

Invited Article

Exploration and reduction of data using principal component analysis

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Abstract

In a data set with two variables only, a scatterplot between the two variables can be easily plotted to represent the data visually. When the number of variables in the data set is large, however, it is more difficult to represent visually. The method of principal component analysis (PCA) can sometimes be used to represent the data faithfully in few dimensions (eg. three or less), with little or no loss of information.

This reduction in dimensionality is best achieved when the original variables are highly correlated, positively or negatively. In this case, it is quite conceivable that 20 or 30 original variables can be adequately represented by two or three new variables, which are suitable combinations of the original ones, and which are called principal components. Principal components are uncorrelated between themselves, so that each component describes a different dimension of the data. The principal components can also be arranged in descending order of their variance. The first component has the largest variance, and is the most important, followed by the second component with the second largest variance, and so on. The first two components can then be evaluated for each case in the data set and plotted against each other in a scattergraph, the score for the first component being plotted along the horizontal axis, the score of the second component being plotted on the vertical axis. This scatterplot is a parsimonious two-dimensional picture of the variables and cases in the original data set. We illustrate the method by applying it to simulated datasets, and to a dataset containing national track record times for males and females in various countries.

Keywords

Matrix plots, correlation, correlation matrix; spheres, ellipsoids; rotation of coordinates, principal components, factors, eigenvalues, scree plot, factor loadings for variables, factor scores of cases, uses of principal component analysis such as exploration of data, dimension reduction, regrouping of variables and ordering of data; application to a data set containing national track record times for males and females in various countries.

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Introduction

Suppose we have a very detailed database on a random sample of 1000 sixth form students, containing information on their academic performance, medical details, body measurements etc. We also assume that there are no missing values. For simplicity's sake we will take different subsets of variables at a time to illustrate how the scatterplots of the variables are affected by the correlation between them.

We will start off at first with a data set of 1000 cases (rows) each containing four variables (or columns):

- id, identification number of student (1-1000),
- V1, mark attained by student in Mathematics test (0 to 100),
- V2, height of student in metres,
- V3, systolic blood pressure of student in mm Hg.

Since the interval variables V1 to V3 have widely disparate means and standard deviations, it is often convenient to standardize the variables. So if the mean mark of Mathematics is 58.2 and its standard deviation is 8.5, the standardised variable z_1 corresponding to V1 is defined as $z_1 = (V_1 - 58.2) / 8.5$, ie. $z_1 = (\text{original variable} - \text{its mean}) / \text{its standard deviation}$. The other interval variables can be standardised in a similar manner. All standardised variables are dimensionless (ie. do not carry any units) and have mean = 0 and standard deviation = 1. Besides, one does not have to worry about the scale of standardized variables in scatterplots: they are all centred about the origin and most of the readings lie between 2.0 and -2.0 if the original variables are normally distributed. Very often, variables are used in their standardized form in multivariate statistics.

The variables mentioned above, namely V1, V2 and V3 are examples of *uncorrelated* variables. (The height of a student does not usually affect his performance in the Mathematics test! etc). Knowledge of one variable does not help to predict the other variables. The degree of association or correlation between two variables in a data set is measured by Pearson's coefficient of correlation, r^{4-5} . When two variables are uncorrelated, the coefficient of correlation r is near zero, and a scatterplot of the two variables in their standardized version will be roughly *circular* in shape. One cannot predict the value of one variable from values of the other. In this case the variables are said to be *independent* or *uncorrelated*. There is no relation between them.

To illustrate this we simulated ^{6,7} a data file of 1000 cases each having three variables, which had little or no correlation between them, just like the variables V1, V2 and V3 above. The correlation between these three variables, which we shall also call V1, V2, V3, can be summarized in a correlation matrix², R_1 , as follows:

$$R_1 = \begin{array}{c} \\ \mathbf{V1} \\ \mathbf{V2} \\ \mathbf{V3} \end{array} \begin{array}{ccc} \mathbf{V1} & \mathbf{V2} & \mathbf{V3} \\ 1 & -0.031 & -0.018 \\ -0.031 & 1 & -0.063 \\ -0.018 & -0.063 & 1 \end{array}$$

Like all correlation matrices, R_1 has 1's down the diagonal signifying perfect correlation between a variable and itself! (in this case, V1 with V1, V2 with V2, and V3 with V3). The matrix is also symmetric ie. the correlation between V1 and V2 is -0.031 , which is identical to the correlation between V2 and V1; and so on for the other variables. As expected from the way we simulated the data, all the off diagonal correlations are very small or near to zero (-0.031 for the correlation between V1 and V2, -0.018 for V1-V3, and -0.063 for V2-V3). It is now instructive to examine the geometric scatter of these uncorrelated variables.

The bivariate scatterplots of each pair of variables can be succinctly summarised by a *matrix plot*^{8,9}. Essentially this plot is analogous to the correlation matrix, except that each correlation is replaced by the corresponding scatterplot. For example, the plot in the 1st row and 2nd column is the scatterplot between V1 (in its standardised version) on the vertical axis against V2 (again standardized) on the horizontal axis. The matrix plot can be considered to be a pictorial representation of the correlation matrix itself. The plots on the main diagonal are perfect straight lines since here, a given variable is plotted against itself. For this reason the plots on the main diagonal are often left empty in such matrix plots.

The matrix plot corresponding to the correlation matrix R_1 for our simulated data set with the variables V1, V2 and V3 are shown in Figure 1a. These three variables are uncorrelated to one other so that the scatterplot for the 1000 cases between each pair of variables looks circular in shape as explained above. When the three variables V1, V2 and V3 are plotted simultaneously in a three dimensional plot, the scatterplot assumes a spherical shape. Please refer to Figure 1b. Whatever the angle at which one chooses to look at the scatter, it always looks like a three dimensional sphere. This is the standard geometry for uncorrelated variables with equal variances.

The effect of high correlation on the geometry of the scatter

So far we have discussed the case when there was little or no correlation between the variables. We now discuss the scatter when there is a high correlation between the variables.

The coefficient of correlation coefficient r between two variables is always in the range -1 to $+1$, and cannot lie outside this range. The coefficient attains the value $+1$ or -1 if a *perfect* straight line is obtained in a scatterplot of the two relevant variables (or equivalently their standardised versions). The $+1$ is obtained when the slope of the line is positive, whilst -1 is obtained when the slope is negative. In these cases, one has a

perfect linear relationship between the two variables. One variable can be perfectly predicted from the other and vice-versa.

For intermediate values of the correlation r , one gets an *elliptical* scatterplot for two variables in their standardised form. If we imagine the correlation between the two variables increasing gradually from 0 to 1, the scatterplot will gradually change from the circular shape when r is zero to elliptical for intermediate values of r . As the correlation increases, the eccentricity of the elliptical scatter will increase, i.e. the ellipse becomes thinner and longer, until r attains the maximum value of $+1$, when the ellipse flattens out to a perfect straight line as we have already mentioned above. The same thing happens when the correlation decreases from 0 to the minimum value of -1 .

To see the effect of correlation on the geometry of the scatter, we now consider the set of variables W1, W2 and W3, where W1 and W2 are identical to V1 and V2 above, (ie. mark in Mathematics test and height respectively), whilst W3 is now the mark in the Statistics test. Logically, one would expect that W1 and W3 are substantially correlated to each other, whilst W2, the height, is not correlated to either. As before, therefore, we simulated a data file of 1000 cases each having three variables with correlation structure similar to W1, W2 and W3. The correlation between these three variables, which we shall also call W1, W2, W3, were then calculated from the simulated data set and are summarized in the correlation matrix R_2 as follows:

$$R_2 = \begin{array}{c} \\ \mathbf{W1} \\ \mathbf{W2} \\ \mathbf{W3} \end{array} \begin{array}{ccc} \mathbf{W1} & \mathbf{W2} & \mathbf{W3} \\ 1 & -0.031 & 0.948 \\ -0.031 & 1 & -0.010 \\ 0.948 & -0.010 & 1 \end{array}$$

As can be observed, the correlation between W1 and W3 is now over 0.9, whilst the correlations between W1 and W2, and W2 and W3 are very near 0. It is now instructive to examine the scatter of the variables W1, W2 and W3.

The matrix plot corresponding to the correlation matrix R_2 for W1, W2 and W3 is shown in Figure 2a. In this case there is a strong correlation (0.948) between W1 and W3. This can be observed both from the top left or bottom right entries in matrix R_2 , as also from the corresponding scatterplots in Figure 2a. The scatterplot between W1 and W3 is not circular, but has the shape of a very long, thin ellipse with high eccentricity, and is practically a straight line. Because of the strong relationship between W1 and W3, the corresponding scatterplot is practically one dimensional in nature. On the other hand the correlations for the other pairs of variables (W1 - W2 and W3 - W2) are all small, and so their corresponding scatterplots look circular in nature. When the three variables are plotted simultaneously in three dimensions, it can be noticed that the scatter is no longer spherical, but is ellipsoidal in nature. Please refer to Figure 2b and Figure 2c. It is important to note however, that although the scatter is still three dimensional, the points are disposed mostly on a two dimensional plane which contains the W2 axis and makes about 45° with both the W1 and W3 axes. This plane can be seen end on in Figure 2b. There is very little variation normal to this plane, ie. in a direction going from left to right in Figure 2b. If in this Figure, one looks at the scatter from a

different angle, say from the right or the left, rather than from the front, one can observe the scatter observed in Figure 2c. It is clear that most of the scatter (or *variation*) occurs in this plane, whilst little variation occurs in a direction perpendicular to this plane, as was shown in Figure 2b. The three dimensional spherical scatter of the uncorrelated variables W_1, W_2 and W_3 in Figure 1b has now changed to the very flat ellipsoidal scatter of W_1, W_2 and W_3 in Figures 2b and 2c, because of the large correlation between W_1 and W_3 . This ellipsoid looks like a very flat rugby ball and is essentially two-dimensional!

An ellipsoid¹⁰ is characterised by its three principal axes and their corresponding direction in space, the principal directions. Analogously to the two-dimensional ellipse, the ellipsoid has a major axis, where it is longest, an intermediate axis, and a minor axis, where it is thinnest. These axes occur at right angles to one another, and their respective orientations in three dimensional space are called the principal directions. In Figures 2b and 2c, one is looking at different projections of the ellipsoidal scatter representing W_1, W_2 and W_3 . Figure 2b clearly shows the smallest axis of the ellipsoid, whilst Figure 2c displays the intermediate and major axes of the ellipsoid. The orientations of the largest, intermediate and smallest axes relative to the (W_1, W_2, W_3) coordinate axes are called P_1, P_2 and P_3 respectively. In mathematical jargon, these three orthogonal directions are called *principal components*, or *factors*, or even *eigenvectors*, whilst the square of the lengths of their corresponding axes are referred to as *eigenvalues*, usually denoted by λ_1, λ_2 , and λ_3 respectively. The eigenvalue of a principal component is equal to the variance explained by that component. Consequently, the larger the eigenvalue, the more important is the associated principal component. As above, the principal components are usually arranged in descending order of eigenvalue, that is $\lambda_1 > \lambda_2 > \lambda_3$. For our data set of the variables W_1, W_2 and W_3 , principal component analysis can be readily summarized in Table 1.

Looking at the right hand side of the table, one can note that the sum of the variances (eigenvalues) of the three components $1.95+1.00+0.05$ add up to 3.00, which is exactly equal to the total variance of the 3 variables W_1, W_2, W_3 in their standardised form. (Note that each standardised variable has a variance of 1). Further, it is clear that the first two eigenvalues (1.95 and 1.00) are considerably larger than the eigenvalue of the last component (0.05), showing that our ellipsoid is very flat and that the third dimension can be ignored. In fact the first two components P_1 and P_2 explain $(1.95+1.00)/3$ or 98% of the total variation in the data.

The orientations of the principal components with respect to the W_1, W_2, W_3 axes are given in the penultimate column. Thus the relation $P_1 = .987*W_1 + .987*W_3$ shows that P_1 is a

line (or direction) lying in the $W_2=0$ plane and making 45° with the positive directions of the W_1 and W_3 axes. The number 0.987 is called the *loading*^{11,12} of W_1 (or even W_3 in this case) on P_1 . The loading can be considered to be the correlation between a given variable and the component, whilst the square of the loading, 0.987^2 or .974, implies that the component P_1 explains 97.4% of the variation in W_1 . In fact, if one sums the squares of the loadings of a given component, one obtains the eigenvalue of that component, which stands for the total variation explained by the component. Thus for example, for $P_1, 0.987^2 + 0.987^2$ is equal to 1.95, the eigenvalue of P_1 . The fact that the loadings of P_1 are close to 1, imply that there is near perfect correlation of this factor with W_1 and W_3 .

Similarly for the smallest axis, the relation $P_3 = 0.16*W_1 - 0.16*W_3$ shows that P_3 is a line (or direction) lying in the $W_2=0$ plane and making 45° with the positive direction of the W_1 axis and the negative direction of W_3 axis. As stated previously, however, this is a weak component, with small loadings, and a small eigenvalue.

The second component satisfies $P_2 = 1.000*W_2$, making the second component practically identical to W_2 . This is not at all surprising since W_2 did not have any loadings on P_1 and P_3 . It was completely 'overlooked' by these two components which in turn explained *all* the variation in W_1 and W_3 . The factor P_2 would therefore have to account solely for all the variation in W_2 , making it identical to this variable. This phenomenon happened because originally, W_2 was constructed to be uncorrelated to W_1 or W_3 .

The principal components P_1, P_2 and P_3 are orthogonal (perpendicular) to each other, and form a set of rectangular Cartesian axes just like W_1, W_2 and W_3 . In fact, principal component analysis can be considered to be a rotation from the W_1, W_2, W_3 coordinate system to the system of principal components P_1, P_2 and P_3 . In our case, since P_2 is identical to W_2 , the rotation takes place in the W_1 - W_3 plane with the W_2 (equivalently P_2) axis fixed. The rotation from the set of original variables (W_1, W_3) to the principal components (P_1, P_3) is illustrated in Figure 3. This is essentially a replot of the top left graph in the matrix plot of Figure 2a, ie a plot between W_3 and W_1 . The axis W_2 (equivalently P_2) is perpendicular to the plane of the diagram in Figure 3. The axes corresponding to W_1 and W_3 are rotated anticlockwise in the plane of the diagram itself through an angle of 45° , so that they now point in the directions of the principal axes of the scatter. These directions, shown as dashed lines in Figure 3, are the two principal components P_1 and P_3 , pointing respectively along the longest axis and the smallest axis respectively. The intermediate axis of the ellipsoid is normal to the diagram, along P_2 (equivalently W_2), and is not shown. One can visualise the geometry of this

Table 1: Principal component Analysis

Principal Component	Axis of ellipsoid	Principal Components <i>Orientation relative to W_1, W_2, W_3</i>	Eigenvalues <i>Variance explained or length squared of axis</i>
P1:	along major axis:	$P_1 = .987*W_1 + .987*W_3$	$\lambda_1 = 1.95$
P2:	intermediate axis:	$P_2 = 1.000*W_2$	$\lambda_2 = 1.00$
P3:	along minor axis:	$P_3 = .160*W_1 - .160*W_3$	$\lambda_3 = 0.05$

situation by imagining a flattened rugby-ball with the longest side pointing along P₁, its width pointing along P₂, and its flattened thickness pointing along P₃.

It is clear that since the axis corresponding to P₃ is so small compared to the others, one can effectively ignore P₃ and consider the ellipsoid to be a two dimensional ellipse in the P₁, P₂ plane. Since the equations relating P₁ and P₂ in terms of W₁, W₂ and W₃ are known, the values of P₁ and P₂ can be computed for every case in the data-set. These *factor scores*^{11,12} can then be plotted on a two-dimensional scatter-plot with P₁ and P₂ as axes. The old variables W₁, W₂, W₃ have been adequately represented by a two-dimensional scatterplot in P₁ and P₂. The dimensionality of the system has been reduced from 3 to 2 with little or no loss of information. We have thus achieved a parsimonious description of our data set by means of a suitable rotation of coordinates. In this case, this was possible because the high correlation between the variables W₁ and W₃ makes one of them practically redundant.

Principal component analysis works best when there is substantial correlation between the variables. When variables are uncorrelated, like the set of variables V₁, V₂ and V₃ given above, the spherical structure of the scatter (please refer to Figure 1b) ensures that no reduction in dimensionality can occur in this case, whatever rotation is performed. Principal component analysis would not be appropriate here. Since the variables are uncorrelated, there are no redundancies in the variables, and all dimensions (variables) have to be retained in this case.

A practical example of principal component analysis

As a practical example of principal component analysis, we now present a data set containing the national track records for females and males in 55 countries^{11,12,13}.

The data set consists of 55 rows, each containing the following variables:

country : name of country;
 f100 : female record in seconds in the 100 metres event;
 f200 : female record in seconds in the 200 metres event;
 f400 : female record in seconds in the 400 metres event;
 f800 : female record in minutes in the 800 metres event;
 f1500 : female record in minutes in the 1500 metres event;
 f3000 : female record in minutes in the 3000 metres event;
 fmar : female record in minutes in the marathon;
 m100 : male record in seconds in the 100 metres event;
 m200 : male record in seconds in the 200 metres event;
 m400 : male record in seconds in the 400 metres event;
 m800 : male record in minutes in the 800 metres event;
 m1500 : male record in minutes in the 1500 metres event;
 m5000 : male record in minutes in the 5000 metres event;
 m10000 : male record in minutes in the 10000 metres event;
 mmar : male record in minutes in the marathon.

The numerical data therefore consists of a matrix of 55 rows and 15 columns, ie. 55 cases of 15 variables each. We would like to discover relationships between the various countries and the various events and to represent these relationships on suitable plots. For this end, principal component analysis will be used to elucidate the structure in the data. The first step in a principal component analysis is the calculation of the correlation matrix (Table 2).

The correlation matrix could be very awkward and cumbersome to present when there are many variables. For this reason, many programs present also a *sorted and shaded* correlation matrix^{6,7}, which represents the correlation matrix succinctly in little space. The variables are sorted so that those with higher correlations are grouped together. The correlations are then represented by symbols: the denser the symbol, eg, the multiplication sign (X), closely followed by the addition sign (+), the larger is the magnitude of the correlation between two variables. Conversely, sparser symbols like the dash (-), the dot (.) and the space () represent progressively smaller correlations. In our case one representation of the above correlation matrix is given by:

Table 2: Correlation Matrix of Track Records Data

	f100	f200	f400	f800	f1500	f3000	fmar	m100	m200	m400	m800	m1500	m5000	m10000	mmar
f100	1														
f200	.95	1													
f400	.83	.86	1												
f800	.73	.72	.90	1											
f1500	.73	.70	.79	.90	1										
f3000	.74	.71	.78	.86	.97	1									
fmar	.69	.69	.71	.78	.88	.90	1								
m100	.67	.73	.67	.63	.55	.60	.62	1							
m200	.77	.81	.73	.72	.66	.70	.71	.92	1						
m400	.80	.83	.81	.76	.70	.71	.66	.84	.85	1					
m800	.81	.82	.78	.79	.85	.86	.82	.76	.81	.87	1				
m1500	.79	.77	.77	.84	.88	.89	.83	.70	.78	.84	.92	1			
m5000	.73	.71	.74	.82	.86	.87	.81	.62	.70	.78	.86	.93	1		
m10000	.72	.72	.74	.82	.87	.87	.82	.63	.70	.79	.87	.93	.97	1	
mmar	.66	.63	.69	.78	.82	.82	.77	.52	.60	.71	.81	.87	.93	.94	1

Absolute Values of Correlations in Sorted and Shaded form

```
mmar      X
f1500     XX
m10000    XXX
f3000     XXXX
m5000     XXXXX
fmar      +XXXXX
m1500     XXXXXXX
f800      +XXXX+XX
m800      XXXXXXXXX
m100      -++++++X
m200      ++++++++XXX
f200      ++++++++X+XX
m400      ++X+++X+XXXXX
f100      ++++++X+X+XXX
f400      +X+++++XX+XXXX
```

The absolute values of the matrix entries have been printed above in shaded form according to the following scheme:

```
LESS THAN OR EQUAL TO    0.195
.  0.195 TO AND INCLUDING 0.390
-  0.390 TO AND INCLUDING 0.585
+  0.585 TO AND INCLUDING 0.780
X  GREATER THAN          0.780
```

One can note here that all the correlations are quite high (>0.5) as evidenced by the dense symbols (X and +). The highest correlations however are observed between times for similar distances for males and females together. Thus for events longer than or equal to 800 metres (mmar, f1500, m10000, f3000, m5000, fmar, m1500, f800, m800) most of the correlations between them are represented by X, and are greater than 0.78. Similarly for the shorter events (m100, m200, f200, m400, f100, f400) most of the correlations exceed 0.78 and are therefore again represented by (X). These two groups of variables are still appreciably correlated together, as evidenced by the many plus signs (+) in the lower left part of the sorted and shaded correlation matrix.

The eigenvalues of the correlation matrix

A very important part of the output of any factor analysis is the list of eigenvalues of the correlation matrix, since these give the square of the lengths of the principal axes of the ellipsoidal scatter. A 'histogram' of the eigenvalues, also called a *scree plot*^{11, 14}, is usually also given. This is a plot of the *i*th largest eigenvalue l_i against *i*. This plot is given so that one can visualise the relative sizes of the eigenvalues. The eigenvalues in this case, along with the associated scree plot, are given next:

Histogram of eigenvalues (Scree Plot)

Eigenvalue	Histogram
1 11.9394	*****
2 1.1458	*****
3 0.5431	***
4 0.4133	**
5 0.3195	**
6 0.1599	*
7 0.1130	*
8 0.0764	
9 0.0671	
10 0.0634	
11 0.0501	
12 0.0434	
13 0.0309	
14 0.0197	
15 0.0150	

This and remaining eigenvalues are too small to appear

There are as many eigenvalues as there are variables originally, 15 in this case. Since each standardized variable has by definition a variance of 1.0, the total variance of these 15 variables (in standardised form) is exactly equal to 15. It can be shown mathematically that the sum of the eigenvalues is exactly equal to the total variance of the system, which is unaffected by any rotation of coordinates. In fact, the 15 eigenvalues given above have a sum of 15.0000 as predicted by the theory.

The eigenvalue of a given principal component is equal to the variance explained by that component. In this case, the largest eigenvalue, that is the eigenvalue of the first principal component, is 11.94 and therefore explains 11.94/15 or nearly 80% of the variation in the original data. Similarly the eigenvalue of the second principal component explains 1.15/15 or about 8% of the variation, whilst the third eigenvalue explains 0.54/15 or about 3% of the total variation. The first three factors together therefore account for 91% of the variance in the original data. This information is usually summarized as follows:

Factor	Variance Explained	Cumulative Proportion of Variance		Carmines Theta
		in Data Space	in Factor Space	
1	11.9394	0.7960	0.8761	0.9817
2	1.1458	0.8724	0.9601	
3	0.5431	0.9086	1.0000	
4	0.4133	0.9361		
5	0.3195	0.9574		
6	0.1599	0.9681		
7	0.1130	0.9756		
8	0.0764	0.9807		
9	0.0671	0.9852		
10	0.0634	0.9894		
11	0.0501	0.9927		
12	0.0434	0.9956		
13	0.0309	0.9977		
14	0.0197	0.9990		
15	0.0150	1.0000		

The variance explained by each factor is the eigenvalue for that factor. Total variance is defined as the sum of the positive eigenvalues of the correlation matrix.

In the second column of the table above, the cumulative proportion of the total variance is the sum of the variance explained (eigenvalues) up to and including the factor, divided by the sum of all the eigenvalues. Thus the first factor explains 80% of the variance, the first two factors explain 87%, the first three explain 91% as shown above, and so on. The fourth column is similarly obtained by dividing by the cumulative sum of the eigenvalues by the sum of the first three eigenvalues only, rather than by the sum of all the eigenvalues (15.0). This is because we had requested the program to give us the first *three* principal components. (This will be discussed further on). The last column gives Carmines' theta, a parameter ranging from 0 to 1. The fact that its value of 0.9817 is very near to 1 implies that the factor analysis on our data set was successful, as a large proportion of the variance was explained by very few factors. A similar parameter is Cronbach's alpha, a measure of internal consistency of the variables in a data set. In our case alpha is

found to be 0.9815 which is very near the maximum value of 1. This again implies that there are high correlations between our variables, and that parsimony is successfully achieved with the first few components.

The principal components or factors

A principal component is described by the *factor loadings*^{11,14} or, equivalently, *loadings* of the original variables on it. As explained above, a principal component can be uniquely specified by these loadings. The loading of an original variable on a principal component can be interpreted as the correlation between them. The loadings for the first three principal components are given in the following table:

Unrotated Factor Loadings for Principal Components

Variable	Factor 1	Factor 2	Factor 3
f100	0.868	0.253	0.283
f200	0.869	0.335	0.242
f400	0.881	0.138	0.343
f800	0.903	- 0.111	0.191
f1500	0.913	- 0.294	0.129
f3000	0.920	- 0.252	0.073
fmar	0.874	- 0.199	- 0.029
m100	0.777	0.491	- 0.311
m200	0.853	0.403	- 0.177
m400	0.890	0.291	- 0.122
m800	0.945	0.030	- 0.094
m1500	0.954	- 0.109	- 0.110
m5000	0.925	- 0.247	- 0.138
m10000	0.930	- 0.249	- 0.158
mmar	0.865	- 0.353	- 0.139
VP	11.939	1.146	0.543

The VP is the variance explained by the Factor. It is computed as the sum of squares for the elements of the factor's column in the factor loading matrix.

As pointed out above, the sum of the squares of the loadings of a given component is equal to the square of the length of the associated principal axis, that is, its eigenvalue or the variance explained by that component. Thus, for the first component under the heading 'Factor 1', we have that $.868^2 + .869^2 + \dots + .865^2$ is equal to 11.939, the eigenvalue of the first principal component. This component is defined by the equation

$P_1 = 0.868*f_{100} + 0.869*f_{200} + \dots + 0.865*m_{mar}$, where all the variables on the right hand side are in their standardised version. The second and third components, P2 and P3, are similarly defined from the two columns on the right.

Since the correlations (loadings) of P1 with the 15 variables are all positive, one can interpret P1 to be a measure of the overall athletic prowess of a country. Countries with above average times on the majority of events tend to have high positive scores on this component. Conversely, countries which are strong in track events, and have shorter times, tend to have high negative scores on this component.

The second principal component under the heading 'Factor 2' has smaller loadings on the variables, but is still readily interpretable. It has positive loadings on the short distance events (m100, m200, m400 f100, f200 and f400) and negative

loadings on the longer events. It therefore contrasts sprints with the longer distance times. Countries which are poor in sprint but do better in the longer distances tend to have high positive scores on this factor, whilst countries which do better in the sprint than in the long distances will have high negative scores.

The third component under the heading 'Factor 3' has positive loadings on most female events, and negative loadings on the male events. This factor therefore differentiates between those countries where females do worse than males from those countries where females do better. Countries where females fare worse than males have a high positive score on this factor, whilst countries where females do relatively better than males have a high negative score.

In the above we decided to retain only the first three components. In general, how does one decide how many components are needed to provide an adequate summary of the given data set? There are various *ad hoc* rules for this, the most common being:^{11,14}

- i) Retain only the components with eigenvalues larger than 1: components with eigenvalues less than one account for less variation than an original standardised variable. This is the default method in most computer programs.
- ii) Include just enough components to explain some relatively large percentage of the total variation. Figures between 70% and 90% have been suggested although this will become smaller as the number of variables increases.
- iii) The scree plot of the eigenvalues is inspected for a possible 'elbow' in the curve. Eigenvalues above this elbow are considered large and their principal components are retained.

In our case, we decided to retain three components because they account for 91% of the variation, and all three components can be readily interpreted.

The factor scores

The three most important principal components P1, P2 and P3 have now been extracted from our data, and we have explicit equations for them in terms of the original standardised variables. It is therefore possible to find the values of P1, P2 and P3 for every case (country) in the data set. The values of P1, P2 and P3 obtained for each case are known as *factor scores*^{11,14}. The factor scores of each country for the first three principal components are listed in the following table:

Country	Symbol	Factor Scores		
		1	2	3
The United States	usa	-1.38	-1.22	-0.12
East Germany	eg	-1.24	-0.83	-1.64
Russia	rus	-1.24	-0.66	-1.00
Great Britain	gb	-1.19	-0.53	-0.23
West Germany	wg	-1.12	-0.55	-0.75
Italy	it	-1.00	-0.55	0.75
Poland	po	-0.95	-0.68	-1.08
Australia	aus	-0.93	-0.50	-0.13
Czechoslovakia	cz	-0.89	-0.51	-2.32
Canada	ca	-0.88	-0.70	-0.99
France	fra	-0.83	-0.63	0.05
Finland	fin	-0.79	-0.05	-1.02
Belgium	bel	-0.73	0.19	0.23

Sweden	swe	-0.70	-0.10	-0.29
Netherlands	net	-0.69	0.48	-0.55
Rumania	rum	-0.65	0.81	-0.86
New Zealand	nze	-0.64	0.87	-0.07
Switzerland	swi	-0.61	0.42	0.42
Hungary	hun	-0.55	-0.07	-0.22
Kenya	ke	-0.55	0.57	1.08
Norway	nor	-0.47	1.33	-0.30
Denmark	de	-0.46	0.58	0.15
Austria	aut	-0.44	0.26	-0.84
Ireland	ire	-0.41	1.01	-0.15
Spain	spa	-0.39	0.83	1.16
Brazil	bra	-0.35	-1.09	0.88
Japan	jap	-0.27	0.57	1.43
Portugal	por	-0.24	1.63	1.09
Mexico	mex	-0.16	0.86	0.76
Colombia	co	-0.05	0.41	0.61
Chile	chl	0.03	0.44	1.39
Israel	isr	0.06	0.66	-0.39
Greece	gre	0.08	-0.36	1.31
Taiwan	tai	0.10	-0.57	-1.77
Argentina	arg	0.16	-0.64	0.27
India	ind	0.17	0.55	0.71
China	chi	0.21	0.80	0.77
Bermuda	ber	0.24	-1.88	-0.23
South Korea	skor	0.29	0.16	1.64
Luxemburg	lux	0.30	0.42	1.71
Turkey	tur	0.37	1.36	1.19
North Korea	nkor	0.45	2.16	-1.00
Burma	bur	0.75	0.56	0.15
Philippines	phi	0.76	-0.64	-0.79
Dominican Republic	dom	0.82	-2.18	1.01
Malaysia	mal	0.83	-1.39	0.99
Thailand	tha	0.97	-1.30	0.04
Costa Rica	cri	1.00	1.39	0.76
Indonesia	ndo	1.00	-0.60	-0.33
Singapore	sin	1.05	-0.78	0.05
Guatemala	gua	1.20	0.45	0.07
Papua & New Guinea	pap	1.61	-0.12	0.01
Mauritius	mau	1.74	-0.07	-0.91
Samoa	sam	3.18	-2.77	0.53
Cook Islands	cook	3.43	2.21	-3.19

Although the original data set was in alphabetical order of country, we decided to list the countries in ascending order of the first factor score. As pointed out above, the first principal component measures the overall athletic prowess of a country with weaker countries having higher scores on this component. The above list is therefore *ordered*, with the stronger nations like the United States, East Germany, Russia, Great Britain etc. at the top of the list, right down to the weaker nations. We have therefore achieved a ranking of the countries in the original data set.

Similarly, the second factor score differentiates countries of similar ability according to whether they are better in the sprints than in the long distance events. Countries who are relatively stronger in sprints have negative scores on the second factor, whilst those who do relatively better in long distances have positive scores on this factor. To make this point clearer, one can plot the first two factor scores (given above) against each other. This scatterplot is given in Figure 4. Here, the factor

score of the second principal component (Factor 2) is plotted on the vertical axis against the factor score of the first principal component (Factor 1) on the horizontal axis. In this figure, therefore, stronger nations appear to the left whilst the weaker nations appear to the right. Countries who do relatively better in sprints appear in the lower half of the plot, whilst those who do relatively better in the longer distances appear in the upper half.

Thus for example on the right hand side of the plot in Figure 4, the Cook Islands (labeled as *cook*) and Samoa (labeled as *sam*) both have high positive scores for Factor 1, indicating the relatively low overall athletic standard in these two countries. These are however differentiated by the second factor score. The Cook Islands have a high positive score on the second component, so that they do better in the long distance events. Conversely, Samoa has a high negative score, so it does relatively better in the sprints than in the long distances. Similarly looking at the central third of the table from bottom to top, one has the Dominican Republic, Bermuda, Malaysia and Thailand (*dom, ber, mal, tha*) who have very similar record profiles with a higher standard of sprinting, right up to North Korea, Turkey and Costa Rica (*nkor, tur, cri*) who are of comparable strength as the previous group, but are relatively better in the longer distances. In a similar way, the stronger nations in the left hand third of the table are separated by the second component into countries where the sprints predominate as in the United States, East Germany, Russia, Canada and Brazil (*usa, eg, rus, ca, bra*) and those who are stronger in the longer distances like Portugal, Norway, Ireland and New Zealand (*por, nor, ire, nze*). The factor scores of the first two principal components, which have been given above and plotted in Figure 4, therefore provide an effective two-dimensional summary of the original data set.

Similar plots can be given for Factor 3 scores versus Factors scores 1 or 2. They could even be plotted simultaneously in a three dimensional scatterplot. The third factor score, Factor 3, is the last column in the ranking list of nations given above. As pointed out above, the third factor distinguishes countries of comparable standards by the relative performance of males and females. Considering the stronger nations in the above list, females do relatively better than males in countries with negative third factor scores, such as East Germany, Russia, Poland, Czechoslovakia and Canada. In the USA, Australia and France, this score is nearly zero, so the two genders have comparable performances in these countries. Conversely, males do better than females in countries like Italy with a high positive score on the third factor.

As can be seen from the above considerations, our original data set with 15 variables has been successfully summarized by just three components and has been effectively represented by a scatterplot between the scores on the first two components. Data reduction has been efficiently performed on this data set.

Rotation of the principal components

In principal component analyses, rotations are sometimes performed also on the principal components themselves in a bid to obtain simpler factors. The goal is to make the loadings for each factor either large or small, not intermediate. There are two types of rotations ^{6, 8}. In *orthogonal* rotations, the resulting factors are still perpendicular (orthogonal) to one

another and the factors are not correlated together. Alternatively, one could allow these rotations to be *oblique*, rather than orthogonal, so that the factors are allowed to be correlated between themselves. With oblique rotation there is a greater tendency for each variable to be associated with a single factor, thereby simplifying interpretation of the factors. Plots are sometimes drawn of the rotated factor loadings: the loadings of the variables for one factor are plotted against those of another factor in a two-dimensional scatterplot.

As an example, an oblique rotation was performed on the first two principal components of the track data. One factor is made up of the long distance events whilst the second factor comprises the shorter distances, as in the sorted and shaded correlation matrix which was presented above. Since the correlation between these two factors is quite high (0.746), a scatterplot for the factor scores of these oblique factors would not be as informative or easily interpreted as the analogous plot in Figure 4, where the scores were for the necessarily orthogonal (and hence uncorrelated) principal components. Conversely, if the correlation between the two oblique factors was low, the plot for the factor scores would be very similar to the plot given in Figure 4 .

Oblique rotation of factors is very popular in applications of social science and psychology¹⁴, where the emphasis is on the correlational structure of the *variables* rather than on the distinction between the cases.

Principal component analysis, factor analysis and other multivariate techniques

The two terms *principal component analysis* (PCA) and

factor analysis (FA) are sometimes used interchangeably, but this is not exactly correct. In fact, principal component analysis is the simplest type of factor analysis. A default run of the factor analysis option in most computer packages is usually a principal component analysis. For this reason principal components are often referred to as *factors*, but one should not forget that factor analysis embraces a whole range of techniques for extracting factors from data. Of these techniques, principal component analysis is the simplest and the most intuitive. In all techniques of factor analysis, however, suitable rotations of coordinates are performed from the old variables to the final extracted factors.

Many multivariate techniques in statistics, like multidimensional scaling^{8,11}, cluster analysis for variables⁷ and cases^{7,8,11}, and correspondence analysis^{7,11} for frequency tables, resemble principal component analysis^{6,8,11} and factor analysis^{6,8,11} in that they try to achieve a parsimonious and faithful description of the underlying data. Data reduction is a common goal to most procedures in multivariate statistics. These multivariate techniques are described in many standard references, some of which are cited below.

Statistical Analyses

The above statistical analyses were performed with BMDP, the Bio-Medical Data Package^{6,7}. In particular we used program 1D to obtain the simulated data sets, and program 4M for principal component analysis⁶. The sorted and shaded correlation matrix can also be obtained from the program 1M for cluster analysis of variables⁷. The above analyses can also be easily performed with other programs such as SPSS^{8,9,14}, which was used to plot the graphs in Figures 1, 2 and 4.

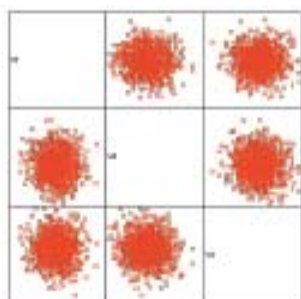


Figure 1a: Matrix plot of the variables V1, V2 and V3. Their correlation matrix R_1 is given in the text. Since the correlation between the variables is low, the scatter plot for each pair of variables assumes a circular shape. When plotted together in a three dimensional plot, the scatter assumes a spherical shape as shown in Figure 1b.

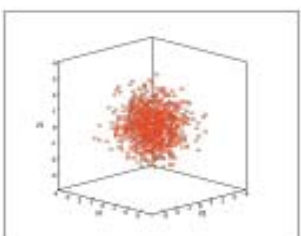


Figure 1b: Three dimensional scatterplot of the variables V1, V2 and V3. Since the variables are not correlated the scatter assumes a spherical shape.

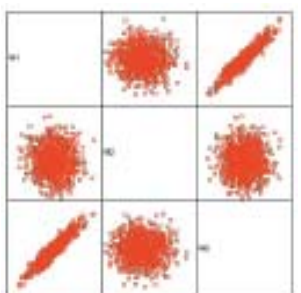


Figure 2c: A glimpse of the scatterplot between W1, W2 and W3 from another point of view, showing more clearly the two-dimensional plane containing most of the scatter. The scatter in this plane is elliptical with the major axis pointing from bottom left to top right, and with the other axis parallel to W2. These directions are called the principal directions or *principal components*. A third principal component exists normal to these two, but this is not important as little variation occurs in this direction, as was shown in Figure 2b.

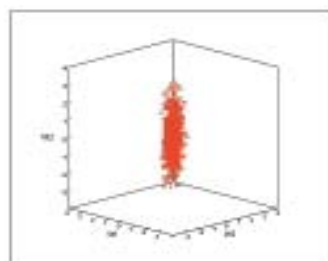


Figure 2b: One view of the ellipsoidal scatter for the three variables W1, W2 and W3. From this view one can appreciate the flat nature of this ellipsoid. The scatter can be safely considered to be two dimensional as there is little scatter or variation in going from left to right in this Figure. Most of the scatter can be observed if one looks from the right or the left. The scatter from this point of view is shown in Figure 2c.

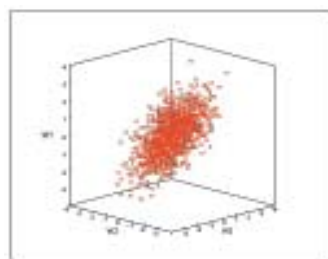


Figure 2a: Matrix plot of the variables W1, W2 and W3. Their correlation matrix R_2 is given in the text. Since the correlation between the variables W1 and W3 is high, the scatterplot for these two variables assumes the shape of a long, thin ellipse as shown in the top right or bottom left scatter plots. When plotted together in a three dimensional plot, the scatter assumes the form of a flat ellipsoid which is nearly two-dimensional. Two views of this ellipsoid are shown in

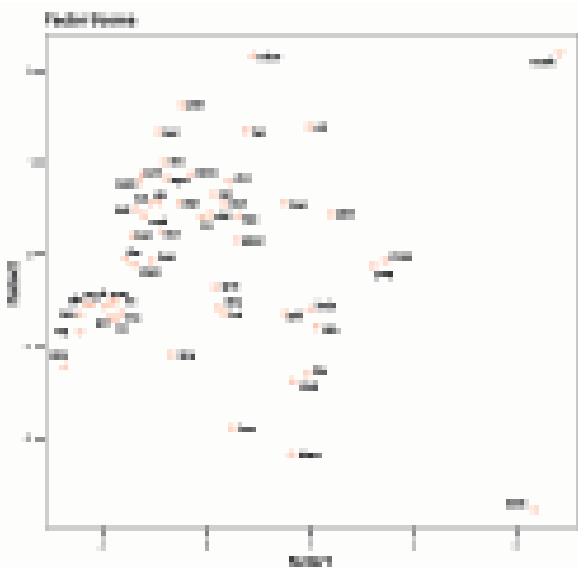


Figure 3: An illustration of the rotation of coordinates in principal component analysis. The scatterplot for W1 and W3 is shown as an ellipse. Please see the corresponding scatterplot in the matrix plot of Figure 2a. This ellipse is inclined at an angle of about 45° to the W1 and W3 axis. The principal components are defined to be the directions defined by the major and minor axes of the ellipse. These are shown as dashed lines and are labeled P1 and P3. The direction W2 is normal to the plane of the paper. Since W2 is not correlated with either W1 or W3, the principal component P2 is identical to W2, and is not affected by the rotation in the W1, W3 plane. Positions of points in the scatter can then be conveniently referred to the P1, P2 and P3 system of coordinates rather than to the old coordinates W1, W2 and W3. As shown in the diagram, little variation occurs along the P3 axis (the direction where the ellipse is very narrow). This component can therefore be safely ignored, retaining only the two components P1 and P2. This leads to a simpler description of the data.

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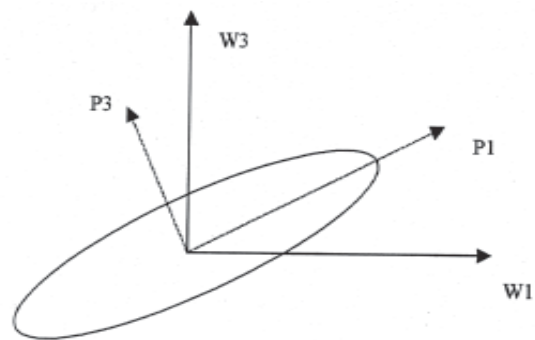


Figure 4: Scatterplot of the first two factor scores for the athletic records data. This is a plot of a set of 15-dimensional observations in the space of the first two principal components. The first principal component represents a measure of the overall athletic prowess of a country: countries with above average times on the majority of events tend to have high scores on this component. So weak nations (eg the Cook Islands and Samoa) appear on the right of the scatterplot, whereas the stronger nations (eg the United States, Russia and East Germany) appear to the left. The second component mainly contrasts performance in sprints with the performance in the middle and long distances. Countries particularly good at the longer distances (eg Norway and Portugal) tend to have high scores on this component, and appear towards the top of the plot. Countries with good sprinting times (eg. the United States, Bermuda, the Dominican Republic) appear towards the bottom of the plot. The original data set with 15 variables has therefore been effectively and parsimoniously represented by a scatterplot of the factor scores derived from its first two principal components.

Suggestions for further reading

The subject of correlation is treated in many elementary textbooks of statistics³, as also in the biostatistical texts^{2,4,5} and in the excellent archaeological text by Shennan¹. This text also has a very readable exposition of principal component analysis and other multivariate techniques, and is strongly recommended for the non-mathematical reader. The reference manuals of statistical software packages like BMDP^{6,7} or SPSS⁸ also describe most multivariate statistical techniques, of which they give many practical examples, along with clear, annotated output. These manuals are an excellent sourcebook of such techniques, and are strongly recommended to the general reader.

The book by Bryman⁹ describes how to use SPSS for Windows to perform numerous statistical techniques, and clearly explains the SPSS output. The book by Tacq¹⁴ also gives very good accounts of many multivariate techniques, how to execute them with the programming language of SPSS, and how to interpret the output of the program. This book gives many interesting examples from the social sciences, and also gives some mathematical and numerical details for the more mathematically oriented reader.

There are many good books on multivariate statistics, of which we cite a few^{11,12,15,16,17}. These texts are more mathematically disposed, those by Flury¹⁶ and Manley¹⁷ being slightly easier than those by Everitt¹¹, Johnson¹² and Morrison¹⁵. These five books contain numerous applications of principal component analysis and other multivariate techniques to different disciplines, and should therefore be of interest even to non-mathematical readers.