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ON THE TIME DECAY OF SOLUTIONS FOR NON-SIMPLE ELASTICITY WITH VOIDS

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Abstract: In this work we consider the non-simple theory of elastic material with voids and we investigate how the coupling of these two aspects of the material affects the behavior of the solutions. We analyze only two kind of different behavior, slow or exponential decay. We introduce four different dissipation mechanisms in the system and we study, in each case, the effect of this mechanism in the behavior of the solutions.

Keywords: non-simple elasticity, exponential decay, localization of solutions.

1. Introduction

The behavior over the time of the solutions of problems determined by partial differential equations is the aim of lots of studies from long ago. Often partial differential equations model thermomechanical situations and, hence, these mathematical studies are of interest in applied sciences and they have been studied for researchers from different fields. In particular, when the elasticity properties of some materials are analyzed, partial differential equations arise. Many times elasticity is combined with other aspects that can also be present in the material as heat or porosity. It is interesting to know if the coupling between these different aspects and elasticity modify the behavior of the solutions. That is: to clarify if the coupling between them is strong or weak.

In this work we consider the coupling between elasticity and porosity or, in other words, we focuss on strain–gradient elastic materials with voids. The classical setting of this theory was established by Nunziato and Cowin [14], Cowin and Nunziato [1] and Cowin [2]. The basic idea of these works is that the bulk density is the product of two scalar fields, the matrix material density and the volume fraction. This theory is deeply discussed in the book of Ieşan [5].

Through the paper, to simplify, we speak about *slow decay* or *exponential decay* of the solutions. We say that the decay of the solutions is exponential if they are exponentially stable and, if they are not, we say that the decay of the solutions is slow. The main difference between these two concepts in a thermomechanical context lies in the fact that, if the decay is exponential, then after a short period of time, the thermomechanical displacements are very small and can be neglected. However, if the decay is slow, then the solutions weaken in a way that thermomechanical displacements could be appreciated in the system after some time. The nature of the solutions highly determines the temporal behavior of the system and, from a thermomechanical point of view, it is relevant to be able to classify them.

Classical elasticity models the macroscopic structure of the material whereas porosity concentrates in the microstructure. Therefore, it is important to understand this coupling and to know if a dissipative

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mechanism in a macroscopic or microscopic level is able to carry the entire structure to an state of quick decay or slow decay of the solutions.

Since Quintanilla [18] showed the slow decay of the solutions when only one mechanism of porous dissipation is considered, other studies have been done to analyze the decay of the solutions when different types of dissipative mechanisms are introduced in the system. The following scheme could help to summarize the answers known up to now.

Thermal effect		Elasticity		Microthermal effect
	$\longrightarrow$	$\updownarrow$	$\longleftarrow$	
Viscoelastic effect		Porosity		Viscoporous effect

If we take simultaneously one effect from the right square and another one from the left square, then we get exponential stability. However, if we consider two simultaneous effects from one square only, then we get slow decay. Pamplona *et al.* [15] proved that when the dissipation mechanisms are of rate type the solutions are analytic. Soufyane [19], Soufyane *et al.* [20, 21] and Pamplona *et al.* [16] showed several situations with more than one dissipative mechanism where the exponential decay holds. Let us also mention that Muñoz-Rivera and Quintanilla [13] obtained polynomial rates of decay form for some problems when the decay is not of exponential type.

Green and Rivlin [8], Mindlin [12] and Toupin [22] proposed the inclusion of higher order gradients in the basic postulates of thermoelasticity. In that case, we usually speak of the *non-simple* theory or strain–gradient elasticity. Higher order derivatives help to modeling in more detail the configurations of the materials and the response of these material to stimuli.

In this work we consider the non-simple theory and ask ourselves the same questions that we investigated for the classical setting when elasticity is coupled with porosity (Magaña and Quintanilla [11]). In a previous work, Fernández-Sare *et al.* [4] analyzed the coupling of non-simple elasticity with heat. They proved the exponential decay of the solutions when the classical Fourier theory is considered and the slow decay if the hyperbolic equation is taking into account.

The paper is based in the theory of “grade consistency” (see Ieşan [6]) and we set our equations supposing that second order derivatives for the displacement could exist in the constitutive equation for the hyperstress, but only first order derivatives in the gradient of the volume fraction. With these basic equations, we have considered four different types of dissipation:

(1) the usual viscoelasticity,
(2) hyperviscoelasticity,
(3) viscoporosity proportional to the time derivative of the porosity, and
(4) viscoporosity proportional to the time derivative of the porosity gradient.

We have analyzed in each case the corresponding system of equations. Let us denote by d the coefficient that links together the effect of the high order gradient of the displacement with the porosity. We have proved that whenever d is different from zero, the decay of solutions in cases (1), (2) and (4) is exponential, whereas in case (3) the decay is slow (the decay is also slow in the other cases when $d = 0$).

In some cases we have sketched the location of the first elements of the spectrum and, even though we do not present a rigorous proof, we think that the semigroup is not analytic. For this reason, we study also the impossibility of localization of the solutions. We prove the impossibility of localization for case (2) following an argument inspired in the one used for the thermoelasticity of type III. Case (4) could be thought in a similar way. Nevertheless, this question remains unsolved for cases (1) and (3).

It is worth noting the big difference between the results we obtain for the non-simple theory and the results previously known for the classical theory. For the later, the exponential decay is obtained only if two dissipation mechanisms are considered together whereas, in our study, there are situations where one mechanism is enough to get the exponential decay.

The structure of the paper is the following. In Section 2 we introduce the basic equations we are going to work with and the conditions that satisfy the involved coefficients. Section 3 is devoted to analyze the effect of viscoelasticity over the solutions, and we have proved that, in this case, the solutions decay exponentially. In Section 4 we study the effect of hyperviscoelasticty (a dissipation mechanism in the second order derivative of the displacement) and we obtain the same conclusion as in the previous section. In Section 5 we show that the usual viscoporosity gives rise to slow decay in the solutions. In Section 6 we introduce a dissipation mechanism in the second order derivative of the fraction volume and we obtain again exponential decay. Section 7 is devoted to show the impossibility of the localization in time of the solutions for the nonsimple viscoelasticity problem (the system analyzed in Section 4). Finally, in Section 8 we summarize the results.

2. PRELIMINARIES

In Ieşan [6] the evolution and constitutive equations which govern the theory we are going to deal with were derived. To be precise, we use (4.20) and (4.7) from the work of Ieşan for the evolution equations, and (4.18) for the constitutive equations.

As we focus our analysis in the one-dimensional problem, the evolution equations become

$$\rho\ddot{u} = \tau_x - \mu_{xx}, \tag{2.1}$$

$$J\ddot{\varphi} = h_x + g. \tag{2.2}$$

Here u is the displacement, φ is the fraction of volume, τ is the stress, μ is the hyperstress, h is the equilibrated stress vector and g the equilibrated body force. The mass density ρ and the product of the mass density by the equilibrated inertia J are assumed to be positive.

The constitutive equations are

$$\tau = au_x + b\varphi, \tag{2.3}$$

$$\mu = cu_{xx} + d\varphi_x \tag{2.4}$$

$$h = du_{xx} + \beta\varphi_x, \tag{2.5}$$

$$g = -\xi\varphi - bu_x, \tag{2.6}$$

Constants a and c are related with the elasticity and hyperelasticity modulus, respectively. b and d are the coupling coefficients between the elastic and porous structures. And β and ξ are constants associated with the porous materials.

The constitutive coefficients satisfy

$$\rho > 0, J > 0, a > 0, c > 0, a\xi - b^2 > 0, c\beta - d^2 > 0. \tag{2.7}$$

These hypothesis guarantee that the inner energy is positive definite, and this fact is related with the well-posedness of the problem (in the sense of Hadamard).

Clearly, system (2.1)-(2.7) is conservative. Later, in each section, we will introduce dissipative mechanisms into the system by modifying the constitutive equations, then try to determine how the solutions decay.

3. VISCOELASTICITY

We first introduce a dissipative mechanism in the elasticity, that is, we will assume that

$$\tau = au_x + b\varphi + \delta\dot{u}_x,$$

with $\delta > 0$. Physically, this new term ($\delta\dot{u}_x$) makes the tension depending on the gradient of the velocity of the displacement. This fact will prevent the material to recover its initial state completely. As the coefficient δ increases, there will be more dissipation of energy and it will be more difficult for the material to recover its initial state.

If we substitute the constitutive equations into the evolution equations we obtain the system of field equations:

$$\begin{cases} \rho\ddot{u} = au_{xx} + b\varphi_x - cu_{xxxx} - d\varphi_{xxx} + \delta\dot{u}_{xx} \\ J\ddot{\varphi} = du_{xxx} + \beta\varphi_{xx} - \xi\varphi - bu_x \end{cases} \tag{3.1}$$

Conditions (2.7) are assumed for the system coefficients.

We assume that the solutions satisfy the following boundary conditions

$$u(0,t) = u(\pi,t) = u_{xx}(0,t) = u_{xx}(\pi,t) = \varphi_x(0,t) = \varphi_x(\pi,t) = 0, \tag{3.2}$$

and the following initial conditions

$$u(x,0) = u_0(x),\ \dot{u}(x,0) = v_0(x),\ \varphi(x,0) = \varphi_0(x),\ \dot{\varphi}(x,0) = \phi_0(x). \tag{3.3}$$

The existence of solutions that do not decay is clear, but if the average of the initial condition for u_0 and φ_0 vanishes, then we avoid this possibility.

It is easy to see that if there exists $n \in \mathbb{N}$ such that $b + dn^2 = 0$, then this system has undamped solutions. Take, for example, $u = 0$ and $\varphi = \sin(\omega t)\cos(nx)$, with $\omega = \sqrt{\frac{\beta n^2 + \xi}{J}}$.

Hence, throughout this section, we will assume that $b + dn^2 \neq 0$ for all $n \in \mathbb{N}$.

We will try to prove that, generically, the solutions of (3.1) decay exponentially, and for this purpose, we will use the semigroup arguments due to Liu and Zheng [10].

We consider the Hilbert space

$$\mathcal{H} = \left\{ (u,v,\varphi,\phi) \in \left(H^2 \cap H_0^1\right) \times L^2 \times H^1 \times L^2, \int_0^\pi \varphi(x)dx = \int_0^\pi \phi(x)dx = 0 \right\}.$$

Taking into account that $\dot{u} = v$ and $\dot{\varphi} = \phi$, and writing $D = \frac{d}{dx}$, we can restate system (3.1) in the following way:

$$\begin{cases} \dot{u} = v, \\ \dot{v} = \frac{1}{\rho}[aD^2u + bD\varphi + \delta D^2 v - cD^4 u - dD^3\varphi], \\ \dot{\varphi} = \phi, \\ \dot{\phi} = \frac{1}{J}[dD^3 u + \beta D^2\varphi - \xi\varphi - bDu]. \end{cases}$$

If we denote $U = (u,v,\varphi,\phi)$, then our initial-boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{A}U, \quad U_0 = (u_0, v_0, \varphi_0, \phi_0),$$

where $\mathcal{A}$ is the following 4×4 matrix:

$$\mathcal{A}=\begin{pmatrix} 0 & I & 0 & 0 \\ \frac{aD^2-cD^4}{\rho} & \frac{\delta D^2}{\rho} & \frac{bD-dD^3}{\rho} & 0 \\ 0 & 0 & 0 & I \\ \frac{dD^3-bD}{J} & 0 & \frac{\beta D^2-\xi}{J} & 0 \end{pmatrix},$$

and I is the identity operator.

It is worth noting that the Hilbert space $\mathcal{H}$ can be decomposed as the direct sum of two subspaces: $\mathcal{H}=\mathcal{K}^N\bigoplus\mathcal{K}$ where $\mathcal{K}^N$ is the finite dimensional subspace generated by the vectors

$$\Omega(h,i,j,k)=(\sin hx,\sin ix,\cos jx,\cos kx)\quad 1\le h,i,j,k\le N.$$

Notice that $\mathcal{K}^N$ is invariant. That means that the solutions starting at $\mathcal{K}^N$ always belong to $\mathcal{K}^N$. And $\mathcal{K}$ is the subspace whose elements have the first N components equal to zero. That means that, for instance, if $u=\sum_{n=1}^{\infty}u_n\sin(nx)$, then $u_n=0$ for $n=1,\dots,N$.

Therefore, the solutions $U(t)$ can be also decomposed in the following way: $U(t)=U_1(t)+U_2(t)$ where $U_1(t)\in\mathcal{K}^N$ and $U_2(t)\in\mathcal{K}$. Thus, to prove the exponential decay of the solutions, it is sufficient to prove the exponential decay of $U_1(t)$ and $U_2(t)$.

As $U_1(t)$ belongs to a finite dimensional subspace, if all the eigenvalues have negative real part, the exponential decay of $U_1(t)$ is guaranteed.

Proposition 3.1. *All the eigenvalues of $\mathcal{A}$ restricted to $\Omega(h,i,j,k)$ have negative real part.*

Proof. Imposing that u and φ in (3.1) are of the form

$$u=K_1\exp(\omega t)\sin(nx),\quad \varphi=K_2\exp(\omega t)\cos(nx),$$

the following homogeneous system on the unknowns K_1 and K_2 is obtained:

$$\begin{pmatrix} cn^4+an^2+\delta\omega n^2+\rho\omega^2 & n(dn^2+b) \\ n\left(dn^2+b\right) & \beta n^2+J\omega^2+\xi \end{pmatrix}\begin{pmatrix} K_1 \\ K_2 \end{pmatrix}=\begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.4}$$

This linear system will have nontrivial solutions if, and only if, the determinant of the coefficients matrix is null. We denote by $p(x)$ the fourth degree polynomial obtained from the determinant of the coefficients matrix once ω is replaced by x:

$$\begin{aligned} p(x)=\rho Jx^4+J\delta n^2x^3+\left(aJn^2+cJn^4+n^2\beta\rho+\xi\rho\right)x^2+\left(n^2\beta+\xi\right)n^2\delta x \\ +(c\beta-d^2)n^6+(a\beta-2bd+c\xi)n^4+(a\xi-b^2)n^2 \end{aligned} \tag{3.5}$$

We denote the coefficients of this polynomial by p_i for $i=0,1,2,3,4$. These coefficients depend on the parameters of system (3.1) and on n:

$$\begin{aligned} p_0 =&\rho J \\ p_1 =&J\delta n^2 \\ p_2 =&cJn^4+(aJ+\beta\rho)n^2+\xi\rho \\ p_3 =&\beta\delta n^4+\xi\delta n^2 \\ p_4 =&n^6\left(c\beta-d^2\right)+n^4(a\beta-2bd+c\xi)+n^2\left(a\xi-b^2\right) \end{aligned}$$

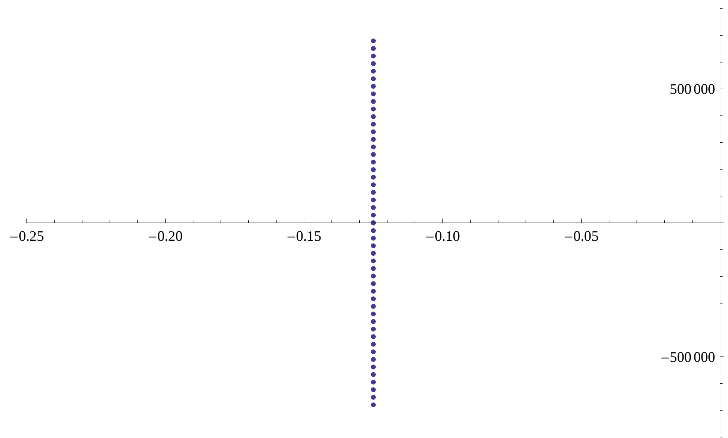

FIGURE 1. Root's behavior of $p_P(x)$.

We use the Routh-Hurwitz theorem (see Dieudonné [3]). It says that, if $p_0 > 0$, then all the roots of polynomial

$$p_0x^4 + p_1x^3 + p_2x^2 + p_3x + p_4 \tag{3.6}$$

have negative real part if, and only if, p_4 and all the leading diagonal minors of matrix

$$\begin{pmatrix} p_1 & p_0 & 0 & 0 \\ p_3 & p_2 & p_1 & p_0 \\ 0 & p_4 & p_3 & p_2 \\ 0 & 0 & 0 & p_4 \end{pmatrix} \tag{3.7}$$

are positive.

Direct calculation show that, in this case, the leading minors Λ_i are all positive:

$$\begin{aligned} \Lambda_1 =& p_1 = J\delta n^2 \\ \Lambda_2 =& J^2n^4\delta\left(a+cn^2\right) \\ \Lambda_3 =& J^2n^6\delta^2\left(b+dn^2\right)^2 \\ \Lambda_4 =& p_4\cdot\Lambda_3 \end{aligned}$$

(By hypothesis, $b+dn^2 \neq 0$ for all $n \in \mathbb{N}$, and a and c are positive.)

$\square$

Remark 3.2. Giving particular values (satisfying conditions (2.7)) to the parameters of system (3.1) we get also a particular polynomial, $p_P(x)$. Take, for instance, $\rho = J = \delta = \beta = \xi = b = d = 1$ and $a = c = 2$. If we compute the roots of $p_P(x)$ for different values of n (for example, we have computed 25 different polynomials beginning with $n = 40000$ and taking an step of 40000 up to $n = 1000000$), it seems that the point spectrum is located along a line parallel to the imaginary axis (see figure 1). This fact let us think that the semigroup is not analytic.

To show the exponential decay of $U_2(t)$ is not so easy.

Now, we define an inner product in $\mathcal{H}$. If $U^* = (u^*, v^*, \varphi^*, \phi^*)$, then

(3.8)

$$\langle U, U^* \rangle = \frac{1}{2}\int_0^\pi \left(\rho v \overline{v}^* + J\phi\overline{\phi}^* + a u_x \overline{u}_x^* + \beta\varphi_x\overline{\varphi}_x{}^* + \xi\varphi\overline{\varphi}^* + b(u_x\overline{\varphi}^* + \overline{u}_x^*\varphi) + d(u_{xx}\overline{\varphi}_x^* + \overline{u}_{xx}^*\varphi_x) + c u_{xx}\overline{u}_{xx}\right) dx$$

Here a superposed bar denotes the conjugate complex number. It is worth recalling that this product is equivalent to the usual product in the Hilbert space $\mathcal{H}$, and notice that the natural restriction of the above defined product to $\mathcal{K}$ is also an inner product.

It can be proved that the general solutions of system (3.1) are given by the semigroup of contractions generated by the operator $\mathcal{A}$.

Direct calculation gives

$$Re\langle \mathcal{A}U, U\rangle = -\frac{1}{2}\int_0^\pi \delta|v_x|^2 dx \leq 0. \tag{3.9}$$

To show the exponential stability we use a result ([7], [17], [9]) which states that a semigroup of contractions on a Hilbert space is exponentially stable if and only if

$$\{i\lambda, \lambda \text{ is real}\} \text{ is contained in the resolvent of } \mathcal{A}, \tag{3.10}$$

and

$$\overline{\lim}_{|\lambda|\to\infty} ||(i\lambda I - \mathcal{A})^{-1}|| < \infty, \tag{3.11}$$

where I denotes the identity matrix.

To prove these conditions we need first the following result.

Lemma 3.3. *Let $\mathcal{A}$ be the above defined matrix. Then, 0 is in the resolvent of $\mathcal{A}$.*

Proof. For any $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ we will find $U \in \mathcal{H}$ such that $\mathcal{A}U = \mathcal{F}$, or equivalently:

$$\left.\begin{aligned} v &= f_1 \\ \tfrac{1}{\rho}[aD^2u + bD\varphi + \delta D^2 v - cD^4 u - dD^3\varphi] &= f_2 \\ \phi &= f_3 \\ \tfrac{1}{J}[dD^3u + \beta D^2\varphi - \xi\varphi - bDu] &= f_4 \end{aligned}\right\} \tag{3.12}$$

It is important to have in mind the domain where we are working. In this case the domain is defined by

$$\mathcal{D} = \{U \in \mathcal{H} : \mathcal{A}U \in \mathcal{H}, u_{xx}(0) = u_{xx}(\pi) = 0\}.$$

The unique solvability of this system is guaranteed using the usual elliptic arguments. To be more precise, we will write

$$u = \sum a_n \sin nx$$
$$v = \sum b_n \sin nx$$
$$\varphi = \sum c_n \cos nx$$
$$\phi = \sum d_n \cos nx$$

and $f_i = \sum f_n^i \sin nx$ for $i = 1,2$ and $f_j = \sum f_n^j \cos nx$ for $j = 3,4$. In this case, f_n^i for $i = 1,2,3,4$ are known and a_n, b_n, c_n and d_n unknown. We also know that $\sum n^4 |f_n^1|^2 < \infty$, $\sum |f_n^2|^2 < \infty$, $\sum n^2 |f_n^3|^2 < \infty$ and $\sum |f_n^4|^2 < \infty$.

From the first and third equations of system (3.12) it is clear that $b_n = f_n^1$ and $d_n = f_n^3$. Therefore, it is clear also that $\sum |b_n|^2 < \infty$ and $\sum |d_n|^2 < \infty$.

From the second and fourth equations of system (3.12) we get

$$a_n = \frac{-f_n^4 Jn(b+dn^2)+\left(n^2\beta+\xi\right)\left(f_n^1 n^2\delta+f_n^2\rho\right)}{n^2\left((b+dn^2)^2-(a+cn^2)\,(n^2\beta+\xi)\right)},$$

and

$$c_n = \frac{f_n^4 Jn(a+cn^2)-\left(b+dn^2\right)\left(f_n^1 n^2\delta+f_n^2\rho\right)}{n\left((b+dn^2)^2-(a+cn^2)\,(n^2\beta+\xi)\right)}.$$

Hence, it is not difficult to see that $\sum n^4|a_n|^2<\infty$ and $\sum n^2|c_n|^2<\infty.$ Notice that in both fractions, the denominators are polynomials on n of degrees 6 and 5, respectively, because $d^2-c\beta\neq 0$ (this is one of our initial assumptions for the constitutive coefficients, see (2.7)).

Finally, taking into account the solutions obtained for a_n, b_n, c_n and d_n, it can be shown that

$$\|U\|_{\mathcal{H}}\leq K\|\mathcal{F}\|_{\mathcal{H}},$$

where K is a constant independent of U. □

The following two lemmas are necessary to prove the exponential decay of vector $U_2\in\mathcal{K}$.

Lemma 3.4. *Let $\mathcal{A}$ be the same matrix as in Lemma 3.3. Then condition (3.10) is satisfied.*

Proof. We split the proof in three steps.

(i) Since 0 is in the resolvent of $\mathcal{A}$, by the contraction mapping theorem, for any real λ such that $|\lambda|<\|\mathcal{A}^{-1}\|^{-1}$, the operator $i\lambda I-\mathcal{A}=\mathcal{A}(i\lambda\mathcal{A}^{-1}-I)$ is invertible. Moreover, $\|(i\lambda I-\mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $\left(-\|\mathcal{A}^{-1}\|^{-1},\|\mathcal{A}^{-1}\|^{-1}\right)$.

(ii) If $\sup\{\|(i\lambda I-\mathcal{A})^{-1}\|,\ |\lambda|<\|\mathcal{A}^{-1}\|^{-1}\}=M<\infty$, then, using the contraction theorem again, the operator

$$i\lambda I-\mathcal{A}=(i\lambda_0 I-\mathcal{A})\left(I+i(\lambda-\lambda_0)(i\lambda_0 I-\mathcal{A})^{-1}\right),$$

is invertible for $|\lambda-\lambda_0|<M^{-1}$. Hence, choosing λ_0 close enough to $\|\mathcal{A}^{-1}\|^{-1}$, the set $\{\lambda,\ |\lambda|<\|\mathcal{A}^{-1}\|^{-1}+M^{-1}\}$ is contained in the resolvent of $\mathcal{A}$ and $\|(i\lambda I-\mathcal{A})^{-1}\|$ is a continuous function of λ in the interval $\left(-\|\mathcal{A}^{-1}\|^{-1}-M^{-1},\|\mathcal{A}^{-1}\|^{-1}+M^{-1}\right)$.

(iii) Suppose that the statement of this lemma is not true. Then, there exists a real number $\sigma\neq 0$ with $\|\mathcal{A}^{-1}\|^{-1}\leq|\sigma|<\infty$ satisfying that the set $\{i\lambda,\ |\lambda|<|\sigma|\}$ is in the resolvent of $\mathcal{A}$ and $\sup\{\|(i\lambda I-\mathcal{A})^{-1}\|,\ |\lambda|<|\sigma|\}=\infty$. In this case, we can find a sequence of real numbers, λ_n, with $\lambda_n\to\sigma$, $|\lambda_n|<|\sigma|$, and a sequence of unit norm vectors in the domain of $\mathcal{A}$, $U_n=(u_n,v_n,\varphi_n,\phi_n)$, such that

$$\|(i\lambda_n I-\mathcal{A})U_n\|\to 0.$$

Writing this condition term by term we get

$$i\lambda_n u_n-v_n\to 0 \text{ in } H^2, \tag{3.13}$$

$$i\lambda_n v_n-\frac{a}{\rho}D^2u_n+\frac{c}{\rho}D^4u_n-\frac{\delta}{\rho}D^2v_n+\frac{d}{\rho}D^3\varphi_n-\frac{b}{\rho}D\varphi_n\to 0 \text{ in } L^2, \tag{3.14}$$

$$i\lambda_n\varphi_n-\phi_n\to 0 \text{ in } H^1, \tag{3.15}$$

$$i\lambda_n\phi_n-\frac{d}{J}D^3u_n+\frac{b}{J}Du_n-\frac{\beta}{J}D^2\varphi_n+\frac{\xi}{J}\varphi_n\to 0 \text{ in } L^2. \tag{3.16}$$

Taking the inner product of $(i\lambda_n I - \mathcal{A})U_n$ with U_n in $\mathcal{H}$ using (3.9) and selecting its real part we obtain $\|Dv_n\|^2 \to 0$. Thus, we have also $v_n \to 0$, and, from (3.13), $u_n \to 0$ in H^1 and, then, $Du_n \to 0$.

Taking the inner product of (3.14) with u_n we obtain

$$i\lambda_n \langle v_n, u_n\rangle - \frac{a}{\rho}\langle D^2 u_n, u_n\rangle + \frac{c}{\rho}\langle D^4 u_n, u_n\rangle - \frac{\delta}{\rho}\langle D^2 v_n, u_n\rangle + \frac{d}{\rho}\langle D^3 \varphi_n, u_n\rangle - \frac{b}{\rho}\langle D\varphi_n, u_n\rangle \to 0 \text{ in } L^2. \tag{3.17}$$

Hence, having in mind that $\langle D^2 u_n, u_n\rangle = -\langle Du_n, Du_n\rangle$ and $-\langle D^2 v_n, u_n\rangle = \langle Dv_n, Du_n\rangle$, we get

$$c\|D^2 u_n\|^2 + d\langle D\varphi_n, D^2 u_n\rangle \to 0 \text{ in } L^2. \tag{3.18}$$

Now, taking the inner product of (3.16) with Du_n and simplifying, we obtain

$$d\|D^2 u_n\|^2 + \beta\langle D\varphi_n, D^2 u_n\rangle \to 0 \text{ in } L^2. \tag{3.19}$$

By hypothesis, $c\beta - d^2 > 0$. Then, conditions (3.18) and (3.19) imply $\|D^2 u_n\| \to 0$ and $\langle D\varphi_n, D^2 u_n\rangle \to 0$.

Removing from (3.14) the terms that tend to zero and, then, taking the inner product of it with $\int_0^x \varphi_n$ (let us write simply $\int \varphi_n$ to simplify the notation) we get

$$c\langle D^4 u_n, \int \varphi_n\rangle - \delta\langle D^2 v_n, \int \varphi_n\rangle + d\langle D^3 \varphi_n, \int \varphi_n\rangle - b\langle D\varphi_n, \int \varphi_n\rangle \to 0 \text{ in } L^2. \tag{3.20}$$

Or, equivalently,

$$c\langle D^2 u_n, D\varphi_n\rangle + \delta\langle Dv_n, \varphi_n\rangle + d\|D\varphi_n\|^2 + b\|\varphi_n\|^2 \to 0 \text{ in } L^2. \tag{3.21}$$

Clearly, from the previous results, $\langle D^2 u_n, D\varphi_n\rangle \to 0$ and $\langle Dv_n, \varphi_n\rangle \to 0$. Then, (3.21) reduces to

$$d\|D\varphi_n\|^2 + b\|\varphi_n\|^2 \to 0 \text{ in } L^2. \tag{3.22}$$

As $\varphi_n \in L^2$, we know that

$$\varphi_n = \sum_{n=n_0+1}^{\infty} a_n(t) \sin nx.$$

Using the Poincaré inequality, we know that

$$\|D\varphi_n\|^2 \geq n_0^2 \|\varphi_n\|.$$

As we can take n_0 as large as we want, we take $n_0 > \sqrt{-b/d}$. Therefore, from (3.22) we have

$$(dn_0^2 + b)\|\varphi_n\|^2 \to 0.$$

This means that $\|\varphi_n\|^2 \to 0$ and (again from (3.22)) $\|D\varphi_n\|^2 \to 0$. Finally, from (3.15), we have $\phi_n \to 0$.

This argument shows that U_n can not be of unit norm, which finishes the proof of this lemma. □

Lemma 3.5. *Let $\mathcal{A}$ be the same matrix as in Lemma 3.3. Then condition (3.11) is satisfied.*

Proof. Suppose that the statement of the lemma is not true. Then, there is a sequence λ_n with $|\lambda_n| \to \infty$ and a sequence of unit norm vectors in the domain of $\mathcal{A}$, $U_n = (u_n, v_n, \varphi_n, \phi_n)$, such that conditions (3.13)–(3.16) hold. Again, the fact that $Dv_n \to 0$ implies $v_n \to 0$ and, from (3.13), $\lambda_n u_n \to 0$. This implies $u_n \to 0$ and, consequently, $Du_n \to 0$.

As in the previous lemma, it is easy to see that $\|D^2 u_n\| \to 0$ and $\langle D\varphi_n, D^2 u_n\rangle \to 0$.

From (3.15) we can conclude that $\lambda_n \int \varphi_n$ is bounded (because ϕ_n is bounded). We divide (3.14) by λ_n and then, we take the inner product of this new expression with $\lambda_n \int \varphi_n$. Therefore, we get again (3.22):

$$d\|D\varphi_n\|^2 + b\|\varphi_n\|^2 \to 0 \text{ in } L^2.$$

Now we multiply (3.16) by φ_n, and we remove the terms that tend to 0:

$$i\lambda_n\langle\phi_n,\varphi_n\rangle + \frac{\beta}{J}\|D\varphi_n\|^2 + \frac{\xi}{J}\|\varphi_n\|^2 \to 0 \text{ in } L^2. \tag{3.23}$$

In this new expression, we can substitute ϕ_n by $i\lambda_n\varphi_n$ and $\|D\varphi_n\|^2$ by $-\frac{b}{d}\|\varphi_n\|^2$, and, therefore, we obtain

$$\Big(-\lambda_n^2 - \frac{b\beta}{dJ} + \frac{\xi}{J}\Big)\|\varphi_n\|^2 \to 0. \tag{3.24}$$

As $|\lambda_n| \to \infty$, expression (3.24) implies $\varphi_n \to 0$ (moreover: $\lambda_n\varphi_n \to 0$). And then, from (3.22) and from (3.15), $D\varphi_n \to 0$ and $\phi_n \to 0$.

This argument shows that U_n can not be of unit norm, which finishes the proof of this lemma. □

Theorem 3.6. *Let (u,φ) be a solution of the problem determined by (3.1), with boundary conditions (3.2) and initial conditions (3.3). Then, (u,φ) decays exponentially.*

Proof. The proof is a direct consequence of Lemmas (3.4) and (3.5). □

In fact, the strong coupling between the equations is given by the term φ_{xxx}. If $d = 0$, then the solutions of system (3.1) decay in a slowly way. As before, we take again

$$u = K_1\exp(\omega t)\sin(nx), \;\; \varphi = K_2\exp(\omega t)\cos(nx). \tag{3.25}$$

In this case we obtain the following homogeneous system:

$$\begin{pmatrix} cn^4 + an^2 + \delta\omega n^2 + \rho\omega^2 & bn \\ bn & \beta n^2 + J\omega^2 + \xi \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{3.26}$$

We denote by $q(x)$ the fourth degree polynomial obtained from the determinant of the coefficients matrix once ω is replaced by x:

$$\begin{aligned} q(x) = \rho Jx^4 + J\delta n^2x^3 + \big(aJn^2 + cJn^4 + n^2\beta\rho + \xi\rho\big)x^2 + \big(n^4\beta\delta + n^2\delta\xi\big)x \\ +an^4\beta + an^2\xi - b^2n^2 + cn^6\beta + cn^4\xi. \end{aligned} \tag{3.27}$$

And now we consider $q(x-\varepsilon)$. We denote by q_i, $i = 0,1,2,3,4$ the coefficients of $q(x-\varepsilon)$:

$$\begin{aligned} q_0 =& \rho J \\ q_1 =& Jn^2\delta - 4J\varepsilon\rho \\ q_2 =& n^2(aJ - 3J\delta\varepsilon + \beta\rho) + cJn^4 + 6J\varepsilon^2\rho + \xi\rho \\ q_3 =& n^2\left(-2aJ\varepsilon + 3J\delta\varepsilon^2 - 2\beta\varepsilon\rho + \delta\xi\right) + n^4(\beta\delta - 2cJ\varepsilon) - 4J\varepsilon^3\rho - 2\varepsilon\xi\rho \\ q_4 =& n^2\left(aJ\varepsilon^2 + a\xi - b^2 - J\delta\varepsilon^3 + \beta\varepsilon^2\rho - \delta\varepsilon\xi\right) + n^4\left(a\beta + cJ\varepsilon^2 + c\xi - \beta\delta\varepsilon\right) + cn^6\beta + J\varepsilon^4\rho + \varepsilon^2\xi\rho \end{aligned}$$

In this case, the third leading minor, Λ_3, is a polynomial on n of degree 10 with coefficients concerning the parameters of system (3.1) and ε:

$$\Lambda_3 = -2c^2J^3\delta\varepsilon n^{10} + R(n),$$

where $R(n)$ is an eighth degree polynomial on n with coefficients concerning also the parameters of system (3.1) and ε.

By taking n large enough, $\Lambda_3 < 0$. Then, the Routh-Hurwitz theorem implies the existence of solution of equation $q(x-\varepsilon) = 0$ with positive real part.

4. Nonsimple viscoelasticity

In this section we will suppose that $\mu = cu_{xx} + d\varphi_x + \alpha\dot{u}_{xx}$, with $\alpha > 0$. That means that we introduce again another dissipation mechanism in the elasticity but in this case it affects the second order derivatives, that is, it is more sensitive in a local level. As we said in the previous section, this mechanism will prevent the material recovers to its initial state completely after an initial displacement. In this case the dissipation is greater than in the case studied before.

The system of field equations is given by

$$\left\{ \begin{array}{l} \rho\ddot{u} = au_{xx} + b\varphi_x - cu_{xxxx} - d\varphi_{xxx} - \alpha\dot{u}_{xxxx} \\ J\ddot{\varphi} = du_{xxx} + \beta\varphi_{xx} - \xi\varphi - bu_x \end{array} \right. \tag{4.1}$$

We assume again the same boundary and initial conditions. And we take once again $b + dn^2 \neq 0$ for all $n \in \mathbb{N}$: if not, undamped solutions can be found.

We will prove that, generically, the solutions of (4.1) decay exponentially. The proof follows the same scheme of the previous section, but we omit some parts.

This initial-boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{B}U, \quad U_0 = (u_0, v_0, \varphi_0, \phi_0),$$

where $\mathcal{B}$ is the following 4×4 matrix:

$$\mathcal{B} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{aD^2 - cD^4}{\rho} & \frac{\alpha D^4}{\rho} & \frac{bD - dD^3}{\rho} & 0 \\ 0 & 0 & 0 & I \\ \frac{dD^3 - bD}{J} & 0 & \frac{\beta D^2 - \xi}{J} & 0 \end{pmatrix}.$$

We decompose $U(t) = U_1(t) + U_2(t)$, as we did before. Again, all the eigenvalues of $\mathcal{B}$ restricted to $\Omega(h,i,j,k)$ have negative real part. In this case, the polynomial we get is the following:

$$\begin{aligned} r(x) = J\rho x^4 + Jn^4\alpha x^3 + \left(aJn^2 + cJn^4 + n^2\beta\rho + \xi\rho\right)x^2 + n^4\alpha\left(n^2\beta + \xi\right)x \\ + n^6\left(c\beta - d^2\right) + n^4(a\beta - 2bd + c\xi) + n^2\left(a\xi - b^2\right) \end{aligned} \tag{4.2}$$

The coefficients are:

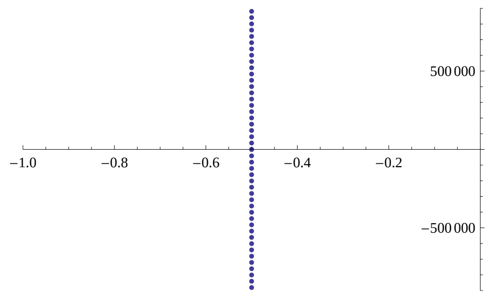

FIGURE 2. Root's behavior of $r_P(x)$.

$$\begin{aligned}
r_0 =& \rho J\\
r_1 =& Jn^4\alpha\\
r_2 =& cJn^4+(aJ+\beta\rho)n^2+\xi\rho\\
r_3 =& n^4\alpha\left(n^2\beta+\xi\right)\\
r_4 =& n^6\left(c\beta-d^2\right)+n^4(a\beta-2bd+c\xi)+n^2\left(a\xi-b^2\right)
\end{aligned}$$

And the leading minors L_i are all positive:

$$\begin{aligned}
L_1 =& r_1 = Jn^4\alpha\\
L_2 =& J^2n^6\alpha\left(a+cn^2\right)\\
L_3 =& J^2n^{10}\alpha^2\left(b+dn^2\right)^2\\
L_4 =& r_4\cdot L_3
\end{aligned}$$

Remark 4.1. Giving particular values (satisfying conditions (2.7)) to the parameters of system (4.1) we get also a particular polynomial, $r_P(x)$. Take, for instance, $\rho = J = \alpha = \beta = \xi = b = d = 1$ and $a = c = 2$. If we compute the roots of $r_P(x)$ for different values of n we get again a line parallel to the imaginary axis (see figure 2).

With the same inner product defined by (3.8), we get

$$Re\langle \mathcal{B}U,U\rangle = -\frac{1}{2}\int_0^\pi \alpha|v_{xx}|^2dx \le 0. \tag{4.3}$$

Lemma 4.2. *Let $\mathcal{B}$ be the above defined matrix. Then,* 0 *is in the resolvent of $\mathcal{B}$.*

Proof. For any $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ we will find $U \in \mathcal{H}$ such that $\mathcal{B}U = \mathcal{F}$, or equivalently:

$$\left.\begin{aligned}
v =& f_1\\
\tfrac{1}{\rho}[aD^2u+bD\varphi+\alpha D^4v-cD^4u-dD^3\varphi] =& f_2\\
\phi =& f_3\\
\tfrac{1}{J}[dD^3u+\beta D^2\varphi-\xi\varphi-bDu] =& f_4
\end{aligned}\right\} \tag{4.4}$$

In this case the domain is defined by

$$\mathcal{D}' = \{U \in \mathcal{H} : \mathcal{B}U \in \mathcal{H}, u_{xx}(0) = u_{xx}(\pi) = 0\}.$$

As in the proof of Lemma 3.3, let us write $u=\sum a_n \sin nx$, $v=\sum b_n \sin nx$, $\varphi=\sum c_n \cos nx$ and $\phi=\sum d_n \cos nx$ and $f_i=\sum f_n^i \sin nx$ for $i=1,2$ and $f_j=\sum f_n^j \cos nx$ for $j=3,4$. Coefficients f_n^i for $i=1,2,3,4$ are known and a_n, b_n, c_n and d_n unknown. We also know that $\sum n^4 |f_n^1|^2<\infty$, $\sum |f_n^2|^2<\infty$, $\sum n^2|f_n^3|^2<\infty$ and $\sum |f_n^4|^2<\infty$.

From the first and third equations of system (4.4) it is clear that $b_n=f_n^1$ and $d_n=f_n^3$. Therefore, it is clear also that $\sum |b_n|^2<\infty$ and $\sum |d_n|^2<\infty$.

From the second and fourth equations of system (4.4) we get

$$a_n=-\frac{f_n^4 Jn(b+dn^2)+\left(n^2\beta+\xi\right)\left(f_n^1 n^4\alpha-f_n^2\rho\right)}{n^2\left((b+dn^2)^2-(a+cn^2)\,(n^2\beta+\xi)\right)},$$

and

$$c_n=\frac{f_n^4 Jn(a+cn^2)+\left(b+dn^2\right)\left(f_n^1 n^4\alpha-f_n^2\rho\right)}{n\left((b+dn^2)^2-(a+cn^2)\,(n^2\beta+\xi)\right)}.$$

Hence $\sum n^4|a_n|^2<\infty$ and $\sum n^2|c_n|^2<\infty$.

□

Lemma 4.3. *Let $\mathcal{B}$ be the same matrix as in Lemma 4.2. Then condition (3.10) is satisfied.*

Proof. The proof follows the same guidelines that the proof of Lemma 3.4, changing a little bit the notation. We concentrate only in the third step of the proof. Let $U_n=(u_n,v_n,\varphi_n,\phi_n)$ be a sequence of unit norm vectors in the domain of $\mathcal{B}$ such that $\|(i\lambda_n I-\mathcal{B})U_n\|\to 0$. Writing this condition term by term we get

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^2, \tag{4.5}$$

$$i\lambda_n v_n-\frac{a}{\rho}D^2u_n+\frac{c}{\rho}D^4u_n-\frac{\alpha}{\rho}D^4v_n+\frac{d}{\rho}D^3\varphi_n-\frac{b}{\rho}D\varphi_n\to 0 \text{ in } L^2, \tag{4.6}$$

$$i\lambda_n\varphi_n-\phi_n\to 0 \text{ in } H^1, \tag{4.7}$$

$$i\lambda_n\phi_n-\frac{d}{J}D^3u_n+\frac{b}{J}Du_n-\frac{\beta}{J}D^2\varphi_n+\frac{\xi}{J}\varphi_n\to 0 \text{ in } L^2. \tag{4.8}$$

Taking the inner product of $(i\lambda_n I-\mathcal{B})U_n$ with U_n in $\mathcal{H}$ using (4.3) and selecting its real part we obtain $\|D^2v_n\|^2\to 0$. Thus, we have also $Dv_n\to 0$ and $v_n\to 0$, and, from (4.5), $u_n\to 0$ in H^2 and, then, $D^2u_n\to 0$.

Removing from (4.6) the terms that tend to zero and, then, taking the inner product of it with $\int\varphi_n$ we get

$$c\langle D^4u_n,\int\varphi_n\rangle-\alpha\langle D^4v_n,\int\varphi_n\rangle+d\langle D^3\varphi_n,\int\varphi_n\rangle-b\langle D\varphi_n,\int\varphi_n\rangle\to 0 \text{ in } L^2. \tag{4.9}$$

Or, equivalently,

$$c\langle D^2u_n,D\varphi_n\rangle-\alpha\langle D^2v_n,D\varphi_n\rangle+d\|D\varphi_n\|^2+b\|\varphi_n\|^2\to 0 \text{ in } L^2. \tag{4.10}$$

Recalling that $D^2v_n\to 0$ and $D\varphi_n$ is bounded, and also the previous results, expression (4.10) reduces to

$$d\|D\varphi_n\|^2+b\|\varphi_n\|^2\to 0 \text{ in } L^2. \tag{4.11}$$

And then, following the same argument as that of Lemma 3.4, $\varphi_n\to 0$ and $D\varphi_n\to 0$. □

Lemma 4.4. *Let $\mathcal{B}$ the same matrix as in Lemma 4.2. Then condition (3.11) is satisfied.*

Proof. The proof is, *mutatis mutandis*, the same as that of Lemma 3.5. □

Theorem 4.5. *Let (u,φ) be a solution of the problem determined by (4.1), with boundary conditions (3.2) and initial conditions (3.3). Then, (u,φ) decays exponentially.*

Proof. The proof is a direct consequence of Lemmas 4.3 and 4.4. □

Once again, the strong coupling between the equations is provided by φ_{xxx}. If $d=0$, the solutions of system (4.1) decay in a slow way. To show that fact, we proceed as in the previous section.

The third leading minor is a polynomial on n of degree 14 with coefficients concerning the parameters of system (4.1) and ε:

$$\Gamma_3 = -2J^2\beta\varepsilon\alpha^3 n^{14} + S(n),$$

where $S(n)$ is a twelfth degree polynomial on n.

By taking n large enough, $\Gamma_3 < 0$.

5. Viscoporosity

We introduce a dissipative mechanism in the porosity, to be precise, it applies to the equilibrated body force and it produces an internal dissipation in the material. Nunziato and Cowin [14] have shown that this internal dissipation results in the material exhibiting a relaxation property similar to that associated with viscoelastic materials.

We will assume that

$$g = -\xi\varphi - bu_x - k\dot{\varphi} \tag{5.1}$$

with $k > 0$.

If we substitute the constitutive equations into the evolution equations, we obtain the field equations for the one-dimensional problem:

$$\begin{cases} \rho\ddot{u} = au_{xx} + b\varphi_x - cu_{xxxx} - d\varphi_{xxx} \\ J\ddot{\varphi} = du_{xxx} + \beta\varphi_{xx} - \xi\varphi - bu_x - k\dot{\varphi} \end{cases} \tag{5.2}$$

We assume the same boundary and initial conditions as in the previous section.

Theorem 5.1. *Let (u,φ) be a solution of the problem determined by (5.2), with boundary conditions (3.2) and initial conditions (3.3). Then (u,φ) decays in a slow way.*

Proof. We will prove that there exists a solution of system (5.2) of the form

$$u = K_1\exp(\omega t)\sin(nx), \quad \varphi = K_2\exp(\omega t)\cos(nx), \tag{5.3}$$

such that $\mathrm{Re}(\omega) > -\varepsilon$ for all positive ε sufficiently small. Hence, a solution ω as near as desired to the imaginary axis can be found. This fact implies that it is impossible to have uniform exponential decay on the solutions of the problem determined by (5.2), with conditions (3.2) and (3.3) and satisfying (2.7).

Imposing that u and φ in (5.2) are as (5.3), the following homogeneous system on the unknown K_1 and K_2 is obtained:

$$\begin{pmatrix} an^2 + cn^4 + \rho\omega^2 & bn + dn^3 \\ bn + dn^3 & \omega(J\omega + k) + n^2\beta + \xi \end{pmatrix} \begin{pmatrix} K_1 \\ K_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{5.4}$$

Our aim is to obtain a nontrivial solution. This linear system will have nontrivial solutions if, and only if, the determinant of the coefficients matrix is null. Let us denote by $t(x)$ the fourth degree polynomial obtained from the determinant of the coefficients matrix once ω is replaced by x:

$$
\begin{aligned}
t(x) = {} & \rho J x^4 + \rho k x^3 + (aJn^2 + cJn^4 + n^2\beta\rho + \xi\rho)x^2 + (a+cn^2)kn^2 x \\
& + (c\beta - d^2)n^6 + (a\beta - 2bd + c\xi)n^4 + (a\xi - b^2)n^2.
\end{aligned} \tag{5.5}
$$

In order to prove that roots of $t(x)$ as near as we want to the complex axis can be found, we will show that for any $\varepsilon > 0$ there are solutions located on the right side of the vertical line $\mathrm{Re}(z) = -\varepsilon$. This fact will be shown if polynomial $t(x-\varepsilon)$ has a root with positive real part.

Let Λ_i for $i = 1,2,3,4$ be the leading diagonal minors of the matrix (3.7). We are going to see that there exists $n \geq 1$ that makes $\Lambda_2 < 0$. Expanding $t(x-\varepsilon)$ and simplifying, we get the coefficients t_i, for $i = 0,1,2,3,4$ which depend on the parameters of system (5.2), n and ε:

$$
\begin{aligned}
t_0 = {} & \rho J \\
t_1 = {} & k\rho - 4J\varepsilon\rho \\
t_2 = {} & aJn^2 + cJn^4 + 6J\varepsilon^2\rho - 3k\varepsilon\rho + n^2\beta\rho + \xi\rho \\
t_3 = {} & -2aJn^2\varepsilon + akn^2 - 2cJn^4\varepsilon + ckn^4 - 4J\varepsilon^3\rho + 3k\varepsilon^2\rho - 2n^2\beta\varepsilon\rho - 2\varepsilon\xi\rho \\
t_4 = {} & J\rho\varepsilon^4 - k\rho\varepsilon^3 + \left(aJn^2 + cJn^4 + n^2\beta\rho + \xi\rho\right)\varepsilon^2 - \left(akn^2 + ckn^4\right)\varepsilon + \\
& + an^4\beta + an^2\xi - b^2n^2 - 2bdn^4 + cn^6\beta + cn^4\xi - d^2n^6
\end{aligned}
$$

Direct computations show that the second leading minor, Λ_2, is a polynomial on n of degree 4 with coefficients concerning the parameters of system (5.2) and ε:

$$
\Lambda_2 = -2cJ^2\rho\varepsilon n^4 + \left(-2aJ^2\varepsilon\rho - 2J\beta\varepsilon\rho^2 + k\beta\rho^2\right)n^2 - 20J^2\varepsilon^3\rho^2 + 15Jk\varepsilon^2\rho^2 - 2J\varepsilon\xi\rho^2 - 3k^2\varepsilon\rho^2 + k\xi\rho^2 \tag{5.6}
$$

By taking n large enough, $\Lambda_2 < 0$. Then, the Routh-Hurwitz theorem implies the existence of solution of equation $t(x-\varepsilon) = 0$ with positive real part.

Therefore, a uniform rate of decay of exponential type for all the solutions of system (5.2) can not be obtained, and so, the decay of the solutions is slow.

□

6. Viscosity in the gradient of φ

Alternatively, it is also possible that the dissipation in the porosity would be given by

$$h = du_{xx} + \beta\phi_x + m\dot{\phi}_x,$$

with $m > 0$. The porous dissipation is imposed to the equilibrated stress tensor and it would be greater than in the previous section.

In this case, the field equations are

$$
\left\{
\begin{array}{l}
\rho\ddot{u} = au_{xx} + b\varphi_x - cu_{xxxx} - d\varphi_{xxx} \\
J\ddot{\varphi} = du_{xxx} + \beta\varphi_{xx} - \xi\varphi - bu_x + m\dot{\varphi}_{xx}
\end{array}
\right. \tag{6.1}
$$

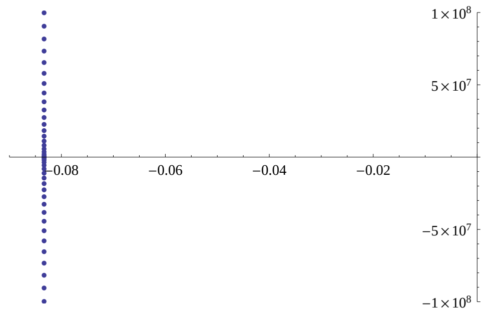

FIGURE 3. Root's behavior of $u_P(x)$.

As before, conditions (2.7) are assumed for the system coefficients. We consider also the same initial and boundary conditions.

We need again $b+dn^2 \neq 0$ for all n. If not, undamped solutions can be found. Take, for instance, $u = \cos(\omega t)\sin(nx)$ and $\varphi = 0$ with $\omega = \sqrt{\frac{n^2(a+cn^2)}{\rho}}$.

We will prove that, generically, the solutions of (6.1) decay exponentially.

This initial-boundary value problem can be written as

$$\frac{dU}{dt} = \mathcal{C}U, \quad U_0 = (u_0, v_0, \varphi_0, \phi_0),$$

where $\mathcal{C}$ is the following 4×4 matrix:

$$\mathcal{C} = \begin{pmatrix} 0 & I & 0 & 0 \\ \frac{aD^2-cD^4}{\rho} & 0 & \frac{bD-dD^3}{\rho} & 0 \\ 0 & 0 & 0 & I \\ \frac{dD^3-bD}{J} & 0 & \frac{\beta D^2-\xi}{J} & \frac{mD^2}{J} \end{pmatrix}.$$

We decompose $U(t) = U_1(t)+U_2(t)$, as we did before. Again, all the eigenvalues of $\mathcal{C}$ restricted to $\Omega(h,i,j,k)$ have negative real part. In this case, the polynomial we get is the following:

$$\begin{aligned} u(x) = {} & J\rho x^4 + mn^2\rho x^3 + \left(aJn^2 + cJn^4 + n^2\beta\rho + \xi\rho\right)x^2 + mn^4\left(a+cn^2\right)x \\ & + (c\beta - d^2)n^6 + (a\beta + c\xi - 2bd)n^4 + (a\xi - b^2)n^2 \end{aligned} \tag{6.2}$$

The corresponding leading minors M_i are all positive:

$$\begin{aligned} M_1 &= mn^2\rho \\ M_2 &= mn^2\rho^2(n^2\beta+\xi) \\ M_3 &= m^2n^6\rho^2\left(b+dn^2\right)^2 \\ M_4 &= u_4 \cdot M_3 \end{aligned}$$

Remark 6.1. Giving particular values (satisfying conditions (2.7)) to the parameters of system (6.1) we get also a particular polynomial, $u_P(x)$. Take, for instance, $\rho = J = \beta = \xi = b = d = m = 1$ and $a = c = 2$. If we compute the roots of $u_P(x)$ for different values of n we obtain, for the third time, a line parallel to the imaginary axis (see figure 3).

With the inner product defined by (3.8), we get

$$Re\langle \mathcal{C}U,U\rangle = -\frac{1}{2}\int_0^\pi m|\phi_x|^2 dx \le 0. \tag{6.3}$$

Lemma 6.2. *Let $\mathcal{C}$ be the above defined matrix. Then, 0 is in the resolvent of $\mathcal{C}$.*

Proof. For any $\mathcal{F} = (f_1, f_2, f_3, f_4) \in \mathcal{H}$ we will find $U \in \mathcal{H}$ such that $\mathcal{C}U = \mathcal{F}$, or equivalently:

$$\left.\begin{aligned} v &= f_1 \\ \tfrac{1}{\rho}[aD^2u + bD\varphi - cD^4u - dD^3\varphi] &= f_2 \\ \phi &= f_3 \\ \tfrac{1}{J}[dD^3u + \beta D^2\varphi - \xi\varphi - bDu + mD^2\phi] &= f_4 \end{aligned}\right\} \tag{6.4}$$

In this case the domain is defined by

$$\mathcal{D}'' = \{U \in \mathcal{H} : \mathcal{C}U \in \mathcal{H}, u_{xx}(0) = u_{xx}(\pi) = 0\}.$$

As in the proof of Lemma 3.3, let us write $u = \sum a_n \sin nx$, $v = \sum b_n \sin nx$, $\varphi = \sum c_n \cos nx$ and $\phi = \sum d_n \cos nx$ and $f_i = \sum f_n^i \sin nx$ for $i = 1,2$ and $f_j = \sum f_n^j \cos nx$ for $j = 3,4$. Coefficients f_n^i for $i = 1,2,3,4$ are known and a_n, b_n, c_n and d_n unknown. We also know that $\sum n^4 |f_n^1|^2 < \infty$, $\sum |f_n^2|^2 < \infty$, $\sum n^2 |f_n^3|^2 < \infty$ and $\sum |f_n^4|^2 < \infty$.

From the first and third equations of system (6.4) it is clear that $b_n = f_n^1$ and $d_n = f_n^3$. Therefore, it is clear also that $\sum |b_n|^2 < \infty$ and $\sum |d_n|^2 < \infty$.

From the second and fourth equations of system (6.4) we get

$$a_n = \frac{f_n^2 \rho\left(n^2\beta + \xi\right) - n\left(b + dn^2\right)\left(f_n^3 mn^2 + f_n^4 J\right)}{n^2\left((b+dn^2)^2 - (a+cn^2)(n^2\beta+\xi)\right)}$$

and

$$c_n = \frac{n\left(a + cn^2\right)\left(f_n^3 mn^2 + f_n^4 J\right) - f_n^2\rho\left(b + dn^2\right)}{n\left((b+dn^2)^2 - (a+cn^2)(n^2\beta+\xi)\right)}.$$

As before, a_n and c_n satisfy the required conditions.

□

Lemma 6.3. *Let $\mathcal{C}$ be the same matrix as in Lemma 6.2. Then conditions (3.10) and (3.11) are satisfied.*

Proof. The proof follows the same guidelines that the proof of Lemma 3.4, changing a little bit the notation. Notice that, in this case, both conditions (λ bounded and $\lambda \to \infty$) are considered together. We concentrate only in the third step of the proof. Let $U_n = (u_n, v_n, \varphi_n, \phi_n)$ be a sequence of unit norm vectors in the domain of $\mathcal{C}$ such that $\|(i\lambda_n I - \mathcal{C})U_n\| \to 0$. Writing this term by term we get

$$i\lambda_n u_n - v_n \to 0 \text{ in } H^2, \tag{6.5}$$

$$i\lambda_n v_n - \frac{a}{\rho}D^2u_n + \frac{c}{\rho}D^4u_n + \frac{d}{\rho}D^3\varphi_n - \frac{b}{\rho}D\varphi_n \to 0 \text{ in } L^2, \tag{6.6}$$

$$i\lambda_n \varphi_n - \phi_n \to 0 \text{ in } H^1, \tag{6.7}$$

$$i\lambda_n \phi_n - \frac{d}{J}D^3u_n + \frac{b}{J}Du_n - \frac{\beta}{J}D^2\varphi_n + \frac{\xi}{J}\varphi_n - \frac{m}{J}D^2\phi_n \to 0 \text{ in } L^2 \tag{6.8}$$

Taking the inner product of $(i\lambda_n I - \mathcal{C})U_n$ with U_n in $\mathcal{H}$ using (4.3) and selecting its real part we obtain $\|D\phi_n\|^2 \to 0$. Thus, we have also $\phi_n \to 0$, and, from (6.7), $\varphi_n \to 0$ in H^1 and, then, $D\varphi_n \to 0$.

We remove from (6.8) the terms that tend to zero. Then, we take the inner product of it with Du_n:

$$\langle i\lambda_n\phi_n, Du_n\rangle - \frac{d}{J}\langle D^3u_n, Du_n\rangle + \frac{b}{J}\langle Du_n, Du_n\rangle - \frac{\beta}{J}\langle D^2\varphi_n, Du_n\rangle - \frac{m}{J}\langle D^2\phi_n, Du_n\rangle \to 0 \text{ in } L^2 \tag{6.9}$$

Applying integration by parts, the last expression is equivalent to

$$-\langle iD\phi_n, \lambda_n u_n\rangle + \frac{d}{J}\|D^2u_n\|^2 + \frac{b}{J}\|Du_n\|^2 + \frac{\beta}{J}\langle D\varphi_n, D^2u_n\rangle + \frac{m}{J}\langle D\phi_n, D^2u_n\rangle \to 0 \text{ in } L^2 \tag{6.10}$$

As $\lambda_n u_n$ and D^2u_n are bounded, this expression becomes

$$d\|D^2u_n\|^2 + b\|Du_n\|^2 \to 0. \tag{6.11}$$

Taking into account the argument used in the proof of Lemma 3.4, we obtain $D^2u_n \to 0$.

Finally, take the inner product of (6.6) with u_n:

$$\langle i\lambda_n v_n, u_n\rangle - \frac{a}{\rho}\langle D^2u_n, u_n\rangle + \frac{c}{\rho}\langle D^4u_n, u_n\rangle + \frac{d}{\rho}\langle D^3\varphi_n, u_n\rangle \to 0 \text{ in } L^2. \tag{6.12}$$

Using again integration by parts and the fact that $i\lambda_n u_n \sim v_n$, we get

$$-\|v_n\|^2 + \frac{c}{\rho}\|D^2u_n\| + \frac{d}{\rho}\langle D\varphi_n, D^2u_n\rangle \to 0 \text{ in } L^2. \tag{6.13}$$

From the previous results, it is clear that $v_n \to 0$. And this proves that U_n can not be of unit norm. □

Theorem 6.4. *Let (u,φ) be a solution of the problem determined by (6.1), with boundary conditions (3.2) and initial conditions (3.3). Then, (u,φ) decays exponentially.*

Proof. The proof is a direct consequence of Lemma 6.3. □

Once again, the strong coupling between the equations is provided by φ_{xxx}. If $d = 0$, the solutions of system (6.1) decay in a slow way. To show that fact, we proceed as in section 3. In this case, the third leading minor is a polynomial on n of degree 10:

$$\Upsilon_3 = -2\rho cm\varepsilon(cJ^2 + m^2\rho)n^{10} + T(n),$$

where $T(n)$ is an eighth degree polynomial on n. By taking n large enough, $\Upsilon_3 < 0$.

7. Impossibility of localization

In this section we investigate the impossibility of the localization in time of the solutions for the non-simple viscoelasticity problem. This means that the only solution that vanishes after a finite time is the null solution. The main idea is to show the uniqueness of solutions for the backward in time problem. Therefore, we consider the following system

$$\begin{cases} \rho\ddot{u} = au_{xx} + b\varphi_x - cu_{xxxx} - d\varphi_{xxx} + \alpha\dot{u}_{xxxx} \\ J\ddot{\varphi} = du_{xxx} + \beta\varphi_{xx} - \xi\varphi - bu_x \end{cases} \tag{7.1}$$

First of all we define several functions and compute their time derivatives. In view of (3.8), we define

$$E_1(t) = \frac{1}{2}\int_0^\pi \left(\rho|\dot{u}|^2 + J|\dot{\varphi}|^2 + a|u_x|^2 + \beta|\varphi_x|^2 + \xi|\varphi|^2 + 2bu_x\varphi + 2du_{xx}\varphi_x + c|u_{xx}|^2\right)dx.$$

If we compute the derivative of this function, we get

$$E_1'(t) = \int_0^\pi \alpha|\dot{u}_{xx}|^2 dx.$$

Using the Lagrange identity method, we know that

$$\frac{\partial}{\partial s}\big(\rho\dot{u}(s)\dot{u}(2t-s)\big) = \rho\ddot{u}(s)\dot{u}(2t-s) - \rho\dot{u}(s)\ddot{u}(2t-s)$$
$$\frac{\partial}{\partial s}\big(J\dot{\varphi}(s)\dot{\varphi}(2t-s)\big) = J\ddot{\varphi}(s)\dot{\varphi}(2t-s) - J\dot{\varphi}(s)\ddot{\varphi}(2t-s)$$

Therefore, it could be seen that

$$\int_0^\pi \big(\rho|\dot{u}|^2 + J|\dot{\varphi}|^2\big)dx = \int_0^\pi \big(a|u_x|^2 + \beta|\varphi_x|^2 + \xi|\varphi|^2 + 2bu_x\varphi + 2du_{xx}\varphi_x + c|u_{xx}|^2\big)dx. \tag{7.2}$$

And, in consequence,

$$E_1(t) = \int_0^\pi \big(\rho|\dot{u}|^2 + J|\dot{\varphi}|^2\big)dx.$$

Let us introduce another function:

$$E_2(t) = \frac{1}{2}\int_0^\pi \big(-\rho|\dot{u}|^2 + J|\dot{\varphi}|^2 - a|u_x|^2 + \beta|\varphi_x|^2 + \xi|\varphi|^2 - c|u_{xx}|^2\big)dx.$$

The derivative of this function is given by

$$E_2'(t) = \int_0^\pi \big(-\alpha|\dot{u}_{xx}|^2 + b\varphi\dot{u}_x - bu_x\dot{\varphi} + d\varphi_x\dot{u}_{xx} - du_{xx}\dot{\varphi}_x\big)dx.$$

We take now

$$E_3(t) = E_2(t) + \int_0^\pi du_{xx}\varphi_x dx + \int_0^\pi bu_x\varphi dx.$$

From the previous relations,

$$E_3'(t) = -\alpha\int_0^\pi |\dot{u}_{xx}|^2 + 2\int_0^\pi d\dot{u}_{xx}\varphi_x dx + 2\int_0^\pi b\dot{u}_x\varphi dx.$$

If we multiply the first equation of system (7.1) by u, the second by φ and, afterwards, we sum both expressions, we get

$$\int_0^\pi (\rho\ddot{u}u + J\ddot{\varphi}\varphi)\,dx = -\int_0^\pi \big(a|u_x|^2 + c|u_{xx}|^2 + \beta|\varphi_x|^2 + \xi|\varphi|^2 + 2bu_x\varphi + 2du_{xx}\varphi_x\big)\,dx + \alpha\int_0^\pi \dot{u}_{xx}u_{xx}dx.$$

Or, equivalently (see (7.2)):

$$\int_0^\pi (\rho\ddot{u}u + J\ddot{\varphi}\varphi)\,dx = -\int_0^\pi \big(\rho|\dot{u}|^2 + J|\dot{\varphi}|^2\big)dx + \alpha\int_0^\pi \dot{u}_{xx}u_{xx}dx.$$

Therefore

$$\frac{d}{dt}\left[\int_0^\pi (\rho\dot{u}u + J\dot{\varphi}\varphi)\,dx - \frac{\alpha}{2}\int_0^\pi |u_{xx}|^2 dx\right] = 0,$$

and, in consequence,

$$\frac{\alpha}{2}\int_0^\pi |u_{xx}|^2 dx = \int_0^\pi (\rho\dot{u}u + J\dot{\varphi}\varphi)\,dx.$$

We consider now the function defined by

$$E(t) = \varepsilon E_1(t) + E_3(t) + \lambda\frac{\alpha}{2}\int_0^\pi |u_{xx}|^2 dx - \lambda\int_0^\pi (\rho\dot{u}u + J\dot{\varphi}\varphi)\,dx,$$

where ε is an arbitrary constant that satisfies $\frac{1}{2} < \varepsilon < 1$, and λ is an arbitrary number as large as necessary.

Matching terms together, we obtain

$$E(t) = \left(\varepsilon - \tfrac{1}{2}\right)\int_0^\pi \rho|\dot{u}|^2dx + \left(\varepsilon + \tfrac{1}{2}\right)\int_0^\pi J|\dot{\varphi}|^2dx + \left(\lambda\tfrac{\alpha}{2} - \tfrac{c}{2}\right)\int_0^\pi |u_{xx}|^2dx - \tfrac{a}{2}\int_0^\pi |u_x|^2dx +$$
$$\tfrac{\beta}{2}\int_0^\pi |\varphi_x|^2dx + \tfrac{\xi}{2}\int_0^\pi |\varphi|^2dx + d\int_0^\pi u_{xx}\varphi_x dx + b\int_0^\pi u_x\varphi dx - \lambda\int_0^\pi \rho\dot{u}u dx - \lambda\int_0^\pi J\dot{\varphi}\varphi dx.$$

Taking λ large enough, there exists $C \in \mathbb{R}$ such that

$$E(t) \geq C\int_0^\pi \left(\rho|\dot{u}|^2 + J|\dot{\varphi}|^2 + a|u_x|^2 + \beta|\varphi_x|^2 + \xi|\varphi|^2 + 2bu_x\varphi + 2du_{xx}\varphi_x + c|u_{xx}|^2\right)dx - \lambda\int_0^\pi \rho\dot{u}u dx - \lambda\int_0^\pi J\dot{\varphi}\varphi dx.$$

Therefore, function

$$\mathcal{E}(t) = \int_0^t E(s)ds$$

satisfies

$$\mathcal{E}(t) \geq C_1\int_0^t E_1(s)ds$$

if t is small enough but $t \in (0,t_0)$, ($t_0 > 0$), with $C_1 \in \mathbb{R}$.

Computing the derivative of $\mathcal{E}(t)$ we obtain

$$\mathcal{E}'(t) = \varepsilon\alpha\int_0^t\int_0^\pi |\dot{u}_{xx}|^2dx - \alpha\int_0^t\int_0^\pi |\dot{u}_{xx}|^2dx + 2d\int_0^t\int_0^\pi \dot{u}_{xx}\varphi_x dx + 2b\int_0^t\int_0^\pi \dot{u}_x\varphi dx \leq C\int_0^t |\varphi_x|^2dx.$$

Hence, a positive constant C_2 can be found such that

$$\mathcal{E}'(t) \leq C_2\mathcal{E}(t).$$

And this implies

$$\mathcal{E}(t) \leq \mathcal{E}(0)\exp(C_2t).$$

If $\mathcal{E}(0) = 0$, then $\mathcal{E}(t) \equiv 0$ for all $t \in (0,t_0)$. Therefore, the only solution to our problem is the null solution. We can summarize this result in the following statement.

Theorem 7.1. *Let (u,φ) be a solution of the problem determined by the system (4.1), the initial conditions (3.3) and the boundary conditions (3.2) such that $u = \varphi \equiv 0$ after a finite time $t_0 > 0$. Then $u = \varphi \equiv 0$ for every $t \geq 0$.*

Remark 7.2. The same analysis works for system (5.2). Nevertheless, to prove the impossibility of localization for system (3.1) is still an open question.

8. Conclusions

In this paper we have analyzed the time decay for the solutions of the partial differential equations that model the behavior of non-simple porous-elastic materials when the following dissipation mechanisms are taken into account:

(1) viscoelasticity,
(2) hyperviscoelasticity,
(3) viscoporosity in the equilibrated body force, and
(4) viscoporosity in the equilibrated stress.

We have seen that if the coupling constant d is different from zero, the solutions decay exponentially in cases (1), (2) and (4). We have also seen the impossibility of localization for (2) (and using the same method, it could be seen also for (4)).

This behavior differs from the one known for the classical theory, where, generically, two dissipation mechanisms are needed (one at the macrostructure level and another one at the microstructure) to guarantee the exponential decay. Therefore, the coupling given by constant d plays an essential role that did not appear in the simple theory.

Finally, we want to remark that, even if d is not zero, the viscoporosity in the equilibrated body force is not strong enough to make the solutions decay exponentially. In the cases slow decay, polynomial stability could be proved. However, we do not plan to get into the details due to the length of this paper.

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