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Two-dimensional Digitized Picture Arrays and Parikh Matrices

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Two-dimensional Digitized Picture Arrays and Parikh Matrices

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Parikh matrix mapping or Parikh matrix of a word has been introduced in the literature to count the scattered subwords in the word. Several properties of a Parikh matrix have been extensively investigated. A picture array is a two-dimensional connected digitized rectangular array consisting of a finite number of pixels with each pixel in a cell having a label from a finite alphabet. Here we extend the notion of Parikh matrix of a word to a picture array and associate with it two kinds of Parikh matrices, called row Parikh matrix and column Parikh matrix. Two picture arrays A and B are defined to be $M-$equivalent if their row Parikh matrices are the same and their column Parikh matrices are the same. This enables to extend the notion of $M-$ambiguity to a picture array. In the binary and ternary cases, conditions that ensure $M-$ambiguity are then obtained.

Keywords: Word; Subword; Parikh matrix; Picture array; Ambiguity

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1. Introduction

The Parikh vector or Parikh mapping [16] of a word w over an alphabet Σ, which has been an important notion in formal language theory [18], counts the number of occurrences of the symbols of Σ in the word w. But the Parikh mapping is not injective and so several words can have the same Parikh vector and hence in passing from words to vectors, much information about a word is lost. An ingenious extension of the Parikh vector, called Parikh matrix mapping or compactly referred to as Parikh matrix, was introduced by Mateescu et al [15]. The Parikh matrix of a word gives more numerical information about a word than a Parikh vector does. There has been a series of studies [1, 2, 3, 8, 13, 14, 19, 20, 21, 22, 23, 24, 25, 26, 27] on the properties of Parikh matrix. One such property is the injectivity of the Parikh matrix mapping. A related complement property is the $M-$ambiguity of words.

Among a number of problems of interest related to digitized picture arrays (see for example [4, 9, 11]), investigation of combinatorial properties of arrays has been done (see for example [5, 6, 17]). A two-dimensional connected digitized rectangular picture array with m rows and n columns or simply, a $m \times n$ picture array is made of a finite number of pixels with each pixel in a cell having a label taken from a finite alphabet Σ. For example, a digitized binary picture array describing the letter T with each pixel having label a or b, with the interpretation that b denotes a blank or empty cell and $a's$ constitute the body of the letter T, is shown in Figure 1.

$$\begin{array}{ccccccc}
a & a & a & a & a & a & a \\
b & b & b & a & b & b & b \\
b & b & b & a & b & b & b \\
b & b & b & a & b & b & b \\
b & b & b & a & b & b & b \\
b & b & b & a & b & b & b
\end{array}$$

Fig. 1: A binary picture array A describing letter T

Here we associate two kinds of matrices with a picture array A, called row Parikh matrix of A and column Parikh matrix of A, with the former matrix counting the subwords in the rows of A and the latter in the columns of A. For example, the row Parikh matrix of the binary picture array A as in Fig. 1, counts the number of 'horizontal' subwords ab in the rows of A whereas the column Parikh matrix of A counts the number of 'vertical' subwords $\begin{smallmatrix} a \\ b \end{smallmatrix}$ in the columns of A, besides both the matrices counting the number of $a's$ and the number of $b's$ in A, with the ordering $a < b$. The notion of $M-$ambiguity or simply ambiguity of a word and in particular, of a binary word and a ternary word, has been extensively investigated [1, 2, 3, 8, 13, 15, 21, 22, 26, 27]. Here we define $M-$equivalence of two picture arrays A and B by requiring their row Parikh matrices to be the same and their column Parikh matrices also to be the same. This enables to extend the notion of $M-$ambiguity to a picture array. We concentrate on binary and ternary picture arrays and obtain conditions for $M-$ambiguity in these cases. A preliminary version [28] of this paper was presented at the 14th International Workshop on Combinatorial Image Analysis (IWCIA'2011).

2. Preliminaries

Let Σ, called an alphabet, be a finite set of symbols. A word over Σ is a finite sequence of symbols from Σ. We denote by Σ^* the set of all words over Σ. The empty word is denoted by λ. For a word $w \in \Sigma^*$, $|w|$ denotes the length of w, which is the number of symbols in w, counting also repetitions.

A word u is called a *subword* (also called scattered subword) of a word w, if there exist words $x_1, \cdots, x_n$ and $y_0, \cdots, y_n$, (some of them possibly empty), such that $u = x_1 \cdots x_n$ and $w = y_0 x_1 y_1 \cdots x_n y_n$. For example if $w = aabbaabab$ is a word over the alphabet $\{a, b\}$, then $ababa$ is a subword of w. The number of occurrences of the word u as a subword of the word w is denoted by $|w|_u$. In particular, if u is a symbol in the alphabet, then $|w|_u$ equals the number of occurrences of the symbol u in w. Two occurrences of a subword are considered different if they differ by at least one position of some letter. A factor u of a word $w \in \Sigma^*$ is also a subword of w such that $w = xuy$, for some $x, y \in \Sigma^*$.

We now recall the definition of a Parik matrix mapping [15], which is a generalization of the Parikh mapping [16]. A triangle matrix is a square matrix $M = (m_{i,j})_{1 \le i,j \le n}$, such that $m_{i,j}$ are non-negative integers for all $1 \le i, j \le n, m_{i,j} = 0$, for all $1 \le j < i \le n$, and, moreover, $m_{i,i} = 1$, for all $1 \le i \le n$. The set $\mathcal{M}_n$ of all triangle matrices of dimension $n \ge 1$ is a monoid with respect to multiplication of matrices.

An ordered alphabet $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$ is an alphabet $\{a_1, \cdots, a_k\}$ with the ordering $a_1 < a_2 < \cdots < a_k$. The Parikh vector $(|w|_{a_1}, \cdots |w|_{a_k})$ of a word w over Σ_k counts the number of occurrences of the symbols of the alphabet in the word w.

Definition 1. [15] *Let $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$ be an ordered alphabet. The Parikh matrix mapping is the monoid morphism $\Psi_k : \Sigma_k^* \to \mathcal{M}_{k+1}$, defined by the condition: $\Psi_k(\lambda) = I_{k+1}$, the $(k+1) \times (k+1)$ unit matrix, and if $\Psi_k(a_q) = (m_{i,j})_{1 \le i,j \le (k+1)}$, then for each $1 \le i \le (k+1), m_{i,i} = 1, m_{q,q+1} = 1$, all other elements of the matrix $\Psi_k(a_q)$ are 0, $1 \le q \le k$.*
For a word $w = a_{i_1} a_{i_2} \cdots a_{i_m}, a_{i_j} \in \Sigma_k$ for $1 \le j \le m$, we have

$$\Psi_k(w) = \Psi_k(a_{i_1}) \Psi_k(a_{i_2}) \cdots \Psi_k(a_{i_m}).$$

In other words $\Psi_k(w)$ is computed by multiplication of matrices and the triangle matrix $\Psi_k(w)$ is called the Parikh matrix of w.

For example, if $\Sigma_3 = \{a < b < c\}$ and $w = acabaccb$, then

$$\Psi_3(a) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Psi_3(b) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \Psi_3(c) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

so that $\Psi_3(acabaccb)$ is a 4×4 triangle matrix and is given by

$$\Psi_3(acabaccb) = \begin{pmatrix} 1 & 3 & 5 & 4 \\ 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Note that the word $acabaccb$ has 5 subwords ab, 2 subwords bc and 4 subwords abc.

We now recall some important facts about Parikh matrices.

Lemma 2. [15] *Let $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$ and $w \in \Sigma_k^*$. The Parikh matrix $\Psi_k(w) = (m_{i,j})_{1 \leq i,j \leq (k+1)}$, has the following properties : i) $m_{i,j} = 0$, for all $1 \leq j < i \leq (k+1)$, ii) $m_{i,i} = 1$, for all $1 \leq i \leq (k+1)$, iii) $m_{i,j+1} = |w|_{a_i a_{i+1} \cdots a_{j-1} a_j}$, for all $1 \leq i \leq j \leq k$.*

The Parikh matrix mapping is not injective (See for example [2, 13, 21]). For example, if the alphabet is $\Sigma_2 = \{a < b\}$, the words $w_1 = ababbabbb$ and $w_2 = aabbbbabb$ have the same Parikh matrix, namely,

$$\begin{pmatrix} 1 & 3 & 14 \\ 0 & 1 & 6 \\ 0 & 0 & 1 \end{pmatrix}$$

Many of the studies (see for example, [1, 2, 3, 8, 13, 21, 25, 26]) in this area deal with this problem of $M-$ambiguity. Let Σ_k and Ψ_k be as in Definition 1. Two words w_1 and w_2 are called $M-$equivalent if $\Psi_k(w_1) = \Psi_k(w_2)$. A word $w \in \Sigma_k^*$ is said to be $M-$unambiguous if there is no word $v \neq w$ such that v is $M-$equivalent to w. Otherwise w is $M-$ambiguous or simply ambiguous.

If M_1 and M_2 are two triangle matrices in $\mathcal{M}_n$, then the triangle matrix $M = M_1 \bigoplus M_2$ is defined [12] as the usual sum of matrices M_1 and M_2 except that all the elements on the main diagonal of M have by definition value 1. The operation $\bigoplus$ is commutative and associative.

Mateescu [12] has considered the matrix $C = A \bigoplus B$ of two Parikh matrices A, B of two binary words x, y respectively, over $\{a < b\}$ and has shown that C is also a Parikh matrix. In fact it is shown in [12] that a preimage of $C = (c_{i,j})_{1 \leq i,j \leq 3}$ is the word $b^t a^p b^q a^r$ where $t + q = |x|_b + |y|_b, p + r = |x|_a + |y|_a$ and $pq = c_{1,3}$.

Note that, in the subsequent sections, we will also denote by $M(w)$, the Parikh matrix of a word w, where the alphabet is understood.

3. Row and Column Parikh matrices of a Picture Array

We extend the notion of Parikh matrix of a word to a picture array. We first recall certain notions related to arrays [9].

Let $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$ be an ordered alphabet and m, n be two positive integers. A $m \times n$ picture array (or array, for short) A over Σ_k is a rectangular arrangement of symbols from Σ_k in m rows and n columns. We will write such an array as follows:

$$A = \begin{array}{c} a_{11} \cdots a_{1n} \\ \cdots \\ \cdots \\ a_{m1} \cdots a_{mn} \end{array}, \quad \text{and the transpose of } A \text{ is } A^t = \begin{array}{c} a_{11} \cdots a_{m1} \\ \cdots \\ \cdots \\ a_{1n} \cdots a_{mn} \end{array},$$

$a_{ij} \in \Sigma$, $1 \le i \le m, 1 \le j \le n$.. For example $\begin{array}{ccccc} a & a & b & a & b \\ b & a & a & b & a \\ b & a & b & b & a \end{array}$ is 3×5 binary array over $\Sigma_2 = \{a < b\}$ and its transpose is $\begin{array}{ccc} a & b & b \\ a & a & a \\ b & a & b \\ a & b & b \\ b & a & a \end{array}$.

We will call the words in the rows of a picture array over Σ_k as horizontal words or simply words and the words in the columns as vertical words. For a word $x = b_1 b_2 \cdots b_n$, $b_i \in \Sigma_k$ for $i = 1, \cdots, n$, we denote by x^t the vertical word $\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_n \end{array}$. Also we define $(x^t)^t = x$. The notion of a subword in a vertical word is analogous to the notion of a subword in a word except that the subword itself is a vertical word.

The set of all picture arrays over Σ_k is denoted by $\mathcal{PA}_k$. We denote respectively by $\circ$ and $\diamond$ the *column concatenation* (also called column product)[9] and *row concatenation* (also called row product)[9] of arrays in $\mathcal{PA}_k$. In contrast to the case of strings, these operations are partially defined, namely, for any $A, B \in \mathcal{PA}_k$, $A \circ B$ is defined if and only if A and B have the same number of rows. Similarly $A \diamond B$ is defined if and only if A and B have the same number of columns. For example, if $A = \begin{array}{ccccc} a & b & b & b & a \\ b & a & a & b & b \\ a & b & a & a & b \end{array}$ and $B = \begin{array}{ccc} a & b & b \\ a & b & b \\ b & b & a \end{array}$, then $A \circ B = \begin{array}{cccccccc} a & b & b & b & a & a & b & b \\ b & a & a & b & b & a & b & b \\ a & b & a & a & b & b & b & a \end{array}$, where as the row concatenation $A \diamond B$ is not defined, since the number of columns in A and B are not equal.

We now introduce the notions of row Parikh matrix and column Parikh matrix of a picture array.

Definition 3. *For $m, n \ge 1$, let $A \in \mathcal{PA}_k$, be a $m \times n$ array over $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$. Let the words in the m rows of A be $x_i, 1 \le i \le m$ and the vertical words* *in the n columns of A be $y_j, 1 \leq j \leq n$. Let the Parikh matrices of x_i and y_j^t be respectively $M(x_i), 1 \leq i \leq m$ and $M(y_j^t), 1 \leq j \leq n$. Then the row Parikh matrix $M_r(A)$ of A is defined as $M_r(A) = M(x_1) \bigoplus \cdots \bigoplus M(x_m)$ and the column Parikh matrix $M_c(A)$ of A is defined as $M_c(A) = M(y_1^t) \bigoplus \cdots \bigoplus M(y_n^t)$.*

Example 4. *Consider the array A in $\mathcal{PA}_3$ over $\{a < b < c\}$ given by*

$$A = \begin{matrix} a & b & a & c & b \\ b & c & c & b & a \\ b & a & b & c & b \\ b & a & a & b & c \\ a & b & c & a & c \end{matrix}$$

The Parikh matrices $M(x_i), 1 \leq i \leq 5$ of the words in the rows

$$x_1 = abacb, x_2 = bccba, x_3 = babcb, x_4 = baabc, x_5 = abcac$$

are respectively $\begin{pmatrix} 1&2&3&1 \\ 0&1&2&1 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&0&0 \\ 0&1&2&2 \\ 0&0&1&2 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&1&2&1 \\ 0&1&3&2 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&2&2&2 \\ 0&1&2&2 \\ 0&0&1&1 \\ 0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&2&1&2 \\ 0&1&1&2 \\ 0&0&1&2 \\ 0&0&0&1 \end{pmatrix}.$

The row Parikh matrix of A is $\begin{pmatrix} 1&8&8&6 \\ 0&1&10&9 \\ 0&0&1&7 \\ 0&0&0&1 \end{pmatrix},$

Likewise, the column Parikh matrix of A is $\begin{pmatrix} 1&8&7&3 \\ 0&1&10&7 \\ 0&0&1&7 \\ 0&0&0&1 \end{pmatrix}.$

4. Ambiguity of Picture Arrays

Extending the notion of $M-$ambiguity (see for example [3, 8, 21, 23]) of a word, we introduce the notion of $M-$ ambiguity of a picture array.

Definition 5. *For $m, n \geq 1$, two $m \times n$ arrays $A, B \in \mathcal{PA}_k$, over $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$ are said to be i) $M-$row equivalent if $M_r(A) = M_r(B)$ and ii) $M-$column equivalent if $M_c(A) = M_c(B)$. The arrays A and B are $M-$equivalent if A and B are both $M-$row equivalent and $M-$column equivalent and in this case we say that A (as well as B) is $M-$ambiguous or simply ambiguous. An array $A \in \mathcal{PA}_k$, is called unambiguous if it is not ambiguous.*

Example 6. *The array $A_1 \in \mathcal{PA}_2$ given by* $A_1 = \begin{matrix} a & b & a & a & b \\ a & a & b & b & a \\ b & a & b & a & b \\ b & b & a & b & a \\ a & a & b & a & b \end{matrix}$ *is $M-$ambiguous since*

$M_r(A_1) = M_r(A_2) = \begin{pmatrix} 1 & 13 & 17 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{pmatrix}$ *and* $M_c(A_1) = M_c(A_2) = \begin{pmatrix} 1 & 13 & 16 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{pmatrix}$ *where the array* $A_2 = \begin{matrix} a & a & b & b & a \\ a & b & a & a & b \\ b & a & b & a & b \\ b & b & b & a & a \\ a & a & a & b & b \end{matrix}$.

Remark 7. *We note that there are arrays* A, B *with* $M_r(A) = M_r(B)$ *but* $M_c(A) \neq M_c(B)$ *or vice versa. For example, if the array* $A_1 \in \mathcal{PA}_2$ *is as in Example 6, and the array* A_3 *is given by* $A_3 = \begin{matrix} a & b & a & a & b \\ a & a & b & b & a \\ a & b & b & b & a \\ b & b & a & b & a \\ a & a & b & a & b \end{matrix}$ *then* $M_r(A_1) = M_r(A_3) = \begin{pmatrix} 1 & 13 & 17 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{pmatrix}$ *but* $M_c(A_1) \neq M_c(A_3)$ *since* $M_c(A_3) = \begin{pmatrix} 1 & 13 & 15 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{pmatrix}$.

In the sequel, we concentrate on binary and ternary picture arrays and obtain some partial results. Making use of the characterizations [8, 13] of equality of Parikh matrices of words over $\Sigma_2 = \{a < b\}$, we obtain conditions for ambiguity of arrays in $\mathcal{PA}_2$.

A characterization of $M-$ambiguity of a binary word over $\Sigma_2 = \{a < b\}$ is well-known [8, 13]. We state this in the following lemma.

Lemma 8. [8, 13] *A word* $w \in \{a < b\}^*$ *is* $M-$*unambiguous if and only if* w *belongs to the language denoted by the regular expression*

$$a^*b^* + b^*a^* + a^*ba^* + b^*ab^* + a^*bab^* + b^*aba^*$$

where a^* *denotes the set of all words over* $\{a\}$ *including the empty word and* $+$ *denotes set union.*

Given $m, n \geq 2$, $A \in \mathcal{PA}_2$, $A = \begin{matrix} a_{11} & \cdots & a_{1n} \\ & \cdots & \\ & \cdots & \\ a_{m1} & \cdots & a_{mn} \end{matrix}$, a 2×2 subarray W of A is of the form $W = \begin{matrix} a_{ij} & a_{ik} \\ a_{lj} & a_{lk} \end{matrix}$, for some i, j, k, l, $1 \leq i < l \leq m, 1 \leq j < k \leq n$. The subarray W is said to occur in A.

Lemma 9. *If the* 2×2 *subarray* $W_1 = \begin{matrix} a & b \\ b & a \end{matrix}$ *occurs in* $A \in \mathcal{PA}_2$ *over* $\Sigma_2 = \{a < b\}$

and is replaced by the 2×2 *subarray* $W_2 = \begin{matrix} b & a \\ a & b \end{matrix}$ *or vice versa, yielding another* $B \in \mathcal{PA}_2$*, then* A *and* B *are* $M-$*equivalent.*

Proof. Let W_1 occur in the $m \times n$ array $A \in \mathcal{PA}_2$. Then A has two rows of the form $w_i = a_{i1} \cdots a_{ij} \cdots a_{ik} \cdots a_{in}$ and $w_l = a_{l1} \cdots a_{lj} \cdots a_{lk} \cdots a_{ln}$ where $1 \leq i < l \leq m$, $1 \leq j < k \leq n$ and $a_{ij} = a_{lk} = a$; $a_{ik} = a_{lj} = b$. When W_1 is replaced by W_2, this implies that the number of subword ab will decrease by $k - j$ in w_i and will increase by $k - j$ in w_l while there is no change in the number of $a's$ and the number of $b's$ in A. Hence the number of subword ab does not change in A so that A and B have the same row Parikh matrix. Likewise by considering the jth and kth columns in A, we can show that A and B have the same column Parikh matrix when W_1 is replaced by W_2 in A. The argument is similar for the case W_2 occuring in A and being replaced by W_1.

The following sufficient condition for $M-$ambiguity of $A \in \mathcal{PA}_2$ is a consequence of Lemma 9.

Theorem 10. *An array* $A \in \mathcal{PA}_2$ *over* $\Sigma_2 = \{a < b\}$*, is* $M-$*ambiguous, if either the* 2×2 *subarray* W_1 *or the* 2×2 *subarray* W_2 *as in Lemma 9, occurs in* A*.*

Remark 11. *(1) Theorem 10 provides only a sufficient condition and is not necessary for the* $M-$*ambiguity of an array in* $\mathcal{PA}_2$*. For instance,* $A = \begin{matrix} a & a & a \\ a & b & b \\ a & b & a \end{matrix}$ *is* M*-ambiguous since* A *and* $B = \begin{matrix} a & b & a \\ a & a & b \\ b & a & a \end{matrix}$ *have the same row Parikh matrix* $\begin{pmatrix} 1 & 6 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ *and the same column Parikh matrix* $\begin{pmatrix} 1 & 6 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{pmatrix}$ *but neither* W_1 *nor* W_2 *as in Theorem 10 occurs in* A*.*

(2) An array $A \in \mathcal{PA}_2$ *may be ambiguous even if the words in all the rows and the vertical words in all the columns of* A *are unambiguous. For instance, if we consider* $A_3 = \begin{matrix} a & a & a & b & b \\ a & b & a & b & b \\ a & a & a & b & b \\ a & a & a & a & b \\ b & a & b & a & a \end{matrix}$*, the words in the rows and the vertical words in the columns of* A_3 *are all unambiguous by Lemma 8 but* A_3 *is ambiguous since the row Parikh matrix of* A_3 *is* $\begin{pmatrix} 1 & 15 & 22 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}$ *and the column Parikh matrix of* A_3

is $\begin{pmatrix} 1 & 15 & 9 \\ 0 & 1 & 10 \\ 0 & 0 & 1 \end{pmatrix}$ *while* $A_4 = \begin{array}{ccccc} a & a & a & b & b \\ a & a & b & b & b \\ a & a & a & b & b \\ a & a & a & a & b \\ b & b & a & a & a \end{array}$ *has the same row Parikh matrix and the same column Parikh matrix, as that of* A_3.

Salomaa [20] has presented a graphic technique that involves two rules $RuleR_F, RuleR_B$, the application of which yields an $M-$equivalent word from a given binary word. We extend this technique to a binary picture array giving again a sufficient condition for $M-$row or $M-$ column equivalence of arrays in $\mathcal{PA}_2$.

The $a-$map of a $m \times n$ array A in $\mathcal{PA}_2$ is a 2D tape of the same size as A with positions of a marked by letters a and other squares being empty. For example, the $a-$map of the array $X = \begin{array}{ccccc} a & a & b & b & a \\ a & b & a & a & b \\ b & a & b & a & b \\ b & b & b & a & a \\ a & a & a & b & b \end{array}$ is given below:

a	a			a
a		a	a	
	a		a	
			a	a
a	a	a		

Fig. 2: The $a-$map of the array X

The following extended rules yield an $M-$row equivalent array from a given array in $\mathcal{PA}_2$. When we say that a is moved one square to the left, right,up or down, we always assume that the target square is empty.
$RuleER_F$: Move a in a row, one square to the left and another a, either in the same row located somewhere to the right of the first one or located somewhere in a different row, one square to the right.
$RuleER_B$: Move a in a row, one square to the right and another a, either in the same row located somewhere to the right of the first one or located somewhere in a different row, one square to the left.
For instance, moving the a in the third square in the second row of X, one square to the left and moving the a in the second square in the third row, one square to the right yields an array $Y = \begin{array}{ccccc} a & a & b & b & a \\ a & a & b & a & b \\ b & b & a & a & b \\ b & b & b & a & a \\ a & a & a & b & b \end{array}$ which is $M-$row equivalent to the array

X, with X and Y having the row Parikh matrix $\begin{pmatrix} 1 & 13 & 17 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{pmatrix}$.

Likewise, the following extended rules yield an $M-$column equivalent array from a given array in $\mathcal{PA}_2$.

RuleER$_U$: Move a in a column, one square up and another a, either in the same column located somewhere below the first one or located somewhere in a different column, one square down.

RuleER$_D$: Move a in a column, one square down and another a, either in the same column located somewhere below the first one or located somewhere in a different column, one square up.

For instance, moving the a in the second square in the second column of Y, one square down and moving the a in the fourth square in the fifth column of Y, one square up yields an array $Z = \begin{matrix} a & a & b & b & a \\ a & b & b & a & b \\ b & a & a & a & a \\ b & b & b & a & b \\ a & a & a & b & b \end{matrix}$ which is $M-$column equivalent to the array X, with X and Z having the column Parikh matrix $\begin{pmatrix} 1 & 13 & 16 \\ 0 & 1 & 12 \\ 0 & 0 & 1 \end{pmatrix}$.

Based on these considerations we have the following Lemma.

Lemma 12. *Any binary picture array Y obtained from X by applying $i)$ the two rules ER_F and ER_B finitely many times is $M-$row equivalent to X and $ii)$ the two rules ER_U and ER_D finitely many times is $M-$column equivalent to X.*

Theorem 13. *Let $A, B \in \mathcal{PA}_2$ be two $M-$equivalent $m \times n$ arrays. Then $i)$ the arrays $A \circ B$ and $B \circ A$ are $M-$equivalent and $ii)$ the arrays $A \diamond B$ and $B \diamond A$ are $M-$equivalent.*

Proof. For $1 \leq i \leq m$, let x_i and y_i be respectively the words in the ith rows of A and B so that the words in the ith rows of $A \circ B$ and $B \circ A$ are respectively $x_i y_i$ and $y_i x_i$.

Suppose the arrays A and B over the alphabet $\Sigma_2 = \{a < b\}$ are $M-$equivalent and hence $M-$row equivalent, we have $\Sigma_{i=1}^m |x_i|_a = \Sigma_{i=1}^m |y_i|_a$, $\Sigma_{i=1}^m |x_i|_b = \Sigma_{i=1}^m |y_i|_b$ and $\Sigma_{i=1}^m |x_i|_{ab} = \Sigma_{i=1}^m |y_i|_{ab}$. Then $\Sigma_{i=1}^m |x_i y_i|_a = \Sigma_{i=1}^m |y_i x_i|_a$, both being equal to $\Sigma_{i=1}^m |x_i|_a + \Sigma_{i=1}^m |y_i|_a$. Likewise $\Sigma_{i=1}^m |x_i y_i|_b = \Sigma_{i=1}^m |y_i x_i|_b$. Also $\Sigma_{i=1}^m |x_i y_i|_{ab} = \Sigma_{i=1}^m |x_i|_{ab} + \Sigma_{i=1}^m |y_i|_{ab} + \Sigma_{i=1}^m |x_i|_a |y_i|_b$. But $\Sigma_{i=1}^m |x_i|_a |y_i|_b = \Sigma_{i=1}^m |x_i|_a (n - |y_i|_a) = \Sigma_{i=1}^m n |x_i|_a - \Sigma_{i=1}^m |x_i|_a |y_i|_a = \Sigma_{i=1}^m n |y_i|_a - \Sigma_{i=1}^m |x_i|_a |y_i|_a = \Sigma_{i=1}^m |y_i|_a (n - |x_i|_a) = \Sigma_{i=1}^m |y_i|_a |x_i|_b$. Hence $\Sigma_{i=1}^m |x_i y_i|_{ab} = \Sigma_{i=1}^m |x_i|_{ab} + \Sigma_{i=1}^m |y_i|_{ab} + \Sigma_{i=1}^m |y_i|_a |x_i|_b = \Sigma_{i=1}^m |y_i x_i|_{ab}$. This shows that $A \circ B$ and $B \circ A$ have the same row Parikh matrix and hence are $M-$row equivalent. Clearly $A \circ B$ and $B \circ A$ have the same column Parikh matrix as both the arrays have the same columns but in a different order. This proves that the arrays $A \circ B$ and $B \circ A$ are $M-$equivalent. The proof of the arrays $A \diamond B$ and $B \diamond A$ being $M-$equivalent is similar.

The notion of a "weak ratio-property" of words is considered in [27]. We extend the notion of weak ratio-property to picture arrays.

Definition 14. *Let $A, B \in \mathcal{PA}_k$ be $m \times n$ and $s \times t$ arrays respectively over the alphabet $\Sigma_k = \{a_1 < a_2 < \cdots < a_k\}$. Let the number of occurrences of $a_i, (1 \leq i \leq k)$, in A and B be respectively $|A|_{a_i}$ and $|B|_{a_i}$. Then A and B are said to satisfy the weak ratio property if*

$$\frac{|A|_{a_1}}{|B|_{a_1}} = \frac{|A|_{a_2}}{|B|_{a_2}} = \cdots = \frac{|A|_{a_k}}{|B|_{a_k}} = \alpha$$

where $\alpha \neq 0$, is a constant.

Theorem 15. *Let $A, B \in \mathcal{PA}_2$ be arrays of sizes $m \times n$ and $m \times l$ respectively. Then the following statements are equivalent:*
i) A and B satisfy weak ratio property
ii) $M_r(A \circ B) = M_r(B \circ A)$, $M_c(A \circ B) = M_c(B \circ A)$ and as a consequence $A \circ B$ and $B \circ A$ are $M-$ equivalent.

Proof. For $1 \leq i \leq m$, let x_i and y_i be respectively the words in the ith rows of A and B so that the words in the ith rows of $A \circ B$ and $B \circ A$ are respectively $x_i y_i$ and $y_i x_i$.
Suppose the arrays A and B over the alphabet $\Sigma_2 = \{a < b\}$ satisfy weak ratio-property so that $|A|_a = \alpha |B|_a$, $|A|_b = \alpha |B|_b$ where $\alpha \neq 0$, is a constant. This implies that $\Sigma_{i=1}^m |x_i|_a = \alpha \Sigma_{i=1}^m |y_i|_a$ and $\Sigma_{i=1}^m |x_i|_b = \alpha \Sigma_{i=1}^m |y_i|_b$. Also $mn = |A|_a + |A|_b = \alpha(|B|_a + |B|_b) = \alpha ml$ so that $n = \alpha l$. Clearly $|A \circ B|_a = \Sigma_{i=1}^m |x_i y_i|_a = \Sigma_{i=1}^m |y_i x_i|_a = |B \circ A|_a$. Likewise $|A \circ B|_b = |B \circ A|_b$. Now $|A \circ B|_{ab} = \Sigma_{i=1}^m |x_i y_i|_{ab} = \Sigma_{i=1}^m |x_i|_{ab} + \Sigma_{i=1}^m |y_i|_{ab} + \Sigma_{i=1}^m |x_i|_a |y_i|_b$. But $\Sigma_{i=1}^m |x_i|_a |y_i|_b = \Sigma_{i=1}^m |x_i|_a (l - |y_i|_a) = l\alpha \Sigma_{i=1}^m |y_i|_a - \Sigma_{i=1}^m |x_i|_a |y_i|_a = \Sigma_{i=1}^m (l\alpha - |x_i|_a)|y_i|_a = \Sigma_{i=1}^m |y_i|_a (n - |x_i|_a) = \Sigma_{i=1}^m |y_i|_a |x_i|_b$. Hence $|A \circ B|_{ab} = \Sigma_{i=1}^m |x_i|_{ab} + \Sigma_{i=1}^m |y_i|_{ab} + \Sigma_{i=1}^m |y_i|_a |x_i|_b = |B \circ A|_{ab}$. This proves $M_r(A \circ B) = M_r(B \circ A)$. The equality $M_c(A \circ B) = M_c(B \circ A)$ follows from the fact that both the arrays have the same columns but in a different order. Conversely, if the arrays $A \circ B$ and $B \circ A$ are $M-$ row equivalent, then $M_r(A \circ B) = M_r(B \circ A)$ so that $\Sigma_{i=1}^m |x_i y_i|_{ab} = \Sigma_{i=1}^m |y_i x_i|_{ab}$ which implies that $\Sigma_{i=1}^m |x_i|_a |y_i|_b = \Sigma_{i=1}^m |x_i|_b |y_i|_a$. But $\Sigma_{i=1}^m |x_i|_a |y_i|_b = \Sigma_{i=1}^m |x_i|_a (l - |y_i|_a)$ and $\Sigma_{i=1}^m |x_i|_b |y_i|_a = \Sigma_{i=1}^m (n - |x_i|_a)|y_i|_a$. This implies that $l\Sigma_{i=1}^m |x_i|_a = n\Sigma_{i=1}^m |y_i|_a$. Likewise $l\Sigma_{i=1}^m |x_i|_b = n\Sigma_{i=1}^m |y_i|_b$. This shows that the weak ratio property holds for A and B.

Corollary 16. *If A and B satisfy the weak ratio-property, then $i)$ the binary array $A \circ B$ (as well as $B \circ A$) is $M-$ambiguous and $ii) M_r(A^s \circ B^s) = M_r((A \circ B)^s)$, $M_c(A^s \circ B^s) = M_c((A \circ B)^s)$ where $A^s = A \circ A \circ \cdots \circ A$ (s times).*

Remark 17. $i)$ *Corresponding results similar to Theorem 15 and Corollary 16 hold good for $A \diamond B$ where $A, B \in \mathcal{PA}_2$ are arrays of sizes $n \times m$ and $l \times m$ respectively.*

ii) A particular case of weak ratio property of arrays $A, B \in \mathcal{PA}_2$ *of sizes* $m \times n$ *and* $m \times l$ *respectively, is obtained by requiring the words in the corresponding rows of* A *and* B *to satisfy the weak ratio property. If the words in the* m *rows of* A *in order are* $x_i, 1 \leq i \leq m$ *and in the* m *rows of* B *in order are* $y_i, 1 \leq i \leq m$ *and if the Parikh matrices* $M(x_i) = \begin{pmatrix} 1 & p_{i1} & r_{i1} \\ 0 & 1 & q_{i1} \\ 0 & 0 & 1 \end{pmatrix}$ *and* $M(y_i) = \begin{pmatrix} 1 & p_{i2} & r_{i2} \\ 0 & 1 & q_{i2} \\ 0 & 0 & 1 \end{pmatrix}$ *for* $1 \leq i \leq m$ *satisfy*

$$\frac{p_{i1}}{p_{i2}} = \frac{q_{i1}}{q_{i2}} = \alpha_i,\ 1 \leq i \leq m,$$

where $\alpha_i \neq 0$ *is a constant, then the binary arrays* A *and* B *are said to satisfy weak row-ratio property. Since* $\alpha_i = \frac{p_{i1}+q_{i1}}{p_{i2}+q_{i2}} = \frac{n}{l} = \frac{p_{j1}+q_{j1}}{p_{j2}+q_{j2}} = \alpha_j$, *we can take* $\alpha_i = \alpha$ *for all* i, $1 \leq i \leq m$, *where* α *is a constant. We note that* A *and* B *in* $\mathcal{PA}_2$ *over the alphabet* $\Sigma_2 = \{a < b\}$ *satisfy weak ratio property, if they satisfy weak row-ratio property, since* $\frac{|A|_a}{|B|_a} = \frac{\Sigma_{i=1}^m p_{i1}}{\Sigma_{i=1}^m p_{i2}} = \frac{\Sigma_{i=1}^m q_{i1}}{\Sigma_{i=1}^m q_{i2}} = \frac{|A|_b}{|B|_b}$ *and hence in this case, Theorem 15 holds.*
iii) We can analogously define a weak column ratio-property by considering the vertical words in the columns of A *and* B, *but requiring* A *to be a* $m \times n$ *array and* B, *a* $l \times n$ *array.*

In [27], certain sufficient condition for ambiguity of a word over $\Sigma_3 = \{a < b < c\}$, based on a 'ratio property', is obtained.

Definition 18. *Two words* w_1, w_2 *are said to satisfy the ratio property, written* $w_1 \sim_r w_2$, *if*

$$\Psi_3(w_1) = \begin{pmatrix} 1 & p_1 & p_{1,2} & p_{1,3} \\ 0 & 1 & p_2 & p_{2,3} \\ 0 & 0 & 1 & p_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and

$$\Psi_3(w_2) = \begin{pmatrix} 1 & q_1 & q_{1,2} & q_{1,3} \\ 0 & 1 & q_2 & q_{2,3} \\ 0 & 0 & 1 & q_3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

satisfy the condition

$$p_i = s \cdot q_i \quad (i = 1, 2, 3), \quad p_{i,i+1} = s \cdot q_{i,i+1}, \quad (i = 1, 2),$$

where $s > 0$ *is a constant integer.*

Lemma 19. [27] *If* w_1, w_2 *are any two words over* Σ_3 *such that* $w_1 \sim_r w_2$, *and* $x = \alpha w_1 w_2 \beta w_2 w_1 \gamma, y = \alpha w_2 w_1 \beta w_1 w_2 \gamma$, *then* $M(x) = M(y)$.

We now make use of Lemma 19 to construct ambiguous ternary picture arrays.

Theorem 20. *Let W_1, W_2 be any two arrays in $\mathcal{PA}_3$ with the same number of rows such that $W_1 = x_1 \diamond \cdots \diamond x_m, W_2 = y_1 \diamond \cdots \diamond y_m$, where for $1 \leq i \leq m$, the word x_i in the ith row of W_1 and the word y_i in the ith row of W_2 satisfy $x_i \sim_r y_i$. Let X, Y be two arrays in $\mathcal{PA}_3$ with the same number of rows such that $X = \xi \circ W_1 \circ W_2 \circ \eta \circ W_2 \circ W_1 \circ \zeta, Y = \xi \circ W_2 \circ W_1 \circ \eta \circ W_1 \circ W_2 \circ \zeta$, (for some arrays ξ, η, ζ). Then $M_r(X) = M_r(Y)$ and $M_c(X) = M_c(Y)$. As a consequence the ternary array X (as well as Y) is $M-$ambiguous.*

Proof. Since the words in the corresponding rows of the ternary arrays W_1, W_2 satisfy ratio property, it follows from Lemma 19 that $M_r(X) = M_r(Y)$. Clearly the words in the columns of X, Y are the same and hence $M_c(X) = M_c(Y)$.

The notion of "weak ratio property" of binary words has been considered for ternary words in [10].

Definition 21. *Two ternary words w_1, w_2 over Σ_3 are said to satisfy the weak ratio property and we write $w_1 \sim_{wr} w_2$, if the Parikh matrices of w_1 and w_2, namely,* $M(w_1) = \begin{pmatrix} 1 & p_1 & s_1 & z_1 \\ 0 & 1 & q_1 & t_1 \\ 0 & 0 & 1 & r_1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $M(w_2) = \begin{pmatrix} 1 & p_2 & s_2 & z_2 \\ 0 & 1 & q_2 & t_2 \\ 0 & 0 & 1 & r_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}$, $p_2 \neq 0 \, q_2 \neq 0, r_2 \neq 0$ *satisfy the following equality of the ratios:*

$$\frac{p_1}{p_2} = \frac{q_1}{q_2} = \frac{r_1}{r_2}.$$

In the case of ternary words over $\Sigma_3 = \{a < b < c\}$, the weak ratio property is not a sufficient condition for the words uv and vu to have the same Parikh matrix. For example [10] consider the words $u = abacbc$ and $v = bac$ over the alphabet Σ_3, with the property that $|u|_a = 2|v|_a$, $|u|_b = 2|v|_b$, $|u|_c = 2|v|_c$ and

$$M(u) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M(v) = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

but $M(uv) \neq M(vu)$ since,

$$M(uv) = \begin{pmatrix} 1 & 3 & 5 & 9 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M(vu) = \begin{pmatrix} 1 & 3 & 5 & 7 \\ 0 & 1 & 3 & 6 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

But conditions for words u, v over alphabets of size 3, to have the same Parikh matrix for the products uv and vu are known [10]. Two of these are stated in Lemmas 22 and 23.

Lemma 22. [10] *Let $u = xmi(x)$, $v = mi(x)x$ for some nonempty word $x \in \Sigma_3$, where $mi(x)$ is the mirror image or reverse of the word x. Then $M(uv) = M(vu)$.*

Lemma 23. [10] *Let $\Sigma_3 = \{a < b < c\}$ and let $u, v \in \Sigma^*$. Then $M(uv) = M(vu)$ if and only if $u \sim_{wr} v$ and $|uv|_{abc} = |vu|_{abc}$.*

Making use of these Lemmas 22 and 23 , we construct arrays X, Y in $\mathcal{PA}_3$ over $\Sigma_3 = \{a < b < c\}$ so that their row (or column) products have the same row (respy. column) Parikh matrix.
For an array X in $\mathcal{PA}_3$, let X_v, X_h respectively denote the arrays obtained by reflecting X on its rightmost column and bottommost row. For example, if

$$X = \begin{matrix} a & b & c & b \\ a & c & c & a \\ b & a & b & c \\ c & a & a & b \end{matrix}, \text{ then } X_v = \begin{matrix} b & c & b & a \\ a & c & c & a \\ c & b & a & b \\ b & a & a & c \end{matrix}, \ X_h = \begin{matrix} c & a & a & b \\ b & a & b & c \\ a & c & c & a \\ a & b & c & b \end{matrix}.$$

Theorem 24. *Given arrays X, Y in $\mathcal{PA}_3$,*
i) if the array $A = X \circ X_v$ and the array $B = X_v \circ X$, then the arrays $A \circ B$ and $B \circ A$ have the same row Parikh matrix and the same column Parikh matrix.
ii) if the array $C = Y \diamond Y_h$ and the array $D = Y_h \diamond Y$, then the arrays $C \diamond D$ and $D \diamond C$ have the same row Parikh matrix and the same column Parikh matrix.

Proof. If the array X has m rows and for $1 \leq i \leq m$, x_i is the word in the ith row of X, then every row of A is of the form $xmi(x)mi(x)x$ for $x \in \Sigma_3^*$ and every row of B is of the form $mi(x)xxmi(x)$ for $x \in \Sigma_3^*$, so that by Lemma 22, $A \circ B$ and $B \circ A$ have the same row Parikh matrix. Clearly $A \circ B$ and $B \circ A$ have the same column Parikh matrix as both the arrays have the same columns but in a different order. The proof of the second statement is similar.

Theorem 25. *i)Let A, B be two arrays in $\mathcal{PA}_3$ over $\Sigma_3 = \{a < b < c\}$ with the same number m of rows such that $A = x_1 \diamond \cdots \diamond x_m$, $B = y_1 \diamond \cdots \diamond y_m$, $x_i, y_i \in \Sigma_3^*, 1 \leq i \leq m$. Let $x_i \sim_{wr} y_i$ and $|x_i y_i|_{abc} = |y_i x_i|_{abc}$ for all $i, 1 \leq i \leq m$. Then the arrays $A \circ B$ and $B \circ A$ have the same row Parikh matrix and the same column Parikh matrix. ii)Let C, D be two arrays $\mathcal{PA}_3$ over $\Sigma_3 = \{a < b < c\}$ with the same number n of columns such that $C = u_1 \circ \cdots \circ u_n$, $B = v_1 \circ \cdots \circ v_n$, $u_j^t, v_j^t \in \Sigma_3^*, 1 \leq j \leq n$. Let $u_j^t \sim_{wr} v_j^t$ and $|u_j^t v_j^t|_{abc} = |v_j^t u_j^t|_{abc}$ for all $j, 1 \leq j \leq n$. Then the arrays $C \diamond D$ and $D \diamond C$ have the same row Parikh matrix and the same column Parikh matrix.*

Proof. The theorem follows from Lemma 23 by noting that the corresponding rows of A and B and the corresponding columns of C and D satisfy the conditions in Lemma 23.

5. Conclusion

The notion of Parikh matrix of a word has been extended to picture arrays. Properties of such matrices are obtained in the case of binary and ternary arrays, some of which in the binary case are specific to arrays. But many conditions obtained here for $M-$ambiguity of a picture array are only sufficient and so it remains to examine whether characterizations can be obtained. Also, we have defined $M-$equivalence by considering picture arrays of the same size. We could also consider $M-$equivalence of two picture arrays of different dimensions and examine its properties. For example, if $A = \begin{matrix} a & b & b & a \\ b & b & a & a \end{matrix}$ and $A = \begin{matrix} a & b \\ b & b \\ a & b \\ a & a \end{matrix}$, then $M_r(A) = M_r(B) = \begin{pmatrix} 1 & 4 & 2 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$ and $M_c(A) = M_c(B) = \begin{pmatrix} 1 & 4 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 1 \end{pmatrix}$. Also, in the area of discrete tomography, reconstruction of binary images has been extensively studied (See for example $[4, 11]$). It is worth examining whether the theoretical properties obtained here can be used in the problem of reconstruction of picture arrays or images.

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References

[1] A. Atanasiu, Binary amiable words, Intern. J. Found. Comput. Sci. 18, 2007: 387–400.
[2] A. Atanasiu, C. Martin-Vide, A. Mateescu, On the injectivity of the Parikh matrix mapping, Fund. Inform. 46, 2001: 1–11.
[3] A. Atanasiu, R. Atanasiu, I. Petre, Parikh matrices and amiable words, Theor. Comput. Sci. 390, 2008: 102–109.
[4] P. Bal*áz*s, Discrete tomographic reconstruction of binary images with disjoint components using shape information, Int. J. Shape Modeling, 14(2), 2008: 189–207.
[5] V. E. Brimkov and R. P. Barneva, Plane digitization and related combinatorial problems, Discrete Appl. Math. 147, 2005: 169-186.
[6] V. E. Brimkov, R. P. Barneva, Combinatorial approach to image analysis, Discrete Appl. Math. 157(16), 2009: 3359–3361.
[7] S.Z. Fazekas, B. Nagy, Scattered subword complexity of non-primitive words, J. Automata, Languages and Combinatorics 13, 2008: 233–247.
[8] S. Fosse, G. Richomme, Some characterizations of Parikh matrix equivalent binary words, Inf. Process. Lett. 92, 2004: 77–82.
[9] D. Giammarresi and A. Restivo, Two-dimensional languages, in: " Handbook of Formal Languages" (Eds. G.Rozenberg and A. Salomaa), Springer-Verlag, Vol. 3, 1997: 215–267.

[10] K. Mahalingam and K.G. Subramanian, Product of Parikh Matrices and Commutativity, Int. J. Found. Comput. Sci. 23, 2012: 207–223.

[11] V. Masilamani, K. Krithivasan, K. G. Subramanian, A. Miin Huey, Efficient Algorithms for Reconstruction of 2D-Arrays from Extended Parikh Images, In:Advances in Visual Computing, Lecture Notes in Comp. Sci. 5359, 2008: 1137–1146.

[12] A. Mateescu, Algebraic aspects of Parikh matrices, in: J. Karhumaki et al (Eds.), Theory is Forever, Springer, Berlin, 2004: 170-180.

[13] A. Mateescu, A. Salomaa, Matrix indicators for subword occurrences and ambiguity, Int. J. Found. Comput. Sci. 15, 2004: 277–292.

[14] A. Mateescu, A. Salomaa, S. Yu, Subword histories and Parikh matrices, J. Comp. System Sci. 68, 2004: 1–21.

[15] A. Mateescu, A. Salomaa, K. Salomaa, S. Yu, A Sharpening of the Parikh Mapping, Theoret. Informatics Appl. 35, 2001: 551–564.

[16] R.J. Parikh, On context-free languages, J. Assoc. Comput. Mach. 13, 1966: 570–581.

[17] H. J. Ryser, Combinatorial properties of matrices of zeros and ones, Canadian Journal of Mathematics, 9, 1957: 371-377.

[18] A. Salomaa, Formal Languages, Academic Press, New York, 1973.

[19] A. Salomaa, Subword histories and associated matrices, Theor. Comput. Sci. 407, 2008: 233–247.

[20] A. Salomaa, Characteristic words for Parikh matrices, In: Z. esik and Z.Fulop (eds.) Automata, Formal Languages and Related Topics, Inst. of Informatics, Univ. of Szeged, 2009:117-127.

[21] A. Salomaa, On the injectivity of Parikh matrix mappings, Fundam. Inf. 64, 2005: 391–404.

[22] A. Salomaa, Criteria for the matrix equivalence of words, Theor. Comp. Sci. 411, 2010: 1818–1827.

[23] A. Salomaa, S. Yu, Subword occurrences, Parikh matrices and Lyndon images, Int. J. Found. Comput. Sci. 21, 2010: 91–111.

[24] T.-F. Serbanuta, Extending Parikh matrices, Theor. Comput. Sci. 310, 2004: 233–246.

[25] V. N. Serbanuta,T.F. Serbanuta, Injectivity of the Parikh matrix mappings revisited, Fundam. Inf. 73, 2006: 265–283.

[26] V. N. Serbanuta, On Parikh Matrices, Ambiguity, and Prints, Intern. J. Found. Comput. Sci. 20, 2009: 151–165.

[27] K.G. Subramanian, A. Miin Huey, A. K. Nagar, On Parikh Matrices, Int. J. Found. Comput. Sci. 20(2), 2009: 211–219.

[28] K. G. Subramanian, K. Mahalingam, R. Abdullah, A.K. Nagar, Binary images, $M-$vectors and ambiguity, Lecture Notes in Comp. Sci. 6636, 2011: 248–260.