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ON AUTOMORPHISMS OF THE AFFINE CREMONA GROUP

by Hanspeter KRAFT & Immanuel STAMPFLI (*)

Abstract. — We show that every automorphism of the group $\mathcal{G}_n := \mathrm{Aut}(\mathbb{A}^n)$ of polynomial automorphisms of complex affine n-space $\mathbb{A}^n = \mathbb{C}^n$ is inner up to field automorphisms when restricted to the subgroup $T\mathcal{G}_n$ of tame automorphisms. This generalizes a result of Julie Deserti who proved this in dimension $n = 2$ where all automorphisms are tame: $T\mathcal{G}_2 = \mathcal{G}_2$. The methods are different, based on arguments from algebraic group actions.

Résumé. — Nous montrons que tous les automorphismes du groupe des automorphismes polynomiaux de l'espace affine $\mathbb{C}^n$ sont des automorphismes intérieurs modulo des automorphismes du corps $\mathbb{C}$, si nous nous restreignons au sous-groupe des automorphismes modérés. Ceci généralise un résultat de Julie Déserti traitant le cas de la dimension $n = 2$. Dans ce cas, tous les automorphismes polynomiaux sont modérés. Nos méthodes sont différentes de celles de Julie Déserti et sont basées sur des arguments d'actions de groupes algébriques.

1. Notation

Let $\mathcal{G}_n := \mathrm{Aut}(\mathbb{A}^n)$ denote the group of polynomial automorphisms of complex affine n-space $\mathbb{A}^n = \mathbb{C}^n$. For an automorphism $\mathbf{g}$ we use the notation $\mathbf{g} = (g_1, g_2, \ldots, g_n)$ if

$$\mathbf{g}(a) = (g_1(a_1, \ldots, a_n), \ldots, g_n(a_1, \ldots, a_n)) \quad \text{for } a = (a_1, \ldots, a_n) \in \mathbb{A}^n$$

where $g_1, \ldots, g_n \in \mathbb{C}[x_1, \ldots, x_n]$. Moreover, we define the degree of $\mathbf{g}$ by $\deg \mathbf{g} := \max(\deg g_1, \ldots, \deg g_n)$. The product of two automorphisms is denoted by $\mathbf{f} \circ \mathbf{g}$.

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The automorphisms of the form $(g_1, \ldots, g_n)$ where $g_i = g_i(x_i, \ldots, x_n)$ depends only on $x_i, \ldots, x_n$, form the *Jonquière subgroup* $\mathcal{J}_n \subset \mathcal{G}_n$. Moreover, we have the inclusions $D_n \subset \mathrm{GL}_n \subset \mathrm{Aff}_n \subset \mathcal{G}_n$ where D_n is the group of *diagonal automorphisms* $(a_1x_1, \ldots, a_nx_n)$ and Aff_n the group of *affine transformations* $\mathbf{g} = (g_1, \ldots, g_n)$ where all g_i have degree 1. The group Aff_n is the semidirect product of GL_n with the commutative unipotent subgroup $\mathcal{T}_n$ of translations. The subgroup $T\mathcal{G}_n \subset \mathcal{G}_n$ generated by $\mathcal{J}_n$ and Aff_n is called the group of *tame* automorphisms.

Main Theorem. — *Let θ be an automorphism of $\mathcal{G}_n$. Then there is an element $\mathbf{g} \in \mathcal{G}_n$ and a field automorphism $\tau\colon \mathbb{C} \to \mathbb{C}$ such that*

$$\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1}) \text{ for all tame automorphisms } \mathbf{f} \in T\mathcal{G}_n.$$

After some preparation in the following sections the proof is given in Section 7. For $n = 2$ where $T\mathcal{G}_2 = \mathcal{G}_2$ this result is due to Julie Deserti [3]. In fact, she proved this for any uncountable field K of characteristic zero. Our methods work for any algebraically closed field of characteristic zero.

2. Ind-group structure and locally finite automorphisms

The group $\mathcal{G}_n$ has the structure of an ind-group given by $\mathcal{G}_n = \bigcup_{d\geqslant 1} (\mathcal{G}_n)_d$ where $(\mathcal{G}_n)_d$ are the automorphisms of degree $\leqslant d$ (see [8]). Each $(\mathcal{G}_n)_d$ is an affine variety and $(\mathcal{G}_n)_d \subset (\mathcal{G}_n)_{d+1}$ is closed for all d. This defines a topology on $\mathcal{G}_n$ where a subset $X \subset \mathcal{G}_n$ is closed (resp. open) if and only if $X \cap (\mathcal{G}_n)_d$ is closed (resp. open) in $(\mathcal{G}_n)_d$ for all d. All subgroups mentioned above are closed subgroups.

In addition, multiplication $\mathcal{G}_n \times \mathcal{G}_n \to \mathcal{G}_n$ and inverse : $\mathcal{G}_n \to \mathcal{G}_n$ are morphisms of ind-varieties where for the latter one has to use the fact that $\deg \mathbf{f}^{-1} \leqslant (\deg \mathbf{f})^{n-1}$. This seems to be a classical result for birational maps of $\mathbb{P}^n$ based on Bézout's Theorem (see [1, Corollary (1.4) and Theorem (1.5)]). It follows that for every subgroup $G \subset \mathcal{G}_n$ the closure $\bar{G}$ in $\mathcal{G}_n$ is also a subgroup.

A closed subgroup G contained in some $(\mathcal{G}_n)_d$ is called an *algebraic subgroup*. In fact, such a G is an affine algebraic group which acts faithfully on $\mathbb{A}^n$, and for every algebraic group H acting on $\mathbb{A}^n$ the image of H in $\mathcal{G}_n$ is an algebraic subgroup.

A subset $X \subset \mathcal{G}_n$ is called *bounded constructible*, if X is a constructible subset of some $(\mathcal{G}_n)_d$.

LEMMA 2.1. — *Let $G \subset \mathcal{G}_n$ be a subgroup and let $X \subset G$ be a subset which is dense in G and bounded constructible. Then G is an algebraic subgroup, and $G = X \circ X$.*

Proof. — By assumption $G \subset \bar{X} \subset (\mathcal{G}_n)_d$ for some d and so $\bar{G} = \bar{X}$ is an algebraic subgroup. Moreover, there is a subset $U \subset X$ which is open and dense in $\bar{G}$. Then $U \circ U = \bar{G}$, and so $\bar{G} = G = X \circ X$. □

An element $\mathbf{g} \in \mathcal{G}_n$ is called *locally finite* if it induces a locally finite automorphism of the algebra $\mathbb{C}[x_1, \ldots, x_n]$ of polynomial functions on $\mathbb{A}^n$. This is equivalent to the condition that the linear span of $\{(\mathbf{g}^m)^*(f) \mid m \in \mathbb{Z}\}$ is finite dimensional for all $f \in \mathbb{C}[x_1, \ldots, x_n]$.

More generally, an action of a group G on an affine variety X is called *locally finite* if the induced action on the coordinate ring $\mathcal{O}(X)$ is locally finite, *i.e.*, for all $f \in \mathcal{O}(X)$ the linear span $\langle Gf \rangle$ is finite dimensional. It is easy to see that the image of G in $\mathrm{Aut}(X)$ is dense in an algebraic group $\bar{G}$ which acts algebraically on X. In fact, one first chooses a finite dimensional G-stable subspace $W \subset \mathcal{O}(X)$ which generates $\mathcal{O}(X)$, and then defines $\bar{G} \subset \mathrm{GL}(W)$ to be the closure of the image of G inside $\mathrm{GL}(W)$.

The next result will be used in the following section. We start again with an action of a group G on an affine variety X and assume that $x_0 \in X$ is a fixed point. Then we obtain a representation $\tau \colon G \to \mathrm{GL}(T_{x_0}X)$ on the tangent space at x_0, given by $\tau(g) := d_{x_0}g$.

LEMMA 2.2. — *Let G act faithfully on an irreducible affine variety X. Assume that $x_0 \in X$ is a fixed point and that there is a G-stable decomposition $\mathfrak{m}_{x_0} = V \oplus \mathfrak{m}_{x_0}^2$. Then the tangent representation $\tau \colon G \to \mathrm{GL}(T_{x_0}X)$ is faithful.*

Proof. — Let $g \in \ker \tau$. Then g acts trivially on V, hence on all powers V^j of V. This implies that the action of g on $\mathcal{O}(X)/\mathfrak{m}_{x_0}^k$ is trivial for all $k \geqslant 1$. Since $\bigcap_k \mathfrak{m}_{x_0}^k = \{0\}$ the claim follows. □

We remark that a G-stable decomposition $\mathfrak{m}_{x_0} = V \oplus \mathfrak{m}_{x_0}^2$ like in the lemma above always exists if G is a reductive algebraic group.

3. Tori and centralizers

For the convenience of the reader we recall two important results about fixed point sets of group actions which we will need below. A complex variety X is called *$\mathbb{Z}/p\mathbb{Z}$-acyclic* if $H_j(X, \mathbb{Z}/p\mathbb{Z}) = 0$ for $j > 0$ and $H_0(X, \mathbb{Z}/p\mathbb{Z}) = \mathbb{Z}/p\mathbb{Z}$. The first result goes back to P. A. Smith [11].

Proposition 3.1 (Corollary to Theorem 7.5 in [10]). — *Let G be a finite p-group and let X be an affine G-variety. If X is $\mathbb{Z}/p\mathbb{Z}$-acyclic, then so is X^G.*

The second result is due to Fogarty and describes the *tangent cone* $C(X^G, x)$ of the fixed point set X^G.

Proposition 3.2 (Theorem 5.2 in [4]). — *Let G be a reductive group. If X is an affine G-variety, then for each point $x \in X$ we have $C(X^G, x) = C(X, x)^G$.*

Define $\mu_k := \{\mathbf{g} \in D_n \mid \mathbf{g}^k = \mathrm{id}\}$. We have $\mu_k \simeq (\mathbb{Z}/k)^n$, and $\mu_\infty := \bigcup_k \mu_k \subset D_n$ is the subgroup of elements of finite order where $\mu_\infty \simeq (\mathbb{Q}/\mathbb{Z})^n$. The next lemma about the centralizer of μ_k is easy.

Lemma 3.3. — *For every $k > 1$ we have $\mathrm{Cent}_{\mathcal{G}_n}(\mu_k) = \mathrm{Cent}_{\mathrm{GL}_n}(\mu_k) = D_n$.*

The following result is crucial for the proof of the main theorem.

Proposition 3.4. — *Let $\mu \subset \mathcal{G}_n$ be a finite subgroup isomorphic to μ_2. Then the centralizer $\mathrm{Cent}_{\mathcal{G}_n}(\mu)$ is a diagonalizable algebraic subgroup of $\mathcal{G}_n$, i.e., isomorphic to a closed subgroup of a torus. Moreover $\dim \mathrm{Cent}_{\mathcal{G}_n}(\mu) \leqslant n$.*

Proof. — We first remark that $\mathrm{Cent}_{\mathcal{G}_n}(\mu)$ is a closed subgroup of $\mathcal{G}_n$. By Theorem 3.1 the fixed point set $F := (\mathbb{A}^n)^{\mu'}$ of every subgroup $\mu' \subset \mu$ is $\mathbb{Z}/2$-acyclic, in particular non-empty and connected. We also know that F is smooth and that $T_a F = (T_a \mathbb{A}^n)^{\mu'}$ since μ' is linearly reductive (see Theorem 3.2). If $a \in (\mathbb{A}^n)^\mu$, then the tangent representation of μ on $T_a\mathbb{A}^n$ is faithful, by Lemma 2.2 above, and so a is an isolated fixed point. Hence, $(\mathbb{A}^n)^\mu = \{a\}$.

Choose generators $\sigma_1, \ldots, \sigma_n$ of μ such that the images in $\mathrm{GL}(T_a\mathbb{A}^n)$ are reflections, *i.e.*, have a single eigenvalue -1, and set $H_i := (\mathbb{A}^n)^{\sigma_i}$. The tangent representation shows that H_i is a hypersurface, hence defined by an irreducible polynomial $f_i \in \mathbb{C}[x_1, \ldots, x_n]$. Moreover, $\sigma_i^*(f_i) = -f_i$ and $\sigma_i^*(f_j) = f_j$ for $j \neq i$. It follows that the linear subspace $V := \mathbb{C}f_1 \oplus \cdots \oplus \mathbb{C}f_n \subset \mathbb{C}[x_1, \ldots, x_n]$ is μ-stable. In addition, any $\mathbf{g} \in G := \mathrm{Cent}_{\mathcal{G}_n}(\mu)$ fixes a and stabilizes all $\mathbb{C}f_i$ and so, by the following Lemma 3.6 applied to the morphism $\varphi := (f_1, \ldots, f_n)\colon \mathbb{A}^n \to \mathbb{A}^n$, the action of G on $\mathbb{A}^n$ is locally finite. Since G is a closed subgroup of $\mathcal{G}_n$, it follows that it is an algebraic subgroup of $\mathcal{G}_n$, and its image in $\mathrm{GL}(V)$ is a closed subgroup contained in a maximal torus, hence a diagonalizable group.

Finally, $\mathfrak{m}_a = V \oplus \mathfrak{m}_a^2$, and thus the homomorphism $G \to \mathrm{GL}(T_a\mathbb{A}^n)$ is injective, by Lemma 2.2. Hence the claim. □

Remark 3.5. — It is not difficult to show that the proposition holds for every finite commutative subgroup μ of rank n. In fact, the proof carries over to subgroups isomorphic to μ_p where p is a prime, and every finite commutative subgroup μ of rank n contains such a group.

Lemma 3.6. — *Let $G \subset \mathrm{Aut}(\mathbb{A}^n)$ be a subgroup and let $\varphi\colon \mathbb{A}^n \to X$ be a dominant morphism such that $\dim X = n$. Assume that $\varphi^*(\mathcal{O}(X))$ is a G-stable subalgebra and that the induced action of G on X is locally finite. Then the same holds for the action of G on $\mathbb{A}^n$.*

Proof. — Put $A := \varphi^*(\mathcal{O}(X)) \subset \mathbb{C}[x_1, \dots, x_n]$ and denote by $R \subset \mathbb{C}[x_1, \dots, x_n]$ the integral closure of A. We first claim that the action of G on R is locally finite. In fact, let $f \in R$ and let $f^m + a_1 f^{m-1} + \cdots + a_m = 0$ be an integral equation of f over A. By assumption, the spaces $\langle Ga_i \rangle$ are all finite dimensional, and so there is a $d \in \mathbb{N}$ such that $\deg ga_i < d$ for all $g \in G$ and all a_i. Since gf satisfies the equation $(gf)^m + (ga_1)(gf)^{m-1} + \cdots + (ga_m) = 0$ we get $\deg(gf) < d$ for all $g \in G$, hence the claim.

Therefore, we can assume that X is normal and that $\varphi\colon \mathbb{A}^n \to X$ is birational. Choose an open set $U \subset \mathbb{A}^n$ such that $\varphi(U) \subset X$ is open and φ induces an isomorphism $U \xrightarrow{\sim} \varphi(U)$. Define $Y := \bigcup_{g \in G} gU \subset \mathbb{A}^n$. Then the induced morphism $\psi := \varphi|_Y \colon Y \to \varphi(Y)$ is G-equivariant and a local isomorphism. Since X is quasi-compact the fibers of ψ are finite, and since ψ is birational and $\varphi(Y)$ normal we get that ψ is a G-equivariant isomorphism.

By assumption, the action of G on X is locally finite, and so G is dense in an algebraic group $\bar{G}$ which acts regularly on X. Clearly, the open set $\varphi(Y)$ is $\bar{G}$-stable and thus the action of $\bar{G}$ on $\mathcal{O}(\varphi(Y))$ is locally finite. Now the claim follows, because $\mathbb{C}[x_1, \dots, x_n] \subset \mathcal{O}(Y)$ is a G-stable subalgebra. □

The proposition above has an interesting consequence for the linearization problem for finite group actions on affine 3-space $\mathbb{A}^3$. In this case it is known that every faithful action of a non-finite reductive group on $\mathbb{A}^3$ is linearizable (Kraft-Russell, see [6]).

Corollary 3.7. — *Let $\mu \subset \mathcal{G}_3$ be a commutative subgroup of rank three. If the centralizer of μ is not finite, then μ is conjugate to a subgroup of D_3.*

4. D_n-stable unipotent subgroups

Recall that every commutative unipotent group U has a natural structure of a $\mathbb{C}$-vector space, given by the exponential map $\exp\colon T_eU \xrightarrow{\sim} U$. Thus $\mathrm{Aut}(U) = \mathrm{GL}(U)$ and every action of an algebraic group on U by group automorphisms is given by a linear representation.

A (non-zero) locally nilpotent vector field $\delta = \sum_{i=1}^n h_i \frac{\partial}{\partial x_i}$ defines a (non-trivial) $\mathbb{C}^+$-action on $\mathbb{A}^n$, hence a one-dimensional unipotent subgroup

$$U_\delta = \{(\exp(t\delta)(x_1), \ldots, \exp(t\delta)(x_n)) \mid t \in \mathbb{C}^+\} \subseteq \mathcal{G}_n,$$

and $U_\delta = U_{\delta'}$ if and only if δ' is a scalar multiple of δ. In the following we denote by $e_1, \ldots, e_n$ the standard basis of $\mathbb{Z}^n$, and by $\varepsilon_1, \ldots, \varepsilon_n$ the standard basis of the character group of D_n.

Lemma 4.1. — *Let $U = U_\delta \subset \mathcal{G}_n$ be a one-dimensional unipotent subgroup. Then U_δ is normalized by D_n if and only if δ is of the form $cx^\gamma \frac{\partial}{\partial x_i}$, where*

$$x^\gamma = x_1^{\gamma_1} \cdots x_{i-1}^{\gamma_{i-1}} x_{i+1}^{\gamma_{i+1}} \cdots x_n^{\gamma_n}$$

and $c \in \mathbb{C}^*$. In particular, $U_\delta = \{\delta(s) := (x_1, \ldots, x_i + s(cx^\gamma), \ldots, x_n) \mid s \in \mathbb{C}\}$, and $\mathbf{d} \circ \delta(s) \circ \mathbf{d}^{-1} = \delta(t^{e_i - \gamma} s)$ for $\mathbf{d} = (t_1x_1, \ldots, t_nx_n) \in D_n$.

Proof. — If U_δ is normalized by D_n, then $\mathbf{d} \circ \delta \circ \mathbf{d}^{-1} \in \mathbb{C}^*\delta$ for all $\mathbf{d} \in D_n$. Writing $\delta = \sum_i h_i \frac{\partial}{\partial x_i}$ it follows that each h_i is a monomial of the form $h_i = a_i x^{\beta + e_i}$ for some $\beta \in \mathbb{Z}^n$. If $\beta_i \geqslant 0$ an induction on m shows that, for all $m \geqslant 1$, we have

$$\delta^m(x_i) = b_m^{(i)} x^{m\beta + e_i}, \quad \text{where } b_m^{(i)} = a_i \prod_{l=1}^{m-1} (lb + a_i) \ \text{ and } \ b = \sum_{j=1}^n a_j \beta_j.$$

Assume that $\beta_i \geqslant 0$ for all i. Since δ is locally nilpotent there is a minimal $m_i \geqslant 0$ such that $b_{m_i+1}^{(i)} = 0$. This implies $a_i = -m_i b$. Since $\delta \neq 0$, we get

$$0 \neq b = \sum_{i=1}^n a_i \beta_i = -b \sum_{i=1}^n m_i \beta_i,$$

and so $\sum m_i \beta_i = -1$. But this is a contradiction, because $m_i, \beta_i \geqslant 0$ for all i. Therefore $a_i x^{\beta + e_i} \neq 0$ implies that $\beta_j \geqslant 0$ for all $j \neq i$, and $\beta_i = -1$. Thus there is only one term in the sum, *i.e.*, $\delta = a_i x^\gamma \frac{\partial}{\partial x_i}$ where $\gamma := \beta + e_i$ has the claimed form. ☐

Remark 4.2. — This lemma can also be expressed in the following way: *There is a bijective correspondence between the D_n-stable one-dimensional unipotent subgroups $U \subset \mathcal{G}_n$ and the characters of D_n of the form $\lambda =$*

$\sum_j \lambda_j\varepsilon_j$ *where one* λ_i *equals* 1 *and the others are* $\leqslant 0$. We will denote this set of characters by $X_u(D_n)$:

$$X_u(D_n) := \{\lambda = \sum \lambda_j\varepsilon_j \mid \exists\, i \text{ such that } \lambda_i = 1 \text{ and } \lambda_j \leqslant 0 \text{ for } j \neq i\}.$$

If $\lambda \in X_u(D_n)$, then U_λ denotes the corresponding one-dimensional unipotent subgroup normalized by D_n.

Remark 4.3. — In [9, Theorem 1] Alvaro Liendo shows that the locally nilpotent derivations normalized by the torus $D_n' := D_n \cap \mathrm{SL}_n$ have exactly the same form.

LEMMA 4.4. — *The subgroup* $\mathcal{T}_n$ *of translations is the only commutative unipotent subgroup normalized by* GL_n.

Proof. — If $U \subset \mathcal{G}_n$ is a commutative unipotent subgroup normalized by GL_n, then all the weights of the representation of GL_n on $T_eU \simeq U$ must belong to $X_u(D_n)$. The dominant weights of GL_n are $\sum_i \lambda_i\varepsilon_i$ where $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, and only those of the form $\lambda = \varepsilon_1 + \sum_{i>1}\lambda_i\varepsilon_i$ where $0 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ occur in $X_u(D_n)$. If $\lambda \neq \varepsilon_1$, *i.e.*, $\lambda = \varepsilon_1 + \lambda_k\varepsilon_k + \lambda_{k+1}\varepsilon_{k+1} + \cdots$ where $\lambda_k < 0$, then the weight $\lambda' := (\lambda_k + 1)\varepsilon_k + \lambda_{k+1}\varepsilon_{k+1} + \cdots$ is dominant and $\lambda' \prec \lambda$. Therefore λ' appears in the irreducible representation of GL_n of highest weight λ, but $\lambda' \notin X_u(D_n)$. Thus U and $\mathcal{T}_n$ are isomorphic as GL_n-modules, hence contain the same D_n-stable one-dimensional unipotent subgroups, and so $U = \mathcal{T}_n$. □

5. Maximal tori

It is clear that $D_n \subset \mathcal{G}_n$ is a maximal commutative subgroup of $\mathcal{G}_n$ since it coincides with its centralizer, see Lemma 3.3. Moreover, Białynicki-Birula proved in [2] that a faithful action of an n-dimensional torus on $\mathbb{A}^n$ is linearizable (*cf.* [7, Chap. I.2.4, Theorem 5]).Thus we have the following result.

LEMMA 5.1. — D_n *is a maximal commutative subgroup of* $\mathcal{G}_n$. *Moreover, every algebraic subgroup of* $\mathcal{G}_n$, *which is isomorphic to* D_n *is conjugate to* D_n.

Now let $G \subset \mathcal{G}_n$ be an algebraic subgroup which is normalized by D_n. Then the non-zero weights of the representation of D_n on the Lie algebra $\operatorname{Lie} G$ belong to $X_u(D_n)$, and the weight spaces are one-dimensional. It follows that the non-zero weight spaces of $\operatorname{Lie} G$ are in bijective correspondence with the D_n-stable one-dimensional unipotent subgroups of G.

Lemma 5.2. — *Let $G \subset \mathcal{G}_n$ be an algebraic subgroup normalized by a torus $D \subset \mathcal{G}_n$ of dimension n, let $U_1, \dots, U_r$ be the D-stable one-dimensional unipotent subgroups of G, and put $X := U_1 \circ \cdots \circ U_r \subset G$.*

(a) *If G is unipotent, then $G = X \circ X$ and $\dim G = r$.*

(b) *If $D \subset G$, then $G^0 = D \circ X \circ D \circ X$ and $\dim G = r + n$.*

Proof. — (a) The canonical map $U_1 \times \cdots \times U_r \to G$ is dominant, and so $X \subset G$ is constructible and dense. Thus $X \circ X = G$, by Lemma 2.1, and $\dim G = \dim \operatorname{Lie} G = r$.

(b) Similarly, we see that $D \circ X \subset G^0$ is constructible and dense, and therefore $D \circ X \circ D \circ X = G^0$, and $\dim G = \dim \operatorname{Lie} G = \dim \operatorname{Lie} D + r$. □

6. Images of algebraic subgroups

The next two propositions are crucial for the proof of our main theorem.

Proposition 6.1. — *Let θ be an automorphism of $\mathcal{G}_n$. Then*

(a) *$D := \theta(D_n)$ is a torus of dimension n, conjugate to D_n.*

(b) *If U is a D_n-stable unipotent subgroup, then $\theta(U)$ is a D-stable unipotent subgroup of the same dimension.*

(c) *$\mathcal{T} := \theta(\mathcal{T}_n)$ is a commutative unipotent subgroup of dimension n, normalized by D, and the representation of D on $\mathcal{T}$ is faithful.*

Proof. — (a) We have $D_n = \operatorname{Cent}_{\mathcal{G}_n}(\mu_2)$, by Lemma 3.3, and thus $D = \theta(D_n) = \operatorname{Cent}_{\mathcal{G}_n}(\theta(\mu_2))$. Proposition 3.4 implies that D is a diagonalizable algebraic subgroup with $\dim D \leqslant n$, hence $D = D^0 \times F$ for some finite group F. If k is prime to the order of F, then $\theta(\mu_k) \subset D^0$ and so $\dim D^0 = n$, because $\mu_k \simeq (\mathbb{Z}/k)^n$. Hence $D = D^0$ is an n-dimensional torus which is conjugate to D_n, by Lemma 5.1.

(b) First assume that $\dim U = 1$. Then U consists of two D_n-orbits, $O := U \setminus \{\mathrm{id}\}$ and $\{\mathrm{id}\}$. It follows that $\theta(U)$ consists of the two D-orbits $\theta(O)$ and $\{\mathrm{id}\}$, and so $\theta(U)$ is bounded constructible and thus a commutative algebraic group normalized by D. Since it does not contain elements of finite order it is unipotent, and since it consists of only two D-orbits it is of dimension 1.

Now let U be arbitrary, $\dim U = r$, and let $U_1, \dots, U_r$ be the different D_n-stable one-dimensional unipotent subgroups of U. Then $X := U_1 \circ U_2 \circ \cdots \circ U_r \subset U$ is dense and constructible and $U = X \circ X$, by Lemma 5.2(a). Applying θ implies that $\theta(X) = \theta(U_1) \circ \cdots \circ \theta(U_r)$ is bounded constructible and connected, as well as $\theta(U) = \theta(X) \circ \theta(X)$, and thus $\theta(U)$ is a connected algebraic subgroup of $\mathcal{G}_n$ normalized by D. Since every element of $\theta(U)$ has infinite order, $\theta(U)$ must be unipotent. Moreover, $\dim \theta(U) \geqslant r$, since $\theta(U)$ contains the D-stable one-dimensional unipotent subgroups $\theta(U_i)$, $i = 1, \ldots, r$. The same argument applied to θ^{-1} finally gives $\dim \theta(U) = r$.

(c) This statement follows from (b) and the fact that $\mathcal{T}_n$ contains a dense D_n-orbit with trivial stabilizer. □

The same arguments, this time using Lemma 5.2(b), gives the next result.

PROPOSITION 6.2. — *Let θ be an automorphism of $\mathcal{G}_n$ and let $G \subset \mathcal{G}_n$ be an algebraic subgroup which contains a torus D of dimension n.*

(a) *The image $\theta(G)$ is an algebraic subgroup of $\mathcal{G}_n$ of the same dimension* $\dim G$*.*
(b) *We have $\theta(G^0) = \theta(G)^0$. In particular, $\theta(G)$ is connected if G is connected.*
(c) *If G is reductive, then so is $\theta(G)$, and then $\theta(G)$ is conjugate to a closed subgroup of* GL_n*.*

Proof. — As above, let $U_1, \ldots, U_r$ be the different D-stable one-dimensional unipotent subgroups of G, and put $X := U_1 \circ \cdots \circ U_r$. Then $D \circ X$ is constructible in G^0, and $D \circ X \circ D \circ X = G^0$, by Lemma 5.2(b). Applying θ we see that $\theta(D \circ X \circ D \circ X) = \theta(D) \circ \theta(X) \circ \theta(D) \circ \theta(X)$ is bounded constructible and connected, and so $\theta(G^0)$ is a connected algebraic subgroup of $\mathcal{G}_n$, of finite index in $\theta(G)$. Since the $\theta(U_i)$ are different $\theta(D)$-stable one-dimensional unipotent subgroups of $\theta(G)$ we have $\dim \theta(G) \geqslant \dim \theta(D) + r = \dim G$. Using θ^{-1} we get equality. This proves (a) and (b).

For (c) we remark that if G contains a normal unipotent subgroup U, then $\theta(U)$ is a normal unipotent subgroup of $\theta(G)$. Moreover, a reductive subgroup G containing a torus of dimension n has no non-constant invariants, and so G is linearizable (see [5, Proposition 5.1]). □

7. Proof of the Main Theorem

Let θ be an automorphism of $\mathcal{G}_n$. It follows from Proposition 6.2 that there is a $\mathbf{g} \in \mathcal{G}_n$ such that $\mathbf{g} \circ \theta(\mathrm{GL}_n) \circ \mathbf{g}^{-1} \subset \mathrm{GL}_n$. Therefore we can assume that $\theta(\mathrm{GL}_n) = \mathrm{GL}_n$. The subgroup $\mathcal{T}_n$ of translations is the only commutative unipotent subgroup normalized by GL_n, by Lemma 4.4. Therefore, $\theta(\mathcal{T}_n) = \mathcal{T}_n$ and so $\theta(\mathrm{Aff}_n) = \mathrm{Aff}_n$. Now the theorem follows from the next proposition. □

Proposition 7.1.

(a) *Every automorphism θ of Aff_n with $\theta(\mathrm{GL}_n) = \mathrm{GL}_n$ and $\theta(\mathcal{T}_n) = \mathcal{T}_n$ is of the form $\theta(\mathbf{f}) = \tau(\mathbf{g} \circ \mathbf{f} \circ \mathbf{g}^{-1})$ where $\mathbf{g} \in \mathrm{GL}_n$ and τ is an automorphism of the field $\mathbb{C}$.*

(b) *If θ is an automorphism of $\mathcal{G}_n$ such that $\theta|_{\mathrm{Aff}_n} = \mathrm{Id}_{\mathrm{Aff}_n}$, then $\theta|_{\mathcal{J}_n} = \mathrm{Id}_{\mathcal{J}_n}$.*

Proof. — (a) It is enough to prove that $\theta(f) = \mathbf{g} \circ \tau(\mathbf{f}) \circ \mathbf{g}^{-1}$ for some $\mathbf{g} \in \mathrm{GL}_n$ and some automorphism $\tau\colon \mathbb{C} \to \mathbb{C}$ of the field $\mathbb{C}$. Let $Z = \mathbb{C}^* \subseteq \mathrm{GL}_n$ be the center of GL_n and define $\theta_0 := \theta|_Z\colon Z \to Z$, $\theta_1 := \theta|_{\mathcal{T}_n}\colon \mathcal{T}_n \to \mathcal{T}_n$. It follows that θ_0 and θ_1 are abstract group homomorphisms of $\mathbb{C}^*$ and $\mathcal{T}_n$ respectively, and for all $c \in \mathbb{C}^*$ we get $\mathbf{t} \in \mathcal{T}_n$

$$(*) \qquad \theta_1(c \cdot \mathbf{t}) = \theta_1(c \circ \mathbf{t} \circ c^{-1}) = \theta_0(c) \circ \theta_1(\mathbf{t}) \circ \theta_0(c)^{-1} = \theta_0(c) \cdot \theta_1(\mathbf{t}),$$

where “ $\cdot$ ” denotes scalar multiplication. We claim that $\tau\colon \mathbb{C} \to \mathbb{C}$ defined by $\tau|_{\mathbb{C}^*} = \theta_0$, $\tau(0) = 0$, is an automorphism of the field $\mathbb{C}$. Indeed, using $(*)$ one sees that $\tau(a+b) = \tau(a) + \tau(b)$ for all $a, b \in \mathbb{C}^*$ such that $a + b \neq 0$. As $\theta_0(-1) = -1$ it follows that $\tau(-a) = -\tau(a)$ for all $a \in \mathbb{C}^*$ and so $\tau(a + (-a)) = \tau(a) + \tau(-a)$. This implies the claim.

Thus we can assume that $\theta_0 = \mathrm{id}_{\mathbb{C}^*}$. Using $(*)$ again, it follows that θ_1 is linear. Considering θ_1 as an element of GL_n we have $\theta_1(\mathbf{t}) = \theta_1 \circ \mathbf{t} \circ \theta_1^{-1}$, and thus we can assume that $\theta_1 = \mathrm{id}_{\mathcal{T}_n}$. But this implies that $\theta(\mathbf{g}) = \mathbf{g}$ for all $\mathbf{g} \in \mathrm{GL}_n$, because

$$\mathbf{g} \circ \mathbf{t} \circ \mathbf{g}^{-1} = \theta(\mathbf{g} \circ \mathbf{t} \circ \mathbf{g}^{-1}) = \theta(\mathbf{g}) \circ \mathbf{t} \circ \hat{\mathrm{E}}\theta(\mathbf{g})^{-1}$$

for all $\mathbf{t} \in \mathcal{T}_n$.

(b) Let $U \subset \mathcal{G}_n$ be a one-dimensional unipotent D_n-stable subgroup. We first claim that $\theta(U) = U$ and that $\theta|_U$ is linear. In fact, $U' := \theta(U)$ is a one-dimensional unipotent D_n-stable subgroup, by Proposition 6.1(b), and the characters λ and λ' associated to U and U' (see Remark 4.2) have the same kernel, because

$$(**) \quad \theta(\lambda(\mathbf{d}) \cdot u) = \theta(\mathbf{d} \circ u \circ \mathbf{d}^{-1}) = \mathbf{d} \circ \theta(u) \circ \mathbf{d}^{-1} = \lambda'(\mathbf{d}) \cdot \theta(u) \quad \text{for } \mathbf{d} \in D_n,\ u \in U.$$

Hence $\lambda = \pm\lambda'$. If $\lambda = -\lambda'$, then $U \subseteq \mathrm{GL}_n$ and so $U' = U$, since $\theta|_{\mathrm{GL}_n} = \mathrm{Id}_{\mathrm{GL}_n}$, hence a contradiction. Thus $\lambda = \lambda'$, and so $U = U'$ and $(**)$ shows that $\theta|_U$ is linear, proving our claim.

As a consequence, $\theta|_{U_\lambda} = a_\lambda \, \mathrm{Id}_{U_\lambda}$ for all $\lambda \in X_u(D_n)$, with suitable $a_\lambda \in \mathbb{C}^*$. If $\lambda_i = 1$ put $\gamma_i := 0$ and $\gamma_j := -\lambda_j$. Then $\mathbf{f} = (x_1, \ldots, x_i + x^\gamma, \ldots, x_n) \in U_\lambda$, see Lemma 4.1. Conjugation with the translation $\mathbf{t}\colon x \mapsto$

$x - \sum_{j \neq i} e_j$ gives

$$\mathbf{t}\circ\mathbf{f}\circ\mathbf{t}^{-1} = (x_1, \ldots, x_i + h_\gamma, \ldots, x_n) \text{ where } h_\gamma := (x_1+1)^{\gamma_1}(x_2+1)^{\gamma_2}\cdots(x_n+1)^{\gamma_n}.$$

Now we apply θ to get $\theta(\mathbf{t}\circ\mathbf{f}\circ\mathbf{t}^{-1}) = \mathbf{t}\circ\theta(\mathbf{f})\circ\mathbf{t}^{-1}$. Since all the monomials $x^{\gamma'}$ with $\gamma' \leqslant \gamma$ appear in h_γ it follows that the corresponding coefficients $a_{\lambda'}$ must all be equal. In particular, $a_\lambda = a_{\varepsilon_i} = 1$ since $U_{\varepsilon_i} \subset \mathcal{T}_n$. This shows that $\theta|_{\mathcal{J}_n} = \mathrm{Id}_{\mathcal{J}_n}$. □

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