

THE PERIODOGRAM ENVELOPE IN PARAMETRIC AND
NON-PARAMETRIC SPECTRAL ESTIMATION

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ABSTRACT

This work reports the role of periodogram envelope as a non-parametric approach for spectral estimation and its interest in parametric spectral estimation using variational approaches.

The spectral envelope, or its prior estimate the periodogram envelope, represents a very good candidate to start any approach for spectral estimation due to its smooth character, its robustness against flat noise spectra and its statistical stability.

Setting envelope constraints in variational spectral estimation the resulting estimate gains in statistical stability yet preserving the high resolution of the best reported methods.

Together with the non-parametric approaches, the case of line spectra is briefly described to evidence the potential of envelope associated matrix in line spectrum. This allows the designer to convert SUD and MUSIC methods to the new starting function.

1. INTRODUCTION

It is assumed that given a signal $x(n)$, its envelope and its instantaneous frequency provides a better description of the involved physical phenomena.

This work deals with the role of the spectral envelope $E_p(\omega)$ defined as the envelope of the power spectral density $S_p(\omega)$ of a given random process $\{x\}$. To be more specific it will be the square of the envelope $E_p^2(\omega)$ the function under study; and, the closeness it can provide to the accurate estimation of the actual power spectral density. The point of special interest around will be up to what degree the periodogram envelope E_p^2 , defined as the envelope of the magnitude of DFT $|x(n)|$, could be used as a starting point to obtain the distribution of the signal power in the frequency domain. The reader will notice that due to the non-linear character of the spectral estimation problem any candidate added to the basic a priori information (i.e. the periodogram) will have a considerable potential to become useful to our final goal which is the accurate estimation of the power spectral density.

The periodogram envelope is a fourth order function and, in this sense this work can be encompassed as how to use high order moments of the signal in spectral estimation. However, the concept of envelope results by itself self explanatory in many senses, and it seems to be better to concentrate efforts in this special fourth order function

without a general background or framework for other non-linear functions.

2. THE SPECTRAL ENVELOPE

Given the signal periodogram $P(\omega)$, we define the spectral envelope as the magnitude of the function $A_p(\omega)$, named as analytical periodogram because its inverse Fourier transform is a causal function.

$$A_p(\omega) = P(\omega) + j H_p(\omega) \quad (1.a)$$

$$F^{-1}[A_p(\omega)] = 0; \quad n < 0 \quad (1.b)$$

The inverse Fourier transform is as denoted in (2).

$$F^{-1}[A_p(\omega)] = \begin{matrix} 0 & ; & n < 0 \\ r(0)/2 & ; & n = 0 \\ r(n) & ; & n > 0 \end{matrix} \quad (2)$$

Note that the analytic periodogram is in correspondence with the analytic signal and its inverse Fourier transform obeys to the concept of causal autocorrelation function introduced some years ago by J. Cadzow.

Working over, or estimating, the analytic spectrum $A_s(\omega)$ represents somehow the same problem; the spectral estimate $S(\omega)$ can be derived from the analytic estimate just taking its real part.

$$S(\omega) = R_e[A_s(\omega)] \quad (3)$$

The difference resides in the powerful mathematical framework behind analytic functions which appears in the analytic periodogram and not in the original function.

Once again the procedure of J.P. Burg should be mentioned in one of the most unknown aspect it has. Maximum entropy or AR modelling of data time series was reported in order to smooth the data periodogram; in fact this was the main goal when, with a few parameters, the authors (Parzen, Burg, etc.) represented a data record power distribution. The high performance of MEM in short data records was an additional result, but as concerns resolution for long data records MEM is not in competition with the periodogram.

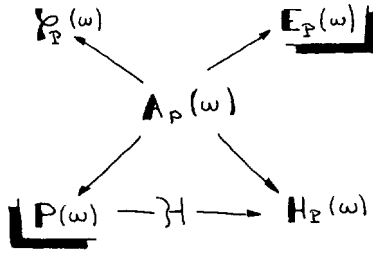
It is easy to show that the periodogram envelope is smoother than the periodogram. Using the magnitude/phase representation of the analytic spectrum (E_p/ϕ_p) and (1) we can arrive to (4).

$$P(\omega) = E_p(\omega) \cdot \cos\phi_p(\omega) \quad (4)$$

Note in (4) that the periodogram has the variations of

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the phase which introduces true peaky character in $P(\omega)$ when compared with the periodogram envelope.



Also it is interesting to check that $\phi(\omega)$ is the minimum phase associated to the magnitude (i.e. the spectral envelope $E_P(\omega)$). This reveals that given $E_P(\omega)$ we can obtain $P(\omega)$ and viceversa. To prove this last sentence, note that the real part of $A_P(s)$ is always positive and, as a consequence, its inverse shares the same property. Because $A_P(s)$ and $A_P^{-1}(s)$ have positive real parts we can conclude that both zeros and poles are inside the unit circle and, in conclusion, $A_P(s)$ is a minimum phase function.

In summary, at the expense of further comments, the spectral envelope does not represent any loss of spectral information. The only differences to be expected between using the periodogram or its envelope as starting information in a spectral estimation procedure is the non-linear character involved in such procedures.

3. NON-PARAMETRIC SPECTRAL ENVELOPE ESTIMATION

The periodogram envelope can be derived from the analytic spectrum taking into account that its Fourier inverse is just the causal autocorrelation function,

$$A_P(\omega) = \underline{S}^T \begin{bmatrix} r_0/2 & r_1 & \dots & r_Q \\ 0 & r_0/2 & \dots & r_{Q-1} \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & r_0/2 \end{bmatrix} \underline{S} = \underline{S}^T \underline{B}_c \underline{S} \quad (5)$$

where \underline{S} is the steering vector defined as $S_i = \exp(ji\omega)$ and r_0, r_1, \dots, r_Q are the first Q estimated autocorrelation samples.

The spectral envelope is derived taking the magnitude of the analytic spectrum.

The main advantage, and this is the difference with the periodogram, is that the spectral envelope is always positive and does not depend on the data or lag window used to estimate r_Q in this respect.

In Figure 1 the reader can see that, in the problem of line spectra, the periodogram envelope provides a smoother behavior than the periodogram and, what is going to be of capital importance in next section, more statistical stability than the periodogram.

Other formulation of interest to derive the periodogram envelope arises from considering the expressions of the periodogram and its Hilbert transform as quadratic forms.

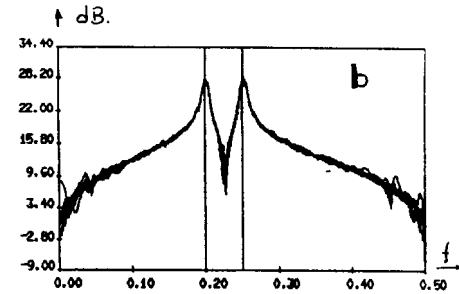
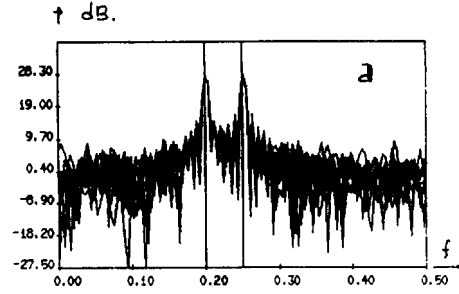


Figure 1. (a) Ten periodograms and in (b) the corresponding envelopes. Data records of 128 samples with two sinusoids at normalized frequencies of 0.2 and 0.25 in white noise at SNR equal to 10dB.

$$P(\omega) = \underline{S}^T \underline{B} \underline{S} \quad (6.a)$$

$$jH_P(\omega) = \underline{S}^T \underline{B}_a \underline{S} \quad (6.b)$$

where $\underline{B} + \underline{B}_a$ is the upper triangular matrix of (5). Thus, from (6) we can derive the magnitude of the analytic spectrum as it is shown in (7), where vector \underline{S} includes a $-\pi/2$ phase in every component.

$$E_P^2(\omega) = (\underline{S}^T \underline{B} \underline{S})^2 + \left(\frac{1}{j} \underline{S}^T \underline{B}_a \underline{S}\right)^2 = (\underline{S}^T \underline{B} \underline{S})^2 + (\underline{S}^T \underline{B}_a \underline{S})^2 \quad (7)$$

Now, using the ideas reported by Lagunas [5] in the topic of Rayleigh estimates for spectral estimation, a new spectral envelope estimate can be derived from (8).

$$E_S^2(\omega) = \underline{S}^T \underline{\Phi} \underline{S} = (\underline{S}^T \underline{B}^2 \underline{S}) + (\underline{S}^T \underline{B}_a^2 \underline{S}) \quad (8)$$

Where matrix $\underline{\Phi}$ can be easily constructed from \underline{B} and \underline{B}_a or directly from the causal autocorrelation function as shown below:

- Given $r(\cdot)$ = causal autocorrelation function
- Compute the autocorrelation of $r(\cdot)$ and denote as $\phi(\cdot)$ the corresponding values
- Form $\underline{\Phi}$ in a Toeplitz fashion with the first Q values of the sequence $\phi(\cdot)$

Both E_P and E_S can be used as an alternative to the classical periodogram with the following advantages: (a) $E(\cdot)$ is always positive; (b) $E(\cdot)$ has not the peaky or noisy

character of the periodogram; (c) $E(\cdot)$ has very good statistical stability and results very robust for practical applications.

4. PARAMETRIC SPECTRAL ENVELOPE ESTIMATION

From the methods reported currently, the only one which provides a variational framework was due to Lagunas et al. [1] proving that MEM as objective with correlation and cepstrum constraints results in an ARMA model for the process under analysis. The algorithm was solved by Musicus [2] which converted the cepstrum constraints in first order information and after using the ideas of Mullis and Roberts [4] for mixed first and second order information the ARMA parameters were obtained.

ARMA models for the analytic spectrum are derived from the z-transform of the causal autocorrelation $r_a(q)$.

$$A_S(z) = \frac{r_0}{2} + \sum_{q=1}^{\infty} r_q \cdot z^{-q} \quad (9)$$

Naming $N(z)$ and $B(z)$ the zero-pole function of the analytic spectrum the expression for the spectral envelope and the spectral density follows :

$$A_S(z) = \frac{N(z)}{B(z)} ; S(z) = \frac{N(z)}{B(z)} + \frac{N(1/z)}{B(1/z)} \quad (10.a)$$

$$E(z) E(1/z) = N(z)N(1/z)/B(z)B(1/z) \quad (10.b)$$

Note that envelope constraints are a nice candidate for variational spectral estimation because differs from the cepstrum in many important aspects. The envelope obeys to a rational model with the same poles that the actual spectral estimate. Due to the previous comment there is no bandwidth extension as in cepstrum. In using a small set of constraints the statistical stability of the selected ones will provide this behavior to the associated spectral estimate yet preserving the involved high resolution of ARMA estimates from a variational approach. The envelope or the squared envelope is a fourth order function that, used together with the classic second order a.c.f., represents a steep ahead in spectral estimation techniques.

Using correlation and envelope constraints in variational spectral estimation is done by finding an extremum for $\phi(0)$ constrained with Q lags of a.c.f. and Q lags of function $\phi(\cdot)$ excluding the zero lag. Thus, using that the envelope is the magnitude of the analytic spectrum the problem could be stated as it is shown in (11).

Objective :

$$\int_{-\pi}^{\pi} |A_S(\omega)|^2 d\omega | \text{EXTREMUM} = \phi^0(0) \quad (11.a)$$

Correlation constraints (causal a.c.f.)

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} A_S(\omega) \cdot \exp(jn\omega) d\omega = r^c(n) ; n=0, Q \quad (11.b)$$

Envelope constraints

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |A_S(\omega)|^2 \exp(jm\omega) d\omega = \phi(m) ; m = 1, Q \quad (11.c)$$

Where

$$\phi(m) = \frac{1}{2\pi} \int_{-\pi}^{\pi} E_P^2(\omega) \exp(jm\omega) d\omega \quad (12)$$

Using the Lagrange multipliers and setting the derivatives with respect $A_S(\cdot)$ equal to zero an ARMA model is obtained of the analytic spectrum [3].

$$A_S(z) = \frac{N(z)}{B(z)}$$

$$N(z) = \sum c(j) z^{-j} \quad (13.a)$$

$$B(z) = \sum b(j) z^j \quad (13.b)$$

The algorithm to find vectors of coefficients \underline{c} and \underline{b} starts with the autocorrelation constraints.

$$\underline{c} = \underline{B}_C \underline{b} \quad (14)$$

where \underline{B}_C is the upper diagonal matrix of (5).

Then, the envelope constraints evolve to (15), where the asterisk in $\phi(\cdot)$ denotes that the zero value is unknown because it is the objective.

$$\frac{N(z)}{B(z)} \frac{N(1/z)}{B(1/z)} = \phi^*(z) = \sum_{-\infty}^{\infty} \phi(m) z^{-m} \quad (15)$$

$N(1/z)/B(1/z)$ can be translated to the sample domain using the autocorrelation constraints.

$$N(z) \frac{N(1/z)}{B(1/z)} = \phi^*(z) \cdot B(z) \quad (16.a)$$

$$\underline{B}_C^T \underline{c} = \underline{\phi}^* \underline{b} \quad (16.b)$$

Finally, introducing (14) in (16.b) the eigenvalue problem results.

$$(\underline{B}_C^T \underline{B}_C) \underline{b} = \underline{\phi}^* \underline{b} \quad (17)$$

It is worthwhile to note that $\underline{\phi}^*$ includes the value which is the objective $\phi_0(0)$. Naming the difference between $\phi(0)$ (obtained from the periodogram envelope) and $\phi^0(0)$ as Δ , it is easy to prove that :

$$\underline{\phi}^* = \underline{\phi} - \Delta \underline{1} \quad (18)$$

Thus, solving (19) for the minimum eigenvalue will result in the desired solution.

$$(\underline{\phi}^* - \underline{B}_C^T \underline{B}_C) \underline{b} = \lambda \underline{b} \quad (19)$$

see [4] for details in solving (17) or (19).

Figure 2 shows the comparison between ARMA spectral estimation using correlation and cepstrum constraints (a) and correlation and envelope constraints (b). The superiority of the herein reported approach becomes evident.

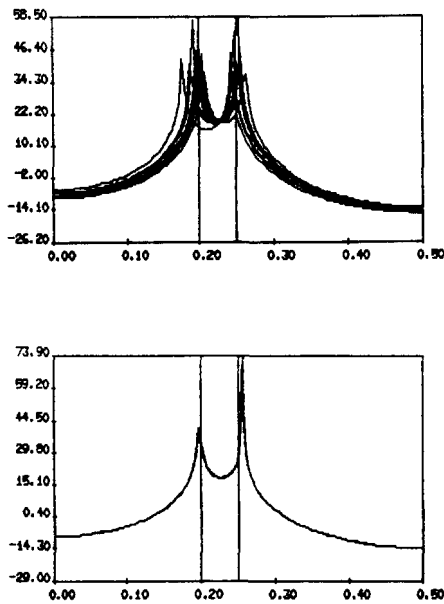


Fig. 2. Cepstrum and correlation constraints (a); and, correlation envelope constraints (b). The signal records are the same as in Figure 1 with 64 samples.

5. LINE SPECTRA FROM THE ENVELOPE MATRIX

This section is devoted to complete the role of envelope in spectral estimation, and more concretely in the problem of line spectra and SVD or MUSIC, by showing the superiority of matrix $\hat{\Phi}$ over the classic matrix \hat{R} as starting function in these procedures.

As an example of use, in Figure 3 the inverse Fourier transform of the eigenvector associated to the minimum eigenvalue of \hat{R} and $\hat{\Phi}$ is represented for 10 records of the same signal that in figure 2. Again, and perhaps more dramatic than before, the superiority of the envelope matrix results clear in front of the periodogram matrix.

$$\begin{array}{llll} E_p^2(\omega) & \text{Inv. F. Transform.} & \phi(\cdot) & \hat{\Phi} \\ P(\omega) & \text{Inv. F. Transform.} & r(\cdot) & \hat{R} \end{array} \quad (20)$$

The reasons for this performance surely are based, without further work, in the statistical stability of the periodogram envelope. As intuitive reasoning it can be said that the Hilbert transform contribution to the envelope in (21) support an enhancement of the signal zones at the expense of reducing the flat noise spectra. Also note that squaring a function tends to reduce and provide better separation between the noise and the signal subspace.

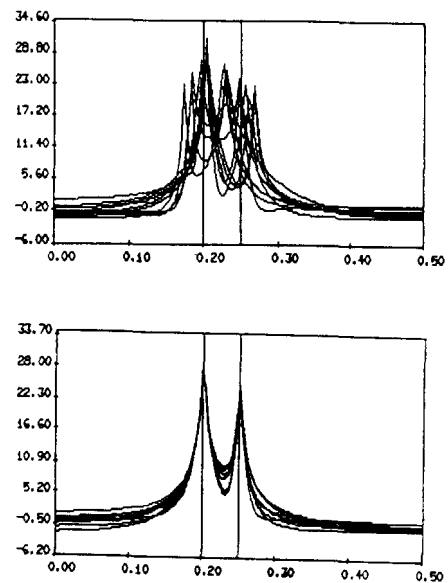


Fig. 3. Inverse of the magnitude of the minimum eigenvalue eigenvector (a) for the periodogram associated matrix \hat{R} and for the envelope associated matrix $\hat{\Phi}$ in (b). The signal records (10 records) are the same that in figure 2.

6. REFERENCES

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