Triangles, Ruler and Compass

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Abstract

A triangle can be specified by giving three independent geometric relationships defined between its elements. Usually, these relationships are distances between two vertices, angles between two sides, and heights. For each triangle specified by a set of three of such relationships, we present a procedure that constructs the triangle using ruler and compass alone.

1 Introduction

Constraints are an important enabling technology for interactive graphics applications. One of such applications where the use of constraints is growing more rapidly is in geometric modelling. The goal is to develop sketching systems based on geometric constaints solvers in which the user defines a rough sketch, annotated with dimensions and constraints, and the system builds an instance that satisfies all constraints.

Several approaches to the geometric constraint solving problem have been reported in the literature. Among them, the constructive technique is one of the most promising approaches. In this class of constraint solvers the constraints are satisfied constructively, by placing geometric elements in some order. They are based on the fact that most configurations in an engineering drawing are solvable by using a rather small set of tools like ruler, compass and protractor.

Basic geometrical constructions with ruler and compass encompasses, among others, constructing congruent angles, lines that are perpendicular or parallel to a given line, lines at a given angle, and lines tangent to circles. Although procedures to perform these constructions with ruler and compass can be found in standard books of geometry and technical drawing for undergraduate courses, the situation is quite different when talking about procedures for more complex constructions. For example, procedures to build triangles o quadrilaterals reported in the literature usually deal only with trivial cases. This work focuses in procedures to build triangles.

A triangle can be specified by giving three independent geometric relationships defined between its elements. Usually, these relationships are distances between two vertices, angles between two sides, and heights. For each triangle specified by a set of three of such relationships, we present a procedure that constructs the triangle using ruler and compass alone.

2 Basic Constructions of Perpendiculars and Parallels

Basic geometrical constructions encompasses, among others, constructing congruent angles, lines that are perpendicular or parallel to a given line, lines at a given angle, and lines tangents to circles. Procedures to perform these constructions with ruler and compass can be found in standard books of geometry and technical drawing for undergraduate courses; we include some of them here just for the sake of completeness. In this Section we shall review the basic procedures to build perpendicular and parallel lines; next Section is devoted to
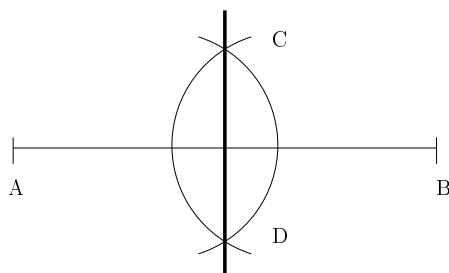
Figure 1: Straight bisector of a given segment.

basic angle constructions, and in Section 4 we shall recall some constructions of straightlines tangents to circles.

2.1 Straight bisector of a segment

Let AB be the given segment in Figure 1. With center on A draw an arc of circle of radius larger than half of the segment length. With the same radius, repeat the process now with center on B. The points C and D where the two arcs meet define the straight bisector perpendicular to the given segment.

2.2 Line perpendicular to a given line

Assume that in Figure 2, l is the given straight line. With center on an arbitrary point P of line l draw an arc of circle of arbitrary radius and let A be one of the points where the arc intersects line l. With center on A and radius greater than the radius of the first arc draw a second arc of circle such that intersects the first arc at B. With the same radius and center on B draw a third arc that intersects the first one at C. Finally, repeat the same operation with center on C to draw the fourth arc such that intersects the third one at D. The line r defined by points D and P gives the normal to the given line l.

2.3 Line perpendicular to a given line through a given point

Let l be the given straight line and P the given point (Figure 3). With center on P draw an arc of circle such that intersects l at two diferent points, A and B. Now with center on A and on B, respectively, draw two arcs of circle which meet at C. The line defined by the given point P and D is the normal to the given line l through point P.
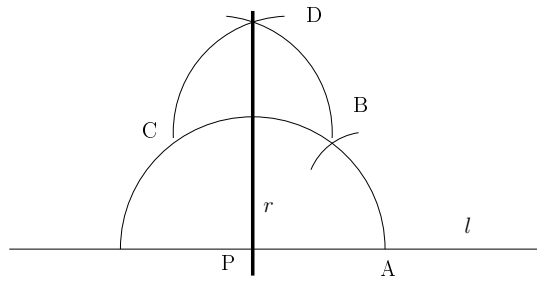
Figure 2: Line perpendicular to a given line.

2.4   Line through a point parallel to a given line

Again let l be the given straight line and P the given point, Figure 4. First, with center on an arbitrary point Q on the line l, draw an arc through point P. Now mark off the smallest arc SP on the other side of the semicircle giving the point C. The straight line r defined by the given point P and point C is the line parallel to l through point P.

2.5   Line parallel to a given line at a given distance

Let d be de given distance. See Figure 5. First drop a line r perpendicular to the given line l through an arbitrary point A on l. Mark off on r a segment $AB = d$. Now drop the perpendicular s to the straight line r through B.
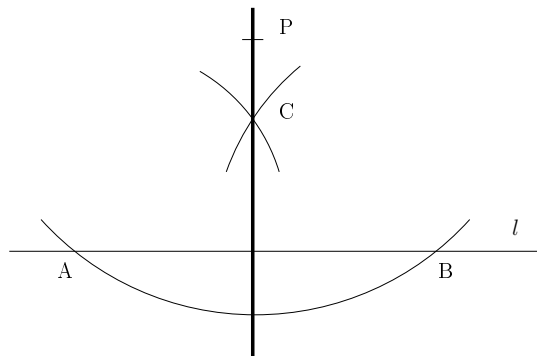

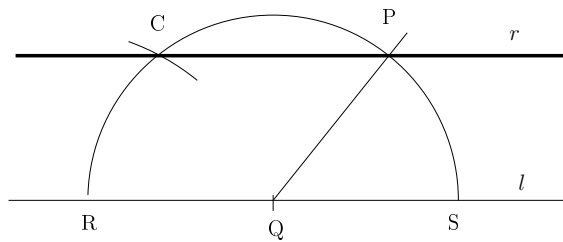Figure 3: Line perpendicular to a given line through a given point.

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Figure 4: Line through a point and parallel to a given line.

3 Basic Constructions of Angles

3.1 Angle congruent to a given angle

Let BAC in Figure 6 be the given angle. First draw a halfline $A'B'$. Then with center on the vertex of the given angle A draw an arc of arbitrary radius which intersects the angle sides at P and Q. Now with the same radius, draw another arc with center on the origin A' of the halfline. Let P' be the point where this arc intersects the halfline. Finally, with center on P' mark off an arc equal to PQ, giving point Q'. The halfline $A'Q'$ defines the desired angle $B'A'C'$.

3.2 Addition and difference of two given angles

The problem of adding two given angles reduces to build two adjacent angles each of them congruent with one of the given angles. See Figure 7.

Similarly, for the difference we build two angles each of them congruent with one of the given angles but this time the angles are built starting at the same
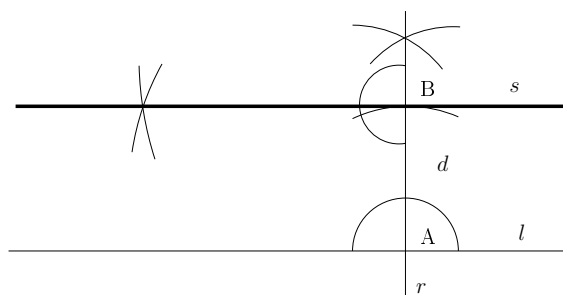

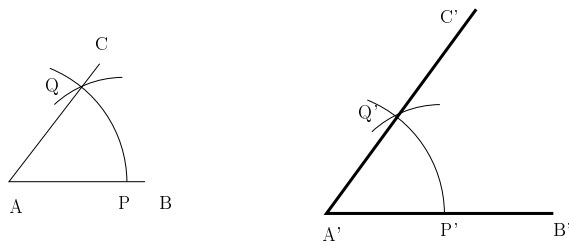Figure 5: Line parallel to a given line at a given distance d.

Figure 6: Angle congruent to a given one.

origin. See Figure 8.

3.3   Straight bisector of an angle

Let BAC be the given angle, Figure 9. With center on the angle vertex A and
arbitrary radius, draw an arc of circle which intersects the angle sides at P and
Q. With center on P and Q, and constant radius, draw two arcs which intersect
each other at R. The line defined by A and R is the bisector.

3.4   Trisection of the right angle

It is well known that the trisection of the angle by ruler and compass alone is in
general impossible. Among those angles for which trisection is possible are 90°
and 180°. Let us see now the procedure for the 90° case depicted in Figure 10.

With center on the angle vertex A, and arbitrary radius draw an arc of circle
which intersects the angle sides at P and Q. With center on P and Q, and
radius equal to that of the first arc, draw two arcs which intersect the first arc
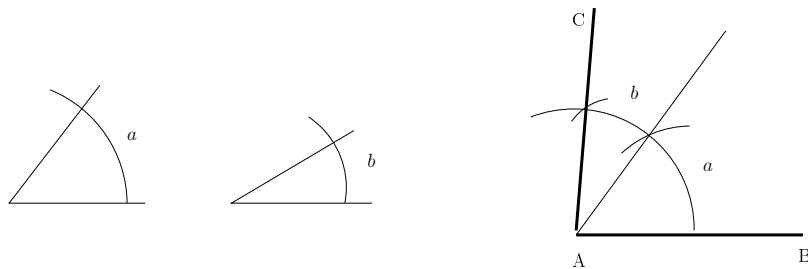at R and S. Lines AR and AS define the trisection of the right angle. Trisecting
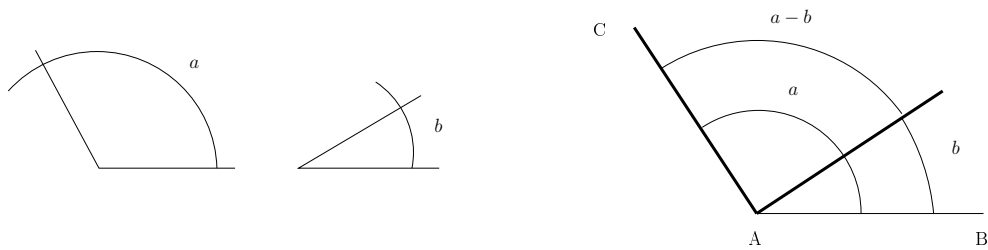

Figure 7: Addition of two angles.

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Figure 8: Difference between two angles.

the 180° angle is now straightforward.

4 Tangencies Between Straightlines and Circles

Among all the problems of tangencies between straightlines and circles, we are now interested only in those we will use in the construction of triangles with ruler and compass.

4.1 Straightline tangencial to a circle through a point on it

The straight line tangencial to a circle γ through a point on it P is the perpendicular l to the radius through the given point, OP. Figure 11.

4.2 Straightline tangent to a circle through an external point

This construction is based on the following theorem related with the operation of inversion in a Euclidean circle.
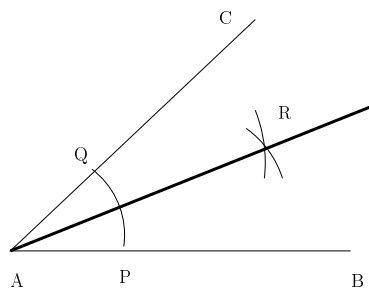

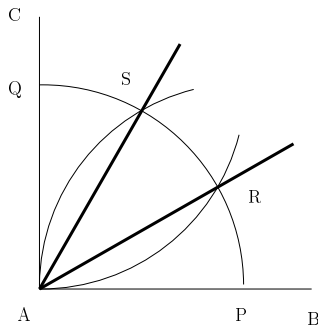Figure 9: Straight bisector of a given angle

Figure 10: Trisection of the right angle.

Theorem 4.1 Let γ be a circle with center O and let P be a point outside γ. Let Q be the midpoint of segment OP and σ be the circle with center Q and radius $OQ = QP$. Then σ cuts γ in two points T and U, and PT and PU are tangent to γ.

The construction is now obvious. See Figure 12.

5 Algebraic Operations and Geometric Constructions

Some of the simplest algebraic operations correspond to elementary geometrical constructions. If two segments are given with lenghts a and b, mesured by an already defined unit segment, and r is any given rational anumber, then it is easy to construct $a + b$, $a - b$, ra, ab, and a/b.
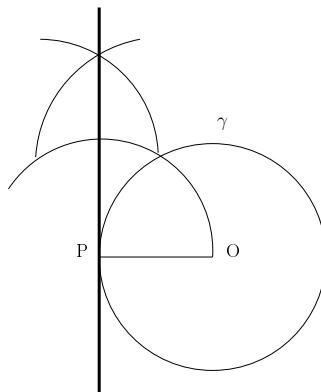

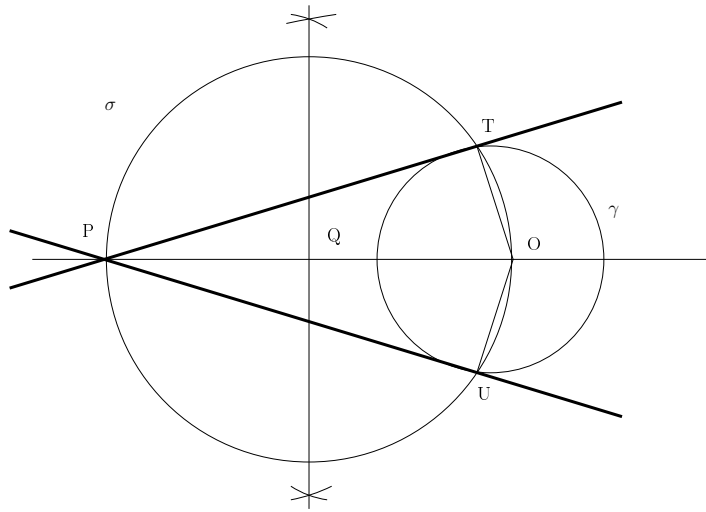Figure 11: Line tangential to a circle through a point on it.

Figure 12: Line tangent to a circle through an external point.

To construct $a + b$ (Figure 13) we draw a straight line and on it mark off with the compass the distance $OA = a$ and $AB = b$. Then $OB = a + b$.

Similarly, for $a - b$ we mark off $OA = a$ and $AB = b$, but now AB is marked in the oposite direction from OA. See Figure 14. The difference $a - b$ is given by OB.

Assume that p and q are integers. Then, to construct pa we simply add $a + \overset{p \, times}{\cdots} + a$. We construct a/q by the following device. See Figure 15. We mark off $OA = a$ on one line, and draw an arbitrary second line through O. On this line we mark off an arbitrary segment $OC = c$, and construct $OD = qc$. Now we connect A and D, and draw a line throug C parallel to AD, intersecting OA at B. The triangles OBC and OAD are similar; hence $OB/a = OB/OA = OC/OD = 1/q$; that is, $OB = a/q$. By performing this operation on the segment pa, we can thus construct ra where $r = p/q$ is any rational number.
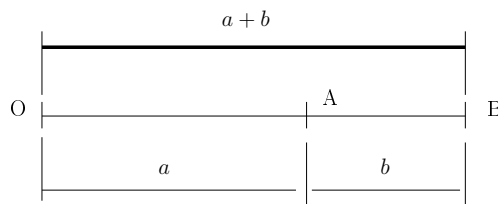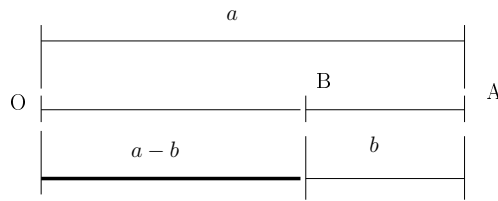

Figure 13: Construction of $a + b$.

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Figure 14: Construction of $a - b$.

The construction of ab can be performed as follows. See Figure 16. We mark off $OA = a$ on one line, and on the same line we mark off a segment $OB = 1$. Draw an arbitrary second line through O, and on it mark off $OC = b$. We connect B and C, and draw a line through A parallel to BC, intersecting OA at D. From the triangles OAD and OBC, $1/b = a/OD$, and $OD = ab$.

To construct a/b, Figure 17 we mark off $OB = b$ and $OA = a$ on the sides of any angle O, and on OB we mark off $OD = 1$. Through D we draw a line parallel to AB meeting OA in C. Then OC will have the length a/b.

It is clear that the rational algebraic processes –addition, subtraction, multiplication, and division of given quantities– can be performed by geometrical constructions. From any given segments, measured by real numbers $a, b, c, \ldots$, we can, by successive application of these simple constructions, construct any quantity that is expressible in terms of $a, b, c, \ldots$ in a rational way, i.e. by repetitive application of addition, subtraction, multiplication and division. The totality of quantities that can be obtained in this way from $a, b, c, \ldots$, constitute what is called a *number field*, a set of numbers such that any rational operations applied to two or more members of the set again yield a number of the set. The
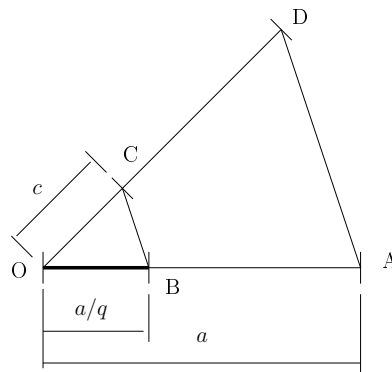

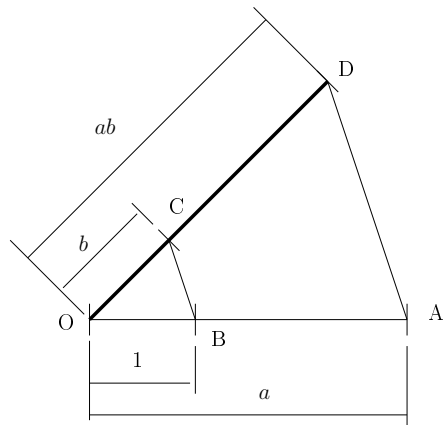Figure 15: Construction of a/q where q is an integer.

Figure 16: Construction of ab.

field is said to be *generated* by the given numbers $a, b, c, \ldots$.

As we will see later, there is a case where, in order to be able to build a triangle, we will need to compute a square root. This means that we need a field wider than that just obtained above. Given a segment a then $\sqrt{a}$ can also be constructed by using only ruler and compass. See Figure 18. On a straight line we mark off $OA = a$ and $AB = 1$. We draw a circle with the segment OB as its diameter and construct the perpendicular to OB through A. This perpendicular meets the circle in C. The triangle OBC has a right angle at C, by the theorem of elementary geometry which states that an angle inscribed in a semicircle is a right angle. Hence, $\angle OCA = \angle ABC$, the right triangles OAC and CAB are
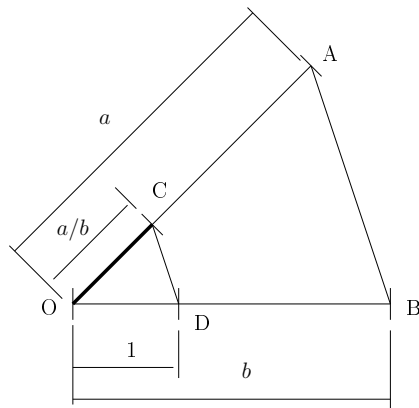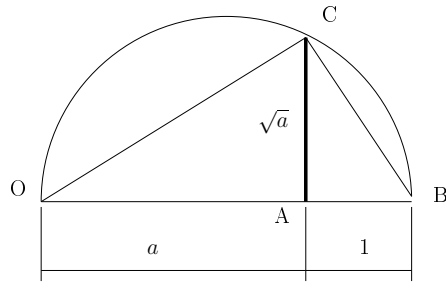

Figure 17: Construction of a/b.

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Figure 18: Construction of the square root of a.

similar, and we have for $x = AC$,

$$\frac{a}{x} = \frac{x}{1}, \qquad x^2 = a, \qquad x = \sqrt{a}$$

6 Construction of Triangles

It is well known that given n points in the Euclidian bidimensional space, a set of $2n - 3$ independent relations defined on the points fixes their relative positions. Hence, a triangle can be defined by giving three relations between any combination of three elements taken from, for example, distance between two vertices, angles between two sides, and distances between a vertex and a side.

Let d stand for the distance between two vertices of a triangle, it is, a sidelength, a stand for the angle between two sides, and h stand for the height between a vertex and the opposite side. Taking into account what has been already said, a triangle can be defined by giving one of the following nine different sets of relations: *ddd*, *dda*, *daa*, *dah*, *ddh*, *dhh*, *aah*, *ahh*, and *hhh*. Becasue in Euclidian geometry the three angles of any triangle sum up to $180°$, the *aaa* case either does not define a triangle or defines many infinite triangles holding the relations.

Let us show how the triangle defined in each of the first eigth cases can be built directly using just ruler and compass.

6.1 ddd

Draw an arbitrary line l and on it mark off a segment AB of length d_1 as depicted in Figure 19. With center on A and on B draw arcs of circle of radius d_3 and
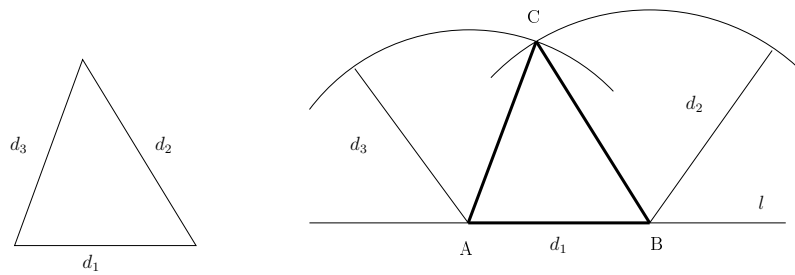
Figure 19: Triangle defined by *ddd*.

d_2 respectively. Point C where the two arcs meet completes the triangle.

6.2   dda

Now there are two different possible constructions depending on whether the given angle a is or is not defined by the two sides constrained in length.

Let us assume that angle a is the angle defined by the two triangle sides constrained in length, d_1 and d_2. Figure 20. Draw an arbitrary line l and on it mark off a segment AB of length d_1. Through point A draw a line r forming an angle a with the first line l. Now on line r and starting from point A mark off a point C such that $AC = d_2$.

Let us now assume that angle a is the angle defined by one side constrained in length and the free side, Figure 21. First draw an arbitrary line l and on it mark off a segment AB of length d_1. With center on point A draw an arc of circle γ of radius d_2. Now draw a line r through point B forming an angle a with the first line l. Point C where line r and arc γ meet defines the third vertex.
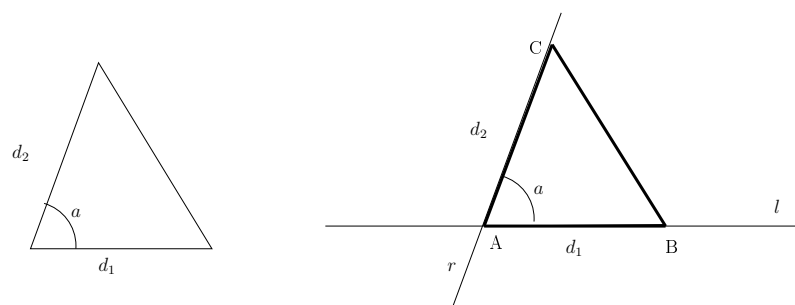

Figure 20: Triangles defined by *dda*.

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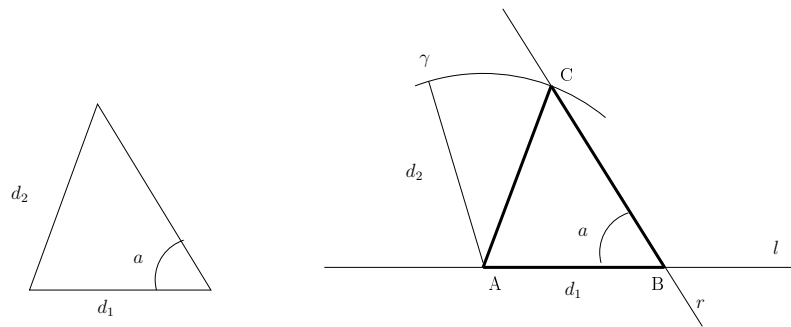
Figure 21: Triangles defined by *dda*.

6.3 daa

Since in Euclidian geometry the three angles of a triangle sum up 180°, we can consider that the only situation we need to solve corresponds to the two angles being adjacent to the side constrained in length. Figure 22.

Draw an arbitrary line l and on it mark off a segment AB of length d. Draw a line r through point A forming an angle a_1 with the first line l. Draw a line s through point B forming an angle a_2 with the first line l. Lines r and s intersect at point C.

6.4 dah

Depending on the adjacencies of the constrained elements, four different cases can arise. Let us consider first the case where the given data is the height corresponding to the side constrained in length and the angle adjacent to the

Figure 22: Triangle defined by *daa*.

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Figure 23: Triangles defined by *dah*.
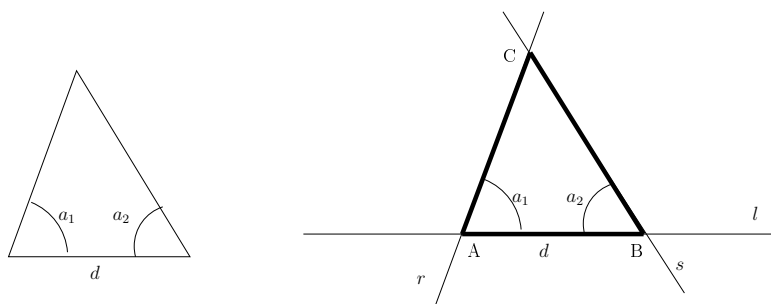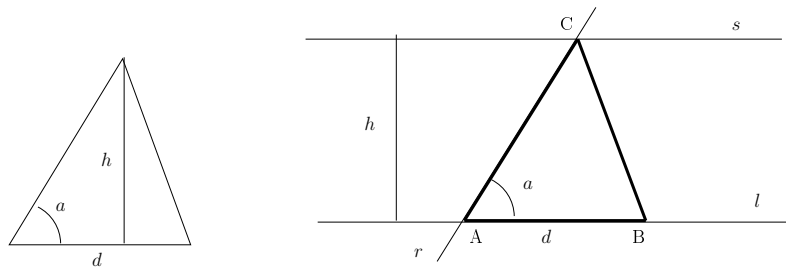
constrained side. Figure 23. Draw an arbitrary line l and on it mark off a segment AB of length d. Draw a line r through point A forming an angle a with the first line l. Now draw a line s parallel to l and at a distance h. The intersection of lines r and s is the third vertex C.

The given data in the second case is the altitud corresponding to the side constrained in length and the angle opposite to the constrained side. This construction is based upon two well known theorems of elementary geometry.

Theorem 6.1 Let γ be a circle with center O, and let P, Q and R be three points on γ. If P and Q are diametrically opposite, then $\angle PRQ$ is a right angle. Figure 24a.

Theorem 6.2 If two angles inscribed in a circle subtend the same arc, then they are congruent. Figure 24b.

The procedure is as follows. See Figure 25. Draw an arbitrary line l and on it mark off a segment $AB = d$. The center of the subtended arc should be equidistant from A and B; so, drop r, the bisector of segment AB. Next, with origin on A draw the angle CAB equal to the given angle a. Now drop the
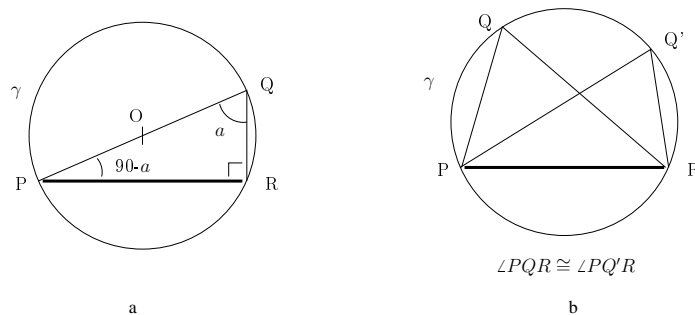

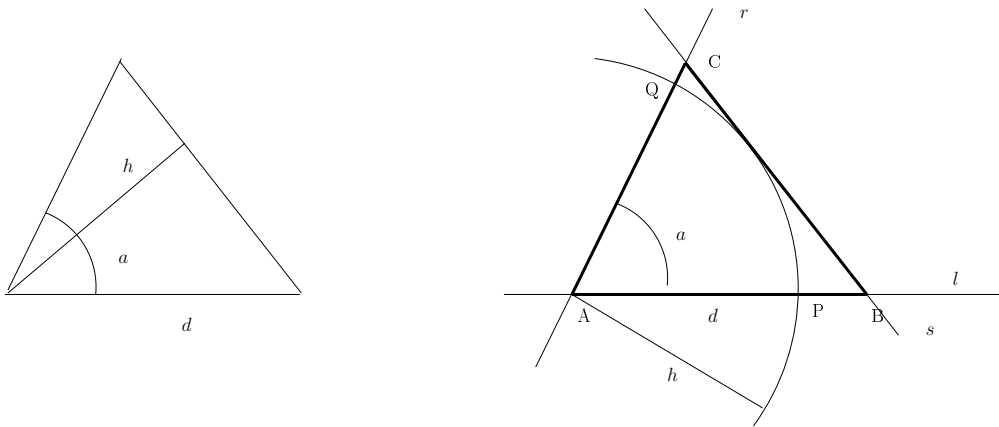Figure 24: The angles inscribed on an arc subtended by a segment are congruent.

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Figure 25: Triangles defined by *dah*.

perpendicular s to the segment AC through A. Since the angle between s and segment AB is equal to 90-a, line s is the limit line of the constant angles subtended by AB such that the angle and the center of the arc are on the same side of AB. See Figure 24a. Hence line s defines a diameter of the subtended arc the center of which is the intersection of lines s and r. Draw the circle γ with center on O and radius equal to OA. Draw the line v parallel to line l at a distance h. The intersection of line v and circle γ gives two points P_1 and P_2. Each of these points along with A and B defines a triangle holding the given constraints.

Another case corresponds to the situation where the height corresponding to a side not constrained in length, and the angle defined by the side constrained in length and the free side are given. See Figure 26. On an arbitrary line l mark off a segment AB with length equal to that of the given side length d. Then draw a second line r through point A and at an angle a with l. With center on point A and radius h draw a circle which intersects l and r at P and Q respectively. Now draw the tangent s to the circle through point B. Lines r and s meet at C, the third point of the triangle.

Finally, we can find the case depicted in Figure 27 where the data is the height corresponding to a side not constrained in length, and an angle defined by the side constrained in length and the side constrained in height. In this case two different situations can arise. If the relation $h/d = \sin a$ holds, the number of independent relations in the set *dah* is only two and there are many infinite triangles holding them. On the other hand, if the relation above does not hold, then the set *dah* does not define a triangle.

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Figure 26: Triangles defined by *dah*.

6.5   ddh

There are two diferent cases. Let us consider first the triangle in Figure 28 where the given height corresponds to a side constrained in length. Draw an arbitrary line l and on it mark off a segment AB with length d_1 equal to the triangle side constrained in height. Now draw a line r parallel to l at a distance h from l. With center on A and radius equal to d_2, draw an arc of circle which intersects r in C and C'. ABC and ABC' are the two triangles solution.

Assume that the set of constraints is the one shown in Figure 29. Now the height correponds to the side not constrained in length. Start the construction by drawing an arbitrary line l and a parallel r at a distance h from l. Now with center on an arbitrary point A on r and radius d_1 and d_2, draw two arcs of circle which intersect r in B, B' and C, C' respectively. ABC and ABC' are triangles that hold the required constraints. Triangles $AB'C'$ and $AB'C$ are the
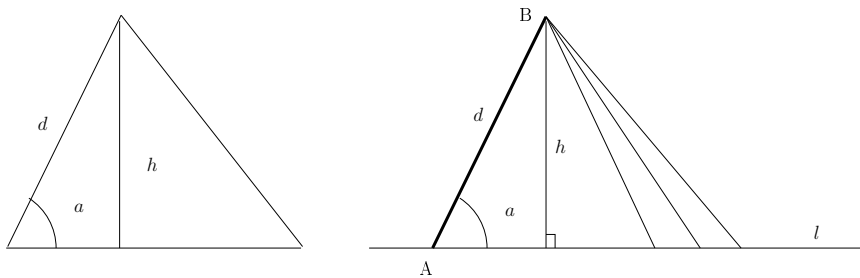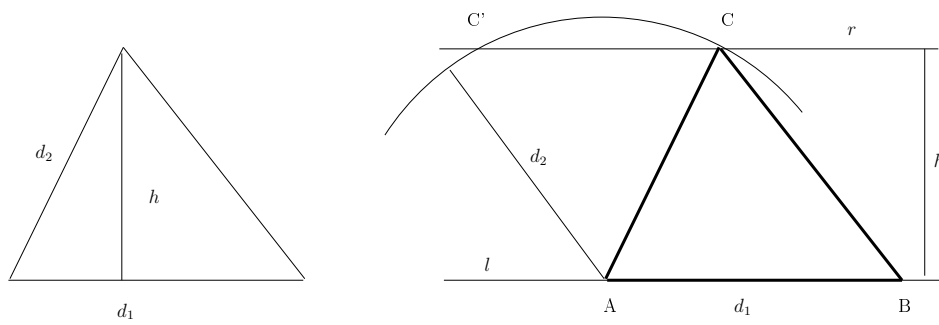

Figure 27: Triangles defined by *dah*.

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Figure 28: Triangle defined by *ddh*.

symmetric solution.

6.6 dhh

Again two different cases should be solved. Assume that the side constrained in length is also constrained in height. See Figure 30. Draw a line l and mark off on it a segment AB with length equal to the constrained side length, d. Draw a line r parallel to l and at a distance equal to the height h_1 which constrains the side constrained in length. With center on A and radius the second given heigth, h_2, draw a circle. Now build the tangent to this circle through point B. Point C where this tangent and line r intersect is the third vertex of the triangle.

In the second case the side constrained in length is not constrained in height, as shown in Figure 31. On an arbitrary line l mark off the segment AB with
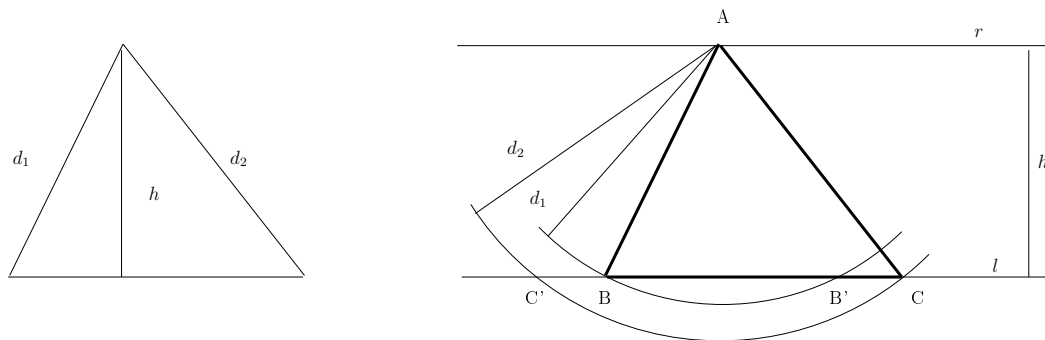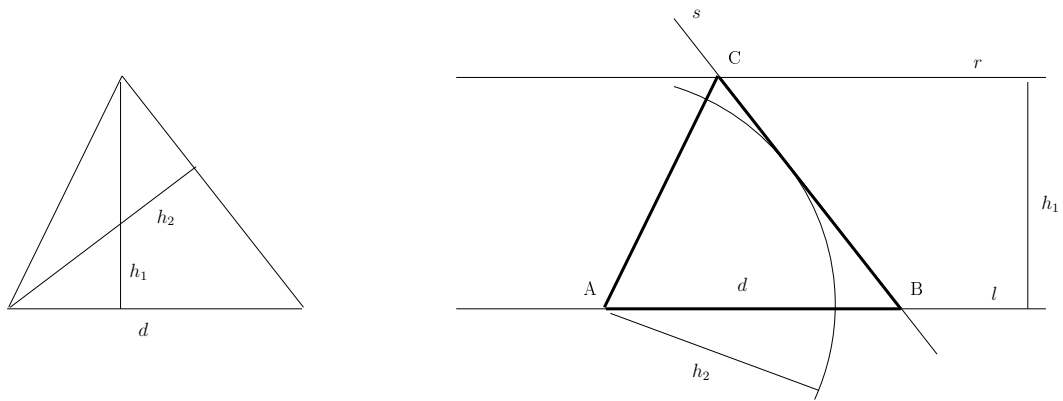

Figure 29: Triangle defined by *ddh*.

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Figure 30: Triangle defined by *dhh*.

length equal to the given side. With center on A and radius equal to the height h_1 draw an arc; the side constrained in height with respect vertex A should be on the tangent to this arc through B. With center on B and radius equal to the height h_2 draw an arc; the side constrained in height with respect vertex B should be on the tangent to this arc through A. The intersection C of the tangents gives the third vertex.

6.7   aah

Since in Euclidian geometry the three angles of a triangle sum up 180°, we only need to solve one case. Let us assume that one of the given angles corresponds
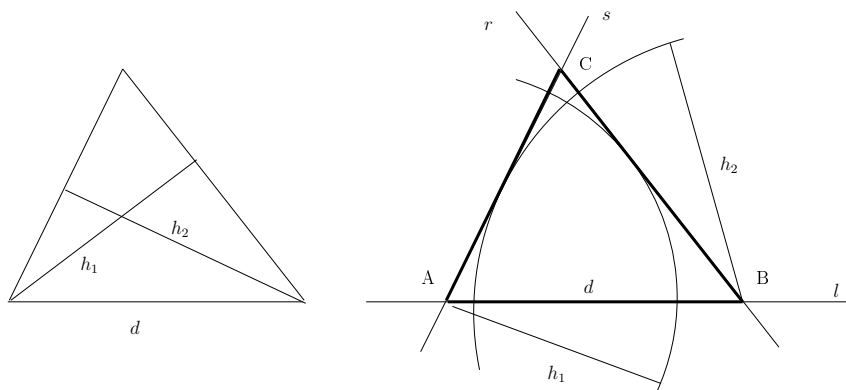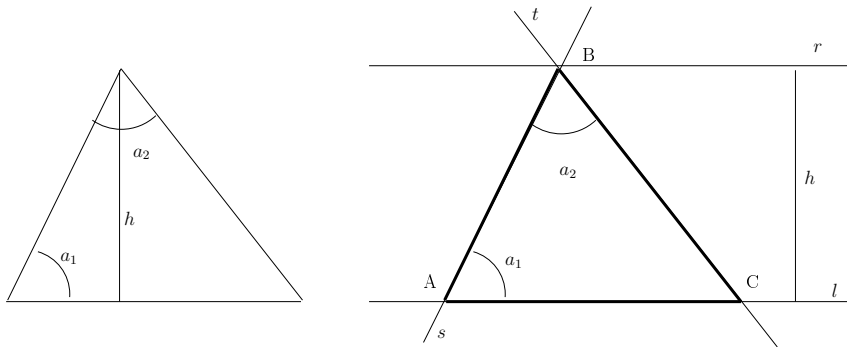

Figure 31: Triangle defined by *dhh*.

Figure 32: Triangle defined by *aah*.

to the vertex constrained in height. See Figure 32.

Draw two lines l and r parallel and at a distance equal to the given height h. Through an arbitrary point A on l and forming an angle a_1 with l draw a line s which meets r at B. Now draw a line t through B forming an angle a_2 with s. Point C where line t intersects l completes the triangle.

6.8   ahh

Here there are again two different situations depending on the constraints adjacency. Figure 33 shows the case where the given heights constrain those sides which define the given angle. Start by drawing two lines, l and r, forming the given angle a. Then draw a line s parallel to l and at a distance h_1 apart from it. Next draw a line t parallel to r and at a distance h_2. The desired triangle ABC is made up from the intersections of the pairs of lines l and r, r and s, and t and l.

Figure 34 shows the case where the given angle and the side opposite are constrained in height. As in the previous case, start by drawing two lines, l and r, forming the given angle a, and let A be the intersecting point. Next draw a line s parallel to r at a distance equal to the second given height, h_2. Lines s and l intersect at point B. Now, the side opposite to the given angle should be tangent to the circle γ with center on A and radius equal to the height which constraints the angle and the side opposite to it. Hence, point C, intersection of line t tangent to γ through B with line r, completes the triangle.
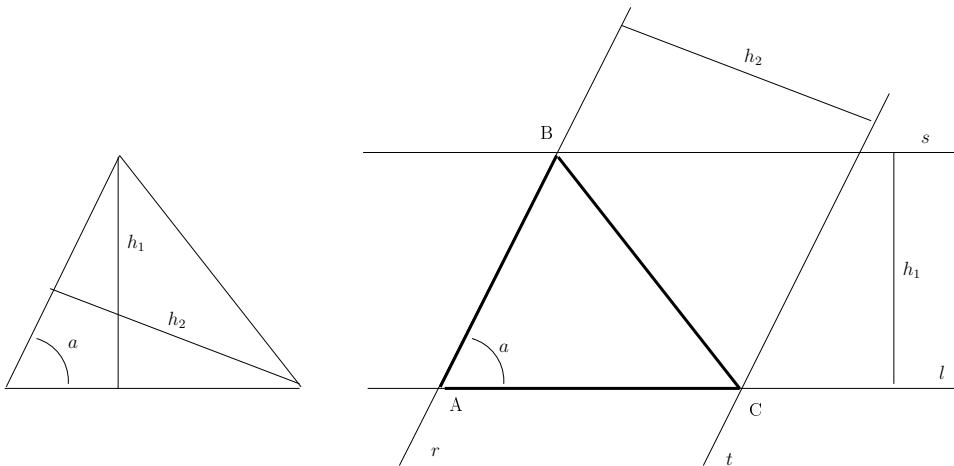
Figure 33: Triangle defined by *ahh*.

6.9 hhh

We shall give a construction based on the following theorem illustrated in Figure 35.

Theorem 6.3 Let O be a point which does not lie on circle γ. If two lines through O intersect γ in pairs of points (P_1, P_2), and (Q_1, Q_2), respectively, then $OP_1 \times OP_2 = OQ_1 \times OQ_2$.
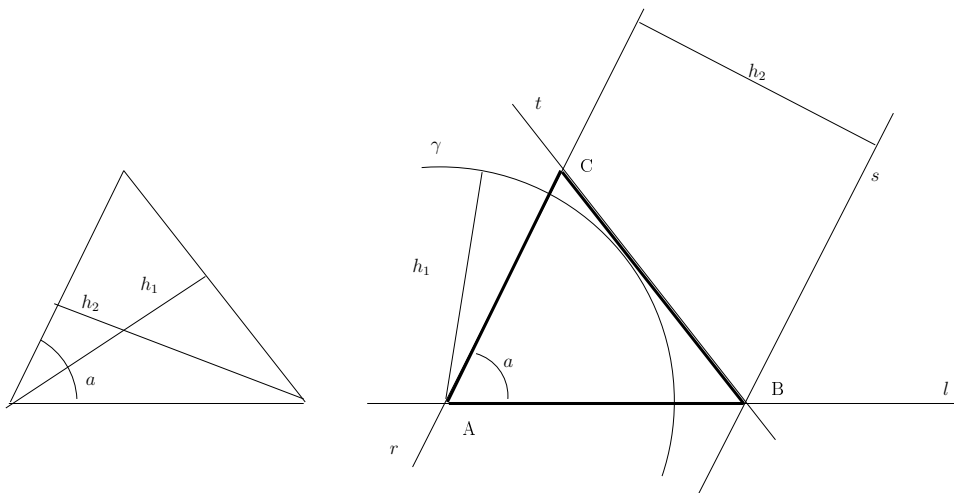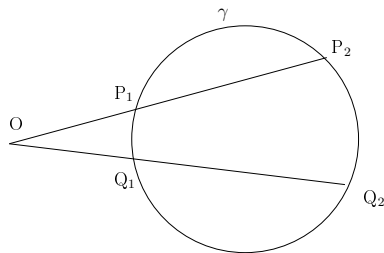

Figure 34: Triangle defined by *ahh*.

Figure 35: Power of a point with respect to a circle.

The *power* of O with respect to γ is this product when O is outside γ and minus this product when O is inside γ.

Let the given data be as depicted in Figure 36. Start by drawing three segments, sharing a common origin O, lenghts given by $OP_1 = h_a, OP_2 = h_b$ and $OP_3 = h_c$, and each of them laying on a different straightline (Figure 37a). Draw the circle γ defined by points P_1, P_2 and P_3. Now extend the straightlines supporting the segments OP_1, OP_2 and OP_3. These lines meet γ at Q_1, Q_2 and Q_3 respectively. Taking into account the relation between the area S, the altitudes and the length sides of a triangle, $2S = ah_a = bh_b = ch_c$, and the definition of power given above, we have that $h_b OQ_1 = h_c OQ_2 = h_a OQ_3 = kS$, that is, OQ_1, OQ_2 and OQ_3, are the sides of a triangle similar to the one defined by the given data. Construct now this triangle $A'B'C'$ as indicated in Figure 37b. Let us take $B = B'$ as the first vertex of the desired triangle. Now drop a perpendicular to $B'C'$ through B' and mark off the segment $B'D = h_a$. Then draw through D the line l parallel to $B'C'$ giving the second vertex A. Finally draw through A a parallel to $A'C'$ which intersects $B'C'$ at C.
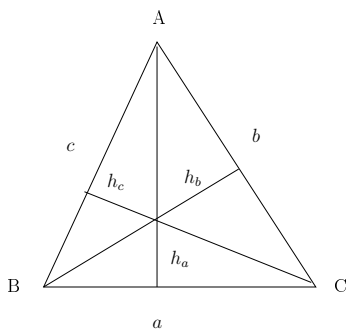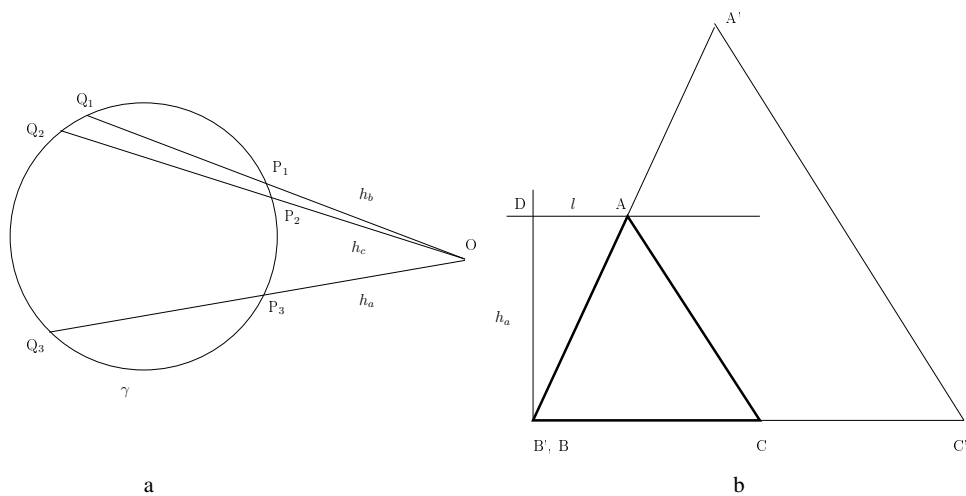

Figure 36: Triangle defined by hhh. Given data.

Figure 37: Triangle defined by *hhh*. a) Computing the power. b) Building the triangle.

7   Summary

Geometric constraints solving is emerging as a powerful tool upon which to build the new generation of Computer Aided Design systems. However, today's constraint-based systems exhibits a number of limitations.

When designing a geometric constraint solver following the constructive approach a crucial step is to select the appropiate rules to be implemented because the solver domain, that is, the range of shapes the constraint solver understands, will depend on the selected set of rules. Althoug it is not possible to implement an infinity of rules in order to solve every possible set of constraints, it is apparent that any geometric constraint solver should be able to solve the simplest constraint-based design: the triangle.

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