Gravitational Swarm Optimizer for Global Optimization

Anupam Yadav[a], Kusum Deep[b], Joong Hoon Kim[c,*], Atulya K Nagar[d]

[a] *Department of Sciences and Humanites*
National Institute of Technology Uttarakhand
Srinagar-246174, Uttarakhand
India
[b] *Department of Mathematics*
Indian Institute of Technology Roorkee, Roorkee-247667
India
[c] *School of Civil, Environmental and Architectural Engineering*
Korea University, Seoul, South. Korea 136-713
[d] *Department of Mathematics and Computer Science*
Liverpool Hope University, Liverpool, U.K.

Abstract

In this article, a new meta-heuristic method is proposed by combining particle swarm optimization (PSO) and gravitational search in a coherent way. The advantage of swarm intelligence and the idea of a force of attraction between two particles are employed collectively to propose an improved meta-heuristic method for constrained optimization problems. Excellent constraint handling is always required for the success of any constrained optimizer. In view of this, an improved constraint-handling method is proposed which was designed in alignment with the constitutional mechanism of the proposed algorithm. The design of the algorithm is analyzed in many ways and the theoretical convergence of the algorithm is also established in the article. The efficiency of the proposed technique was assessed by solving a set of 24 constrained problems and 15 unconstrained problems which have been proposed in IEEE-CEC sessions 2006 and 2015, respectively. The results are compared with 11 state-of-the-art algorithms for constrained problems and 6 state-of-the-art algorithms for unconstrained problems. A variety of ways are considered to examine the ability of the proposed algorithm in terms of its converging ability, success, and statistical behavior. The performance of the proposed constraint-handling method is judged by analyzing its ability to produce a feasible population. It was concluded that the proposed algorithm performs efficiently with good results as a constrained optimizer.

Keywords: Particle Swarm Optimization, Gravitational Search Algorithm, Constrained Optimization, Shrinking hypersphere, constrained handling

*corresponding author

Email addresses: `anupuam@gmail.com` (Anupam Yadav), `kusumfma@iitr.ernet.in` (Kusum Deep), `jaykim@korea.ac.kr` (Joong Hoon Kim), `nagara@hope.ac.uk` (Atulya K Nagar)

1. Introduction

Constrained optimization problems (COPs) are an important class of problems in the field of optimization, because many real-life problems arising in engineering, computer science, finance, and business science can be modeled as nonlinear constrained optimization problems. The formulation of any real-life problem into a constrained optimization problem involves the use of many parameters. Determination of the optimal value of each of these parameters is very important because together these values provide the solution to the problems.

Mathematically, a constrained optimization problem can be formulated in the form of an objective function, which is constrained by some linear and nonlinear constraints. The following model provides the mathematical description of a nonlinear constrained optimization problem:

$$\text{Min. or Max.} f(x), \quad x = [x_1, x_2, x_3, ..., x_D], \tag{1}$$

subject to a set of inequality constraints

$$g_k(x) \leqslant 0, \quad k = 1, 2, 3, ...q, \tag{2}$$

as well as equality constraints

$$h_k(x) = 0, \quad k = q + 1, q + 2, ...m, \tag{3}$$

where objective function f is defined over subspace of a D dimensional real vector space $S \subseteq R^n$ and x is a member of D-dimensional vector space. A set of q inequality constraints and $m - q$ equality constraints, define the feasible region $F \subseteq S$. $L_i \leqslant x_i \leqslant U_i$ are the lower and upper bounds of the decision variables in the domain S, where $i = 1 : 1 : D$.

A class of optimization techniques is available in the literature for use with COPs. In principle, both deterministic and nondeterministic techniques are being used to solve COPs. Unfortunately, selected predefined assumptions of deterministic techniques restrict their applicability to a specific class of problems. This restriction directed us to focus on nondeterministic techniques, among which differential-free, nature-inspired optimization techniques have become very popular, because of their applicability to a wide range of optimization problems. This article focuses on the development of a new meta-heuristic method for constrained optimization problems.

In recent years, many nature-inspired optimization techniques have been developed to solve constrained optimization problems. Initially, these techniques were only used to solve unconstrained optimization problems. Particle Swarm Optimization (PSO) [7, 28, 44], Differential Evolution (DE) [5, 33, 8], and the Gravitational Search Algorithm (GSA) [45] are known to deliver excellent performance for unconstrained optimization problems, but have been found to perform variedly with COPs. In particular, the performance is strongly affected when problems have to be solved at a high level of complexity. The complexity in COPs mainly occurs when the ratio of the search region to the feasible region is very small [29]. This level of complexity in problems requires the combined use of different classes of algorithms to provide a more powerful constrained optimizer. Many modifications and hybridizations intended to improve the efficiency and robustness of the algorithms have appeared in the literature. Banks et al. [3] provide detailed information about the possible improvements of the PSO algorithm by hybridization, and they exhaustively discussed the major benefits of this development. Huang [23] improved the availability of the DE algorithm by evolving two subpopulation. Lwin and Qu [31] proposed a hybrid algorithm by integrating population-based incremental learning and DE for the solution of constrained portfolio selections.

The success of any constrained optimization algorithm mostly depends on the strength of the constraint-handling technique, the design of which has to be customized for an individual optimization algorithm. A few good constraint-handling mechanisms, that are capable of performing well, have been proposed. For example, Deb [19] proposed an efficient constraint-handling approach for genetic algorithms, whereas Coello and Carlos [14] published a comprehensive survey on constraint-handling approaches for a large number of optimization algorithms. Mezura-Montes and Coello [35] also furnished a detailed report in which they presented the future scope and trends of constraint-handling mechanisms. In [18] the design of a very good constraint-handling method for multiple swarm-based cultural PSO is described. The constraint-handling method discussed in [1] was successfully embedded within DE based on a penalty function. The FPBRM constraint-handling method proposed by Mun and Cho [40] for a modified harmony search algorithm also produced good results for optimization problems. The advantage of these algorithms lies in the fact that they were specifically designed for the technique being used for the optimization. This kind of constraint-handling mechanism is naturally compatible with the algorithms and enhances the performance of the optimizer. The overall message emerging from these studies is that an effective constraint-handling method should be based on the individual algorithm in which this constraint-handling method will be utilized. This inspired the authors of this article to propose a new constraint-handling mechanism that is appropriate for and compatible with the proposed optimization algorithm.

This research extends the concept of the recently proposed shrinking hypersphere PSO (SHPSO) [51] for unconstrained optimization and engineering design problems, as opposed to the method in [52], which was extended for constrained optimization problems. The performance of the SHPSO approach was improved using the GSA [45] and a global constrained optimizer was established with theoretical proof of its convergence and stability.

The organization of the paper is as follows. Section 2 briefly provides the concept of the GSA, and in sections 2.1 and section 2.2 the principles of PSO and SHPSO are discussed. Subsequently, in section 3, the proposed SHPSO-GSA is presented and section 4 contains a detailed theoretical and experimental analysis of the proposed algorithm. Section 5 discusses the proposed constraint-handling method and in section 6 the experimental results are discussed followed by the conclusions. A flow chart of the article is depicted in Fig. 2.

2. Gravitational Search Algorithm

The GSA [45] is a recent meta-heuristic algorithm for solving nonlinear optimization problems. It is inspired by Newton's basic physical theory that states that a force of attraction works between every particle in the universe and this force is directly proportional to the product of their masses and inversely proportional to the square of the distance between their positions. In the GSA, each particle is equipped with four kinds of properties: position, mass, active gravitational mass (M_{ai}), and passive gravitational masses (M_{pi}) . The position of the mass provides the solution of the problem. Gravitational mass and inertial mass can be evaluated using a fitness function. Each kind of mass follows the basic laws of physics:

i. *Law of gravity*: Each particle attracts every other particle and the gravitational force between two particles is directly proportional to the product of their respective mass and inversely proportional to the square of the distance between them [42].

ii. *Law of motion*: The current velocity of any mass is equal to the sum of the fractions of its previous velocities and the variations in the velocity. Variation in the velocity of any mass is equal to the force exerted upon the system divided by the mass of inertia.

Inspired by the definitions above, we are able to define the physics of the GSA. Let the position of the i^{th} particle at any instant t in a D-dimensional search space be $X_i^t(x_{i1}^t, x_{i2}^t, ..., x_{iD}^t)$ for $i = 1...n$. The force of attraction (F_{ijD}^t) on the i^{th} particle to j^{th} particle is defined as in the following equation

$$F_{ijD}^t = G^t \times \frac{M_{pi}^t \times M_{aj}^t}{R_{ij}^t} \times (x_{id}^t - x_{jd}^t) \tag{4}$$

where $d = 1, 2, ...D$, M_{pi}^t is the passive gravitational mass related to i^{th} particle at time t, M_{ai}^t is the active gravitational mass related to j^{th} particle at time t, G^t is the gravitational constant at time t and R_{ij}^t is the Euclidean distance between the two particles i and j.

$$R_{ij}^t = ||X_i^t, X_j^t||^2 \tag{5}$$

The value of gravitational constant G^t can be calculated by Eq. 6

$$G^t = G^{t_0} \times \exp((-\alpha \frac{iter}{itermax})) \tag{6}$$

where α and G_0 are descending coefficient and initial value, respectively. *iter* is the current iteration and *itermax* is the predefined maximum number of iterations.

The total force of attraction exerted by the i^{th} particle at any instant t in a D-dimensional space is given by Eq. 7

$$F_{id}^t = \sum_{i=1, i \neq j}^{ps} rand() F_{ijd}^t \tag{7}$$

where $d = 1, 2, ...D$, $rand()$ is the uniform random number generator in [0,1] and ps **is the number of agents**. which is introduced to provide the stochastic nature to the algorithm. By using the law of motion the acceleration of i^{th} particle is given by the following equation:

$$ac_{id}^t = \frac{F_{id}^t}{M_{ii}^t} \tag{8}$$

where M_{ii}^t is the inertial mass of the i^{th} particle. The velocity and position of particles are calculated as follow:

$$V_{id}^{t+1} = rand() \times V_{id}^t + ac_{id}^t \tag{9}$$

$$x_{id}^{t+1} = x_{id}^t + V_{id}^{t+1} \tag{10}$$

where $rand()$ is uniform random number generator in [0,1]. The gravitational and inertial masses are simply calculated by the fitness evaluations. A greater mass can be treated as better particle and having higher force of attraction so that it can influence other particles with high level of attraction. The gravitational and inertial mass will be updated with the help of following equations:

$$M_{ai} = M_{pi} = M_{ii} \quad \text{for} \quad i = 1, 2...ps \tag{11}$$

$$m_i^t = \frac{\text{fit}_i^t - \text{worst}^t}{\text{best}^t - \text{worst}^t} \tag{12}$$

$$M_i^t = \frac{m_i^t}{\sum_{i=1}^{ps} m_i^t} \tag{13}$$

where fit_i^t represents the fitness value of the i^{th} particle at time t. best^t and worst^t may be defined by the following equations:

$$\text{best}^t = \min(\text{fit}_j^t), j \in \{1, ...ps\} \tag{14}$$

$$\text{worst}^t = \max(\text{fit}_j^t), j \in \{1, ...ps\} \tag{15}$$

The exhaustive procedure of GSA is explained in Algorithm 1.

Algorithm 1 Pseudo code of Gravitational Search algorithm

```
Initialization
Randomly initialize (X_1^t, X_2^t, ..., X_ps^t) of size ps in search range [X_min, X_max]
Evaluate the fitness values (fit_1^t, fit_2^t, ..., fit_ps^t) of X(Agent)
Set iteration t=0
Reproduction and Updating
while Stopping Criterion is not satisfied do
    Calculate G^t, best^t, worst^t and M_i^t
    Calculate the total force in each direction F_i^t
    Calculate the ac_i^t and velocity
    Calculate the degree of infeasibility of each particle
    for i=1: ps do
        V_i^{t+1} = rand() × V_i^t + ac_i^t
        X_i^{t+1} = X_i^t + V_i^{t+1}
        Evaluate the fitness values (fit_i^t) of X(Agent)
    end for
end while
```

In recent times, the GSA has been improved in various ways such as the development of Binary GSA [53], prototype classifier-based GSA [2], and improved GSA [27] for optimal shape design. These developments and applications based on the GSA have inspired the continued investigation of this algorithm for the benefit of the science and engineering community. The study of Rashedi et al. [45] shows that the performance of the

GSA is very promising for unconstrained optimization problems; however, studies of the performance of the GSA over constrained optimization problems have indicated that, in terms of memory-less functionality, the performance of the algorithm is disappointing. This lack of intelligence provides the motivation to assemble the memory in the GSA by hybridizing it with PSO. In the next section, PSO is briefly explained.

2.1. Particle Swarm Optimization

Particle Swarm Optimization (PSO) is a nature-inspired stochastic population-based optimization search technique, inspired by the social behavior of fish and birds. PSO was first introduced by Kennedy and Eberhart in 1995 [26]. It uses the learning, information sharing, and position-updating strategy of each solution. It is very simple to implement on mathematical and computational platforms. The technique behaves analogously with bird flocking in that the search space is analogous to the flocking area of the bird; hence, each bird may be treated as a particle in the search space equipped with communicating abilities.

Let the position of the i^{th} particle in a D-dimensional search space be $X_i^t(x_{i1}^t, x_{i2}^t, ..., x_{iD}^t)$ with a flag of velocity $V_i^t(v_{i1}^t, v_{i2}^t, ..., v_{id}^t)$ at any moment t, where $i = 1$ to ps, where ps is the swarm size. Let $Pbest_i^t$ and $Gbest_i^t$ denote the latest best position of the particle(personal best) and global best at the moment t. Initially $Pbest_i^t$ and X_i^t are same. According to the theory of PSO, the position of the particle can be considered to have changed when the linear combination of the aforementioned three influences has occurred. Mathematically, this can be expressed as:

$$\text{Influence of}(V_i^t) + \text{Influence of}(Gbest_i^t - X_i^t) + \text{Influence of}(Pbest_i^t - X_i^t) \tag{16}$$

The formulated Eq. (16) is the velocity update equation, which can be written as:

$$V_i^{t+1} = \omega V_i^t + c_1 r_1 (X_i^t - Gbest_i^t) + c_2 r_2 (X_i^t - Pbest_i^t) \tag{17}$$

ω is the inertia weight, r_1 and r_2 are the uniform random number generators between 0 to 1. c_1 and c_2 are the PSO parameters their values are mentioned in Table 1.

$$X_i^{t+1} = X_i^t + V_i^{t+1} \tag{18}$$

Finally the velocity update equation in its formal expression is written in Eq. (17). The updated position of particles may be obtained by applying V_i^{t+1} on the position X_i^t (Eq. (18)). The exhaustive procedure of PSO is explained in Algorithm 2.

Algorithm 2 Pseudo code of Particle Swarm Optimization Algorithm

Initialization
Randomly initialize $(X_1^t, X_2^t, ..., X_{ps}^t)$ of size ps in search range $[X_{min}, X_{max}]$
Initialize the velocity $(V_1^t, V_2^t, , V_{ps}^t)$ in the range $[V_{min}, V_{max}]$
Set iteration t=0
Evaluate the fitness values $(fit_1^t, fit_2^t, ..., fit_{ps}^t)$ of X
Set X_i^t to be $Pbest = (Pbest_1^t, Pbest_2^t, ..., Pbest_{ps}^t)$ for each particle
Set the particle with best fitness value $Gbest$
Reproduction and Updating
while Stopping Criterion is not satisfied **do**
  for i=1: ps **do**
    $V_i^{t+1} = \omega V_i^t + c_1 r_1 (Pbest_i^t - X_i^t) + c_2 r_2 (Gbest^t - X_i^t)$
    $X_i^{t+1} = X_i^t + V_i^{t+1}$
    $Pbest_i^{t+1} = Pbest_i^t$
    $Gbest^{t+1} = Gbest^t$
    Evaluate fitness(X_i^{t+1})
    if (fitness($Pbest_i^{t+1}$) <fitness(X_i^{t+1})) **then**
      X_i^{t+1}=$Pbest_i^{t+1}$
    end if
    if (fitness($Gbest^{t+1}$) <fitness($Pbest_i^{t+1}$)) **then**
      ($Pbest_i^{t+1}$)= $Gbest^{t+1}$
    end if
  end for
end while

Since it was initially proposed, the PSO technique has seen many modifications; for example, the first version [26] was later improved [47] by incorporating the concept of inertia weight into PSO. Subsequently, Eberhart and Shi [21] again proposed a method for tuning the parameters of the PSO algorithm. Clerc [9, 11, 10] introduced many major modifications to increase the usefulness of PSO. Many hybridized and modified versions of PSO were also proposed, such as Compact PSO [41], ICA-PSO [24], teaching and peer-learning PSO [30]. These hybridizations were implemented by incorporating the other heuristic techniques into PSO. A detailed study of the trajectory of PSO particles was discussed by Van den Bergh and Engelbrecht [50], and Zhang et al. [55] provided a theoretical analysis of PSO. Pluhacek et al. [43] designed a modified PSO driven by a multi-chaotic number generator. A new method referred to as SHPSO was proposed by Yadav and Deep [51, 52] based on the idea of the shrinking hypersphere versus particle generation. Additional noteworthy optimization algorithms, such as two-swarm COPSO [48], DSS-MDE [54], COMDE [39], DECV [36], time adaptive hybrid PSO [4], and A-DDE [37] were also presented in the literature.

These improvements to PSO inspired the design of a new hybrid PSO method in combination with the

Table 1: Parameter selection for the SHPSO

ω	c_1	c_2	c_3
0.9	1.49618	1.49618	1.01

GSA. This involved the proposal of a new equation for updating the velocity of PSO by incorporating the principles of GSA with the aim of improving the procedure for updating the positions of the particles. In the section 2.2, the important aspects of SHPSO [51, 52] are discussed.

2.2. Shrinking Hypersphere based PSO (SHPSO)

SHPSO was recently proposed by Yadav and Deep [51, 52]. This algorithm uses the shrinking hypersphere strategy in the search space to update the positions of the particles. SHPSO functions as follows

Let *NGbest* be the best (fittest) particle in the global hypershere [51] with the global best as the center. Further, let *NPbest* be the best (fittest) particle in the hypersphere with *Pbest* as the center. Compare *NGbest* & *NPbest* and select the fittest. Now apply both the velocity update and position update equations in such a way that at each iteration, the radius of the hypersphere decreases from r to 0, which is proportional to the total number of iterations as specified by the user. Thus, if a sufficiently large number of iterations are performed, then the algorithm will converge to a point hypersphere. A description of the SHPSO is provided in Algorithm 3. The fine tuned parameter values [51] is listed in Table 1. In the next section the proposed SHPSO-GSA algorithms is discussed.

Recently, Hsing-Chih et al. [22] designed a hybrid algorithm by combining PSO with the GSA for unconstrained nonlinear optimization problems, which they tested against a benchmark of 14 problems. The difference between the proposed approach and that of Hsing-Chih et al. [22] is that theirs involved unconstrained optimization, whereas this paper focuses on constrained optimization. Secondly, Hsing-Chih et al. [22] used standard PSO to hybridize GSA, in contrast to the SHPSO [51] used in this work. Finally, the benchmark functions used by Hsing-Chih et al. [22] only had two categories, i.e., uni-modal and multi-modal, whereas SHPSO [51] has already been tested over more complex functions proposed in IEEE CEC sessions. In this work, the IEEE CEC 2006 benchmark set of constrained optimization problems was employed to test the proposed SHPSO-GSA algorithm. In addition, the design of the mechanism of the proposed hybrid algorithm is completely different from the version presented by Hsing-Chih et al. [22].

Another concept of hybridization of PSO and GSA is proposed by Mirjalili et. al. [38]. The major attractions of this hybridization was, it is designed with Standard PSO and original GSA. The idea of PSOGSA [38]

Algorithm 3 Pseudo code of the shrinking hypersphere based PSO

Initialization

Randomly initialize $(X_1^t, X_2^t, ..., X_{ps}^t)$ of size ps in search range $[X_{min}, X_{max}]$

Initialize the velocity $(V_1^t, V_2^t, ..., V_{ps}^t)$ in the range $[V_{min}, V_{max}]$

Set iteration t=0

Evaluate the fitness values $(fit_1^t, fit_2^t, ..., fit_{ps}^t)$ of X

Set X_i^t to be $Pbest = (Pbest_1^t, Pbest_2^t, ..., Pbest_{ps}^t)$ for each particle

Sort the Swarm based on the feasibility based rule [29]

i. The feasible solutions are listed in front of the infeasible solutions.

ii. The feasible solutions are sorted in ascending order of their objective function values.

iii. The infeasible solutions are sorted in ascending order of their degree of constrained violations.

Set the first particle in the second list to $Gbest$

Generation of Hypersphere

Generate a hypersphere of radius r centered at $Gbest$, generate ps particle inside the hypersphere

Evaluate the fitness value of each generated particle and choose the best feasible, name it $NGbest$

Again generate ps particles $Hbest = (Hbest_1^t, Hbest_2^t, ..., Hbest_{ps}^t)$ in the hypershpere of $Pbest$ and compute fitness of $Hbest$

$Pbest = Best(Pbest, Hbest)$

Reproduction and Updating

while Stopping Criterion is not satisfied **do**

    for i **do**=1: ps

        $V_i^{t+1} = \omega V_i^t + c_1 r_1 (Pbest_i^t - X_i^t) + c_2 r_2 (Gbest^t - X_i^t) + c_3 r_3 (NGbest^t - X_i^t)$

        $X_i^{t+1} = X_i^t + V_i^{t+1}$

        $Pbest_i^{t+1} = Pbest_i^t$

        $Gbest^{t+1} = Gbest^t$

        Evaluate fitness(X_i^{t+1});

        if fitness($Pbest_i^{t+1}$) <fitness(X_i^{t+1}) **then**

            X_i^{t+1}=$Pbest_i^{t+1}$

        end if

        if fitness($Gbest^{t+1}$) <fitness($Pbest_i^{t+1}$) **then**

            ($Pbest_i^{t+1}$)= $Gbest^{t+1}$, Using feasibility based rule

        end if

        Again sort the swarm

        Goto 12 Update $Hbest$ and $NGbest$. Using feasibility based rule

    end for

end while

seems to similar to the proposed article (SHPSO-GSA) but on careful observation both the articles have significant difference both in idea as well as content. Table 2 presents a crisp comparison and differences between these two algorithms Overall, the major difference between these algorithms are the idea of implementation

Table 2: Comparison between SHPSO-GSA and PSOGSA

Sr. No.	SHPSO-GSA	PSOGSA [38]
1.	Hybridized with SHPSO and GSA	Hybridized with SPSO and GSA
2.	Proper justification and need of hybridization is discussed	Not provided in the paper
3.	The reason for each kind of influence is discussed before proposing the algorithm	No discussion and justification is provided, the authors have abruptly replaced the pbest with acceleration term of GSA
5.	A detailed theoretical analysis is provided for the convergence and stability of the algorithm	Not discussed
6.	The idea is justified with many experiments	Only few unconstrained problems are solved
7.	Constrained problems of CEC 2006 and unconstrained problems of CEC 2015 is solved	Only simple unconstrained problems are solved with very less dimension
8.	Compatible constraint handling mechanism is proposed	Not applicable

and the current one is applicable for constrained as well as unconstrained problems while PSOGSA is applied only for unconstrained problems. The results of original SHPSO [51] in comparison to PSOGSA, the results are far better than PSOGSA. Therefore an outcome of the results of PSOGSA and SHPSO-GSA can be analyzed easily and a conclusion can be drawn that SHPSO-GSA is better than PSOGSA based on experimental studies.

3. The Proposed Hybridized SHPSO-GSA Algorithm

3.1. Motivation of Hybridization

The fundamental motivation for designing the SHPSO-GSA algorithm was to utilize the memory-enabled behavior of PSO in the memory-less approach of GSA; i.e., GSA does not keep track of the path of any individual particle in its memory. The memory functionality was assembled into GSA by using it jointly with the recently proposed SHPSO. The advantage of the GSA is the constitutional diversity in the algorithm, which originates from the fundamental concept of defined acceleration of a particle. Because the acceleration of a particle is a function of the force experienced by the existing gravitational field due to the other particles and its mass. On the other hand, when processing COPs, the performance of the GSA is strongly affected by the design of the constraints and the shape of the search region. The problems in which the intersection of the feasible and search regions is very small, the GSA does not behave well, because the mass values of particles behave similarly, thereby diluting the effect of the gravitational field over the particles. This results in the particles ceasing their movement at places of less importance. In contrast, the ability of PSO to memorize the trajectory of a particle avoids the looping of the particles in some interval. This discussion is experimentally justified as follows. Plots of the GSA and the proposed SHPSO-GSA (in the next section) have been applied to solve the problems $g01$ and $g02$ [29], and the fitness value and infeasibility of the last population (for a population size of 60), achieved after 4000 iterations, are compared. The results of the final population are depicted in Fig 3. The red stars in the figure show the fitness value of each particle in the last iteration and for each red star there is a corresponding blue circle vertically above the red star, to show the degree of infeasibility of each particle. The degree of infeasibility is calculated with the following equation:

$$G(x_i) = \sum_{j=1}^{m} G_j(x_i)$$

where $G(x_i)$ is the degree of infeasibility of the i^{th} particle x_i and j is the number of constraints. For the function $g01$ the global fitness is '-15' [29] and the particles of the GSA are unable to reach the global minimum even when the infeasibility of the particles is very high. However, when the same problem is solved by the SHPSO-GSA the result is excellent, with every particle able to reach the global minima with zero degree of infeasibility. Fig. 4 depicts the same analysis for the problem $g02$ and again the particles of the GSA are not able to reach to the global minima, whereas the SHPSO-GSA again produces an excellent result with 100% feasibility and optimum fitness. The major idea was to use exploitation capacity of the SHPSO and exploration ability of GSA algorithm together. In further sections it has been discussed that the designed hybrid performs in a good capacity in comparison to individual performances.

Apart from the above discussions the possible hybrid of SHPSO and GSA will increase the exploration ability. This is due to the exponential behavior of the acceleration term (See Eq. 29 and the idea is to utilize the exploratory nature of GSA. In order to measure the increased exploration ability the potential search volume [28] of the SHPSO and SHPSO-GSA. The Potential search ranges for i^{th} particle of SHPSO is given by:

$$r_{1i}^{d} = |Gbest^{d} - X_i^{d}| + |Pbest_i^{d} - X_i^{d}| + |NGbest - X_i^{d}| \tag{19}$$

The potential search ranges of the proposed SHPSO-GSA is designed based on the velocity update equation proposed in Eq. 23. Which is presented in Eq. 20

$$r_{2i}^{d} = ac_i^t + |Gbest^t - X_i^d| + |Pbest_i^t - X_i^d| + |NGbest^d - X_i^d| \tag{20}$$

The potential search volumes for the i^{th} particle, corresponding to SHPSO and SHPSO-GSA is given by R_1 and R_2 respectively.

$$R_1 = \prod_i^d r_{1i} \tag{21}$$

$$R_2 = \prod_i^d r_{2i} \tag{22}$$

To understand the changed exploration ability of the proposed algorithm due to incorporation of the acceleration, the potential search volume is plotted against iterations, the total explored potential search volume of SHPSO and SHPSO-GSA are presented in Fig.5 and 6. These figures are plotted for sphere function over $[-100, 100]^{25}$. Multiple runs are performed for this experiment and median curve is presented in the article. It can be observed from the figures that the proposed hybrid has a better exploration ability in comparison to the original SHPSO algorithm.

3.2. Idea of Hybridization

The idea for implementing hybridization is to jointly influence the particles of the swarm by GSA as well as by SHPSO. The implementation of hybridization requires the movement of the particles to be controlled by the acceleration calculated from the gravitational search, i.e., the personal best (Pbest) of each particle and $NGbest$. These three influences are employed collectively to obtain the new position of the particles. As in SHPSO, the position of the particle is updated by the newly calculated velocity, which is a combination of the global best, personal best,$NGbest$, and its previous velocity. The global best influence of the SHPSO is replaced with the calculated acceleration of the particle. The removal of global best would not affect the converging ability of the algorithm, due to the way $NGbest$ is designed; hence, it will effectively fulfill the need of global best. Therefore, the proposed velocity update equation would be the combination of acceleration, personal best, and the value of $NGbest$.

Mathematically it can be formulated as: let $X_i^t(x_{i1}^t, x_{i2}^t, ..., x_{iD}^t)$ is the position of the i^{th} particle in the population at any instant t. It is proposed that to update the position of the i^{th}particle four choices are sampled, these choices are:

i. To move towards its current direction. .

ii. To move towards the particles suggested acceleration by gravitational search.

iii. To move towards the personal best of the particles.

iv. To move towards the NGbest of the population.

A combined influence is used, instead of moving toward any one of the direction listed above. Mathematically which can be expressed as

$$\omega \times V_i^t + ac_i^t + (Pbest_i^t - X_i^t) + (NGbest - X_i^t)$$

with the help of Eq. 23, a new velocity and position update equation can be defined as:

$$V_i^{t+1} = \omega \times V_i^t + c_1' \times r_1 \times ac_i^t + c_2' \times r_2 \times (Pbest_i^t - X_i^t) + c_3' \times r_3 \times (NGbest - X_i^t) \tag{23}$$

where ω is the inertia weight, c_1', c_2' and c_3' are the parameters whereas, r_1, r_2 and r_3 are uniform random number in the interval [0,1]. Using Eq. 23, the position update equation may be written as:

$$X_i^{t+1} = X_i^t + V_i^{t+1} \tag{24}$$

An exhaustive procedure of SHPSO-GSA is presented in Algorithm 4. A logical flow chart of the proposed algorithm is also depicted in Fig. 7.

4. A Detailed Analysis of the Proposed Algorithm

This section presents a rigorous analysis of the proposed SHPSO-GSA. The effect of the modified velocity updated equation, theoretical convergence, and converging ability is discussed in detail.

4.1. Effect of Proposed Velocity Update Equation

The effect of the proposed velocity update equation was studied by comparing its behavior with that of standard PSO by plotting the velocity of an individual particle against particle generation. The proposed equation can be observed to provide improved exploration followed by excellent convergence. Fig. 8 shows the behavior of the velocity update equation against the generation of particles. The velocity update equation of the SHPSO-GSA algorithm was plotted under similar conditions and is shown in Fig. 9. The effect of

Algorithm 4 Pseudo code of SHPSO-GSA

Initialization
Randomly initialize $(X_1^t, X_2^t, ..., X_{ps}^t)$ of size ps in search range $[X_{min}, X_{max}]$
Set iteration t=0
Evaluate the fitness values $(fit_1^t, fit_2^t, ..., fit_{ps}^t)$ of X
for i=1: ps
Set X_i^t to be $Pbest = (Pbest_1^t, Pbest_2^t, ..., Pbest_{ps}^t)$ for each particle
Calculate G^t, best^t, worst^t and M_i^t for i = 1,2,. . .,ps
Calculate the total force in each direction F_i^t
Calculate the $accl_i^t$ and velocity.
Calculate degree of infeasibility for $G(X_i^t)$ each particle X_i^t
Generation of Hypersphere
Generate a hypersphere of radius r centered at $Gbest$
generate ps particle inside the hypersphere
Evaluate the fitness value of each generated particle
Choose $NGbest$ as a best feasible particle
Generate ps particles $Hbest = (Hbest_1^t, Hbest_2^t, ..., Hbest_{ps}^t)$ in the hypershpere of $Pbest$
Evaluate fitness of $Hbest$
$Pbest = Best(Pbest, Hbest)$
Replace each infeasible X_i^t to each feasible in the hypersphere of $Pbest$ or replace with less infeasibility
Reproduction and Updating
while Stopping Criterion is not satisfied
$V_i^{t+1} = \omega V_i^t + c_1' r_1 ac_i^t + c_2' r_2 (Pbest_i^t - X_i^t) + c_3' r_3 (NGbest^t - X_i^t)$
$X_i^{t+1} = X_i^t + V_i^{t+1}$
Evaluate the fitness values (fit_i^t) of X
Update G_i^t, best$_i^t$, worst$_i^t$ M_i^t
Calculate the total force in each direction F_i^t
Calculate the ac_i^t and velocity
Again sort the swarm
Go to Step(2) Update $Hbest$ and $NGbest$, Using feasibility based rule
end while
end for

the incorporation of acceleration in SHPSO-GSA is obvious when these two figures are compared. In Fig. 8 the velocity is seen to undergo damping at the end of the generations, which means the algorithm has good exploration abilities, although convergence is not achieved; however, in Fig. 9 the influence of acceleration on the velocity leads to enhanced exploration abilities as well as a good convergence for SHPSO-GSA. This comparative analysis of velocities is able to explain the benefit of incorporation of acceleration in the velocity update equation.

4.2. Convergence Condition for SHPSO-GSA

In this section, a theoretical analysis is performed for ensuring the convergence of the SHPSO-GSA algorithm. Let $X(t)$ be the position of any typical particle of the swarm at t^{th} iteration. The velocity update and position update equation are defined as:

$$V_i^{t+1} = \omega V_i^t + c_1' r_1 ac_i^t + c_2' r_2 (Pbest_i^t - X_i^t) + c_3' r_3 (NGbest^t - X_i^t) \tag{25}$$

$$X_i^{t+1} = X_i^t + V_i^{t+1} \tag{26}$$

Solving these two equations for a general particle

$$X^{t+1} - X(t) = \omega(X^t - X^{t-1}) + c_1' r_1 \times ac^t + c_2' r_2 (Pbest^t - X^t) + c_3' r_3 (NGbest^t - X^t) \tag{27}$$

$$\Rightarrow X^{t+1} - (1 + \omega - c_2' r_2 - c_3' r_3) X^t - \omega X^{t-1} = c_1' r_1 \times ac^t + c_2' r_2 Pbest^t + c_3' r_3 Gbest^t \tag{28}$$

Eq. 8 defines the acceleration of a particle which is a function for Gravitational Constant (G^t), Mass (M) and unit direction of $(X_i^t - X_j^t)$, the way all the entities are defined the acceleration (ac^t) becomes a scalar multiple of e^{-t},

$$\text{i.e. } ac^t \approx Ce^{-t} \tag{29}$$

where C is a constant. Replacing the value of ac^t from Eq. 29 to Eq. 28, we have

$$X^{t+1} - (1 + \omega - c_2' r_2 - c_3' r_3) X^t - \omega X^{t-1} = c_1' r_1 C e^{-t} + c_2' r_2 Pbest^t + c_3' r_3 Gbest^t \tag{30}$$

The characteristic equation of Eq. 30 is given by

$$\lambda^2 - (1 + \omega - c_2' r_2 - c_3' r_3)\lambda + \omega = 0 \tag{31}$$

$$\lambda_1 = \nu + \sqrt{\nu^2 - 4\omega} \tag{32}$$

$$\lambda_2 = \nu - \sqrt{\nu^2 - 4\omega} \tag{33}$$

where $\nu = (1 + \omega - c_2' r_2 - c_3' r_3)$, without loss of generality we may assume that at $t = 0$, $X(t) = X_0$ and at $t = 1$, $X(t) = X_1$. Using these initial conditions, the complementary solution of Eq. 30 will be

$$X_*^t = A\lambda_1^t + B\lambda_2^t \tag{34}$$

where $A = \frac{X_0\lambda_2 - X_1}{\lambda_1 - \lambda_2}$ and $B = \frac{\lambda_1 X_0 - X_1}{\lambda_1 - \lambda_2}$.

The particular solution will be

$$X_p^t = \frac{c_1' r_1 C e^{-t}}{1 + \omega + \nu} + \frac{c_2' r_2 Pbest^t - c_3' r_3 NGbest^t}{c_2' r_2 + c_3' r_3} \tag{35}$$

The general solution of the Eq. 30 will be

$$X^t = X_*^t + X_p^t \tag{36}$$

Since

$$\lim_{t\to\infty} \frac{c_1' r_1 C e^{-t}}{1 + \omega + \nu} \to 0$$

, therefore X^t will converge to a limit point if and only if

$$\lim_{t\to\infty} A\lambda_1^t + B\lambda_2^t = 0 \tag{37}$$

When $\nu^2 - 4\omega \geq 0$ then, λ_1 and λ_2 must be real. In this case a necessary and sufficient condition for the convergence will be $max \| \lambda_1 \|, \| \lambda_2 \| < 1$. In other case when they are not real then $\lambda_1 = Re^{i\theta}$ and $\lambda_2 = Re^{-i\theta}$, where $R = \sqrt{\omega}$ and $\theta = arctan \frac{\sqrt{4\omega - \nu^2}}{\nu}$, ω is a constant and in this case the necessary and sufficient condition for the convergence will be $\| R \| < 1$, i.e. $max \| \lambda_1 \|, \| \lambda_2 \| < 1$.

Hence in both the cases the presented algorithm converges with the above condition. Here r_1, r_2 and r_3 are the uniform random variables, therefore the expected values of the $c'_1 r_1$, $c'_2 r_2$ and $c'_3 r_3$ may be calculated with the help of the probability density function of uniform random variable.
therefore

$$E[c'_1 r_1] = c'_1 \int_0^1 \frac{x}{1-0} dx = \frac{c'_1}{2} \tag{38}$$

similarly,

$$E[c'_2 r_2] = \frac{c'_2}{2} \tag{39}$$

$$E[c'_3 r_3] = \frac{c'_3}{2} \tag{40}$$

Hence using the expected values the limit becomes

$$\lim_{t \to \infty} X^t = \frac{c'_2 Pbest + c'_3}{c_2 + c_3} \tag{41}$$

In general for arbitrary values of c'_2 and c'_3 any typical particle will converge to the following point

$$\frac{c'_2 Pbest + c'_3 NGbest}{c_2 + c_3}$$

This formally proves the trajectory of a particle of SHPSO-GSA will converge to a point which is a convex combination of $Pbest$ and $NGbest$.

4.3. Converging Ability

The converging ability of the proposed SHPSO-GSA algorithm was tested by applying it over the spherical function for a thirty-dimensional environment and the positions of the particles were observed after 200, 500, 800 and 1200 iterations are observed. A three dimensional environment of the above experiment is depicted in Fig. 10. Fig. 10(a) shows the initial population, In the Fig. 10(b) the position of the points are depicted after 200 iterations and the position of the points reaches up to the order of 10^{-11}. After 500 iterations the position of the points converges to the order of 10^{-41}, see Fig. 10(c). At the level of 800 iterations the position converges to the order of 10^{-50} as shown by the positions indicated in Fig. 10(d). Finally, after 1200 iterations the convergence of the positions approaches zero (10^{-76}), see Fig. 10(e).

5. A New Constraint Handling Method

A parameter-free constraint-handling approach is used to ensure the feasibility of the particles. The degree of constrained violation is evaluated by using Eq. 42 and the total degree of violation of an individual x is evaluated by taking the sum of violations at each constraint, i.e. $G(x) = \sum_{j=1}^{m} G_j(x)$. In each iteration the swarm is sorted in the following three ways:

i. The feasible solutions are listed in front of the infeasible solutions.

ii. The feasible solutions are sorted in ascending order of their objective function values. The first particle of the list will be the $Gbest$ and the best feasible particle in the global hypersphere will be the $NGbest$.

iii. The infeasible solutions are sorted in ascending order of their degree of constrained violations.

A schematic diagram is presented in Fig. 11 to facilitate understanding of the sorting method applied to the swarm. After sorting the swarm in the above manner, a new feasibility-based rule is added to process the swarm in the next iteration.

1. If all the particles in the swarm are feasible then process them to the next iteration after sorting in ascending order of their objective function value.
2. If there is an infeasible particle in the swarm, then the particles in the hypersphere of that individual will considered for the replacement. For example, if at any iteration t the i^{th} particle of the swarm is infeasible then the particles of hypersphere of $Pbest_i^t$ will be sorted in ascending order of their feasibility, if any feasible particle(s) is available in the hypersphere of $Pbest_i^t$, it will replace the i^{th} particle having the least objective function value. In case there is no feasible particle inside the hypersphere of $Pbest_i^t$, the particle having the least constraint violation will be replaced with the i^{th} particle.

The process will be repeated for all the particles in the main swarm for each constraint. Depending upon the feasibility-based rule discussed above, the total degree of constraint violation for each particle will be evaluated by using Eq. 42,

$$G(x_i) = \sum_{j=1}^{m} G_j(x_i) \tag{42}$$

where m is the number of constraints involved in the problem. In the next section, the results of the computer simulation are discussed and a comparison with other methods is presented. A schematic diagram was designed (Fig. 12) to aid the understanding of the working procedure for processing infeasible particles.

Table 3: Fine tuned parameter values for SHPSO-GSA

ω	c'_1	c'_2	c'_3	$popsize$
0.9	1.487	1.487	1.08	60

Table 4: State-of-the-art algorithms for comparison

Sr. No.	Algorithm	Sr. No.	Algorithm
1	Basic PSO [47]	7	DSS-MDE[a] [54]
2	Trelea Type I [49]	8	COMDE[b] [39]
3	Trelea Type II; [49]	9	DECV[c] [36]
4	Clerc PSO [13]	10	A-DDE[d] [37]
5	Standard PSO (SPSO 2011) [12]	11	GSA [45]
6	SHPSO [51]		

[a]Multimember DE with DSS
[b]Constrained optimization based on modified differential evolution algorithm
[c]Differential Evolution Combined Variants
[d]Adaptive Diversity Differential Evolution

5.1. Parameter Setting

The parameter values involved in the SHPSO-GSA has been determined from a rigorous fine tuning. The fine tuning of the parameters are inspired from the original SHPSO algorithm, but still since in the SHPSO-GSA a different component of acceleration is added to the algorithm, therefore based on the current scenario a rigorous fine tuning has been performed for each choice of set of parameter values. After fine tuning it has been concluded that the ranges $1.47 \leq c_1 \leq 1.49$, $1.47 \leq c_2 \leq 1.49$ *and* $1.00 \leq c_3 \leq 1.11$ *for* c_1, c_2 *and* c_3 *are suitable for most of the test problems.* $\omega = 0.9$ *came out as a best choice. On the above ranges of parameter values many experiments are done and the optimal value of the parameters are listed in Table 3.*

In order to understand the SHPSO-GSA execution, a simulation process of an example is presented in Fig. 13. This provides a clear understanding of the proposed algorithm.

6. Experimental Analysis and Results

The proposed SHPSO-GSA algorithm is tested on twenty four benchmark problems proposed in IEEE CEC 2006 [29]. The results are compared with eleven state of the algorithms, ; The experiments were performed by using the following experimental setup

Table 5: System Configuration

PC Configuration	Software Platform
Windows 7 Professional	Matlab 7.10.0.499
Intel(R), Xenon(R) CPU	(R2010a)
X5660, 2.80 GHz	License No. 301017

6.1. Experimental Setup

All the algorithms were coded on the MATLAB platform. The results were obtained after 100 runs of the experiment were performed and analyzed. The results were evaluated for a maximum of 4000 iterations with an error tolerance of 0.00001 and the MATLAB-based seed function was used to generate the same initial population for each algorithm to justify the comparison. The experiments were conducted by using the system configuration listed in Table 5.

6.2. Benchmark Criteria

The results of SHPSO-GSA were compared with the other algorithms by recording the best, worst, mean, standard deviation, and feasibility index for each function, where the feasibility index is the ratio of the number of feasible particles in the last population to the total number of particles. Detailed results for each independent experiment are listed in Table 7. The feasibility rate, success rate, and success performance are calculated using the following formulae.

i. Feasible Run: A run during which atleast one feasible solution is recorded.

   Feasibility Rate = $\frac{\text{No. of Feasible Runs}}{\text{Total Runs}}$

ii. Successful Run(Sruns): A run during which atleast one feasible solution is recorded meeting with $|\text{fit}(x) - \text{fit}(x^*)| \leq .0001$, where $f(x)$ is the known global minima and $f(x^*)$ is the obtained minima.

   Success Rate = $\frac{\text{No. of Successful Runs}}{\text{Total Runs}}$

iii. Success Performance (SP) = $\frac{\text{Avg. function evaluation of Sruns} \times \text{Total runs}}{\#\ \text{Successful runs}}$

To compare the results with other algorithms due care is taken as suggested by M. Črepinšek et. al. [16, 15, 20, 17].

6.3. Results and Discussion

The experiment was run 100 times for each problem and the results were recorded in the form of the best, mean, worst, standard deviation (STDEV), and mean infeasibility values. The maximum number of iterations was fixed at 4000 with an error tolerance of 0.00001. Table 8 presents the attempt that was made to solve the 24 test problems taken from IEEE CEC 2006 [29] using the following approaches: basic PSO, Trelea I, Trelea II, Clerc PSO, SPSO 2011, GSA, SHPSO, and SHPSO-GSA. Out of the 24 problems, none of the three algorithms were able to successfully solve two of the problems, namely $g20$ and $g22$. Interestingly, the infeasibility rate in these two cases was the lowest when SHPSO-GSA was used. For seven of the problems, namely $g05$, $g13$, $g14$, $g15$, $g17$, $g21$ and $g23$, neither the GSA nor SHPSO were able to provide a feasible solution; however, the SHPSO-GSA could provide an optimal solution with an infeasibility rate of 0. For seven of the problems, namely $g01$, $g06$, $g07$, $g10$, $g11$, $g16$ and $g18$, the GSA was unable to reach a feasible solution, whereas SHPSO and SHPSO-GSA provided an optimal solution with 0 mean infeasibility. For the problems$g01$, $g04$ to $g16$, $g18$, $g19$, $g21$, and $g24$ the SHPSO-GSA determined the best fitness value in comparison with other algorithms. For the problems $g14$ and $g21$ the SHPSO-GSA was the only algorithm capable of providing a feasible solution for the problem. Thus, for 19 out of the 24 problems, the SHPSO-GSA produced a 100% feasible population in its last iteration, thereby confirming the effectiveness of the designed constraint-handling approach. The standard deviation of the results provided by SHPSO-GSA is '0' for 17 of the problems, which justifies the converging ability of the algorithm. There are four problems in which the SHPSO-GSA outperforms the COMDE, and it has an edge in 11 problems in comparison to A-DDE and is significantly more successful than DSS-MDE and DECV. The average number of successful runs in the function evaluation also exceeded those of COMDE, A-DDE, DSS-MDE, and DECV. Hence, we were able to conclude that the behavior of SHPSO-GSA is superior to the other PSO variants and the GSA based on the benchmark of selected problems.

6.4. Success

Table 9 presents the number of successful evaluations of the average function out of 100 runs. These results show that the SHPSO-GSA successfully solved 14 of the problems with a 100% success rate, whereas the other PSO variants were only able to solve one, five, six, five, five, three, and one of the problems, respectively, with 100% success. Table 8 shows the results of the successful runs out of 100 runs for each algorithm. Finally, in Table 10, the successful performance of the SHPSO-GSA is compared with that of the other algorithms and it is clear that, for most of the problems, the performance of SHPSO-GSA is superior.

6.5. Convergence Plots

The convergence ability of SHPSO-GSA was compared with those of the other algorithms by plotting the convergence graphs for the selected problems $g01$, $g02$, $g04$, $g06$, $g07$, $g08$, $g09$ and $g10$, as shown in Figs. 14 and 15. The convergence of the proposed SHPSO-GSA is shown to be more rapid and more accurate in comparison to that of the SHPSO algorithm and the GSA.

6.6. Wilcoxon Signed Rank Test

In order to test the statistical validity of the results, pairwise Wilcoxon signed rank test [20] is used for SHPSO-GSA Vs other algorithms. The test has been conducted with $\alpha = 0.05$ significance level. The result of the test is presented in Table 11. In the comparative table +, 0 and − shows the SHPSO-GSA has outperformed, equally performed and poorly in comparison to other algorithms respectively. The result justify that for majority of the problems SHPSO-GSA has better ranking than others.

6.7. ECD Plot

The algorithms were ranked on the basis of their successful performance by plotting the empirical cumulative distribution of SP/SP_{best} listed in Table 10, where SP is the successful performance of the i^{th} problem and SP_{best} is the best successful performance of the i^{th} individual algorithm on the problem. The empirical cumulative distribution (ECD) plots compare the ability of the algorithms to perform successfully, with the algorithm that reaches the top of the plot quickly will be considered as a best performer. Fig. 16 shows the ECD plots of all the algorithms, from which the available data representing successful performance, i.e., the problems none of the algorithms succeeded in solving, are excluded. From Fig. 16 it is clear that the SHPSO-GSA performs excellently in comparison to the other algorithms because it reached the top of the graph quickly in comparison to the others.

7. Performance of SHPSO-GSA on Unconstrained Problems

In order to study the performance of SHPSO-GSA over unconstrained optimization problems. It is applied to solve CEC 2015 benchmark [6] expensive optimization test problems. All the 15 problems are solved and the results are compared with the state-of-the-art algorithms listed in Table 6

The results are listed in a form of best, worst, mean and standard deviation (stdev) of the fitness values of the corresponding problem in Table 12 and 13. The best result for each algorithm is presented in bold faces. The results are recorded for 10D and 30D structure. For 10D results, the major observations are, for F_1,

Table 6: State of the art algorithms for comparing the results of the unconstrained optimization problems CEC 2015 [6]

Sr. No.	Algorithm	Sr. No.	Algorithm
1.	MVMO WO[a] [25]	4.	MVMO WL[b] [25]
2.	AncDE[c] [46]	5.	iSRPSO[d] [32]
3.	CMAES-R[e] [34]	6.	CMAES-D[f] [34]

[a]Mean-variance mapping optimization with out Local Search
[b]Mean-variance mapping optimization with Local Search
[c]An Ancestor based Extension to Differential Evolution
[d]Improved SRPSO Algorithm
[e]Covariance Matrix Adaptation Evolution Strategy with specialized parameters
[f]Covariance Matrix Adaptation Evolution Strategy with default parameters

F_2,F_3, $F4$, F_5, F_8,F_9,F_{10}, F_{11}, F_{12}, F_{13} and F_{15} all the recorded values of SHPSOGSA is better from other six algorithms. For the functions F_7 and F_{14}, except standard deviation all other entries of SHPSOGSA is better than others. In case of 30D category, for the functions F_1,F_4, F_10, F_{12}, F_13, F_14 and F_15 all the recorded values of SHPSOGSA is better than other six algorithms. For function F_2, F_3,F_5,F_7, F_11 the SHPSOGSA records best fitness value than others. There is only one function F_6 for which the results of the SHPSOGSA are competitive with other algorithms. Except this functions the output in all the streams starting form best fitness to standard deviation the performance of SHPSOGSA is excellent in comparison to other algorithms.

8. Algorithm Complexity

The time complexity of SHPSO-GSA is studies based on the strategy defined in CEC 2015 benchmark [6]. To measure the complexity of the strategy employed is presented in Algorithm 5.

Algorithm 5 Strategy for the calculation of algorithm complexity

Run the test program below:

for i=1:1000000 **do**

$x = 0.55 + (double)i;$

$x = x + x; x = x/2; x = x * x; x = sqrt(x); x = log(x); x = exp(x); x = x/(x + 2);$

end for

Computing time for the above=T_0;

The average complete computing time for the algorithm = T_1

The complexity of the algorithm is measured by: $\frac{T1}{T0}$

The complete computing time for the algorithm with 200000 evaluations of the same

D dimensional Function is T_2

Execute step c five times and get five T_2 values. $T_2 = Mean(T_2)$

The complexity of the algorithm is reflected by: T_2, T_1, T_0, and $\frac{(T_2-T_1)}{T_0}$

The results of the algorithm complexity is listed in Table 14. The comparative analysis of the time complexity of SHPSO-GSA with MVMO WO and MVMO WL is presented in Table 15. The results are compared for 10D and 30D, it has been observed that the time complexity of SHPSOGSA is significantly less than other two algorithms. For most of the problems the time complexity of SHPSOGSA is 10 times lesser than other two algorithms. Therefore in terms of time complexity the proposed SHPSOGSA outperforms than other two existing algorithms.

A similar Wilcoxon signed rank test with 0.05 significance lever is performed for CEC 2015 problems, the outcome of the results are presented in Table 16 and 17. The test results suggests a good ranking of SHPSO-GSA over other algorithms for CEC 2015 problems.

9. Conclusion

This paper presents a new algorithm named SHPSO-GSA, which was developed by hybridizing shrinking hypersphere-based PSO with the GSA, to produce an optimizer capable of improved constraint. The need for and design of the proposed algorithm is well established and justified in various respects. The validity of the designed hybrid was tested in multiple ways with positive output. An effective constraint-handling technique, which is compatible with the proposed algorithm, was defined to ensure the feasibility of the method. The proposed constraint-handling methods were shown to work efficiently by solving 24 state-of-the-art problems and by comparing the results with other PSO variants known to deliver good results and the standard GSA. 15 state-of-the-art unconstrained optimization problems are solved. The presented results are explained and discussed in a variety of ways. The converging ability and statistical significance of the SHPSO-GSA were also established by conducting multiple experiments to prove that the proposed hybrid algorithm is an excellent constrained optimizer as justified by the theoretical as well as experimental results. The optimization ability of the SHPSO-GSA is expected to find application across a wide range of constrained optimization problems.

Acknowledgement

This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korean government (MSIP) (No. 2013R1A2A1A01013886) and National Institute of Technology Uttarakhand, India. We would like to express our gratitude toward the unknown potential reviewers who have agreed to review this article and who have provided valuable suggestions to improve the quality of the article.

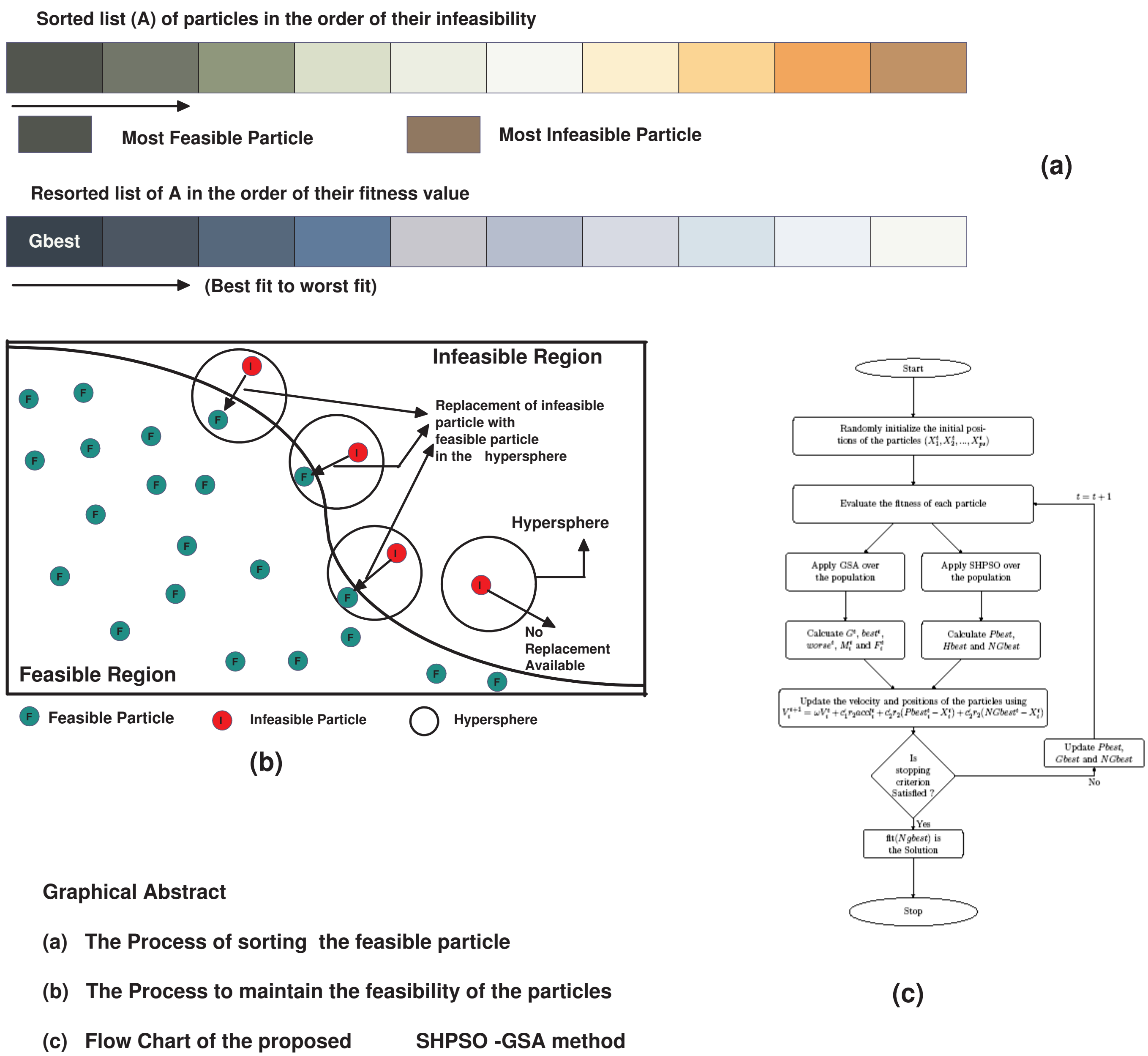


Figure 1: Graphical Abstract

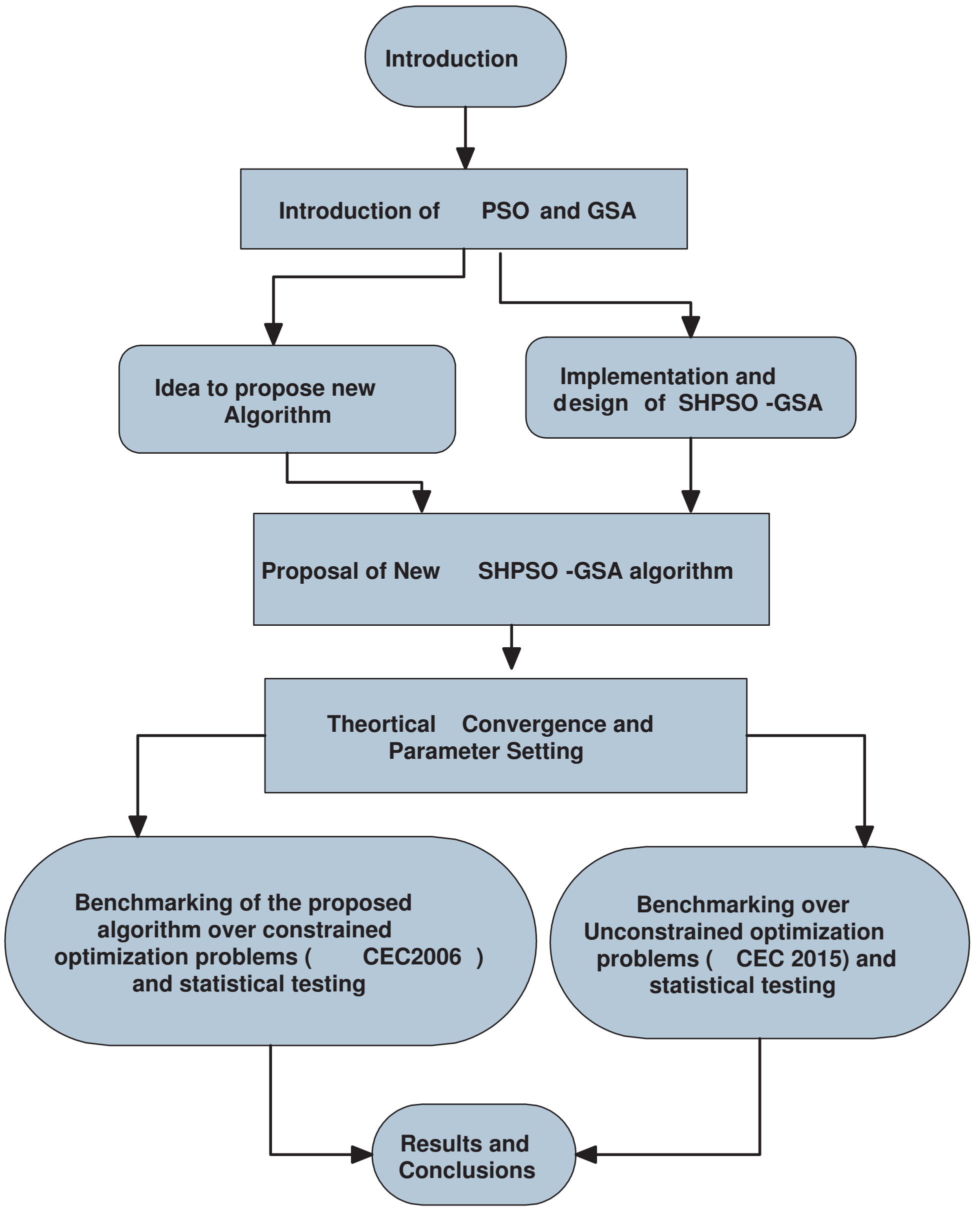


Figure 2: Flow chart of the article

Table 7: Comparative results of objective function values

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
g01	Basic PSO	-12.13134	-11.73167	-11.27491	0.29468	1
	Trelea I	**-15**	-15	-15	2.93E-08	1
	Trelea II	-14.97134	-14.94020	-14.88136	0.029358	1
	Clerc PSO	**-15**	-14.86231	-13.80655	0.35855	1
	SPSO 2011	-12.93994	-11.30763	-9.15690	1.11479	1
	GSA	-	-	-	-	0
	SHPSO	-14.9997	-14.9865	-14.6063	0.0718	1
	SHPSO-GSA	**-15.0000**	-15.0000	-15.0000	0.0000	1
	COMDE	**-15.0000**	-15.0000	-15.0000	1.97E-13	-
	A-DDE	**-15.0000**	-15.0000	-15.0000	7.00E-06	-
	DSS-MDE	**-15.0000**	-15.0000	-15.0000	1.30E-10	-
	DECV	**-15.0000**	-14.8550	-13.000	4.59E-01	
g02	Basic PSO	-0.38956	-0.34746	-0.30882	0.02347	1
	Trelea I	-0.71391	-0.50644	-0.40666	0.10567	1
	Trelea II	-0.46005	-0.40574	-0.36998	0.02962	1
	Clerc PSO	-0.78936	-0.73797	-0.61741	0.05469	1
	SPSO 2011	-0.47152	-0.42630	-0.37556	0.02880	1
	GSA	-0.24230	-0.0983	-0.0465	0.0280	1
	SHPSO	-0.78630	-0.7595	-0.3089	0.0853	1
	SHPSO-GSA	**-0.80369**	-0.80315	-0.8024	1.3E-6	1
	COMDE	**-0.80369**	-0.80361	-0.80123	5.0E-03	-
	A-DDE	-0.80360	-0.771090	-0.60985	3.66E-02	-
	DSS-MDE	-0.80361	-0.78697	-0.72853	1.50E-02	-
	DECV	-0.70400	-0.56945	-0.23820	9.51E-02	-
g03	Basic PSO	-0.53934	-0.24110	-0.06084	0.14432	0.077
	Trelea I	-0.59083	-0.34578	-0.11216	0.17087	0.13

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	Trelea II	-0.66021	-0.29456	-0.09025	0.17138	0.11
	Clerc PSO	-0.60807	-0.42095	-0.25371	0.11676	0.17
	SPSO 2011	-0.79984	-0.59903	-0.43980	0.10907	0.28
	GSA	-	-	-	-	0.2
	SHPSO	-0.59400	-0.17800	-0.00440	0.1877	0.6
	SHPSO-GSA	**-1.00000**	-1.00000	-0.99239	0.367E-10	**0.95**
	COMDE	**-1.00000**	-1.00000	-0.99999	0.302E-8	-
	A-DDE	**-1.00000**	-1.00000	-1.00000	**9.30E-12**	-
	DSS-MDE	**-1.00000**	-1.00000	-1.00000	1.90E-08	-
	DECV	-0.46100	-0.13400	-0.00200	1.17E-01	-
g04	Basic PSO	-30647.57136	-30630.49048	-30619.48588	8.97067	1
	Trelea I	-30665.53900	-30665.53816	-30665.5378	0.000277	1
	Trelea II	-30665.53800	-30665.53444	-30665.52728	0.00303	1
	Clerc PSO	-30644.89244	-30576.40504	-30458.78388	55.85832	1
	SPSO 2011	-30565.58060	-30424.19684	-30307.45752	83.34507	1
	GSA	-29510.0520	-27327.3890	-23560.6830	2109.8120	0.5
	SHPSO	-30665.5380	-30665.3600	-30660.1850	0.9773	1
	SHPSO-GSA	**-30665.5390**	-30665.5390	-30665.5390	0.0000	1
	COMDE	**-30665.5390**	-30665.5390	-30665.5390	0.0000	-
	A-DDE	**-30665.5390**	-30665.5390	-30665.5390	3.20E-13	-
	DSS-MDE	**-30665.5390**	-30665.5390	-30665.5390	2.70E-11	-
	DECV	**-30665.5390**	-30665.5390	-30665.5390	1.56E-06	-
g05	Basic PSO	-	-	-	-	0
	Trelea I	-	-	-	-	0.06
	Trelea II	-	-	-	-	0
	Clerc PSO	5129.32261	5178.94962	5315.66708	63.49032	1
	SPSO 2011	5127.74289	-	5435.07466	-	0.76

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	GSA	-	-	-	-	0
	SHPSO	-	-	-	-	0.5
	SHPSO-GSA	**5126.4967**	5126.4967	5126.4967	0.0000	**1**
	COMDE	5126.4981	5126.4981	5126.4981	0.0000	-
	A-DDE	5126.4970	5126.4970	5126.4970	2.10E-11	-
	DSS-MDE	5126.4970	5126.4970	5126.4970	0.000	-
	DECV	5126.4970	5126.4970	5126.4970	0.000	-
g06	Basic PSO	-6933.28680	-6897.26958	-6855.47497	29.82939	1
	Trelea I	-6961.80842	-6961.79465	-6961.76673	0.013544	1
	Trelea II	-6960.29072	-6956.66901	-6952.08561	2.523567	1
	Clerc PSO	-6883.87603	-6464.55368	-5784.23680	371.61882	1
	SPSO 2011	-6891.97980	-5935.52414	-4666.83552	743.4924772	1
	GSA	-3744.2747	2.7931E+05	1.0208E+06	2.4217E+05	0
	SHPSO	-6961.8136	-6961.8132	-6961.8090	0.0008	1
	SHPSO-GSA	-6961.8139	-6961.8139	-6961.8139	0.0000	1
	COMDE	-6961.8138	-6961.8138	-6961.8138	0.0000	-
	A-DDE	**-6961.8140**	-6961.8140	-6961.8140	2.11E-12	-
	DSS-MDE	**-6961.8140**	-6961.8140	-6961.8140	0.0000	-
	DECV	**-6961.8140**	-6961.8140	-6961.8140	0.0000	-
g07	Basic PSO	82.04177	121.49313	173.21687	29.27512	0.97933
	Trelea I	25.61511	26.216681	27.01475	0.44408	1
	Trelea II	38.51871	41.405807	44.83348	1.95376	1
	Clerc PSO	24.61889	25.193279	26.54658	0.59327	1
	SPSO 2011	41.01492	108.01448	320.67014	96.72589	1
	GSA	850.4374	2103.3302	4824.1543	1076.3890	0.2
	SHPSO	24.9019	25.1638	27.9587	0.5415	1
	SHPSO-GSA	**24.3062**	24.3062	24.3062	0.0000	1

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	COMDE	**24.3062**	24.3062	24.3062	4.7E-07	-
	A-DDE	**24.3060**	24.3060	24.3060	4.20E-05	-
	DSS-MDE	**24.3060**	24.3060	24.3060	7.50E-07	-
	DECV	**24.3060**	24.7940	29.5110	1.37E+00	-
g08	Basic PSO	-0.09582	-0.09582	-0.09582	3.44E-14	1
	Trelea I	-0.09582	-0.09582	-0.09582	1.38E-17	1
	Trelea II	-0.09582	-0.09582	-0.09582	1.50E-17	1
	Clerc PSO	-0.09582	-0.09582	-0.09582	1.61E-17	1
	SPSO 2011	-0.09582	-0.09582	-0.09582	1.60E-17	1
	GSA	**-0.09580**	0.5679	53.5663	6.4742	0.7
	SHPSO	**-0.09580**	-0.0958	-0.0958	0.0000	1
	SHPSO-GSA	**-0.09580**	-0.09580	-0.09580	0.0000	1
	COMDE	**-0.09580**	-0.0958	-0.0958	9.00E-18	-
	A-DDE	**-0.09580**	-0.0958	-0.0958	9.10E-10	-
	DSS-MDE	**-0.09580**	-0.0958	-0.0958	4.00E-17	-
	DECV	**-0.09580**	-0.0958	-0.0958	4.23E-17	-
g09	Basic PSO	711.16899	730.88847	747.02641	11.59183	1
	Trelea I	680.74916	680.86869	680.99542	0.072776	1
	Trelea II	684.51697	686.22695	688.10022	1.218367	1
	Clerc PSO	680.67590	680.74817	680.83062	0.044645	1
	SPSO 2011	680.87510	689.21798	732.40846	15.15200	1
	GSA	836.16420	1.4309E+06	9.5128E+06	2.2672E+06	0.8
	SHPSO	680.64850	680.69560	680.7703	0.0234	1
	SHPSO-GSA	**680.63000**	680.63000	680.63000	0.0000	1
	COMDE	680.63001	680.63001	680.63001	4.071E-13	-
	A-DDE	680.63001	680.63001	680.63001	1.15E-10	-
	DSS-MDE	680.63001	680.63001	680.63001	2.90E-13	-

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	DECV	680.63001	680.63001	680.63001	3.45E-07	-
g10	Basic PSO	10785.19568	11901.53736	13316.83004	830.65975	0.8
	Trelea I	7831.87690	8203.07587	8951.75208	318.29829	1
	Trelea II	9184.23861	10081.46538	10707.05976	536.45609	0.9
	Clerc PSO	7063.0057	7208.54728	7522.17599	151.91782	1
	SPSO 2011	7702.34856	11559.18985	16060.10486	2948.23499	1
	GSA	-	-	-	-	0.2
	SHPSO	7331.9543	7590.6267	12288.9460	888.4462	1
	SHPSO-GSA	**7049.2480**	7049.2480	7049.2480	0.0000	1
	COMDE	**7049.2480**	7049.2480	7049.2480	1.5E-04	-
	A-DDE	**7049.2480**	7049.2480	7049.2480	3.23E-04	-
	DSS-MDE	**7049.2480**	7049.2490	7049.2550	1.40E-03	-
	DECV	**7049.2480**	7103.5480	7808.9800	1.48E+02	-
g11	Basic PSO	0.75090	0.76917	0.82647	0.02183	0.1
	Trelea I	0.75073	0.77771	0.82684	0.02708	0.37
	Trelea II	0.75058	0.75904	0.78710	0.01103	0.21
	Clerc PSO	0.75293	0.77908	0.82047	0.02635	0.59
	SPSO 2011	0.75033	0.77466	0.88049	0.04422	0.68
	GSA	0.84520	1.6221	4.5674	1.16550	0.9
	SHPSO	0.75000	0.8220	0.9997	0.08270	1
	SHPSO-GSA	0**.74990**	0.74990	0.74990	0.0000	1
	COMDE	0.74991	0.74991	0.74991	0.0000	-
	A-DDE	0.75000	0.75000	0.75000	5.35E-15	-
	DSS-MDE	0.74991	0.74991	0.74991	0.0000	-
	DECV	0.75000	0.75000	0.75000	1.12E-16	-
g12	Basic PSO	-0.99999	-0.99998	-0.99996	1.15E-05	1
	Trelea I	-1.00000	-1.00000	-1.00000	0.00000	1

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	Trelea II	-1.00000	-1.00000	-1.00000	0.00000	1
	Clerc PSO	-1.00000	-1.00000	-1.00000	0.00000	1
	SPSO 2011	-1.00000	-1.00000	-1.00000	0.00000	1
	GSA	-0.99990	-0.82040	-0.87030	0.00290	0.99
	SHPSO	-1.00000	-1.00000	-1.00000	0.00000	1
	SHPSO-GSA	-1.00000	-1.00000	-1.00000	0.00000	1
	COMDE	-1.00000	-1.00000	-1.00000	0.00000	-
	A-DDE	-1.00000	-1.00000	-1.00000	4.10E-09	-
	DSS-MDE	-1.00000	-1.00000	-1.00000	0.00000	-
	DECV	-1.00000	-1.00000	-1.00000	0.00000	-
g13	Basic PSO	-	-	-	-	0
	Trelea I	-	-	-	-	0.07066
	Trelea II	-	-	-	-	0
	Clerc PSO	0.74535	-	0.99986	-	0.58
	SPSO 2011	0.51468	-	1.76553	-	0.96
	GSA	-	-	-	-	0.5
	SHPSO	-	-	-	-	0.9
	SHPSO-GSA	0.05390	0.05390	0.05390	0.00000	**1**
	COMDE	0.05390	0.05390	0.05390	1.4E-17	-
	A-DDE	0.05390	0.07620	0.43880	1.00E-13	-
	DSS-MDE	0.05390	0.05390	0.05390	1.00E-13	-
	DECV	0.05970	0.38240	0.99900	2.68E-01	-
g14	Basic PSO	-	-	-	-	0
	Trelea I	-	-	-	-	0
	Trelea II	-	-	-	-	0
	Clerc PSO	-	-	-	-	0
	SPSO 2011	-	-	-	-	0.26

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	GSA	-	-	-	-	0.2
	SHPSO	-	-	-	-	0.029
	SHPSO-GSA	**-47.7649**	-47.7649	-47.7649	0.0000	**1**
	COMDE	-47.7648	-47.7648	-47.7648	0.0000	-
	A-DDE	-47.7648	-47.7641	-47.7640	9.0E-06	-
	DSS-MDE	-	-	-	-	-
	DECV	-47.7648	-47.7225	-47.0365	1.62E-01	-
g15	Basic PSO	-	-	-	-	0
	Trelea I	-	-	-	-	0.00133
	Trelea II	-	-	-	-	0
	Clerc PSO	962.10129	963.84503	967.04043	1.57568	0.89133
	SPSO 2011	961.82955	964.09145	969.87642	2.95027	1
	GSA	963.95780	811.60760	983.46470	97.7669	0.1
	SHPSO	967.09500	966.81230	969.91590	0.80730	0.3
	SHPSO-GSA	961.71500	961.71500	961.71500	0.00000	1
	COMDE	961.71500	961.71500	961.71500	0.00000	-
	A-DDE	961.71500	961.71500	961.71500	0.00000	-
	DSS-MDE	-	-	-	-	-
	DECV	961.71500	961.71500	961.7150	2.31E-01	-
g16	Basic PSO	-1.89386	-1.88638	-1.87975	0.00379	1
	Trelea I	-1.90515	-1.90515	-1.90515	2.73E-09	1
	Trelea II	-1.90493	-1.90482	-1.90468	8.09E-05	1
	Clerc PSO	-1.90515	-1.88138	-1.79158	0.0397	1
	SPSO 2011	-1.80035	-1.63310	-1.45734	0.1115	1
	GSA	-	-	-	-	0.1
	SHPSO	**-1.90520**	-1.90520	-1.90520	0.00000	1
	SHPSO-GSA	**-1.90520**	-1.90520	-1.90520	0.00000	1

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	COMDE	-1.90510	-1.90510	-1.90510	0.00000	-
	A-DDE	-1.90510	-1.90510	-1.90510	0.00000	-
	DSS-MDE	-	-	-	-	-
	DECV	-1.9051	-1.9051	-1.9051	1.10E-06	-
g17	Basic PSO	-	-	-	-	0
	Trelea I	-	-	-	-	0
	Trelea II	-	-	-	-	0
	Clerc PSO	8900.87142	8988.16628	9146.99254	87.12748	1
	SPSO 2011	-	-	-	-	0.28
	GSA	-	-	-	-	0.1
	SHPSO	-	-	-	-	0.5
	SHPSO-GSA	8853.54019	8853.23453	8887.23422	0.0034	**1**
	COMDE	**8853.54002**	8856.50010	8859.84000	1.35E-02	-
	A-DDE	**8853.54002**	8854.66400	8858.87400	1.43E+00	-
	DSS-MDE	-	-	-	-	-
	DECV	8853.5412	8919.9363	8938.5710	2.59E+01	-
g18	Basic PSO	-	-	-	-	0
	Trelea I	-0.78433	-0.64646	-0.55436	0.068323	1
	Trelea II	-0.55439	-0.45036	-0.38724	0.054303	0.55
	Clerc PSO	-0.86285	-0.70750	-0.53972	0.106072	1
	SPSO 2011	-	-	-	-	0.56
	GSA	-0.29280	0.5234	193.3305	58.4214	0.33
	SHPSO	-0.64890	-0.6080	-0.1744	0.0846	1
	SHPSO-GSA	**-0.86600**	-0.86600	-0.86600	0.00000	1
	COMDE	**-0.86600**	-0.86600	-0.86600	0.00000	-
	A-DDE	**-0.86600**	-0.86600	-0.86600	0.00000	-
	DSS-MDE	-	-	-	-	-

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	DECV	**-0.86600**	-0.85960	-0.67490	3.48E-02	-
g19	Basic PSO	116.34307	149.73991	178.53261	16.00399	1
	Trelea I	52.327558	56.735485	64.54874	3.759886	1
	Trelea II	75.831398	85.461093	94.23523	5.437420	1
	Clerc PSO	39.222218	43.781070	46.80910	2.678964	1
	SPSO 2011	98.088985	131.92794	221.03201	32.37753	1
	GSA	954.23940	18959.77900	41292.36700	8886.98110	0.8
	SHPSO	40.56930	45.12000	92.89390	9.18240	1
	SHPSO-GSA	**32.65550**	32.65550	32.65550	0.00000	1
	COMDE	**32.65550**	32.6555	32.6555	1.41E-19	-
	A-DDE	**32.65550**	32.65800	32.66500	1.72E-03	-
	DSS-MDE	-	-	-	-	-
	DECV	**32.65550**	32.66050	32.78530	2.37E-02	-
g21	Basic PSO	-	-	-	-	0
	Trelea I	-	-	-	-	0
	Trelea II	-	-	-	-	0
	Clerc PSO	-	-	-	-	0
	SPSO 2011	-	-	-	-	0
	GSA	-	-	-	-	0
	SHPSO	-	-	-	-	0.2
	SHPSO-GSA	193.7245	193.7245	193.7245	0.0000	1
	COMDE	193.7245	193.7245	193.7245	1.39E-18	-
	A-DDE	193.7245	193.7245	193.7260	2.60E-04	-
	DSS-MDE	-	-	-	-	-
	DECV	193.7245	198.0905	324.7028	2.39E+01	-
g23	Basic PSO	-	-	-	-	0
	Trelea I	-	-	-	-	0

Continued on next page

Table 7 – *Continued from previous page*

Problem	Algorithm	Best	Mean	Worse	STDEV	Feas. Index
	Trelea II	-	-	-	-	0
	Clerc PSO	-	-	-	-	0
	SPSO 2011	-	-	-	-	0.04
	GSA	-	-	-	-	0
	SHPSO	-	-	-	-	0.1
	SHPSO-GSA	-400.0551	-400.0547	-400.0546	0.0000	1
	COMDE	-400.0551	-399.4551	396.2345	1.91E-01	-
	A-DDE	-400.0551	-391.4150	-367.4520	9.13E+00	-
	DSS-MDE	-	-	-	-	-
	DECV	-400.0550	-392.0296	-342.5245	1.23E+01	-
g24	Basic PSO	-5.50801	-5.50800	-5.50799	5.27E-06	1
	Trelea I	-5.50801	-5.50801	-5.50801	7.02E-16	1
	Trelea II	-5.50801	-5.50801	-5.50801	7.02E-16	1
	Clerc PSO	-5.50801	-5.50801	-5.50801	7.02E-16	1
	SPSO 2011	-5.50801	-5.50436	-5.48873	0.006172	1
	GSA	-5.42720	-3.70590	-1.20990	1.23290	0.8
	SHPSO	-5.50800	-5.50800	-5.50800	0.00000	1
	SHPSO-GSA	**-5.50801**	-5.50801	-5.50801	0.00100	1
	COMDE	-5.50801	-5.50801	-5.50801	**0.0000**	-
	A-DDE	-5.50801	-5.50801	-5.50801	3.12E-14	-
	DSS-MDE	-	-	-	-	-
	DECV	-5.50801	-5.50801	-5.50801	2.71E-15	-

Table 8: Successful Runs

Problem	Basic PSO	SPSO2011	Clerc	Trelea I	Trelea II	SHPSO	GSA	SHPSO-GSA	COMDE	A-DDE	DSS-MDE	DECV
g01	0	**100**	**100**	**100**	**100**	94	0	**100**	**100**	98	90	**100**
g02	0	0	0	0	0	0	0	**96**	90	0	0	0
g03	0	0	0	0	0	0	0	**100**	**100**	90	0	0
g04	0	0	**100**	**100**	**100**	0	0	**100**	**100**	98	95	92
g05	0	**100**	0	0	0	0	0	**100**	**100**	0	0	0
g06	60	0	**100**	**100**	**100**	0	0	**100**	**100**	**100**	**100**	**100**
g07	0	0	0	0	0	0	0	**100**	**100**	**100**	0	0
g08	100	100	100	100	100	100	100	100	100	100	100	100
g09	0	0	0	0	0	0	0	**100**	**100**	**100**	**100**	**100**
g10	0	0	0	0	0	0	0	**100**	99	95	80	70
g11	80	**100**	70	75	55	90	0	**100**	**100**	**100**	**100**	**100**
g12	80	**100**	**100**	**100**	**100**	**100**	95	**100**	**100**	**100**	**100**	**100**
g13	0	0	0	0	0	0	0	97	99	98	99	0
g14	0	0	0	0	0	0	0	100	100	0	-	0
g15	0	0	**100**	0	0	0	0	**100**	**100**	**100**	-	90
g16	0	0	80	75	70	89	0	**100**	**100**	**100**	-	**100**
g17	0	0	0	0	0	0	0	91	**100**	**100**	-	90
g18	0	80	0	0	0	0	0	**100**	**100**	90	-	0
g19	0	0	0	0	0	0	0	90	90	89	-	70
g20	0	0	0	0	0	0	0	0	-	-	-	-
g21	0	0	0	0	0	0	0	**100**	**100**	90	-	0
g22	0	0	0	0	0	0	0	0	0	0	-	0
g23	0	0	0	0	0	0	0	**100**	90	80	-	70
g24	80	90	90	85	80	**100**	0	**100**	**100**	**100**	-	**100**

Table 9: Average function evaluations of successful runs

Problem	Basic PSO	SPSO2011	Clerc	Trelea I	Trelea II	SHPSO	GSA	SHPSO-GSA	COMDE	DSS-MDE	DECV
g01	-	122166	97756	75171	56092	254808	-	128700	130000	130458	**33770**
g02	-	-	-	-	-	-	-	150000	200000	**109107**	-
g03	-	-	-	-	-	-	-	**120000**	150000	-	-
g04	-	-	62230	60163	55696	-	-	47960	50000	53444	**21687**
g05	-	114054	-	-	-	-	-	**27960**	200000	48832	64530
g06	49628	-	95190	55795	50387	-	-	**6900**	12000	19076	18429
g07	-	-	-	-	-	-	-	130000	200000	129200	**59828**
g08	50496	50726	50857	50979	51136	**1200**	2580	**1200**	4000	3101	1832.67
g09	-	-	-	-	-	-	-	**22260**	70000	42496	27866
g10	-	-	-	-	-	-	-	78420	200000	**51686**	
g11	53222	50009	50784	50540	50345	185733.3	-	**17820**	50000	498553	31771.77
g12	48902	12038	16201	20534	-	**4620**	18947.4	6600	6000	30615	7330
g13	-	-	-	-	-	-	-	169224	150000	**4061**	316734
g14	-	-	-	-	-	-	-	**49440**	50000	53073	95154.08
g15	-	-	64518	-	-	-	-	**30420**	50000	479584	316734
g16	-	-	72486	61251	53225	98831.5	-	**13620**	30000	39992	184372
g17	-	-	-	-	-	-	-	**199800**	200000	-	-
g18	-	143251	-	-	-	-	-	51000	60000	68984	**42043**
g19	-	-	-	-	-	-	-	127108	200000	138025	**86005**
g20	-	-	-	-	-	-	-	-	-	-	-
g21	-	-	-	-	-	-	-	**13000**	200000	-	-
g22	-	-	-	-	-	-	-	-	-	-	-
g23	-	-	-	-	-	-	-	**127320**	200000	401084	182492.77
g24	50121	58649	54902	53776	53788	13980	-	**3900**	5000	8744	4669

Table 10: Success Performance

Problem	Basic PSO	SPSO2011	Clerc	Trelea I	Trelea II	SHPSO	GSA	SHPSO-GSA	COMDE	DSS-MDE	DECV
g01	-	122166	97756	75171	56092	271073	-	128700	130000	144953	33770
g02	-	-	-	-	-	-	-	157894	222222	-	-
g03	-	-	-	-	-	-	-	120000	150000	-	-
g04	-	-	62230	60163	55696	-	-	47960	50000	56256	23572
g05	-	114054	-	-	-	-	-	27960	200000	-	-
g06	82713	-	95190	55795	50387	-	-	6900.0	12000	19076	18429
g07	-	-	-	-	-	-	-	13000	200000	-	-
g08	50496	50726	50857	50979	51136	1200	2580	1200	4000	3101	1832
g09	-	-	-	-	-	-	-	22260	70000	42296	27866
g10	-	-	-	-	-	-	-	78420.0	202020	64607	114285
g11	66528	50009	72549	67387	91536	206370	-	17820	50000	498553	31771
g12	61128	12038	16201	20534	-	4620	19945	6600	6000	30615	7330
g13	-	-	-	-	-	-	-	172678.1	151515	4102	-
g14	-	-	-	-	-	-	-	49440	50000	-	-
g15	-	-	64518	-	-	-	-	30420	50000	-	351926
g16	-	-	90608	81668	76036	111047	-	13620	30000	-	184372
g17	-	-	-	-	-	-	-	222000	200000	-	-
g18	-	179064	-	-	-	-	-	51000	60000	-	-
g19	-	-	-	-	-	-	-	138161	22222	-	122864
g20	-	-	-	-	-	-	-	-	-	-	-
g21	-	-	-	-	-	-	-	130000	200000	-	-
g22	-	-	-	-	-	-	-	-	-	-	-
g23	-	-	-	-	-	-	-	127320	22222	-	260704
g24	62651	65166	61002	63266	67235	13980	-	3900	5000	-	4669

Table 11: Wilcoxon signed rank test results of single-problem analysis with a significance level of α = 0.05 for CEC 2006 problems

	Pairwise Comparison of SHPSO-GSA Versus									
Prob.	Basic PSO	Trelea I	Trelea II	Clerc PSO	SPSO	GSA	SHPSO	COMDE	DSS-MDE	DECV
g01	+	+	+	+	+	+	+	0	+	0
g02	+	+	+	+	+	+	+	0	+	+
g03	+	+	+	+	+	+	+	0	+	+
g04	+	+	+	+	+	+	+	0	+	0
g05	+	+	+	0	0	+	0	0	+	+
g06	+	+	+	+	+	+	+	0	0	+
g07	+	+	+	+	−	+	+	0	0	+
g08	+	+	+	+	+	+	+	0	+	+
g09	+	+	+	+	+	+	+	0	+	+
g10	+	+	+	+	+	+	+	0	+	0
g11	−	+	−	+	+	+	+	0	+	+
g12	+	+	+	+	+	0	0	+	0	∗
g13	+	+	+	+	+	0	0	∗	+	∗
g14	0	+	0	0	+	0	0	∗	+	∗
g15	+	+	+	0	+	0	0	∗	+	∗
g16	+	+	+	+	+	0	0	∗	+	∗
g17	0	+	0	0	+	0	0	∗	+	∗
g18	+	+	+	+	+	0	0	∗	+	∗
g19	+	+	+	+	+	0	0	∗	+	∗
g23	+	+	0	0	+	0	0	∗	+	∗
g24	+	0	+	+	+	0	0	∗	∗	∗

∗ represents that there no comparison available

Table 12: Comparative results of objective function values for 10D

Algorithm	Best	Worse	Mean	Median	STDEV	Algorithm	Best	Worse	Mean	Median	STDEV
	F1					F2					
MVMO WO	1.52E+05	8.70E+06	2.08E+06	1.80E+06	2.09E+06	MVMO WO	1.31E+04	6.89E+04	4.36E+04	4.19E+04	1.63E+04
MVMO WL	2.28E-01	1.95E+03	1.93E+02	2.86E-01	5.92E+02	MVMO WL	1.16E-02	1.82E-02	1.68E-02	1.78E-02	1.91E-03
AncDE	2.16E+06	8.87E+07	1.78E+07	8.84E+06	2.25E+07	AncDE	1.41E+04	5.91E+04	3.53E+04	3.78E+04	1.49E+04
iSRPSO	4.01E+05	3.39E+07	7.40E+06	3.79E+06	9.56E+06	iSRPSO	6.71E+03	4.82E+04	3.19E+04	3.16E+04	1.06E+04
CMAES-R	4.91E+06	6.70E+08	9.26E+07	1.49E+05	1.48E+08	CMAES-R	1.52E+04	2.06E+05	6.23E+04	3.76E+04	4.45E+04
CMAES-D	1.11E+03	4.43E+07	2.49E+06	4.28E+07	9.85E+06	CMAES-D	9.48E+03	6.44E+04	3.73E+04	4.69E+04	1.67E+04
SHPSO-GSA	1.20E-11	1.20E-11	**1.20E-11**	1.20E-11	0.00E+00	SHPSO-GSA	2.83E-14	1.70E-11	2.62E-13	1.11E-12	0.00E+00
	F3						F4				
MVMO WO	3.19E+00	1.17E+01	8.11E+00	8.52E+00	2.05E+00	MVMO WO	1.38E+02	8.84E+02	4.71E+02	4.70E+02	2.27E+02
MVMO WL	7.49E+00	1.15E+01	9.40E+00	9.47E+00	1.06E+00	MVMO WL	1.30E+02	9.24E+02	4.65E+02	4.42E+02	2.42E+02
AncDE	2.48E+00	1.06E+01	5.73E+00	5.84E+00	2.11E+00	AncDE	1.03E+03	2.12E+03	1.63E+03	1.66E+03	3.21E+02
iSRPSO	3.55E+00	1.03E+01	6.60E+00	6.35E+00	1.74E+00	iSRPSO	3.67E+02	2.00E+03	9.25E+02	8.33E+02	4.54E+02
CMAES-R	3.03E+02	3.11E+02	3.07E+02	3.06E+02	2.11E+00	CMAES-R	1.42E+03	2.56E+03	2.19E+03	2.01E+03	3.06E+02
CMAES-D	3.03E+02	3.10E+02	3.07E+02	3.06E+02	1.84E+00	CMAES-D	8.83E+02	2.67E+03	1.91E+03	2.28E+03	5.72E+02
SHPSO-GSA	8.51E-02	3.60E+00	2.81E-01	1.11E+00	1.01E-01	SHPSO-GSA	2.19E+00	3.87E+00	2.49E+00	3.05E+00	2.01E-01
	F5						F6				
MVMO WO	1.18E+00	3.88E+00	2.61E+00	2.68E+00	6.68E-01	MVMO WO	2.88E-01	9.11E-01	5.66E-01	6.15E-01	1.51E-01
MVMO WL	5.76E-01	2.45E+00	1.13E+00	9.38E-01	5.91E-01	MVMO WL	7.51E-02	1.20E+00	3.26E-01	2.17E-01	2.63E-01
AncDE	1.50E+00	4.05E+00	2.60E+00	2.66E+00	6.18E-01	AncDE	2.85E-01	8.20E-01	5.50E-01	5.55E-01	1.43E-01
iSRPSO	1.25E+00	3.92E+00	2.46E+00	2.50E+00	8.04E-01	iSRPSO	3.40E-01	8.10E-01	5.29E-01	5.35E-01	1.20E-01
CMAES-R	5.02E+02	5.04E+02	5.03E+02	5.03E+02	5.83E-01	CMAES-R	6.00E+02	6.02E+02	6.01E+02	6.01E+02	2.67E-01

Continued on next page

Table 12 – *Continued from previous page*

Algorithm	Best	Worse	Mean	Median	STDEV	Algorithm	Best	Worse	Mean	Median	STDEV
CMAES-D	5.01E+02	5.04E+02	5.03E+02	5.03E+02	8.44E-01	CMAES-D	6.00E+02	6.01E+02	6.01E+02	6.01E+02	1.25E-01
SHPSO-GSA	1.44E-01	5.40E-01	1.45E-01	2.13E-01	0.00E+00	SHPSO-GSA	1.02E-01	4.36E-01	2.86E-01	3.02E-01	2.10E-01
	F7						F8				
MVMO WO	3.53E-01	1.28E+00	7.91E-01	8.67E-01	3.43E-01	MVMO WO	2.50E+00	1.04E+01	5.46E+00	5.40E+00	2.25E+00
MVMO WL	2.33E-01	1.33E+00	6.37E-01	4.56E-01	3.48E-01	MVMO WL	2.44E+00	2.29E+02	4.14E+01	1.34E+01	6.20E+01
AncDE	2.94E-01	1.29E+00	6.35E-01	5.34E-01	2.91E-01	AncDE	4.46E+00	9.02E+00	6.26E+00	6.32E+00	1.12E+00
iSRPSO	2.99E-01	1.54E+00	5.71E-01	4.62E-01	3.47E-01	iSRPSO	2.55E+00	7.52E+00	5.03E+00	4.83E+00	1.25E+00
CMAES-R	7.00E+02	7.04E+02	7.01E+02	7.01E+02	7.46E-01	CMAES-R	8.05E+02	9.54E+02	8.15E+02	8.05E+02	3.30E+01
CMAES-D	7.00E+02	7.06E+02	7.01E+02	7.01E+02	1.31E+00	CMAES-D	8.04E+02	3.69E+03	9.50E+02	8.07E+02	6.45E+02
SHPSO-GSA	1.49E-01	4.06E-01	2.42E-01	3.12E-01	5.00E-01	SHPSO-GSA	4.31E-01	1.84E+00	8.12E-01	6.00E-01	5.00E-02
	F9						F10				
MVMO WO	2.97E+00	4.26E+00	3.83E+00	3.88E+00	2.97E-01	MVMO WO	8.89E+03	4.75E+05	1.73E+05	1.03E+05	1.68E+05
MVMO WL	2.95E+00	4.56E+00	4.01E+00	4.13E+00	4.42E-01	MVMO WL	2.07E+02	8.70E+02	4.97E+02	4.45E+02	1.96E+02
AncDE	3.64E+00	4.35E+00	3.97E+00	3.93E+00	2.02E-01	AncDE	3.10E+04	6.99E+05	2.62E+05	1.98E+05	2.09E+05
iSRPSO	3.46E+00	4.51E+00	3.95E+00	3.93E+00	2.74E-01	iSRPSO	7.16E+03	1.97E+06	3.53E+05	2.53E+05	4.37E+05
CMAES-R	9.04E+02	9.04E+02	9.04E+02	9.04E+02	2.72E-01	CMAES-R	1.39E+05	2.95E+06	7.20E+05	4.01E+05	8.05E+05
CMAES-D	9.04E+02	9.05E+02	9.04E+02	9.04E+02	2.00E-01	CMAES-D	2.68E+04	3.71E+06	8.76E+05	4.44E+05	1.01E+06
SHPSO-GSA	2.84E+00	3.32E+00	3.27E+00	3.20E+00	1.00E-01	SHPSO-GSA	1.38E+02	6.30E+02	1.48E+02	1.44E+02	2.36E-02
	F11						F12				
MVMO WO	3.79E+00	1.92E+01	8.40E+00	7.72E+00	3.38E+00	MVMO WO	3.59E+01	4.51E+02	2.16E+02	1.84E+02	1.18E+02
MVMO WL	4.89E+00	3.01E+01	1.17E+01	9.23E+00	6.81E+00	MVMO WL	3.75E+01	4.26E+02	2.00E+02	2.01E+02	9.28E+01
AncDE	3.58E+00	1.05E+01	6.65E+00	6.39E+00	2.00E+00	AncDE	9.10E+01	4.44E+02	2.24E+02	2.18E+02	9.90E+01
iSRPSO	4.15E+00	1.22E+01	7.26E+00	6.60E+00	2.23E+00	iSRPSO	3.89E+01	3.64E+02	1.82E+02	1.96E+02	9.61E+01

Continued on next page

Table 12 – *Continued from previous page*

Algorithm	Best	Worse	Mean	Median	STDEV	Algorithm	Best	Worse	Mean	Median	STDEV
CMAES-R	1.11E+03	1.12E+03	1.11E+03	1.11E+03	2.16E+00	CMAES-R	1.30E+03	1.66E+03	1.48E+03	1.42E+03	1.14E+02
CMAES-D	1.10E+03	1.11E+03	1.11E+03	1.11E+03	2.79E+00	CMAES-D	1.24E+03	1.70E+03	1.43E+03	1.47E+03	1.19E+02
SHPSO-GSA	1.70E+00	5.93E+00	4.51E+00	4.50E+00	2.34E-01	SHPSO-GSA	2.19E+01	1.61E+02	2.99E+01	1.04E+02	3.54E-01
	F13						F14				
MVMO WO	3.18E+02	3.42E+02	3.26E+02	3.25E+02	6.03E+00	MVMO WO	1.93E+02	2.21E+02	2.05E+02	2.06E+02	7.29E+00
MVMO WL	3.15E+02	3.18E+02	3.16E+02	3.16E+02	7.13E-01	MVMO WL	1.94E+02	2.15E+02	2.06E+02	2.06E+02	5.37E+00
AncDE	3.13E+02	3.39E+02	3.25E+02	3.24E+02	6.91E+00	AncDE	1.96E+02	2.15E+02	2.04E+02	2.03E+02	5.33E+00
iSRPSO	3.18E+02	3.64E+02	3.31E+02	3.27E+02	1.14E+01	iSRPSO	1.94E+02	2.11E+02	2.02E+02	2.02E+02	4.90E+00
CMAES-R	1.62E+03	1.73E+03	1.65E+03	1.63E+03	2.84E+01	CMAES-R	1.60E+03	1.62E+03	1.61E+03	1.60E+03	5.34E+00
CMAES-D	1.62E+03	2.02E+03	1.66E+03	1.64E+03	8.79E+01	CMAES-D	1.59E+03	1.62E+03	1.60E+03	1.61E+03	6.74E+00
SHPSO-GSA	3.15E+02	3.15E+02	3.15E+02	3.15E+02	0.00E+00	SHPSO-GSA	1.91E+02	2.06E+02	2.00E+02	2.02E+02	1.79E-02
	F15										
MVMO WO	1.90E+01	5.55E+02	4.48E+02	4.53E+02	1.13E+02						
MVMO WL	2.37E+01	6.25E+02	4.76E+02	4.91E+02	1.27E+02						
AncDE	9.93E+00	5.03E+02	3.59E+02	4.06E+02	1.50E+02						
iSRPSO	1.23E+01	4.78E+02	3.00E+02	4.07E+02	1.78E+02						
CMAES-R	1.56E+03	2.06E+03	1.94E+03	1.91E+03	1.01E+02						
CMAES-D	1.51E+03	2.08E+03	1.90E+03	1.95E+03	1.44E+02						
SHPSO-GSA	7.06E+00	1.03E+01	9.05E+00	1.00E+01	1.01E-02						

Table 13: Comparative results of objective function values for 30D

Algorithm	Best	Worse	Mean	Median	STDEV	Algorithm	Best	Worse	Mean	Median	STDEV
Problems:	F1						F2				
MVMO WO	3.11E+06	1.02E+08	3.01E+07	1.80E+07	2.86E+07	MVMO WO	6.85E+04	2.08E+05	1.39E+05	1.40E+05	3.42E+04
MVMO WL	4.72E-01	1.89E+04	2.09E+03	4.89E+01	4.97E+03	MVMO WL	6.06E-03	7.20E-03	6.93E-03	7.10E-03	**3.24E-04**
AncDE	1.12E+08	1.49E+09	5.21E+08	4.45E+08	3.54E+08	AncDE	7.64E+04	1.76E+05	1.18E+05	1.15E+05	2.49E+04
iSRPSO	1.63E+08	1.35E+09	7.19E+08	6.02E+08	3.12E+08	iSRPSO	4.89E+04	1.05E+05	7.67E+04	7.73E+04	1.37E+04
CMAES-R	2.78E+07	3.74E+08	1.22E+08	2.26E+05	7.57E+07	CMAES-R	1.02E+05	2.18E+05	1.50E+05	1.42E+05	3.35E+04
CMAES-D	9.69E+03	1.48E+07	1.29E+06	1.23E+08	3.35E+06	CMAES-D	8.25E+04	2.08E+05	1.42E+05	1.57E+05	3.04E+04
SHPSO-GSA	3.67E-01	3.67E-01	3.67E-01	3.67E-01	0.00E+00	SHPSO-GSA	6.96E-01	5.22E+04	6.12E+02	3.15E+03	2.11E+01
	F3						F4				
MVMO WO	2.96E+01	4.22E+01	3.54E+01	3.48E+01	3.89E+00	MVMO WO	4.32E+02	1.77E+03	9.68E+02	9.55E+02	2.86E+02
MVMO WL	2.99E+01	4.34E+01	3.79E+01	3.86E+01	3.85E+00	MVMO WL	7.26E+02	2.82E+03	1.43E+03	1.35E+03	5.75E+02
AncDE	2.05E+01	3.96E+01	2.67E+01	2.59E+01	4.48E+00	AncDE	5.90E+03	7.62E+03	6.78E+03	6.81E+03	4.94E+02
iSRPSO	1.81E+01	3.36E+01	2.57E+01	2.60E+01	3.42E+00	iSRPSO	3.93E+03	6.68E+03	5.41E+03	5.45E+03	8.03E+02
CMAES-R	3.11E+02	3.23E+02	3.18E+02	3.25E+02	3.38E+00	CMAES-R	7.20E+03	8.98E+03	8.10E+03	5.44E+03	5.24E+02
CMAES-D	3.19E+02	3.33E+02	3.25E+02	3.18E+02	3.92E+00	CMAES-D	3.21E+03	8.93E+03	5.77E+03	8.15E+03	1.91E+03
SHPSO-GSA	1.80E+01	3.84E+02	1.83E+01	1.19E+02	9.03E-01	SHPSO-GSA	7.21E+00	5.69E+01	4.07E+1	4.49E+01	6.92E+01
	F5						F6				
MVMO WO	2.32E+00	5.47E+00	3.91E+00	4.03E+00	8.13E-01	MVMO WO	4.74E-01	8.95E-01	6.34E-01	6.16E-01	1.24E-01
MVMO WL	1.03E+00	2.91E+00	1.68E+00	1.62E+00	5.05E-01	MVMO WL	3.06E-01	8.15E-01	5.20E-01	5.00E-01	1.32E-01
AncDE	3.18E+00	5.34E+00	4.24E+00	4.27E+00	5.91E-01	AncDE	3.30E-01	9.31E-01	5.89E-01	5.58E-01	1.39E-01
iSRPSO	2.93E+00	5.24E+00	4.24E+00	4.41E+00	6.84E-01	iSRPSO	3.78E-01	8.07E-01	6.35E-01	6.40E-01	1.05E-01
CMAES-R	5.03E+02	5.05E+02	5.04E+02	5.04E+02	5.58E-01	CMAES-R	6.01E+02	6.01E+02	6.01E+02	6.01E+02	1.61E-01

Continued on next page

Table 13 – *Continued from previous page*

Algorithm	Best	Worse	Mean	Median	STDEV	Algorithm	Best	Worse	Mean	Median	STDEV
CMAES-D	5.01E+02	5.05E+02	5.04E+02	5.04E+02	1.07E+00	CMAES-D	6.00E+02	6.01E+02	6.01E+02	6.01E+02	1.51E-01
SHPSO-GSA	1.06E+00	9.84E+01	2.10E+00	2.12E+00	5.55E-01	SHPSO-GSA	5.26E-01	1.43E+00	7.45E-01	1.10E+00	4.62E-01
	F7						F8				
MVMO WO	1.97E-01	8.29E-01	4.57E-01	4.13E-01	1.80E-01	MVMO WO	4.48E+01	1.48E+03	1.75E+02	9.71E+01	3.12E+02
MVMO WL	3.32E-01	7.80E-01	4.39E-01	4.09E-01	9.94E-02	MVMO WL	7.19E+01	9.26E+02	4.03E+02	3.21E+02	2.64E+02
AncDE	3.31E-01	1.15E+00	5.77E-01	5.43E-01	2.08E-01	AncDE	3.93E+01	2.16E+03	4.48E+02	2.43E+02	5.51E+02
iSRPSO	2.57E-01	1.13E+00	5.68E-01	4.88E-01	2.26E-01	iSRPSO	1.12E+02	1.58E+03	6.26E+02	5.15E+02	4.77E+02
CMAES-R	7.00E+02	7.01E+02	7.01E+02	7.00E+02	3.00E-01	CMAES-R	8.19E+02	9.45E+02	8.43E+02	8.26E+02	3.11E+01
CMAES-D	7.00E+02	7.01E+02	7.01E+02	7.01E+02	3.11E-01	CMAES-D	8.20E+02	8.41E+02	8.27E+02	8.28E+02	6.64E+00
SHPSO-GSA	1.36E-01	3.23E+00	1.76E+00	1.82E+00	3.11E-01	SHPSO-GSA	2.78E+00	7.95E+00	4.23E+00	5.32E+00	1.04E-04
	F9						F10				
MVMO WO	1.32E+01	1.42E+01	1.37E+01	1.37E+01	2.90E-01	MVMO WO	2.46E+06	1.66E+07	5.82E+06	4.81E+06	3.24E+06
MVMO WL	1.21E+01	1.40E+01	1.34E+01	1.35E+01	5.20E-01	MVMO WL	1.71E+03	6.03E+05	9.29E+04	9.08E+03	1.77E+05
AncDE	1.33E+01	1.41E+01	1.38E+01	1.38E+01	2.03E-01	AncDE	6.97E+06	3.83E+07	1.81E+07	1.83E+07	7.95E+06
iSRPSO	1.32E+01	1.40E+01	1.36E+01	1.35E+01	2.54E-01	iSRPSO	1.79E+06	1.48E+07	6.83E+06	6.48E+06	3.98E+06
CMAES-R	9.13E+02	9.14E+02	9.14E+02	9.14E+02	2.37E-01	CMAES-R	4.75E+06	4.87E+07	2.01E+07	3.12E+06	1.28E+07
CMAES-D	9.14E+02	9.14E+02	9.14E+02	9.14E+02	2.18E-01	CMAES-D	1.22E+06	1.15E+07	4.12E+06	1.55E+07	2.97E+06
SHPSO-GSA	1.32E+01	8.11E+01	2.17E+01	2.21E+00	1.32E-02	SHPSO-GSA	3.29E+01	5.18E+02	7.13E+01	1.20E+02	1.67E-01
	F11						F12				
MVMO WO	2.10E+01	1.55E+02	6.42E+01	4.23E+01	4.57E+01	MVMO WO	3.50E+02	1.00E+03	7.05E+02	7.31E+02	1.99E+02
MVMO WL	4.12E+01	2.19E+02	1.43E+02	1.51E+02	5.40E+01	MVMO WL	2.99E+02	1.66E+03	8.60E+02	9.12E+02	2.95E+02
AncDE	2.79E+01	1.86E+02	4.24E+01	3.42E+01	3.44E+01	AncDE	1.09E+03	1.62E+03	1.37E+03	1.33E+03	1.51E+02
iSRPSO	2.00E+01	1.32E+02	5.09E+01	3.15E+01	3.69E+01	iSRPSO	3.17E+02	1.29E+03	7.36E+02	7.09E+02	2.64E+02

Continued on next page

Table 13 – *Continued from previous page*

Algorithm	Best	Worse	Mean	Median	STDEV	Algorithm	Best	Worse	Mean	Median	STDEV
CMAES-R	1.12E+03	1.17E+03	1.13E+03	1.12E+03	1.02E+01	CMAES-R	2.04E+03	2.87E+03	2.42E+03	1.79E+03	2.35E+02
CMAES-D	1.12E+03	1.24E+03	1.14E+03	1.13E+03	3.45E+01	CMAES-D	1.48E+03	2.50E+03	1.86E+03	2.41E+03	2.50E+02
SHPSO-GSA	1.96E+01	1.34E+02	8.02E+00	8.09E+00	3.61E-01	SHPSO-GSA	1.01E+02	2.09E+02	1.61E+02	1.90E+02	2.05E-01
	F13						F14				
MVMO WO	3.32E+02	4.13E+02	3.66E+02	3.60E+02	2.25E+01	MVMO WO	2.30E+02	3.39E+02	2.80E+02	2.72E+02	3.20E+01
MVMO WL	3.30E+02	3.63E+02	3.44E+02	3.47E+02	1.13E+01	MVMO WL	2.21E+02	3.63E+02	2.76E+02	2.70E+02	4.10E+01
AncDE	3.53E+02	3.87E+02	3.68E+02	3.66E+02	9.11E+00	AncDE	2.42E+02	3.20E+02	2.71E+02	2.66E+02	2.24E+01
iSRPSO	3.71E+02	4.25E+02	4.00E+02	4.00E+02	1.67E+01	iSRPSO	2.39E+02	2.90E+02	2.66E+02	2.66E+02	1.47E+01
CMAES-R	1.68E+03	2.03E+03	1.79E+03	1.69E+03	8.35E+01	CMAES-R	1.63E+03	1.71E+03	1.67E+03	1.63E+03	2.13E+01
CMAES-D	1.67E+03	1.75E+03	1.69E+03	1.77E+03	1.62E+01	CMAES-D	1.62E+03	1.67E+03	1.64E+03	1.66E+03	1.18E+01
SHPSO-GSA	3.27E-02	2.91E+01	1.03E+00	1.06E+00	3.45E-03	SHPSO-GSA	1.01E+02	2.17E+02	1.13E+02	1.64E+02	2.50E-02
	F15										
MVMO WO	5.12E+02	1.25E+03	9.62E+02	1.10E+03	2.70E+02						
MVMO WL	7.12E+02	1.40E+03	1.19E+03	1.22E+03	1.51E+02						
AncDE	7.80E+02	1.19E+03	9.71E+02	9.53E+02	1.23E+02						
iSRPSO	7.62E+02	1.13E+03	9.51E+02	9.76E+02	1.13E+02						
CMAES-R	2.08E+03	2.50E+03	2.30E+03	2.35E+03	1.08E+02						
CMAES-D	1.94E+03	2.53E+03	2.30E+03	2.34E+03	1.71E+02						
SHPSO-GSA	1.02E+02	8.10E+02	3.10E+02	4.21E+02	6.91E-01						

Table 14: Time Complexity of SHPSO-GSA on each expensive problem for 10D and 30D

		10D			30D		
	T_0	T_1	T_2	$\frac{T_2-T_1}{T_0}$	T_1	T_2	$\frac{T_2-T_1}{T_0}$
F1	4.7500	2.9696	13.9176	2.3048	4.8073	59.0677	11.4232
F2	4.7500	2.9696	13.3056	2.1760	4.8073	55.7915	10.7335
F3	4.7500	2.9696	74.2582	15.0081	4.8073	404.1140	84.0646
F4	4.7500	2.9696	14.7977	2.4901	4.8073	57.7969	11.1557
F5	4.7500	2.9696	41.5659	8.1255	4.8073	438.6437	91.3340
F6	4.7500	2.9696	49.2907	9.7518	4.8073	55.7731	10.7296
F7	4.7500	2.9696	50.8997	10.0905	4.8073	56.2839	10.8372
F8	4.7500	2.9696	52.4701	10.4212	4.8073	58.8311	11.3734
F9	4.7500	2.9696	49.1310	9.7182	4.8073	59.3111	11.4745
F10	4.7500	2.9696	195.2342	40.4768	4.8073	61.9702	12.0343
F11	4.7500	2.9696	71.3128	14.3880	4.8073	126.5688	25.6340
F12	4.7500	2.9696	57.1012	11.3961	4.8073	80.7357	15.9849
F13	4.7500	2.9696	61.7599	12.3769	4.8073	103.0586	20.6845
F14	4.7500	2.9696	59.3629	11.8723	4.8073	94.5932	18.9023
F15	4.7500	2.9696	200.1465	41.5109	4.8073	451.5639	94.0540

Table 15: Comparison of Time Complexity $\frac{(T_2-T_1)}{T_0}$

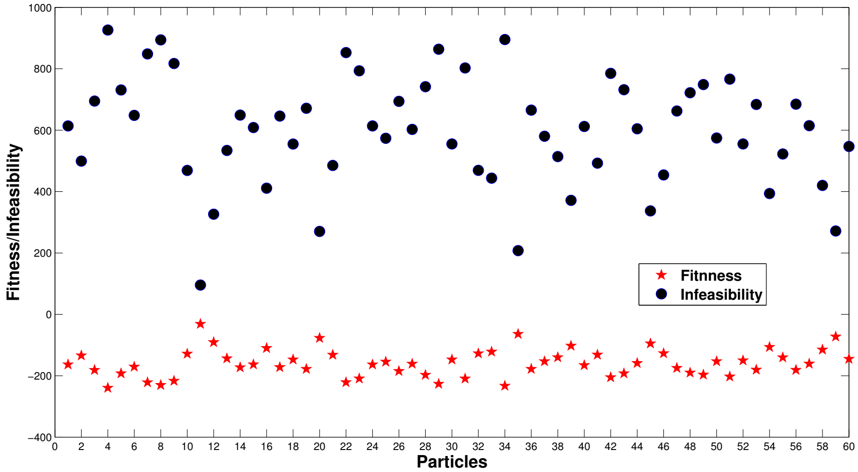
Func.	MVMO WO	MVMO WL	SHPSOGSA	MVMO WO	MVMO WL	SHPSOGSA
	(10D)	(10D)	(10D)	(30D)	(30D)	(30D)
F1	32.03	42.91	2.30	46.48	52.11	11.42
F2	38.48	52.29	2.18	65.03	74.9	10.73
F3	44.79	61.23	15.01	84.66	100.04	84.06
F4	51.14	69.17	2.49	105.53	122.76	11.16
F5	56.61	78.79	8.13	123.16	144.76	91.33
F6	62.64	87.32	9.75	140.92	169.29	10.73
F7	69.11	95.35	10.09	159.63	193.02	10.84
F8	74.56	102.92	10.42	184.37	215.07	11.37
F9	80.34	110.72	9.72	201.37	233.74	11.47
F10	86.09	118.12	40.48	224.29	251.32	12.03
F11	92.14	126.06	14.39	247.41	273.15	25.63
F12	97.69	135.37	11.4	268.59	296.29	15.98
F13	104.26	142.41	12.38	293.76	318.7	20.68
F14	111.64	149.99	11.87	319.77	338.19	18.9
F15	117.41	158.86	41.51	346.42	365.31	94.05

Table 16: Wilcoxon signed rank test results of single-problem analysis with a significance level of $\alpha = 0.05$ CEC 2015 10D problems

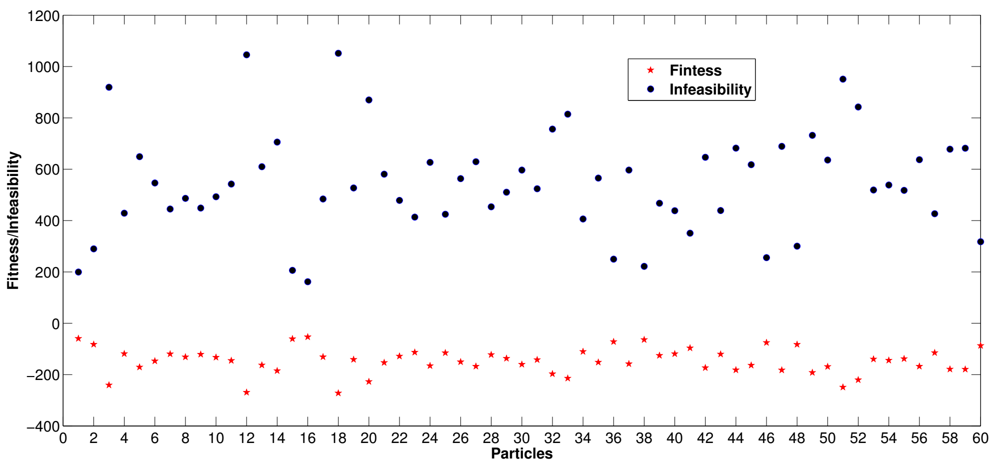
	Function Pairwise Comparison of SHPSO-GSA Versus					
Function	MVMO WO	MVMO WL	AncDE	iSRPSO	CMAES-R	CMAES-D
F1	+	0	+	+	+	+
F2	+	0	+	+	+	+
F3	+	+	+	+	+	+
F4	+	+	+	+	+	+
F5	0	0	+	+	+	+
F6	0	−	0	0	+	+
F7	0	0	0	0	0	+
F8	+	+	+	+	+	+
F9	0	0	0	0	+	+
F10	+	0	+	0	+	+
F11	0	0	0	0	+	+
F12	0	0	0	0	+	+
F13	0	0	−	+	0	0
F14	0	−	0	0	+	+
F15	+	+	0	+	+	+
+/0/−	7/8/0	4/9/2	7/7/1	8/7/0	13/2/0	14/1/0

Table 17: Wilcoxon signed rank test results of single-problem analysis with a significance level of $\alpha = 0.05$ CEC 2015 30D problems

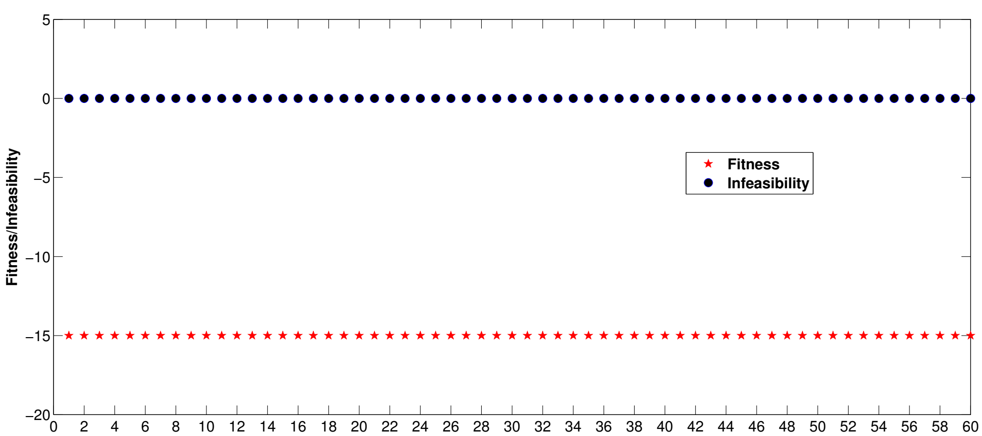
	Pairwise Comparison of SHPSO-GSA Versus					
Function	MVMO WO	MVMO WL	AncDE	iSRPSO	CMAES-R	CMAES-D
F1	+	0	+	+	+	+
F2	+	−	+	+	+	+
F3	0	0	0	0	+	+
F4	+	+	+	+	+	+
F5	0	0	0	0	+	+
F6	0	0	0	0	+	+
F7	0	0	0	0	+	+
F8	0	0	0	+	+	+
F9	0	−	0	0	+	+
F10	+	−	+	+	+	+
F11	0	0	0	0	+	+
F12	0	0	+	0	+	+
F13	+	+	+	+	+	+
F14	0	0	0	0	+	+
F15	0	0	0	0	+	+
+/0/−	5/10/0	2/10/3	6/9/0	6/9/0	15/0/0	15/0/0



(a) Initial position of fitness and infeasibility of last population for the problem g01

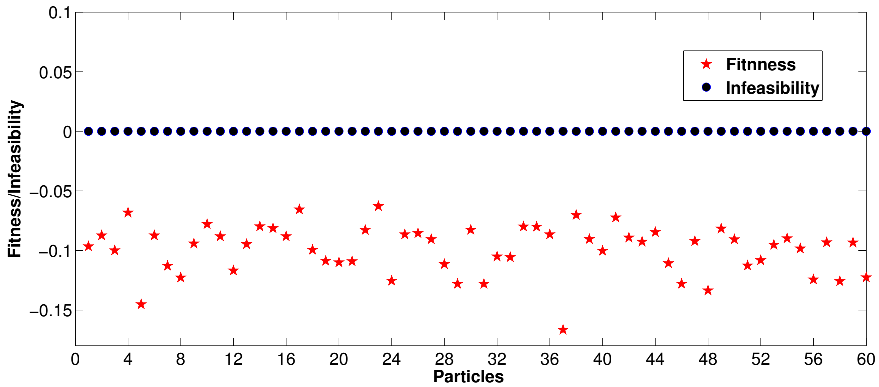


(b) Behavior of fitness and infeasibility of last population by GSA for the problem g01

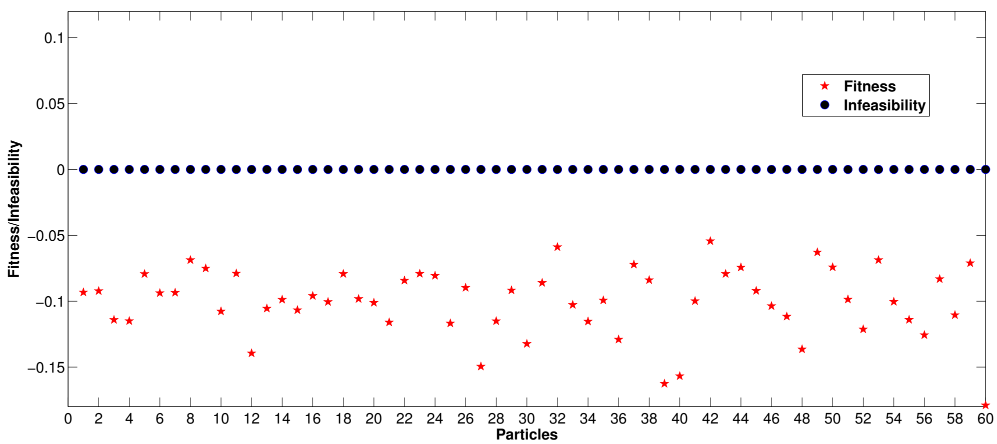


(c) Behavior of fitness and infeasibility by SHPSOGSA for the problem g01

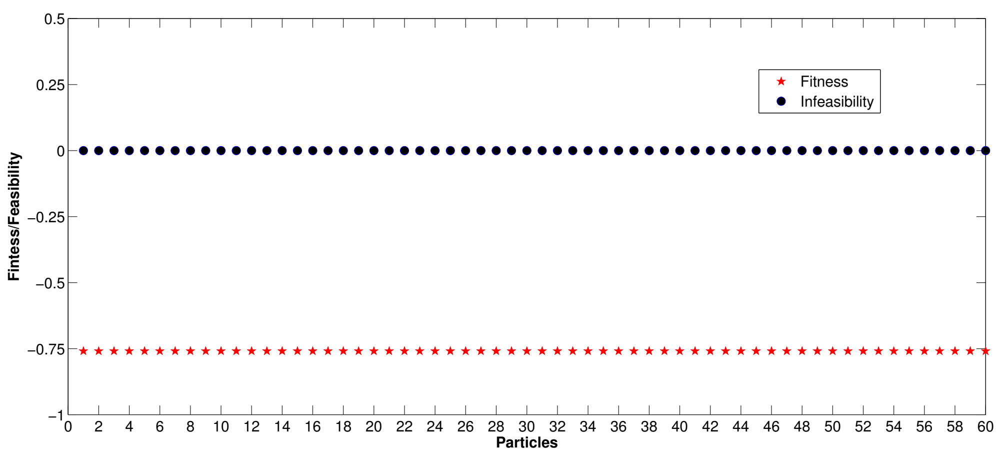
Figure 3: Comparison of fitness and infeasibility of the last population for 4000 iterations for the problem $g01$



(a) Initial position of fitness and infeasibility for the problem g02



(b) Behavior of fitness and infeasibility of last population by GSA for the problem g02



(c) Behavior of fitness and infeasibility of last population by SHPSOGSA for the problem g02

Figure 4: Comparison of fitness and infeasibility of the last population for 4000 iterations for the problem g02

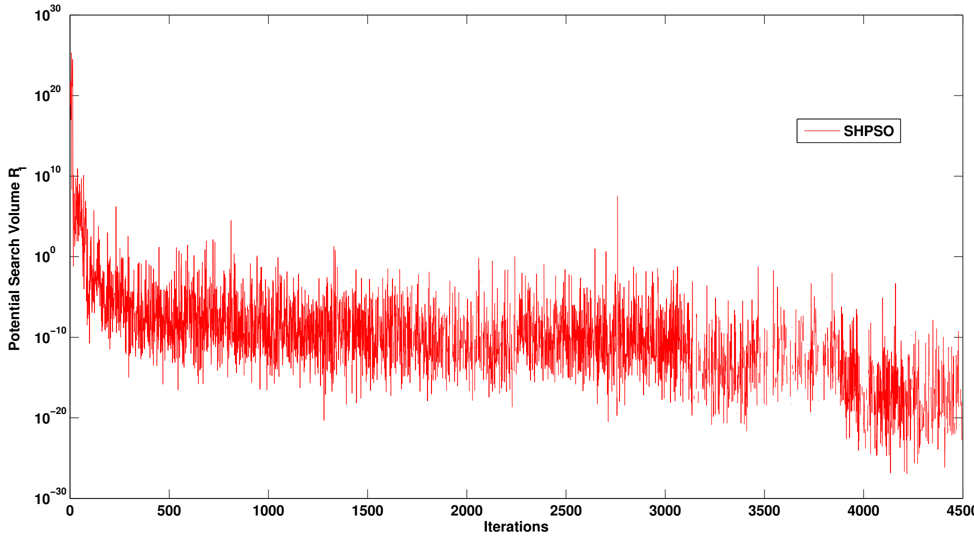


Figure 5: Potential Search Range of SHPSO against iterations

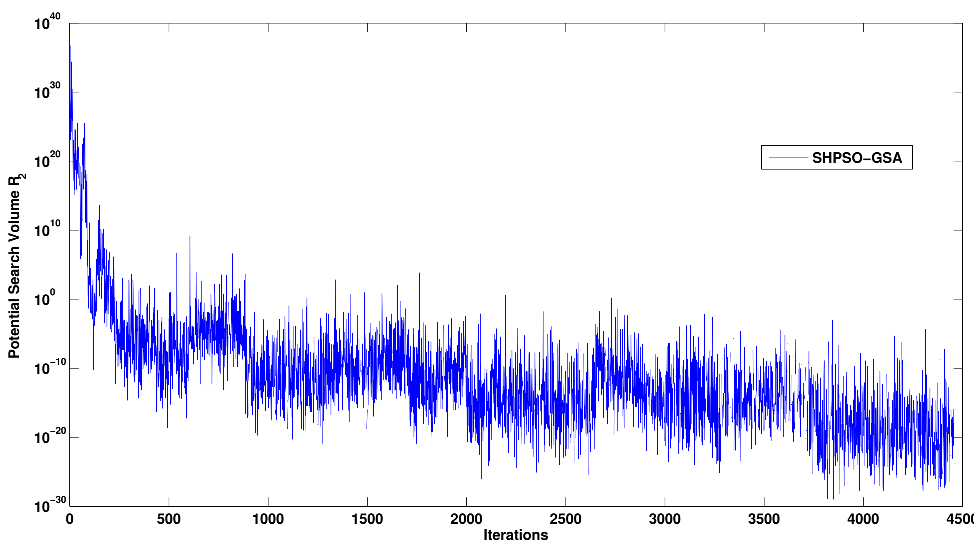
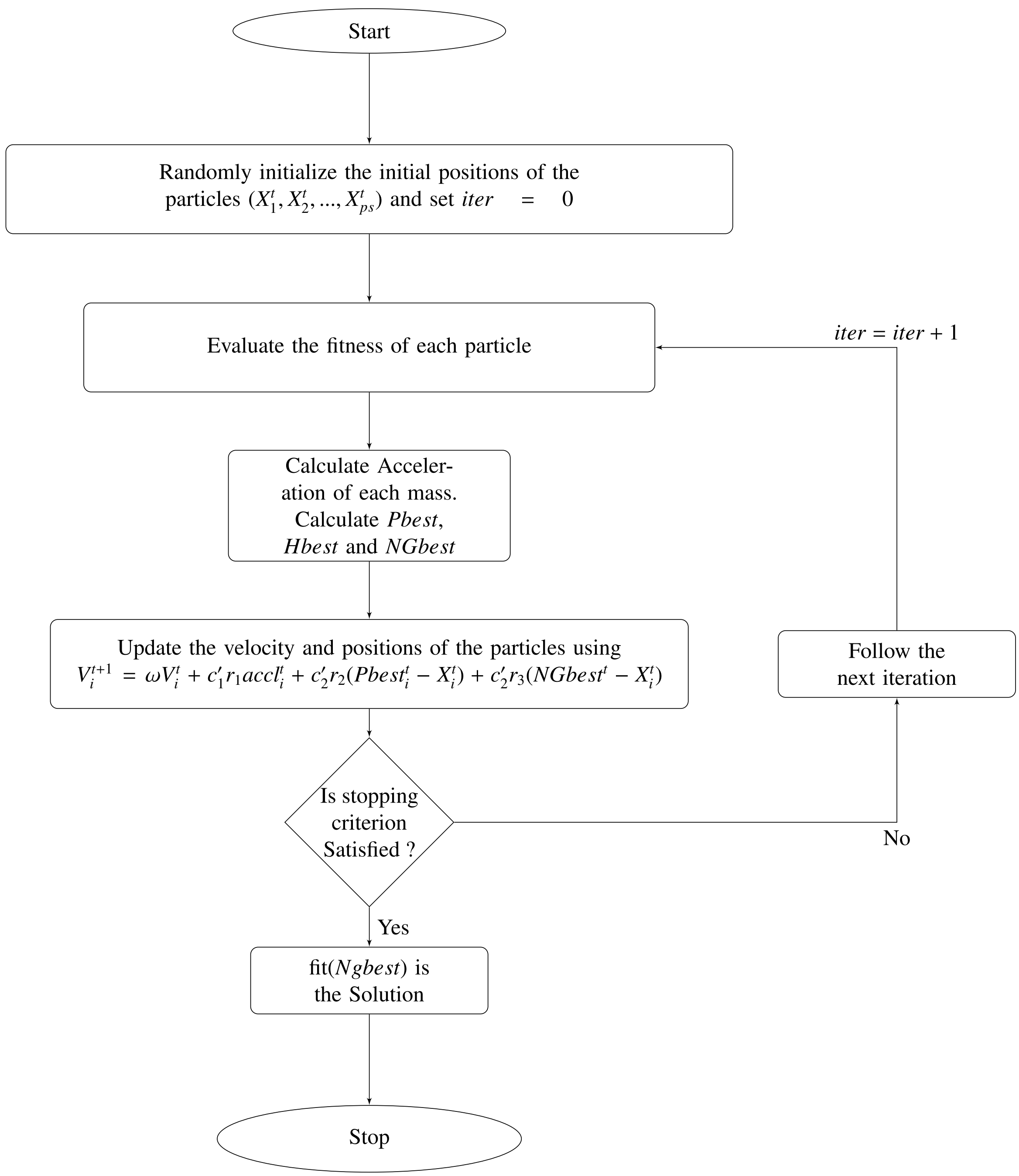


Figure 6: Potential Search Range of SHPSO-GSA against iterations

Figure 7: The flow chart of SHPSO-GSA



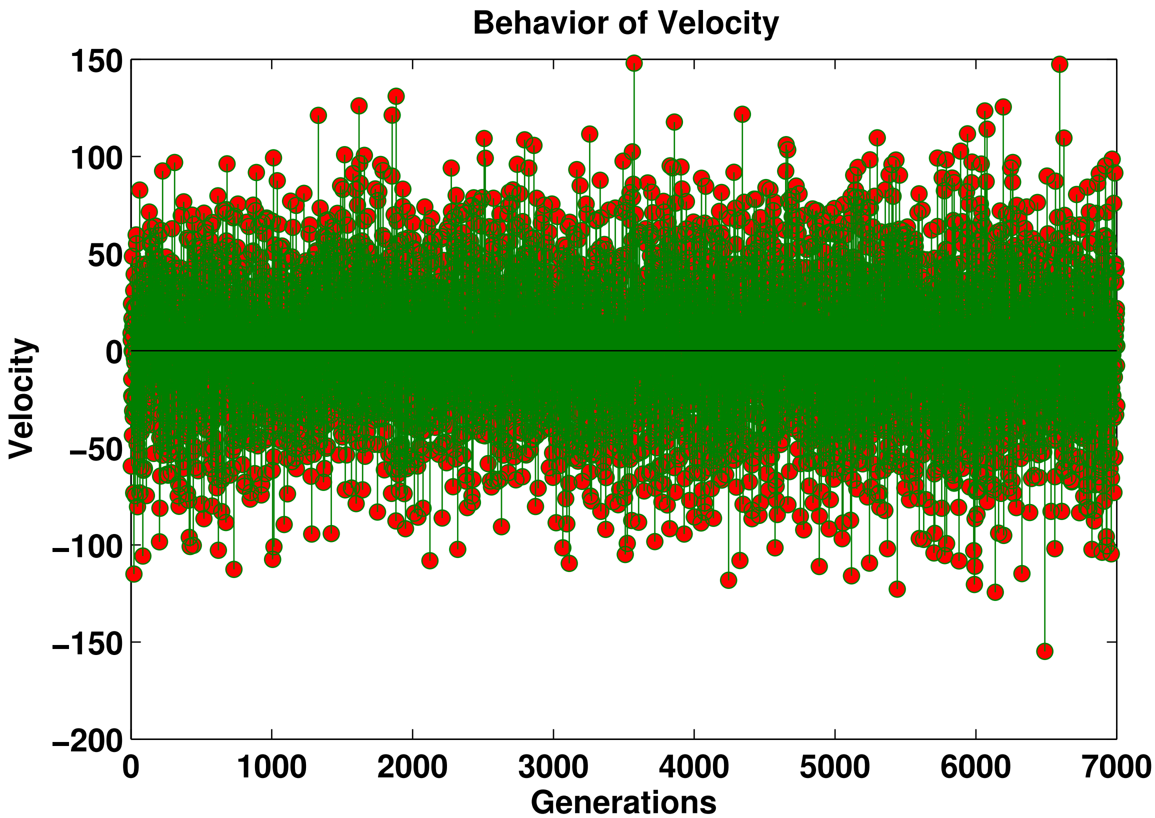


Figure 8: The behavior of velocity update equation of original PSO

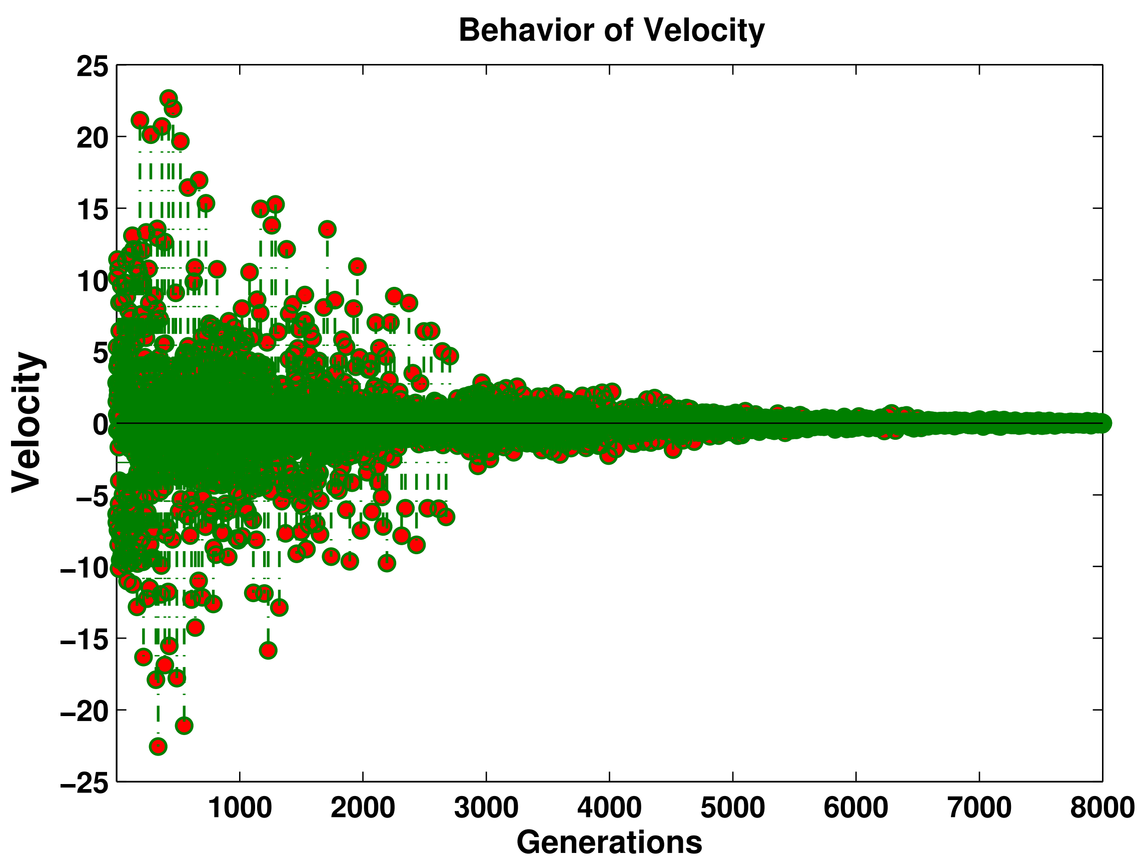
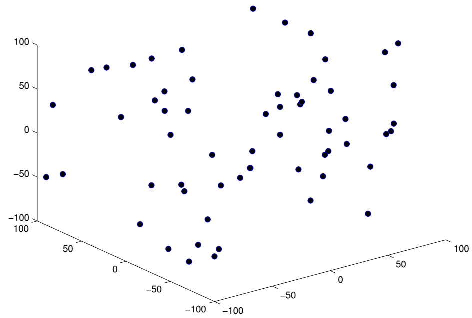
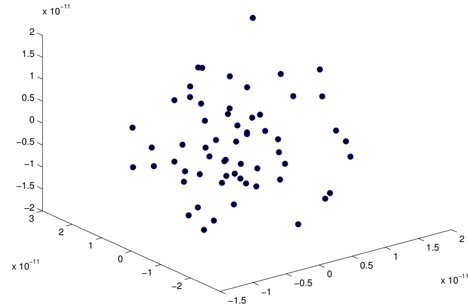


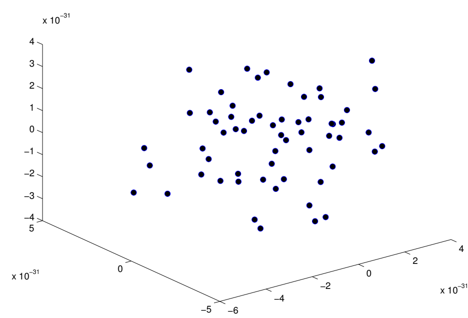
Figure 9: The behavior of velocity update equation of the proposed SHPSO-GSA



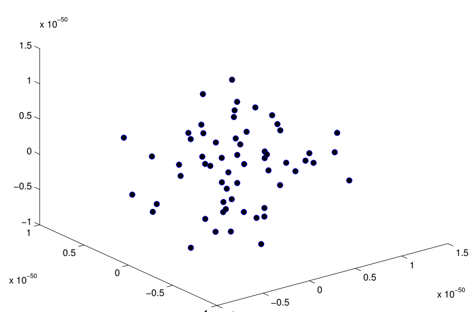
(a) Particles position at initialization



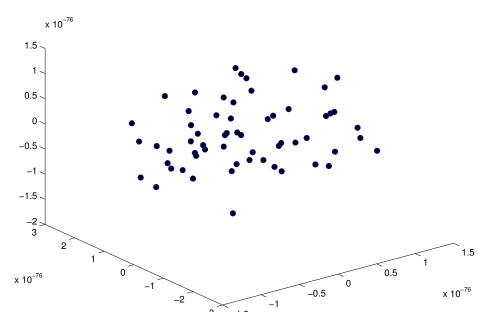
(b) Particles positions after 200 iterations



(c) Particles positions after 500 iterations



(d) Particles positions after 800 iterations



(e) Particles positions after 1200 iterations

Figure 10: Iteration wise position of the particles in the search space for the Sphere function

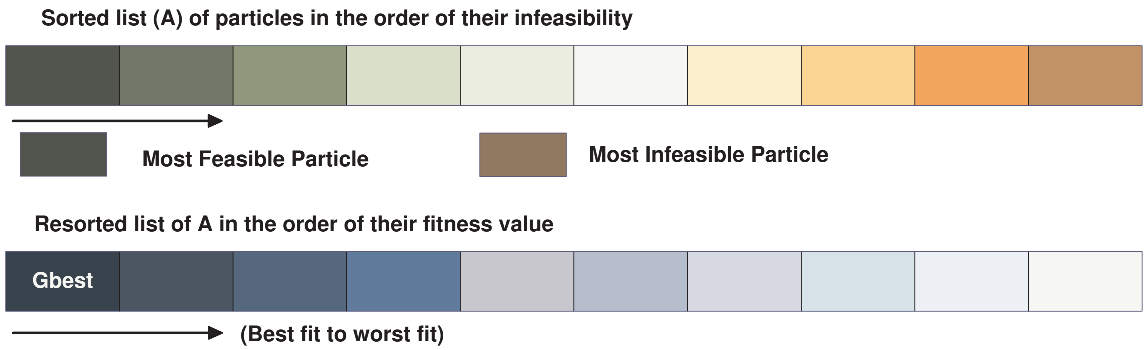


Figure 11: Schematic diagram of Sorting of the swarm for the identification of Gbest

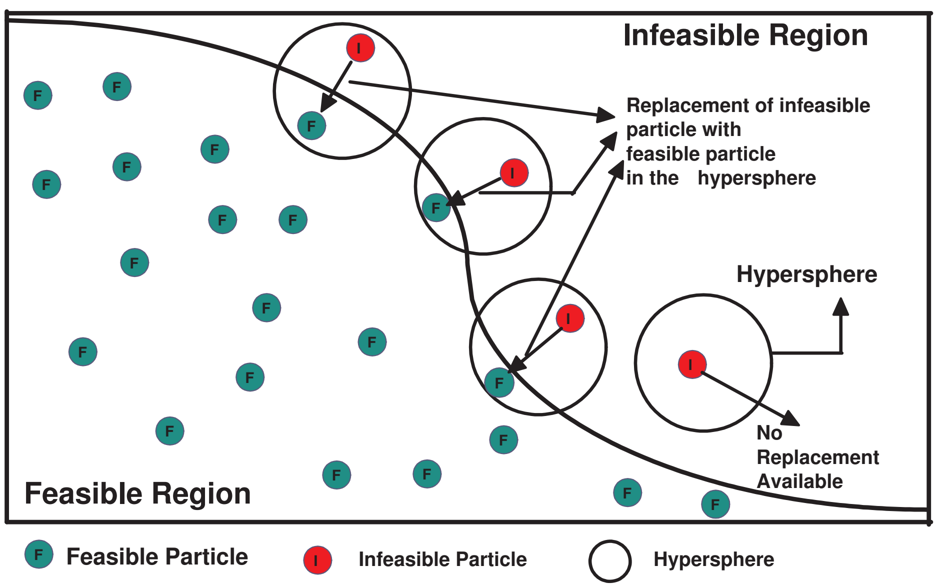


Figure 12: Schematic diagram of working procedure of the constrained handling

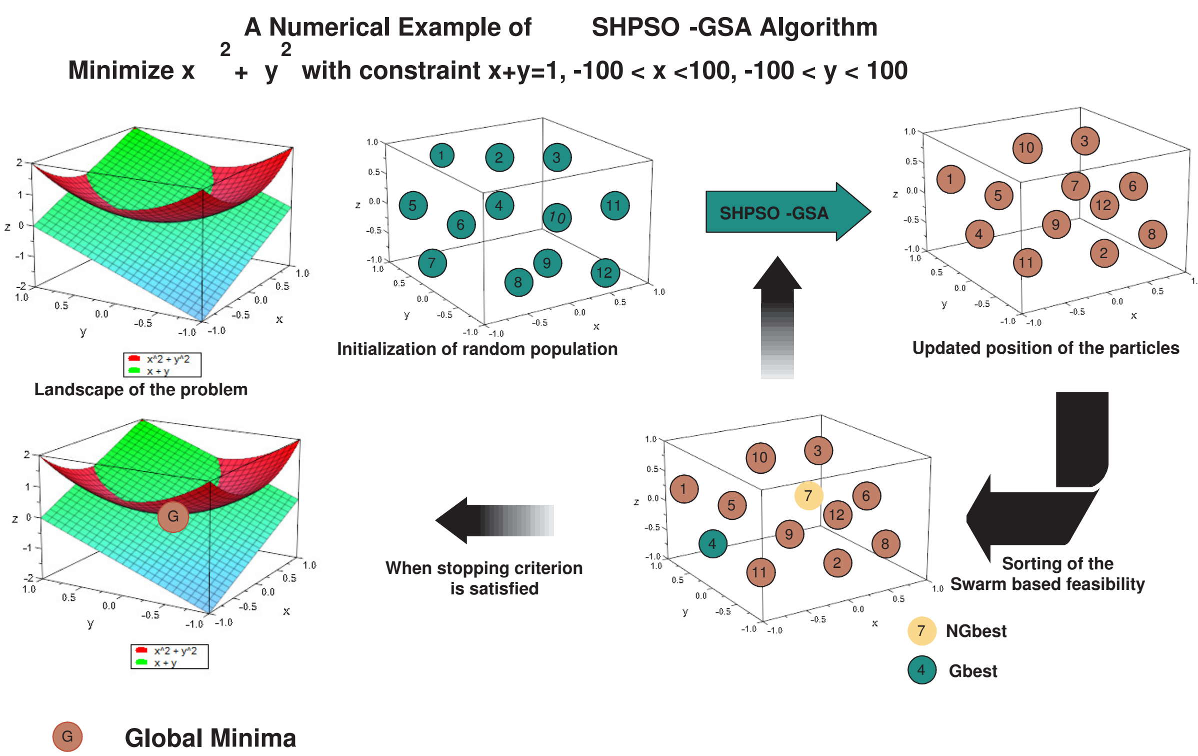


Figure 13: Numerical Example of SHPSO-GSA process

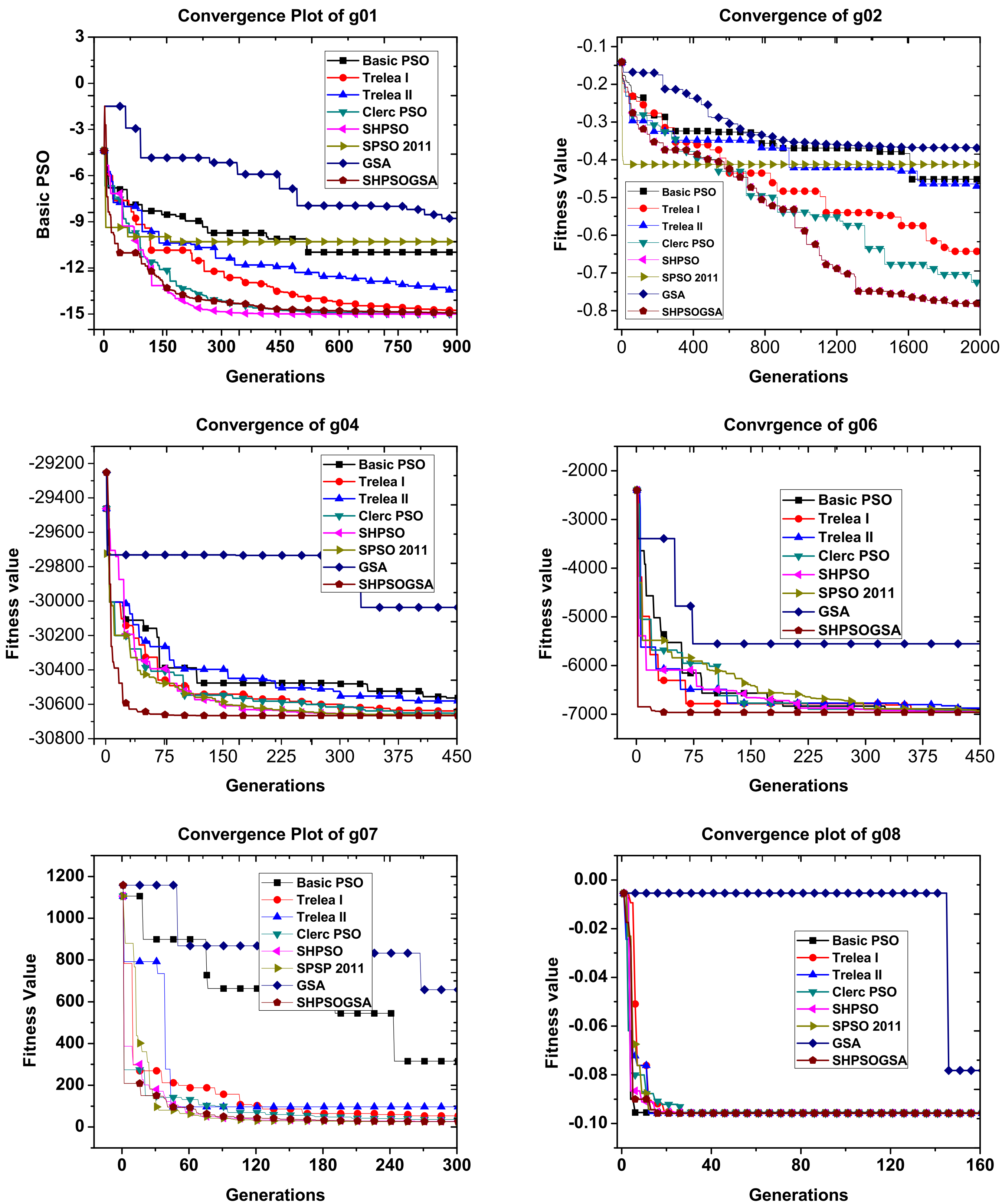


Figure 14: Convergence behavior of the algorithms

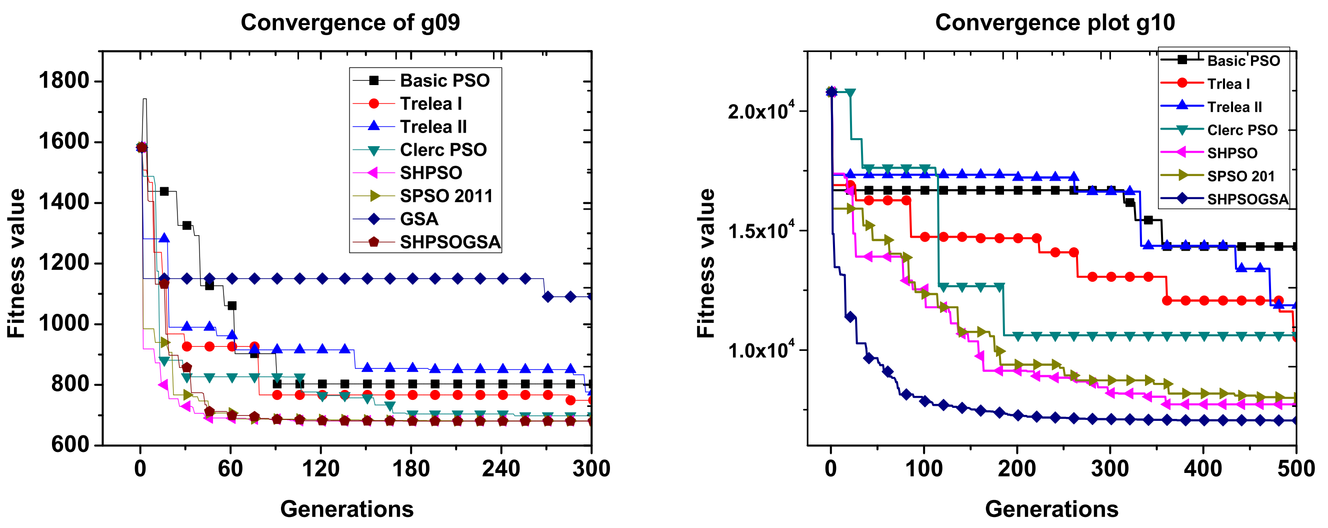


Figure 15: Convergence behavior of the algorithms

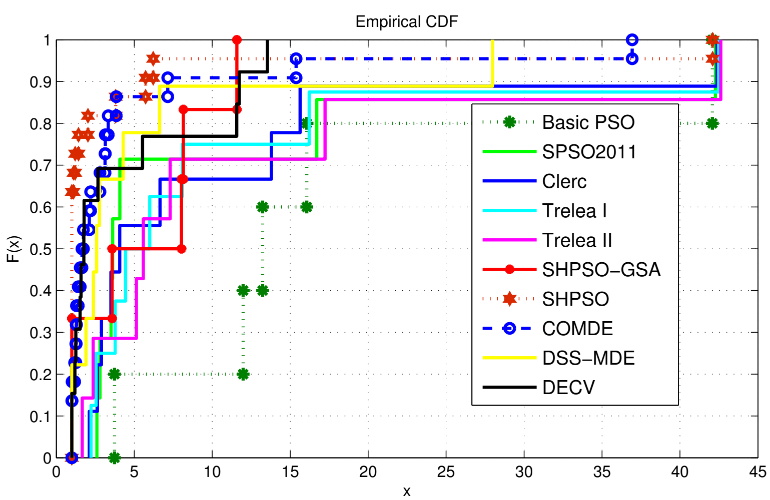


Figure 16: Empirical cumulative distribution plot all the algorithms

References

[1] Ali, M. M., Zhu, W., 2012. A penalty function-based differential evolution algorithm for constrained global optimization. Comput Optim Appl 54 (2), 707–739.

[2] Bahrololoum, A., Nezamabadi-Pour, H., Bahrololoum, H., Saeed, M., 2012. A prototype classifier based on gravitational search algorithm. Appl Soft Comput 12 (2), 819–825.

[3] Banks, A., Vincent, J., Anyakoha, C., 2008. A review of particle swarm optimization. part ii: hybridisation, combinatorial, multicriteria and constrained optimization, and indicative applications. Nat Comp Ser 7 (1), 109–124.

[4] Bonyadi, M. R., Li, X., Michalewicz, Z., 2014. A hybrid particle swarm with a time-adaptive topology for constrained optimization. Swarm and Evolutionary Computation 18, 22–37.

[5] Brest, J., Greiner, S., Boskovic, B., Mernik, M., Zumer, V., 2006. Self-adapting control parameters in differential evolution: A comparative study on numerical benchmark problems. Evolutionary Computation, IEEE Transactions on 10 (6), 646–657.

[6] Chen, Q., Liu, B., Zhang, Q., Liang, J., Suganthan, P., Qu, B., May 2015. Problem definitions and evaluation criteria for cec 2015 special session on bound constrained single-objective computationally expensive numerical optimization. Tech. rep.

[7] Chen, W., Zhang, J., Chung, H., Zhong, W., Wu, W., Shi, Y., 2010. A novel set-based particle swarm optimization method for discrete optimization problems. Evolutionary Computation, IEEE Transactions on 14 (2), 278–300.

[8] Cheng, J., Zhang, G., Neri, F., 2013. Enhancing distributed differential evolution with multicultural migration for global numerical optimization. Inform Sciences 247, 72–93.

[9] Clerc, M., 1998. Some math about particle swarm optimization. Online available at http://clerc. maurice. free. fr/PSO/PSOmathstuff/PSOmathstuff.htm.

[10] Clerc, M., 1999. The swarm and the queen: towards a deterministic and adaptive particle swarm optimization. In: Proceedings of the Congress on Evolutionary Computation, CEC 99. Vol. 3. pp. 1951–1957.

[11] Clerc, M., 2001. Think locally, act locally: The way of life of cheap-PSO, an adaptive PSO. Tech. rep.

[12] Clerc, M., 2011. Standard particle swarm optimisation. available online: http://clerc.maurice.free.fr/PSO/PSOmathstuff/PSOmathstuff. htm.

[13] Clerc, M., Kennedy, J., 2002. The particle swarm-explosion, stability, and convergence in a multidimensional complex space. IEEE Transactions on Evolutionary Computation, 6 (1), 58–73.

[14] Coello Coello, C. A., 2002. Theoretical and numerical constraint-handling techniques used with evolutionary algorithms: a survey of the state of the art. Comput Method Appl M 191 (11), 1245–1287.

[15] Črepinšek, M., Liu, S.-H., Mernik, L., 2012. A note on teaching–learning-based optimization algorithm. Information Sciences 212, 79–93.

[16] Črepinšek, M., Liu, S.-H., Mernik, L., Mernik, M., 2016. Is a comparison of results meaningful from the inexact replications of computational experiments? Soft Computing 20 (1), 223–235.

[17] Črepinšek, M., Liu, S.-H., Mernik, M., 2014. Replication and comparison of computational experiments in applied evolutionary computing: common pitfalls and guidelines to avoid them. Applied Soft Computing 19, 161–170.

[18] Daneshyari, M., Yen, G. G., 2012. Constrained multiple-swarm particle swarm optimization within a cultural framework. Systems, Man and Cybernetics, Part A: Systems and Humans, IEEE Transactions on 42 (2), 475–490.

[19] Deb, K., 2000. An efficient constraint handling method for genetic algorithms. Comput Method Appl M 186 (2), 311–338.

[20] Derrac, J., García, S., Molina, D., Herrera, F., 2011. A practical tutorial on the use of nonparametric statistical tests as a methodology for comparing evolutionary and swarm intelligence algorithms. Swarm and Evolutionary Computation 1 (1), 3–18.

[21] Eberhart, R., Shi, Y., 2000. Comparing inertia weights and constriction factors in particle swarm optimization. In: Proceedings of the Congress on Evolutionary Computation. Vol. 1. IEEE, pp. 84–88.

[22] Hsing-Chih Tsai, Yaw-Yauan Tyan, Y.-W. W. Y.-H. L., 2013. Gravitational Particle Swarm. Appl Math Comput 219, 9106–9117.

[23] Huang, F.-z., Wang, L., He, Q., 2007. An effective co-evolutionary differential evolution for constrained optimization. Appl Math Comput 186 (1), 340–356.

[24] Idoumghar, L., Chérin, N., Siarry, P., Roche, R., Miraoui, A., 2013. Hybrid ica–pso algorithm for continuous optimization. Appl Math Comput 219 (24), 11149–11170.

[25] José L. Rueda, I. E., May 2015. Mvmo for bound constrained single-objective computationally expensive numerical optimization. In: IEEE Congress on Evolutionary Computation (CEC 2015). IEEE, pp. 1–7.

[26] Kennedy, J., Eberhart, R., 1995. Particle swarm optimization. In: Proceedings of IEEE International Conference on Neural Networks. Vol. 4. IEEE, pp. 1942–1948.

[27] Khatibinia, M., Khosravi, S., 2014. A hybrid approach based on an improved gravitational search algorithm and orthogonal crossover for optimal shape design of concrete gravity dams. Appl Soft Comput 16, 223–233.

[28] Liang, J., Qin, A., Suganthan, P., Baskar, S., 2006. Comprehensive learning particle swarm optimizer for global optimization of multimodal functions. IEEE Transactions on Evolutionary Computation, 10 (3), 281–295.

[29] Liang, J., Runarsson, T., Mezura-Montes, E., Clerc, M., Suganthan, P., Coello, C., Deb, K., 2006. Problem definitions and evaluation criteria for the CEC 2006 special session on constrained real-parameter optimization. Technical Report presented at CEC' 2006.

[30] Lim, W. H., Mat Isa, N. A., 2014. Teaching and peer-learning particle swarm optimization. Appl Soft Comput 18, 39–58.

[31] Lwin, K., Qu, R., 2013. A hybrid algorithm for constrained portfolio selection problems. Appl Intell 39 (2), 251–266.

[32] M. R. Tanweer, S. Suresh, N. S., May 2015. Improved srpso algorithm for solving cec 2015 computationally expensive numerical optimization problems. In: IEEE Congress on Evolutionary Computation (CEC 2015). 1–7. IEEE, Sendai, Japan.

[33] Mallipeddi, R., Suganthan, P., Pan, Q., Tasgetiren, M., 2011. Differential evolution algorithm with ensemble of parameters and mutation strategies. Appl Soft Comput 11 (2), 1679–1696.

[34] Martin Andersson, Sunith Bandaru, A. H. N.-A. S., May 2015. Parameter tuned cma-es on the cecŠ15 expensive problems. In: IEEE Congress on Evolutionary Computation (CEC 2015). IEEE, Sendai, Japan, pp. 1–8.

[35] Mezura-Montes, E., Coello Coello, C. A., 2011. Constraint-handling in nature-inspired numerical optimization: past, present and future. Swarm and Evolutionary Computation 1 (4), 173–194.

[36] Mezura-Montes, E., Miranda-Varela, M. E., del Carmen Gómez-Ramón, R., 2010. Differential evolution in constrained numerical optimization: an empirical study. Information Sciences 180 (22), 4223–4262.

[37] Mezura-Montes, E., Palomeque-Ortiz, A. G., 2009. Parameter control in differential evolution for constrained optimization. In: Evolutionary Computation, 2009. CEC'09. IEEE Congress on. IEEE, pp. 1375–1382.

[38] Mirjalili, S., Hashim, S. Z. M., 2010. A new hybrid psogsa algorithm for function optimization. In: Computer and information application (ICCIA), 2010 international conference on. IEEE, pp. 374–377.

[39] Mohamed, A. W., Sabry, H. Z., 2012. Constrained optimization based on modified differential evolution algorithm. Information Sciences 194, 171–208.

[40] Mun, S., Cho, Y.-H., 2012. Modified harmony search optimization for constrained design problems. Expert Syst Appl 39 (1), 419–423.

[41] Neri, F., Mininno, E., Iacca, G., 2013. Compact particle swarm optimization. Inform Sciences 239, 96–121.

[42] Newton, I., 1714. Philosophiae Naturalis Principia Mathematica. Sumptibus Societatis.

[43] Pluhacek, M., Senkerik, R., Zelinka, I., 2014. Particle swarm optimization algorithm driven by multichaotic number generator. Soft Comput 18 (4), 631–639.

[44] Poli, R., Kennedy, J., Blackwell, T., 2007. Particle swarm optimization. Swarm intelligence 1 (1), 33–57.

[45] Rashedi, E., Nezamabadi-pour, H., Saryazdi, S., 2009. GSA: a gravitational search algorithm. Inform Sciences 179 (13), 2232–2248.

[46] Rushikesh Sawant, Donagh Hatton, D. P. O., May 2015. An ancestor based extension to differential evolution (ancde) for single-objective computationally expensive numerical optimization. In: IEEE Congress on Evolutionary Computation (CEC 2015). IEEE, Sendai, Japan, pp. 1–7.

[47] Shi, Y., Eberhart, R., 1998. A modified particle swarm optimizer. In: IEEE World Congress on Computational Intelligence. IEEE, pp. 69–73.

[48] Sun, S., Li, J., 2014. A two-swarm cooperative particle swarms optimization. Swarm and Evolutionary Computation 15, 1–18.

[49] Trelea, I., 2003. The particle swarm optimization algorithm: convergence analysis and parameter selection. Inform. Process. Lett. 85 (6), 317–325.

[50] Van den Bergh, F., Engelbrecht, A., 2006. A study of particle swarm optimization particle trajectories. Information Sciences, 176 (8), 937–971.

[51] Yadav, A., Deep, K., 2013. Shrinking hypersphere based trajectory of particles in pso. Appl Math Comput 220, 246–267.

[52] Yadav, A., Deep, K., 2014. A shrinking hypersphere pso for engineering optimisation problems. J Exp Theor Artif In, 1–33(DOI:10.1080/0952813X.2014.924589).

[53] Yuan, X., Ji, B., Zhang, S., Tian, H., Hou, Y., 2014. A new approach for unit commitment problem via binary gravitational search algorithm. Appl Soft Comput 22, 249–260.

[54] Zhang, M., Luo, W., Wang, X., 2008. Differential evolution with dynamic stochastic selection for constrained optimization. Information Sciences 178 (15), 3043–3074.

[55] Zhang, Y., Gong, D.-w., Sun, X.-y., Geng, N., 2013. Adaptive bare-bones particle swarm optimization algorithm and its convergence analysis. Soft Comput 18, 1337–1352.