \mathcal{S}\left(\mathbb{R}^d\right)$ uniformly in t as in (7.6.1).*

PROOF.

- The existence has already been proved in Lemma 7.4.
- To show uniqueness, let us suppose there exist two solutions $u(\mathbf{x},t)$ and $v(\mathbf{x},t)$ satisfying the conditions of the statement. Let us consider $w(\mathbf{x},t) = u(\mathbf{x},t) - v(\mathbf{x},t)$. It holds that $w \in C^\infty\left(\mathbb{R}^d \times (0,+\infty)\right) \cap C^0\left(\mathbb{R}^d \times [0,+\infty)\right)$ and $w(\cdot,t) \in \mathcal{S}\left(\mathbb{R}^d\right)$ uniformly in t. Moreover:

$$\begin{cases} w_t(\mathbf{x},t) - \triangle w(\mathbf{x},t) = 0 & \mathbf{x} \in \mathbb{R}^d,\ t \in (0,+\infty) \\ w(\mathbf{x},0) = 0 & \mathbf{x} \in \mathbb{R}^d,\ t = 0 \end{cases}$$

We will use the Energy Method to see that $w(\mathbf{x},t) = 0$ on $\mathbb{R}^d \times [0,+\infty)$. Let us define the energy at time $t \in [0,+\infty)$ by:

$$E(t) = \int_{\mathbb{R}^d} |w(\mathbf{x},t)|^2 \, d\mathbf{x} = \int_{\mathbb{R}^d} w(\mathbf{x},t)\overline{w(\mathbf{x},t)} d\mathbf{x}$$

The assumptions on w allow us to differentiate $E(t)$ under the integral sign. We get that:

$$\frac{d}{dt}E(t) = \int_{\mathbb{R}^d} \frac{\partial}{\partial t}\left(w(\mathbf{x},t)\overline{w(\mathbf{x},t)}\right) d\mathbf{x} = \int_{\mathbb{R}^d}\left(w_t(\mathbf{x},t)\overline{w(\mathbf{x},t)} + w(\mathbf{x},t)\overline{w_t(\mathbf{x},t)}\right) d\mathbf{x} =$$

$$= \int_{\mathbb{R}^d}\left(\triangle w(\mathbf{x},t)\overline{w(\mathbf{x},t)} + w(\mathbf{x},t)\overline{\triangle w(\mathbf{x},t)}\right) d\mathbf{x}$$

Using Green's identity (Theorem A.12), we get:

$$\frac{d}{dt}E(t) = \lim_{R\longrightarrow+\infty}\int_{|\mathbf{x}|\le R}\left(\triangle w(\mathbf{x},t)\overline{w(\mathbf{x},t)} + w(\mathbf{x},t)\overline{\triangle w(\mathbf{x},t)}\right) d\mathbf{x} = -2\lim_{R\longrightarrow+\infty}\int_{|\mathbf{x}|\le R} \nabla w(\mathbf{x},t)\nabla\overline{w(\mathbf{x},t)} d\mathbf{x}$$

$$+ \lim_{R\longrightarrow+\infty}\int_{|\mathbf{x}|=R}\overline{w(\mathbf{x},t)}\left(\nabla w(\mathbf{x})\cdot n\right) dS + \lim_{R\longrightarrow+\infty}\int_{|\mathbf{x}|=R} w(\mathbf{x},t)\left(\overline{\nabla w(\mathbf{x})}\cdot n\right) dS$$

Note that as $\overline{w(\cdot,t)}$, $\nabla w(\cdot,t)\cdot n \in \mathcal{S}\left(\mathbb{R}^d\right)$ for all $t \in (0,+\infty)$, it holds that $\lim_{|\mathbf{x}|\longrightarrow+\infty} |\mathbf{x}|^{2d}\left|\overline{w(\mathbf{x},t)}\right| = \lim_{|\mathbf{x}|\longrightarrow+\infty} |\mathbf{x}|^{2d}\left|\nabla w(\mathbf{x},t)\cdot n\right| = 0$. Then,

1. There exists $M_1 > 0$ such that $|\mathbf{x}|^{2d}\left|\overline{w(\mathbf{x},t)}\right| < 1$ for all $|\mathbf{x}| \ge M_1$
2. There exists $M_2 > R$ such that $|\mathbf{x}|^{2d}\left|\nabla w(\mathbf{x},t)\cdot n\right| < 1$ for all $|\mathbf{x}| \ge M_2$

Let $M = \max\{M_1, M_2\}$, applying this and Property A.1. in the Spherical coordinates section in the appendix, we get that for all $R > M$

$$\left|\int_{|\mathbf{x}|=R}\overline{w(\mathbf{x},t)}\left(\nabla w(\mathbf{x})\cdot n\right) dS\right| \le \int_{|\mathbf{x}|=R}\left|\overline{w(\mathbf{x},t)}\right|\left|\nabla w(\mathbf{x})\cdot n\right| dS \le \int_{|\mathbf{x}|=R}\frac{1}{|\mathbf{x}|^{4d}} dS = \frac{1}{R^{4d}}\int_{|\mathbf{x}|=R} dS =$$

$$= \frac{1}{R^{4d}} SA\left(B(0,R)\right) = \frac{1}{R^{4d}} d\alpha(d)R^{d-1} = \frac{1}{R^{3d+1}} d\alpha(d) \overset{R\longrightarrow+\infty}{\longrightarrow} 0$$

Doing the same argument, we get that $\lim_{R\longrightarrow+\infty}\int_{|\mathbf{x}|=R} w(\mathbf{x},t)\left(\overline{\nabla w(\mathbf{x})}\cdot n\right) dS = 0$. Hence:

$$\frac{d}{dt}E(t) = -2\lim_{R\longrightarrow+\infty}\int_{|x|\le R}\nabla w(\mathbf{x},t)\nabla\overline{w(\mathbf{x},t)} d\mathbf{x} = -2\int_{x\in\mathbb{R}}\left|\nabla w(\mathbf{x},t)\right|^2 d\mathbf{x} \le 0 \quad \forall t > 0$$

Hence, as $E(t) \ge 0$, $E(0) = 0$ and $E(t)$ is decreasing in $(0,+\infty)$, we conclude $E(t) = \int_{\mathbb{R}^d}|w(\mathbf{x},t)|^2 d\mathbf{x} = 0$ for all $t \ge 0$. As w is continuous, then $w(\mathbf{x},t) = 0$ for all $\mathbf{x} \in \mathbb{R}^d$ and $t \ge 0$. Thus, it holds that $u(\mathbf{x},t) = v(\mathbf{x},t)$ and the solution is unique.

Maximum Principle for the Heat Equation

We will prove what is known as the maximum principle for the heat equation. We will then use this principle to prove uniqueness of solutions to the initial-value problem for the heat equation in a more general case than in Theorem 7.4.

Theorem 7.5. (Maximum Principle on Bounded Domains) *Let Ω be an open, bounded set in $\mathbb{R}^d$. Let $\Omega_T := \Omega \times (0,T)$ and $\Gamma_T := \left(\overline{\Omega} \times \{0\}\right) \cup \left(\partial\Omega \times (0,T]\right)$. Assume $u \in C_1^2\left(\Omega_T\right) \cap C\left(\overline{\Omega}_T\right)$ satisfies $u_t - \triangle u \leq 0$ in Ω_T. Then:*

$$\max_{(\mathbf{x},t)\in\overline{\Omega}_T} u(\mathbf{x},t) = \max_{(\mathbf{x},t)\in\Gamma_T} u(\mathbf{x},t)$$

PROOF.
Note first that u is a continuous function in the compact $\overline{\Omega}_T$. Hence, u has a maximum in $\overline{\Omega}_T$.
• Step 1: first we proof the Theorem for the case $u_t\left(\mathbf{x},t\right) - \triangle u\left(\mathbf{x},t\right) < 0$ for all $(\mathbf{x},t) \in \Omega_T$. Let us suppose that $\max_{(\mathbf{x},t)\in\overline{\Omega}_T} u(\mathbf{x},t) = u(\mathbf{x}_0,t_0)$ and $\mathbf{x}_0 \in \Omega$, $t_0 \in (0,T]$, i.e. $(\mathbf{x}_0,t_0) \notin \Gamma_T$.

1. Define $G(\mathbf{x}) := u(\mathbf{x},t_0)$ for $\mathbf{x} \in \Omega$. Note that, as u has a maximum at $(\mathbf{x}_0,t_0)$, then $G(\mathbf{x})$ has a maximum in $\mathbf{x}_0$ in the open set Ω. Consequently, $\triangle G\left(\mathbf{x}_0\right) = \triangle u(\mathbf{x}_0,t_0) \leq 0$.
2. Define $F(t) := u(\mathbf{x_0},t)$ for $t \in (0,T]$. Note that, as u has a maximum at $(\mathbf{x}_0,t_0)$, then $F(t)$ has a maximum in t_0. We have the following cases:

- If $0 < t_0 < T$, then $F(t)$ has a maximum in t_0 in the open interval $(0,T)$. Hence, $F'(t) = u_t(\mathbf{x}_0,t_0) = 0$.
- If $t_0 = T$, by Taylor expansion around T and $h > 0$: $F(T-h) = F(T) - F'(T)h + \mathcal{O}(h^2)$, or equivalently, $F'(T) = \frac{F(T)-F(T-h)+\mathcal{O}(h^2)}{h}$. Note that $F(T) > F(T-h)$. Hence, taking h small enough, we can conclude that $F'(T) = u_t(\mathbf{x}_0,T) \geq 0$.

Hence, we have seen that $u_t\left(\mathbf{x}_0,t_0\right) - \triangle u\left(\mathbf{x}_0,t_0\right) \geq 0$, which is a contradiction. Thus, the $(\mathbf{x}_0,t_0) \in \Gamma_T$.

• Step 2: let us now proof the theorem when $u_t\left(\mathbf{x},t\right) - \triangle u\left(\mathbf{x},t\right) = 0$ for all $(\mathbf{x},t) \in \Omega_T$. Let us define $u_\epsilon\left(\mathbf{x},t\right) = u\left(\mathbf{x},t\right) - \epsilon t$ for some $\epsilon > 0$. Note that $u_\epsilon \in C_1^2\left(\Omega_T\right) \cap C\left(\overline{\Omega}_T\right)$ and satisfies:

$$\frac{\partial}{\partial t} u_\epsilon\left(\mathbf{x},t\right) - \triangle u_\epsilon\left(\mathbf{x},t\right) = u_t\left(\mathbf{x},t\right) - \epsilon - \triangle u\left(\mathbf{x},t\right) = -\epsilon < 0 \quad \text{for all} \quad (\mathbf{x},t) \in \Omega_T$$

Hence we can apply Step 1 to u_ϵ

$$u\left(\mathbf{x},t\right) - \epsilon t = u_\epsilon\left(\mathbf{x},t\right) \leq \max_{(\mathbf{x},t)\in\overline{\Omega}_T} u_\epsilon(\mathbf{x},t) = \max_{(\mathbf{x},t)\in\Gamma_T} u_\epsilon(\mathbf{x},t) \quad \text{for all} \quad (\mathbf{x},t) \in \overline{\Omega}_T$$

Moreover, as for all $t \geq 0$, $u_\epsilon\left(\mathbf{x},t\right) \leq u(\mathbf{x},t)$, then $\max_{(\mathbf{x},t)\in\Gamma_T} u_\epsilon(\mathbf{x},t) \leq \max_{(\mathbf{x},t)\in\Gamma_T} u(\mathbf{x},t)$. Thus:

$$u\left(\mathbf{x},t\right) \leq \max_{(\mathbf{x},t)\in\overline{\Omega}_T} u_\epsilon(\mathbf{x},t) + \epsilon t \leq \max_{(\mathbf{x},t)\in\Gamma_T} u(\mathbf{x},t) + \epsilon t \quad \text{for all} \quad (\mathbf{x},t) \in \overline{\Omega}_T$$

And as we have shown it for an arbitrary $\epsilon > 0$, we can conclude

$$\max_{(\mathbf{x},t)\in\overline{\Omega}_T} u(\mathbf{x},t) = \max_{(\mathbf{x},t)\in\Gamma_T} u(\mathbf{x},t)$$

Corollary 7.1. *Let Ω be an open, bounded set in $\mathbb{R}^d$. Let $\Gamma_T := \left(\overline{\Omega} \times \{0\}\right) \cup (\partial\Omega \times (0,T])$ and $\Omega_T := \Omega \times (0,T)$. Assume $u \in C_1^2(\Omega_T) \cap C\left(\overline{\Omega}_T\right)$. Then, the following statements hold:*

1. *If $u_t - \triangle u \geq 0$ in Ω_T. Then $\min\limits_{(\mathbf{x},t)\in\overline{\Omega}_T} u(\mathbf{x},t) = \min\limits_{(\mathbf{x},t)\in\Gamma_T} u(\mathbf{x},t)$.*
2. *If $u_t - \triangle u = 0$ in Ω_T. Then $\min\limits_{(\mathbf{x},t)\in\overline{\Omega}_T} u(\mathbf{x},t) = \min\limits_{(\mathbf{x},t)\in\Gamma_T} u(\mathbf{x},t)$ and $\max\limits_{(\mathbf{x},t)\in\overline{\Omega}_T} u(\mathbf{x},t) = \max\limits_{(\mathbf{x},t)\in\Gamma_T} u(\mathbf{x},t)$.*

Theorem 7.6. (Maximum Principle on $\mathbb{R}^d$) *Suppose $u \in C_1^2\left(\mathbb{R}^d \times (0,T]\right) \cap C\left(\mathbb{R}^d \times [0,T]\right)$ with $T > 0$ solves (7.3.1) and satisfies the growth estimate $u(\mathbf{x},t) \leq Ae^{a|\mathbf{x}|^2}$ ($\mathbf{x} \in \mathbb{R}^d$, $0 \leq t \leq T$) for constants A, $a > 0$. Then:*

$$\sup_{(\mathbf{x},t)\in\mathbb{R}^d\times[0,T]} u(\mathbf{x},t) = \sup_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$$

PROOF.

• Step 1: first, let us suppose that $4aT < 1$. Therefore, there exists $\epsilon > 0$ such that $4a(T+\epsilon) < 1$. Now fix $\mathbf{y} \in \mathbb{R}^d$ and $\mu > 0$ and define:

$$v(\mathbf{x},t) := u(\mathbf{x},t) - \frac{\mu}{(T+\epsilon-t)^{\frac{d}{2}}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon-t)}}$$

Using the fact that u is a solution of the heat equation, it is straightforward to show that v is a solution of the heat equation, that is

$$v_t(\mathbf{x},t) = \triangle v(\mathbf{x},t), \qquad (\mathbf{x},t) \in \mathbb{R}^d \times (0,T]$$

Now, let us fix $r > 0$ and define $U := B(\mathbf{y},r)$, $U_T := U \times (0,T)$ and $\Gamma_T := \left(\overline{B(\mathbf{y},r)} \times \{0\}\right) \cup (\partial B(\mathbf{y},r) \times (0,T])$. From Theorem 7.5., we know that:

$$\max_{(\mathbf{x},t)\in\overline{U}_T} v(\mathbf{x},t) = \max_{(\mathbf{x},t)\in\Gamma_T} v(\mathbf{x},t)$$

We will show now that $\max\limits_{(\mathbf{x},t)\in\Gamma_T} v(\mathbf{x},t) \leq \sup\limits_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$. Let us consider the following cases:

1. If $(\mathbf{x},t) \in \overline{B(\mathbf{y},r)} \times \{0\}$, i.e. $t = 0$ and $\mathbf{x} \in \overline{B(\mathbf{y},r)}$. Then:
$$v(\mathbf{x},0) = u(\mathbf{x},0) - \frac{\mu}{(T+\epsilon)^{\frac{d}{2}}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon)}} \leq u(\mathbf{x},0) = g(\mathbf{x}) \leq \sup_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$$
2. If $(\mathbf{x},t) \in \partial B(\mathbf{y},r) \times [0,T]$, i.e. $|\mathbf{x}-\mathbf{y}| = r$ and $t \in [0,T]$. Then:
$$v(\mathbf{x},t) = u(\mathbf{x},t) - \frac{\mu}{(T+\epsilon-t)^{\frac{d}{2}}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon-t)}} \leq Ae^{a|\mathbf{x}|^2} - \frac{\mu}{(T+\epsilon-t)^{\frac{d}{2}}} e^{\frac{r^2}{4(T+\epsilon-t)}} \leq Ae^{a(|\mathbf{y}|+r)^2} - \frac{\mu}{(T+\epsilon)^{\frac{d}{2}}} e^{\frac{r^2}{4(T+\epsilon)}}$$
Now, by assumption, $4a(T+\epsilon) < 1$. Therefore, $\frac{1}{4(T+\epsilon)} = a + \gamma$ for some $\gamma > 0$. Thus:
$$v(\mathbf{x},t) \leq Ae^{a(|\mathbf{y}|+r)^2} - \mu\,(4(a+\gamma))^{\frac{d}{2}} e^{(a+\gamma)r^2} \leq A_1 e^{(a+\gamma)r^2} e^{-\gamma r^2 + 2a|\mathbf{y}|r} - A_2 e^{(a+\gamma)r^2} =$$
$$= e^{(a+\gamma)r^2}\left(A_1 e^{-\gamma r^2 + 2a|\mathbf{y}|r} - A_2\right) \overset{r\longrightarrow+\infty}{\longrightarrow} -\infty$$
where we have defined $A_1 := Ae^{a|\mathbf{y}|^2} > 0$ and $A_2 := \mu\,(4(a+\gamma))^{\frac{d}{2}} > 0$. Hence, taking r sufficiently large,
$$v(\mathbf{x},t) \leq \sup_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$$

Hence, for r large enough we have that $v(\mathbf{x},t) \leq \sup_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$ for all $(\mathbf{x},t) \in \overline{U}_T$. In particular, $v(\mathbf{y},t) \leq \sup_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$ for all $t \in [0,T]$. Note that we have made this argument for an arbitrary $\mathbf{y} \in \mathbb{R}^d$. Hence we can conclude that $v(\mathbf{y},t) \leq \sup_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$ for all $t \in [0,T]$ and for all $\mathbf{y} \in \mathbb{R}^d$. That is:

$$v(\mathbf{x},t) \;=\; u(\mathbf{x},t) - \frac{\mu}{(T+\epsilon-t)^{\frac{d}{2}}} e^{\frac{|\mathbf{x}-\mathbf{y}|^2}{4(T+\epsilon-t)}} \;\leq\; \sup_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$$

And taking the limit as $\mu \longrightarrow 0^+$ in both sides of the inequality we conclude that $u(\mathbf{x},t) \leq \sup_{\mathbf{x}\in\mathbb{R}^d} g(\mathbf{x})$ for all $t \in [0,T]$ and for all $\mathbf{x} \in \mathbb{R}^d$.

• Step 2: if $4aT \geq 1$. We divide the interval $[0,T]$ into some subintervals $[0,T_1]$, $[T_1, 2T_1]$,..., where $T_1 = \frac{1}{8a}$ and perform the same calculations of the step 1 on each of these subintervals.

Theorem 7.7. (Uniqueness of solutions on $\mathbb{R}^d$) *Let $g \in C\left(\mathbb{R}^d\right)$ and $f \in C\left(\mathbb{R}^d \times [0,T]\right)$ with $T > 0$. Then, there exists at most one solution $u \in C_1^2\left(\mathbb{R}^d \times (0,T]\right) \cap C\left(\mathbb{R}^d \times [0,T]\right)$ of (7.5.1) satisfying the growth estimate $|u(\mathbf{x},t)| \leq Ae^{a|\mathbf{x}|^2}$ ($\mathbf{x} \in \mathbb{R}^d$, $0 \leq t \leq T$) for constants A, $a > 0$.*

PROOF.

Let us suppose there exist two solutions u, $v \in C_1^2\left(\mathbb{R}^d \times (0,T]\right) \cap C\left(\mathbb{R}^d \times [0,T]\right)$ satisfying the growth estimate.

Let us consider $w(\mathbf{x},t) = u(\mathbf{x},t) - v(\mathbf{x},t)$. Note that $w \in C_1^2\left(\mathbb{R}^d \times (0,T]\right) \cap C\left(\mathbb{R}^d \times [0,T]\right)$. Moreover, w satisfies:

$$\begin{cases} w_t(\mathbf{x},t) - \triangle w(\mathbf{x},t) = 0 & \mathbf{x} \in \mathbb{R}^d,\ t \in (0,T) \\ w(\mathbf{x},0) = 0 & \mathbf{x} \in \mathbb{R}^d,\ t = 0 \end{cases}$$

Finally, $w(\mathbf{x},t) \leq |w(\mathbf{x},t)| = |u(\mathbf{x},t) - v(\mathbf{x},t)| \leq 2Ae^{a|\mathbf{x}|^2}$. Thus, we can apply Theorem 7.6. and conclude that:

$$w(\mathbf{x},t) \;\leq\; \sup_{(\mathbf{x},t)\in\mathbb{R}^d\times[0,T]} w(\mathbf{x},t) \;=\; \sup_{\mathbf{x}\in\mathbb{R}^d} 0 \;=\; 0$$

Hence, $u(\mathbf{x},t) \leq v(\mathbf{x},t)$ for all $\mathbf{x} \in \mathbb{R}^d$, $0 \leq t \leq T$.

Now, we consider $\widetilde{w}(\mathbf{x},t) = v(\mathbf{x},t) - u(\mathbf{x},t)$ and we apply exactly the same argument as above. We get that $v(\mathbf{x},t) \leq u(\mathbf{x},t)$ for all $\mathbf{x} \in \mathbb{R}^d$, $0 \leq t \leq T$.

Hence, we conclude $u(\mathbf{x},t) = v(\mathbf{x},t)$ for all $\mathbf{x} \in \mathbb{R}^d$, $0 \leq t \leq T$. That is, there exists at most one solution.

7.7 The steady-state heat equation in the upper half-plane

The equation we are now concerned with is:

$$\begin{cases} \triangle u(x,y) \;=\; \frac{\partial^2}{\partial x^2}u(x,y) \;+\; \frac{\partial^2}{\partial y^2}u(x,y) \;=\; 0 & (x,y) \in \mathbb{R}^2_+ \\ u(x,0) \;=\; g(x) & x \in \mathbb{R} \end{cases} \tag{7.7.1}$$

where we have defined $\mathbb{R}^2_+ := \big\{(x,y) \in \mathbb{R}^2 : x \in \mathbb{R},\, y > 0\big\}$. The operator $\triangle$ is the Laplacian and the above partial differential equation describes the steady-state heat distribution in $\mathbb{R}^2_+$ subject to $u = g$ on the boundary. We will give a probabilistic interpretation to the problem (7.7.1).

Probabilistic interpretation

Let us discretize the domain $\mathbb{R}^2_+$ with step h and let us consider the random walk of a particle along the domain. Let the rule of the movement be: the particle can jump to left, right, up or down with distance h equally likely, that is with probability $\frac{1}{4}$. Suppose that if the particle impacts on $y = 0$ at the point (x,y), then, we win $g(x)$ euros.

We are interested in determine the expected gain if we are in the point $(x,y) \in \mathbb{R}^2_+$. We will denote it by $u(x,y)$. Notice that the expected gain if we are in a point of the boundary is $u(x,0) = g(x)$.

Assume the particle is at a point $(x,y) \in \mathbb{R}^2_+$. As the particle can move in four directions, if we apply conditional expectation, we get that the expected gain in (x,y) is

$$u(x,y) \;=\; \frac{1}{4}\left[u(x+h,y) \;+\; u(x-h,y) \;+\; u(x,y+h) \;+\; u(x,y-h)\right]$$

Note that this expression can be written in this alternative way:

$$(u(x+h,y) \;+\; u(x-h,y) \;-\; 2u(x,y)) - (u(x,y+h) \;+\; u(x,y-h) - 2u(x,y)) \;=\; 0$$

Now taking Taylor expansions for $u(x+h,y)$, $u(x-h,y)$, $u(x,y+h)$ and $u(x,y-h)$

$$u_{xx}(x,y)h^2 \;+ \mathcal{O}(h^2) + \; u_{yy}(x,y)h^2 \;+ \mathcal{O}(h^2) \;=\; 0$$

Finally, dividing by h^2 and taking the limit $h \longrightarrow 0$, we get the Laplace's equation

$$\triangle u(x,y) \;=\; u_{xx}(x,y) \;+\; u_{yy}(x,y) \;=\; 0 \quad \text{with} \quad u(x,0) \;=\; g(x)$$

The Poisson kernel for $\mathbb{R}^2_+$

Definition 7.2. *We define the Poisson Kernel for the upper half-plane $\mathbb{R}^2_+$ as the family of functions $\{\mathcal{P}_y(x)\}_{y>0}$ $(x \in \mathbb{R})$ defined by:*

$$\mathcal{P}_y(x) \;:=\; \frac{1}{\pi}\frac{y}{x^2+y^2} \qquad \textit{with} \quad x \in \mathbb{R} \quad \textit{and} \quad y > 0$$

Proposition 7.2. *The Poisson Kernel for the upper half-plane satisfies the following statements:*

1. $\mathcal{P}_y(x) \in L^1(\mathbb{R})$ *for all* $y > 0$.
2. $\int_{\mathbb{R}} e^{-2\pi|\xi|y} e^{2\pi i\xi x} d\xi = \mathcal{P}_y(x)$ *for all* $y > 0$. *We deduce that* $\widehat{\mathcal{P}}_y(\xi) = e^{-2\pi|\xi|y}$.
3. $\{\mathcal{P}_y(x)\}_{y>0}$ *is a family of good kernels.*

PROOF.

• Proof 1: let $y > 0$, we note that:

$$\int_{\mathbb{R}} |\mathcal{P}_y(x)|\, dx = \frac{2}{\pi}\int_0^{+\infty} \frac{y}{x^2+y^2} dx = \frac{2}{\pi y}\int_0^{+\infty} \frac{1}{\left(\frac{x}{y}\right)^2+1} dx = \frac{2}{\pi y}\left[y \arctan\left(\frac{x}{y}\right)\right]_{x=0}^{x=\infty} = \frac{2}{\pi y}\left(\frac{y\pi}{2}\right) = 1$$

• Proof 2: the formula is fairly straightforward since we can split the integral from $-\infty$ to 0 and 0 to $+\infty$. Then, since $y > 0$:

1. $\int_0^{\infty} e^{-2\pi\xi y} e^{2\pi i\xi x} d\xi = \int_0^{\infty} e^{2\pi i\xi(x+iy)} d\xi = \left[\frac{e^{2\pi i\xi(x+iy)}}{2\pi i(x+iy)}\right]_0^{\infty} = -\frac{1}{2\pi i(x+iy)}$
2. $\int_{-\infty}^{0} e^{2\pi\xi y} e^{2\pi i\xi x} d\xi = \int_{-\infty}^{0} e^{2\pi i\xi(x-iy)} d\xi = \left[\frac{e^{2\pi i\xi(x-iy)}}{2\pi i(x-iy)}\right]_{-\infty}^{0} = \frac{1}{2\pi i(x-iy)}$

Therefore, we get:

$$\int_{\mathbb{R}} e^{-2\pi|\xi|y} e^{2\pi i\xi x} d\xi = \frac{1}{2\pi i(x-iy)} - \frac{1}{2\pi i(x+iy)} = \frac{1}{\pi}\frac{y}{x^2+y^2} = \mathcal{P}_y(x)$$

Moreover, we deduce that $\widehat{\mathcal{P}}_y(\xi) = e^{-2\pi|\xi|y}$. In effect, as $\mathcal{P}_y(x)$, $e^{-2\pi|\xi|y} \in L^1(\mathbb{R}) \cap C(\mathbb{R})$ for all $y > 0$, we can apply the Fourier inversion Theorem (Theorem 1.1.) to the previous equality:

$$\int_{\mathbb{R}} e^{-2\pi|\xi|y} e^{2\pi i\xi x} d\xi = \overline{\widehat{\left[e^{-2\pi|\xi|y}\right]}}(x) = \mathcal{P}_y(x) \implies e^{-2\pi|\xi|y} = \widehat{\mathcal{P}}_y(\xi)$$

• Proof 3: we have seen in Proof 1 that $\int_{\mathbb{R}} |\mathcal{P}_y(x)|\, dx = \int_{\mathbb{R}} \mathcal{P}_y(x) dx = 1$ for all $y > 0$. It remains to check the last property of good kernels. Given a fixed $\delta > 0$:

$$\int_{|x|>\delta} |\mathcal{P}_y(x)|\, dx = 2\int_{\delta}^{+\infty} \frac{1}{\pi}\frac{y}{x^2+y^2} dx = \frac{2}{\pi}\left[\arctan\left(\frac{x}{y}\right)\right]_{x=\delta}^{x=\infty} = 1 - \frac{2}{\pi}\arctan\left(\frac{\delta}{y}\right)$$

Thus, we get that this quantity goes to 0 as $y \longrightarrow 0$.

Existence of solution

We are going to proceed as in the case of the time-dependent heat equation. First, we suppose we can apply all theorems which we proved for the Fourier Transform. And later, we will formalize it by stating the theorem of existence of solution.

Hence, let us take the Fourier transform of the equation $\triangle u(x, y) = 0$ respect the variable x. Using Proposition 1.1.2 we get:

$$-4\pi\xi^2 \hat{u}(\xi, y) + \frac{\partial^2}{\partial y^2}\hat{u}(\xi, y) = 0$$

with the boundary condition $\hat{u}(\xi, 0) = \hat{g}(\xi)$, i.e. we get an ordinary differential equation in y (with ξ fixed). Its general solution is:

$$\hat{u}(\xi, y) = A(\xi)\, e^{-2\pi|\xi|y} + B(\xi)\, e^{2\pi|\xi|y}$$

If we disregard the second term because of its rapid exponential increase we find, after setting $y = 0$, that:

$$\hat{u}(\xi, y) = \hat{g}(\xi)\, e^{-2\pi|\xi|y}$$

Now using that the Fourier transform of the Poisson Kernel is $e^{-2\pi|\xi|y}$ (Proposition 7.2.2) , Proposition 5.2.1 and the Fourier inversion Theorem:

$$\hat{u}(\xi, y) = \hat{g}(\xi)\, e^{-2\pi|\xi|y} = \hat{g}(\xi)\, \widehat{\mathcal{P}}_y(\xi) \overset{Prop.5.2.1}{=} \widehat{(g * \mathcal{P}_y)}(\xi)$$

$$\overset{Inversion\,Theor.}{\Longrightarrow} u(x, y) = (g * \mathcal{P}_y)(x)$$

Theorem 7.8. (Existence of solution) *Let $g \in \mathcal{S}(\mathbb{R})$ and let $u(x, y) = (g * \mathcal{P}_y)(x)$. Then, the following statements hold:*

1. *$u(x, y)$ is C^∞ in $\mathbb{R}^2_+$ and $\triangle u = 0$.*
2. *$u(x, y) \longrightarrow g(x)$ uniformly as $y \longrightarrow 0$. That is: $\lim\limits_{y \longrightarrow 0} \|u(\cdot, y) - g\|_\infty = 0$. Thus, if $u(x, 0) = g(x)$ then $u \in C^\infty(\mathbb{R}^2_+) \cap \overline{C(\mathbb{R}^2_+)}$.*
3. *$u(x, y) \longrightarrow g(x)$ as $y \longrightarrow 0$ in $L^2(\mathbb{R})$. That is: $\lim\limits_{y \longrightarrow 0} \|u(\cdot, y) - g\|_2 = 0$.*
4. *u vanishes at infinity in the sense that $u(x, y) \longrightarrow 0$ as $|x| + y \longrightarrow +\infty$.*

PROOF.

• Proof 1: applying the Differentiation under the integral sign Theorem, it is not difficult to see that $u(x, y) = (g * \mathcal{P}_y)(x)$ is $C^\infty(\mathbb{R}^2_+)$ and:

$$\frac{\partial^{k+l}}{\partial x^k \partial y^l} u(x, y) = \int_{\mathbb{R}} \frac{\partial^{k+l}}{\partial x^k \partial y^l} \mathcal{P}_y(t)\, g(x - t) dt \qquad \text{for all } k, l \in \mathbb{N}$$

In particular, it holds that $\triangle u(x, t) = \int_{\mathbb{R}} \triangle (\mathcal{P}_y(t)\, g(x - t))\, dt = 0$ by straightforward calculation.

• Proof 2: as $g \in \mathcal{S}(\mathbb{R})$ and $\{\mathcal{P}_y(x)\}_{y>0}$ is a family of good kernels, we can apply Proposition 6.9 and we get that $u(x, y) \longrightarrow g(x)$ uniformly in x as $y \longrightarrow 0$.

• Proof 3: as $\mathcal{P}_y(x) \in L^1(\mathbb{R})$ for all $y > 0$ and $g \in \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, then by Proposition 3.4. we get that $u(\cdot, y) = g * \mathcal{P}_y$ is in $L^2(\mathbb{R})$ for all $y > 0$. Hence, applying Theorem 6.4. (Plancherel in $L^2(\mathbb{R})$) we get that:

$$\|u(\cdot, y) - g\|_2 = \|\hat{u}(\cdot, y) - \hat{g}\|_2$$

The rest of the proof is similar to the proof of Theorem 7.1.5.

• Proof 4: we start showing the result when $|x| \longrightarrow +\infty$. Note that

$$|u(x,y)| \leq \int_{\mathbb{R}} |\mathcal{P}_y(t)\, g(x-t)|\, dt = \int_{|t|\leq \frac{|x|}{2}} |\mathcal{P}_y(t)\, g(x-t)|\, dt + \int_{|t|\geq \frac{|x|}{2}} |\mathcal{P}_y(t)\, g(x-t)|\, dt$$

- We start checking the integral when $|t| \leq \frac{|x|}{2}$. First of all note that it holds

$$|x-t| \geq |x| - |t| \geq \frac{|x|}{2} \implies \left(1 + |x-t|^2\right) |g(x-t)| \geq \left(1 + \frac{|x|^2}{4}\right) |g(x-t)|$$

Using the fact that $g \in \mathcal{S}(\mathbb{R})$, when we take the limit in both sides of the previous inequality we get

$$\lim_{|x|\longrightarrow+\infty} \left(1 + \frac{|x|^2}{4}\right) |g(x-t)| \leq \lim_{|x-t|\longrightarrow+\infty} \left(1 + |x-t|^2\right) |g(x-t)| = 0$$

Then, for $\epsilon = 1$, there exists $M > 0$ such that $|g(x-t)| \leq \frac{4}{4+|x|^2}$ for all $|x| > M$. Using this and that $\|\mathcal{P}_y(\cdot)\|_1 = 1$ for all $y > 0$:

$$\int_{|t|\leq \frac{|x|}{2}} |\mathcal{P}_y(t)\, g(x-t)|\, dt \leq \frac{4}{4+|x|^2} \overset{|x|\longrightarrow+\infty}{\longrightarrow} 0$$

- Now, we consider the integral when $|t| \geq \frac{|x|}{2}$. Note that $\mathcal{P}_y(t) = \frac{1}{\pi}\frac{y}{t^2+y^2} \leq \frac{1}{\pi}\frac{y}{\left(\frac{x}{2}\right)^2+y^2}$. Hence, using this and the fact that $g \in L^1(\mathbb{R})$, we get:

$$\int_{|t|\geq \frac{|x|}{2}} |\mathcal{P}_y(t) g(x-t)|\, dt \leq \frac{1}{\pi}\frac{y}{\left(\frac{x}{2}\right)^2 + y^2} \|g\|_1 \overset{|x|\longrightarrow+\infty}{\longrightarrow} 0$$

- Hence, we conclude that $|u(x,y)| \overset{|x|\longrightarrow+\infty}{\longrightarrow} 0$ for all $y > 0$. On the other hand, if $y \longrightarrow +\infty$:

$$|u(x,y)| \leq \int_{\mathbb{R}} \left|\frac{1}{\pi}\frac{y}{t^2+y^2} g(x-t)\right| dt \leq \frac{1}{\pi y}\|g\|_1 \overset{y\longrightarrow+\infty}{\longrightarrow} 0 \quad \text{for all} \quad x \in \mathbb{R}$$

Uniqueness of solution

We have proved existence of solution of the problem (7.7.1) in the case $g \in \mathcal{S}(\mathbb{R})$. Now, we will prove that in fact this solution is unique. The proof relies on a basic fact about harmonic functions, which are functions satisfying $\triangle u = 0$. The fact is that the value of an harmonic function at a point equals its average value around any circle centered at that point.

Theorem 7.9. (Mean-value formulas for Laplace's equation) *Suppose Ω is an open set in $\mathbb{R}^d$. Let u be a function in $C^2(\Omega)$ such that $\triangle u = 0$. Then, for each ball of radius $R > 0$ centered at $\mathbf{x}_0 \in \mathbb{R}^d$ such that $\overline{B_R(\mathbf{x}_0)} \subset \Omega$ holds:*

$$u(\mathbf{x}_0) = \fint_{B_R(\mathbf{x}_0)} u(\mathbf{x}) d\mathbf{x} = \fint_{\partial B_R(\mathbf{x}_0)} u(\mathbf{x}) dS(\mathbf{x})$$

where we have introduced the notation: $\fint_A f(\mathbf{x}) d\mathbf{x} = \frac{1}{|A|}\int_A f(\mathbf{x}) d\mathbf{x}$.

PROOF.

• We start defining the following function for $r \in (0, R+\epsilon)$:

$$\psi(r) := \fint_{\partial B_r(\mathbf{x}_0)} u(\mathbf{x}) dS(\mathbf{x}) = \frac{1}{|\partial B_r(\mathbf{x}_0)|} \int_{\partial B_r(\mathbf{x}_0)} u(\mathbf{x}) dS(\mathbf{x}) = \begin{bmatrix} \mathbf{x} = \mathbf{x}_0 + r\mathbf{y} \\ dS(\mathbf{x}) = r^{d-1} dS(\mathbf{y}) \end{bmatrix} =$$

$$= \frac{1}{|S^{d-1}|} \int_{S^{d-1}} u(\mathbf{x}_0 + r\mathbf{y}) dS(\mathbf{y})$$

Now, we differentiate $\psi(r)$ and we notice that $\mathbf{y} = n$ (normal vector of S^{d-1}). Hence, using Green's identity (Theorem A.12):

$$\psi'(r) = \frac{1}{|S^{d-1}|} \int_{S^{d-1}} \nabla u(\mathbf{x}_0 + r\mathbf{y}) \cdot \mathbf{y} dS(\mathbf{y}) \overset{Green}{=} \frac{r}{|S^{d-1}|} \int_{B_1(0)} \triangle u(\mathbf{x}_0 + r\mathbf{y}) d\mathbf{y} = 0$$

Hence, as ψ is continuous, we get that is constant in $(0, R+\epsilon)$. Moreover:

$$\lim_{r \longrightarrow 0} \psi(r) = \lim_{r \longrightarrow 0} \fint_{\partial B_r(\mathbf{x}_0)} u(\mathbf{x}) dS(\mathbf{x}) = u(\mathbf{x}_0)$$

Consequently, we conclude that $u(\mathbf{x}_0) = \fint_{\partial B_R(\mathbf{x}_0)} u(\mathbf{x}) dS(\mathbf{x})$.

• We note that it holds:

$$\int_{B_R(\mathbf{x}_0)} u(\mathbf{x}) d\mathbf{x} = \int_0^R \int_{\partial B_r(\mathbf{x}_0)} u(\mathbf{x}) dS(\mathbf{x}) dr = \int_0^R |\partial B_r(\mathbf{x}_0)| \, u(\mathbf{x}_0) dr = |B_R(\mathbf{x}_0)| \, u(\mathbf{x}_0)$$

Theorem 7.10. (Uniqueness of solution) *There exists an unique solution $u(x,y)$ of the problem (7.7.1) with $g \in \mathcal{S}(\mathbb{R})$ such that $u \in C^2(\mathbb{R}^2_+) \cap C\left(\overline{\mathbb{R}^2_+}\right)$ and vanishes at infinity (in the sense of Theorem 7.8.4). Consequently, this unique solution is given by $u(x,y) = (g * \mathcal{P}_y)(x)$.*

PROOF.

Let us suppose there exist two solutions u and v satisfying the conditions of the statement. Let us define $w(x,y) := u(x,y) - v(x,y)$. Then, it holds that $w \in C^2(\mathbb{R}^2_+) \cap C\left(\overline{\mathbb{R}^2_+}\right)$ solves the problem (7.7.1) with $w(x,0) = 0$ for all $x \in \mathbb{R}$ and vanishes at infinity.

Considering separately the real and imaginary parts of w we may suppose that u itself is real-valued. Let us suppose $u \neq v$. Then there exists a point $(x_0, y_0) \in \mathbb{R}^2_+$ such that $w(x_0, y_0) > 0$ (otherwise we consider $w = v - u$). We shall see that this leads to a contradiction.

First, since w vanishes at infinity, we can find a large semi-disc of radius R, $D_R^+ := \{(x,y) : x^2 + y^2 \leq R, \, y \geq 0\}$ outside of which $w(x,y) \leq \frac{1}{2} w(x_0, y_0)$. Note that it implies $(x_0, y_0) \in D_R^+$.

Next, since w is continuous in D_R^+, it attains its maximum M there, so there exists a point $(x_1, y_1) \in D_R^+$ with $w(x_1, y_1) = M$. Note that it holds that $w(x,y) \leq M$ in the semi-disc but also outside since $w(x,y) \leq \frac{1}{2} w(x_0, y_0) \leq \frac{M}{2}$. Hence, $w(x,y) \leq M$ throughout the entire upper half-plane.

Now, the mean-value property for harmonic functions implies:

$$w(x_1, y_1) = \frac{1}{2\pi} \int_{\partial B_\rho(x_1, y_1)} w(x,y) dS$$

whenever the circle of integration lies in the upper half-plane. In particular, this equation holds if $0 < \rho < y_1$. Since $w(x_{1,}y_1)$ equals the maximum value M and $w(x, y) \leq M$, it follows by continuity that $w(x, y) = M$ on $\partial B_\rho(x_1, y_1)$. In effect, otherwise $w(x, y) \leq M - \epsilon$ on an arc of length $\delta > 0$ on the circle, and this would give:

$$M = w(x_1, y_1) = \frac{1}{2\pi} \int_{\partial B_\rho(x_1, y_1)} w(x, y) dS \leq M - \frac{\epsilon\delta}{2\pi} < M$$

which is a contradiction. Now, letting $\rho \longrightarrow y_1$, and using the continuity of w again, we see that this implies $w(x_1, 0) = M > 0$ which contradicts the fact that $w(x, 0) = 0$ for all $x \in \mathbb{R}$.

Hence we conclude that $w(x, t) = u(x, t) - v(x, t) = 0$ on $\mathbb{R}^2_+$ and the solution is unique.

Chapter 8

The Wave Equation

8.1 Derivation of the wave equation

Let us suppose we are given a multidimensional elastic solid modeled by $U \subset \mathbb{R}^d$. Let us define $u(\mathbf{x}, t)$ as the displacement in some direction of the point $\mathbf{x} \in \Omega$ at time $t \in \mathbb{R}$. Let V represent any smooth subregion of U. The acceleration within V is then:

$$\frac{d^2}{dt^2} \int_V u(\mathbf{x}, t) d\mathbf{x} = \int_V u_{tt}(\mathbf{x}, t) d\mathbf{x}$$

The net contact force is $-\int_{\partial V} F(\mathbf{x}, t) \cdot n dS$ where F denotes the force acting on V through ∂V and the mass density is taken to be unity. Newton's law asserts the mass times the acceleration equals to the net force. Hence, we get:

$$\int_V u_{tt}(\mathbf{x}, t) d\mathbf{x} = -\int_{\partial V} F(\mathbf{x}, t) \cdot n dS \overset{Diverg.Th.}{=} -\int_V div(F(\mathbf{x}, t)) d\mathbf{x}$$

This identity obtains for each subregion V and so $u_{tt}(\mathbf{x}, t) = -div(F(\mathbf{x}, t))$ for all $\mathbf{x} \in U$. For elastic bodies, F is a function of the displacement gradient $\nabla u(\mathbf{x}, t)$. Hence:

$$u_{tt}(\mathbf{x}, t) = -div\left(F\left(\nabla u(\mathbf{x}, t)\right)\right) \quad \text{for all } (\mathbf{x}, t) \in U \times \mathbb{R}$$

For small ∇u, the linearization $F(\nabla u) \approx -a \nabla u$ is often appropriate and so we finally get the wave equation:

$$u_{tt}(\mathbf{x}, t) + div(-a\nabla u(\mathbf{x}, t)) = 0 \implies u_{tt}(\mathbf{x}, t) - a\triangle u(\mathbf{x}, t) = 0 \quad \text{for all} \quad (\mathbf{x}, t) \in U \times \mathbb{R}$$

Note that, rescaling the variables of the equation in the same way we did for the Heat equation, we can suppose that $a = 1$. Moreover, this physical interpretation strongly suggests it will be mathematically appropriate to specify two initial conditions, on the displacement u and the velocity u_t at $t = 0$. We are interested in solving the wave equation in $U = \mathbb{R}^d$. Hence, we will consider the homogeneous initial-value problem:

$$\begin{cases} u_{tt}(\mathbf{x}, t) = \triangle u(\mathbf{x}, t) & (\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R} \\ u(\mathbf{x}, 0) = g(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \\ u_t(\mathbf{x}, 0) = h(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{8.1.1}$$

8.2 Solution in terms of Fourier Transforms

First of all, let us suppose that the solution u and the functions g and h satisfy the hypothesis of the theorems proved in the previous chapters and that we can apply it freely like we did for the the heat equation. Later we will formalize it.

Let us take the Fourier transform respect $\mathbf{x}$ in both sides of the wave equation. On the left hand, we get:

$$\widehat{[u_{tt}]}(\boldsymbol{\xi},t) = \int_{\mathbb{R}^d} e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{x}} u_{tt}(\mathbf{x},t)d\mathbf{x} = \int_{\mathbb{R}^d} \frac{\partial^2}{\partial t^2}\left(e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}u(\mathbf{x},t)\right)d\mathbf{x} \overset{Theor\ A.5.}{=}$$
$$= \frac{\partial^2}{\partial t^2}\left(\int_{\mathbb{R}^d} e^{-2\pi i\boldsymbol{\xi}\cdot\mathbf{x}}u(\mathbf{x},t)d\mathbf{x}\right) = \hat{u}_{tt}(\boldsymbol{\xi},t)$$

On the right hand side, we apply Proposition 6.5.4. with $\alpha_i = (0,..,0,2,0,..,0)$.

$$\widehat{[\triangle u]}(\boldsymbol{\xi},t) = \left[\widehat{\sum_{i=1}^{n}\left(\frac{\partial}{\partial\mathbf{x}}\right)^{\alpha_i} u}\right](\boldsymbol{\xi},t) = \sum_{i=1}^{n}(2\pi i\boldsymbol{\xi})^{\alpha_i}\,\hat{u}(\boldsymbol{\xi},t) = -4\pi^2\sum_{i=1}^{n}\xi_i^2\hat{u}(\boldsymbol{\xi},t) = -4\pi^2|\boldsymbol{\xi}|^2\,\hat{u}(\boldsymbol{\xi},t)$$

Hence, fixing $\boldsymbol{\xi}$, we get an ordinary differential equation of the Fourier transform of u with respect the temporal variable $\hat{u}_{tt}(\boldsymbol{\xi},t) = -4\pi^2|\boldsymbol{\xi}|^2\,\hat{u}(\boldsymbol{\xi},t)$ whose solution is given by

$$\hat{u}(\boldsymbol{\xi},t) = A(\boldsymbol{\xi})\cos(2\pi|\boldsymbol{\xi}|t) + B(\boldsymbol{\xi})\sin(2\pi|\boldsymbol{\xi}|t)$$

where for each $\boldsymbol{\xi}$, $A(\boldsymbol{\xi})$ and $B(\boldsymbol{\xi})$ are unknown constants to be determined by the initial conditions. In fact, taking the Fourier transform (in $\mathbf{x}$) of the initial conditions yields

$$\hat{u}(\boldsymbol{\xi},0) = \hat{g}(\boldsymbol{\xi}) \qquad \text{and} \qquad \hat{u}_t(\boldsymbol{\xi},0) = \hat{h}(\boldsymbol{\xi})$$

We may now solve for $A(\boldsymbol{\xi})$ and $B(\boldsymbol{\xi})$ to obtain $A(\boldsymbol{\xi}) = \hat{f}(\boldsymbol{\xi})$ and $2\pi|\boldsymbol{\xi}|B(\boldsymbol{\xi}) = \hat{g}(\boldsymbol{\xi})$. Therefore we find that:

$$\hat{u}(\boldsymbol{\xi},t) = \hat{g}(\boldsymbol{\xi})\cos(2\pi|\boldsymbol{\xi}|t) + \hat{h}(\boldsymbol{\xi})\frac{\sin(2\pi|\boldsymbol{\xi}|t)}{2\pi|\boldsymbol{\xi}|}$$

and the solution of u is given by taking the inverse Fourier transform in the $\boldsymbol{\xi}$ variables. This formal derivation then leads to a precise existence theorem for our problem.

Theorem 8.1. *Let g and h be in $\mathcal{S}\left(\mathbb{R}^d\right)$. Then, a classical solution for the problem (8.1.1) is given by*

$$u(\mathbf{x},t) = \int_{\mathbb{R}^d}\left[\hat{g}(\boldsymbol{\xi})\cos(2\pi|\boldsymbol{\xi}|t) + \hat{h}(\boldsymbol{\xi})\frac{\sin(2\pi|\boldsymbol{\xi}|t)}{2\pi|\boldsymbol{\xi}|}\right]e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi} \tag{8.2.1}$$

Moreover, $u \in C^\infty\left(\mathbb{R}^d\times\mathbb{R}\right)$ and $u(\cdot,t)\in\mathcal{S}\left(\mathbb{R}^d\right)$ for all $t\in\mathbb{R}$.

PROOF.

• This proof is straightforward once we note that we can differentiate in $\mathbf{x}$ and t under the integral sign (because g and h are both Schwartz functions, we can apply similar arguments as the given for the heat equation). Since we conclude that u is $C^\infty\left(\mathbb{R}^d\times\mathbb{R}\right)$. Now, on the one hand we differentiate the exponential with respect to the $\mathbf{x}$ variables to get

$$\triangle u(\mathbf{x},t) = \int_{\mathbb{R}^d}\left[\hat{g}(\boldsymbol{\xi})\cos(2\pi|\boldsymbol{\xi}|t) + \hat{h}(\boldsymbol{\xi})\frac{\sin(2\pi|\boldsymbol{\xi}|t)}{2\pi|\boldsymbol{\xi}|}\right]\left(-4\pi^2|\boldsymbol{\xi}|^2\right)e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}$$

while on the other hand we differentiate the terms in brackets with respect to t twice to get:

$$\frac{\partial^2}{\partial t^2}u(\mathbf{x},t) \;=\; \int_{\mathbb{R}^d}\left[-4\pi^2\,|\boldsymbol{\xi}|^2\,\hat{g}\,(\boldsymbol{\xi})\cos\left(2\pi\,|\boldsymbol{\xi}|\,t\right)\;-\;4\pi^2\,|\boldsymbol{\xi}|^2\,\hat{h}\,(\boldsymbol{\xi})\,\frac{\sin\left(2\pi\,|\boldsymbol{\xi}|\,t\right)}{2\pi\,|\boldsymbol{\xi}|}\right]e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi}$$

This shows that u solves the wave equation. Now, setting $t = 0$, we get by the Inversion Theorem (Theorem 6.2.)

$$u(\mathbf{x},0) \;=\; \int_{\mathbb{R}^d}\hat{g}\,(\boldsymbol{\xi})\,e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi} \;=\; g(\mathbf{x})$$

Finally differentiating once with respect to t, setting $t = 0$, and using the Fourier inversion shows that

$$\frac{\partial}{\partial t}u(\mathbf{x},0) \;=\; \int_{\mathbb{R}^d}\hat{h}\,(\boldsymbol{\xi})\,e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi} \;=\; h(\mathbf{x})$$

Thus u also verifies the initial conditions.

- Finally, we shall prove that $u(\cdot,t)\in\mathcal{S}\left(\mathbb{R}^d\right)$ for all $t\in\mathbb{R}$. Note that

$$u(\mathbf{x},t) \;=\; \int_{\mathbb{R}^d}\left[\hat{g}\,(\boldsymbol{\xi})\cos\left(2\pi\,|\boldsymbol{\xi}|\,t\right)\;+\;\hat{h}\,(\boldsymbol{\xi})\,\frac{\sin\left(2\pi\,|\boldsymbol{\xi}|\,t\right)}{2\pi\,|\boldsymbol{\xi}|}\right]e^{2\pi i\mathbf{x}\cdot\boldsymbol{\xi}}d\boldsymbol{\xi} \;=\; \left[\begin{array}{c}\boldsymbol{\xi}\;=\;h(\boldsymbol{\beta})\;=\;-\boldsymbol{\beta}\\ (Dh)\,(\boldsymbol{\beta})\;=\;-Id\implies|\det\left(Dh\right)(\boldsymbol{\beta})|\;=1\end{array}\right]$$

$$=\; \int_{\mathbb{R}^d}\left[\hat{g}\,(-\boldsymbol{\beta})\cos\left(2\pi\,|\boldsymbol{\beta}|\,t\right)\;+\;\hat{h}\,(-\boldsymbol{\beta})\,\frac{\sin\left(2\pi\,|\boldsymbol{\beta}|\,t\right)}{2\pi\,|\boldsymbol{\beta}|}\right]e^{-2\pi i\mathbf{x}\cdot\boldsymbol{\beta}}d\boldsymbol{\beta}$$

As $\cos\left(2\pi\,|\boldsymbol{\beta}|\,t\right)$ and $\frac{\sin(2\pi|\boldsymbol{\beta}|t)}{2\pi|\boldsymbol{\beta}|t}$ are infinitely times differentiable, then $\hat{g}\,(-\boldsymbol{\beta})\cos\left(2\pi\,|\boldsymbol{\beta}|\,t\right)$ and $\hat{h}\,(-\boldsymbol{\beta})\,\frac{\sin(2\pi|\boldsymbol{\beta}|t)}{2\pi|\boldsymbol{\beta}|}$ are in $\mathcal{S}\left(\mathbb{R}^d\right)$ for all $t\in\mathbb{R}$. Thus, $u\,(\cdot,t)$ can be seen as the Fourier Transform of a function in $\mathcal{S}\left(\mathbb{R}^d\right)$ for all $t\in\mathbb{R}$. Then, by Theorem 6.1., we conclude that $u(\cdot,t)\in\mathcal{S}\left(\mathbb{R}^d\right)$ for all $t\in\mathbb{R}$.

Lemma 8.1. *Let a and b be complex numbers and α be real. Then:*

$$|a\cos\left(\alpha\right)+b\sin\left(\alpha\right)|^2\;+\;|-a\sin\left(\alpha\right)+b\cos\left(\alpha\right)|^2\;=\;|a|^2\;+\;|b|^2$$

PROOF.

Let us define $e_1 := (\cos(\alpha),\sin(\alpha))$, $e_2 := \left(-\sin\left(\alpha\right),\cos\left(\alpha\right)\right)$ and $Z := (a,b)$. As $\{e_1,e_2\}$ form an orthonormal basis, we can write $Z = Z_{e_1} + Z_{e_2}$ where $Z_{e_1} := \langle Z,e_1\rangle\,e_1$ and $Z_{e_2} := \langle Z,e_2\rangle\,e_2$ respectively. Thus, by Pythagorean theorem it holds:

$$|Z|^2\;=\;|Z_{e_1}|^2\;+\;|Z_{e_2}|^2\;=\;|\langle Z,e_1\rangle|^2\;+\;|\langle Z,e_2\rangle|^2\;\implies\;|a|^2\;+\;|b|^2\;=$$

$$=\;|a\cos\left(\alpha\right)+b\sin\left(\alpha\right)|^2\;+\;|-a\sin\left(\alpha\right)+b\cos\left(\alpha\right)|^2$$

Theorem 8.2. *Let us define the energy of a solution of (8.1.1) as:*

$$E(t)\;=\;\int_{\mathbb{R}^d}\left(\left|\frac{\partial u}{\partial t}\,(\mathbf{x},t)\right|^2\;+\;\left|\frac{\partial u}{\partial x_1}\,(\mathbf{x},t)\right|^2\;+\;...\;+\;\left|\frac{\partial u}{\partial x_d}\,(\mathbf{x},t)\right|^2\right)d\mathbf{x} \tag{8.2.2}$$

If u is the solution given in Theorem 8.1. by (8.2.1), then $E(t)$ is conserved in time, that is, $E(t)\;=\;E(0)$ for all $t\in\mathbb{R}$.

PROOF.
Note first that $\hat{u}(\boldsymbol{\xi},t) = \hat{g}(\boldsymbol{\xi})\, cos\,(2\pi\,|\boldsymbol{\xi}|\,t) + \hat{h}(\boldsymbol{\xi})\,\frac{sin(2\pi|\boldsymbol{\xi}|t)}{2\pi|\boldsymbol{\xi}|}$. As $u(\cdot,t) \in \mathcal{S}\left(\mathbb{R}^d\right)$ for all $t \in \mathbb{R}$, then $\frac{\partial}{\partial x_j}u(\cdot,t)$ and $\frac{\partial}{\partial t}u(\cdot,t) = \triangle u(\cdot,t)$ are in $\mathcal{S}\left(\mathbb{R}^d\right)$ for all $t \in \mathbb{R}$. Hence, we can apply Plancherel Theorem in $\mathcal{S}\left(\mathbb{R}^d\right)$ (Theorem 6.3.)

$$\int_{\mathbb{R}^d}\left|\frac{\partial}{\partial t}u(\mathbf{x},t)\right|^2 d\mathbf{x} = \int_{\mathbb{R}^d}\left|\frac{\partial}{\partial t}\hat{u}(\boldsymbol{\xi},t)\right|^2 d\boldsymbol{\xi} = \int_{\mathbb{R}^d}\left|-2\pi\,|\boldsymbol{\xi}|\,\hat{g}(\boldsymbol{\xi})\sin(2\pi\,|\boldsymbol{\xi}|\,t) + \hat{h}(\boldsymbol{\xi})\cos(2\pi\,|\boldsymbol{\xi}|\,t)\right|^2 d\boldsymbol{\xi}$$

$$\int_{\mathbb{R}^d}\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}u(\mathbf{x},t)\right|^2 d\mathbf{x} = \int_{\mathbb{R}^d}\sum_{j=1}^{d}\left|\widehat{\left[\frac{\partial}{\partial x_j}u\right]}(\boldsymbol{\xi},t)\right|^2 d\boldsymbol{\xi} = \int_{\mathbb{R}^d}\left|2\pi\,|\boldsymbol{\xi}|\,\hat{g}(\boldsymbol{\xi})\cos(2\pi\,|\boldsymbol{\xi}|\,t) + \hat{h}(\boldsymbol{\xi})\sin(2\pi\,|\boldsymbol{\xi}|\,t)\right|^2 d\boldsymbol{\xi}$$

We apply Lemma 8.1. with $a = 2\pi\,|\boldsymbol{\xi}|\,\hat{g}(\boldsymbol{\xi})$, $b = \hat{h}(\boldsymbol{\xi})$ and $\alpha = 2\pi\,|\boldsymbol{\xi}|\,t$, and we get:

$$E(t) = \int_{\mathbb{R}^d}\left|\frac{\partial}{\partial t}u(\mathbf{x},t)\right|^2 d\mathbf{x} + \int_{\mathbb{R}^d}\sum_{j=1}^{d}\left|\frac{\partial}{\partial x_j}u(\mathbf{x},t)\right|^2 d\mathbf{x} = \int_{\mathbb{R}^d}\left(4\pi^2\,|\boldsymbol{\xi}|^2\,|\hat{g}(\boldsymbol{\xi})|^2 + \left|\hat{h}(\boldsymbol{\xi})\right|^2\right) d\boldsymbol{\xi}$$

We note that $E(t)$ is independent of the variable t. Hence, we conclude that the energy remains constant in time and in particular, $E(t) = E(0)$ for all $t \in \mathbb{R}$.

8.3 Solution by spherical means

The drawback with the formula (8.2.1), which does give the solution of the wave equation, is that it is quite indirect, involving the calculation of the Fourier transforms of g and h, and then the inverse Fourier transform. However, for every dimension d there is a more explicit formula. Generally, the formula is "elementary" whenever d is odd, and more complicated when d is even.

The overall plan will be to study first the average of u over certain spheres in $\mathbb{R}^d$. These averages turn out to solve the Euler-Poisson-Darboux equation. Later we will transform this equation into the usual one-dimensional wave equation, which has a simple solution. Finally, we will be able to find the solution u.

Solution for $d = 1$, d'Alembert's formula

Lemma 8.2. *Let us consider the nonhomogeneous initial-value problem of the transport equation in $\mathbb{R}^d$ given by*

$$\begin{cases} u_t(\mathbf{x},t) + b\cdot\nabla u(\mathbf{x},t) = f(\mathbf{x},t) & \text{in } \mathbb{R}^d\times(0,+\infty) \\ u(\mathbf{x},0) = g(\mathbf{x}) & \text{on } \mathbb{R}^d\times\{t=0\} \end{cases}$$

where $b \in \mathbb{R}^d$, $g \in C^1(\mathbb{R}^d)$ and $f \in C^1(\mathbb{R}^d\times[0,+\infty))$. Then the solution is given by

$$u(\mathbf{x},t) = g(\mathbf{x}-tb) + \int_0^t f(\mathbf{x}+(s-t)b,s)ds$$

for all $\mathbf{x} \in \mathbb{R}^d$ and $t \geq 0$ and is in $C^1\left(\mathbb{R}^d\times[0,+\infty)\right)$.

PROOF.
Let us parameterize the line through $(\mathbf{x}, t) \in \mathbb{R}^d \times (0, +\infty)$ with direction $(b, 1)$ by $\Gamma := \{(\mathbf{x}+sb, t+s)$ s.t. $s > -t\}$. And let us define $z(s) := u(\mathbf{x} + sb, t + s)$. Hence:

$$\dot{z}(s) = \nabla u(\mathbf{x} + sb, t + s) \cdot b + u_t(\mathbf{x} + sb, t + s) = f(\mathbf{x} + sb, t + s)$$

Consequently, integrating between 0 and $-t$ in both sides of the equality

$$z(0) - z(-t) = \int_{-t}^{0} f(\mathbf{x} + sb, t + s)ds \implies u(\mathbf{x}, t) = g(\mathbf{x} - tb) + \int_0^t f(\mathbf{x} + (s - t)b, s)ds$$

Moreover $u \in C^1\left(\mathbb{R}^d \times [0, +\infty)\right)$ since $g \in C^1(\mathbb{R}^d)$ and $f \in C^1(\mathbb{R}^d \times [0, +\infty))$.

D'Alembert's formula Now, we focus our attention on the initial-value problem for one-dimensional wave equation in $\mathbb{R}$:

$$\begin{cases} u_{tt}(x,t) = \triangle u(x,t) & (x,t) \in \mathbb{R} \times (0, +\infty) \\ u(x, 0) = g(x) & x \in \mathbb{R} \\ u_t(x, 0) = h(x) & x \in \mathbb{R} \end{cases} \tag{8.3.1}$$

We will derive a formula for u in terms of g and h. First, we will do it assuming g and h are smooth enough and later we will formalize the result. We start noticing that the wave equation can be "factored" as

$$\left(\frac{\partial}{\partial t} + \frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x,t) = u_{tt}(x,t) + u_{xx}(x,t) = 0$$

Defining $v(x, t) := \left(\frac{\partial}{\partial t} - \frac{\partial}{\partial x}\right) u(x, t)$ we get that $v_t(x, t) + v_x(x, t) = 0$, i.e. v satisfies the homogeneous one-dimensional transport equation with $b = 1$. Hence, for an arbitrary initial condition $a(x)$ and using Lemma 8.2. we get that $v(x, t) = a(x - t)$.

But, by the definition of v, we get $u_t(x, t) - u_x(x, t) = a(x - t)$, i.e. u satisfies the inhomogeneous one-dimensional transport equation with $b = -1$. Hence, for an arbitrary initial condition $b(x)$ and again by Lemma 8.2. we get that

$$u(x,t) = b(x+t) + \int_0^t a(x + (t-s) - s)ds = b(x+t) + \frac{1}{2}\int_{x-t}^{x+t} a(y)dy$$

Now, we will use the initial conditions to compute a and b. The first initial condition gives that $g(x) = u(x, 0) = b(x)$. The second initial condition gives

$$u_t(x,t) = b^{'}(x+t) + \frac{1}{2}a(x+t) + \frac{1}{2}a(x-t) \implies h(x) = u_t(x,0) = g^{'}(x) + a(x)$$

$$\implies a(x) = h(x) - g^{'}(x)$$

Hence, using the the expressions of a and b we get the d'Alembert's formula:

$$u(x,t) = g(x+t) + \frac{1}{2}\int_{x-t}^{x+t}\left(h(y) - g^{'}(y)\right)dy = \frac{1}{2}\left(g(x+t) + g(x-t)\right) + \frac{1}{2}\int_{x-t}^{x+t} h(y)dy \tag{8.3.2}$$

Theorem 8.3. (Solution of wave equation $d = 1$) *Assume $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$. Let us define u given by d'Alembert's formula (8.3.2). Then, u solves the initial-value problem for the one-dimensional wave equation (8.3.1) and is in $C^2(\mathbb{R} \times [0, +\infty))$.*

PROOF. The proof is a straightforward calculation similar to the proofs we did for the heat equation.

Remark 8.1. *In general, it holds that if* $g \in C^k(\mathbb{R})$ *and* $h \in C^{k-1}(\mathbb{R})$*, then* $u \in C^k(\mathbb{R} \times [0, +\infty))$*, but is not in general smoother. Thus, the wave equation does not cause instantaneous smoothing of the initial data, as does the heat equation.*

Spherical means

Let us take $d \geq 2$ and $m \geq 2$. Let us suppose $u \in C^m\left(\mathbb{R}^d \times (0, +\infty)\right)$ solves the initial-value problem of the wave equation:

$$\begin{cases} u_{tt}(\mathbf{x}, t) - \triangle u(\mathbf{x}, t) = 0 & \text{in } \mathbb{R}^d \times (0, +\infty) \\ u(\mathbf{x}, 0) = g(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \\ u_t(\mathbf{x}, 0) = h(\mathbf{x}) & \mathbf{x} \in \mathbb{R}^d \end{cases} \tag{8.3.3}$$

with $g \in C^m(\mathbb{R}^d)$ and $h \in C^{m-1}(\mathbb{R}^d)$. We intend to derive an explicit formula of u in terms of g and h as we did for the one-dimensional case. We define for $\mathbf{x} \in \mathbb{R}^d$ fixed, for all $t > 0$ and $r > 0$:

$$U(\mathbf{x}; r, t) := \fint_{\partial B(\mathbf{x}, r)} u(\mathbf{y}, t) dS(\mathbf{y}), \quad G(\mathbf{x}; r) := \fint_{\partial B(\mathbf{x}, r)} g(\mathbf{y}) dS(\mathbf{y}) \quad \text{and} \quad H(\mathbf{x}; r) := \fint_{\partial B(\mathbf{x}, r)} h(\mathbf{y}) dS(\mathbf{y})$$

Lemma 8.3. (Euler-Poisson-Darboux equation) *Fix* $\mathbf{x} \in \mathbb{R}^d$*. Then* $U \in C^m\left(\overline{\mathbb{R}}_+ \times [0, +\infty)\right)$ *and*

$$\begin{cases} U_{tt}(\mathbf{x}; r, t) - U_{rr}(\mathbf{x}; r, t) - \frac{d-1}{r} U_r(\mathbf{x}; r, t) = 0 & (r, t) \in \mathbb{R}_+ \times (0, +\infty) \\ U(\mathbf{x}; r, 0) = G(\mathbf{x}; r) & r \in \mathbb{R}_+ \\ U_t(\mathbf{x}; r, 0) = H(\mathbf{x}; r) & r \in \mathbb{R}_+ \end{cases} \tag{8.3.4}$$

PROOF.

• As $u(\mathbf{x}, t) \in C^m\left(\mathbb{R}^d \times [0, +\infty)\right)$, it is clear from Theorem A.5. that $U(\mathbf{x}; r, t)$ is m times continuously differentiable respect the variable t. Moreover, as $\lim_{r \longrightarrow 0+} U(\mathbf{x}; r, t) = u(\mathbf{x}, t)$, we see that $U(\mathbf{x}; r, t) \in C\left(\overline{\mathbb{R}}_+ \times [0, +\infty)\right)$. Now, note that doing exactly the same argument as we did in the proof of Theorem 7.9. we get:

$$U_r(\mathbf{x}; r, t) = \frac{r}{d\alpha(d)} \int_{B(0,1)} \triangle u(\mathbf{x} + r\mathbf{z}, t) d\mathbf{z} \tag{8.3.5}$$

and we see that $\lim_{r \longrightarrow 0+} U_r(\mathbf{x}; r, t) = 0$. Consequently, we conclude that $U_r(\mathbf{x}; r, t) \in C\left(\overline{\mathbb{R}}_+ \times [0, +\infty)\right)$. We next differentiate U_r and we get

$$U_{rr}(\mathbf{x}; r, t) = \frac{1}{d\alpha(d)} \int_{B(0,1)} \triangle u(\mathbf{x} + r\mathbf{z}, t) d\mathbf{z} + \frac{r}{d\alpha(d)} \int_{B(0,1)} \nabla(\triangle u(\mathbf{x} + r\mathbf{z}, t)) \cdot \mathbf{z} d\mathbf{z} \tag{8.3.6}$$

and we see that $\lim_{r \longrightarrow 0+} U_{rr}(\mathbf{x}; r, t) = \frac{1}{d\alpha(d)} \int_{B(0,1)} \triangle u(\mathbf{x}, t) d\mathbf{z} = \frac{\triangle u(\mathbf{x}, t)}{d}$. Thus, $U_{rr} \in C\left(\overline{\mathbb{R}}_+ \times [0, +\infty)\right)$. Similarly, we can compute U_{rrr}, etc., and we verify that U is m times continuously differentiable respect r. Thus, we get that $U(\mathbf{x}; r, t) \in C^m\left(\overline{\mathbb{R}}_+ \times [0, +\infty)\right)$.

• Now let us work the expression (8.3.6). We define the function $v(\mathbf{z},t) = \frac{1}{2}\left(z_1^2 + z_2^2 + ... + z_d^2\right)$. Note that $\nabla v(\mathbf{z},t) = \mathbf{z}$ and $\triangle v(\mathbf{z},t) = d$. Hence, using Green's identity (Theorem A.12):

$$\int_{B(0,1)} r\nabla(\triangle u(\mathbf{x}+r\mathbf{z},t))\cdot\mathbf{z}d\mathbf{z} = \int_{B(0,1)} r\nabla(\triangle u(\mathbf{x}+r\mathbf{z},t))\cdot\nabla v(\mathbf{z},t)d\mathbf{z} \overset{Green\,Id.}{=} -\int_{B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z},t)\triangle v(\mathbf{z},t)d\mathbf{z}+$$

$$+\int_{\partial B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z},t)\left(\nabla v(\mathbf{z},t)\cdot\mathbf{z}\right)dS(\mathbf{z}) = -d\int_{B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z},t)d\mathbf{z} + \int_{\partial B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z},t)dS(\mathbf{z}) \quad (8.3.7)$$

Hence, substituting (8.3.7) in (8.3.6) we get

$$U_{rr}(\mathbf{x};r,t) = \left(\frac{1}{d}-1\right)\frac{1}{\alpha(d)}\int_{B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z},t)d\mathbf{z} + \frac{1}{d\alpha(d)}\int_{\partial B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z},t)dS(\mathbf{z}) \quad (8.3.8)$$

Finally, using the expressions of U_r and U_{rr} given by (8.3.5) and (8.3.8) we conclude that U satisfies the following equation:

$$U_{rr}(\mathbf{x};r,t) + \frac{d-1}{r}U_r(\mathbf{x};r,t) = \frac{1}{d\alpha(d)}\int_{\partial B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z},t)dS(\mathbf{z}) =$$

$$= \frac{1}{d\alpha(d)}\int_{\partial B(0,1)} u_{tt}(\mathbf{x}+r\mathbf{z},t)dS(\mathbf{z}) = U_{tt}(\mathbf{x};r,t)$$

• It remains to show that U satisfies the initial conditions of the problem (8.3.4). But using that u satisfies the initial conditions of the problem (8.3.3), we immediately get

$$U(\mathbf{x};r,0) = ⨍_{\partial B(\mathbf{x},r)} u(\mathbf{y},0)dS(\mathbf{y}) = ⨍_{\partial B(\mathbf{x},r)} g(\mathbf{y})dS(\mathbf{y}) = G(\mathbf{x};r)$$

$$U_t(\mathbf{x};r,0) = ⨍_{\partial B(\mathbf{x},r)} u_t(\mathbf{y},0)dS(\mathbf{y}) = ⨍_{\partial B(\mathbf{x},r)} h(\mathbf{y})dS(\mathbf{y}) = H(\mathbf{x};r)$$

8.4 Solution for $d = 3$: the Kirchhoff's Formula

Let us take $d = 3$ and suppose $u \in C^2\left(\mathbb{R}^3 \times [0,+\infty)\right)$ solves the problem (8.3.3) with $g \in C^2\left(\mathbb{R}^2\right)$ and $h \in C^1(\mathbb{R})$. We recall definitions U, G, H and then we set $\widetilde{U}(\mathbf{x};r,t) := rU(\mathbf{x};r,t)$, $\widetilde{G}(\mathbf{x};r,t) := rG(\mathbf{x};r,t)$ and $\widetilde{H}(\mathbf{x};r,t) = rH(\mathbf{x};r,t)$.

Lemma 8.4. *Fix $\mathbf{x} \in \mathbb{R}^3$. Then, function $\widetilde{U}$ solves the one-dimensional wave equation on the half-line with Dirichlet boundary condition*

$$\begin{cases} \widetilde{U}_{tt}(\mathbf{x};r,t) - \widetilde{U}_{rr}(\mathbf{x};r,t) = 0 & (r,t) \in \mathbb{R}_+ \times (0,+\infty) \\ \widetilde{U}(\mathbf{x};r,0) = \widetilde{G}(\mathbf{x};r) & r \in \mathbb{R}_+ \\ \widetilde{U}_t(\mathbf{x};r,0) = \widetilde{H}(\mathbf{x};r) & r \in \mathbb{R}_+ \\ \widetilde{U}(\mathbf{x};0,t) = 0 & t \in (0,+\infty) \end{cases} \quad (8.4.1)$$

Moreover,

$$\widetilde{U}(\mathbf{x};r,t) = \begin{cases} \frac{1}{2}\left(\widetilde{G}(\mathbf{x};r+t) + \widetilde{G}(\mathbf{x};r-t)\right) + \frac{1}{2}\int_{r-t}^{r+t} \widetilde{H}(\mathbf{x};y)dy & \text{if } r \geq t \geq 0 \\ \frac{1}{2}\left(\widetilde{G}(\mathbf{x};r+t) - \widetilde{G}(\mathbf{x};t-r)\right) + \frac{1}{2}\int_{-r+t}^{r+t} \widetilde{H}(\mathbf{x};y)dy & \text{if } 0 \leq r \leq t \end{cases} \quad (8.4.2)$$

PROOF.

• Note that using Lemma 8.3. it holds: $\widetilde{U}_{rr} = (rU)_{rr} = 2U_r + rU_{rr} = rU_{tt} = \widetilde{U}_{tt}$ in $\mathbb{R}_+ \times (0, +\infty)$. Moreover $\widetilde{U}(\mathbf{x}; r, 0) = rU(\mathbf{x}; r, 0) = rG(\mathbf{x}; r) = \widetilde{G}(\mathbf{x}; r)$ and similarly $\widetilde{U}_t(\mathbf{x}; r, 0) = \widetilde{H}(\mathbf{x}; r)$. Finally, $\widetilde{U}(\mathbf{x}; 0, t) = 0U(\mathbf{x}; 0, t) = 0$.

• To derive the solution of the equation (8.4.1), we will use a reflection method. Let us define the odd reflections of $\widetilde{U}$, $\widetilde{G}$ and $\widetilde{H}$

$$\widetilde{U}_{ref}(\mathbf{x}; r, t) = \begin{cases} \widetilde{U}(\mathbf{x}; r, t) & \text{if } r \geq 0\,, t \geq 0 \\ -\widetilde{U}(\mathbf{x}; -r, t) & \text{if } r \leq 0\,, t \geq 0 \end{cases}, \quad \widetilde{G}_{ref}(\mathbf{x}; r) = \begin{cases} \widetilde{G}(\mathbf{x}; r) & \text{if } r \geq 0 \\ -\widetilde{G}(\mathbf{x}; -r) & \text{if } r \leq 0 \end{cases}$$

$$\text{and} \quad \widetilde{H}_{ref}(\mathbf{x}; r) = \begin{cases} \widetilde{H}(\mathbf{x}; r) & \text{if } r \geq 0 \\ -\widetilde{H}(\mathbf{x}; -r) & \text{if } r \leq 0 \end{cases}$$

Then, problem (8.4.1) becomes

$$\begin{cases} \left(\widetilde{U}_{ref}\right)_{tt}(\mathbf{x}; r, t) - \left(\widetilde{U}_{ref}\right)_{rr}(\mathbf{x}; r, t) = 0 & (r, t) \in \mathbb{R} \times (0, +\infty) \\ \widetilde{U}_{ref}(\mathbf{x}; r, 0) = \widetilde{G}_{ref}(\mathbf{x}; r) & r \in \mathbb{R} \\ \left(\widetilde{U}_{ref}\right)_t(\mathbf{x}; r, 0) = \widetilde{H}_{ref}(\mathbf{x}; r) & r \in \mathbb{R} \end{cases}$$

Hence, we can apply d'Alembert's formula (8.3.2) and we get

$$\widetilde{U}_{ref}(\mathbf{x}; r, t) = \frac{1}{2}\left(\widetilde{G}_{ref}(\mathbf{x}; r + t) + \widetilde{G}_{ref}(\mathbf{x}; r - t)\right) + \frac{1}{2}\int_{r-t}^{r+t} \widetilde{H}_{ref}(\mathbf{x}; y)dy$$

Finally, recalling the definitions of $\widetilde{U}_{ref}$, $\widetilde{G}_{ref}$ and $\widetilde{H}_{ref}$, we can transform this expression to read for $r \geq 0$ and $t \geq 0$ and we get the expression (8.4.2).

Kirchhoff's formula Now, note that $u(\mathbf{x}, t) = \lim_{r \longrightarrow 0^+} U(\mathbf{x}; r, t)$. Hence, using Lemma 8.4.

$$u(\mathbf{x}, t) = \lim_{r \longrightarrow 0^+} \frac{\widetilde{U}(\mathbf{x}; r, t)}{r} = \lim_{r \longrightarrow 0^+} \left[\frac{1}{2r}\left(\widetilde{G}(\mathbf{x}; r + t) - \widetilde{G}(\mathbf{x}; t - r)\right) + \frac{1}{2r}\int_{-r+t}^{r+t} \widetilde{H}(\mathbf{x}; y)dy\right] =$$

$$= \widetilde{G}'(\mathbf{x}; t) + \widetilde{H}(\mathbf{x}; t)$$

Hence, recalling the definitions of $\widetilde{G}$ and $\widetilde{H}$ we deduce

$$u(\mathbf{x}, t) = \frac{\partial}{\partial t}\left(t \fint_{\partial B(\mathbf{x}, t)} g(\mathbf{y})dS(\mathbf{y})\right) + t \fint_{\partial B(\mathbf{x}, t)} h(\mathbf{y})dS(\mathbf{y}) \tag{8.4.3}$$

But notice that making a change of variables we get that

$$\fint_{\partial B(\mathbf{x}, t)} g(\mathbf{y})dS(\mathbf{y}) = \fint_{\partial B(0,1)} g(\mathbf{x} + t\mathbf{z})dS(\mathbf{z}) \implies \frac{\partial}{\partial t}\left(\fint_{\partial B(\mathbf{x}, t)} g(\mathbf{y})dS(\mathbf{y})\right) =$$

$$= \fint_{\partial B(0,1)} \nabla g(\mathbf{x} + t\mathbf{z}) \cdot \mathbf{z} dS(\mathbf{z}) = \fint_{\partial B(\mathbf{x}, t)} \nabla g(\mathbf{y}) \cdot \left(\frac{\mathbf{y} - \mathbf{x}}{t}\right) dS(\mathbf{y})$$

Returning to (8.4.3), the Kirchhoff's formula for the solution of the initial-value problem (8.3.3) in three dimensions is

$$u(\mathbf{x}, t) = \fint_{\partial B(\mathbf{x}, t)} \left(th(\mathbf{y}) + g(\mathbf{y}) + \nabla g(\mathbf{y}) \cdot (\mathbf{y} - \mathbf{x})\right) dS(\mathbf{y}) \tag{8.4.4}$$

8.5 Solution for $d = 2$: the Poisson's Formula

We will use Kirchoff's formula for the solution of the wave equation in three dimensions to derive the solution of the wave equation in two dimensions. This technique is known as the method of descent.

Let us consider the initial-value problem (8.3.3) for $n = 2$ and simply regard it as a problem for $n = 3$, in which the third variable x_3 does not appear.

Indeed, let us assume $u \in C^2(\mathbb{R}^2 \times [0, +\infty))$ solves (8.3.3) for $n = 2$. We will denote with a bar $\bar{\cdot}$ the extensions we will do to $\mathbb{R}^3$. Hence, let us define

$$\overline{u}\,(x_1, x_2, x_3, t) \,:=\, u(x_1, x_2, t), \qquad \overline{g}\,(x_1, x_2, x_3) \,:=\, g(x_1, x_2) \quad \text{and} \quad \overline{h}\,(x_1, x_2, x_3) \,:=\, h(x_1, x_2)$$

Then, as u satisfies (8.3.3) then

$$\begin{cases} \bar{u}_{tt}\,(x_1, x_2, x_3, t) \,-\, \triangle\overline{u}\,(x_1, x_2, x_3, t) \,=\, 0 & \text{in } \mathbb{R}^3 \times (0, +\infty) \\ \bar{u}\,(x_1, x_2, x_3, 0) \,=\, \overline{g}\,(x_1, x_2, x_3) & \text{on } \mathbb{R}^3 \times \{t = 0\} \\ \bar{u}_t\,(x_1, x_2, x_3, 0) \,=\, \overline{h}\,(x_1, x_2, x_3) & \text{on } \mathbb{R}^3 \times \{t = 0\} \end{cases}$$

Let us write $\mathbf{x} := (x_1, x_2) \in \mathbb{R}^2$, $\bar{\mathbf{x}} = (x_1, x_2, 0) \in \mathbb{R}^3$, let $\overline{B}\,(\bar{\mathbf{x}}, t)$ be the ball in $\mathbb{R}^3$ with center $\bar{\mathbf{x}}$ and radius $t > 0$ and let $d\overline{S}$ be the two-dimensional surface measure on $\partial\overline{B}\,(\bar{\mathbf{x}}, t)$. Applying Kirchhoff's formula in the form (8.4.3), we get

$$u(\mathbf{x}, t) \,=\, \bar{u}(\overline{\mathbf{x}}, t) \,=\, \frac{\partial}{\partial t}\left(t \fint_{\partial\overline{B}(\overline{\mathbf{x}}, t)} \overline{g}(\mathbf{y}) d\overline{S}(\mathbf{y}) \right) \,+\, t \fint_{\partial\overline{B}(\overline{\mathbf{x}}, t)} \overline{h}(\mathbf{y}) d\overline{S}(\mathbf{y}) \tag{8.5.1}$$

Now, note that we can simplify this expression by observing that:

$$\fint_{\partial\overline{B}(\overline{\mathbf{x}}, t)} \overline{g}(\mathbf{y}) d\overline{S}(\mathbf{y}) \;=\; \frac{1}{4\pi t^2} \int_{\partial\overline{B}(\overline{\mathbf{x}}, t)^+} \overline{g}(y_1, y_2, 0) d\overline{S}(\mathbf{y}) \,+\, \frac{1}{4\pi t^2} \int_{\partial\overline{B}(\overline{\mathbf{x}}, t)^-} \overline{g}(y_1, y_2, 0) d\overline{S}(\mathbf{y}) \tag{8.5.2}$$

where we have divided $\partial\overline{B}(\overline{\mathbf{x}}, t)$ in the superior hemisphere $\partial\overline{B}(\overline{\mathbf{x}}, t)^+$ and the inferior hemisphere $\partial\overline{B}(\overline{\mathbf{x}}, t)^-$. Now, we parameterize $\partial\overline{B}(\overline{\mathbf{x}}, t)^+$ and $\partial\overline{B}(\overline{\mathbf{x}}, t)^-$ respectively by

$$\gamma^+(z_1, z_2) \,=\, \left(z_1, z_2, \sqrt{t^2 - (z_1 - x_1)^2 - (z_2 - x_2)^2} \right)$$

$$\gamma^-(z_1, z_2) \,=\, \left(z_1, z_2, -\sqrt{t^2 - (z_1 - x_1)^2 - (z_2 - x_2)^2} \right)$$

where $\mathbf{z} := (z_1, z_2) \in B(\mathbf{x}, t)$. As $\left| \frac{\partial\gamma^{\pm}(\mathbf{z})}{\partial z_1} \times \frac{\partial\gamma^{\pm}(\mathbf{z})}{\partial z_2} \right| \,=\, \frac{t}{\sqrt{t^2 - |\mathbf{z} - \mathbf{x}|^2}}$, we get

$$\int_{\partial\overline{B}(\overline{\mathbf{x}}, t)^+} \overline{g}(y_1, y_2, 0) d\overline{S}(\mathbf{y}) \,=\, \int_{B(\mathbf{x}, t)} \overline{g}(\gamma^+(\mathbf{z})) \left| \frac{\partial\gamma^+(\mathbf{z})}{\partial z_1} \times \frac{\partial\gamma^+(\mathbf{z})}{\partial z_2} \right| d\mathbf{z} \,=\, \int_{B(\mathbf{x}, t)} g(\mathbf{z}) \frac{t}{\sqrt{t^2 - |\mathbf{z} - \mathbf{x}|^2}} d\mathbf{z}$$

$$\int_{\partial\overline{B}(\overline{\mathbf{x}}, t)^-} \overline{g}(y_1, y_2, 0) d\overline{S}(\mathbf{y}) \,=\, \int_{B(\mathbf{x}, t)} \overline{g}(\gamma^-(\mathbf{z})) \left| \frac{\partial\gamma^-(\mathbf{z})}{\partial z_1} \times \frac{\partial\gamma^-(\mathbf{z})}{\partial z_2} \right| d\mathbf{z} \,=\, \int_{B(\mathbf{x}, t)} g(\mathbf{z}) \frac{t}{\sqrt{t^2 - |\mathbf{z} - \mathbf{x}|^2}} d\mathbf{z}$$

Thus, substituting this in (8.5.2), we get

$$\fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{g}(\mathbf{y}) d\overline{S}(\mathbf{y}) = \frac{1}{2\pi t}\int_{B(\mathbf{x},t)} g(\mathbf{z})\frac{1}{\sqrt{t^2-|\mathbf{z}-\mathbf{x}|^2}}d\mathbf{z} = \left[\begin{array}{r} \mathbf{z} = T(\mathbf{y}) = \mathbf{x}+t\mathbf{y} \\ (DT)(\mathbf{y}) = tId \Longrightarrow |det\,(DT)(\mathbf{y})| = t^2 \end{array}\right]$$

$$= \frac{1}{2\pi}\int_{B(0,1)} g(\mathbf{x}+t\mathbf{y})\frac{1}{\sqrt{1-|\mathbf{y}|^2}}d\mathbf{y} \tag{8.5.3}$$

The previous expression also holds for the function $\overline{h}$, hence

$$\fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{h}(\mathbf{y}) d\overline{S}(\mathbf{y}) = \frac{1}{2\pi t}\int_{B(\mathbf{x},t)} h(\mathbf{z})\frac{1}{\sqrt{t^2-|\mathbf{z}-\mathbf{x}|^2}}d\mathbf{z} = \frac{t}{2}\fint_{B(\mathbf{x},t)} h(\mathbf{z})\frac{1}{\sqrt{t^2-|\mathbf{z}-\mathbf{x}|^2}}d\mathbf{z}$$

Finally, using the expression deduced in (8.5.3), we can write

$$\frac{\partial}{\partial t}\left(t\fint_{\partial \overline{B}(\overline{\mathbf{x}},t)} \overline{g}(\mathbf{y}) d\overline{S}(\mathbf{y})\right) = \frac{1}{2\pi}\int_{B(0,1)} g(\mathbf{x}+t\mathbf{y})\frac{1}{\sqrt{1-|\mathbf{y}|^2}}d\mathbf{z} + \frac{t}{2\pi}\int_{B(0,1)} \nabla g(\mathbf{x}+t\mathbf{y})\cdot\mathbf{y}\frac{1}{\sqrt{1-|\mathbf{y}|^2}}d\mathbf{y} =$$

$$\overset{\mathbf{y}=\frac{1}{t}(\mathbf{z}-\mathbf{x})}{=} \frac{t}{2}\fint_{B(\mathbf{x},t)} g(\mathbf{z})\frac{1}{\sqrt{t^2-|\mathbf{z}-\mathbf{x}|^2}}d\mathbf{z} + \frac{t}{2}\fint_{B(\mathbf{x},t)} \nabla g(\mathbf{z})\cdot(\mathbf{z}-\mathbf{x})\frac{1}{\sqrt{t^2-|\mathbf{z}-\mathbf{x}|^2}}d\mathbf{z}$$

Thus, formula (8.5.1) becomes the Poisson's formula for the solution of the initial-value of the wave equation in two dimensions

$$u(\mathbf{x},t) = \frac{1}{2}\fint_{B(\mathbf{x},t)} \frac{tg(\mathbf{z}) + t\nabla g(\mathbf{z})\cdot(\mathbf{z}-\mathbf{x}) + t^2 h(\mathbf{z})}{\sqrt{t^2-|\mathbf{z}-\mathbf{x}|^2}}d\mathbf{z} \tag{8.5.4}$$

8.6 Solution of the wave equation for odd dimensions

We will generalize the ideas we used to find the Kirchhoff's formula in the case $d = 3$. We will solve the Euler-Poisson-Darboux PDE for odd $d \geq 3$. In this section, we set $d = 2k+1$ for $k \geq 1$. Moreover, we suppose that $u \in C^{k+1}\left(\mathbb{R}^d \times [0,+\infty)\right)$ solves the initial-value problem (8.3.3). Let us consider

$$U(\mathbf{x};r,t) := \fint_{\partial B(\mathbf{x},r)} u(\mathbf{y},t)dS(\mathbf{y}), \quad G(\mathbf{x};r) := \fint_{\partial B(\mathbf{x},r)} g(\mathbf{y})dS(\mathbf{y}) \quad \text{and} \quad H(\mathbf{x};r) := \fint_{\partial B(\mathbf{x},r)} h(\mathbf{y})dS(\mathbf{y})$$

Note that by Lemma 8.3. the function $U \in C^{k+1}\left(\overline{\mathbb{R}}_+ \times [0,+\infty)\right)$ for each $\mathbf{x} \in \mathbb{R}^d$ fixed. Hence, let us fix $\mathbf{x} \in \mathbb{R}^d$ and define for $r > 0$ and $t \geq 0$

$$\begin{cases} \widetilde{U}(r,t) := & \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}U(\mathbf{x};r,t)\right] \\ \widetilde{G}(r) := & \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}G(\mathbf{x};r)\right] \\ \widetilde{H}(r) := & \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}H(\mathbf{x};r)\right] \end{cases}$$

Note that it holds that $\widetilde{U}(r,0) = \widetilde{G}(r)$ and $\widetilde{U}_t(r,0) = \widetilde{H}(r)$.

Lemma 8.5. *Let $\phi : \mathbb{R} \longrightarrow \mathbb{R}$ be $C^{k+1}(\mathbb{R})$. Then, the following identities hold for $k = 1, 2, ...$:*

1. $\left(\frac{d^2}{dr^2}\right)\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[r^{2k-1}\phi(r)\right] = \left(\frac{1}{r}\frac{d}{dr}\right)^{k}\left[r^{2k}\frac{d}{dr}\phi(r)\right]$

2. $\left(\frac{1}{r}\frac{d}{dr}\right)^{k-1}\left[r^{2k-1}\phi(r)\right] = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{d^j}{dr^j}\phi(r)$, *where* β_j^k $(j = 0, ..., k-1)$ *are independent of* ϕ.

3. $\beta_0^k = 1 \cdot 3 \cdot 5 \cdots (2k-1)$

PROOF. The proof can be done by induction and can be found in [Evans2010].

Lemma 8.6. *The function* $\widetilde{U}$ *solves the one-dimensional wave equation on the half-line with Dirichlet boundary conditions*

$$\begin{cases} \widetilde{U}_{tt}(r,t) - \widetilde{U}_{rr}(r,t) = 0 & (r,t) \in \mathbb{R}_+ \times (0,+\infty) \\ \widetilde{U}(r,0) = \widetilde{G}(r) \ \text{ and } \ \widetilde{U}_t(r,0) = \widetilde{H}(r) & r \in \mathbb{R}_+ \\ \widetilde{U}(0,t) = 0 & t \in (0,+\infty) \end{cases}$$

PROOF.

We combine Lemma 8.3. and the identities provided by Lemma 8.5. Note that

$$\widetilde{U}_{rr}(r,t) = \left(\frac{\partial^2}{\partial r^2}\right)\left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}U(\mathbf{x};r,t)\right] \overset{Lemma\,8.5.1}{=} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k}\left[r^{2k}U_r(\mathbf{x};r,t)\right] =$$

$$= \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}U_{rr}(\mathbf{x};r,t) + 2kr^{2k-2}U_r(\mathbf{x};r,t)\right] = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}\left(U_{rr}(\mathbf{x};r,t) + \frac{n-1}{r}U_r(\mathbf{x};r,t)\right)\right]$$

$$\overset{Lemma\,8.3.}{=} \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}U_{tt}(\mathbf{x};r,t)\right] = \widetilde{U}_{tt}(r,t)$$

It is clear by definition that $\widetilde{U}(r,0) = \widetilde{G}(r)$ and $\widetilde{U}_t(r,0) = \widetilde{H}(r)$. Finally, as $U(\mathbf{x};0,t) = u(\mathbf{x},t)$, we have by Lemma 8.5.2

$$\widetilde{U}(0,t) = \left(\frac{1}{r}\frac{\partial}{\partial r}\right)^{k-1}\left[r^{2k-1}U(\mathbf{x};0,t)\right] = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{\partial^j}{\partial r^j}U(\mathbf{x};0,t) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{\partial^j}{\partial r^j}u(\mathbf{x},t) = 0$$

Derivation of the solution

We have seen in Lemma 8.6. that $\widetilde{U}$ solves the the one-dimensional wave equation on the half-line with Dirichlet boundary conditions. In Lemma 8.4. we saw that the solution is given by

$$\widetilde{U}(r,t) = \begin{cases} \frac{1}{2}\left(\widetilde{G}(r+t) + \widetilde{G}(r-t)\right) + \frac{1}{2}\int_{r-t}^{r+t}\widetilde{H}(y)dy & \text{if } r \geq t \geq 0 \\ \frac{1}{2}\left(\widetilde{G}(r+t) - \widetilde{G}(t-r)\right) + \frac{1}{2}\int_{-r+t}^{r+t}\widetilde{H}(y)dy & \text{if } 0 \leq r \leq t \end{cases} \tag{8.6.1}$$

Moreover, Lemma 8.5.2. asserts $\widetilde{U}(r,t) = \sum_{j=0}^{k-1}\beta_j^k r^{j+1}\frac{\partial^j}{\partial r^j}U(\mathbf{x};r,t)$. Consequently, $\frac{\widetilde{U}(r,t)}{\beta_0^k r} = U(\mathbf{x};r,t) +$ $+ \sum_{j=1}^{k-1}\frac{\beta_j^k}{\beta_0^k}r^j\frac{\partial^j}{\partial r^j}U(\mathbf{x};r,t)$. Recall that $u(\mathbf{x},t) = \lim_{r\longrightarrow 0^+}U(\mathbf{x};r,t)$. Thus

$$\lim_{r\longrightarrow 0}\frac{\widetilde{U}(r,t)}{\beta_0^k r} = \lim_{r\longrightarrow 0}U(\mathbf{x};r,t) + \lim_{r\longrightarrow 0}\sum_{j=1}^{k-1}\frac{\beta_j^k}{\beta_0^k}r^j\frac{\partial^j}{\partial r^j}U(\mathbf{x};r,t) = u(\mathbf{x},t)$$

Then, using the expression of $\widetilde{U}(r,t)$ given by the solution (8.6.1)

$$u(\mathbf{x},t) = \lim_{r\longrightarrow 0}\frac{\widetilde{U}(r,t)}{\beta_0^k r} = \frac{1}{\beta_0^k}\lim_{r\longrightarrow 0}\left[\frac{1}{2r}\left(\widetilde{G}(r+t)-\widetilde{G}(t-r)\right) + \frac{1}{2r}\int_{-r+t}^{r+t}\widetilde{H}(y)dy\right] =$$

$$= \frac{1}{\beta_0^k}\left[\widetilde{G}'(t) + \widetilde{H}(y)\right]$$

Hence, recalling the definitions of $\widetilde{G}$ and $\widetilde{H}$ and recalling that $d = 2k+1$, we get the representation formula for $x\in\mathbb{R}^d$ and $t>0$

$$u(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}⨍_{\partial B(\mathbf{x},t)} g(\mathbf{y})\,dS(\mathbf{y})\right] + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}⨍_{\partial B(\mathbf{x},t)} h(\mathbf{y})\,dS(\mathbf{y})\right]\right) \tag{8.6.2}$$

where d is odd and $\gamma_d = 1\cdot 3\cdot 5\cdots(d-2)$.

Theorem 8.4. (Solution of wave equation in odd dimensions) *Assume d is an odd integer, $d\geq 3$, and suppose $g\in C^{m+1}\left(\mathbb{R}^d\right)$ and $h\in C^m\left(\mathbb{R}^d\right)$, for $m = \frac{d+1}{2}$. Define u by (8.6.2). Then, the following statements hold:*

1. $u\in C^2\left(\mathbb{R}^d\times[0,+\infty)\right)$
2. $u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t) = 0$ *in* $\mathbb{R}^d\times(0,+\infty)$
3. $\lim_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u(\mathbf{x},t) = g(\mathbf{x}_0)$ *and* $\lim_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u_t(\mathbf{x},t) = h(\mathbf{x}_0)$ *for each point* $\mathbf{x}_0\in\mathbb{R}^d$.

PROOF.

• Proof 1: note first that making a change of variables we can write (8.6.2) as

$$u(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}⨍_{\partial B(0,1)} g(\mathbf{x}+t\mathbf{z})\,dS(\mathbf{z})\right] + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}⨍_{\partial B(0,1)} h(\mathbf{x}+t\mathbf{z})\,dS(\mathbf{z})\right]\right) \tag{8.6.3}$$

From this expression, we immediately get

$$u_t(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\left(\frac{\partial}{\partial t}\right)^2\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}⨍_{\partial B(0,1)} g(\mathbf{x}+t\mathbf{z})\,dS\right] + \left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}⨍_{\partial B(0,1)} h(\mathbf{x}+t\mathbf{z})\,dS\right]\right) \tag{8.6.4}$$

$$u_{tt}(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\left(\frac{\partial}{\partial t}\right)^3\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}⨍_{\partial B(0,1)} g(\mathbf{x}+t\mathbf{z})\,dS\right] + \left(\frac{\partial}{\partial t}\right)^2\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}⨍_{\partial B(0,1)} h(\mathbf{x}+t\mathbf{z})\,dS\right]\right)$$

These three functions are continuous in $\mathbb{R}^d\times(0,+\infty)$. In effect, the higher order of partial derivative respect t which appears in g is $\frac{d-3}{2}+3 = m+1$ and we know that $g\in C^{m+1}\left(\mathbb{R}^d\right)$. Similarly, the higher order of partial derivative respect t that appears in h is $\frac{d-3}{2}+2 = m$ and we know that $h\in C^m\left(\mathbb{R}^d\right)$. Therefore, we conclude that u is two times continuously differentiable respect t in $\mathbb{R}^d\times(0,+\infty)$.

Furthermore, using the differentation under the integral sign theorem in expressions (8.6.3) and (8.6.4), we see that u and u_t are continously differentiable respect $\mathbf{x}$. Thus, we conclude that $u\in C^2\left(\mathbb{R}^d\times(0,+\infty)\right)$. In proof 3 we will see that in fact $u\in C^2\left(\mathbb{R}^d\times[0,+\infty)\right)$.

• Proof 2: we will show the result for the case $g = 0$. In a similar way, one proves the result when $h = 0$. On the one hand, we differentiate under the integral sign (8.6.3) respect $\triangle_x$:

$$\triangle u(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}\fint_{\partial B(0,1)} \triangle h\left(\mathbf{x}+t\mathbf{z}\right)dS\left(\mathbf{z}\right)\right]$$

On the other hand, differentiating (8.6.3) respect $\left(\frac{\partial^2}{\partial t^2}\right)$ and using Lemma 8.5.1

$$\left(\frac{\partial^2}{\partial t^2}\right)u(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-1}{2}}\left[\frac{t^{d-1}}{d\alpha(d)}\int_{\partial B(0,1)} \nabla h\left(\mathbf{x}+t\mathbf{z}\right)\cdot\mathbf{z}dS\left(\mathbf{z}\right)\right]$$

By Divergence Theorem (Theorem A.11.),

$$\frac{\partial^2}{\partial t^2}u(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-1}{2}}\left[\frac{t^{d}}{d\alpha(d)}\int_{B(0,1)} \triangle h\left(\mathbf{x}+t\mathbf{z}\right)d\mathbf{z}\right] = \frac{1}{\gamma_d}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[\frac{t^{d-2}}{\alpha(d)}\int_{B(0,1)} \triangle h\left(\mathbf{x}+t\mathbf{z}\right)d\mathbf{z}\right.$$
$$\left.+\frac{t^{d-2}}{d\alpha(d)}\int_{B(0,1)} t\nabla\left(\triangle h\left(\mathbf{x}+t\mathbf{z}\right)\right)d\mathbf{z}\right]$$

Finally, for the second term of the previous expression we use the same strategy of Lemma 8.3. to derive expression (8.3.7) and we conclude

$$\left(\frac{\partial^2}{\partial t^2}\right)u(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[\frac{t^{d-2}}{\alpha(d)}\int_{B(0,1)} \triangle h\left(\mathbf{x}+t\mathbf{z}\right)d\mathbf{z} - \frac{t^{d-2}}{\alpha(d)}\int_{B(0,1)} \triangle h\left(\mathbf{x}+t\mathbf{z}\right)d\mathbf{z}+\right.$$
$$\left.+\frac{t^{d-2}}{d\alpha(d)}\int_{\partial B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z})dS(\mathbf{z})\right] = \frac{1}{\gamma_d}\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-3}{2}}\left[t^{d-2}\fint_{\partial B(0,1)} \triangle u(\mathbf{x}+r\mathbf{z})dS(\mathbf{z})\right]$$

Thus, we obtain $\left(\frac{\partial^2}{\partial t^2}\right)u(\mathbf{x},t) = \triangle u(\mathbf{x},t)$ in $\mathbb{R}^d\times(0,+\infty)$.

• Proof 3: first we show that $\lim_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u(\mathbf{x},t) = g(\mathbf{x}_0)$. We apply Lemma 8.5.2 to (8.6.3)

$$u(\mathbf{x},t) = \left(\frac{1}{\gamma_d}\left(\frac{\partial}{\partial t}\right)\sum_{j=0}^{m-2}\beta_j^{m-1}t^{j+1}\frac{\partial^j}{\partial t^j}\fint_{\partial B(0,1)} g\left(\mathbf{x}+t\mathbf{z}\right)dS\left(\mathbf{z}\right) + \frac{1}{\gamma_d}\sum_{j=0}^{m-2}\beta_j^{m-1}t^{j+1}\frac{\partial^j}{\partial t^j}\fint_{\partial B(0,1)} h\left(\mathbf{x}+t\mathbf{z}\right)dS\left(\mathbf{z}\right)\right)$$
$$= \left(\frac{1}{\gamma_d}\sum_{j=0}^{m-2}\beta_j^{m-1}(j+1)t^j\frac{\partial^j}{\partial t^j}\fint_{\partial B(0,1)} g\left(\mathbf{x}+t\mathbf{z}\right)dS\left(\mathbf{z}\right) + \frac{1}{\gamma_d}\sum_{j=0}^{m-2}\beta_j^{m-1}t^{j+1}\frac{\partial^{j+1}}{\partial t^{j+1}}\fint_{\partial B(0,1)} g\left(\mathbf{x}+t\mathbf{z}\right)dS\left(\mathbf{z}\right)+\right.$$
$$\left.+\frac{1}{\gamma_d}\sum_{j=0}^{m-2}\beta_j^{m-1}t^{j+1}\frac{\partial^j}{\partial t^j}\fint_{\partial B(0,1)} h\left(\mathbf{x}+t\mathbf{z}\right)dS\left(\mathbf{z}\right)\right) \tag{8.6.5}$$

Taking the limit as $(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)$ in the previous expression we conclude

$$\lim_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u(\mathbf{x},t) = \frac{1}{\gamma_d}\beta_0^{m-1}\fint_{\partial B(0,1)} g\left(\mathbf{x}_0\right)dS\left(\mathbf{z}\right) = g(\mathbf{x}_0)$$

Finally we shall prove that $\lim_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u_t(\mathbf{x},t) = h(\mathbf{x}_0)$. We differentiate respect t (8.6.5)

$$u_t(\mathbf{x},t) = \left(\frac{1}{\gamma_d}\sum_{j=1}^{m-2}\beta_j^{m-1}(j+1)jt^{j-1}\frac{\partial^j}{\partial t^j}\fint_{\partial B(0,1)} g\left(\mathbf{x}+t\mathbf{z}\right)dS\left(\mathbf{z}\right) + \frac{1}{\gamma_d}\sum_{j=0}^{m-2}\beta_j^{m-1}t^{j+1}\frac{\partial^{j+2}}{\partial t^{j+2}}\fint_{\partial B(0,1)} g\left(\mathbf{x}+t\mathbf{z}\right)dS\left(\mathbf{z}\right)+\right.$$

$$+\frac{2}{\gamma_d}\sum_{j=0}^{m-2}\beta_j^{m-1}(j+1)t^j\frac{\partial^{j+1}}{\partial t^{j+1}}\fint_{\partial B(0,1)} g(\mathbf{x}+t\mathbf{z})\,dS(\mathbf{z}) + \frac{1}{\gamma_d}\sum_{j=0}^{m-2}\beta_j^{m-1}(j+1)t^j\frac{\partial^{j}}{\partial t^{j}}\fint_{\partial B(0,1)} h(\mathbf{x}+t\mathbf{z})\,dS(\mathbf{z}) +$$
$$+\frac{1}{\gamma_d}\sum_{j=0}^{m-2}\beta_j^{m-1}t^{j+1}\frac{\partial^{j+1}}{\partial t^{j+1}}\fint_{\partial B(0,1)} h(\mathbf{x}+t\mathbf{z})\,dS(\mathbf{z})\Big)$$

Taking the limit as $(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)$ in the previous expression we conclude

$$\lim_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u_t(\mathbf{x},t) = \frac{2}{\gamma_d}\beta_1^{m-1}\frac{\partial}{\partial t}\fint_{\partial B(0,1)} g(\mathbf{x}_0)\,dS(\mathbf{z}) + \frac{2}{\gamma_d}\beta_0^{m-1}\frac{\partial}{\partial t}\fint_{\partial B(0,1)} g(\mathbf{x}_0)\,dS(\mathbf{z})$$
$$+\frac{1}{\gamma_d}\beta_0^{m-1}\fint_{\partial B(0,1)} h(\mathbf{x}_0)\,dS(\mathbf{z}) = h(\mathbf{x}_0)$$

Remark 8.2. *Notice that to compute $u(\mathbf{x},t)$ we need only have information on g, h and their derivatives on the sphere $\partial B(\mathbf{x},t)$ and not on the entire ball $B(\mathbf{x},t)$.*

Remark 8.3. *Comparing formula (8.6.2) with d'Alembert's formula (8.3.2), we observe that the latter does not involve the derivatives of g. This suggest that for $d>1$, a solution of the wave equation (8.3.3) need not for times $t>0$ be as smooth as its initial value g: irregularities in g may focus at times $t>0$, thereby causing u to be less regular.*

8.7 Solution of the wave equation for even dimensions: the Method of Descent

Now we suppose that $d\geq 2$ is even. We will apply the idea that we used to derive the Poisson's Formula (case $d=2$), i.e. we will use formula (8.6.2) for $d+1$ (odd) to derive the solution for the d-dimensional case. This technique is known as the method of descent.

Derivation of the solution

• Let us take $m=\frac{d+2}{2}$ and suppose $u\in C^m\left(\mathbb{R}^d\times[0,+\infty)\right)$ is a solution of (8.3.3). In this section, we will denote with a bar $\bar{\cdot}$ the extensions we will do to $\mathbb{R}^{d+1}$. Let us define:

$$\begin{cases}\overline{u}(x_1,..,x_{d+1},t) := u(x_1,..,x_d,t)\\ \overline{g}(x_1,..,x_{d+1}) := g(x_1,..,x_d)\\ \overline{h}(x_1,..,x_{d+1}) := h(x_1,..,x_d)\end{cases}$$

As u satisfies (8.3.3), we conclude that $\bar{u}$ is the solution of the following initial problem:

$$\begin{cases}\bar{u}_{tt}(x_1,..,x_{d+1},t) - \triangle\overline{u}(x_1,..,x_{d+1},t) = 0 & \text{in } \mathbb{R}^{d+1}\times(0,+\infty)\\ \bar{u}(x_1,..,x_{d+1},0) = \overline{g}(x_1,..,x_{d+1}) & \text{on } \mathbb{R}^{d+1}\times\{t=0\}\\ \bar{u}_t(x_1,..,x_{d+1},0) = \overline{h}(x_1,..,x_{d+1}) & \text{on } \mathbb{R}^{d+1}\times\{t=0\}\end{cases} \tag{8.7.1}$$

Let us write $\mathbf{x} := (x_1, ..., x_d) \in \mathbb{R}^d$, $\bar{\mathbf{x}} = (\mathbf{x}, 0) \in \mathbb{R}^{d+1}$, let $\overline{B}(\bar{\mathbf{x}}, t)$ be the ball in $\mathbb{R}^{d+1}$ with center $\bar{\mathbf{x}}$ and radius $t > 0$ and let $d\overline{S}$ be the d-dimensional surface measure on $\partial\overline{B}(\bar{\mathbf{x}}, t)$. Applying (8.6.2) we get

$$\overline{u}(\mathbf{x},t) = \frac{1}{\gamma_{d+1}}\left(\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}}\left[t^{d-1}⨍_{\partial\overline{B}(\bar{\mathbf{x}},t)}\overline{g}(\mathbf{y})\,d\overline{S}(\mathbf{y})\right] + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}}\left[t^{d-1}⨍_{\partial\overline{B}(\bar{\mathbf{x}},t)}\overline{h}(\mathbf{y})\,d\overline{S}(\mathbf{y})\right]\right) \tag{8.7.2}$$

Note now that $\partial\overline{B}(\bar{\mathbf{x}},t) = \partial\overline{B}(\bar{\mathbf{x}},t)^+ \cup \partial\overline{B}(\bar{\mathbf{x}},t)^-$, where $\partial\overline{B}(\bar{\mathbf{x}},t)^+ := \partial\overline{B}(\bar{\mathbf{x}},t) \cap \{y_{d+1} \geq 0\}$ is the graph of the function $\gamma(\mathbf{y}) = \sqrt{t^2 - |\mathbf{y}-\mathbf{x}|^2}$ for $\mathbf{y} \in B(\mathbf{x},t) \subset \mathbb{R}^d$. Likewise, $\partial\overline{B}(\bar{\mathbf{x}},t)^- := \partial\overline{B}(\bar{\mathbf{x}},t) \cap \{y_{d+1} \leq 0\}$ is the graph of $-\gamma$. Moreover, note that $\sqrt{1 + |\nabla\gamma(\mathbf{y})|^2} = \frac{t}{\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2}}$. Consequently

$$⨍_{\partial\overline{B}(\bar{\mathbf{x}},t)}\overline{g}(\mathbf{y})\,d\overline{S}(\mathbf{y}) = \frac{1}{(d+1)\alpha(d+1)t^d}\int_{\partial\overline{B}(\bar{\mathbf{x}},t)}\overline{g}(\mathbf{y})\,d\overline{S}(\mathbf{y}) = \frac{2}{(d+1)\alpha(d+1)t^d}\int_{B(\mathbf{x},t)}g(\mathbf{y})\sqrt{1+|\nabla\gamma(\mathbf{y})|^2}d\mathbf{y}$$
$$= \frac{2}{(d+1)\alpha(d+1)t^d}\int_{B(\mathbf{x},t)}g(\mathbf{y})\frac{t}{\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2}}d\mathbf{y} = \frac{2t\alpha(d)}{(d+1)\alpha(d+1)}⨍_{B(\mathbf{x},t)}g(\mathbf{y})\frac{1}{\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2}}d\mathbf{y} \tag{8.7.3}$$

Similarly, we get the same formula for h

$$⨍_{\partial\overline{B}(\bar{\mathbf{x}},t)}\overline{h}(\mathbf{y})\,d\overline{S}(\mathbf{y}) = \frac{2t\alpha(d)}{(d+1)\alpha(d+1)}⨍_{B(\mathbf{x},t)}h(\mathbf{y})\frac{1}{\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2}}d\mathbf{y} \tag{8.7.4}$$

Thus, we insert formula (8.7.3) and (8.7.4) in (8.7.2) and we find

$$u(\mathbf{x},t) = \frac{1}{\gamma_{d+1}}\frac{2\alpha(d)}{(d+1)\alpha(d+1)}\left(\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}}\left[⨍_{B(\mathbf{x},t)}\frac{t^d g(\mathbf{y})}{\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2}}d\mathbf{y}\right] + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}}\left[⨍_{B(\mathbf{x},t)}\frac{t^d h(\mathbf{y})}{\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2}}\right]\right)$$

• Let us define $\gamma_d = 2 \cdot 4 \cdot 6 \cdots (d-2)\,d$. We are going to see that $\frac{1}{\gamma_{d+1}}\frac{2\alpha(d)}{(d+1)\alpha(d+1)} = \frac{1}{\gamma_d}$. Note that, as we defined in (8.6.2), $\gamma_{d+1} = 1 \cdot 3 \cdot 5 \cdots (d-1)$ and $\alpha(d) = \frac{\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d+2}{2}\right)}$ (volume of the unit ball). It follows

1. As d is even: $\Gamma\left(\frac{d+2}{2}\right) = \frac{d}{2}\Gamma\left(\frac{d}{2}\right) = \frac{d}{2}\frac{d-2}{2}\Gamma\left(\frac{d-2}{2}\right) = ... = \frac{d}{2}\frac{d-2}{2}\cdots\frac{4}{2}\frac{2}{2}$
2. Analogously: $\Gamma\left(\frac{d+3}{2}\right) = \frac{d+1}{2}\Gamma\left(\frac{d+1}{2}\right) = \frac{d+1}{2}\frac{d-1}{2}\Gamma\left(\frac{d-1}{2}\right) = ... = \frac{d+1}{2}\frac{d-1}{2}\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi}$

Using this we get

$$\frac{\alpha(d)}{\alpha(d+1)} = \frac{\pi^{\frac{d}{2}}\Gamma\left(\frac{d+3}{2}\right)}{\Gamma\left(\frac{d+2}{2}\right)\pi^{\frac{d+1}{2}}} = \frac{\frac{d+1}{2}\frac{d-1}{2}\cdots\frac{3}{2}\frac{1}{2}\sqrt{\pi}}{\sqrt{\pi}\frac{d}{2}\frac{d-2}{2}\cdots\frac{4}{2}\frac{2}{2}} = \frac{(d+1)(d-1)\cdots 3}{2d(d-2)\cdots 4\cdot 2} = \frac{(d+1)\gamma_{d+1}}{2\gamma_d}$$

And finally we conclude

$$\frac{1}{\gamma_{d+1}}\frac{2\alpha(d)}{(d+1)\alpha(d+1)} = \frac{2}{\gamma_{d+1}(d+1)}\frac{(d+1)\gamma_{d+1}}{2\gamma_d} = \frac{1}{\gamma_d}$$

• Hence the resulting representation formula for d even is

$$u(\mathbf{x},t) = \frac{1}{\gamma_d}\left(\left(\frac{\partial}{\partial t}\right)\left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}}\left[t^d⨍_{B(\mathbf{x},t)}\frac{g(\mathbf{y})}{\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2}}d\mathbf{y}\right] + \left(\frac{1}{t}\frac{\partial}{\partial t}\right)^{\frac{d-2}{2}}\left[t^d⨍_{B(\mathbf{x},t)}\frac{h(\mathbf{y})}{\sqrt{t^2-|\mathbf{y}-\mathbf{x}|^2}}\right]\right) \tag{8.7.5}$$

where $\gamma_d = 2 \cdot 4 \cdot 6 \cdots (d-2)\,d$, for $\mathbf{x} \in \mathbb{R}^d$ and $t > 0$.

Theorem 8.5. (Solution of wave equation in even dimensions) *Assume d is an odd integer, $d \geq 2$, and suppose $g \in C^{m+1}\left(\mathbb{R}^d\right)$ and $h \in C^m\left(\mathbb{R}^d\right)$, for $m = \frac{d+2}{2}$. Define u by (8.7.5). Then, the following statements hold:*

1. $u \in C^2\left(\mathbb{R}^d \times [0,+\infty)\right)$
2. $u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t) = 0$ *in* $\mathbb{R}^d \times (0,+\infty)$
3. $\lim\limits_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u(\mathbf{x},t) = g(\mathbf{x}_0)$ *and* $\lim\limits_{(\mathbf{x},t)\longrightarrow(\mathbf{x}_0,0)} u_t(\mathbf{x},t) = h(\mathbf{x}_0)$ *for each point* $\mathbf{x}_0 \in \mathbb{R}^d$.

PROOF. This theorem follows from Theorem 8.4. In effect, we only have to consider the extensions of u, g and h defined in (8.7.1) and apply Theorem 8.4.

Remark 8.4 In contrast to the solution for d odd (formula (8.6.2)), to compute $u(\mathbf{x},t)$ for d even, we need information on $u = g$ and $u_t = h$ on all of $B(\mathbf{x},t)$, and not just on $\partial B(\mathbf{x},t)$.

8.8 Uniqueness of solution: the Energy Method

As we did with the heat equation, we will use the Energy Method to show that there exists an unique solution of problem (8.3.3) such that $u(\cdot,t) \in \mathcal{S}\left(\mathbb{R}^d\right)$ for all $t \geq 0$ and $u \in C^2(\mathbb{R}^d \times [0,+\infty))$. In particular the initial conditions g and h should be in $\mathcal{S}\left(\mathbb{R}^d\right)$.

In Theorem 8.2. we proved that the energy of the solution given by (8.2.1) is conserved in time. We will generalize this result by showing that in fact it is true for any solution satisfying $u(\cdot,t) \in \mathcal{S}(\mathbb{R}^d)$ for all $t \geq 0$ and $u \in C^2(\mathbb{R}^d \times [0,+\infty))$. From this result, we will be able to conclude that the solution is unique.

Theorem 8.6. *Let $u \in C^2(\mathbb{R}^d \times [0,+\infty))$ s.t. $u(\cdot,t) \in \mathcal{S}(\mathbb{R}^d)$ for all $t \geq 0$ be a solution of (8.3.3) with g and h in $\mathcal{S}\left(\mathbb{R}^d\right)$. Then, its energy $E(t)$ (given by (8.2.2)) is conserved in time, that is, $E(t) = E(0)$.*

PROOF.

Differentiating under the integral sign, we get that

$$\dot{E}(t) = \frac{1}{2}\int_{\mathbb{R}^d} \frac{\partial}{\partial t}\left(|u_t(\mathbf{x},t)|^2 + |\nabla u(\mathbf{x},t)|^2\right) d\mathbf{x} = \int_{\mathbb{R}^d} \left(u_t(\mathbf{x},t)u_{tt}(\mathbf{x},t) + \nabla u(\mathbf{x},t)\cdot\nabla u_t(\mathbf{x},t)\right) d\mathbf{x} \tag{8.8.1}$$

Let us work the second term of the previous expression. We will use Green Identity:

$$\int_{\mathbb{R}^d} \nabla u(\mathbf{x},t)\cdot\nabla u_t(\mathbf{x},t)d\mathbf{x} = \lim_{R\longrightarrow+\infty}\int_{|\mathbf{x}|\leq R} \nabla u(\mathbf{x},t)\cdot\nabla u_t(\mathbf{x},t)d\mathbf{x} = -\lim_{R\longrightarrow+\infty}\int_{|\mathbf{x}|\leq R} \triangle u(\mathbf{x},t)u_t(\mathbf{x},t)d\mathbf{x}$$

$$+\lim_{R\longrightarrow+\infty}\int_{|\mathbf{x}|=R} u_t(\mathbf{x},t)\nabla u(\mathbf{x},t)\cdot n dS(\mathbf{x}) = -\int_{\mathbb{R}^d} \triangle u(\mathbf{x},t)u_t(\mathbf{x},t)d\mathbf{x}$$

where we have used that $\lim\limits_{R\longrightarrow+\infty}\int_{|\mathbf{x}|=R} u_t(\mathbf{x},t)\nabla u(\mathbf{x},t)\cdot n dS(\mathbf{x}) = 0$ by doing a similar argument to the proof of Theorem 7.4., since $u_t(\cdot,t)$ and $\nabla u(\cdot,t)\cdot n$ are in $\mathcal{S}\left(\mathbb{R}^d\right)$ for all $t \geq 0$. Using this in (8.8.1)

$$\dot{E}(t) = \int_{\mathbb{R}^d} u_t(\mathbf{x},t)\left(u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t)\right) d\mathbf{x} = 0$$

Hence, $E(t)$ is constant and as is continuous on $t \geq 0$, we conclude that $E(t) = E(0)$ for all $t \geq 0$.

Theorem 8.7. (Uniqueness for wave equation) *There exists an unique solution* $u \in C^2(\mathbb{R}^d \times [0,+\infty))$ *of (8.3.3) with* g *and* h *in* $\mathcal{S}\left(\mathbb{R}^d\right)$ *such that* $u(\cdot,t) \in \mathcal{S}(\mathbb{R}^d)$ *for all* $t \geq 0$.

PROOF.

We have proved the existence of solution in Theorem 8.1. To prove uniqueness, we suppose there exist two solutions u and v satisfying the conditions of the statement. Then, we consider $w(\mathbf{x},t) = u(\mathbf{x},t) - v(\mathbf{x},t)$ and we notice that $w \in C^2(\mathbb{R}^d \times [0,+\infty))$, $w(\cdot,t) \in \mathcal{S}(\mathbb{R}^d)$ for all $t \geq 0$ and

$$\begin{cases} w_{tt}(\mathbf{x},t) - \triangle w(\mathbf{x},t) = 0 & \text{in } \mathbb{R}^d \times (0,+\infty) \\ w(\mathbf{x},0) = 0 & \mathbf{x} \in \mathbb{R}^d \\ w_t(\mathbf{x},0) = 0 & \mathbf{x} \in \mathbb{R}^d \end{cases}$$

Using theorem 8.6, the energy $E(t) = E(0)$ and $E(0) = \frac{1}{2}\int_{\mathbb{R}^d}\left(|w_t(\mathbf{x},0)|^2 + |\nabla w(\mathbf{x},0)|^2\right)d\mathbf{x} = 0$. Thus,

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d}\left(|w_t(\mathbf{x},t)|^2 + |\nabla w(\mathbf{x},t)|^2\right)d\mathbf{x} = 0 \quad \text{for all } t \geq 0$$

Consequently $w_t(\mathbf{x},t) = 0$ and $\nabla w(\mathbf{x},t) = 0$, and hence $w(\mathbf{x},t) = cte$ in $\mathbb{R}^d \times [0,+\infty)$. Finally as $w(\mathbf{x},0) = 0$ we conclude that $w(\mathbf{x},t) = 0$ in $\mathbb{R}^d \times [0,+\infty)$ and the solution is unique.

8.9 Huygens' Principle and finite propagation speed

Definition 8.1. *We define the domain of dependence of a solution* u *of the wave equation at a point* $(\mathbf{x}_0,t_0) \in \mathbb{R}^d \times [0,+,\infty)$ *as the set of points* $D(\mathbf{x}_0,t_0) \subset \mathbb{R}^d \times [0,+\infty)$ *upon which* $u(\mathbf{x}_0,t_0)$ *depends. In other words,* $u(\mathbf{x}_0,t_0)$ *depends on everything that has happened in the domain of dependence.*

Definition 8.2. *We define the domain of influence of a solution* u *of the wave equation at a point* $(\mathbf{x}_0,t_0) \in \mathbb{R}^d \times [0,+,\infty)$ *as the set of points* $I(\mathbf{x}_0,t_0) \subset \mathbb{R}^d$ *in which the solution* u *is influenced by* $u(\mathbf{x}_0,t_0)$. *In other words,* $u(\mathbf{x}_0,t_0)$ *influences the solution at all points in the range of influence.*

• **Case** $d = 1$

We observe from Alembert's formula (8.3.2) that the value of the solution at a point (x,t) depends only on the values of f and g in the interval centered at x of length $2t$.

- For the case $h = 0$, the domain of dependence of the solution at a point (x_0,t_0) and the domain of influence of the initial conditions at a point x are respectively

$$D(x_0,t_0) = \{(\mathbf{x},t) \in \mathbb{R}^d \times [0,+\infty) : 0 \leq t \leq t_0,\ |x - x_0| = t_0 - t\}$$

$$I(\mathbf{x}_0,0) = \{(\mathbf{x},t) \in \mathbb{R}^d \times [0,+\infty) : |\mathbf{x} - \mathbf{x}_0| = t\}$$

In red, we have plotted these respective domains:

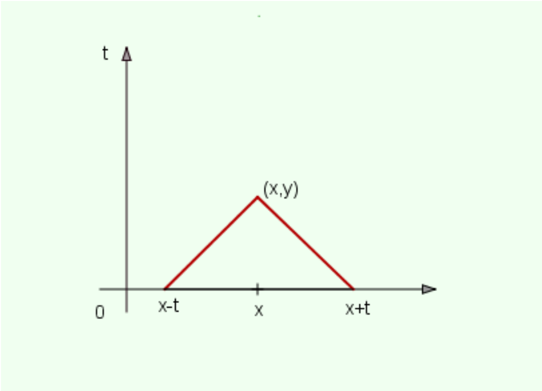


Figure 8.9.1: Domain of dependence

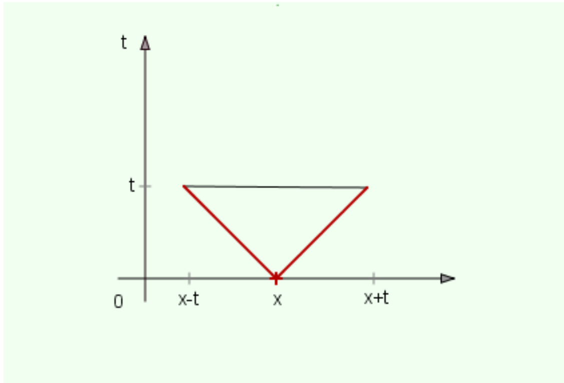


Figure 8.9.2: Domain of influence

- For the case $h \neq 0$, the domain of dependence of the solution at a point (x_0, t_0) and the domain of influence of the initial conditions at a point x are respectively

$$D(x_0, t_0) \;=\; \{(\mathbf{x}, t) \in \mathbb{R}^d \times [0, +\infty) : \, 0 \leq t \leq t_0, \, |x - x_0| \;\leq\; t_0 - t\}$$

$$I(\mathbf{x}_0, 0) \;=\; \{(\mathbf{x}, t) \in \mathbb{R}^d \times [0, +\infty) : \, |\mathbf{x} - \mathbf{x}_0| \;\leq\; t\}$$

In red, we have plotted these respective domains:

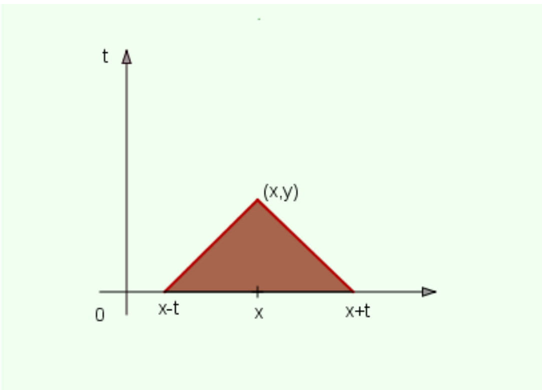


Figure 8.9.3: Domain of dependence

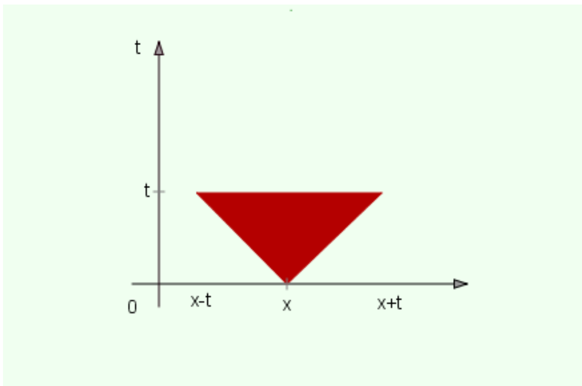


Figure 8.9.4: Domain of influence

• Case d odd

We observe from solution (8.6.2) for the case d odd ($d \geq 3$) that the solution at a point $\mathbf{x}_0$ and at time t_0 depends only on the initial data in an immediate neighborhood of the sphere $\partial B(\mathbf{x}_0, t_0)$. This is a statement of Huygens principle. In particular, it says that information from a point source travels in the form of a sphere. The wavefront is thus sharp, with a sudden onset, at the start, and sudden cutoff at the end.

We see that the domain of dependence of the solution at a point $(\mathbf{x}_0, t_0)$ is the cone (called the backward light cone)

$$D(\mathbf{x}_0, t_0) \;=\; \{(\mathbf{x}, t) \in \mathbb{R}^d \times [0, +\infty) : \, 0 \leq t \leq t_0, \, |\mathbf{x} - \mathbf{x}_0| \;=\; t_0 - t\}$$

In red it is plotted the domain of dependence.

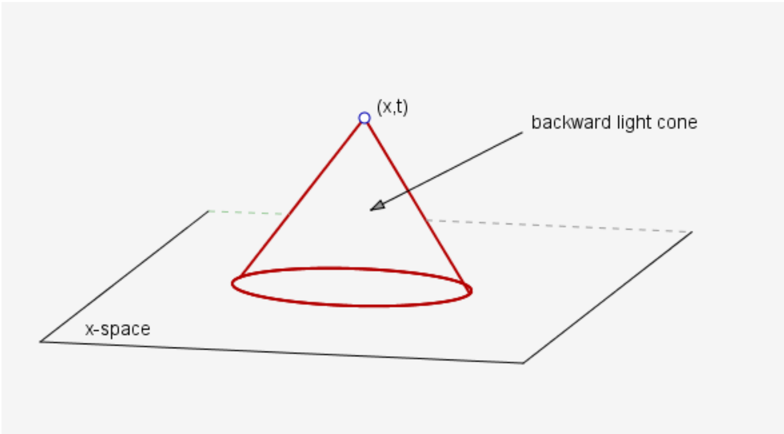


Figure 8.9.5: Dependence domain

In the same way, the domain of influence of the initial conditions in a point $\mathbf{x}_0$ is the cone (called the forward light cone)

$$I(\mathbf{x}_0, 0) = \{(\mathbf{x}, t) \in \mathbb{R}^d \times [0, +\infty) : |\mathbf{x} - \mathbf{x}_0| = t\}$$

In red it is plotted the domain of influence.

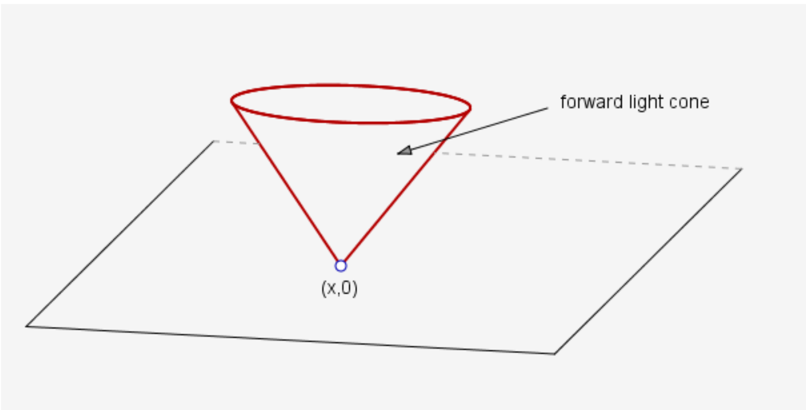


Figure 8.9.6: Influence Domain

- **Case d even**

We observe from the solution (8.7.5) for the case d even that the solution at a point $\mathbf{x_0}$ and at time t_0, depends only on the initial data on the entire $B(\mathbf{x}_0, t_0)$, and not just on $\partial B(\mathbf{x}_0, t_0)$. Hence, wavefronts do have a sharp onset, but they decay with a long tail. Huygens principle is not true in even dimensions. We see that in this case, the domain of dependence of the solution at a point $(\mathbf{x}_0, t_0)$ is

$$D(\mathbf{x}_0, t_0) = \{(\mathbf{x}, t) \in \mathbb{R}^d \times [0, +\infty) : 0 \leq t \leq t_0, |\mathbf{x} - \mathbf{x}_0| \leq t_0 - t\}$$

while the domain of influence of the initial conditions in a point $\mathbf{x}_0$ is

$$I(\mathbf{x}_0, 0) = \{(\mathbf{x}, t) \in \mathbb{R}^d \times [0, +\infty) : |\mathbf{x} - \mathbf{x}_0| \leq t\}$$

• Finite propagation speed

Another important aspect of the wave equation connected with these considerations is that of the finite speed of propagation. This means that if we have an initial disturbance localized at $\mathbf{x} = \mathbf{x}_0$, then after a finite time t, its effects will have propagated only inside the ball centered at $\mathbf{x}_0$ of radius t.

We already know this from the representation formulas (8.6.2) and (8.7.5). However, we don't need these formulas to conclude that. Energy methods provide a much simpler proof.

Theorem 8.8. (Finite propagation speed) *Let $u \in C^2\left(\mathbb{R}^d \times [0,+\infty)\right)$ be a solution of problem (8.3.3). If $u \equiv u_t \equiv 0$ on $B(\mathbf{x}_0, t_0) \times \{t = 0\}$, then $u \equiv 0$ within the cone*

$$C \quad = \quad \{(\mathbf{x}, t) \in \mathbb{R}^d \times [0, +\infty) : \, 0 \leq t \leq t_0, \, |\mathbf{x} - \mathbf{x}_0| \leq t_0 - t\}$$

In particular, we see that any "disturbance" originating outside $B(\mathbf{x}_0, t_0)$ has no effect on the solution within C, and consequently has finite propagation speed.

PROOF. The proof is similar to the proof of Theorem 8.7. We define

$$e(t) \quad := \quad \frac{1}{2} \int_{B(\mathbf{x}_0, t_0 - t)} \left(u_t^2(\mathbf{x}, t) + |\nabla u(\mathbf{x}, t)|^2 \right) d\mathbf{x} \quad \textit{for all} \quad 0 \leq t \leq t_0$$

and we work this expression to see that $\dot{e}(t) \leq 0$. Then, we see that $e(t) \leq e(0) = 0$ for all $0 \leq t \leq t_0$. Consequently $u_t = 0$ and $\nabla u = 0$ within the cone C and we conclude that $u = 0$ in C. For further details, see [Evans2010, pages 84-85].

Part III

Fourier Transform of distributions: application to filters

Chapter 9

Introduction to distributions

9.1 Definition of a distribution

Definition 9.1. *We define the space of test functions as the space $\mathcal{D}(\mathbb{R})$ which we defined as*

$$\mathcal{D}(\mathbb{R}) := \{\varphi : \mathbb{R} \to \mathbb{C} \ s.t \ \varphi \in C^{\infty}(\mathbb{R}), \ supp(\varphi) \text{ is bounded}\}$$

Definition 9.2. *We define a distribution as a continuous linear functional $T : \mathcal{D}(\mathbb{R}) \longrightarrow \mathbb{C}$. The value of T at φ will be denoted in either of two ways: $T(\varphi)$ or $< T, \varphi >$.*

Remark 9.1. *The continuity of T means the following: let $\{\varphi_n\}_{n\in\mathbb{N}} \in \mathcal{D}(\mathbb{R})$ such that $\varphi_n \overset{n\longrightarrow+\infty}{\longrightarrow} \varphi$ in $\mathcal{D}(\mathbb{R})$, then $T(\varphi_n) \overset{n\longrightarrow+\infty}{\longrightarrow} T(\varphi)$. We need to define the concept of convergence in $\mathcal{D}(\mathbb{R})$.*

Definition 9.3. *A sequence of elements $\{\varphi_n\}_{n\in\mathbb{N}} \in \mathcal{D}(\mathbb{R})$ tends to 0 in $\mathcal{D}(\mathbb{R})$ if the following hold:*

1. *The supports of all the φ_n are contained in a fixed compact interval.*
2. *$\{\varphi_n\}$ as well as all of the derived sequences tend to 0 uniformly on $\mathbb{R}$ as $n \longrightarrow +\infty$.*

Definition 9.4. *A sequence of elements $\{\varphi_n\}_{n\in\mathbb{N}} \in \mathcal{D}(\mathbb{R})$ tends to φ in $\mathcal{D}(\mathbb{R})$ if $\{\varphi_n - \varphi\}_{n\in\mathbb{N}}$ tends to 0 in $\mathcal{D}(\mathbb{R})$.*

Remark 9.2. *There does not exist a distance function, much less a norm, on $\mathcal{D}(\mathbb{R})$ that gives this notion of convergence. There is, however, a well-defined topology on $\mathcal{D}(\mathbb{R})$. It is sufficient for our purposes to have the notion of convergence.*

Proposition 9.1. *If a sequence of elements $\{\varphi_n\}$ in $\mathcal{D}(\mathbb{R})$ tends to 0 in $\mathcal{D}(\mathbb{R})$, then the sequence $\{\varphi_n'\}$ is in $\mathcal{D}(\mathbb{R})$ and tends to 0 in $\mathcal{D}(\mathbb{R})$.*

PROOF. Since $\{\varphi_n\}$ is $C^{\infty}(\mathbb{R})$ and has compact support, then $\{\varphi_n'\}$ is $C^{\infty}(\mathbb{R})$ and has compact support. Hence, $\{\varphi_n'\}$ is in $\mathcal{D}(\mathbb{R})$. Moreover, $\{\varphi_n\}$ tends to 0 in $\mathcal{D}(\mathbb{R})$, then the supports of all the φ_n are contained in a fixed compact interval and $\{\varphi_n\}$ and all of the derived sequences tend to 0 uniformly on $\mathbb{R}$ as $n \longrightarrow +\infty$. Thus, all these properties are inherited by $\{\varphi_n'\}$ and we conclude that $\{\varphi_n'\}$ tends to 0 in $\mathcal{D}(\mathbb{R})$.

Definition 9.5. *The set of continuous linear functionals on* $\mathcal{D}(\mathbb{R})$ *is called the topological dual of* $\mathcal{D}(\mathbb{R})$*. We denote it by* $\mathcal{D}^*(\mathbb{R})$*.*

Example 9.1. *We introduce three basic distributions that will play an important role in our future analysis.*

• *Point distributions:* let a be a real number. Then, the mapping defined on $\mathcal{D}(\mathbb{R})$ by $\delta_a(\varphi) = \varphi(a)$ is a distribution. It is clear that δ_a is linear. Moreover, it is continuous since if $\{\varphi_n\} \in \mathcal{D}(\mathbb{R})$ tends to φ in $\mathcal{D}(\mathbb{R})$, then $\{\varphi_n - \varphi\}_{n\in\mathbb{N}}$ tends uniformly to 0. Thus, $\delta_a(\varphi_n) = \varphi_n(a) \overset{n\longrightarrow+\infty}{\longrightarrow} \varphi(a) = \delta_a(\varphi)$.

• *Dirac delta* δ*:* it is the particular case of the point distribution with $a = 0$. We will denote $\delta = \delta_0$.

• *Dirac's comb:* let $(\lambda_n)_{n\in\mathbb{Z}}$ be a complex sequence and let $a > 0$. Then, the linear functional defined by $T = \sum_{n=-\infty}^{+\infty} \lambda_n \delta_{na}$ is a distribution. Note that it is well-defined since $\varphi \in \mathcal{D}(\mathbb{R})$ has compact support and the sum $T(\varphi)$ will be finite for each φ. Moreover, T is continuous. Let $\{\varphi_p\} \in \mathcal{D}(\mathbb{R})$ tends to φ in $\mathcal{D}(\mathbb{R})$. The supports of all the φ_p are contained in a fixed compact interval $[A, B]$. Then

$$T(\varphi_p) = \sum_{n=-\infty}^{+\infty} \lambda_n \varphi_p(na) = \sum_{A\le na\le B} \lambda_n \varphi_p(na) \overset{p\longrightarrow+\infty}{\longrightarrow} \sum_{A\le na\le B} \lambda_n \varphi(na) = T(\varphi)$$

9.2 Distributions as generalized functions

Proposition 9.2. *Let* f *be a locally integrable function on* $\mathbb{R}$*, then the functional* T_f *defined by*

$$T_f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx \quad \textit{for all} \quad \varphi \in \mathcal{D}(\mathbb{R})$$

is a distribution.

PROOF. Note that T_f is clearly a linear functional. It remains to show the continuity. Let $\{\varphi_n\} \in \mathcal{D}(\mathbb{R})$ tend to φ in $\mathcal{D}(\mathbb{R})$. In particular, $\{\varphi_n\}$ tends to φ uniformly. Assume $\operatorname{supp}(\varphi_n) = \operatorname{supp}(\varphi) \subset (a, b)$. Thus

$$|T_f(\varphi_n) - T_f(\varphi)| \le \int_a^b |f(x)|\,|\varphi_n(x) - \varphi(x)|\,dx \le \|\varphi_n - \varphi\|_\infty \int_a^b |f(x)|\,dx \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

We conclude that T_f is continuous. Thus, T_f is a distribution.

Proposition 9.3. *Let us define the mapping* $i : L^1_{loc}(\mathbb{R}) \longrightarrow \mathcal{D}^*(\mathbb{R})$ *by* $i(f) = T_f$*. Then,* i *is well-defined, is linear and is 1-to-1.*

PROOF.

Note that $i(f) = T_f \in \mathcal{D}^*(\mathbb{R})$ for all $f \in L^1_{loc}(\mathbb{R})$ (by proposition 9.2.). Moreover if $f, g \in L^1_{loc}(\mathbb{R})$ are equal almost everywhere, then $T_f = T_g$. Hence, i is well-defined.

Furthermore, i is clearly linear. It remains to show that it is 1-to-1. We will see that if $T_f = 0$ then $f = 0$ a.e. In effect, if $T_f = 0$ then $T_f(\varphi) = \int_{\mathbb{R}} f(x)\varphi(x)dx = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R})$ and appling Lemma 4.1. we get that $f = 0$ almost everywhere.

Remark 9.3. *From Proposition 9.3. we conclude that each distribution T_f is uniquely identified with the locally integrable function f. Hence, we can make the identification $f \leftrightarrow T_f$. With this identification, and doing an abuse of notation, we see that $L^1_{loc}(\mathbb{R}) \subset \mathcal{D}^*(\mathbb{R})$.*

Definition 9.6. *Let $f \in L^1_{loc}(\mathbb{R})$. The distributions T_f defined in Proposition 9.3. are called generalized functions or regular distributions.*

Example 9.2. *We introduce two basic generalized functions*

1. *Unit step function (Heaviside's function):* let u be the Heaviside's function. Its associated distribution is defined by
$$T_u(\varphi) \;=\; \int_0^{+\infty} \varphi(x)dx$$
2. *The constant function $f = K \in \mathbb{C}$:* Its associated generalized function is given by
$$T_f(\varphi) \;=\; K\int_{\mathbb{R}} \varphi(x)dx$$

9.3 Elementary Operations on Distributions

Definition 9.7. (reflection) *The reflection T_σ of a distribution T is defined by $T_\sigma(\varphi) = T(\varphi_\sigma)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. It holds that T_σ is a well-defined distribution. A distribution T is said to be even if $T_\sigma = T$ and odd if $T_\sigma = -T$.*

Example 9.3. *The Dirac Delta is even. In effect, $\delta_\sigma(\varphi) = \delta(\varphi_\sigma) = \varphi_\sigma(0) = \varphi(0) = \delta(\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. Hence, $\delta_\sigma = \delta$.*

Definition 9.8. (translation) *Let $a \in \mathbb{R}\setminus\{0\}$. The translate $\tau_a T$ of a distribution T is defined by $\tau_a T(\varphi) = T(\tau_{-a}\varphi)$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. It holds that $\tau_a T$ is a well-defined distribution. A distribution T is said to be periodic with period $a \neq 0$ if $\tau_a T = T$.*

Example 9.4. *The Dirac's comb with coefficients $\lambda_n = 1$ is a periodic. In effect, we see that*

$$\tau_a T(\varphi) \;=\; \sum_{n=-\infty}^{+\infty} \delta_{na}(\tau_{-a}\varphi) = \sum_{n=-\infty}^{+\infty} \varphi((n+1)a) = \sum_{n=-\infty}^{+\infty} \varphi(na) = T(\varphi)$$

Definition 9.9. *A distribution is said to be null (be zero, vanish) on an open set Ω if $T(\varphi) = 0$ for all test functions φ such that $supp(\varphi) \subset \Omega$.*

Definition 9.10. (support) *The support of a distribution T, supp(T), is defined to be the complement of the largest open set on which T is null.*

Example 9.5. *$Supp\left(\sum_{i=1}^{n} \lambda_i\delta_{a_i}\right) = \{a_1, ..., a_n\}$ with $\lambda_i \in \mathbb{C}\setminus\{0\}$ and $a_i \in \mathbb{R}$.*

Proposition 9.4. *If T_f is a generalized function, then supp(T_f) = supp(f).*

PROOF. The proof can be found in [C. Gasquet2010, page 253].

Definition 9.11. *The space of distributions whose supports lie to the right of some finite point is denoted by $\mathcal{D}^*_+(\mathbb{R})$. That is*

$$\mathcal{D}^*_+(\mathbb{R}) = \{T \in \mathcal{D}^*(\mathbb{R}) \ \ s.t. \ supp(T) \subset [t_0, +\infty) \ \ for\ some\ t_0 \in \mathbb{R}\}$$

Definition 9.12. (product of a distribution and a function) *The product of a distribution T by an infinitely differentiable function g, denoted by gT, is defined by $< gT, \varphi > = < T, g\varphi >$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. It holds that gT is a well-defined distribution.*

Example 9.6. *Let $g \in C^\infty(\mathbb{R})$. Note that $< g\delta_a, \varphi > = < \delta_a, g\varphi > = g(a)\varphi(a) = < g(a)\delta_a, \varphi >$. Hence the product of the point distribution by g is $g\delta_a = g(a)\delta_a$.*

9.4 The derivative of a distribution

Definition 9.13. (derivative of a distribution) *The derivative of a distribution T, denoted by T', is defined by $< T', \varphi > = - < T, \varphi' >$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.*

Proposition 9.5. *The derivative of a distribution is a distribution.*

PROOF.

Clearly T' is a linear functional defined on $\mathcal{D}(\mathbb{R})$ since $\varphi' \in \mathcal{D}(\mathbb{R})$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. It remains to prove the continuity. Let us take $\{\varphi_n\}_{n\in\mathbb{N}} \in \mathcal{D}(\mathbb{R})$ such that $\varphi_n \overset{n\longrightarrow+\infty}{\longrightarrow} \varphi$ in $\mathcal{D}(\mathbb{R})$. By proposition 9.1. it holds that $\{\varphi'_n\}_{n\in\mathbb{N}} \in \mathcal{D}(\mathbb{R})$ and $\varphi'_n \overset{n\longrightarrow+\infty}{\longrightarrow} \varphi'$ in $\mathcal{D}(\mathbb{R})$. Hence, using the continuity of T

$$< T', \varphi_n > = - < T, \varphi'_n > \overset{n\longrightarrow+\infty}{\longrightarrow} - < T, \varphi' > = < T', \varphi >$$

and this proves the continuity of T'.

Proposition 9.6. *Each distribution T is infinitely times differentiable, and the nth derivative of T satisfies the relation $< T^{(n)}, \varphi > = (-1)^n < T, \varphi^{(n)} >$ for all $\varphi \in \mathcal{D}(\mathbb{R})$.*

PROOF.

It is clear from Definition 9.13. and Proposition 9.5. We just have to apply induction.

Example 9.7. *We introduce the following two examples:*

- *Derivative of the point distribution:* note that $\delta'_a(\varphi) = < \delta'_a, \varphi > = - < \delta_a, \varphi' > = -\varphi'(a)$.
- *Derivative of the unit step u:* we will compute its derivative in the sense of distribution, i.e. the derivative of its generalized function T_u. Note that $< T'_u, \varphi > = - < T_u, \varphi' >$. Using example 9.2.1

$$< T'_u, \varphi > = - \int_0^{+\infty} \varphi'(x)dx = -[\varphi(x)]_0^{+\infty} = \varphi(0) = < \delta, \varphi >$$

Thus, we get that the derivative of the unit step u in the sense of distribution is the Dirac Delta δ.

Proposition 9.7. *Let f be a function that is absolutely continuous on all compact intervals $[a,b]$ (in particular $f \in L^1_{loc}(\mathbb{R})$). Then the derivative of f in the sense of distributions agrees a.e. with the associated distribution of the usual, or ordinary, derivative. That is, $T_f^{'} = T_{f'}$.*

PROOF.

Note that as f is absolutely continuous on all compact intervals we can conclude that f is continuous. Hence, we will be able to apply integration by parts

$$< T_f^{'}, \varphi > \; = \; - < T_f, \varphi^{'} > \; = \; - \int_{\mathbb{R}} f(x)\varphi^{'}(x)dx \overset{(parts)}{=} \int_{\mathbb{R}} f^{'}(x)\varphi(x)dx \; = \; < T_{f'}, \varphi >$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. Hence, by Lemma 4.1. it follows that $T_f^{'} = T_{f'}$.

Proposition 9.8. (derivative in the sense of a discontinuous function) *Let f be continuously differentiable on the invervals $(-\infty, a) \cup (a, +\infty)$ such that has finite left and right limits at $a \in \mathbb{R}$, which we denote by $f(a^-)$ and $f(a^+)$ respectively. Then, $T_f^{'} = T_{f'} + (f(a^+) - f(a^-))\,\delta_a$.*

PROOF.

As f is in $C^1((-\infty, a) \cup (a, +\infty))$, we will be able to apply integration by parts

$$< T_f^{'}, \varphi >= - \int_{-\infty}^{a} f(x)\varphi^{'}(x)dx - \int_{a}^{+\infty} f(x)\varphi^{'}(x)dx \overset{(parts)}{=} -f(a^-)\varphi(a) + \int_{-\infty}^{a} f^{'}(x)\varphi(x)dx + f(a^+)\varphi(a)$$

$$+ \int_{a}^{+\infty} f^{'}(x)\varphi(x)dx = \int_{\mathbb{R}} f^{'}(x)\varphi(x)dx + \left(f(a^+) - f(a^-)\right)\varphi(a) =< T_{f'}, \varphi > + \left(f(a^+) - f(a^-)\right)\delta_a(\varphi)$$

Hence, we conclude that $T_f^{'} = T_{f'} + (f(a^+) - f(a^-))\,\delta_a$.

9.5 Convergence of a sequence of distributions

Definition 9.14. (convergence) *A sequence of distributions $(T_n)_{n\in\mathbb{N}}$ is said to converge to a distribution T if*

$$< T_n, \varphi > \overset{n \longrightarrow \infty}{\longrightarrow} < T, \varphi > \qquad \textit{for all} \;\; \varphi \in \mathcal{D}(\mathbb{R})$$

Example 9.8. *Let $(a_n)_{n\in\mathbb{N}}$ be a real sequence which converges to a. Then, using the continuity of $\varphi \in \mathcal{D}(\mathbb{R})$ we get that $< \delta_{a_n}, \varphi >= \varphi(a_n) \overset{n\longrightarrow\infty}{\longrightarrow} \varphi(a) =< \delta_a, \varphi >$. Thus, we conclude that δ_{a_n} converges to δ_a.*

Theorem 9.1. (continuity of derivation) *If the sequence of distributions $(T_n)_{n\in\mathbb{N}}$ converges to the distribution T, then the sequence of derivatives $(T_n^{'})_{n\in\mathbb{N}}$ converges to $T^{'}$.*

PROOF. Note that this follows directly from definition 9.13. and definition 9.14. In effect, for all $\varphi \in \mathcal{D}(\mathbb{R})$.

$$< T_n^{'}, \varphi > \; = \; - < T_n, \varphi^{'} > \overset{n\longrightarrow+\infty}{\longrightarrow} - < T, \varphi^{'} > \; = \; < T^{'}, \varphi >$$

Proposition 9.9. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of integrable functions that converges in $L^1(\mathbb{R})$ to $f \in L^1(\mathbb{R})$. Then (f_n) converges to f in the sense of distributions.*

PROOF. The result is a consequence of the following inequalities

$$|< T_{f_n}, \varphi > - < T_f, \varphi >| = \left| \int_{\mathbb{R}} (f_n(x) - f(x))\, \varphi(x) dx \right| \leq \int_{\mathbb{R}} |f_n(x) - f(x)|\, |\varphi(x)|\, dx \leq$$

$$\leq \|f_n - f\|_1 \, \|\varphi\|_\infty \overset{n \longrightarrow +\infty}{\longrightarrow} 0$$

Proposition 9.10. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence of square-integrable functions that converges in $L^2(\mathbb{R})$ to $f \in L^2(\mathbb{R})$. Then (f_n) converges to f in the sense of distributions.*

PROOF. We recall that $\mathcal{D}(\mathbb{R}) \subset L^2(\mathbb{R})$. Hence, applying Hölder's inequality we get

$$|< T_{f_n}, \varphi > - < T_f, \varphi >| = \left| \int_{\mathbb{R}} (f_n(x) - f(x))\, \varphi(x) dx \right| \leq \int_{\mathbb{R}} |f_n(x) - f(x)|\, |\varphi(x)|\, dx \leq$$

$$\leq \|f_n - f\|_2 \, \|\varphi\|_2 \overset{n \longrightarrow +\infty}{\longrightarrow} 0$$

Remark 9.4. *Note that we see in Proposition 9.9. and Proposition 9.10. that the convergence of a sequence of functions in the sense of distribution is generally "weaker" than the notion of uniform convergence.*

9.6 Primitives of a Distribution

Definition 9.15. (primitive) *Let $T \in \mathcal{D}^*(\mathbb{R})$, we say that $U \in \mathcal{D}^*(\mathbb{R})$ is a primitive of T if $U' = T$.*

Theorem 9.2. *The derivative of a distribution U is the zero element of $\mathcal{D}^*(\mathbb{R})$ if and only if U is a constant, that is, if and only if there exists $K \in \mathbb{C}$ such that $U = T_K$.*

PROOF.

First, let us make the following observation. If $\varphi \in \mathcal{D}(\mathbb{R})$, then $\varphi' \in \mathcal{D}(\mathbb{R})$ and $\int_{\mathbb{R}} \varphi'(x)dx = 0$. Conversely, if $\psi \in \mathcal{D}(\mathbb{R})$ such that $\int_{\mathbb{R}} \psi(x)dx = 0$, then the function $\varphi(x) = \int_{-\infty}^{x} \psi(t)dt$ is in $\mathcal{D}(\mathbb{R})$ and is a primitive of ψ. Thus, let us define

$$\mathcal{D}_0(\mathbb{R}) = \left\{ \varphi' \text{ s.t. } \varphi \in \mathcal{D}(\mathbb{R}) \right\} = \left\{ \psi \text{ s.t. } \psi \in \mathcal{D}(\mathbb{R}), \int_{\mathbb{R}} \psi(x)dx = 0 \right\}$$

• To prove the direct implication, we assume that $U' = 0$, that is $< U', \varphi > = - < U, \varphi' > = 0$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. Then, for all $\psi \in \mathcal{D}_0(\mathbb{R})$, $< U, \psi > = 0$. Let us consider $\rho \in \mathcal{D}(\mathbb{R})$ such that

$$\int_{\mathbb{R}} \rho(x)dx = 1 \quad \text{and} \quad \rho(x) = 0 \quad \text{if} \quad |x| \geq 1$$

Now, for each $\varphi \in \mathcal{D}(\mathbb{R})$ we define $\psi_\varphi := \varphi - I(\varphi)\rho$, where $I(\varphi) = \int_{\mathbb{R}} \varphi(x)dx$. Note that

$$\int_{\mathbb{R}} \psi_\varphi(x)dx = 0 \quad \text{and} \quad \psi_\varphi \in \mathcal{D}(\mathbb{R})$$

hence, $\psi_\varphi \in \mathcal{D}_0(\mathbb{R})$ and consequently $< U, \psi_\varphi >= 0$. Thus

$$< U, \varphi > \; = \; < U, \psi_\varphi + I(\varphi)\rho > \; = \; I(\varphi) < U, \rho > \; = \; K \cdot \int_{\mathbb{R}} \varphi(x)dx \; = < T_K, \varphi >$$

where $K = < U, \rho >$ is a constant which belongs to $\mathbb{C}$. Thus, we conclude that $U = T_K$.

• Conversely, assume that there exists $K \in \mathbb{C}$ s.t. $U = T_K$. Then, we conclude that $U' = 0$. In effect:

$$< U', \varphi > \; = \; - < U, \varphi' > \; = \; - < T_K, \varphi' > \; = \; -K \int_{\mathbb{R}} \varphi'(x)dx \; = 0 \quad \text{for all } \varphi \in \mathcal{D}(\mathbb{R})$$

Theorem 9.3. *Every distribution T has a primitive $U \in \mathcal{D}^*(\mathbb{R})$ and all the primitives of T are of the form $U + T_C$, where C is some constant in $\mathbb{C}$.*

PROOF. The proof of this theorem can be found in [C. Gasquet2010, pages 276, 277].

Chapter 10

The Fourier Transform of distributions

In this chapter, we will extend the Fourier transform to distributions. We motivate it by studying the case of generalized functions. Let us consider $f \in L^1(\mathbb{R})$ and let $\varphi \in \mathcal{D}(\mathbb{R})$. We note that by applying Fubini's Theorem we get

$$< T_{\hat{f}}, \varphi > = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} f(x) e^{-2\pi i x \xi} dx \right) \varphi(\xi) d\xi = \int_{\mathbb{R}} f(x) \left(\int_{\mathbb{R}} e^{-2\pi i x \xi} \varphi(\xi) d\xi \right) dx = \int_{\mathbb{R}} f(x) \hat{\varphi}(x) dx = < T_f, \hat{\varphi} >$$

This suggest that the Fourier transform for a distribution T should by defined by $< \widehat{T}, \varphi > = < T, \hat{\varphi} >$. But this expression only makes sense if $\hat{\varphi} \in \mathcal{D}(\mathbb{R})$. It holds that $\hat{\varphi} \in C^\infty(\mathbb{R})$ but we will see that $\hat{\varphi}$ has never compact support. We have, however, shown that the Schwartz Space $\mathcal{S}(\mathbb{R})$ is invariant under the Fourier transform. This leads to the introduction of the subspace of tempered distribution.

10.1 The Space $\mathcal{S}^*(\mathbb{R})$ of tempered distributions

Definition 10.1. *We define the space of tempered distributions $\mathcal{S}^*(\mathbb{R})$ as the vector space of continuous linear functionals T defined on $\mathcal{S}(\mathbb{R})$. Note that the continuity of T means*

$$\text{If } \varphi_n \overset{n \longrightarrow \infty}{\longrightarrow} \varphi \quad \text{in } \mathcal{S}(\mathbb{R}) \quad \text{then} \quad < T, \varphi_n > \overset{n \longrightarrow \infty}{\longrightarrow} < T, \varphi > \quad \text{in } \mathbb{C}$$

Remark 10.1. *Let $T \in \mathcal{S}^*(\mathbb{R})$. Since $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ then $T(\varphi)$ is well-defined for all $\varphi \in \mathcal{D}(\mathbb{R})$. Furthermore, convergence in $\mathcal{D}(\mathbb{R})$ implies convergence in $\mathcal{S}(\mathbb{R})$. Consequently, the elements of $\mathcal{S}^*(\mathbb{R})$ restricted to $\mathcal{D}(\mathbb{R})$ are distributions. Since $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$, we can identify $\mathcal{S}^*(\mathbb{R})$ with a subspace of $\mathcal{D}^*(\mathbb{R})$.*

Definition 10.2. *The elements of $\mathcal{S}^*(\mathbb{R})$ are called tempered distributions.*

Proposition 10.1. *Suppose that T is a distribution, i.e. $T \in \mathcal{D}^*(\mathbb{R})$. Then T is a tempered distribution, $T \in \mathcal{S}^*(\mathbb{R})$, if and only if T is continuous on $\mathcal{D}(\mathbb{R})$ in the topology of $\mathcal{S}(\mathbb{R})$.*

PROOF.

Clearly the condition is necessary since if $T \in \mathcal{S}^*(\mathbb{R})$ then for $\varphi_n \in \mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$ it holds that if $\varphi_n \overset{n \longrightarrow \infty}{\longrightarrow} \varphi$ in $\mathcal{S}(\mathbb{R})$ then $< T, \varphi_n > \overset{n \longrightarrow \infty}{\longrightarrow} < T, \varphi >$.

The proof in the other direction depends on the fact that $\mathcal{D}(\mathbb{R})$ is dense in $\mathcal{S}(\mathbb{R})$. With a full understanding of the topologies of $\mathcal{D}(\mathbb{R})$ and $\mathcal{S}(\mathbb{R})$ the result follows directly. However, we have not introduced these topologies since we are mainly interested in convergence of sequences. For this reason, we will not make the proof. It can be found in [Khoan1972].

Definition 10.3. **(convergence of sequences in $\mathcal{S}^*(\mathbb{R})$)** *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence in $\mathcal{S}^*(\mathbb{R})$. We say that $\{T_n\}_{n\in\mathbb{N}}$ tends to T in $\mathcal{S}^*(\mathbb{R})$ if*

$$\lim_{n\longrightarrow+\infty} < T_n, \varphi > \; = \; < T, \varphi > \quad \textit{for all} \;\; \varphi \in \mathcal{S}(\mathbb{R})$$

Remark 10.2. *The convergence of a sequence in $\mathcal{S}^*(\mathbb{R})$ implies convergence in $\mathcal{D}^*(\mathbb{R})$ since $\mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R})$.*

Proposition 10.2. *If T is a tempered distribution, then the following statements hold for each $k \in \mathbb{N}$:*

1. *$x^k T$ is in $\mathcal{S}^*(\mathbb{R})$ and the mapping $T \longrightarrow x^k T$ is continuous from $\mathcal{S}^*(\mathbb{R})$ to $\mathcal{S}^*(\mathbb{R})$.*
2. *The derivative $T^{(k)}$ is in $\mathcal{S}^*(\mathbb{R})$ and the mapping $T \longrightarrow T^{(k)}$ is continuous from $\mathcal{S}^*(\mathbb{R})$ to $\mathcal{S}^*(\mathbb{R})$.*

PROOF.

• Proof 1: by definition 9.12. $x^k T$ is a distribution. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathbb{R})$ which converges to $\varphi \in \mathcal{D}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$, then by proposition 2.4.2, $x^k \varphi_n(x) \overset{n\longrightarrow\infty}{\longrightarrow} x^k \varphi(x)$ in $\mathcal{S}(\mathbb{R})$. Hence, as $T \in \mathcal{S}^*(\mathbb{R})$:

$$\left| < x^k T, \varphi_n > - < x^k T, \varphi > \right| \; = \; \left| < T, x^k (\varphi_n - \varphi) > \right| \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

Thus, $x^k T$ is continuous on $\mathcal{D}(\mathbb{R})$ with the topology of $\mathcal{S}(\mathbb{R})$ and by proposition 10.1. we conclude that $x^k T \in \mathcal{S}^*(\mathbb{R})$. To prove the continuity of the mapping $T \longrightarrow x^k T$, let us take $\{T_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}^*(\mathbb{R})$ such that tends to T in $\mathcal{S}^*(\mathbb{R})$. Then for all $\varphi \in \mathcal{S}(\mathbb{R})$

$$< x^k T_n, \varphi > \; = \; < T_n, x^k \varphi > \overset{n\longrightarrow\infty}{\longrightarrow} < T, x^k \varphi > \; = \; < x^k T, \varphi >$$

• Proof 2: by proposition 9.5. $T^{(k)}$ is a distribution. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathbb{R})$ which converges to $\varphi \in \mathcal{D}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$, then by proposition 2.4.1, $\varphi_n^{(k)}(x) \overset{n\longrightarrow\infty}{\longrightarrow} \varphi^{(k)}(x)$ in $\mathcal{S}(\mathbb{R})$. Hence, as $T \in \mathcal{S}^*(\mathbb{R})$:

$$\left| < T^{(k)}, \varphi_n > - < T^{(k)}, \varphi > \right| \; = \; \left| < T, \varphi_n^{(k)} - \varphi^{(k)} > \right| \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

Thus, $T^{(k)}$ is a continuous on $\mathcal{D}(\mathbb{R})$ with the topology of $\mathcal{S}(\mathbb{R})$ and by proposition 10.1. we conclude that $T^{(k)} \in \mathcal{S}^*(\mathbb{R})$. To prove the continuity of the mapping $T \longrightarrow T^{(k)}$ we apply the same technique we have used in 1.

Proposition 10.3. *Let f be in $L^p(\mathbb{R})$, $p \geq 1$. Then its generalized function T_f is a tempered distribution.*

PROOF.

Note that it holds that $L^p(\mathbb{R}) \subset L^1_{loc}(\mathbb{R})$. Hence, for any element $f \in L^p(\mathbb{R})$, T_f is a distribution. We will apply Proposition 10.1. Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathbb{R})$ which converges to $\varphi \in \mathcal{D}(\mathbb{R})$ in $\mathcal{S}(\mathbb{R})$. Note that

$$|< T_f, \varphi_n > - < T_f, \varphi >| \leq \int_{\mathbb{R}} |f(x)| \, |\varphi_n(x) - \varphi(x)| \, dx \leq \|f\|_p \, \|\varphi_n - \varphi\|_q = \|f\|_p \, \|(\varphi_n - \varphi)^q\|_1^{\frac{1}{q}} \tag{10.1.1}$$

where we have used Hölder's inequality since $f \in L^p(\mathbb{R})$ and $(\varphi_n - \varphi) \in \mathcal{D}(\mathbb{R}) \subset \mathcal{S}(\mathbb{R}) \subset L^q(\mathbb{R})$. Moreover, note that $(\varphi_n - \varphi)^q \stackrel{n\to\infty}{\longrightarrow} 0$ in $\mathcal{S}(\mathbb{R})$ and by proposition 2.4.3 we have that $\|(\varphi_n - \varphi)^q\|_1 \stackrel{n\to\infty}{\longrightarrow} 0$. Hence, $|< T_f, \varphi_n > - < T_f, \varphi >| \stackrel{n\to\infty}{\longrightarrow} 0$. Thus, T_f is continuous on $\mathcal{D}(\mathbb{R})$ with the topology of $\mathcal{S}(\mathbb{R})$ and by proposition 10.1. we conclude that $T_f \in \mathcal{S}^*(\mathbb{R})$.

10.2 The Fourier transform on $\mathcal{S}^*(\mathbb{R})$

Definition 10.4. *Let $T \in \mathcal{S}^*(\mathbb{R})$. The Fourier transform of T, which we will denote by $\widehat{T}$, is defined*

$$< \widehat{T}, \varphi > = < T, \widehat{\varphi} > \quad \textit{for all} \quad \varphi \in \mathcal{S}(\mathbb{R})$$

Remark 10.3. *The definition makes sense since the Fourier transform is invariant in $\mathcal{S}(\mathbb{R})$.*

Remark 10.4. *$\widehat{T}$ is a tempered distribution because the Fourier transform is a continuous operator on $\mathcal{S}(\mathbb{R})$ (Theorem 2.2.).*

Proposition 10.4. *Let f be in $L^1(\mathbb{R})$ then $\widehat{T_f} = T_{\hat{f}}$. Analogously, if f is $L^2(\mathbb{R})$ then $\widehat{T_f} = T_{\mathcal{F}f}$.*

PROOF.

Note first that by Proposition 10.3. T_f is a tempered distribution in both cases.

• Case $f \in L^1(\mathbb{R})$: by property 1.6. we have that $\hat{f} \in L^\infty(\mathbb{R})$. Hence, by Proposition 10.3. $T_{\hat{f}}$ is a tempered distribution. Moreover, by Property 1.8.

$$< T_{\hat{f}}, \varphi > = \int_{\mathbb{R}} \hat{f}(\xi) \, \varphi(\xi) \, d\xi = \int_{\mathbb{R}} f(x)\widehat{\varphi}(x)dx = < T_f, \widehat{\varphi} > = < \widehat{T_f}, \varphi >$$

• Case $f \in L^2(\mathbb{R})$: as by Theorem 4.1. $\mathcal{F}f \in L^2(\mathbb{R})$ then $T_{\mathcal{F}f}$ is a tempered distribution. Moreover as $\varphi \in \mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R})$, we can apply Proposition 4.4.

$$< T_{\mathcal{F}f}, \varphi > = \int_{\mathbb{R}} \mathcal{F}f(\xi) \, \varphi(\xi) \, d\xi = \int_{\mathbb{R}} f(x)\widehat{\varphi}(x)dx = < T_f, \widehat{\varphi} > = < \widehat{T_f}, \varphi >$$

Theorem 10.1. *The Fourier transform is a linear, 1-to-1, bicontinuous mapping from $\mathcal{S}^*(\mathbb{R})$ to $\mathcal{S}^*(\mathbb{R})$. The inverse mapping $\check{\cdot}$ is defined by*

$$< \check{T}, \varphi > = < T, \check{\varphi} > \quad \textit{for all} \quad \varphi \in \mathcal{S}(\mathbb{R})$$

Moreover, for all $T \in \mathcal{S}^*(\mathbb{R})$ it holds that $\check{\hat{T}} = \hat{\check{T}} = T$.

PROOF.
The mapping $\hat{}$: $\mathcal{S}^*(\mathbb{R}) \longrightarrow \mathcal{S}^*(\mathbb{R})$ defined by $<\hat{T},\varphi> = <T,\widehat{\varphi}>$ is clearly linear. It is also continuous, since if $(T_n)_{n\in\mathbb{N}}$ in $\mathcal{S}^*(\mathbb{R})$ converges to T in $\mathcal{S}^*(\mathbb{R})$ then:

$$<\widehat{T_n},\varphi> \;=\; <T_n,\widehat{\varphi}> \overset{n\longrightarrow+\infty}{\longrightarrow} <T,\widehat{\varphi}> \;=\; <\widehat{T},\varphi> \quad \text{for all} \quad \varphi\in\mathcal{S}(\mathbb{R})$$

i.e. $\widehat{T_n}$ converges to $\widehat{T}$ in $\mathcal{S}^*(\mathbb{R})$. The same argument works for $\check{}$. Finally, using Theorem 2.2.

$$<\check{\hat{T}},\varphi> \;=\; <T,\hat{\check{\varphi}}> \;=\; <T,\check{\hat{\varphi}}> \;=\; <\hat{\check{T}},\varphi> \;=\; <T,\varphi> \quad \text{for all} \quad \varphi\in\mathcal{S}(\mathbb{R})$$

Thus $\check{\hat{T}} = \hat{\check{T}} = T$, $\hat{}$ is 1-to-1 and its inverse mapping is $\check{}$.

Proposition 10.5. *Let T be a tempered distribution. Then, the following statements hold:*

1. *For all $k\in\mathbb{N}$, $\widehat{T}^{(k)} = [\widehat{(-2i\pi x)^k T}]$ and $\widehat{T^{(k)}} = (2i\pi\xi)^k\widehat{T}$.*
2. *For $a\in\mathbb{R}$, $\tau_a\widehat{T} = [\widehat{e^{2i\pi ax}T}]$ and $\widehat{\tau_a T} = e^{-2i\pi a\xi}\widehat{T}$.*

PROOF.
• Proof 1: note that by Proposition 10.2.2 $x^kT\in\mathcal{S}^*(\mathbb{R})$. Thus, $\widehat{x^kT}$ exists and satisfies for all $\varphi\in\mathcal{S}(\mathbb{R})$

$$<\widehat{x^kT},\varphi> \;=\; <x^kT,\widehat{\varphi}> \;=\; <T,x^k\widehat{\varphi}> \;=\; \frac{1}{(2i\pi)^k} <T,\widehat{\varphi^{(k)}}>$$

where we have used that by Proposition 1.1.2 $\widehat{\varphi^{(k)}}(x) = (2i\pi x)^k\,\widehat{\varphi}(x)$. Finally, using the definition of derivative of a distribution

$$<\widehat{x^kT},\varphi> \;=\; \frac{1}{(2i\pi)^k} <\widehat{T},\varphi^{(k)}> \;=\; \frac{(-1)^k}{(2i\pi)^k} <\widehat{T}^{(k)},\varphi> \quad \text{for all} \quad \varphi\in\mathcal{S}(\mathbb{R})$$

hence we conclude that $\widehat{T}^{(k)} = [\widehat{(-2i\pi x)^k T}]$. The relation $\widehat{T^{(k)}} = (2i\pi\xi)^k\widehat{T}$ is obtained in a similar way using Proposition 1.1.1.

• Proof 2: the function $x\longmapsto e^{2\pi iax}$ is $C^\infty(\mathbb{R})$. Proceeding similarly as the proof of Proposition 10.2. we get $e^{2\pi iax}T\in\mathcal{S}^*(\mathbb{R})$. Thus $\widehat{e^{2\pi iax}T}$ exists and satisfies

$$<\widehat{e^{2\pi iax}T},\varphi> \;=\; <e^{2\pi iax}T,\widehat{\varphi}> \;=\; <T,e^{2\pi iax}\widehat{\varphi}> \quad \text{for all} \quad \varphi\in\mathcal{S}(\mathbb{R})$$

Now, using Property 1.2. we get that $e^{2\pi iax}\widehat{\varphi}(x) = \widehat{\tau_{-a}\varphi}(x)$. Hence:

$$<\widehat{e^{2\pi iax}T},\varphi> \;=\; <T,\widehat{\tau_{-a}\varphi}> \;=\; <\widehat{T},\tau_{-a}\varphi> \;=\; <\tau_a\widehat{T},\varphi> \quad \text{for all} \quad \varphi\in\mathcal{S}(\mathbb{R})$$

hence we conclude that $\tau_a\widehat{T} = [\widehat{e^{2i\pi ax}T}]$. Analogously, we prove that $\widehat{\tau_aT} = e^{-2i\pi a\xi}\widehat{T}$.

Proposition 10.6. *Let T be a tempered distribution. Then, the following statements hold:*

1. $[\widehat{T_\sigma}] = \left(\hat{T}\right)_\sigma = \check{T}$

2. $\widehat{\widehat{T}} = T_\sigma$

PROOF. The proof is similar to the one made in Proposition 10.5. See [C. Gasquet2010, page 289].

Example 10.1. **(Fourier Transform of δ_a)** *It is easy to show that δ_a is a tempered distribution. Hence, $\widehat{\delta_a}$ exists and for all $\varphi \in \mathcal{S}(\mathbb{R})$*

$$< \widehat{\delta_a}, \varphi > \; = \; < \delta_a, \widehat{\varphi} > \; = \; \widehat{\varphi}(a) \; = \; \int_{\mathbb{R}} e^{-2i\pi ax} \varphi(x) dx \; = \; < T_{e^{-2i\pi ax}}, \varphi >$$

Thus we conclude that $\widehat{\delta_a} = T_{e^{-2i\pi ax}}$. In particular we have that the Fourier transform of the Dirac's distribution is $\widehat{\delta} = T_1$.

10.3 The space $\mathcal{K}^*(\mathbb{R})$ of distributions with compact support

We saw in Proposition 1.1.3 that if $f \in L^1(\mathbb{R})$ has compact support then $\hat{f} \in C^\infty(\mathbb{R})$. We will show that $\widehat{T}$ is infinitely times continuously differentiable for distributions with compact support.

Definition 10.5. *We define the space $\mathcal{K}^*(\mathbb{R})$ as the subspace of $\mathcal{D}^*(\mathbb{R})$ of those distributions that have compact support.*

Remark 10.5. *The space $\mathcal{K}^*(\mathbb{R})$ can be seen as the dual of $C^\infty(\mathbb{R})$ when $C^\infty(\mathbb{R})$ is endowed with the following topology: a sequence φ_n tends to 0 if and only if for each $p \in \mathbb{N}$, $\varphi_n^{(p)}$ tends to 0 uniformly on every compact set $K \subset \mathbb{R}$.*

Proposition 10.7. *The space $\mathcal{K}^*(\mathbb{R})$ is a linear subspace of $\mathcal{S}^*(\mathbb{R})$. Hence, the Fourier transform of a distribution with compact support is well-defined.*

PROOF. The proof can be found in [Khoan1972].

Theorem 10.2. **(representation of $\mathcal{K}^*(\mathbb{R})$)** *If $T \in \mathcal{K}^*(\mathbb{R})$ and the support of T is in the interior of some compact set K, then there exist positive integers $n_1, n_2, ..., n_p$ and continuous functions $f_1, f_2, ..., f_p$ whose supports are in K such that $T = \sum_{j=1}^{p} T_{f_j}^{(n_j)}$.*

PROOF. The proof can be found in [Khoan1972, Schwartz1965].

Definition 10.6. *A function $f : \mathbb{R} \longrightarrow \mathbb{C}$ is said to be slowly increasing if there exists $C > 0$ and $N \in \mathbb{N}$ such that*

$$|f(x)| \leq C(1 + x^2)^N \quad \textit{for all} \;\; x \in \mathbb{R}$$

Theorem 10.3. *Let $T \in \mathcal{K}^*(\mathbb{R})$. Then, the following statement hold:*

1. *$\widehat{T}$ and all its derivatives can be seen as the generalized distribution of slowly increasing functions in $C^\infty(\mathbb{R})$. That is, there exist slowly increasing functions $g_k \in C^\infty(\mathbb{R})$ such that $\widehat{T^{(k)}} = T_{g_k}$ for all $k \in \mathbb{N}$.*

2. *If we define $f := g_0$, then $g_k = f^{(k)}$ for all $k \in \mathbb{N}$. In particular, $f^{(k)}$ is slowly increasing.*

PROOF.
• Proof 1: assume that the support of $T \in \mathcal{K}^*(\mathbb{R})$ is in the interior of some compact set K. By theorem 10.2., $T = \sum_{j=1}^{p} T_{f_j}^{(n_j)}$ for some positive integers $n_1, n_2, ..., n_p$ and continuous functions $f_1, f_2, ..., f_p$ whose supports are in K. Thus,

$$< \widehat{T}^{(k)}, \varphi > \; = \; (-1)^k < \widehat{T}, \varphi^{(k)} > \; = \; (-1)^k < T, \widehat{\varphi^{(k)}} > \; = \; (-1)^k \sum_{j=1}^{p} < T_{f_j}^{(n_j)}, \widehat{\varphi^{(k)}} >$$

$$= \; (-1)^k \sum_{j=1}^{p} < T_{f_j}, \left(\widehat{\varphi^{(k)}}\right)^{(n_j)} > \qquad \textit{for all} \quad \varphi \in \mathcal{S}(\mathbb{R}) \tag{10.3.1}$$

Now, let us work the term $\left(\widehat{\varphi^{(k)}}\right)^{(n_j)}$. Applying proposition 1.1.2. and proposition 1.1.1. we get

$$\left(\widehat{\varphi^{(k)}}\right)^{(n_j)}(\xi) \; = \; \left((2i\pi\xi)^k \,\widehat{\varphi}(\xi)\right)^{(n_j)} \; = \sum_{i=0}^{n_j} A_i \begin{pmatrix} n_j \\ i \end{pmatrix} \xi^{(k-i,0)_+} \left(\widehat{\varphi}(\xi)\right)^{(n_j-i)} \; =$$

$$= \sum_{i=0}^{n_j} A_i \begin{pmatrix} n_j \\ i \end{pmatrix} \xi^{(k-i,0)_+} (-2\pi i)^{n_j-i} \,[\widehat{x^{n_j-i}\varphi}]\,(\xi)$$

for some constants $A_i \in \mathbb{C}$ and we have denoted $(k-i,0)_+ = \max(k-i,0)$. Thus, using this in (10.3.1)

$$< \widehat{T}^{(k)}, \varphi > \; = \; (-1)^k \sum_{j=1}^{p} < T_{f_j}, \sum_{i=0}^{n_j} A_i \begin{pmatrix} n_j \\ i \end{pmatrix} \xi^{(k-i,0)_+} (-2\pi i)^{n_j-i} \,[\widehat{x^{n_j-i}\varphi}]\,(\xi) > \; =$$

$$= \; (-1)^k \sum_{j=1}^{p}\sum_{i=0}^{n_j} A_i \begin{pmatrix} n_j \\ i \end{pmatrix} (-2\pi i)^{n_j-i} \int_{\mathbb{R}} f_j(\xi)\xi^{(k-i,0)_+} \left(\int_{\mathbb{R}} x^{n_j-i}\varphi(x)e^{-2\pi i x\xi}dx\right) d\xi$$

Finally, after using Fubini's Theorem and a straightforward calculation we obtain

$$< \widehat{T}^{(k)}, \varphi > \; = \; \int_{\mathbb{R}} \varphi(x) g_k(x) dx \; = \; < T_{g_k}, \varphi > \quad \textit{for all} \quad \varphi \in \mathcal{S}(\mathbb{R})$$

where we have defined for all $k \in \mathbb{N}$ the following functions

$$g_k(x) \; := \; (-1)^k \sum_{j=1}^{p}\sum_{i=0}^{n_j} A_i \begin{pmatrix} n_j \\ i \end{pmatrix} (-2\pi i)^{n_j-i} x^{n_j-i} \int_{\mathbb{R}} f_j(\xi)\xi^{(k-i,0)_+} e^{-2\pi i x\xi} d\xi$$

Note that by Theorem A.5. we get that $g_k \in C^\infty(\mathbb{R})$. Moreover, using that f_j has compact support in K and is continuous for all j and after manipulating this expression we can find a constant $C > 0$ and $N \in \mathbb{N}$ such that $|g_k(x)| \leq C(1+x^2)^N$. In conclusion, we have that $\widehat{T}^{(k)} = T_{g_k}$ where $g_k \in C^\infty(\mathbb{R})$ is a slowly increasing function.

• Proof 2: we will show that $g_1 = f'$. The general result is proved by induction. Let us take $\varphi \in \mathcal{D}(\mathbb{R})$. On the one hand we have:

$$< \widehat{T}', \varphi > \; = \; < T_{g_1}, \varphi >$$

On the other hand, we use that by proof 1, $\widehat{T} = T_{g_0} = T_f$. Thus

$$< \widehat{T}', \varphi > \; = \; < T_f', \varphi > \; = \; - < T_f, \varphi' > \; = \; -\int_{\mathbb{R}} f(x)\varphi'(x)dx \; = \; \int_{\mathbb{R}} f'(x)\varphi(x)dx \; = \; < T_{f'}, \varphi >$$

Hence we get that $T_{g_1} = T_{f'}$ on $\mathcal{D}^*(\mathbb{R})$ and from proposition 9.3. we conclude that $g_1 = f'$ a.e.

Theorem 10.4. (the Paley-Wiener theorem) *Let $T \in \mathcal{K}^*(\mathbb{R})$. Assume that supp$(T) \subset [-M, +M]$ for some $M > 0$. Then the function $g_0 \in C^\infty(\mathbb{R})$ given by Theorem 10.3. s.t. $\widehat{T} = T_{g_0}$ can be extended to a holomorphic function $\tilde{g_0} : \mathbb{C} \longrightarrow \mathbb{C}$ that satisfies the following estimate: there exist $C > 0$ and $m \in \mathbb{N}$ such that*

$$|\tilde{g_0}(z)| \;\leq\; C(1+|z|^2)^{\frac{m}{2}} e^{2\pi M|Im(z)|} \quad \textit{for all} \;\; z \in \mathbb{C}$$

PROOF.

We have shown in the proof of Theorem 10.3. that

$$g_0(x) \;:=\; \sum_{j=1}^{p} \int_{\mathbb{R}} f_j(\xi) e^{-2\pi i x\xi} d\xi \sum_{i=0}^{n_j} \binom{n_j}{i} (-2\pi i)^{n_j - i} x^{n_j - i} \;=\; \sum_{j=1}^{p} (1-2\pi i x)^{n_j} \int_{\mathbb{R}} f_j(\xi) e^{-2\pi i x\xi} d\xi$$

Let us extend this function on $\mathbb{C}$ by defining

$$\tilde{g_0}(z) \;=\; \sum_{j=1}^{p} (1-2\pi i z)^{n_j} \int_{\mathbb{R}} f_j(\xi) e^{-2\pi i z\xi} d\xi \quad \text{for all} \quad z \in \mathbb{C}$$

One shows by direct computation that $\tilde{g_0}$ is holomorphic on $\mathbb{C}$, i.e. it is infinitely times differentiable on $\mathbb{C}$. In the same way, one proves the estimate of $\tilde{g_0}$.

Proposition 10.8. *The Fourier transform of a distribution T with compact support ($T \neq 0$) cannot have compact support.*

PROOF. If $T \in \mathcal{K}^*(\mathbb{R})$ has compact support, the function g_0 given by Theorem 10.3. s.t. $\widehat{T} = T_{g_0}$ is the restriction to $\mathbb{R}$ of the holomorphic extension $\widetilde{g_0}$ on $\mathbb{C}$ given by Theorem 10.4. If $\widehat{T} = T_{g_0}$ had compact support, then g_0 would also have compact support and it would vanish on some nonempty open interval. But being $\widetilde{g_0}$ analytic, this implies that g_0 vanishes everywhere. Thus, $\widehat{T}$ cannot have compact support.

Chapter 11

Convolution of distributions and Fourier Transform

11.1 The convolution of a distribution and a $C^{\infty}(\mathbb{R})$ function

Proposition 11.1. *Let φ be a function and T be a functional and assume that satisfy one of the following three conditions*

1. $\varphi \in \mathcal{D}(\mathbb{R})$ *and* $T \in \mathcal{D}^*(\mathbb{R})$
2. $\varphi \in \mathcal{S}(\mathbb{R})$ *and* $T \in \mathcal{S}^*(\mathbb{R})$
3. $\varphi \in C^{\infty}(\mathbb{R})$ *and* $T \in \mathcal{K}^*(\mathbb{R})$

Then, the function ψ defined by $\psi(x) = < \tau_x T, \varphi >$ is infinitely times differentiable and

$$\psi^{(k)}(x) \; = \; < \tau_x T, \varphi^{(k)} > \tag{11.1.1}$$

PROOF.

• Proof 1: we saw in definition 9.8. that $\tau_x T \in \mathcal{D}^*(\mathbb{R})$ and when $\varphi \in \mathcal{D}(\mathbb{R})$, the expression $\psi(x) = < \tau_x T, \varphi >$ is well-defined for all $x \in \mathbb{R}$. We should show that it is differentiable. Let $(h_n)_n$ be a sequence of nonzero reals that tends to 0 as $n \longrightarrow \infty$. Let us define

$$\alpha_n(y) \; := \; \frac{1}{h_n}\left(\varphi(y+x+h_n) - \varphi(x+y)\right) \; = \; \frac{1}{h_n}\left(\tau_{-x-h_n}\varphi(y) - \tau_{-x}\varphi(y)\right) \tag{11.1.2}$$

Note that it holds

$$< T, \alpha_n > \; = \; \frac{1}{h_n}\left(< T, \tau_{-x-h_n}\varphi > - < T, \tau_{-x}\varphi >\right) \; = \; \frac{1}{h_n}\left(\psi(x+h_n) - \psi(x)\right) \tag{11.1.3}$$

To prove that ψ is differentiable, we will show that $\alpha_n \overset{n\longrightarrow+\infty}{\longrightarrow} \tau_{-x}\varphi'$ in $\mathcal{D}(\mathbb{R})$. Then, as T is continuous on $\mathcal{D}(\mathbb{R})$, it will hold that:

$$\psi'(x) \; = \; \lim_{n\longrightarrow\infty} \frac{1}{h_n}\left(\psi(x+h_n) - \psi(x)\right) \; = \; \lim_{n\longrightarrow\infty} < T, \alpha_n > \; = \; < T, \tau_{-x}\varphi' > \; = \; < \tau_x T, \varphi' >$$

Let us suppose that $\text{supp}(\varphi) \subset [-M, M]$ and $|h_n| \leq 1$, then $\text{supp}(\alpha_n) \subset [-x-M-1, x+M+1]$ which is a fixed compact interval that we will denote by K. Now, note that by the mean value problem, there exists $\theta_n \in (0,1)$ such that:

$$\alpha_n^{(q)}(y) \; = \; \frac{1}{h_n}\left(\varphi^{(q)}(y+x+h_n) - \varphi^{(q)}(x+y)\right) \; = \; \varphi^{(q+1)}(y+x+\theta_n h_n)$$

Applying again the mean value theorem, there exists $\gamma_n \in (0,1)$ such that

$$\left|\alpha_n^{(q)}(y) - \varphi^{(q+1)}(y+x)\right| = \left|\varphi^{(q+1)}(y+x+\theta_n h_n) - \varphi^{(q+1)}(y+x)\right| = \left|\theta_n h_n \varphi^{(q+2)}(y+x+\gamma_n\theta_n h_n)\right|$$

$$\leq |h_n| \left\|\varphi^{(q+2)}\right\|_\infty \overset{n\longrightarrow+\infty}{\longrightarrow} 0$$

Hence, we conclude that $\alpha_n^{(q)}$ converges uniformly to $\tau_{-x}\varphi^{(q+1)}$ for all $q \in \mathbb{N}$. Consequently, $\alpha_n \overset{n\longrightarrow+\infty}{\longrightarrow} \tau_{-x}\varphi^{'}$ in $\mathcal{D}(\mathbb{R})$. By induction and repeating the same argument, one proves that $\psi \in C^\infty(\mathbb{R})$ and $\psi^{(k)}(x) = <\tau_x T, \varphi^{(k)}>$.

• Proof 2: similarly to Proposition 10.2. one proves that $\tau_x T \in \mathcal{S}^*(\mathbb{R})$ and $\psi(x) = <\tau_x T, \varphi>$ is well-defined for all $x \in \mathbb{R}$. Let us define α_n as in (11.1.2). Note that (11.1.3) also holds. Hence, arguing as in proof 1, it is sufficient to verify that $\alpha_n \overset{n\longrightarrow+\infty}{\longrightarrow} \tau_{-x}\varphi^{'}$ in $\mathcal{S}(\mathbb{R})$ to show that ψ is differentiable and that $\psi^{'}(x) = <\tau_x T, \varphi^{'}>$. As before, the mean value theorem leads to the inequality

$$\left|y^p\left(\alpha_n^{(q)}(y) - \tau_{-x}^{(q+1)}\varphi(y)\right)\right| \leq |h_n| \left|y^p\varphi^{(q+2)}(y+x+\rho_n h_n)\right| =$$

$$= \frac{|h_n|\,|y|^p}{1+|x+y+\rho_n h_n|^p}\left[\varphi^{(q+2)}(y+x+\rho_n h_n) + |x+y+\rho_n h_n|^p\,\varphi^{(q+2)}(y+x+\rho_n h_n)\right]$$

where $0 < \rho_n < 1$. Let us denote $C := \sup\limits_{y\in\mathbb{R}}\left(\frac{|y|^p}{1+|x+y+\rho_n h_n|^p}\right)$. Consequently

$$\sup_{y\in\mathbb{R}}\left|y^p\left(\alpha_n^{(q)}(y) - \tau_{-x}^{(q+1)}\varphi(y)\right)\right| \leq C\,|h_n|\left(\left\|\varphi^{(q+2)}\right\|_\infty + \sup_{t\in\mathbb{R}}\left|t^p\varphi^{(q+2)}(t)\right|\right)$$

Hence, since $\varphi \in \mathcal{S}(\mathbb{R})$, it follows that $\alpha_n \overset{n\longrightarrow+\infty}{\longrightarrow} \tau_{-x}\varphi^{'}$ in $\mathcal{S}(\mathbb{R})$. By induction, one proves that $\psi \in C^\infty(\mathbb{R})$ and $\psi^{(k)}(x) = <\tau_x T, \varphi^{(k)}>$.

• Proof 3: note that by Remark 10.5. $\tau_x T \in \mathcal{K}^*(\mathbb{R})$ and $\psi(x) = <\tau_x T, \varphi>$ is well-defined for all $x \in \mathbb{R}$ when $\varphi \in C^\infty(\mathbb{R})$. Arguing as before, it is sufficient to show that $\alpha_n \overset{n\longrightarrow+\infty}{\longrightarrow} \tau_{-x}\varphi^{'}$ in $C^\infty(\mathbb{R})$ with the topology defined in Remark 10.5. That is, $(\alpha_n)^{(p)} \overset{n\longrightarrow+\infty}{\longrightarrow} \left(\tau_{-x}\varphi^{'}\right)^{(p)}$ uniformly on all compact subsets of $\mathbb{R}$. This is done using inequalities similar to those used in the proofs of 1 and 2.

Definition 11.1. *Let φ be a function and T be a functional and assume that satisfy one of the conditions of Proposition 11.1. The convolution of φ and T is the function $\varphi * T$ defined by*

$$(\varphi * T)(x) = <T^y, \varphi(x-y)>$$

where we denote T^y to emphasize that we apply the functional T to the function $y \longmapsto \varphi(x-y)$.

Remark 11.1. *Note that in definition 11.1, we can write the convolution as*

$$(\varphi * T)(x) = <T, \tau_x\varphi_\sigma> = <\tau_{-x}T, \varphi_\sigma> \tag{11.1.4}$$

Thus, by proposition 11.1., we know the meaning of the convolutions $\mathcal{D} * \mathcal{D}^*$, $\mathcal{S} * \mathcal{S}^*$, $C^\infty * \mathcal{K}^*$.

Proposition 11.2. (derivation) *Let φ be a function and T be a functional and assume that satisfy one of the conditions of Proposition 11.1. Then $\varphi * T \in C^{\infty}(\mathbb{R})$ and*

$$(\varphi * T)^{(k)} \;=\; \varphi^{(k)} * T \;=\; \varphi * T^{(k)} \qquad \textit{for } k \;=\; 1, 2...$$

PROOF.

Let us define $\psi(x) \; =< \tau_x T, \varphi_\sigma >$. By proposition 11.1. ψ is well-defined and is infinitely times differentiable. Now, from (11.1.4), it holds that $(\varphi * T)(x) \;=\; \psi(-x)$. Hence, we conclude that $\varphi * T \in C^{\infty}(\mathbb{R})$. Moreover, from (11.1.1)

$$(\varphi * T)^{(k)}(x) \;=\; (-1)^k \psi^{(k)}(-x) \;=< \tau_{-x} T, \left(\varphi^{(k)}\right)_\sigma > \;=\; \left(\varphi^{(k)} * T\right)(x)$$

Finally, on the other hand

$$\left(\varphi * T^{(k)}\right)(x) \overset{(11.1.4)}{=} < \tau_{-x} T^{(k)}, \varphi_\sigma > \;=\; (-1)^k < T, (\tau_x \varphi_\sigma)^{(k)} > \;=\; < T, \tau_x \left(\varphi^{(k)}\right)_\sigma > \;=$$

$$= < \tau_{-x} T, \left(\varphi^{(k)}\right)_\sigma > \overset{(11.1.4)}{=} \left(\varphi^{(k)} * T\right)(x)$$

Proposition 11.3. (support) *Let $\varphi \in C^{\infty}(\mathbb{R})$ and $T \in \mathcal{K}^*(\mathbb{R})$. Then, supp$(\varphi * T) \subset$ supp$(T) +$ supp(φ), where $+$ denotes the algebraic sum of the two sets.*

PROOF.

Since supp(T) is compact and supp(φ) is closed then supp$(T) +$ supp(φ) is closed. Let us define $\Omega := \mathbb{R} \backslash (\text{supp}(T) + \text{supp}(\varphi))$ and let $x \in \Omega$. Note that for all $y \in \text{supp}(\varphi)$ then $(x - y) \notin \text{supp}(T)$. Consequently

$$(\varphi * T)(x) \;=< \tau_{-x} T, \varphi_\sigma > \;=\; < T^y, \varphi(x - y) > \;=\; 0$$

and we conclude that $x \notin \text{supp}(\varphi * T)$. Hence, $\text{supp}(\varphi * T) \subset \text{supp}(T) + \text{supp}(\varphi)$.

Corollary 11.1. *Let $\varphi \in \mathcal{D}(\mathbb{R})$ and $T \in \mathcal{K}^*(\mathbb{R})$. Then, the convolution $\varphi * T$ has compact support.*

PROOF. Since supp(T) and supp(φ) is compact then supp$(T) +$ supp(φ) is also compact. Moreover, as supp$(\varphi * T)$ is closed and is contained in a compact, we conclude that supp$(\varphi * T)$ is compact.

11.2 The convolution $\mathcal{K}^*(\mathbb{R}) * \mathcal{D}^*(\mathbb{R})$

Theorem 11.1. **($\mathcal{K}^*(\mathbb{R}) * \mathcal{D}^*(\mathbb{R})$)** *Let $S \in \mathcal{K}^*(\mathbb{R})$ and $T \in \mathcal{D}^*(\mathbb{R})$. Then, it holds*

1. *There exists a distribution $S * T$ called the convolution of S and T such that for all $\varphi \in \mathcal{D}(\mathbb{R})$*

$$< S * T, \varphi > \;=\; < S^t, < T^x, \varphi(x + t) >> \;=\; < T^u, < S^x, \varphi(x + u) >> \qquad (11.2.1)$$

2. *The mapping $(S, T) \longmapsto S * T$ from $\mathcal{K}^*(\mathbb{R}) \times \mathcal{D}^*(\mathbb{R})$ to $\mathcal{D}^*(\mathbb{R})$ is continuous with respect to each variable.*

PROOF.
• Proof 1: first of all, note that we can write (11.2.1) as

$$< S * T, \varphi > \ = \ < S, (\varphi_\sigma * T)_\sigma > \ = \ < T, (\varphi_\sigma * S)_\sigma > \qquad \text{for all} \quad \varphi \in \mathcal{D}(\mathbb{R}) \tag{11.2.2}$$

Note that by proposition 11.2., $(\varphi_\sigma * T)_\sigma \in C^\infty(\mathbb{R})$ when $T \in D^*(\mathbb{R})$. Thus, the expression $< S^t, < T^x, \varphi(x+t) >> = < S, (\varphi_\sigma * T)_\sigma >$ is well-defined when $S \in \mathcal{K}^*(\mathbb{R})$.

Similarly, as $\varphi_\sigma \in \mathcal{D}(\mathbb{R})$ and $S \in \mathcal{K}^*(\mathbb{R})$ we can apply corollary 11.1. and conclude that $(\varphi_\sigma * T)_\sigma \in \mathcal{D}(\mathbb{R})$. Thus, the expression $< T^u, < S^x, \varphi(x+u) >> = < T, (\varphi_\sigma * S)_\sigma >$ makes sense for all $T \in \mathcal{D}^*(\mathbb{R})$. It is not clear, however , that $< S, (\varphi_\sigma * T)_\sigma > \ = \ < T, (\varphi_\sigma * S)_\sigma >$. This is, in fact true. But we will not prove this result. It can be found in [Schwartz1965].

• Proof 2: first we show that the mapping is continuous respect to the first variable. Let $(S_n)_n$ be a sequence in $\mathcal{K}^*(\mathbb{R})$ such that $S_n \overset{n\to\infty}{\longrightarrow} 0$ in $\mathcal{K}^*(\mathbb{R})$, that is $< S_n, \varphi > \overset{n\to\infty}{\longrightarrow} 0$ for all $\varphi \in C^\infty(\mathbb{R})$. Then, using this and the continuity of T

$$< S_n * T, \varphi > \ = \ < T^u, < (S_n)^x, \varphi(x+u) >> \overset{n\to\infty}{\longrightarrow} < T^u, 0 >= 0 \qquad \text{for all} \quad \varphi \in \mathcal{D}(\mathbb{R})$$

And we conclude that the mapping is continuous respect to the first variable. One shows that the mapping is continuous with respect to the second variable by doing a similar argument.

Proposition 11.4. (Dirac distributions) *Let $T \in \mathcal{D}^*(\mathbb{R})$. Then, the following statements hold*

1. $\delta_a * T \ = \ T * \delta_a \ = \ \tau_a T$ *for all* $a \in \mathbb{R}$. *In particular, δ acts like a unit element for convolution.*
2. $\delta^{(k)} * T \ = \ T * \delta^{(k)} \ = \ T^{(k)}$ *for all* $k \ = \ 0, 1, 2, 3...$

PROOF.
• Proof 1: first of all note that $\delta_a \in \mathcal{D}^*(\mathbb{R})$ (example 9.1.1) and $\text{supp}(\delta_a) \ = \ \{a\}$. Thus, it follows that $\delta_a \in \mathcal{K}^*(\mathbb{R})$. As $T \in \mathcal{D}^*(\mathbb{R})$ we get by Theorem 11.1 that $\delta_a * T \ = \ T * \delta_a$ is well defined and is a distribution. Moreover, using (11.2.1)

$$< \delta_a * T, \varphi > \ = \ < T^u, < \delta_a, \varphi(x+u) >> \ = \ < T^u, \varphi(a+u) > \ = \ < \tau_a T, \varphi >$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. Thus, $\delta_a * T \ = \ T * \delta_a \ = \ \tau_a T$.

• Proof 2: similarly to 1, we show that $\delta^{(k)} * T \ = \ T * \delta^{(k)}$ is well defined and is a distribution. Moreover, using (11.2.1)

$$< \delta^{(k)} * T, \varphi > \ = \ < T^u, < \delta^{(k)}, \varphi(x+u) >> \ = \ (-1)^k < T^u, \varphi^{(k)}(u) > \ = \ < T^{(k)}, \varphi >$$

for all $\varphi \in \mathcal{D}(\mathbb{R})$. Thus, $\delta^{(k)} * T \ = \ T * \delta^{(k)} \ = \ T^{(k)}$.

Proposition 11.5. (Derivatives) *Let $S \in \mathcal{K}^*(\mathbb{R})$ and $T \in \mathcal{D}^*(\mathbb{R})$ Then $(S * T)^{(k)} \ = \ S^{(k)} * T \ = \ S * T^{(k)}$.*

PROOF. Note that by Theorem 11.1. $S * T \in \mathcal{D}^*(\mathbb{R})$. Hence, for all $\varphi \in \mathcal{D}(\mathbb{R})$, it holds

$$< (S * T)^{(k)}, \varphi > \ = \ (-1)^k < S * T, \varphi^{(k)} > \overset{(11.2.1)}{=} (-1)^k < S^t, < T^x, \varphi^{(k)}(x+t) >> \ =$$

$$= < S^t, < \left(T^{(k)}\right)^x, \varphi(x+t) >> \; = \; < S * T^{(k)}, \varphi >$$

Thus, $(S*T)^{(k)} = S * T^{(k)}$. Similarly, one proves that $(S*T)^{(k)} = S^{(k)} * T$.

Proposition 11.6. (support) *Let $S \in \mathcal{K}^*(\mathbb{R})$ and $T \in \mathcal{D}^*(\mathbb{R})$. Then,*

$$supp(S*T) \subset supp\,(S) + supp(T)$$

PROOF.

Since supp(S) is compact, supp$\,(S)$ + supp(T) is closed. Let us take $\Omega = \mathbb{R} \setminus \{\text{supp}\,(S) + \text{supp}(T)\}$ and $\varphi \in \mathcal{D}(\mathbb{R})$ with supp$\,(\varphi) \subset \Omega$. We will show that $< S*T, \varphi >= 0$. Then, we will be able to conclude that supp$(S*T) \subset$ supp$\,(S)$ + supp(T). Note that from (11.2.2)

$$< S*T, \varphi > \; = \; < T, (\varphi_\sigma * S)_\sigma >$$

Hence, it is sufficient to prove that supp$\,((\varphi_\sigma * S)_\sigma) \cap$ supp$(T) = \emptyset$. Let us suppose that there exists $u \in$ supp$\,((\varphi_\sigma * S)_\sigma) \cap$ supp(T), then $-u \in$ supp$\,(\varphi_\sigma * S) \subset$ supp$\,(\varphi_\sigma)$ + supp(S). This means that $-u = y + x$ with $-y \in$ supp$\,(\varphi)$ and $x \in$ supp$\,(S)$. But then, $-y = u + x$, thus

$$\text{supp}\,(\varphi) \cap (\text{supp}\,(S) + \text{supp}(T)) \neq \emptyset$$

which is a contradiction since supp$\,(\varphi) \subset \Omega = \mathbb{R} \setminus \{\text{supp}\,(S) + \text{supp}(T)\}$.

Corollary 11.2. *Let $S, T \in \mathcal{K}^*(\mathbb{R})$. Then, $S*T \in \mathcal{K}^*(\mathbb{R})$.*

PROOF. If $S, T \in \mathcal{K}^*(\mathbb{R})$, then supp$(S)$ and supp(T) are compact. Thus, supp$(S*T) \subset$ supp$\,(S)$ + supp(T) is also compact.

11.3 The convolution $\mathcal{K}^*(\mathbb{R}) * \mathcal{S}^*(\mathbb{R})$

Note that the convolution $\mathcal{K}^*(\mathbb{R}) * \mathcal{S}^*(\mathbb{R})$ is a particular case of the convolution $\mathcal{K}^*(\mathbb{R}) * \mathcal{D}^*(\mathbb{R})$. But we will see that in this case the distribution that one obtains is tempered.

Proposition 11.7. *Let $S \in \mathcal{K}^*(\mathbb{R})$ and let us consider the mapping $A : \mathcal{S}(\mathbb{R}) \longrightarrow \mathcal{S}(\mathbb{R})$ defined by $A(\varphi) = \alpha$ where $\alpha(u) = < S^x, \varphi(x+u) >$. It holds that A is well-defined and is continuous.*

PROOF.

• First, we shall show that A is well-defined, i.e. $A(\varphi) = \alpha \in \mathcal{S}(\mathbb{R})$ when $\varphi \in \mathcal{S}(\mathbb{R})$. As $S \in \mathcal{K}^*(\mathbb{R})$, we can use Theorem 10.2. and write $S = \sum_{j=1}^{p} T_{f_j}^{(n_j)}$ for some positive integers n_j and continuous functions f_j which support in some compact set K. Thus

$$\alpha(u) \; = \sum_{j=1}^{p} < T_{f_j}^{(n_j)}, \varphi(x+u) > \; = \; \sum_{j=1}^{p} (-1)^{n_j} \int_K f_j(x) \varphi^{(n_j)}(x+u) dx \qquad (11.3.1)$$

From this expression and using that $\varphi \in \mathcal{D}(\mathbb{R})$, it is not difficult to verify that $\alpha \in \mathcal{S}(\mathbb{R})$. For more details, see [Khoan1972].

• To prove continuity of the mapping A, we consider a sequence $\varphi_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{S}(\mathbb{R})$ and show that $A(\varphi_n) = \alpha_n$ converges to $A(0) = 0$ in $\mathcal{S}(\mathbb{R})$. From (11.3.1) we see

$$\alpha_n(u) \;=\; \sum_{j=1}^{p} (-1)^{n_j} \int_K f_j(x)\varphi_n^{(n_j)}(x+u)dx$$

Differentiating under the integral sign

$$\left|u^m \alpha_n^{(q)}(u)\right| \;=\; \left|\sum_{j=1}^{p} (-1)^{n_j} \int_K u^m f_j(x)\varphi_n^{(n_j+q)}(x+u)dx\right| \leq \sum_{j=1}^{p} \int_K |f_j(x)| \left|u^m\varphi_n^{(n_j+q)}(x+u)\right| dx \;\leq$$

$$\leq \sum_{j=1}^{p} \int_K |f_j(x)| \frac{|u|^m}{1+|x+u|^m}\left(1+|x+u|^m\right)\left|\varphi_n^{(n_j+q)}(x+u)\right| dx \;\leq\; C\sum_{j=1}^{p}\left(\left\|\varphi_n^{(n_j+q)}\right\|_\infty + \sup_{t\in\mathbb{R}}\left|t^m\varphi_n^{(n_j+q)}(t)\right|\right)$$

for some constant $C > 0$. This shows that if $\varphi_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{S}(\mathbb{R})$ then $A(\varphi_n) = \alpha_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{S}(\mathbb{R})$. Thus, we conclude that A is continuous.

Proposition 11.8. *Let $S \in \mathcal{K}^*(\mathbb{R})$ and $T \in \mathcal{S}^*(\mathbb{R})$. Then, the convolution $S * T$ is a tempered distribution, i.e. $S * T \in \mathcal{S}^*(\mathbb{R})$.*

PROOF.

We know that $S*T \in \mathcal{D}^*(\mathbb{R})$ from Theorem 11.1. Thus, it is sufficient to show that $S*T$ is continuous on $\mathcal{D}(\mathbb{R})$ with the topology of $\mathcal{S}(\mathbb{R})$ (Proposition 10.1.). In effect, let $(\varphi_n)_n$ be a sequence in $\mathcal{D}(\mathbb{R})$ which converges to 0 in $\mathcal{S}(\mathbb{R})$. From (11.2.1), $< S*T, \varphi_n > \;=\; < T^u, < S^x, \varphi_n(x+u) >>$. Moreover, by Proposition 11.7. $< S^x, \varphi_n(x+u) > = A(\varphi_n) \in \mathcal{S}(\mathbb{R})$ and $A(\varphi_n) \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{S}(\mathbb{R})$. Hence, using the continuity of $T \in \mathcal{S}^*(\mathbb{R})$

$$< S*T, \varphi_n > \;=\; < T, A(\varphi_n) > \overset{n\longrightarrow\infty}{\longrightarrow} < T, 0 > \;= 0$$

Proposition 11.9. *Let $S \in \mathcal{K}^*(\mathbb{R})$ and $T \in \mathcal{S}^*(\mathbb{R})$. The mapping $(S,T) \longmapsto S*T$ is continuous with respect to each variable. That is:*

1. *Let $(S_n)_n$ be a sequence in $\mathcal{K}^*(\mathbb{R})$ that converges to 0 in $\mathcal{K}^*(\mathbb{R})$, i.e. $< S_n, \varphi > \overset{n\longrightarrow\infty}{\longrightarrow} 0$ for all $\varphi \in C^\infty(\mathbb{R})$. Then $S_n * T \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{S}^*(\mathbb{R})$.*
2. *Let $(T_n)_n$ be a sequence in $\mathcal{S}^*(\mathbb{R})$ that converges to 0 in $\mathcal{S}^*(\mathbb{R})$, i.e. $< T_n, \varphi > \overset{n\longrightarrow\infty}{\longrightarrow} 0$ for all $\varphi \in \mathcal{S}(\mathbb{R})$. Then $S * T_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{S}^*(\mathbb{R})$.*

PROOF.

• Proof 1: let $\varphi \in \mathcal{S}(\mathbb{R})$. As $T \in \mathcal{S}^*(\mathbb{R})$, it follows from Proposition 11.2. that $(\varphi_\sigma * T)_\sigma \in C^\infty(\mathbb{R})$. Thus, for all $\varphi \in \mathcal{S}(\mathbb{R})$

$$< S_n * T, \varphi > \;=\; < S_n, (\varphi_\sigma * T)_\sigma > \overset{n\longrightarrow\infty}{\longrightarrow} \; 0$$

• Proof 2: let $\varphi \in \mathcal{S}(\mathbb{R})$. As $S \in \mathcal{K}^*(\mathbb{R})$, it follows from Proposition 11.7. that $A(\varphi) = < S^x, \varphi(x+u) > \in \mathcal{S}(\mathbb{R})$. Thus, for all $\varphi \in \mathcal{S}(\mathbb{R})$

$$< S * T_n, \varphi > \; = \; < T_n, < S^x, \varphi(x+u) >> \overset{n \longrightarrow \infty}{\longrightarrow} \; 0$$

11.4 The convolution $\mathcal{D}^*_+(\mathbb{R}) * \mathcal{D}^*_+(\mathbb{R})$

In the previous sections we have studied the convolution of two distributions where at least one of them has compact support. Without this condition on the support, the convolution is not generally defined. However, the convolution is defined when both distributions are in $\mathcal{D}^*_+(\mathbb{R})$ or $\mathcal{D}^*_-(\mathbb{R})$ (these spaces were defined in definition 9.11.).

Proposition 11.10. *Let $T \in \mathcal{D}^*_+(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$. Assume that supp$(T) \subset [a, +\infty)$ and supp$(\varphi) \subset (-\infty, b]$. Then, $< T, \varphi >_s$ defined by*

$$< T, \varphi >_s \; = \; < T, \theta\varphi >$$

where θ is a function in $\mathcal{D}(\mathbb{R})$ equal to 1 on an interval $[-M,\, M]$ containing a and b in its interior, is well-defined.

PROOF.

Note that $\theta\varphi \in \mathcal{D}(\mathbb{R})$, thus $< T, \theta\varphi >$ makes sense. We should show that the definition of $< T, \varphi >_s$ does not depend on the choice of θ. In effect, let θ_1 be another function in $\mathcal{D}(\mathbb{R})$ equal to 1 on $[-M_1,\, M_1]$ containing a and b. Then, $(\theta - \theta_1)\,\varphi$ vanishes on $[-m, +\infty)$, where $m = \min\{M, M_1\}$. Thus, supp$((\theta - \theta_1)\,\varphi) \subset (-\infty, -m)$. Now, since supp$(T) \subset [a, +\infty)$ we have

$$\text{supp}\,((\theta - \theta_1)\,\varphi) \cap \text{supp}(T) \;\subset\; (-\infty, -m) \cap [a, +\infty) \;=\; \emptyset$$

Thus, $< T, (\theta - \theta_1)\,\varphi >= 0$, and consequently $< T, \theta\varphi > = < T, \theta_1\varphi >$.

Proposition 11.11. *Let $T \in \mathcal{D}^*_+(\mathbb{R})$ and $\varphi \in C^\infty(\mathbb{R})$. Assume that supp$(T) \subset [a, +\infty)$ and supp$(\varphi) \subset (-\infty, b]$. Then, $\psi(t) := < T^x, \varphi(x+t) >_s$ is defined for all $t \in \mathbb{R}$, supp$(\psi) \subset (-\infty, b-a]$ and $\psi \in C^\infty(\mathbb{R})$.*

PROOF.

Note first that $\tau_{-t}\varphi \in C^\infty(\mathbb{R})$ and supp$(\tau_{-t}\varphi) \subset (-\infty, b-t]$. Thus, by proposition 11.10, $\psi(t) := < T^x, \varphi(x+t) >_s$ is well-defined. Now, note that $\psi(t) = 0$ if supp$(\tau_{-t}\varphi) \cap \text{supp}(T) = \emptyset$ which happens when $b - t < a$. Hence, supp$(\psi) \subset (-\infty, b-a]$. Finally, by proposition 11.10

$$\psi(t) \; = < T^x, \varphi(x+t) >_s \; = \; < T, \theta\tau_{-t}\varphi > \; = \; < \tau_t T, (\tau_t\theta)\,\varphi >$$

and as $T \in \mathcal{D}^*(\mathbb{R})$ and $(\tau_t\theta)\,\varphi \in \mathcal{D}(\mathbb{R})$, we conclude by Proposition 11.2. that $\psi \in C^\infty(\mathbb{R})$.

Theorem 11.2. (convolution $\mathcal{D}_+^*(\mathbb{R}) * \mathcal{D}_+^*(\mathbb{R})$) *Let T and S be in $\mathcal{D}_+^*(\mathbb{R})$. The following statements hold:*

1. *There exists a distribution called the convolution of S and T, denoted by $S*T$ such that*
$$< S*T, \varphi > \;=\; < S^t, < T^x, \varphi(x+t) >_s >_s \;=\; < T^u, < S^x, \varphi(x+u) >_s >_s \qquad \text{for all} \quad \varphi \in \mathcal{D}(\mathbb{R})$$
2. $(S*T)^{(k)} \;=\; S^{(k)} * T \;=\; S * T^{(k)}$
3. *The mapping $(S,T) \longmapsto S*T$ of $\mathcal{D}_+^*(\mathbb{R}) \times \mathcal{D}_+^*(\mathbb{R})$ into $\mathcal{D}^*(\mathbb{R})$ is continuous with respect to each variable. The convergence of $(S_n)_n \in \mathcal{D}_+^*(\mathbb{R})$ to 0 in $\mathcal{D}_+^*(\mathbb{R})$ means that $S_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{D}^*(\mathbb{R})$ and that there exists a constant c such that $supp(S_n) \subset [c, +\infty)$ for all n.*

PROOF.

• Proof 1: first we show that $< S^t, < T^x, \varphi(x+t) >_s >_s$ is well-defined. As $\varphi \in \mathcal{D}(\mathbb{R})$ and $T \in \mathcal{D}_+^*(\mathbb{R})$, then by Proposition 11.11. $\psi(t) \;=< T^x, \varphi(x+t) >_s$ makes sense and has a support of the form $(-\infty, M]$, for some $M > 0$. Finally, as $S \in \mathcal{D}_+^*(\mathbb{R})$ we apply again Proposition 11.11. and conclude that $< S^t, \psi(t) >_s =< S^t, < T^x, \varphi(x+t) >_s >_s$ is well-defined. Analogously $< T^u, < S^x, \varphi(x+u) >_s >_s$ is well-defined.

It also holds that $< S^t, < T^x, \varphi(x+t) >_s >_s \;=\; < T^u, < S^x, \varphi(x+u) >_s >_s$, however we will not prove this fact. For more details see [Khoan1972].

Next, we should show that $S*T$ is continuous to prove that $S*T \in \mathcal{D}^*(\mathbb{R})$. Let $(\varphi_n)_n \in \mathcal{D}(\mathbb{R})$ s.t. $\varphi_n \overset{n\longrightarrow\infty}{\longrightarrow} \varphi$ in $\mathcal{D}(\mathbb{R})$. As T is a distribution, it holds that

$$\lim_{n\longrightarrow\infty} \psi_n(t) \;=\; \lim_{n\longrightarrow\infty} < T^x, \varphi_n(x+t) >_s \;=\; < T^x, \varphi(x+t) >_s \;=\; \psi(t)$$

Finally, as S is a distribution, we conclude that

$$< S*T, \varphi_n > \;=\; < S^t, \psi_n(t) >_s \;=\; < S^t, \theta\psi_n(t) > \overset{n\longrightarrow\infty}{\longrightarrow} < S^t, \lim_{n\longrightarrow\infty} \theta\psi_n(t) > \;=$$
$$=\; < S^t, \theta\psi(t) > \;=\; < S^t, \psi(t) >_s \;=\; < S*T, \varphi >$$

Thus $S*T$ is a continuous functional on $\mathcal{D}(\mathbb{R})$ and consequently we get that it is a distribution.

• Proof 2: we have seen in 1 that $S*T \in \mathcal{D}^*(\mathbb{R})$. Hence, for all $\varphi \in \mathcal{D}(\mathbb{R})$ hold,

$$< (S*T)^{(k)}, \varphi > \;=\; (-1)^k < S*T, \varphi^{(k)} > \;=\; (-1)^k < S^t, < T^x, \varphi^{(k)}(x+t) >_s >_s \;=$$
$$=\; < S^t, < \left(T^{(k)}\right)^x, \varphi(x+t) >_s >_s \;=\; < S*T^{(k)}, \varphi >$$

Thus, $(S*T)^{(k)} \;=\; S*T^{(k)}$. Similarly, one proves that $(S*T)^{(k)} \;=\; S^{(k)}*T$.

• Proof 3: let $\varphi \in \mathcal{D}(\mathbb{R})$ and let $(S_n)_n \in \mathcal{D}_+^*(\mathbb{R})$ to 0 in $\mathcal{D}_+^*(\mathbb{R})$. As we mentioned in the statement, it holds that $S_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{D}^*(\mathbb{R})$ and that there exists a constant a such that $\text{supp}(S_n) \subset [a, +\infty)$ for all n. Assume $\text{supp}(\varphi) \subset (-\infty, b]$, consequently by Proposition 11.11. it follows that $\text{supp}(\psi_n) \subset (-\infty, b-a]$ for all n, where we have defined $\psi_n(u) \;=< (S_n)^x, \varphi(x+u) >_s$. Moreover, using that $S_n \overset{n\longrightarrow\infty}{\longrightarrow} 0$ in $\mathcal{D}^*(\mathbb{R})$

$$\psi_n(u) \;=\; < (S_n)^x, \varphi(x+u) >_s \overset{n\longrightarrow\infty}{\longrightarrow} 0$$

Finally, using the continuity of the distribution T

$$< S_n * T, \varphi > \; = \; < T^u, \psi_n(u) >_s \; = \; < T^u, \theta\psi_n(u) > \overset{n\longrightarrow\infty}{\longrightarrow} \; < T^u, \lim_{n\longrightarrow\infty} \theta\psi_n(u) > \; = \; 0$$

We have proved the continuity with respect the first variable. Analogously, we prove the continuity with respect the second variable.

Proposition 11.12. **(support)** *Let T and S be in $\mathcal{D}_+^*(\mathbb{R})$ with supp(S) $\subset [a_1, +\infty)$ and supp(T) $\subset [a_2, +\infty)$ then supp($S * T$) $\subset [a_1 + a_2, +\infty)$ and consequently $S * T \in \mathcal{D}_+^*(\mathbb{R})$.*

PROOF.

Take $\varphi \in \mathcal{D}(\mathbb{R})$ with $\operatorname{supp}(\varphi) \subset (-\infty, a_1 + a_2)$. The support of $\psi(t) \; = \; < T^x, \varphi(x+t) >_s$ is in $(-\infty, a_1)$ by Proposition 11.11. Thus

$$\operatorname{supp}(\psi) \cap \operatorname{supp}(S) \; \subseteq \; (-\infty, a_1) \cap [a_1, +\infty) \; = \; \emptyset$$

Hence $< S * T, \varphi > = < S, \psi >_s = 0$. This proves that $\operatorname{supp}(S * T) \subset [a_1 + a_2, +\infty)$.

11.5 The associativity of convolution

In the previous sections, we have defined the convolution of two distributions in several cases and we have seen that it is a commutative operation. If we wish to convolve three or more distributions, we run into two problems: existence and associativity. It can be seen that the convolution is not associative in general. Nevertheless, we will give two results (without proof) which show that convolution is associative in some specific cases.

Proposition 11.13. *The convolution of n distributions of which at least $n-1$ have compact support is associative and commutative.*

PROOF. The proof of this statement can be found in [C. Gasquet2010, page 306].

Proposition 11.14. *The convolution in $\mathcal{D}_+^*(\mathbb{R})$ is associative.*

PROOF. The proof of this statement can be found in [C. Gasquet2010, page 307].

11.6 The Convolution and the Fourier Transform of Distributions

We saw that for functions, the Fourier transform interchanges convolution and multiplication. We wish to determine under what conditions the relations $\widehat{T * U} \; = \; \widehat{T} \cdot \widehat{U}$ and $\widehat{T \cdot U} \; = \; \widehat{T} * \widehat{U}$ are also true for distributions.

Theorem 11.3. (Representation of $\mathcal{S}^*(\mathbb{R})$) *Let $T \in \mathcal{S}^*(\mathbb{R})$. Then there exist positive integers $n_1, n_2, ..., n_p$ and slowly increasing continuous functions $f_1, f_2, ..., f_p$ such that*

$$T = \sum_{k=1}^{p} T_{f_k}^{(n_k)}$$

PROOF. The proof of this statement can be found in [Khoan1972].

Lemma 11.1. (Convolution $\mathcal{S}(\mathbb{R}) * \mathcal{S}^*(\mathbb{R})$) *Let $\varphi \in \mathcal{S}(\mathbb{R})$ and $T \in \mathcal{S}^*(\mathbb{R})$. Then the convolution $\varphi * T$ and all its derivatives are slowly increasing C^∞ functions.*

PROOF.

By Theorem 11.3. we can write $T = \sum_{k=1}^{p} T_{f_k}^{(n_k)}$ where f_k is slowly decreasing. Thus, by definition 11.1.

$$(\varphi * T)(x) = < T^y, \varphi(x-y) > = <\sum_{k=1}^{p} T_{f_k}^{(n_k)}, \varphi(x-y) > = \sum_{k=1}^{p} < T_{f_k}, \varphi^{(n_k)}(x-y) > =$$

$$= \sum_{k=1}^{p} \int_{\mathbb{R}} f_k(y)\varphi^{(n_k)}(x-y)dy = \sum_{k=1}^{p} \int_{\mathbb{R}} f_k(x-y)\varphi^{(n_k)}(y)dy$$

Now, as f_k are slowly increasing continuous functions, there exist $C_k > 0$ and integers N_k such that $|f_k(x)| \leq C_k \left(1+x^2\right)^{N_k}$. Thus:

$$|(\varphi * T)(x)| \leq \sum_{k=1}^{p} C_k \int_{\mathbb{R}} \left(1+(x-y)^2\right)^{N_k} \left|\varphi^{(n_k)}(y)\right| dy \leq \sum_{k=1}^{p} C_k \int_{\mathbb{R}} \left(2y^2 + \left(1+2x^2\right)\right)^{N_k} \left|\varphi^{(n_k)}(y)\right| dy$$

$$= \sum_{k=1}^{p} C_k \int_{\mathbb{R}} \sum_{j=0}^{N_k} \binom{N_k}{j} \left(1+2x^2\right)^{N_k - j} (2y)^{2j} \left|\varphi^{(n_k)}(y)\right| dy \leq$$

$$\leq \left(1+2x^2\right)^{\max\limits_{1\leq k\leq p}\{N_k\}} \left[\sum_{k=1}^{p} C_k \int_{\mathbb{R}} \sum_{j=0}^{N_k} \binom{N_k}{j} (2y)^{2j} \left|\varphi^{(n_k)}(y)\right| dy\right]$$

As $\varphi \in \mathcal{S}(\mathbb{R})$ then $\sum_{j=0}^{N_k} \binom{N_k}{j} (2y)^{2j} \left|\varphi^{(n_k)}(y)\right| dy$ will be in $\mathcal{S}(\mathbb{R}) \subset L^1(\mathbb{R})$. Thus, we finally get that

$$|(\varphi * T)(x)| \leq C \left(1+2x^2\right)^{\max\limits_{1\leq k\leq p}\{N_k\}} \quad \text{where} \quad C := \sum_{k=1}^{p} C_k \int_{\mathbb{R}} \sum_{j=0}^{N_k} \binom{N_k}{j} (2y)^{2j} \left|\varphi^{(n_k)}(y)\right| dy$$

which proves that $\varphi * T$ is slowly increasing. One obtains similar estimates for the derivatives of $\varphi * T$ by repeating the computation for $(\varphi * T)^{(k)}(x) = < T_y, \varphi^{(k)}(x-y) >$.

Lemma 11.2. *Let f be a slowly increasing function. Then T_f is a tempered distribution.*

PROOF.

As f is slowly increasing function, then $|f(x)| \leq C(1+x^2)^N$ and $f \in L^1_{loc}(\mathbb{R})$. Consequently $T_f \in \mathcal{D}^*(\mathbb{R})$. We will use Proposition 10.1. to show that $T_f \in \mathcal{S}^*(\mathbb{R})$. Let $(\varphi_n)_n \in \mathcal{D}(\mathbb{R})$ be a sequence that tends to 0 in $\mathcal{S}(\mathbb{R})$. Thus,

$$|< T_f, \varphi_n >| \leq \int_{\mathbb{R}} |f(x)|\,|\varphi_n(x)|\,dx = \int_{\mathbb{R}} \frac{|f(x)|\,(1+x^2)}{(1+x^2)}\,|\varphi_n(x)|\,dx \leq \int_{\mathbb{R}} \frac{C\cdot(1+x^2)^{N+1}}{(1+x^2)}\,|\varphi_n(x)|\,dx$$

$$\leq C\cdot\sup_{x\in\mathbb{R}}\left|(1+x^2)^{N+1}\varphi_n(x)\right|\int_{\mathbb{R}}\frac{1}{(1+x^2)}dx = C\pi\cdot\sup_{x\in\mathbb{R}}\left|(1+x^2)^{N+1}\varphi_n(x)\right| \overset{n\to+\infty}{\longrightarrow} 0$$

where we have used that $(1+x^2)^{N+1}\varphi_n(x)$ tends to 0 in $\mathcal{S}(\mathbb{R})$ by proposition 2.4.2. Hence, we conclude that $T_f \in \mathcal{S}^*(\mathbb{R})$.

Proposition 11.15. (Fourier Transform of $\mathcal{S}(\mathbb{R}) * \mathcal{S}^*(\mathbb{R})$) *Let $\psi \in \mathcal{S}(\mathbb{R})$ and $T \in \mathcal{S}^*(\mathbb{R})$. Then $\widehat{T_{\psi*T}} = \widehat{\psi}\cdot\widehat{T}$ and $\widehat{\psi\cdot T} = T_{\widehat{\psi}*\widehat{T}}$.*

PROOF.

• We start showing that $\widehat{T_{\psi*T}} = \widehat{\psi}\cdot\widehat{T}$. Note that, by Lemma 11.1., $\psi*T$ is a slowly increasing function and hence, by lemma 11.2., $T_{\psi*T}$ is a tempered distribution. Thus, for all $\varphi\in\mathcal{S}(\mathbb{R})$

$$< \widehat{T_{\psi*T}}, \varphi > = < T_{\psi*T}, \widehat{\varphi} > = \int_{\mathbb{R}} (\psi*T)(x)\widehat{\varphi}(x)dx = \int_{\mathbb{R}} < T^y, \psi(x-y) > \widehat{\varphi}(x)dx =$$

$$= < T^y, \int_{\mathbb{R}} \psi(x-y)\widehat{\varphi}(x)dx > = < T^y, \int_{\mathbb{R}} \psi_\sigma(y-x)\widehat{\varphi}(x)dx > = < T, \psi_\sigma * \widehat{\varphi} >$$

On the other hand, since $\widehat{\psi} \in \mathcal{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$ and $\widehat{T} \in \mathcal{S}^*(\mathbb{R}) \subset \mathcal{D}^*(\mathbb{R})$, we saw in definition 9.12. that the product $\widehat{\psi}\cdot\widehat{T}$ makes sense and

$$< \widehat{\psi}\cdot\widehat{T}, \varphi > = < \widehat{T}, \widehat{\psi}\cdot\varphi > = < T, \widehat{\widehat{\psi}\cdot\varphi} >$$

Applying Proposition 5.3.2 and Theorem 2.2., we see that

$$\widehat{\widehat{\psi}\cdot\varphi} = \widehat{\widehat{\psi}} * \widehat{\varphi} = (\hat{\hat{\psi}}_\sigma) * \widehat{\varphi} = \psi_\sigma * \widehat{\varphi}$$

Thus, we get that $< \widehat{\psi}\cdot\widehat{T}, \varphi > = < T, \psi_\sigma * \widehat{\varphi} > = < \widehat{T_{\psi*T}}, \varphi >$ for all $\varphi\in\mathcal{S}(\mathbb{R})$. Hence, we conclude that $\widehat{T_{\psi*T}} = \widehat{\psi}\cdot\widehat{T}$.

• Next, we will show that $\widehat{\psi\cdot T} = T_{\widehat{\psi}*\widehat{T}}$. Note that $\widehat{\psi}\in\mathcal{S}(\mathbb{R})$ and $\widehat{T}\in\mathcal{S}^*(\mathbb{R})$. Thus, $\widehat{\psi}*\widehat{T}$ is slowly increasing (Lemma 11.1.) and, by Lemma 11.2., we get that $T_{\widehat{\psi}*\widehat{T}} \in \mathcal{S}^*(\mathbb{R})$. It can be seen that $\widehat{T_{\psi*T}} = \widehat{\psi}\cdot\widehat{T}$ also holds changing the operator $\hat{\cdot}$ by $\check{\cdot}$. Thus

$$\check{T}_{\widehat{\psi}*\widehat{T}} = \check{\widehat{\psi}}\cdot\check{\widehat{T}} = \psi\cdot T$$

where we have used that $\hat{\cdot}$ is an isomorphism on $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}^*(\mathbb{R})$ (with the corresponding definitions in each case). Thus, applying $\hat{\cdot}$ to the previous expression, we finally obtain

$$T_{\widehat{\psi}*\widehat{T}} = \widehat{\psi\cdot T}$$

Definition 11.2. *Let $f \in C^{\infty}(\mathbb{R})$ and let $T \in \mathcal{D}^*(\mathbb{R})$. We define the product of T_f and T by $< T_f \cdot T, \varphi > = < T, f\varphi >$ for all $\varphi \in \mathcal{D}(\mathbb{R})$. It can be easily seen that $T_f \cdot T$ defines a distribution.*

Proposition 11.16. (Fourier Transform of $\mathcal{K}^*(\mathbb{R}) * \mathcal{S}^*(\mathbb{R})$) *Let $S \in \mathcal{K}^*(\mathbb{R})$ and $T \in \mathcal{S}^*(\mathbb{R})$. Then $\widehat{S * T} = \widehat{S} \cdot \widehat{T}$.*

PROOF.

Note first of all that as $\widehat{S} \in \mathcal{K}^*(\mathbb{R})$, then there exists a slowly increasing function function $f \in C^{\infty}(\mathbb{R})$ such that $\widehat{S} = T_f$ (Theorem 10.3.). Then, by definition 11.2. $\widehat{S} \cdot \widehat{T} = T_f \cdot \widehat{T}$ makes sense. Let $\varphi \in \mathcal{S}(\mathbb{R})$

$$< \widehat{S} \cdot \widehat{T}, \varphi > \; = \; < T_f \cdot \widehat{T}, \varphi > \; = \; < \widehat{T}, f\varphi >$$

As f and all its derivatives are slowly increasing C^{∞} functions, one can easily check that $f\varphi \in \mathcal{S}(\mathbb{R})$. Then

$$< \widehat{S} \cdot \widehat{T}, \varphi > \; = \; < T, \widehat{f\varphi} > \tag{11.6.1}$$

Now, we are going to show that $\widehat{f\varphi} = \varphi * S_{\sigma}$. Let $\psi \in \mathcal{D}(\mathbb{R})$, using Property 1.8.

$$< T_{\widehat{f\varphi}}, \psi > \; = \; \int_{\mathbb{R}} \widehat{f\varphi}(\xi)\psi(\xi)d\xi \; = \; \int_{\mathbb{R}} f(x)\varphi(x)\widehat{\psi}(x)dx \; = \; < T_f, \varphi\widehat{\psi} > \; = \; < \varphi\widehat{S}, \widehat{\psi} > \; = \; < \widehat{\varphi\widehat{S}}, \psi >$$

Using Proposition 11.15. (note that $\varphi \in \mathcal{S}(\mathbb{R})$ and $\widehat{S} \in \mathcal{S}^*(\mathbb{R})$) we finally get

$$< T_{\widehat{f\varphi}}, \psi > \; = \; < T_{\widehat{\varphi * \widehat{S}}}, \psi > \overset{Prop.\,10.6.2}{=} < T_{\widehat{\varphi} * S_{\sigma}}, \psi > \qquad \text{for all} \quad \psi \in \mathcal{D}(\mathbb{R})$$

Thus, $T_{\widehat{f\varphi}} = T_{\widehat{\varphi} * S_{\sigma}}$ on $\mathcal{D}^*(\mathbb{R})$ and from Proposition 9.3. we conclude that $\widehat{f\varphi} = \widehat{\varphi} * S_{\sigma}$ a.e. Using this in 11.6.1

$$< \widehat{S} \cdot \widehat{T}, \varphi > \; = \; < T, \widehat{\varphi} * S_{\sigma} > \overset{Def.\,11.1.}{=} < T^u, < (S_{\sigma})^y, \widehat{\varphi}(u-y) >> \; = \; < T^u, < S^y, \widehat{\varphi}(u+y) >>$$

$$= \; < T * S, \widehat{\varphi} > \; = \; < \widehat{T * S}, \varphi > \qquad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R})$$

where we have used that $T * S \in \mathcal{S}^*(\mathbb{R})$ (Proposition 11.8.). Thus, we conclude that $\widehat{S * T} = \widehat{S} \cdot \widehat{T}$.

Chapter 12

Filters and distributions

12.1 Introduction

The purpose of this chapter is to do a brief study of signals and the systems that transmit them. The notion of signal is extensive. The observation of some phenomenon yields certain quantities that depend on time (or on space, on frequency...). These quantities, which are assumed to be measurable, will be called signals. They correspond in mathematics to the notion of function (and more generally, as we will do, the notion of distribution). Some examples of signals are:

- Intensity of an electric current.
- Potential difference between two points in a circuit.
- Gray levels of the points of an image $g(i, j)$
- A sound

Any entity, or apparatus, where one can distinguish input signals and output signals will be called a transmission system. That is, we will have an input signal (distribution) T_x which will be transformed by a transmission system in an output signal (distribution) T_y.

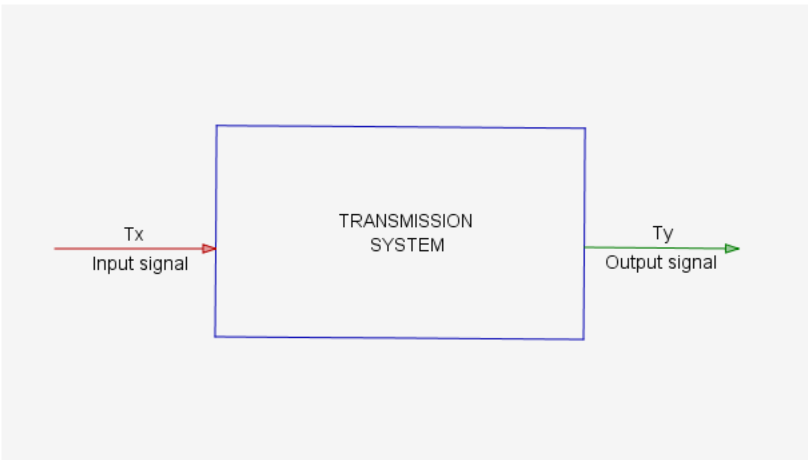


Figure 12.1.1: Diagram of a system

The transmission system will be modeled by an operator A acting on distributions, and we will write $T_y = A(T_x)$ where $T_x \in X$, the set of input signals, and $T_y \in Y$ the set of output signals. Examples of systems are:

- An electric circuit
- An amplifier
- The telephone and the Internet.

In particular, we will study filters which are systems that remove some unwanted components or features from the input signal.

12.2 Filters and properties

Definition 12.1. (Analog Filter) *Let X be a translation-invariant linear subspace of $\mathcal{D}^*(\mathbb{R})$. An analog filter is a mapping $A: X \longrightarrow \mathcal{D}^*(\mathbb{R})$ that is linear, invariant and continuous. That is*

- *A is linear: $A(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 A(T_1) + \lambda_2 A(T_2)$ for all $T_1, T_2 \in X$ and $\lambda_1, \lambda_2 \in \mathbb{R}$.*
- *A is invariant: $A(\tau_a T) = \tau_a A(T)$ for all $T \in X$ and $a \in \mathbb{R}$.*
- *A is continuous: if $(T_n)_n$ is a sequence in X such that $T_n \overset{n\longrightarrow+\infty}{\longrightarrow} T$ in X. Then $A(T_n) \overset{n\longrightarrow+\infty}{\longrightarrow} A(T)$ in $\mathcal{D}^*(\mathbb{R})$.*

Remark 12.1. *We will consider $X \subseteq \mathcal{D}^*(\mathbb{R})$ with a topology that is at least as fine as the topology induced in $\mathcal{D}^*(\mathbb{R})$. We will typically consider X as $\mathcal{K}^*(\mathbb{R})$, $\mathcal{S}^*(\mathbb{R})$, $\mathcal{D}^*_+(\mathbb{R})$, $\mathcal{D}^*(\mathbb{R})$... X will be the space of input signals.*

Proposition 12.1. (convolution filters) *The convolution system $A: X \longrightarrow \mathcal{D}^*(\mathbb{R})$ defined by $A(T) = S * T$ is an analog filter in the following cases:*

1. *$S \in \mathcal{D}^*_+(\mathbb{R})$ and $X \subset \mathcal{D}^*_+(\mathbb{R})$.*
2. *S is a distribution with bounded support, that is, $S \in \mathcal{K}^*(\mathbb{R})$ and X is any subspace of $\mathcal{D}^*(\mathbb{R})$.*

PROOF.

• Proof 1: from Theorem 11.2. and Proposition 11.12. we see that $A(T) = S * T$ makes sense and is in $\mathcal{D}^*_+(\mathbb{R}) \subset \mathcal{D}^*(\mathbb{R})$. We should show that A is linear, invariant and continuous. In effect, A is linear since for all $T_1, T_2 \in \mathcal{D}^*_+(\mathbb{R})$, $\lambda_1, \lambda_2 \in \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R})$

$$< A(\lambda_1 T_1 + \lambda_2 T_2), \varphi > = < S * (\lambda_1 T_1 + \lambda_2 T_2), \varphi > = < S^t, < \lambda_1 T_1 + \lambda_2 T_2, \varphi(x+t) >_s >_s =$$
$$= \lambda_1 < S^t, < T_1, \varphi(x+t) >_s >_s + \lambda_2 < S^t, < T_2, \varphi(x+t) >_s >_s = \lambda_1 < (S * T_1), \varphi > +$$
$$+ \lambda_2 < (S * T_2), \varphi > = < \lambda_1 (S * T_1) + \lambda_2 (S * T_2), \varphi > = < \lambda_1 A(T_1) + \lambda_2 A(T_2), \varphi >$$

Thus, we conclude that $A(\lambda_1 T_1 + \lambda_2 T_2) = \lambda_1 A(T_1) + \lambda_2 A(T_2)$, i.e., A is linear. Moreover, A is invariant since for all $T \in \mathcal{D}^*_+(\mathbb{R})$, $a \in \mathbb{R}$ and $\varphi \in \mathcal{D}(\mathbb{R})$

$$< A(\tau_a T), \varphi > = < S*(\tau_a T), \varphi > = < S^t, < (\tau_a T)^x, \varphi(x+t) >_s >_s = < S^t, < T^x, \tau_{-a}\varphi(x+t) >_s >_s$$

$$= < S * T, \tau_{-a}\varphi > \; = \; < A(T), \tau_{-a}\varphi > \; = \; < \tau_a A(T), \varphi >$$

Thus, we get that $A(\tau_a T) = \tau_a A(T)$, i.e., A is invariant. Finally, the continuity of A follows from Theorem 11.2.3.

• Proof 2: from Theorem 11.1. we see that $A(T) = S * T$ is well-defined. To see that A is linear, we use the same argument as in 1. Moreover, using Proposition 11.13. and Proposition 11.4.

$$\tau_a A(T) \; = \; \tau_a (S * T) \; = \; \delta_a * S * T \; = \; S * \delta_a * T \; = \; S * \tau_a T \; = \; A(\tau_a T) \qquad \text{for all } T \in X \text{ and } a \in \mathbb{R}$$

Thus, A is invariant. Finally, the continuity of A follows from Theorem 11.1.2.

Definition 12.2. (Impulse response) *Let us consider an analog filter $A : X \longrightarrow \mathcal{D}^*(\mathbb{R})$. Assume that $\delta \in X$. We define the impulse response I of A as the response of the system to the Dirac delta, that is $I = A(\delta)$.*

Remark 12.2. *When δ is not in X, for example when $X = L^2(\mathbb{R})$, one can still define the impulse response because in practice all of the filters encountered will be convolution systems.*

Definition 12.3. (Step response) *Let us consider an analog filter $A : X \longrightarrow \mathcal{D}^*(\mathbb{R})$. We define the step response U of A as $U = A(T_u)$, where u is the Heaviside function.*

Definition 12.4. (Transfer function) *Let us consider an analog filter $A : X \longrightarrow \mathcal{D}^*(\mathbb{R})$. Assume that its impulse response is in $\mathcal{S}^*(\mathbb{R})$. We define the transfer function of A as the Fourier transform of its impulse response I. That is, $H = \widehat{I} = \widehat{A(\delta)}$.*

Lemma 12.1. (sinusoidal signals) *Let us define the sinusoidal signals $e_\lambda(t) = e^{2i\pi\lambda t}$ for $\lambda \in \mathbb{R}$. Then T_{e_λ} are tempered distributions for all $\lambda \in \mathbb{R}$. Moreover, $\widehat{T_{e_\lambda}} = \delta_\lambda$.*

PROOF.

• Note that $e_\lambda \in L^1_{loc}(\mathbb{R})$. Consequently $T_{e_\lambda} \in \mathcal{D}^*(\mathbb{R})$. We will use Proposition 10.1. to see that $T_{e_\lambda} \in \mathcal{S}^*(\mathbb{R})$. Let $(\varphi_n)_n$ be a sequence in $\mathcal{D}^*(\mathbb{R})$ such that $\varphi_n \overset{n \longrightarrow +\infty}{\longrightarrow} 0$ in $\mathcal{S}^*(\mathbb{R})$. Thus

$$|< T_{e_\lambda}, \varphi_n >| \; \leq \; \int_{\mathbb{R}} |e_\lambda(t)\varphi_n(t)|\, dt \; = \; \int_{\mathbb{R}} \frac{1}{1+t^2} \left|\left(1+t^2\right)\varphi_n(t)\right| dt \; \leq \; \pi \cdot \sup_{t \in \mathbb{R}} \left|\left(1+t^2\right)\varphi_n(t)\right| \overset{n \longrightarrow +\infty}{\longrightarrow} 0$$

Hence, we conclude by Proposition 10.1. that T_{e_λ} is a tempered distribution.

• Now, as $T_{e_\lambda} \in \mathcal{S}^*(\mathbb{R})$ then $\widehat{T_{e_\lambda}} \in \mathcal{S}^*(\mathbb{R})$ is well-defined and

$$< \widehat{T_{e_\lambda}}, \varphi > = < T_{e_\lambda}, \widehat{\varphi} > = \int_{\mathbb{R}} e^{2i\pi\lambda t}\widehat{\varphi}(t)dt \; = \; \check{\widehat{\varphi}}(\lambda) \; = \; \varphi(\lambda) \; = < \delta_\lambda, \varphi > \qquad \text{for all} \quad \varphi \in \mathcal{S}(\mathbb{R})$$

Thus, we conclude that $\widehat{T_{e_\lambda}} = \delta_\lambda$.

Proposition 12.2. *Let us consider a convolution filter $A : X \longrightarrow \mathcal{D}^*(\mathbb{R})$ defined by $A(T) = I * T$. Assume that $T_{e_\lambda} \in X$ and that the impulse response I of A satisfies one of the following conditions*

1. $I \in \mathcal{K}^*(\mathbb{R})$

2. *There exists* $f \in \mathcal{S}(\mathbb{R})$ *such that* $I = T_f$.

Then, there exists $h \in C^\infty(\mathbb{R})$ *such that* $H = T_h$ *and* $A(T_{e_\lambda}) = h(\lambda)T_{e_\lambda}$ *for all* $\lambda \in \mathbb{R}$. *In other words,* T_{e_λ} *is an eigenfunction of* A *with eigenvalue* $h(\lambda)$.

PROOF.

First of all note that in both cases $I \in \mathcal{S}^*(\mathbb{R})$ and hence its Transfer function $H = \widehat{I}$ is defined.

• Proof 1: as $T_{e_\lambda} \in \mathcal{S}^*(\mathbb{R})$ (Lemma 12.1.) and $I \in \mathcal{K}^*(\mathbb{R})$, we can apply Proposition 11.16.

$$\widehat{A(T_{e_\lambda})} = \widehat{I * T_{e_\lambda}} = \widehat{I} \cdot \widehat{T_{e_\lambda}} \overset{Lemma\ 12.1.}{=} H \cdot \delta_\lambda \quad \text{for all } \lambda \in \mathbb{R} \tag{12.2.1}$$

Now note that as $I \in \mathcal{K}^*(\mathbb{R})$, there exists $h \in C^\infty(\mathbb{R})$ such that $H = \widehat{I} = T_h$ (Theorem 10.3). Thus,

$$< H \cdot \delta_\lambda, \varphi > = < T_h \cdot \delta_\lambda, \varphi > \overset{Def. 11.2}{=} < \delta_\lambda, h\varphi > = h(\lambda)\varphi(\lambda) = < h(\lambda)\delta_\lambda, \varphi >$$

And we conclude that $H \cdot \delta_\lambda = h(\lambda)\delta_\lambda$. Using this in (12.2.1), we see that $\widehat{A(T_{e_\lambda})} = h(\lambda)\delta_\lambda$. Finally, applying the operator $\check{\cdot}$ in both sides of this equality

$$A(T_{e_\lambda}) = h(\lambda)T_{e_\lambda} \quad \text{for all } \lambda \in \mathbb{R}$$

• Proof 2: note that by Proposition 10.4. it holds $H = \widehat{I} = \widehat{T_f} = T_{\widehat{f}}$ where $\widehat{f} \in \mathcal{S}(\mathbb{R}) \subset C^\infty(\mathbb{R})$. Moreover, as $f \in \mathcal{S}(\mathbb{R})$ and $T_{e_\lambda} \in \mathcal{S}^*(\mathbb{R})$, we can apply Proposition 11.15.

$$\widehat{A(T_{e_\lambda})} = \widehat{I * T_{e_\lambda}} = \widehat{T_f} \cdot \widehat{T_{e_\lambda}} \overset{Lemma\ 12.1.}{=} H \cdot \delta_\lambda \quad \text{for all } \lambda \in \mathbb{R}$$

Similarly as before one can see that $H \cdot \delta_\lambda = \widehat{f}(\lambda)\delta_\lambda$. Hence, $\widehat{A(T_{e_\lambda})} = \widehat{f}(\lambda)\delta_\lambda$ and applying the operator $\check{\cdot}$ in both sides of this equality we finally get

$$A(T_{e_\lambda}) = \widehat{f}(\lambda)T_{e_\lambda} \quad \text{for all } \lambda \in \mathbb{R}$$

Definition 12.5. (realizable/causal filters) *Let* $A : X \longrightarrow \mathcal{D}^*(\mathbb{R})$ *be an analog filter. A is said to be realizable (or causal) if supp*$(T) \subseteq [t_0, +\infty)$ *implies that supp*$(A(T)) \subseteq [t_0, +\infty)$ *for all* $T \in X$ *and* $t_0 \in \mathbb{R}$.

Remark 12.3. *This property is completely natural for a physical system in which the variable is time. It says that the response at time t depends only on what has happened before t. In particular, the system does not respond before there is an input.*

Proposition 12.3. *Let us consider a convolution filter* $A : X \longrightarrow \mathcal{D}^*(\mathbb{R})$ *defined by* $A(T) = I * T$. *Then A is realizable if and only if supp*$(I) \subseteq [0, +\infty)$.

PROOF.

• If A is realizable, as supp$(\delta) \subseteq [0, +\infty)$ and $I = A(\delta)$, then supp$(I) \subseteq [0, +\infty)$.

• Conversely, if supp$(I) \subseteq [0, +\infty)$ and supp$(T) \subseteq [t_0, +\infty)$ for some $T \in X$ and $t_0 \in \mathbb{R}$, then I and T are in $\mathcal{D}^*_+(\mathbb{R})$. Consequently, supp$(I * T) \subseteq [t_0, +\infty)$ by Proposition 11.12.

12.3 Tempered solutions of linear differential equations

Let us consider a system $A: X \subset \mathcal{D}^*(\mathbb{R}) \longrightarrow \mathcal{D}^*(\mathbb{R})$ whose input $T \in X$ and output $G = A(T)$ are related by the differential equation

$$\sum_{k=0}^{q} b_k G^{(k)} = \sum_{j=0}^{p} a_j T^{(j)} \qquad \text{with} \quad b_q,\, a_p \neq 0 \tag{12.3.1}$$

Note that (12.3.1) is taken in the sense of distributions. The coefficients a_j and b_k are fixed complex numbers, and we will assume that T is a tempered distribution. We will see that in general (12.3.1) has a unique solution G in $\mathcal{S}^*(\mathbb{R})$, which defines the output of the system. We define the two polynomials

$$P(x) := \sum_{j=0}^{p} a_j x^j \qquad \text{and} \qquad Q(x) := \sum_{k=0}^{q} b_k x^k \tag{12.3.2}$$

Proposition 12.4. *Let $T \in \mathcal{S}^*(\mathbb{R})$ and assume that $\frac{P(x)}{Q(x)}$ has no poles on the imaginary axis. Then, (12.3.1) has a unique solution $G \in \mathcal{S}^*(\mathbb{R})$.*

PROOF.

• Uniqueness: first, let us assume that there exists $G \in \mathcal{S}^*(\mathbb{R})$ solving (12.3.1). Then, we can take the Fourier transform of both sides and using Proposition 10.5.1. we get

$$\widehat{\sum_{k=0}^{q} b_k G^{(k)}} = \widehat{\sum_{j=0}^{p} a_j T^{(j)}} \implies \sum_{k=0}^{q} b_k \widehat{G^{(k)}} = \sum_{j=0}^{p} a_j \widehat{T^{(j)}} \implies \sum_{k=0}^{q} b_k (2i\pi\lambda)^k \widehat{G} = \sum_{j=0}^{p} a_j (2i\pi\lambda)^j \widehat{T} \implies$$

$$\implies Q(2i\pi\lambda)\widehat{G} = P(2i\pi\lambda)\widehat{T} \implies \widehat{G} = \frac{P(2i\pi\lambda)}{Q(2i\pi\lambda)}\widehat{T} \tag{12.3.3}$$

Where we have used that $\frac{P(x)}{Q(x)}$ has no poles on the imaginary axis. Note that $\widehat{T} \in \mathcal{S}^*(\mathbb{R})$ and $\frac{P(2i\pi\lambda)}{Q(2i\pi\lambda)}$ is slowly increasing, thus it is easy to see that $\frac{P(2i\pi\lambda)}{Q(2i\pi\lambda)}\widehat{T} \in \mathcal{S}^*(\mathbb{R})$. This shows that $\widehat{G}$ is uniquely determined and Theorem 10.1. implies that G is uniquely determined. Thus (12.3.1) has at most one solution in $\mathcal{S}^*(\mathbb{R})$.

• Existence: let us define $h(\lambda) := \frac{P(2i\pi\lambda)}{Q(2i\pi\lambda)}$. As it is slowly increasing then $H := T_h$ is a tempered distribution (Lemma 11.2.) and has an inverse Fourier transform $I = \check{H}$ which we can compute by descomposing h into partial fractions. We consider the following cases:

- *Case 1: h has only simple poles.* It holds that h can be written as:

$$h(\lambda) = \sum_{j=0}^{p-q} \alpha_j \,(2i\pi\lambda)^j + \sum_{k=1}^{q} \frac{\beta_k}{2i\pi\lambda - z_k} \qquad \alpha_j,\, \beta_k \in \mathbb{C}$$

where we define $\alpha_j = 0$ if $j < 0$ (the polynomial part is zero if $p < q$) and $z_1, ..., z_q$ are the simple poles of $\frac{P(x)}{Q(x)}$ in $\mathbb{C}$. Now, let us define:

$$K_- := \{k \in \{1, 2, ..., q\} \text{ such that } \operatorname{Re}(z_k) < 0\}$$

$$K_+ := \{k \in \{1, 2, ..., q\} \text{ such that } \operatorname{Re}(z_k) > 0\}$$

Note that it holds

$$H = T_h = \sum_{j=0}^{p-q} \alpha_j T_{(2i\pi\lambda)^j} + \sum_{k\in K_-} \beta_k \tau_{z_k} T_{\frac{1}{2i\pi\lambda}} + \sum_{k\in K_+} \beta_k \tau_{z_k} T_{\frac{1}{2i\pi\lambda}}$$

Taking the inverse Fourier transform of T_h and using the properties defined in chapter 10, it can be seen that

$$I = \check{H} = \sum_{j=0}^{p-q} \alpha_j \delta^{(j)} + \sum_{k\in K_-} \beta_k e^{z_k t} T_u - \sum_{k\in K_+} \beta_k e^{z_k t} T_{u_\sigma}$$

where u is the Heaviside's function.

- *Case 2: h has multiple poles.* Note that for the polynomial part of h we will be able to apply the same argument as in 1. Thus, we can limit ourselves to the case $p < q$. Let us assume that $z_1, z_2, ..., z_l$ are the poles of $\frac{P(x)}{Q(x)}$ in $\mathbb{C}$ $(l \leq q)$ and let $m_1, m_2, ..., m_l$ be their respective multiplicities. Then, we can write

$$h(\lambda) = \sum_{k=1}^{l} \sum_{m=1}^{m_k} \frac{\beta_{k,m}}{(2i\pi\lambda - z_k)^m} \quad \Longrightarrow \quad T_h = \sum_{k=1}^{l} \sum_{m=1}^{m_k} \beta_{k,m} \tau_{z_k} T_{\frac{1}{(2i\pi\lambda)^m}}$$

Taking the inverse Fourier transform of T_h and using the properties defined in chapter 10, it can be seen that

$$I = \check{T}_h = \left(\sum_{k\in K_-} P_k(t) e^{z_k t} \right) T_u - \left(\sum_{k\in K_+} P_k(t) e^{z_k t} \right) T_{u_\sigma} \quad \text{with} \quad P_k(t) := \sum_{m=1}^{m_k} \beta_{k,m} \frac{t^{m-1}}{(m-1)!}$$

- *Conclusion:* from case 1 and case 2, we can conclude that the general form of I will be

$$I = \sum_{j=0}^{p-q} \alpha_j \delta^{(j)} + \left(\sum_{k\in K_-} P_k(t) e^{z_k t} \right) T_u - \left(\sum_{k\in K_+} P_k(t) e^{z_k t} \right) T_{u_\sigma} \quad \text{with} \quad P_k(t) := \sum_{m=1}^{m_k} \beta_{k,m} \frac{t^{m-1}}{(m-1)!}$$

- At this point, we know from (12.3.3) that $\widehat{G} = h(\lambda)\widehat{T} = T_h\widehat{T} = \widehat{I} \cdot \widehat{T}$. We would like to take the inverse Fourier transform in the expression and apply the results of Proposition 11.15. and Proposition 11.16. For this reason, we are going to express I in a different way. Let us take $\theta \in \mathcal{D}^* (\mathbb{R})$ such that

$$\theta = \begin{cases} 1 & \text{if } |t| \leq 1 \\ 0 & \text{if } |t| \geq 2 \end{cases}$$

Note that $I = I\theta + I(1-\theta) = I_1 + I_2$. Consequently, the following facts hold:

- $I_1 = I\theta = \sum_{j=0}^{p-q} \alpha_j \delta^{(j)} + \left(\sum_{k\in K_-} P_k(t) e^{z_k t} \right) T_{\theta u} - \left(\sum_{k\in K_+} P_k(t) e^{z_k t} \right) T_{\theta u_\sigma}$ which is in $\mathcal{K}^* (\mathbb{R})$.
- $I_2 = I(1-\theta) = T_f$ where $f(t) := \sum_{k\in K_-} P_k(t) e^{z_k t}(1-\theta)u - \sum_{k\in K_+} P_k(t) e^{z_k t}(1-\theta)u_\sigma$. It is easy to see that $f \in \mathcal{S}(\mathbb{R})$.

Hence, as $T \in \mathcal{S}^* (\mathbb{R})$, we can apply Proposition 11.15. and Proposition 11.16. and we get:

$$\widehat{I * T} = \widehat{I_1 * T} + \widehat{I_2 * T} = \widehat{I_1} \cdot \widehat{T} + \widehat{I_2} \cdot \widehat{T} = \widehat{I} \cdot \widehat{T}$$

Consequently, we can write $\widehat{G} = \widehat{I} \cdot \widehat{T} = \widehat{I * T}$ and taking the inverse Fourier transform in both sides

$$G = I * T$$

which shows that (12.3.1) has solution and A is a convolution system.

Remark 12.4. *If $\frac{P(x)}{Q(x)}$ has a pole on the imaginary axis, then the solution G in $\mathcal{S}^*(\mathbb{R})$ is no longer unique. For example, let us consider the differential equation*

$$G'' + w^2 G = \delta \qquad \textit{with} \quad w > 0$$

It can be proved by a straightforward calculation that this equation has as solutions in $\mathcal{S}^*(\mathbb{R})$ all the generalized functions T_g where

$$g(t) = \begin{cases} Acos(wt) + Bsin(wt) & \textit{if } t < 0 \\ Acos(wt) + \left(B + \frac{1}{w}\right) sin(wt) & \textit{if } t > 0 \end{cases}$$

One can prove that $T_g \in \mathcal{S}^*(\mathbb{R})$ by computing a similar proof to the one made in Lemma 12.1.

Proposition 12.5. *Let us consider the polynomials P and Q defined in (12.3.2). Assume that $\frac{P(x)}{Q(x)}$ has no poles on the imaginary axis. Then the system $A : \mathcal{S}^*(\mathbb{R}) \longrightarrow \mathcal{S}^*(\mathbb{R})$ defined by the solution of (12.3.1) is an analog filter.*

PROOF. The proof of this proposition can be found in [C. Gasquet2010].

Proposition 12.6. *Under the hypothesis of Proposition 12.5., let $A : \mathcal{S}^*(\mathbb{R}) \longrightarrow \mathcal{S}^*(\mathbb{R})$ be the filter defined by the solution of (12.3.1). Then A is realizable if and only if the real parts of the poles $\frac{P(x)}{Q(x)}$ are strictly negative.*

PROOF.
From Proposition 12.3. A is realizable if and only if supp$(I) \subseteq [0, +\infty)$. Note that in the proof of Proposition 12.4. we got that

$$I = \sum_{j=0}^{p-q} \alpha_j \delta^{(j)} + \left(\sum_{k \in K_-} P_k(t) e^{z_k t} \right) T_u - \left(\sum_{k \in K_+} P_k(t) e^{z_k t} \right) T_{u_\sigma}$$

Thus, we deduce that supp$(I) \subseteq [0, +\infty)$ if and only if the term $\left(\sum_{k \in K_+} P_k(t) e^{z_k t} \right) T_{u_\sigma}$ does not appear in I. And this fact happens if and only if $K_+ = \emptyset$, i.e. if the real parts of the poles $\frac{P(x)}{Q(x)}$ are strictly negative.

12.4 Causal solutions of linear differential equations

We are going to look for the causal solutions of a linear differential equation with constant coefficients. For convenience, we write the equation with $b_q = 1$

$$\sum_{k=0}^{q-1} b_k G^{(k)} + G^{(q)} = \sum_{j=0}^{p} a_j T^{(j)} \qquad \text{with} \quad a_p \neq 0 \tag{12.4.1}$$

We assume $T \in \mathcal{D}_+^*(\mathbb{R})$ and we wish to find a solution $G \in \mathcal{D}_+^*(\mathbb{R})$. Thus, the filter will be realizable. Note that in this case, we cannot use the Fourier transform because T and G are not assumed to be tempered. This lack of restriction is essential, since we will find solutions that grow exponentially.

Existence and uniqueness of a causal solution

We start transforming the equation (12.4.1) into a first-order linear system. Let us introduce the auxiliary distributions $G_1, ..., G_{q-1}$ and define:

$$\begin{cases} G_1 := G' \\ G_2 := G_1' \\ \quad\vdots \\ G_{q-1} := G_{q-2}' \\ S := \sum\limits_{j=0}^{p} a_j T^{(j)} \end{cases}$$

Now, if we define

$$M := \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{q-1} \end{bmatrix}, \quad \overline{G} = \begin{bmatrix} G \\ G_1 \\ \vdots \\ G_{q-2} \\ G_{q-1} \end{bmatrix}, \quad \Phi = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ S \end{bmatrix}$$

Then, (12.4.1) can be written as the matrix equation $\overline{G}' = M\overline{G} + \Phi$. Let us define $e^{tM} := Id + tM + \frac{t^2}{2!}M^2 + \ldots + \frac{t^n}{n!}M^n + \ldots = \sum\limits_{k=0}^{\infty} \frac{t^k}{k!}M^k$. One can proves the following properties

1. e^{tM} is well-defined, i.e., the sequence $\sum\limits_{k=0}^{\infty} \frac{t^k}{k!}M^k$ converges for all real t in the (normed) space of $q \times q$ matrices.
2. e^{tM} is invertible and its inverse is e^{-tM}.
3. The function $t \longmapsto e^{tM}$ is derivable and its derivative is the function $t \longmapsto Me^{tM}$.
4. It holds that $e^{tM}e^{sM} = e^{(s+t)M}$.

Thus, let us define $X(t) := e^{-tM}\overline{G}(t)$. Then, from the matrix equation $\overline{G}' = M\overline{G} + \Phi$ we get that $X'(t) = e^{-tM}\Phi(t)$. The solution of this differential equation is

$$X(t) = X_0 + \int_{-\infty}^{t} e^{-sM}\Phi(s)ds$$

and all the solutions of $\overline{G}' = M\overline{G} + \Phi$ will be of the form

$$\overline{G}(t) = e^{tM}X(t) = e^{tM}X_0 + \int_{-\infty}^{t} e^{(t-s)M}\Phi(s)ds$$

where X_0 is an arbitrary fixed vector. Note that as Φ has the distribution S in one of its components, the integral $\int_{-\infty}^{t} e^{(t-s)M}\Phi(s)ds$ is solved as in the case of functions using Theorem 9.3. Assume that

$\text{supp}(T) \subseteq [t_0, +\infty)$, then $\text{supp}(S) \subseteq [t_0, +\infty)$. Thus, $\Phi(t) = 0$ and $\overline{G}(t) = e^{tM}X_0$ for all $t < t_0$. Moreover, if we impose $G(t)$ to be causal, then necessarily $X_0 = 0$. Consequently, $\overline{G}' = M\overline{G} + \Phi$ has the unique causal solution

$$\overline{G}(t) = \int_{-\infty}^{t} e^{(t-s)M}\Phi(s)ds$$

and therefore (12.4.1) has a unique causal solution G. Moreover, since $\overline{G}(t) = 0$ for $t < t_0$, then $G(t) = 0$ for $t < t_0$ and the system is realizable.

Proposition 12.7. *If $T \in \mathcal{D}_+^*(\mathbb{R})$ then equation (12.4.1) has a unique solution $G \in \mathcal{D}_+^*(\mathbb{R})$. Moreover, the system $B : \mathcal{D}_+^*(\mathbb{R}) \longrightarrow \mathcal{D}_+^*(\mathbb{R})$ defined by the unique solution $G \in \mathcal{D}_+^*(\mathbb{R})$ of (12.4.1) is a convolution system and hence a filter.*

PROOF.

We have already shown in the beginning of this section that (12.4.1) has a unique solution $G \in \mathcal{D}_+^*(\mathbb{R})$ when $T \in \mathcal{D}_+^*(\mathbb{R})$. Hence the system B is well-defined. We are going to show that $B(T) = G$ is given by a convolution.

First of all we compute the impulse response I of the system B which is given by $I = B(\delta)$. That is, I is the unique solution in $\mathcal{D}_+^*(\mathbb{R})$ of (12.4.1) with input $T = \delta \in \mathcal{D}_+^*(\mathbb{R})$. Then

$$\sum_{k=0}^{q-1} b_k I^{(k)} + I^{(q)} = \sum_{j=0}^{p} a_j \delta^{(j)}$$

Now, let $T \in \mathcal{D}_+^*(\mathbb{R})$. Taking the convolution with the input T in both sides of the previous equality and using Theorem 11.2.

$$\sum_{k=0}^{q-1} b_k (I * T)^{(k)} + (I * T)^{(q)} = \sum_{j=0}^{p} a_j (\delta * T)^{(j)} = \sum_{j=0}^{p} a_j T^{(j)}$$

As $I * T \in \mathcal{D}_+^*(\mathbb{R})$ is a solution of (12.4.1) with the input $T \in \mathcal{D}_+^*(\mathbb{R})$ and this solution is unique, we conclude that

$$B(T) = G = I * T \qquad \text{for all} \quad T \in \mathcal{D}_+^*(\mathbb{R}) \tag{12.4.2}$$

Hence, the system B is a convolution system and from Proposition 12.1.1 we conclude that B is an analog filter.

Remark 12.5. *Notice once again, the fact that the differential equation has a unique solution is a consequence of a constraint on G, in this case, it is that G have support limited on the left.*

12.5 Examples of filters

***The** RC **filter** $RCG' + G = T$*

Let us consider the following circuit with a resistor R and a capacitor C. Assume that the input to the circuit is the voltage $x(t)$ and the output is the voltage $v(t)$.

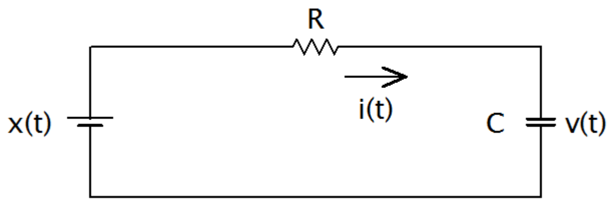


Figure 12.5.1: RC circuit

• The potential difference across a capacitor with charge Q is $v(t) = \frac{Q(t)}{C}$, thus by Ohm's law we get

$$Ri(t) + v(t) = x(t)$$

Moreover, the intensity i in the resistor is the same as the intensity in the capacitor and $Q^{'}(t) = i(t)$. Thus we obtain

$$RCv'(t) + v(t) = x(t)$$

Therefore, we see that this system is governed by a first-order linear differential equation with constant coefficients. We assume $x \in L^1(\mathbb{R})$. Then, solution is given by

$$v(t) = \frac{1}{RC}\int_{-\infty}^{t} e^{-\frac{t-s}{RC}} x(s)ds \tag{12.5.1}$$

• We are going to consider more general input and output signals. We use Proposition 12.7. We consider the system $B : \mathcal{D}_{+}^{*}(\mathbb{R}) \longrightarrow \mathcal{D}_{+}^{*}(\mathbb{R})$ defined by the unique solution $B(T) = G \in \mathcal{D}_{+}^{*}(\mathbb{R})$ of

$$RCG^{'} + G = T \qquad \text{where} \qquad T \in \mathcal{D}_{+}^{*}(\mathbb{R})$$

We are going to find the output signal $B(T) = G$. The strategy will be: first we will find the impulse response I of the system B and later we will use (12.4.2) to obtain the output signal.

1) Impulse response: we shall find the unique solution $I \in \mathcal{D}_{+}^{*}(\mathbb{R})$ of $RCI^{'} + I = \delta$. We will try with a generalized function $I = T_f$ with $f \in \mathcal{D}_{+}(\mathbb{R})$. Then, for $\varphi \in \mathcal{D}(\mathbb{R})$

$$< RCT_f^{'}, \varphi > + < T_f, \varphi > = < \delta, \varphi > \Longrightarrow -\int_{\mathbb{R}} RCf(x)\varphi^{'}(x)dx + \int_{\mathbb{R}} f(x)\varphi(x)dx = \varphi(0)$$

$$\overset{parts}{\Longrightarrow} \int_{\mathbb{R}} \left(RCf^{'}(x) + f(x)\right)\varphi(x)dx = \varphi(0)$$

Then for $\varphi \in \mathcal{D}(\mathbb{R})$ with $\text{supp}(\varphi) \subseteq (0, +\infty)$ we will have $\int_0^{+\infty}\left(RCf^{'}(x) + f(x)\right)\varphi(x)dx = 0$. Thus

$$RCf^{'}(x) + f(x) = 0 \Longrightarrow f(x) = ke^{-\frac{x}{RC}} \qquad k \in \mathbb{R} \quad \text{and} \quad x \in (0, +\infty)$$

Moreover, it should hold that $f \in \mathcal{D}_{+}(\mathbb{R})$ since we want $I = T_f \in \mathcal{D}_{+}^{*}(\mathbb{R})$. This suggests that $f(x) = ke^{-\frac{x}{RC}}u(x)$ is a good candidate (where u is the Heaviside function). Note that:

$$< RCT_f^{'} + T_f, \varphi > = -\int_0^{\infty} RCke^{-\frac{x}{RC}}\varphi^{'}(x)dx + \int_0^{\infty} ke^{-\frac{x}{RC}}\varphi(x)dx = RCk\varphi(0) - \int_0^{\infty} ke^{-\frac{x}{RC}}\varphi(x)dx$$

$$+\int_0^{\infty} ke^{-\frac{x}{RC}}\varphi(x)dx \;=\; < RCk\delta, \varphi >$$

Thus, taking $k = \frac{1}{RC}$ we get that $RCT_f' + T_f = \delta$. Hence, the impulse response is $I = T_f$ with $f(x) = \frac{1}{RC}e^{-\frac{x}{RC}}u(x)$.

2) Output signal: we have seen in (12.4.2) in Proposition 12.7. that the causal solution is given by

$$B(T) \;=\; G \;=\; I * T \;=\; T_f * T$$

Note that when T is a generalized function, i.e. $T = T_g$ with $g \in \mathcal{D}_+(\mathbb{R})$, we have by Theorem 11.2.

$$< B(T_g), \varphi > = < T_f * T_g, \varphi > = < (T_f)^t, < (T_g)^x, \varphi(x+t) >> = \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} g(x)\varphi(x+t)dxdt \overset{x=y-t}{=}$$

$$= \int_{\mathbb{R}} f(t) \int_{\mathbb{R}} g(y-t)\varphi(y)dydt \;=\; \int_{\mathbb{R}} \varphi(y) \int_{\mathbb{R}} g(y-t)f(t)dtdy \;=\; \int_{\mathbb{R}} (f*g)(y)\varphi(y)dy \;= < T_{f*g}, \varphi >$$

Hence $B(T_g) = T_{f*g}$ where we have that

$$(f*g)(t) \;=\; \int_{\mathbb{R}} g(x)f(t-x)dx \;=\; \frac{1}{RC}\int_{-\infty}^{t} e^{-\frac{t-x}{RC}}g(x)dx$$

That is, we have obtained that the output signal is the generalized function of the solution obtained in (12.5.1).

The integrator $G' = T$

We consider the system $B : \mathcal{D}_+^*(\mathbb{R}) \longrightarrow \mathcal{D}_+^*(\mathbb{R})$ defined by the unique solution $B(T) = G \in \mathcal{D}_+^*(\mathbb{R})$ of $G' = T$. The impulse response I is given by the solution of $I' = \delta$. By example 9.7. we know that $I = T_u$ (u is the Heaviside function). Thus the output solution is given by

$$B(T) \;=\; G \;=\; I * T \;=\; T_u * T \qquad \text{for all} \quad T \in \mathcal{D}_+^*(\mathbb{R})$$

Note that when T is a generalized function, i.e. $T = T_g$ with $g \in \mathcal{D}_+(\mathbb{R})$, we have

$$< B(T_g), \varphi > \;=\; < T_u * T_g, \varphi > \;=\; < T_{u*g}, \varphi > \qquad \text{for all} \quad \varphi \in \mathcal{D}^*(\mathbb{R})$$

Consequently we get that $B(T_g) = T_{u*g}$. That is, $B(T_g)$ is the generalized function of

$$(u*g)(x) \;=\; \int_0^{+\infty} g(t-x)dx \overset{y=t-x}{=} \int_{-\infty}^{t} g(y)dy$$

The differentiator $G = T'$

We consider the system $B : \mathcal{D}_+^*(\mathbb{R}) \longrightarrow \mathcal{D}_+^*(\mathbb{R})$ defined by the unique solution $B(T) = G \in \mathcal{D}_+^*(\mathbb{R})$ of $G = T'$. The impulse response I is given by $I = \delta'$. Thus, using Proposition 11.4.2, the output solution is given by

$$B(T) \;=\; G \;=\; I * T \;=\; \delta' * T \;=\; T' \qquad \text{for all } T \in \mathcal{D}_+^*(\mathbb{R})$$

Note that when T is a generalized function, i.e. $T = T_g$ with $g \in \mathcal{D}_+(\mathbb{R})$, we have $B(T_g) = T_g' = T_{g'}$.

Appendix A

Concepts and theorems of Real Analysis

A.1 Lebesgue Integral Theory

Definition A.1. Let X be a set. We define a σ-algebra on the set X as a family χ of subsets of X holding the following three properties:

1. $\emptyset \in \chi$.
2. If $A \in \chi$ then $A^c = X \setminus A \in \chi$.
3. Let $\{A_n\}_{n\in\mathbb{N}}$ s.t. $A_n \in \chi$ for all $n \in \mathbb{N}$, then $\bigcup_{n=1}^{\infty} A_n \in \chi$.

Definition A.2. We define a measurable space as the pair (X, χ), where X is a set and χ is a σ-algebra on X.

Definition A.3. Let X be a set and χ be a σ-algebra on X. We say that the function $f : X \longrightarrow \mathbb{R}$ is χ- measurable if $f^{-1}((\alpha, +\infty)) \in \chi$ for all $\alpha \in \mathbb{R}$.

Definition A.4. A measure defined on a measurable space (X, χ) is a function $\mu : \chi \longrightarrow \mathbb{R}^* := \mathbb{R} \cup \{\pm\infty\}$ satisfying the following properties:

1. $\mu(\emptyset) = 0$.
2. $\mu(E) \geq 0$ for all $E \in \chi$.
3. For all countable collections $\{E_i\}_{i=1}^{\infty} \subseteq \chi$ of pairwise disjoint sets: $\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$.

Definition A.5. A measure space is a triplet (X, χ, μ) where X is a set, χ is a σ-algebra on X and μ is a measure defined in the measurable space (X, χ).

Definition A.6. Let (X, χ) be a measurable space. A function $\varphi : X \longrightarrow \mathbb{R}$ is called simple function if it can be written as

$$\varphi(x) = \sum_{j=1}^{\infty} a_j \mathbf{1}_{E_j}(x)$$

where $a_j \in \mathbb{R}$ and $E_j \in \chi$ for all $j \in \mathbb{N}$.

Definition A.7. Let (X, χ, μ) be a measure space and let $f \in M^+(X, \chi) := \{f : X \longrightarrow \mathbb{R}^* \ \chi-\text{measurable}, f(x) \geq 0 \text{ for all } x \in X\}$. We define the integral of f respect μ as:

$$\int_X f d\mu := \sup_{\varphi} \int_X \varphi d\mu = \sum_{j=1}^{\infty} a_j \mu(E_j) \in [0, +\infty) \cup \{+\infty\}$$

where the supremum is taken with respect to all the simple functions $\varphi \in M^+(X, \chi)$ satisfying $0 \leq \varphi(x) \leq f(x)$ for all $x \in X$.

Remark A.1. If $E \in \chi$, then $f\mathbf{1}_E \in M^+(X, \chi)$ and we define the integral of f respect μ on E as:

$$\int_E f d\mu := \int_X f\mathbf{1}_E d\mu$$

Theorem A.1. **(Monotone Convergence Theorem)** Let (X, χ, μ) be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a non-decreasing sequence ($\forall n \in \mathbb{N}, \forall x \in X,\ f_{n+1}(x) \geq f_n(x)$) of functions of $M^+(X, \chi)$. Suppose f_n converges pointwise to f μ-almost everywhere. Then, it holds:

$$\int_X f d\mu := \lim_{n\to+\infty} \int_X f_n d\mu$$

Definition A.8. Let $f : X \longrightarrow \mathbb{R}$ be a (extended) real-valued function:

1. We define the positive part of f as: $f^+(x) = \max(f(x), 0) = \begin{cases} f(x) & \text{if } f(x) > 0 \\ 0 & \text{otherwise.} \end{cases}$

2. We define the negative part of f as: $f^-(x) = -\min(f(x), 0) = \begin{cases} -f(x) & \text{if } f(x) < 0 \\ 0 & \text{otherwise.} \end{cases}$

Definition A.9. Let (X, χ, μ) be a measure space. We define the set of Lebesgue integral functions $L(X, \chi, \mu)$ as the set of functions $f : X \longrightarrow \mathbb{R}$ such that:

1. f is χ-measurable.

2. f^+ and f^- have finite integral respect μ.

In this case, if $E \in \chi$, we define:

$$\int_X f d\mu := \int_X f^+ d\mu - \int_X f^- d\mu \quad \text{and} \quad \int_E f d\mu := \int_E f^+ d\mu - \int_E f^- d\mu$$

Theorem A.2. **(Dominated Convergence Theorem)** Let (X, χ, μ) be a measure space. Let $\{f_n\}_{n\in\mathbb{N}}$ be a sequence of real-valued, measurable functions. Suppose that the sequence converges pointwise to a function f μ-a.e. and is dominated by some integrable function g in the sense that $|f_n(x)| \leq g(x)$ $\forall n \in \mathbb{N}$, μ-a.e. Then

$$\int_X f d\mu := \lim_{n\to+\infty} \int_X f_n d\mu$$

A.2 Lp Spaces

Definition A.10. Let (X, χ, μ) be a measure space. Let $f, g : X \longrightarrow \mathbb{R}$ be χ-measurable. We say that f and g are μ-equivalent and we will write $f \overset{\mu}{\backsim} g$ if they are equal $\mu-$almost everywhere, i.e. f and g only differ in a measure zero set. It holds that the relation $\overset{\mu}{\backsim}$ defines an equivalence relation.

Definition A.11. Let (X, χ, μ) be a measure space and let $p \in [1, +\infty)$. We define the L_p Space as: $L_p(X, \chi, \mu)$:={classes of μ-equivalence of χ-measurable functions $f : X \longrightarrow \mathbb{R}$ s.t. $|f|^p \in L(X, \chi, \mu)$}

Theorem A.3. Let (X, χ, μ) be a measure space and let $f \in L_p(X, \chi, \mu)$ with $p \in [1, +\infty)$. Let us define $\|f\|_p := \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}}$. It holds that:

1. $\|\cdot\|_p$ defines a norm in $L_p(X, \chi, \mu)$
2. $(L_p(X, \chi, \mu), \|\cdot\|_p)$ is a Banach Space.

Definition A.12. Let (X, χ, μ) be a measure space. We define the L_∞ Space as:
$L_\infty(X, \chi, \mu)$:= {classes of μ-equivalence of χ-measurables functions $f : X \longrightarrow \mathbb{R}$ s.t. are bounded up to a measure zero set}
Let us define $\|f\|_\infty := \inf\{C \geq 0 : |f(x)| \leq C$ for almost every $x\}$. It holds that $\|\cdot\|_\infty$ defines a norm in $L_\infty(X, \chi, \mu)$.

Inequality A.1. (Hölder's inequality) Let (X, χ, μ) be a measure space. Let $f \in L_p(X, \chi, \mu)$ and $g \in L_q(X, \chi, \mu)$ with $p, q \in [1, +\infty]$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then, it holds:

1. $fg \in L_1(X, \chi, \mu)$
2. $\|fg\|_1 \leq \|f\|_p \|g\|_q$

For the case $p = q = 2$ this inequality is called Cauchy-Schwarz-Bunyakovskii's inequality.

Inequality A.2. (Minkowski's inequality) Let (X, χ, μ) be a measure space. Let $f, g \in L_p(X, \chi, \mu)$ with $p \in [1, +\infty]$. Then, it holds:

1. $f + g \in L_p(X, \chi, \mu)$
2. $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

A.3 Integrals that depend on a parameter

In this section, we consider a measure space $(X,\ \chi,\ \mu)$ and an arbitrary interval (a,b), bounded or not, of $\mathbb{R}$. Let $f:(a,b)\times X \longrightarrow \mathbb{R}$ and assume that for all $t\in(a,b)$ the function $x\longmapsto f(t,x)$ is integrable. We define:

$$I(t) := \int_X f(t,x)d\mu(x) \qquad t \in (a,b)$$

Concerning the continuity and differentiability of the function $I(t)$, we have the following results.

Theorem A.4. (Continuity of parameter dependent integrals) If for almost all $x \in X$ the function $t \longmapsto f(t,x)$ is continuous at $t^* \in (a,b)$ and if there exists an integrable function g such that:

$$|f(t,x)| \leq g(x) \quad \text{almost everywhere in } X, \quad \text{for all } t \text{ in a neighborhood of } t^*$$

Then $I(t)$ is continuous at t^*.

Theorem A.5. (Differentiability of parameter dependent integrals) Suppose that V is a neighborhood of $t^* \in (a,b)$, $V \subseteq (a,b)$, such that the following two conditions hold:

1. For almost all $x \in X$, $t \longmapsto f(t,x)$ is continuously differentiable on V.
2. There exists an integrable function g such that $\left|\frac{\partial f}{\partial t}(t,x)\right| \leq g(x)$ for all $t \in V$, a.e. in X.

Then $I(t)$ is differentiable at t^* and $I'(t^*) := \int_X \frac{\partial f}{\partial t}(t^*,x)d\mu(x)$

A.4 Other key theorems

Theorem A.6. (Fubini) Let $(X,\ \chi,\ \mu_1)$ and $(Y,\ \mathcal{Y},\ \mu_2)$ be two measure spaces. Assume that $f: X\times Y \longrightarrow \mathbb{R}^*$ is measurable and that $E\times F$ is a measurable set in $X\times Y$. Then, the following statements hold:

1. If f is nonnegative on $E\times F$, then:

$$\int_{E\times F} f(x,y)dxdy = \int_E \left(\int_F f(x,y)dy\right)dx = \int_F \left(\int_E f(x,y)dx\right)dy.$$

   The three integrals can be equal to $+\infty$.

2. If f is integrable on $E\times F$, then:
   - The function $x \mapsto f(x,y)$ is integrable for almost all $y\in F$.
   - The function $y \mapsto f(x,y)$ is integrable for almost all $x \in E$
   - The three integrals in 1 are finite and equal.

3. f is integrable if and only if $\int_E dx \int_F |f(x,y)|\,dy$ or $\int_F dy \int_E |f(x,y)|\,dx$ is finite.

Theorem A.7. The space of simple functions is dense in $L^1(\mathbb{R}^d)$, i.e there exists a sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ of simple functions such that $\lim_{n\to\infty} \|f-\varphi_n\|_1 = 0$.

Theorem A.8. (Heine Theorem) Let $f : K \longrightarrow \mathbb{R}^n$ be continuous and let K be compact. Then f is uniformly continuous.

Theorem A.9. Let $I \subset \mathbb{R}$ be an open interval (bounded or unbounded). Let $C_c^0(I)$ denote the space of continuous functions that have bounded support in I. Then it holds that $C_c^0(I)$ is dense in $L^p(I)$.

Theorem A.10. (Change of variables) Let A and B be two compact subsets of $\mathbb{R}^d$ and let $g : A \longrightarrow B$ be a diffeomorphism of class C^1. Let $f : \mathbb{R}^d \longrightarrow \mathbb{C}$. If f is continuous on B, then:

$$\int_B f(\mathbf{x})d\mathbf{x} = \int_A f(g(\mathbf{y}))\left|det\left(Dg\right)(\mathbf{y})\right| d\mathbf{y}$$

Where Dg denotes the Jacobian of g.

Theorem A.11. (Divergence Theorem) Let V be subset of $\mathbb{R}^d$ which is compact and has a piecewise smooth boundary ∂V. If $F : V \subset \mathbb{R}^d \longrightarrow \mathbb{R}^d$ is a continuously differentiable vector field defined on a neighborhood of V, then we have:

$$\int_V (\nabla \cdot F)\, dV = \int_{\partial V} (F \cdot n)\, dS$$

Where the left side is a volume integral over the volume V, the right side is the surface integral over the boundary of the volume V.

Theorem A.12. (Green's identity) Let $u, v \in C^2\left(\mathbb{R}^d\right)$ and $\Omega \subset \mathbb{R}^d$. Then, it holds:

$$\int_\Omega \triangle u\left(\mathbf{x}\right) v\left(\mathbf{x}\right) d\mathbf{x} = -\int_\Omega \nabla u\left(\mathbf{x}\right) \nabla v\left(\mathbf{x}\right) d\mathbf{x} + \int_{\partial\Omega} v\left(\mathbf{x}\right)\left(\nabla u\left(\mathbf{x}\right)\cdot n\right) dS$$

Theorem A.13. (Uniform Convergence and continuity) Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $C\left(A\right)$ (with $A \subset \mathbb{R}^d$) such that converges uniformly to f in $A \subset \mathbb{R}^d$. Then $f \in C(A)$.

Theorem A.14. (Uniform Convergence and differentiation) Let $(f_n)_{n\in\mathbb{N}}$ be a $C^1\left(\mathbb{R}\right)$ sequence in an interval $[a, b]$. Assume that there exists $x_0 \in [a, b]$ such that $(f_n(x_0))$ converges. If $\left(f_n^{'}\right)_{n\in\mathbb{N}}$ converges uniformly to a function g in $[a, b]$, then:

1. $(f_n)_{n\in\mathbb{N}}$ converges uniformly to a function $f \in C^1\left((a, b)\right)$.
2. $f^{'} = g$ in $[a, b]$.

Appendix B

Spherical coordinates in $\mathbb{R}^d$

An important application of the change of variables formula is to the case of polar coordinates in $\mathbb{R}^2$, spherical coordinates in $\mathbb{R}^3$ and their generalization in $\mathbb{R}^d$. These are particularly important when the function, or set we are integrating over, exhibit some rotational (or spherical) symmetries.

The spherical coordinates system in $\mathbb{R}^d$ is given by $\mathbf{x} = g(r, \theta_1, ..., \theta_{d-1})$ where:

$$\begin{cases} x_1 & = rsin(\theta_1)\,sin(\theta_2)\,...sin(\theta_{d-2})\,cos(\theta_{d-1}) \\ x_2 & = rsin(\theta_1)\,sin(\theta_2)\,...sin(\theta_{d-2})\,sin(\theta_{d-1}) \\ \vdots & \\ x_{d-1} & = rsin(\theta_1)\,sin(\theta_2) \\ x_d & = rcos(\theta_1) \end{cases}$$

with $0 \leq \theta_i \leq \pi$ for $1 \leq i \leq d-2$ and $0 \leq \theta_{d-1} \leq 2\pi$. The determinant of the Jacobian of this transformation is given by:

$$r^{d-1} sin^{d-2}(\theta_1)\, sin^{d-3}(\theta_2) \ldots sin(\theta_{d-2})$$

Hence, if we integrate over $B(0, N)$, we get:

$$\int_{B(0,N)} f(x)dx = \int_0^{\pi}\int_0^{\pi}\cdots\int_0^{2\pi}\int_0^{N}\left[f(g(r,\theta_1,...,\theta_{d-1}))\, r^{d-1} sin^{d-2}(\theta_1)\, sin^{d-3}(\theta_2)\ldots sin(\theta_{d-2})\right] dr d\theta_{d-1}...d\theta_1$$

Note that each point $\mathbf{x} \in \mathbb{R}^d \setminus \{0\}$ can be written uniquely as $r\gamma$ with $\gamma \in S^{d-1}$, since $x = |x|\frac{x}{|x|} = r\gamma$ and $\frac{x}{|x|} \in S^{d-1}$. Consequently, the previous expression can be written as:

$$\int_{B(0,N)} f(x)dx = \int_0^N \int_{S^{d-1}} r^{d-1} f(r\gamma) d\sigma(\gamma) dr$$

Appendix C

Notation

- ***Fourier transform in*** $L^1(\mathbb{R}^d)$ ***or*** $\mathcal{S}(\mathbb{R}^d)$***:*** let $f \in L^1(\mathbb{R}^d)$ or $\mathcal{S}(\mathbb{R}^d)$, we denote its Fourier transform as $\hat{f}$ and its inverse Fourier transform as $\check{f}$.

- ***Fourier transform in*** $L^2(\mathbb{R}^d)$***:*** let $f \in L^2(\mathbb{R}^d)$, we denote its Fourier transform as $\mathcal{F}(f)$ and its inverse Fourier transform as $\overline{\mathcal{F}}(f)$.

- ***Reflexion:*** let f be a function. We denote its reflexion as $f_\sigma(x) \equiv f(-x)$.

- ***Translation:*** let f be a function and $a \in \mathbb{R}$. We denote the translation of f by a as $\tau_a f(x) \equiv f(x-a)$.

- ***Gradient operator:*** let $f \in C^1(\mathbb{R}^d)$, we denote the gradient of f as $\nabla f(\mathbf{x}) = grad(f) = \left(\frac{\partial}{\partial x_1} f(\mathbf{x}), \quad \cdots \quad , \frac{\partial}{\partial x_n} f(\mathbf{x}) \right)^T$.

- ***Laplace operator (or Laplacian):*** let $f \in C^2(\mathbb{R}^d)$, we denote the Laplacian of f as $\triangle f(\mathbf{x}) = \sum_{i=1}^{d} \frac{\partial^2}{\partial x_i^2} f(\mathbf{x})$.

- ***Divergence operator:*** let $f \in C^1(\mathbb{R}^d)$, we denote the divergence of f as $\nabla \cdot f(\mathbf{x}) = div(f)(\mathbf{x}) = \sum_{i=1}^{d} \frac{\partial}{\partial x_i} f(\mathbf{x})$.

- ***Interior of a set:*** let $A \subset \mathbb{R}^d$. We denote its interior as $\mathring{A}$.

- ***Closure of a set:*** let $A \subset \mathbb{R}^d$. We denote its closure as $\overline{A}$.

- ***Boundary of a set:*** let $A \subset \mathbb{R}^d$. We denote its boundary as ∂A.

- ***Ball of radius*** r ***and center*** $\mathbf{x}$***:*** $B(\mathbf{x}, r)$.

- ***Boundary of the ball of radius*** r ***and center*** $\mathbf{x}$***:*** $\partial B(\mathbf{x}, r)$.

- ***Unit sphere on*** $\mathbb{R}^d$***:*** S^{d-1}.

- ***Volume of unit ball in*** $\mathbb{R}^d$***:*** $\alpha(d)$.

- ***Surface area of unit ball in*** $\mathbb{R}^d$***:*** $d\alpha(d)$

• With this notation, the volume of the ball of radius r and center $\mathbf{x} \in \mathbb{R}^d$, written as $Vol\,(B(\mathbf{x}, r))$ is given by $\alpha(d)r^d$. Analogously, its surface area, written as $SA(B(\mathbf{x}, r))$, is given by $d\alpha(d)r^{d-1}$.

- ***Average of*** f ***over*** $B(\mathbf{x}, r)$***:*** for $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, we define the average of f over $B(\mathbf{x}, r)$ as

$$\fint_{B(\mathbf{x},r)} f(\mathbf{y})d\mathbf{y} \;=\; \frac{1}{Vol(B(\mathbf{x}, r))} \int_{B(\mathbf{x},r)} f(\mathbf{y})d\mathbf{y} \;=\; \frac{1}{\alpha(d)r^d} \int_{B(\mathbf{x},r)} f(\mathbf{y})d\mathbf{y}$$

- ***Average of*** f ***over*** $\partial B(\mathbf{x}, r)$***:*** for $f : \mathbb{R}^d \longrightarrow \mathbb{R}$, we define the average of f over $\partial B(\mathbf{x}, r)$ as

$$\fint_{\partial B(\mathbf{x},r)} f(\mathbf{y})dS(\mathbf{y}) \;=\; \frac{1}{SA(B(\mathbf{x}, r))} \int_{\partial B(\mathbf{x},r)} f(\mathbf{y})dS(\mathbf{y}) \;=\; \frac{1}{d\alpha(d)r^{d-1}} \int_{\partial B(\mathbf{x},r)} f(\mathbf{y})dS(\mathbf{y})$$

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