Propagation modes in lossy cylindrical structures

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ABSTRACT

The purpose of this communication is to calculate the orders of the Bessel functions that satisfies the equation $f_\nu(r) = J_\nu(k_r \cdot r) + A_\nu \cdot Y_\nu(k_r \cdot r)$ when the contour conditions are $f_\nu(a)=f_\nu(b)=0$. We will see that when k_r is a constant then the orders ν may be complex.

I. INTRODUCTION

The solution to the wave equation in cilyndrical coordinates, that is widely used in electromagnetics, has a linear combination of Bessel functions as follows: $f_\nu(r) = J_\nu(k_r \cdot r) + A_\nu \cdot Y_\nu(k_r \cdot r)$. This function must satisfy the boundary condition forced by the geometry of our structure. We will force electric wall conditions in r=a and in r=b. That is $f_\nu(a)=f_\nu(b)=0$.

The bibliography about this topic is scarce. In [1] we have one of the best treatises about Bessel Functions. But there is not any reference to our problem. In [2], a reference from the last century, we have this problem but without resolution. So we are in front of a problem that has not been solved clearly. We will use the method proposed in [3] in order to find a solution.

And, in addition, we will find out that this method allows to calculate the propagation modes in lossy structures. So we can see the wide possibilies of the proposed numerical method.

II. THE WAVE EQUATION

The wave equation in cilyndrical coordinates has the following solution:

$$\begin{gathered} \Delta^2 \Phi(r,\varphi,z) + k^2 \cdot \Phi(r,\varphi,z) = 0 \\ \Downarrow \\ \Phi(r,\varphi,z) = R(r) \cdot F(\varphi) \cdot Z(z) \end{gathered} \tag{1}$$

where F(φ) y Z(z) are:

$$\begin{cases} \dfrac{1}{F(\varphi)} \cdot \dfrac{\partial^2 F(\varphi)}{\partial \varphi^2} = -\nu^2 & \Rightarrow \quad F(\varphi) = e^{j\nu\varphi} + A_\nu \cdot e^{-j\nu\varphi} \\ \dfrac{1}{Z(z)} \cdot \dfrac{\partial^2 Z(z)}{\partial z^2} = -k_z^2 & \Rightarrow \quad Z(z) = e^{jk_z\varphi} + B_{k_z} \cdot e^{-jk_z\varphi} \end{cases} \tag{2}$$

and R(r) is the solution of the Bessel equation:

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r} \cdot \frac{\partial}{\partial r} + k_r^2 - \frac{\nu^2}{r^2} \right) \cdot R(r) = 0 \tag{3}$$

where $k_r^2 = k^2 - k_z^2$. Consquently, R(r) is:

$$R(r) = J_\nu(k_r \cdot r) + C_\nu \cdot Y_\nu(k_r \cdot r) \tag{4}$$

The three functions R(r), F(φ) y Z(z) must satisfy the boundary conditions that are forced by the structure. We will suppose that these conditions for function R(r) are R(a)=R(b)=0. If k_r is a constant we are supposed to find the values for ν and C_ν. This last value is very simple. We only have to force the boundary conditions in r=a or in r=b. So:

$$R(r)=J_\nu(k_r\cdot r)-\frac{J_\nu(k_r\cdot a)}{Y_\nu(k_r\cdot a)}\cdot Y_\nu(k_r\cdot r) \tag{5}$$

The main problem is to calculate the order ν..

III. SOLUTION OF R(r) AS AN EIGENSYSTEM

Following the method proposed in [3], we can write the equation (3) as:

$$L_{k_r}\cdot R(r)=\frac{\nu^2}{r^2}\cdot R(r) \tag{6}$$

where L_{k_r} is an operator defined as:

$$L_{k_r}\doteq\frac{\partial^2}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial}{\partial r}+k_r^2 \tag{7}$$

Then we propose a R(r) solution in a series form:

$$R(r)=\sum_{m=0}^{\infty} d_m^{k_r}\cdot g_m(r) \tag{8}$$

where $d_m^{k_r}$ are unknown coefficients and $g_m(r)$ are a set of basis function that agree with the boundary conditions $g_m(a)=g_m(b)=0$.
Now we define an inner product as:

$$\langle f\mid g\rangle=\int_{r=a}^{b} f(r)\cdot g(r)\cdot r\cdot dr \tag{9}$$

If we substitute equation (8) in the Bessel equation and we use the inner product defined in (9) we reach the following eigensystem:

$$\overline{\overline{P}}\cdot\overline{D}=\nu^2\cdot\overline{\overline{Q}}\cdot\overline{D} \tag{10}$$

where $\overline{\overline{P}}$ and $\overline{\overline{Q}}$ are two matrices which elements are:

$$\begin{cases} Q_{rt}=\langle g_r\mid g_t\rangle \ ; \ r,t=0,1,2,... \\ P_{rt}=\langle L_{k_r}\cdot g_r\mid g_t\rangle \ ; \ r,t=0,1,2,... \end{cases} \tag{11}$$

and $\overline{D}$ is a vector contining the coefficients of R(r).
If we see the equation (10) carefully we discover that $\overline{D}$ y ν^2 are, respectively, the eigenvectors and the eigenvalues of the previous eigensystem. So, in order to obtain the orders ν of the Bessel functions and the coefficients, we are supposed to solve just an eigensystem problem.
As basis functions $g_m(r)$ we can use the following ones:

$$g_m(r)=\sin\left(m\cdot\pi\cdot\frac{r-a}{b-a}\right) \tag{12}$$

The values of Q_{rt} y P_{rt} are:

$$\begin{cases} P_{r,t} = \left(k^2 - \left(\frac{r \cdot \pi}{\Delta} \right)^2 \right) \cdot I_{r,t}^{(1)} + \frac{r \cdot \pi}{\Delta} \cdot I_{r,t}^{(2)} \\ Q_{r,t} = I_{r,t}^{(3)} \end{cases}$$

where each parameter $I_{r,t}^{(1)}$, $I_{r,t}^{(2)}$ y $I_{r,t}^{(3)}$ is:

$$I_{r,t}^{(1)} = \int_a^b \operatorname{sen}\left(r \cdot \pi \cdot \frac{\rho - a}{b - a} \right) \cdot \operatorname{sen}\left(t \cdot \pi \cdot \frac{\rho - a}{b - a} \right) \cdot \rho \cdot d\rho =$$

$$= \begin{cases} -\frac{1}{2} \cdot \left(\frac{b-a}{\pi \cdot (r^2 - t^2)} \right)^2 \cdot \left(4 \cdot r \cdot t + (-1)^{r+t} \cdot (r-t)^2 - (-1)^{r-t} \cdot (r+t)^2 \right) & ; \quad r \neq t \\ \frac{b^2 - a^2}{4} & ; \quad r = t \end{cases} \tag{14}$$

$$I_{r,t}^{(3)} = \int_a^b \operatorname{sen}\left(r \cdot \pi \cdot \frac{\rho - a}{b - a} \right) \cdot \operatorname{sen}\left(t \cdot \pi \cdot \frac{\rho - a}{b - a} \right) \cdot \frac{1}{\rho} \cdot d\rho \tag{15}$$

$$I_{r,t}^{(2)} = \int_a^b \cos\left(r \cdot \pi \cdot \frac{\rho - a}{b - a} \right) \cdot \operatorname{sen}\left(t \cdot \pi \cdot \frac{\rho - a}{b - a} \right) \cdot d\rho =$$

$$= \begin{cases} \frac{(b-a) \cdot t \cdot \left[-1 + (-1)^{r-t} \right]}{\pi \cdot (r^2 - t^2)} & ; \quad r \neq t \\ 0 & ; \quad r = t \end{cases} \tag{16}$$

IV. EXAMPLE

In the first example we have that k_z=0, k=k_r=62, and that a=0.2 m and b=0.3 m. In the following table we have five values for v and their coefficients d_m^v:

v	d				
13.4014	-0.9948	0.1020	0.0018	-0.0006	0.0003
j·2.4542	-0.0311	-0.9986	-0.0424	-0.0031	-0.0040
j·17.4607	0.0010	0.1208	-0.9804	-0.1541	-0.0196
j·26.9161	-0.0063	0.0121	-0.2334	0.9394	0.2465

We can see that only one order v is real. This is the only mode that propagates along the structure consisting of a circular sector with two electric walls in r=a and r=b. The other modes do not propagate. They are cut off modes because the order v is imaginary and they do not transport energy.

We can state that the biggest is the distance between radii a and b, the biggest is the number of modes that can propagate, and viceversa. So we can have an structure without propagating modes.

With this method we can even calculate the orders ν when k (the wave number in free space) is complex (this is a lossy media). Then the order ν is complex, with real and imaginary parts. The former one indicates the propagation and the last the attenuation.
In figure 1 we have the module of E_z component in a circular sector with radii a=0.2 m and b=0.5 m and permittivity $\varepsilon_r=4-j$ when a sinus wave incides in $\varphi=\pi/2$ at frequency f=3 Ghz. Note the attenuation inside the sector because of the losses in the dielectric.
In figure 2 we have the same E_z component when a=0.2 m and b=0.22 m with dielectric $\varepsilon_r=1$. In this case all the modes are cut off. The values of ν are:

ν				
j·30.2103	j·64.5909	j·98.0018	j·131.1858	j·164.2803

V. CONCLUSIONS

To sum up we can say that we have managed to find the modes, real and complex ones, that appear in a cylindrical structure when there are electric wall boundary conditions in two constant radii. The method is fast and simple. We only need to solve a simple eigenvalue and eigenvector problem.

REFERENCIES

[1] G. N. Watson, "A Treatise on the Theory of Bessel Functions". Cambridge University Press, edition 1995.
[2] Maxime Bôcher, "On Some Applications of Bessel's Functions with Pure Imaginary Index". *Annals of Mathematics*, vol. VI, No 6, pp. 137-160, May 1892.
[3] B. Gimeno, Marco Guglielmi, "Multimode Equivalent Network Representation for Junctions between Coaxial and Circular Waveguides", *Microwave and Optical Technology Letters*, vol. 17, pp. 180-193, 1997.

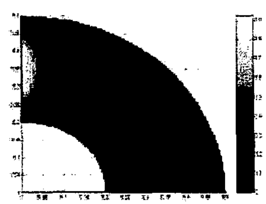

Figure 1

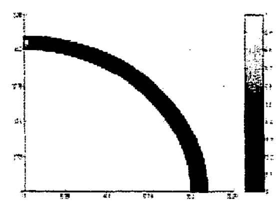

Figure 2