Distance-related Properties of Corona of Certain Graphs

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Abstract

A graph G is called a $m-$eccentric point graph if each point of G has exactly $m \geq 1$ eccentric points. When $m = 1$, G is called a unique eccentric point ($u.e.p$) graph. Using the notion of corona of graphs, we show that there exists a $m-$eccentric point graph for every $m \geq 1$. Also, the eccentric graph G_e of a graph G is a graph with the same points as those of G and in which two points u and v are adjacent if and only if either u is an eccentric point of v or v is an eccentric point of u in G. We obtain the structure of the eccentric graph of corona $G \circ H$ of self-centered or non-self-centered $u.e.p$ graph G with any other graph H and obtain its domination number.

Keywords: *Domination, Eccentricity, Eccentric Graph*

1 Introduction

The notion of distance [2] in graphs has been studied in the context of many applications such as communication networks. The distance related parameter, known as eccentricity of a point in a graph and the associated notions of eccentric points, $m-$eccentric point graphs [1, 3, 6] and in particular, unique eccentric point ($u.e.p$) graphs [4], have also been well investigated. Another kind of graph known as corona [6] $G \circ H$ of two graphs G and H has also been well-studied. Also, the concept of eccentric graph G_e of a graph G was introduced in [5], based on the notion of distance among points in G. Here we show, using the notion of corona of graphs, that there exists a $m-$eccentric point graph for every $m \geq 1$. We also obtain the eccentric graph of corona $G \circ H$ where H is any graph and G is either self-centered $u.e.p$ graph or non-self-centered $u.e.p$ graph and and obtain its domination number.

We recall here certain basic definitions [1, 3, 6] related to graphs. A graph $G = (V, E)$ consists of a finite non-empty set V (also denoted by $V(G)$) whose elements are called points or vertices and another set E (or $E(G)$) of unordered pairs of distinct elements of V, called edges. In a graph G, the distance $d_G(u, v)$ or $d(u, v)$, when G is understood, between two points u and v is the length of the shortest path between u and v. The eccentricity $e_G(u)$ or simply, $e(u)$ of a point u in G is defined as $e(u) = max_{v \in V(G)} d(u, v)$. For two points u, v in G, the point v is an eccentric point of u if $d(u, v) = e(u)$. We denote by $E(v)$, the set of all eccentric points of a point v in G. A graph G is called a $m-$eccentric point graph if $|E(u)|$, the number of elements of $E(u)$ equals m, for all u in $V(G)$. When $m = 1$, G is called a unique eccentric point ($u.e.p$) graph. The radius $r(G)$ and the diameter $diam(G)$ of a graph G are respectively defined as $r(G) = \min\{e(u) \,|\, \text{for all}\, u \in V(G)\}$ and $diam(G) = \max\{e(u) \,|\, \text{for all}\, u \in V(G)\}$. A graph G is called a self-centered graph if $r(G) = diam(G)$.

The eccentric graph [5] G_e of a graph G is a graph with the same points as those of G and in which two points u and v are adjacent if and only if either u is an eccentric point of v or v is an eccentric point of u in G. A graph G and its eccentric graph G_e are shown in Fig. **??**. The corona [6] $G \circ H$ of two graphs G and H is a graph made of one copy of G with points $v_1, \cdots v_n$, $n \geq 1$, and n copies of another graph H such that for every i, $1 \leq i \leq n$, the point v_i is joined with all the points of the i^{th} copy of H.

We also need the following well-known notions. A complete graph K_n on n points, is a graph in which there is an edge between every pair of distinct points. The complement $\overline{G}$ of a graph G is a graph having the same points as those of G and such that two points x and y are adjacent in $\overline{G}$ if and only if x and y are not adjacent in G. The union $G_1 \cup G_2$ of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ is the graph $G_1 \cup G_2 = (V_1 \cup V_2, E_1 \cup E_2)$ and the join $G_1 + G_2$ of G_1 and G_2 is a graph obtained from $G_1 \cup G_2$ by joining every point of G_1 with every other point of G_2. For three or more graphs $G_1, G_2, \cdots, G_n$ the sequential join $G_1 + G_2 + \cdots + G_n$ is the graph $(G_1 + G_2) \cup (G_2 + G_3) \cup \cdots \cup (G_{n-1} + G_n)$. In a graph $G = (V, E)$ a subset $S \subset V$ is called a dominating set if each point u in $V - S$ has a neighbour in S $i.e$ u is adjacent to some point in S. The cardinality of a minimum dominating set of a graph G is called its domination number and it is denoted by $\gamma(G)$.

2 Eccentric Point Graphs

In this section we make use of the notions of corona and $u.e.p$ graphs to show that there exists, for every $m \geq 1$, an $m-$eccentric point graph.

Lemma 2.1 *Let G be a graph whose eccentric points are $p_1, \cdots, p_l$, for some $l \geq 1$. Let H be any other graph. In the corona $G \circ H$, all the points of $G \circ H$ each of which is joined with p_i, for some i, $1 \leq i \leq l$, are the only eccentric points of $G \circ H$.*

Proof. Let $V(G) = \{v_1, v_2, v_3, \cdots, v_{n-l}, p_1, p_2, \cdots, p_l\}$ such that $p_i's$ are eccentric points of G. Let H be any graph on m points. Let

$$V(G \circ H) = \{v_i, p_k, u_j^t | 1 \leq i \leq n-l, 1 \leq k \leq l, 1 \leq t \leq n \,\text{and}\, 1 \leq j \leq m\}$$

such that for a fixed i with $1 \leq i \leq n-l$, the points u_j^i for all $1 \leq j \leq m$, are joined with v_i while for a fixed i, with $n-l+1 \leq i \leq n$, the points u_j^i for all $1 \leq j \leq m$, are joined with p_i.

Then, let us prove that every point u_j^i for all $n-l+1 \leq i \leq n$ and $1 \leq j \leq m$ is an eccentric point. Suppose that for some $n-l+1 \leq i \leq n$ and $1 \leq j \leq m$, u_j^i is not an eccentric point. Then consider the point $v \in V(G)$ whose eccentric point in G is p_i to which the point u_j^i is attached in $G \circ H$. Let for some $n-l+1 \leq k \leq n$ and $1 \leq j \leq m$, u_j^k be the eccentric point of v in $G \circ H$. Then $e_{G \circ H}(v) = d_{G \circ H}(v, u_j^k) = d_G(v, p_k) + 1 < d_G(v, P_i) + 1 = d_{G \circ H}(v, u_j^i)$. That is $e_{G \circ H}(v) < d_{G \circ H}(v, u_j^i)$, which is a contradiction and hence every point u_j^i, $n-l+1 \leq i \leq n$ and $1 \leq j \leq m$ is an eccentric point. Now, it remains

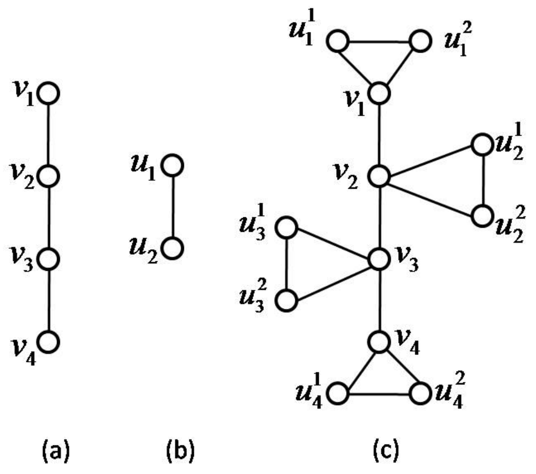

Figure 1: a) Graph G b) Graph H c) Corona $G \circ H$

to prove that (1) no point of G as a point of $G \circ H$ is an eccentric point in $G \circ H$ and (2) no point of u_j^i for $1 \leq i \leq n-l$ and $1 \leq j \leq m$ is an eccentric point in $G \circ H$.

In order to prove (1), suppose that u is an eccentric point of $G \circ H$. Then there exists a point $v \in G \circ H$ for which u is the eccentric point. Then u cannot be any v_i, $1 \leq i \leq n-l$ or any p_i, $n-l+1 \leq i \leq n$ for otherwise $e_{G\circ H}(v) = d_{G\circ H}(u,v) < d_{G\circ H}(u,v)+1 = d_{G\circ H}(v,u_j^i)$ which is a contradiction, due to the fact that any path between v and u_j^i passes through either v_i or p_i. Thus no point of G as a point of $G \circ H$, can be an eccentric point of $G \circ H$.

For proving (2), suppose that u_j^i for some $1 \leq i \leq n-l$, $1 \leq j \leq m$ is an eccentric point of some point v in $G \circ H$, then $e_{G\circ H}(v) = d_{G\circ H}(v,u_j^i) = d_{G\circ H}(v,v_i) + 1 < d_{G\circ H}(v,u_j^k)$. That is $e_{G\circ H}(v) < d_{G\circ H}(v,u_j^k)$ for some $n-l+1 \leq k \leq n$, which is a contradiction.

Remark 2.2 *With the graphs G and H as shown in Fig. 1, the eccentric points of $G \circ H$ are $u_1^1, u_1^2, u_4^1, u_4^2$. It can be noticed that no point v_i, $1 \leq i \leq 4$, of G is an eccentric point of $G \circ H$ and all the points of $G \circ H$ that are joined with the points that are eccentric points of G are the only eccentric points of $G \circ H$.*

Theorem 2.3 *Let G be a u.e.p graph on n points and H be any graph on m points. Then the corona of G and H, $G \circ H$ is a $m-$ eccentric point graph.*

Proof. Let G be a $u.e.p$ graph with n points $v_1, v_2, \cdots, v_n$ and H be any graph on m points. Let the points of $G \circ H$ that are points in the k^{th} copy of H, be u_j^k, $1 \leq j \leq m$. For any two points v_i, v_k of G such that v_k is the only eccentric point in G of v_i, by Lemma 2.1, the points u_j^k, $1 \leq k \leq m$, are the eccentric points in $G \circ H$ of v_i as well as of u_j^i, $1 \leq j \leq m$. No other point u_j^r, for some $r \neq k$, $1 \leq r \leq n$, can be an eccentric point in $G \circ H$ of v_i or u_j^i, $1 \leq j \leq m$, since $d_{G \circ H}(v_i, u_j^r) = d_G(v_i, v_r) + 1 < d_G(v_i, v_k) + 1 = d_{G \circ H}(v_i, u_j^k)$ and $d_{G \circ H}(u_j^i, u_j^r) = d_{G \circ H}(v_i, u_j^r) + 1 < d_{G \circ H}(v_i, u_j^k) + 1 = d_{G \circ H}(u_j^i, u_j^k)$. This implies that $E(v_i) = \{u_1^k, u_2^k, \cdots, u_m^k\}$ if v_k is an eccentric point of v_i in G and $E(u_j^p) = \{u_1^q, u_2^q, \cdots, u_m^q\}$ if v_q is an eccentric point of v_p in G. Therefore, $|E(u)| = m$, for all points u in $G \circ H$ and so $G \circ H$ is a $m-$ eccentric point graph.

As a consequence of the Theorem 2.3, we obtain the following corollary.

Corollary 2.4 *For every $m \geq 1$ there exists a $m-$ eccentric point graph.*

3 Eccentric graph of corona of *u.e.p* graph with any other graph

In this section we obtain the eccentric graph of corona of a $u.e.p$ graph with any other graph.

Theorem 3.1 *Let G be a self-centered u.e.p. graph on $2n$ points and H be a graph on m points. Then the eccentric graph $(G \circ H)_e$ is the union of n copies of $K_1 + \overline{K_m} + \overline{K_m} + K_1$.*

Proof. Let G be a self-centered $u.e.p.$ graph on $2n$ points and H be a graph on m points. Let $V(G) = \{v_1, v_2, v_3, \cdots, v_{2n}\}$ such that v_i and v_{i+n} $(1 \leq i \leq n)$ are eccentric points of each other, in the graph G. Then by lemma 2.1, all the points $u_j^i(1 \leq i \leq 2n; 1 \leq j \leq m)$ are eccentric points in $G \circ H$ because all the points of G are eccentric points in G. This implies that the eccentric points of u_j^i and v_i are $u_j^{i+n}(1 \leq i \leq n, 1 \leq i \leq m)$ and the eccentric points of u_j^{i+n} and v_{i+n} are $u_j^i(1 \leq i \leq n, 1 \leq j \leq m)$. Now, in $(G \circ H)_e$, which has the same point set as $G \circ H$, the point v_i is adjacent with all the points u_j^{i+n}, each of the points u_j^{i+n} is adjacent with every point u_j^i and all the points u_j^i

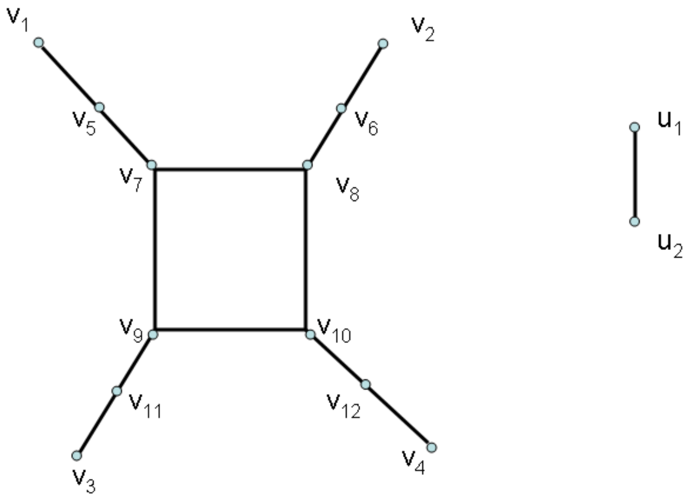

Figure 2: a) A non-self centered $u.e.p$ Graph G b) Graph H

are adjacent with $v_{i+n}(1 \leq i \leq n, 1 \leq j \leq m)$. Therefore, $(G \circ H)_e$ is the union of n copies of $K_1 + \overline{K_m} + \overline{K_m} + K_1$.

Theorem 3.2 *Let H be a graph on m points and G be a non-self-centered $u.e.p$ graph on n points having the properties (i) $P(G) = EP(G)$, (ii) $|P(G)| = 2t, t > 1$, (iii) for every u in $P(G)$ there is at least one v in $V(G) - P(G)$ such that $E(v) = \{u\}$, then $(G \circ H)_e$ is a union of t copies of $\overline{K_{t_i}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}$, for some $t_i \geq 1$ and $t_j \geq 1$, t_i and t_j depending on G and H.*

Proof. Let H be a graph on m points and G be a non-self-centered $u.e.p$ graph on n points having the properties (i) $P(G) = EP(G)$, (ii) $|P(G)| = 2t, t > 1$, (iii) for every u in $P(G)$ there is at least one v in $V(G) - P(G)$ such that $E(v) = \{u\}$. Let $V(G) = \{v_1, v_2, v_3, \cdots, v_n\}$. Let $v_1, v_2, \cdots, v_{2t}$ for some $t \geq 1$ be the peripheral vertices of G, so that $|P(G)| = 2t$. For $1 \leq i \leq t$, let v_i and v_{i+t} be the eccentric points of each other. Let $V(G \circ H) = V(G) \cup \{u_j^i\ 1 \leq j \leq m, 1 \leq i \leq n\}$. Then by Lemma 2.1, all the points $u_j^i, 1 \leq i \leq 2t$, $1 \leq j \leq m$ are the eccentric points of $G \circ H$ because $v_1, v_2, v_3, \cdots, v_{2t}$ are the eccentric points in G. This implies that for $1 \leq i \leq t$, $1 \leq j \leq m$, u_j^{i+t} is the

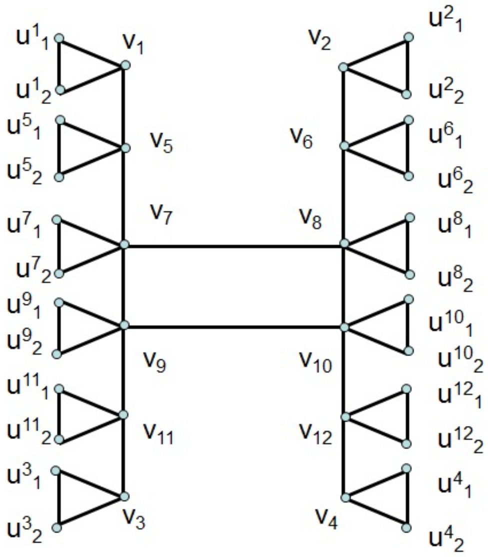

Figure 3: Corona of G and H, $G \circ H$

eccentric point of u_j^i,v_i, v_k as well as u_j^k with $E(v_k) = \{v_i\}$ for $v_k \in V(G)$. Also, $1 \leq i \leq t$, $1 \leq j \leq m$, u_j^i is the eccentric point of u_j^{i+t}, v_{i+t},v_k as well as u_j^k. Since an eccentric graph G_e, of any graph G, is constructed with the same points as those of G and each edge of G_e joins a point x with the eccentric points of x treated as a point of G. Thus, the structure of $(G \circ H)_e$ is clearly, union of t copies of $\overline{K_{t_i}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}$, where for some $t_i \geq 1$ and $t_j \geq 1$. Note that t_i and t_j depend on G and H.

Example 3.3 *A non-self-centered u.e.p graph G and a graph H on $m = 2$ points are shown in Fig. 2 . It is clear that in G, the eccentric points are v_1, v_2, v_3 and v_4 and $E(v_4) = E(v_{10}) = E(v_{12}) = \{v_1\}$; $E(v_1) = E(v_5) = E(v_7) = \{v_4\}$; $E(v_2) = E(v_6) = E(v_8) = \{v_3\}$; $E(v_3) = E(v_9) = E(v_{11}) = \{v_2\}$. The corona of G and H is shown in Fig.3. Note that $v_1^1, v_2^1, v_1^2, v_2^2, v_1^3, v_2^3, v_1^4, v_2^4$ are the eccentric vertices of $(G \circ H)_e$. The eccentric graph,$(G \circ H)_e$ is union of 2 copies of $\overline{K_{t_i}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}$,where $m = 2$; $t_i = 7$ and $t_j = 7$ and it is shown in the Fig.4.*

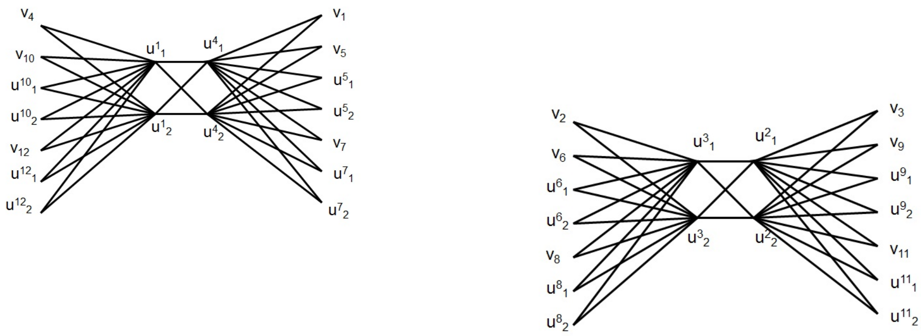

Figure 4: Eccentric Graph of $G \circ H$, $(G \circ H)_e$

Theorem 3.4 *Let G be a self-centered u.e.p. graph on $2n$ points and H be a graph on m points. Then the domination number $\gamma(G \circ H)_e = 2n$.*

Proof. Let G be a self-centered $u.e.p.$ graph on $2n$ points and H be a graph on m points. Now, by Theorem 3.1 $(G \circ H)_e$ is union of n copies of $K_1 + \overline{K_m} + \overline{K_m} + K_1$. In each copy, there are two $v_i's$ dominating the remaining points in that copy. Therefore, $\gamma(G \circ H)_e = 2n$.

Theorem 3.5 *Let H be a graph on m points and G be a non-self-centered u.e.p graph on n points having the properties (i) $P(G) = EP(G)$, (ii) $|P(G)| = 2t, t > 1$, (iii) for every u in $P(G)$ there is at least one v in $V(G) - P(G)$ such that $E(v) = \{u\}$, then the domination number $\gamma(G \circ H)_e = 2t$.*

Proof. Let H be a graph on m points and G be a non-self-centered $u.e.p$ graph on n points having the properties (i) $P(G) = EP(G)$, (ii) $|P(G)| = 2t, t > 1$, (iii) for every u in $P(G)$ there is at least one v in $V(G) - P(G)$ such that $E(v) = \{u\}$. Then by Theorem 3.2, $(G \circ H)_e$ is a union of t copies of $\overline{K_{t_i}} + \overline{K_m} + \overline{K_m} + \overline{K_{t_j}}$, for some $t_i \geq 1$ and $t_j \geq 1$, t_i and t_j depending on G and H. In each copy, there are two points dominating the remaining points in that copy. Therefore, $\gamma(G \circ H)_e = 2t$.

4 Conclusion

The structure of eccentric graph of $m-$ eccentric point graph can be investigated. Also the problem of finding a graph whose eccentric graph is a $m-$ eccentric point graph remains open.

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