

# Multiperiod Portfolio Optimization with Multiple Risky Assets and General Transaction Costs<sup>☆</sup>

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## Abstract

We analyze the optimal portfolio policy for a multiperiod mean-variance investor facing multiple risky assets in the presence of general transaction costs. For proportional transaction costs, we give a closed-form expression for a no-trade region, shaped as a multi-dimensional parallelogram, and show how the optimal portfolio policy can be efficiently computed for many risky assets by solving a single quadratic program. For market impact costs, we show that at each period it is optimal to trade to the boundary of a state-dependent rebalancing region. Finally, we show empirically that the losses associated with ignoring transaction costs and behaving myopically may be large.

*Keywords:* Portfolio optimization; multiperiod utility; no-trade region; market impact.

*JEL Classification:* G11

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## 1. Introduction

Transaction costs play an important role in investment management because they can easily erode the gains from a trading strategy; see, for instance, DeMiguel et al. (2014). Under mild assumptions, it is possible to show that the optimal portfolio policy in the presence of transaction costs is characterized by a *no-trade region*: if the portfolio weights are inside this region, then it is optimal not to trade, and otherwise it is optimal to trade to the boundary of the no-trade region; see Magill and Constantinides (1976); Constantinides (1979, 1986). Unfortunately, standard portfolio selection assumptions such as constant relative risk aversion (CRRA) utility and borrowing constraints render the optimal portfolio problem with transaction costs analytically intractable. Most previous research has only computed the optimal portfolio policy numerically and generally for up to two risky assets; see Davis and Norman (1990); Akian et al. (1996); Leland (2000); Muthuraman and Kumar (2006), and Lynch and Tan (2010).

One exception is Gârleanu and Pedersen (2013) (hereafter G&P), who consider a framework that allows them to provide closed-form expressions for the optimal portfolio policy in the presence of *quadratic* transaction costs.<sup>1</sup> Their investor maximizes the present value of the mean-variance utility of her *wealth changes* at multiple time periods, she has access to unconstrained borrowing, and she faces multiple risky assets with predictable *price changes*. Several features of this framework make it analytically tractable. First, the focus on utility of wealth changes (rather than consumption) plus the access to unconstrained borrowing imply that there is no need to track the investor's total wealth evolution; instead, it is sufficient to track wealth change at each period. Second, the focus on price changes (rather than returns) implies that there is no need to track the risky-asset price evolution; instead, it is sufficient to account for price changes. Finally, the aforementioned features, combined with the use of mean-variance utility and quadratic transaction costs, place the problem in the category of linear quadratic control problems, which are analytically tractable.

In this paper, we use the formulation of G&P to study the optimal portfolio policies for general transaction costs. Our portfolio selection framework is both more general and

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<sup>1</sup> G&P consider *only* quadratic transaction costs; i.e., they do not consider proportional costs.

more specific than that considered by G&P. It is more general because we consider a broader class of transaction costs that includes not only quadratic transaction costs, but also the less analytically tractable proportional and market impact costs.<sup>2</sup> It is more specific because, consistent with most of the literature on proportional transaction costs, we consider the case with constant investment opportunity set, whereas G&P focus on the case with predictability.

Our framework is relevant for active institutional investors who typically operate multiple investment strategies simultaneously. These investors operate each investment strategy only for a few months or quarters, and discontinue the investment strategy once its performance deteriorates. When introducing a new investment strategy, the institutional investor must trade towards new portfolio positions, while balancing the utility benefits from holding the new portfolio with the transaction costs associated with the transition.

We make three contributions to the literature. Our first contribution is to characterize analytically the optimal portfolio policy for the case with many risky assets and *proportional* transaction costs. Specifically, we provide a closed-form expression for the no-trade region, which is shaped as a multi-dimensional parallelogram, and use the closed-form expressions to show how the size of the no-trade region shrinks with the investment horizon and the risk-aversion parameter, and grows with the level of proportional transaction costs and the discount factor. Moreover, we show how the optimal portfolio policy can be computed by solving a quadratic program—a class of optimization problems that can be efficiently solved for cases with up to thousands of risky assets.

Our second contribution is to study analytically the optimal portfolio policy in the presence of market impact costs, which arise when the investor makes large trades that distort market prices.<sup>3</sup> Traditionally, researchers have assumed that the market *price* impact is linear on the amount traded (see Kyle (1985)), and thus that market impact costs are quadratic.

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<sup>2</sup>It is also realistic to assume that there would be both proportional and quadratic transaction costs, but unfortunately this case is not analytically tractable. Specifically, neither the analytical approach employed by G&P nor our analytical approach apply to the case with joint proportional and quadratic costs. However, it is reasonable to assume that in each particular real-world situation one of these two types of transaction costs will dominate. For a small investor, proportional costs will dominate, and for an institutional investor the quadratic or market impact costs will dominate.

<sup>3</sup>This is particularly relevant for optimal execution, where institutional investors have to execute an investment decision within a fixed time interval; see Bertsimas and Lo (1998) and Engle et al. (2012).

Under this assumption, G&P derive closed-form expressions for the optimal portfolio policy within their multiperiod setting. However, Torre and Ferrari (1997), Grinold and Kahn (2000), and Almgren et al. (2005) show that the square root function is more appropriate for modeling market price impact, thus suggesting that market impact costs grow at a rate slower than quadratic. Our contribution is to extend the analysis by G&P to a *general case* where we are able to capture the distortions on market price through a power function with an exponent between one and two. For this general formulation, we show *analytically* that there exists a state-dependent *rebalancing region* for every time period, such that the optimal policy at each period is to trade to the boundary of the corresponding rebalancing region.

Our third contribution is to use an empirical dataset with the prices of 15 commodity futures to evaluate the losses associated with ignoring transaction costs and investing myopically, as well as identify how these losses depend on relevant parameters. We find that the losses associated with either ignoring transaction costs or behaving myopically can be large. Moreover, the losses from ignoring transaction costs increase on the level of transaction costs, and decrease with the investment horizon, whereas the losses from behaving myopically increase with the investment horizon and are unimodal on the level of transaction costs.

Our work is related to Dybvig (2005), who considers a single-period investor with mean-variance utility and proportional transaction costs. For the case with multiple risky assets, he shows that the optimal portfolio policy is characterized by a no-trade region shaped as a parallelogram, but the manuscript does not provide a detailed analytical proof. Like Dybvig (2005), we consider proportional transaction costs and mean-variance utility, but we extend the results to a multi-period setting, and show how the results can be rigorously proven analytically. In addition, we consider the case with market impact costs.

This manuscript is organized as follows. Section 2 describes the multiperiod framework under general transaction costs. Section 3 studies the case with proportional transaction costs, Section 4 the case with market impact costs, and Section 5 the case with quadratic transaction costs. Section 6 evaluates the utility loss associated with ignoring transaction costs and with behaving myopically for an empirical dataset on 15 commodity futures. Section 7 concludes. Appendix A contains the figures, and Appendix B contains the proofs

for the results in the paper.

## 2. General framework

Our framework is closely related to the one proposed by G&P. Like G&P, we consider an investor who maximizes the present value of the mean-variance utility of excess wealth changes (net of transaction costs) by investing in multiple risky assets and for multiple periods, and who has access to unconstrained borrowing. Moreover, like G&P, we make assumptions on the distribution of risky-asset price changes rather than returns. One difference with the framework proposed by G&P is that, consistent with most of the existing transaction cost literature, we focus on the case with a constant investment opportunity set; that is, we consider the case with independently and identically distributed (iid) price changes.

As discussed in the introduction, these assumptions render the model analytically tractable. The assumptions seem reasonable in the context of institutional investors who typically operate many different and relatively unrelated investment strategies. Each of these investment strategies represents only a fraction of the institutional investor's portfolio, and thus focusing on excess wealth changes and assuming unconstrained borrowing is a good approximation. The stationarity of price changes is also a reasonable assumption for institutional investors, who often have shorter investment horizons. This is because they operate each investment strategy only for a few months or quarters, and discontinue the investment strategy once its performance deteriorates.<sup>4</sup>

Finally, there are three main differences between our model and the model by G&P. First, we consider a more general class of transaction costs that includes not only quadratic

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<sup>4</sup> To test whether our results are also meaningful under the assumption of stationary returns, we consider an investor subject to the same investment framework as in our manuscript, but facing a risky asset with *stationary returns, rather than stationary price changes*. This portfolio problem is challenging even from a computational perspective, but we use a numerical approach similar to that in DeMiguel and Uppal (2005) to compute the optimal portfolio policy for small problems with one risky asset and up to ten trading dates. We then evaluate the utility loss associated with following the G&P-type portfolio policy proposed in Theorem 1 in Section 3. We find that the relative utility loss is typically smaller than 0.2% for investment horizons of up to six months, and smaller than 1.25% for investment horizons of up to one year. We thus conclude that the assumption that price changes are stationary is reasonable in the context of institutional investors with short-term horizons of up to one year.

transaction costs, but also proportional and market impact costs. Second, we consider both finite and infinite investment horizons, whereas G&P focus on the infinite horizon case. Finally, we assume iid price changes, whereas G&P focus on the case with predictable price changes. We now rigorously state this assumption.

Let  $r_{t+1}$  denote the vector of price changes (in excess of the risk-free asset price) between times  $t$  and  $t + 1$  for  $N$  risky assets.

**Assumption 1.** *Price changes,  $r_{t+1}$ , are independently and identically distributed (iid) with mean vector  $\mu \in \mathbb{R}^N$  and covariance matrix  $\Sigma \in \mathbb{R}^{N \times N}$ .*

The investor's decision in our framework can be written as:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1 - \rho)^t (x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t) - (1 - \rho)^{t-1} \kappa \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^p \right], \quad (1)$$

where  $x_t \in \mathbb{R}^N$  contains the number of shares of each of the  $N$  risky assets held in period  $t$ ,  $T$  is the investment horizon,  $\rho$  is the discount rate, and  $\gamma$  is the *absolute* risk-aversion parameter.<sup>5</sup>

The term  $\kappa \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^p$  is the transaction cost for the  $t$ th period,<sup>6</sup> where  $\kappa \in \mathbb{R}$  is the transaction cost parameter,  $\Lambda \in \mathbb{R}^{N \times N}$  is the symmetric positive semidefinite transaction cost matrix, and  $\|s\|_p$  is the  $p$ -norm of vector  $s$ ; that is,  $\|s\|_p^p = \sum_{i=1}^N |s_i|^p$ . This term allows us to capture the transaction costs associated with both small and large trades. Small trades typically do not impact market prices, and thus their transaction costs come from the bid-ask spread and other brokerage fees, which are modeled as proportional to the amount traded. Our transaction cost term captures proportional transaction costs for the case with  $p = 1$  and  $\Lambda = I$ , where  $I$  is the identity matrix.

Large trades can have both temporary as well as permanent impact on market prices. Market price impact is temporary when it affects a single transaction, and permanent when it affects every future transaction. For simplicity of exposition, we focus on the case with

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<sup>5</sup> Because the investment problem is formulated in terms of wealth changes, the mean-variance utility is defined in terms of the *absolute* risk aversion parameter, rather than the *relative* risk aversion parameter. Note that the relative risk-aversion parameter equals the absolute risk-aversion parameter times the wealth.

<sup>6</sup>The  $p$ th root of the positive definite matrix  $\Lambda$  can be defined as  $\Lambda^{1/p} = Q^T D^{1/p} Q$ , where  $\Lambda = Q^T D Q$  is the spectral decomposition of a symmetric positive definite matrix.

temporary market impact costs, but our analysis can be extended to the case with permanent impact costs following an approach similar to that in Section 4 of G&P. For market impact costs, Almgren et al. (2005) suggest that transaction costs grow as a power function with an exponent between one and two, and hence we consider in our analysis values of  $p \in (1, 2]$ . The transaction cost matrix  $\Lambda$  captures the distortions to market prices generated by the interaction between multiple assets. G&P argue that it can be viewed as a multi-dimensional version of Kyle's lambda (see Kyle (1985)), and they argue that a sensible choice for the transaction cost matrix is  $\Lambda = \Sigma$ . We consider this case as well as the case with  $\Lambda = I$  to facilitate the comparison with the case with proportional transaction costs.

Note that, like G&P, we also assume the investor can costlessly short the different assets. The advantage of allowing costless shortsales is that it allows us to obtain analytical expressions for the no-trade region for the case with proportional transaction costs, and for the rebalancing regions for the case with market impact costs. We state this assumption explicitly now.

**Assumption 2.** *The investor can costlessly short the risky assets.*

Finally, the multiperiod mean-variance framework proposed by G&P and the closely related framework described in Equation (1) differ from the traditional dynamic mean-variance approach, which attempts to maximize the mean-variance utility of *terminal wealth*. Part 1 of Proposition 1 below, however, shows that the utility given in Equation (1) is equal to the mean-variance utility of the change in excess of *terminal wealth* for the case where the discount rate  $\rho = 0$ . This shows that the framework we consider is not too different from the traditional dynamic mean-variance approach.<sup>7</sup> Also, a worrying feature of multiperiod mean-variance frameworks is that as demonstrated by Basak and Chabakauri (2010), they are often time-inconsistent: the investor may find it optimal to deviate from the ex-ante optimal policy as time goes by. Part 2 of Proposition 1 below, however, shows that the framework we consider is time-consistent.

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<sup>7</sup>Another difference is that the traditional dynamic mean-variance framework relies on the assumption that returns are stationary, whereas our framework relies on the assumption that price changes are stationary. As mentioned in Footnote 4, we find that the relative utility loss associated with assuming stationary price changes rather than stationary returns is small for investment horizons of up to one year.

**Proposition 1.** *Let Assumptions 1 and 2 hold, then the multiperiod mean-variance framework described in Equation (1) satisfies the following properties:*

1. *The utility given in Equation (1) is equivalent to the mean-variance utility of the change in excess terminal wealth for the case where the discount rate  $\rho = 0$ .*
2. *The optimal portfolio policy for the multiperiod mean-variance framework described in Equation (1) is time-consistent.*

### 3. Proportional transaction costs

We now study the case where transaction costs are proportional to the amount traded. This type of transaction cost is appropriate to model small trades, where the transaction cost originates from the bid-ask spread and other brokerage commissions. Section 3.1 characterizes analytically the no-trade region and the optimal portfolio policy, and Section 3.2 shows how the no-trade region depends on the level of proportional transaction costs, the risk-aversion parameter, the discount rate, the investment horizon, and the correlation and variance of asset price changes.

#### 3.1. The no-trade region

The investor's decision for this case can be written as:

$$\max_{\{x_t\}_{t=1}^T} \left\{ \sum_{t=1}^T \left[ (1 - \rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1 - \rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \right\}. \quad (2)$$

The following theorem characterizes the optimal portfolio policy.

**Theorem 1.** *Let Assumptions 1 and 2 hold, then:*

1. *It is optimal not to trade at any period other than the first period; that is,*

$$x_1 = x_2 = \dots = x_T. \quad (3)$$



2. The investor's optimal portfolio for the first period  $x_1$  (and thus for all subsequent periods) is the solution to the following quadratic programming problem:

$$\min_{x_1} (x_1 - x_0)^\top \Sigma (x_1 - x_0), \quad (4)$$

$$s.t. \quad \|\Sigma(x_1 - x^*)\|_\infty \leq \frac{\kappa}{(1 - \rho)\gamma} \frac{\rho}{1 - (1 - \rho)^T}. \quad (5)$$

where  $x_0$  is the starting portfolio, and  $x^* = \Sigma^{-1}\mu/\gamma$  is the optimal portfolio in the absence of transaction costs (the Markowitz or target portfolio).

3. Constraint (5) defines a no-trade region shaped as a parallelogram centered at the target portfolio  $x^*$ , such that if the starting portfolio  $x_0$  is inside this region, then it is optimal not to trade at any period, and if the starting portfolio is outside this no-trade region, then it is optimal to trade at the first period to the point in the boundary of the no-trade region that minimizes the objective function in (4), and not to trade thereafter.

A few comments are in order. First, a counterintuitive feature of our optimal portfolio policy is that it only involves trading in the first period. A related property, however, holds for most of the policies in the literature. Liu (2004), for instance, explains that: “the optimal trading policy involves possibly an initial discrete change (jump) in the dollar amount invested in the asset, followed by trades in the minimal amount necessary to maintain the dollar amount within a constant interval.” The “jump” in Liu’s policy, is equivalent to the first-period investment in our policy. The reason why our policy does not require any rebalancing after the first period is that it relies on the assumption that *prices changes* are iid. As a result, the portfolio and no-trade region in our framework are defined in terms of *number of shares*, and thus no rebalancing is required after the first period because realized price changes do not alter the number of shares held by the investor.

Second, inequality (5) provides a closed-form expression for the no-trade region, which is a multi-dimensional parallelogram centered around the target portfolio. This expression shows that it is optimal to trade only if the marginal increment in utility from trading in one of the assets is larger than the transaction cost parameter  $\kappa$ . To see this, note that

inequality (5) can be rewritten as

$$-\kappa e \leq (\gamma(1 - \rho)(1 - (1 - \rho)^T)/\rho)\Sigma(x_1 - x^*) \leq \kappa e, \quad (6)$$

where  $e$  is the  $N$ -dimensional vector of ones. Moreover, because Part 1 of Theorem 1 shows that it is optimal to trade only at the first period, it is easy to show that the term in the middle of (6) is the gradient (first derivative) of the discounted multiperiod mean variance utility  $\sum_{t=1}^T (1 - \rho)^t (x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t)$  with respect to  $x_1$ . Consequently, it is optimal to trade at the first period only if the *marginal* increase in the present value of the multiperiod mean-variance utility is larger than the transaction cost parameter  $\kappa$ .

Third, the optimal portfolio policy can be conveniently computed by solving the quadratic program (4)–(5). This class of optimization problems can be efficiently solved for cases with up to thousands of risky assets using widely available optimization software. As mentioned in the introduction, most of the existing results for the case with transaction costs rely on numerical analysis for the case with two risky assets. Our framework can be used to deal with cases with proportional transaction costs and hundreds or even thousands of risky assets.

### 3.2. Comparative statics

The following corollary establishes how the no-trade region depends on the level of proportional transaction costs, the risk-aversion parameter, the discount rate, and the investment horizon.

**Corollary 1.** *The no-trade region satisfies the following properties:*

1. *The no-trade region expands as the proportional transaction parameter  $\kappa$  increases.*
2. *The no-trade region shrinks as the risk-aversion parameter  $\gamma$  increases.*
3. *The no-trade region expands as the discount rate parameter  $\rho$  increases.*
4. *The no-trade region shrinks as the investment horizon  $T$  increases.*

Part 1 of Corollary 1 shows that, not surprisingly, the size of the no-trade region grows with the transaction cost parameter  $\kappa$ . The reason for this is that the larger the transaction

costs, the less willing the investor is to trade in order to diversify. This is illustrated in Panel (a) of Figure A.1, which depicts the no-trade regions for different values of the transaction cost parameter  $\kappa$  for the two commodity futures on gasoil and sugar.<sup>8</sup> Note also that (as discussed in Section 3.1) the no-trade regions for different values of the transaction cost parameter are all centered around the target portfolio.

Part 2 of Corollary 1 shows that the size of no-trade region decreases with the risk aversion parameter  $\gamma$ . Intuitively, as the investor becomes more risk-averse, the optimal policy is to move closer to the diversified (safe) position  $x^*$ , despite the transaction costs associated with this. Part 3 of Corollary 1 shows that the size of the no-trade region increases with the discount rate  $\rho$ . This makes sense intuitively because the larger the discount rate, the less important the utility is for future periods and thus the smaller the incentive is to trade today.

Finally, Part 4 of Corollary 1 shows that the size of the no-trade region decreases with the investment horizon  $T$ . To see this intuitively, note that we have shown that the optimal policy is to trade at the first period and hold this position thereafter. Then, a multiperiod investor with shorter investment horizon will be more concerned about the transaction costs incurred at the first stage, compared with the investor who has a longer investment horizon. Finally, when  $T \rightarrow \infty$ , the no-trade region shrinks to the parallelogram bounded by  $\kappa\rho/((1 - \rho)\gamma)$ , which is much closer to the center  $x^*$ . When  $T = 1$ , the multiperiod problem reduces to the single-period problem studied by Dybvig (2005).

The no-trade region also depends on the correlation between assets. Panel (b) of Figure A.1 shows the no-trade regions for different correlations.<sup>9</sup> When the two assets are positively correlated, the parallelogram leans to the left, reflecting the substitutability of the two risky assets, whereas with negative correlation it leans to the right. In the absence of correlations the no-trade region becomes a rectangle.

Finally, the impact of variance on the no-trade region is shown in Panel (c) of Figure A.1,

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<sup>8</sup>Although we illustrate Corollary 1 using two commodity futures, the results apply to the general case with  $N$  risky assets.

<sup>9</sup>Because change in correlation also makes the target shift, in order to emphasize how correlation affects the shape of the region, we change the covariance matrix  $\Sigma$  in a manner so as to keep the Markowitz portfolio (the target) fixed, similar to the analysis in Muthuraman and Kumar (2006).

where for expositional clarity we have considered the case with two uncorrelated symmetric risky assets. Like Muthuraman and Kumar (2006), we find that as variance increases, the no-trade region moves towards the risk-free asset because the investor is less willing to hold the risky assets. Also the size of the no-trade region shrinks as the variance increases because the investor is more willing to incur transaction costs in order to diversify her portfolio.

#### 4. Market impact costs

We now consider the case of large trades that may impact market prices. As discussed in Section 2, to simplify the exposition we focus on the case with temporary market impact costs, but the analysis can be extended to the case with permanent impact costs following an approach similar to that in Section 4 of G&P. Almgren et al. (2005) suggest that market impact costs grow as a power function with an exponent between one and two, and hence we consider a general case, where the transaction costs are given by the  $p$ -norm with  $p \in (1, 2)$ , and where we capture the distortions on market price through the transaction cost matrix  $\Lambda$ . For exposition purposes, we first study the single-period case.

##### 4.1. The single-period case

For the single-period case, the investor's decision is:

$$\max_x (1 - \rho)(x^\top \mu - \frac{\gamma}{2} x^\top \Sigma x) - \kappa \|\Lambda^{1/p}(x - x_0)\|_p^p, \quad (7)$$

where  $1 < p < 2$ . Problem (7) can be solved numerically, but unfortunately it is not possible to obtain closed-form expressions for the optimal portfolio policy. The following proposition, however, shows that the optimal portfolio policy is to trade to the boundary of a *rebalancing region* that depends on the starting portfolio and contains the target or Markowitz portfolio.

**Proposition 2.** *Let Assumptions 1 and 2 hold. Then, if the starting portfolio  $x_0$  is equal to the target or Markowitz portfolio  $x^*$ , the optimal policy is not to trade. Otherwise, it is optimal to trade to the boundary of the following rebalancing region:*

$$\frac{\|\Lambda^{-1/p}\Sigma(x - x^*)\|_q}{p\|\Lambda^{1/p}(x - x_0)\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}, \quad (8)$$

where  $q$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

Comparing Proposition 2 with Theorem 1 we observe that there are three main differences between the cases with proportional and market impact costs. First, for the case with market impact costs it is always optimal to trade (except in the trivial case where the starting portfolio coincides with the target or Markowitz portfolio), whereas for the case with proportional transaction costs it may be optimal not to trade if the starting portfolio is inside the no-trade region. Second, the rebalancing region depends on the starting portfolio  $x_0$ , whereas the no-trade region is independent of it. Third, the rebalancing region contains the target or Markowitz portfolio, but it is not centered around it, whereas the no-trade region is centered around the Markowitz portfolio.

Note that, as in the case with proportional transaction costs, the size of the rebalancing region increases with the transaction cost parameter  $\kappa$ , and decreases with the risk-aversion parameter. Intuitively, the more risk-averse the investor, the larger her incentives are to trade and diversify her portfolio. Also, the rebalancing region grows with  $\kappa$  because the larger the transaction cost parameter, the less attractive to the investor it is to trade to move closer to the target portfolio.

Panel (a) in Figure A.2 depicts the rebalancing region and the optimal portfolio policy for a two-asset example when  $\Lambda = I$ , which is a realistic assumption when the amount traded is small. The graph shows that the rebalancing region is convex and contains the Markowitz portfolio. Moreover, it shows that the optimal policy is to trade to the boundary of the rebalancing region. We have produced a similar figure for the case when  $\Lambda = \Sigma$ , which G&P argue is realistic for larger trades, but the qualitative insights from the figure are similar and thus we do not report it to conserve space.

#### 4.2. The multiperiod case

The investor's decision for this case can be written as:

$$\max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1 - \rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1 - \rho)^{t-1} \kappa \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^p \right]. \quad (9)$$

As in the single-period case, it is not possible to provide closed-form expressions for the optimal portfolio policy, but the following theorem illustrates the analytical properties of the optimal portfolio policy.

**Theorem 2.** *Let Assumptions 1 and 2 hold, then:*

1. *If the starting portfolio  $x_0$  is equal to the target or Markowitz portfolio  $x^*$ , then the optimal policy is not to trade at any period.*
2. *Otherwise it is optimal to trade at every period. Moreover, at the  $t$ th period it is optimal to trade to the boundary of the following rebalancing region:*

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t} \Lambda^{-1/p} \Sigma (x_s - x^*)\|_q}{p \|\Lambda^{1/p} (x_t - x_{t-1})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}, \quad (10)$$

where  $q$  is such that  $\frac{1}{p} + \frac{1}{q} = 1$ .

Theorem 2 shows that for the multiperiod case with market impact costs it is optimal to trade at every period (except in the trivial case where the starting portfolio coincides with the Markowitz portfolio). This is in contrast to the case with proportional transaction costs, where it is not optimal to trade at any period other than the first. The reason for this is that with strictly convex transaction costs,  $p > 1$ , the transaction cost associated with a trade can be reduced by breaking it into several smaller transactions. For the case with proportional transaction costs, on the other hand, the cost of a transaction is the same whether executed at once or broken into several smaller trades, and thus it makes economic sense to carry out all the trading in the first period to take advantage of the utility benefits from the beginning.

Note also that for the case with market impact costs at every period it is optimal to trade to the boundary of a *state-dependent* rebalancing region that depends not only on the starting portfolio, but also on the portfolio for every subsequent period. Finally, note that the size of the rebalancing region for period  $t$ , assuming the portfolios for the rest of the periods are fixed, increases with the transaction cost parameter  $\kappa$  and decreases with the discount rate  $\rho$  and the risk-aversion parameter  $\gamma$ .

The following proposition shows that the rebalancing region for period  $t$  contains the rebalancing region for every subsequent period. Moreover, the rebalancing region converges to the Markowitz portfolio as the investment horizon grows, and thus the optimal portfolio  $x_T$  converges to the target portfolio  $x^*$  in the limit when  $T$  goes to infinity. The intuition behind this result is again that the strict convexity of the transaction cost function for the case with  $p > 1$  implies that it is optimal to trade at every period, but it is never optimal to trade all the way to the Markowitz portfolio. Thus, the investor's portfolio converges to the Markowitz portfolio only as time goes to infinity.

**Proposition 3.** *Let Assumptions 1 and 2 hold, then:*

1. *The rebalancing region for the  $t$ -th period contains the rebalancing region for every subsequent period.*
2. *Every rebalancing region contains the Markowitz portfolio.*
3. *The rebalancing region converges to the Markowitz portfolio in the limit when the investment horizon goes to infinity.*

Panel (b) in Figure A.2 shows the optimal portfolio policy and the rebalancing region for the two commodity futures on gasoil and sugar with an investment horizon  $T = 3$  when  $\Lambda = I$ . The figure shows how the rebalancing region for each period contains the rebalancing region for subsequent periods. Moreover, every rebalancing region contains, but is not centered at, the Markowitz portfolio  $x^*$ . Every period, the optimal policy is to trade to the boundary of the rebalancing region, and thus approach the Markowitz portfolio  $x^*$ .

Finally, we study numerically the impact of the market impact cost growth rate  $p$  on the optimal portfolio policy. Figure A.3 shows the rebalancing regions and trading trajectories for investors with different transaction growth rates  $p = 1, 1.25, 1.5, 1.75, 2$ . When the transaction cost matrix  $\Lambda = I$ , Panel (a) shows how the rebalancing region depends on  $p$ . In particular, for  $p = 1$  we recover the case with proportional transaction costs, and hence the rebalancing region becomes a parallelogram. For  $p = 2$ , the rebalancing region becomes an ellipse. And for values of  $p$  between 1 and 2, the rebalancing regions resemble superellipses<sup>10</sup>

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<sup>10</sup>The general expression for a superellipse is  $|\frac{x}{a}|^m + |\frac{y}{b}|^n = 1$  with  $m, n > 0$ .

but not centered at the target portfolio  $x^*$ . On the other hand, Panel (b) in Figure A.3 shows how the trading trajectories depend on  $p$  for a particular investment horizon of  $T = 10$  days. We observe that, as  $p$  grows, the trading trajectories become more curved and the investor converges towards the target portfolio at a slower rate. To conserve space, we do not provide the figure for the case  $\Lambda = \Sigma$ , but we find that for this case the trajectories are less curved as  $p$  grows, and they become a straight line for  $p = 2$ .

## 5. Quadratic transaction costs

For the case with quadratic transaction costs, our framework differs from that in G&P in two respects only. First, G&P's work focuses on the impact of predictability, whereas consistent with most of the existing literature on transaction costs we assume a *constant investment opportunity set*. Second, G&P consider an infinite horizon, whereas we allow for a finite investment horizon. The next theorem adapts the results of G&P to obtain an explicit characterization of the optimal portfolio policy. The result is a straightforward variation of the results by G&P and thus we do not provide a proof to conserve space.

**Theorem 3.** *Let Assumptions 1 and 2 hold, then:*

1. *The optimal portfolio  $x_t, x_{t+1}, \dots, x_T$  satisfies the following equations:*

$$x_t = A_1 x^* + A_2 x_{t-1} + A_3 x_{t+1}, \quad \text{for } t = 1, 2, \dots, T-1 \quad (11)$$

$$x_t = B_1 x^* + B_2 x_{t-1}, \quad \text{for } t = T, \quad (12)$$

where

$$A_1 = (1 - \rho)\gamma [(1 - \rho)\gamma\Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Sigma,$$

$$A_2 = 2\kappa [(1 - \rho)\gamma\Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Lambda,$$

$$A_3 = 2(1 - \rho)\kappa [(1 - \rho)\gamma\Sigma + 2\kappa\Lambda + 2(1 - \rho)\kappa\Lambda]^{-1} \Lambda,$$



with  $A_1 + A_2 + A_3 = I$ , and

$$B_1 = (1 - \rho)\gamma[(1 - \rho)\gamma\Sigma + 2\kappa\Lambda]^{-1}\Sigma,$$

$$B_2 = 2\kappa[(1 - \rho)\gamma\Sigma + 2\kappa\Lambda]^{-1}\Lambda,$$

with  $B_1 + B_2 = I$ .

2. *The optimal portfolio converges to the Markowitz portfolio as the investment horizon  $T$  goes to infinity.*

Note that the system of equations (11)–(12) can be rewritten as a linear system of equations with  $N \times T$  decisions, where  $N$  is the number of assets and  $T$  is the investment horizon. This linear system can be easily solved numerically for realistic problems. Note also that Theorem 3 shows that the optimal portfolio for each stage is a combination of the Markowitz strategy (the target portfolio), the previous period portfolio, and the next period portfolio. The next corollary provides an interesting property of the optimal portfolio policy for the case where the transaction costs matrix is proportional to the covariance matrix, which G&P argue is realistic.

**Corollary 2.** *Let Assumptions 1 and 2 hold, and let  $\Lambda = \Sigma$ . Then, the optimal portfolios for periods  $t = 1, 2, \dots, T$  lie on a straight line.*

Corollary 2 shows that, when  $\Lambda = \Sigma$ , the solution becomes simpler and easier to interpret than when  $\Lambda = I$ . Note that when  $\Lambda = \Sigma$ , matrices  $A$  and  $B$  in Theorem 3 become scalars, and hence the optimal portfolio at period  $t$  can be expressed as a linear combination of the Markowitz portfolio, the previous period portfolio and the next period portfolio. For this reason, it is intuitive to observe that the optimal trading strategies for all the periods must lie on a straight line.

## 6. Empirical analysis

In this section, we study empirically the losses associated with ignoring transaction costs and investing myopically, as well as how these losses depend on the transaction cost parameter, the investment horizon, the risk-aversion parameter, and the discount rate. We first

consider the case with proportional transaction costs, and then study how the monotonicity properties of the losses change when transaction costs are quadratic. We have also considered the case with market impact costs ( $p = 1.5$ ), but the monotonicity properties for this case are in the middle of those for the cases with  $p = 1$  and  $p = 2$  and thus we do not report the results to conserve space.

For each type of transaction cost (proportional or quadratic), we consider three different portfolio policies. First, we consider the *target* portfolio policy, which consists of trading to the target or Markowitz portfolio in the first period and not trading thereafter. This is the optimal portfolio policy for an investor in the absence of transaction costs. Second, we consider the *static* portfolio policy, which consists of trading at each period to the solution to the single-period problem subject to transaction costs. This is the optimal portfolio policy for a myopic investor who takes into account transaction costs. Third, we consider the *multiperiod* portfolio policy, which is the optimal portfolio policy for a multiperiod investor who takes into account transaction costs.

Finally, we evaluate the utility of each of the three portfolio policies using the appropriate multiperiod framework. That is, when considering proportional transaction costs, we evaluate the investor's utility from each portfolio with the objective function in equation (2); and when considering quadratic transaction costs, we evaluate the investor's utility using the objective function (1). These utilities are in units of wealth, and to facilitate the comparison of the different policies, we report the percentage utility loss of the suboptimal portfolio policies with respect to the utility of the optimal multiperiod portfolio policy. Note that the percentage utility loss is expressed in relative terms. Specifically, the utility loss of any suboptimal portfolio policy with respect to the utility of optimal multiperiod portfolio policy is defined as the difference between the investor's utilities from the optimal multiperiod portfolio policy and the suboptimal portfolio policy divided by the utility from the optimal multiperiod portfolio policy. Note also that, as argued by DeMiguel et al. (2009, Footnote 17), it can be shown that the mean-variance utility is approximately equal to the certainty equivalent of an investor with quadratic utility, and thus although we report relative utility losses in this section, they can be interpreted equivalently as relative certainty equivalent losses.

We consider an empirical dataset similar to the one used by G&P.<sup>11</sup> In particular, the dataset is constructed with 15 commodity futures: Aluminum, Copper, Nickel, Zinc, Lead, and Tin from the London Metal Exchange (LME), Gasoil from the Intercontinental Exchange (ICE), WTI Crude, RBOB Unleaded Gasoline, and Natural Gas from the New York Mercantile Exchange (NYMEX), Gold and Silver from the New York Commodities Exchange (COMEX), and Coffee, Cocoa, and Sugar from the New York Board of Trade (NYBOT). The dataset contains daily data from July 7th, 2004 until September 19th, 2012. For our evaluation, we replace the mean and covariance matrix of price changes with their sample estimators.

## 6.1. Proportional transaction costs

### 6.1.1. Base case

For our base case, we adapt the parameters used by G&P in their empirical analysis to the case with proportional transaction costs. We assume proportional transaction costs of 50 basis points ( $\kappa = 0.005$ ), an absolute risk-aversion parameter of  $\gamma = 10^{-6}$ , which corresponds to a relative risk aversion of one for a small investor managing one million dollars,<sup>12</sup> an annual discount rate of  $\rho = 2\%$ , and an investment horizon of  $T = 22$  days (one month). For all the cases, the investor's initial portfolio is the equally weighted portfolio; that is, the investor splits her one million dollars equally among the 15 assets.

For our base case, we observe that the utility loss associated with investing myopically (that is, the relative difference between the utility of the *multiperiod* portfolio policy and the *static* portfolio policy) is 60.46%. This utility loss is large because the no-trade region corresponding to the static portfolio policy contains the equally-weighted portfolio, and thus the static portfolio policy is to remain at the starting equally-weighted portfolio, which attains a much lower multiperiod mean-variance utility than the multiperiod portfolio policy. The utility loss associated with ignoring transaction costs altogether (that is, the relative difference between the utility of the *multiperiod* portfolio policy and the *target* portfolio

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<sup>11</sup>We thank Alberto Martin-Utrera for making this dataset available to us.

<sup>12</sup>G&P consider a smaller absolute risk aversion  $\gamma = 10^{-9}$ , which corresponds to a larger investor managing  $M = 10^9$  dollars. It makes sense, however, to consider a smaller investor (and thus a larger absolute risk-aversion parameter) in the context of proportional transaction costs because these are usually associated with small trades.

policy) is 49.33%. This loss is large because the target portfolio policy requires a large amount of trading in the first period that results in large transaction costs. Summarizing, we find that the loss associated with either ignoring transaction costs or behaving myopically can be substantial.<sup>13</sup> Section 6.1.2 confirms that this is also true when we change the relevant model parameters.

### 6.1.2. Comparative statics

We study numerically how the utility losses associated with ignoring transaction costs (i.e., with the static portfolio) and investing myopically (i.e., with the target portfolio) depend on the transaction cost parameter, the investment horizon, the risk-aversion parameter, and the discount rate.

Panel (a) in Figure A.4 depicts the utility loss associated with the target and static portfolios for values of the proportional transaction cost parameter  $\kappa$  ranging from 0 to 460 basis points (which is the value of  $\kappa$  for which the optimal multiperiod policy is not to trade). As expected, the utility loss associated with ignoring transaction costs is zero in the absence of transaction costs and increases monotonically with transaction costs. Moreover, for large transaction cost parameters, the utility loss associated with ignoring transaction costs grows linearly with  $\kappa$  and can be very large.<sup>14</sup> The utility losses associated with behaving myopically are unimodal (first increasing and then decreasing) in the transaction cost parameter, being zero for the case with zero transaction costs (because both the single-period and multiperiod portfolio policies coincide with the target or Markowitz portfolio), and for the case with large transaction costs (because both the single-period and multiperiod portfolio policies result in little or no trading). The utility loss of behaving myopically reaches a maximum of 80% for a level of transaction costs of around 5 basis points.

Panel (b) in Figure A.4 depicts the utility loss associated with investing myopically and

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<sup>13</sup> We find that the multiperiod portfolio policy contains aggregate short positions of around 30% of the total wealth invested. We acknowledge that it would be more realistic to assume that there is a cost associated with holding short positions, but Assumption 2 allows us to provide closed-form expressions for the no-trade region and thus helps to keep our exposition simple. We have computed numerically the no-trade region for the case with shortsale constraints and find that the qualitative findings about the shape and size of the no-trade region continue to hold.

<sup>14</sup>In fact, the utility of the target portfolio policy is negative (and thus the utility loss is larger than 100%) for  $\kappa$  larger than 69 basis points because of the high transaction costs associated with this policy.

ignoring transaction costs for investment horizons ranging from  $T = 5$  (one week) to  $T = 260$  (over one year). Not surprisingly, the utility loss associated with behaving myopically grows with the investment horizon. Also, the utility loss associated with ignoring transaction costs is very large for short-term investors, and decreases monotonically with the investment horizon. The reason for this is that the size of the no-trade region for the multiperiod portfolio policy decreases monotonically with the investment horizon, and thus the target and multiperiod policies become similar for long investment horizons. This makes sense intuitively: by adopting the Markowitz portfolio, a multiperiod investor incurs transaction cost losses at the first period, but makes mean-variance utility gains for the rest of the investment horizon. Hence, when the investment horizon is long, the transaction losses are negligible compared with the utility gains.

Finally, we find that the relative utility losses associated with investing myopically and ignoring transaction costs do not depend on the risk-aversion parameter or the discount rate.

## 6.2. Quadratic transaction costs

In this section we study whether and how the presence of quadratic transaction costs (as opposed to proportional transaction costs) affects the utility losses of the static and target portfolios.

### 6.2.1. The base case

Our base case parameters are similar to those adopted in G&P. We assume that the matrix  $\Lambda = \Sigma$  and set the absolute risk-aversion parameter  $\gamma = 10^{-8}$ , which corresponds to an investor with relative-risk aversion of one who manages 100 million dollars,<sup>15</sup> discount rate  $\rho = 2\%$  annually, transaction costs parameter  $\kappa = 1.5 \times 10^{-7}$  (which corresponds to  $\lambda = 3 \times 10^{-7}$  in G&P's formulation), investment horizon  $T = 22$  days (one month), and an equal-weighted initial portfolio.

Similar to the case with proportional transaction costs, we find that the losses associated with either ignoring transaction costs or behaving myopically are substantial. For instance,

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<sup>15</sup>G&P choose a smaller absolute risk aversion parameter  $\gamma = 10^{-9}$ , which corresponds to an investor with relative-risk aversion of one who manages one billion dollars. The insights from our analysis are robust to the use of  $\gamma = 10^{-9}$ , but we choose  $\gamma = 10^{-8}$  because this results in figures that are easier to interpret.

for the base case we find that the utility loss associated with investing myopically is 28.98%, whereas the utility loss associated with ignoring transaction costs is 109.14%. Moreover, we find that the utility losses associated with the target portfolio are relatively larger, compared to those of the static portfolio, for the case with quadratic transaction costs. The explanation for this is that the target portfolio requires large trades in the first period, which are penalized heavily in the context of *quadratic* transaction costs. The static portfolio, on the other hand, results in smaller trades over successive periods and this will result in smaller overall *quadratic* transaction costs.

Finally, we have carried out comparative static analyses similar to those for proportional transaction costs, but the qualitative insights we have obtained are similar to those for proportional transaction costs, and thus we do not report the results to conserve space.

## 7. Conclusions

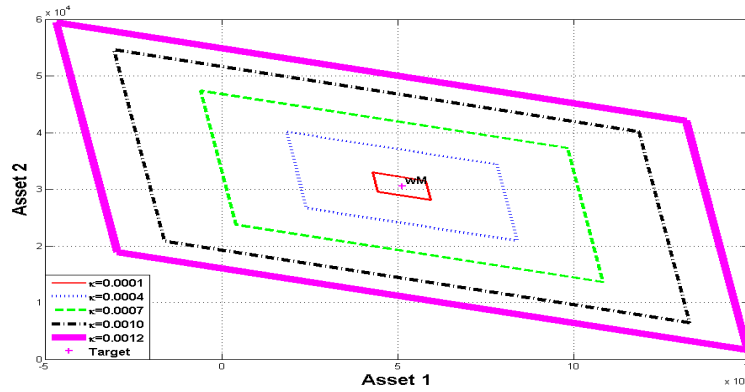
We study the optimal portfolio policy for a multiperiod mean-variance investor facing multiple risky assets subject to proportional, market impact, or quadratic transaction costs. For the case with proportional transaction costs, we provide a closed-form expression for a no-trade region shaped as a parallelogram, and use this closed-form expression to show how the no-trade region shrinks with the investment horizon and the risk-aversion parameter, and grows with the level of proportional transaction costs and the discount rate. Moreover, we show that the optimal portfolio policy can be conveniently computed by solving a single quadratic program for problems with up to thousands of risky assets. For the case with market impact costs, the optimal portfolio policy is to trade to the boundary of a state-dependent rebalancing region. In addition, the rebalancing region converges to the Markowitz portfolio as the investment horizon grows. We also show numerically that the losses associated with ignoring transaction costs or investing myopically may be large.

## Appendix A. Figures

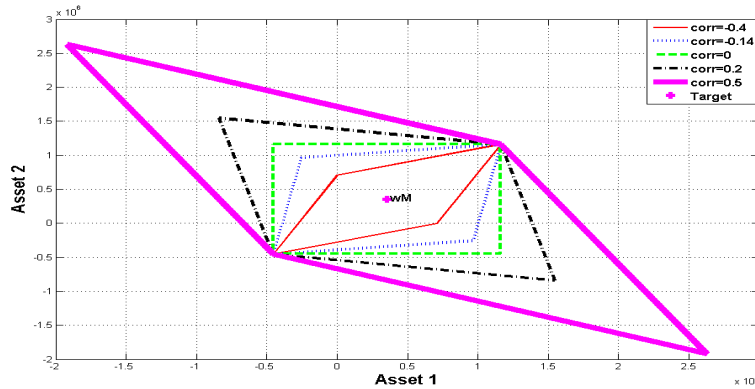
**Figure A.1: No-trade region: comparative statics.**

This figure shows how the no-trade region for a multiperiod investor subject to proportional transaction costs depends on relevant parameters. For the base case, we consider a proportional transaction cost parameter  $\kappa = 0.005$ , annual discount factor  $\rho = 2\%$ , absolute risk-aversion parameter  $\gamma = 10^{-6}$ , and mean and covariance matrix of price changes equal to the sample estimators for the commodity futures on gasoil and sugar.

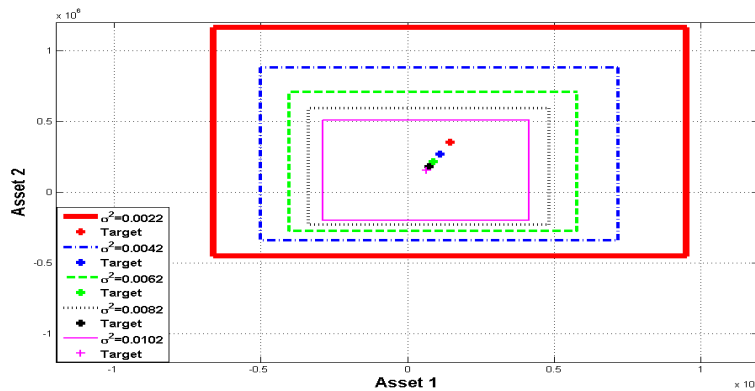
(a) No-trade regions for different  $\kappa$



(b) No-trade regions for different correlations



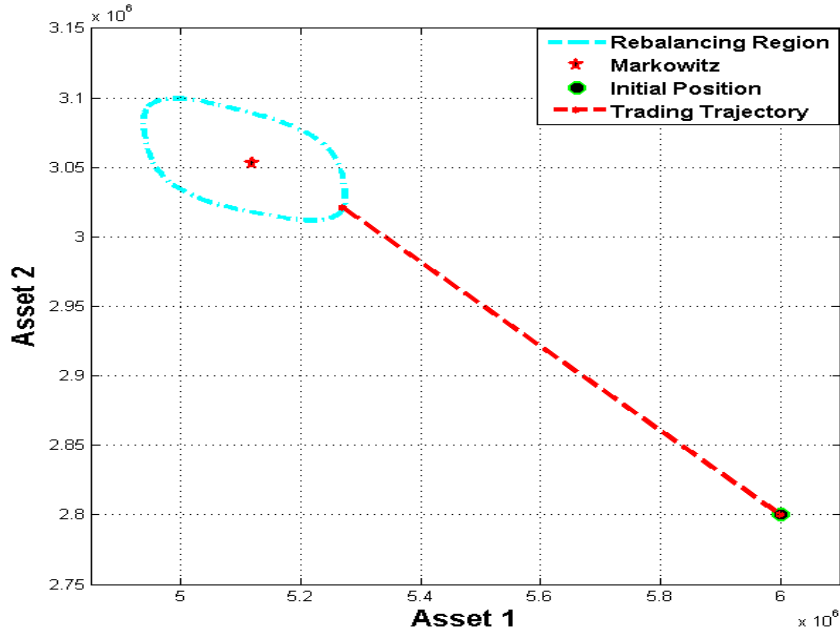
(c) No-trade regions for different  $\sigma^2$



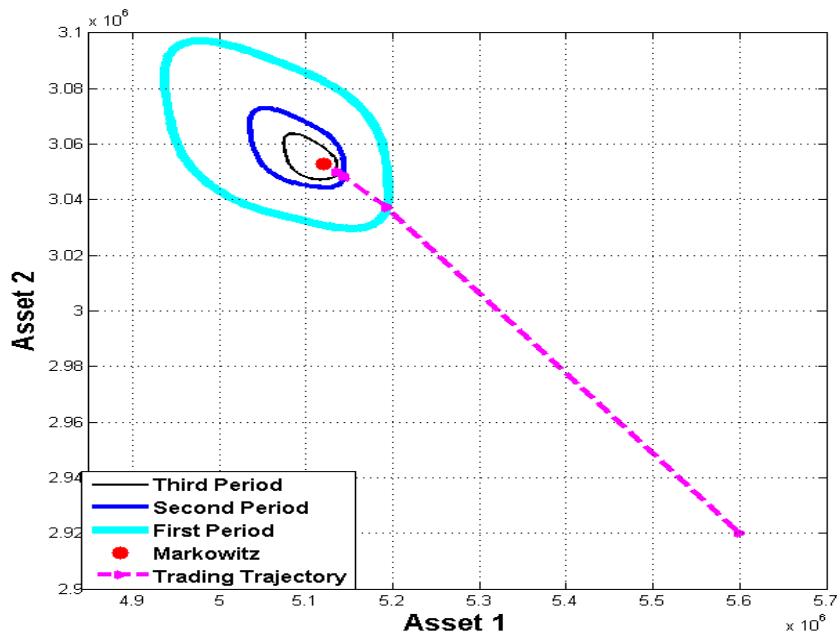
**Figure A.2: Rebalancing region for market impact costs.**

This figure depicts the rebalancing region for an investor subject to market impact costs. We consider an absolute risk aversion parameter  $\gamma = 10^{-7}$ , annual discount rate  $\rho = 2\%$ , transaction cost matrix  $\Lambda = I$ , the exponent of the power function is  $p = 1.5$ , the transaction cost parameter  $\kappa = 1.5 \times 10^{-8}$ , and the mean and covariance matrix of price changes are equal to the sample estimators for the commodity futures on gasoil and sugar. Panel (a) depicts the rebalancing region for a single-period investor, and Panel (b) for a multiperiod investor.

(a) Rebalancing region for single-period investor



(b) Rebalancing region for multiperiod investor

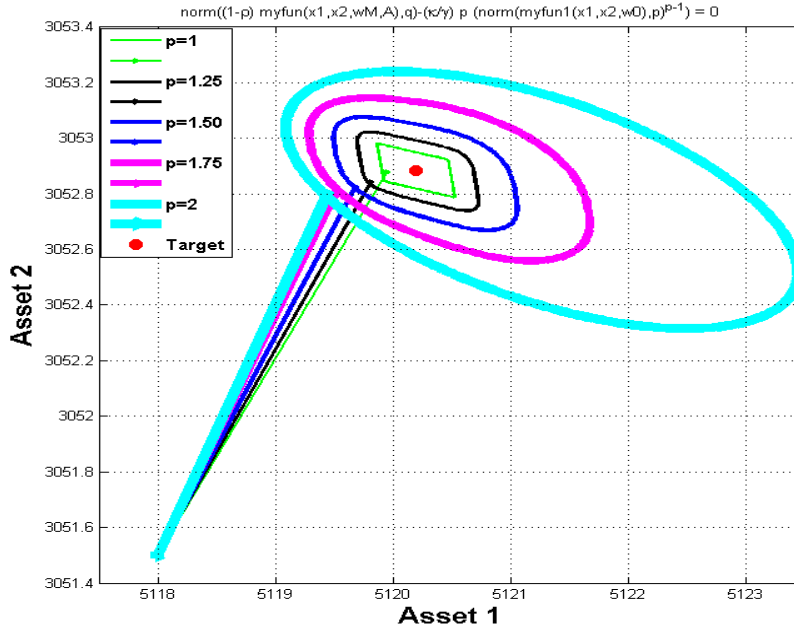




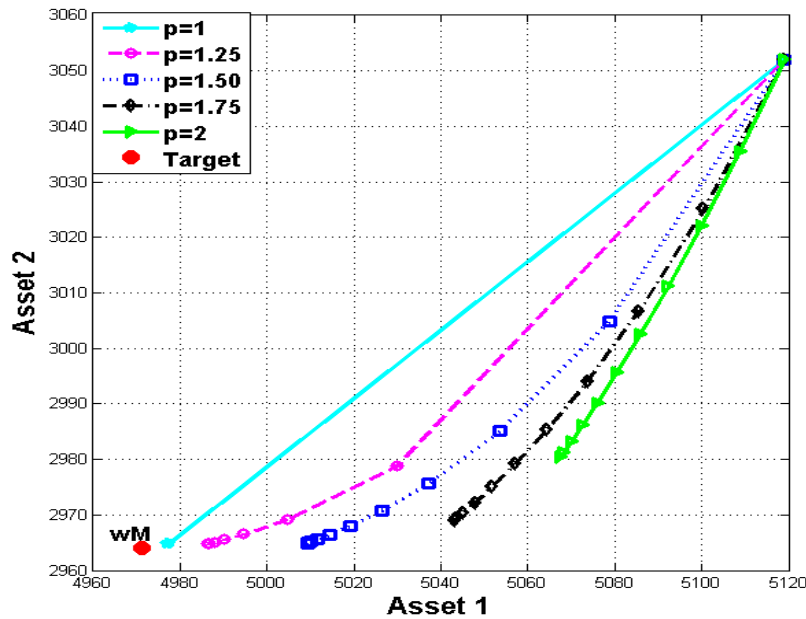
**Figure A.3: Rebalancing regions and trading trajectories for different  $p$ .**

This figure shows how the rebalancing regions and trading trajectories for the market impact costs model change with the exponent of the transaction cost function  $p$ . Panel (a) depicts the rebalancing regions for the single-period investor, with transaction cost parameter  $\kappa = 1.5 \times 10^{-8}$  and annual discount rate  $\rho = 50\%$ . Panel (b) depicts the multiperiod optimal trading trajectories when the investment horizon  $T = 10$ , with transaction cost parameter  $\kappa = 5 \times 10^{-6}$ , and annual discount rate  $\rho = 5\%$ . In both cases, we consider transaction costs matrix  $\Lambda = I$ , the risk-aversion parameter  $\gamma = 10^{-4}$ , and mean and covariance matrix of price changes equal to the sample estimators for the commodity futures on gasoil and sugar.

(a) Rebalancing regions depending on exponent  $p$ .



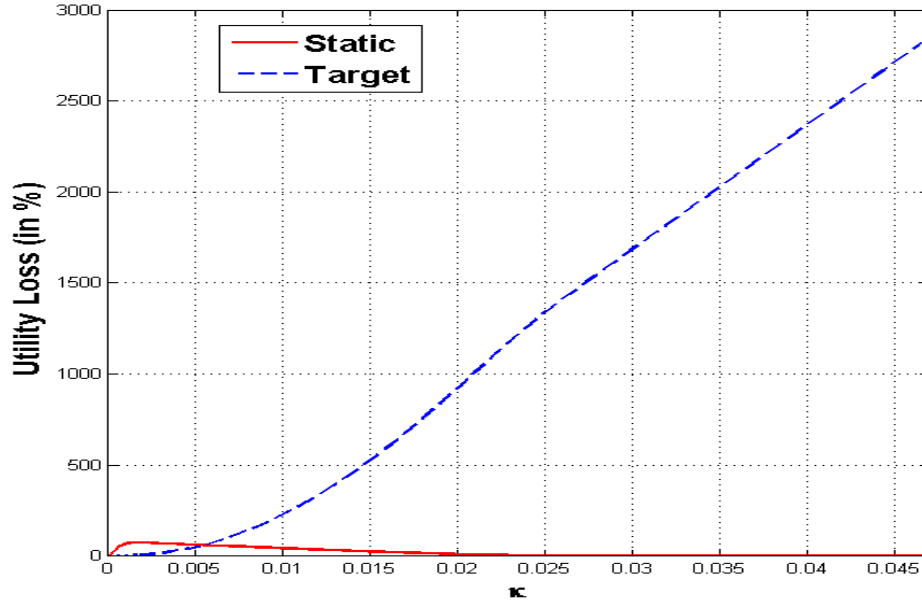
(b) Trading trajectories depending on exponent  $p$ .



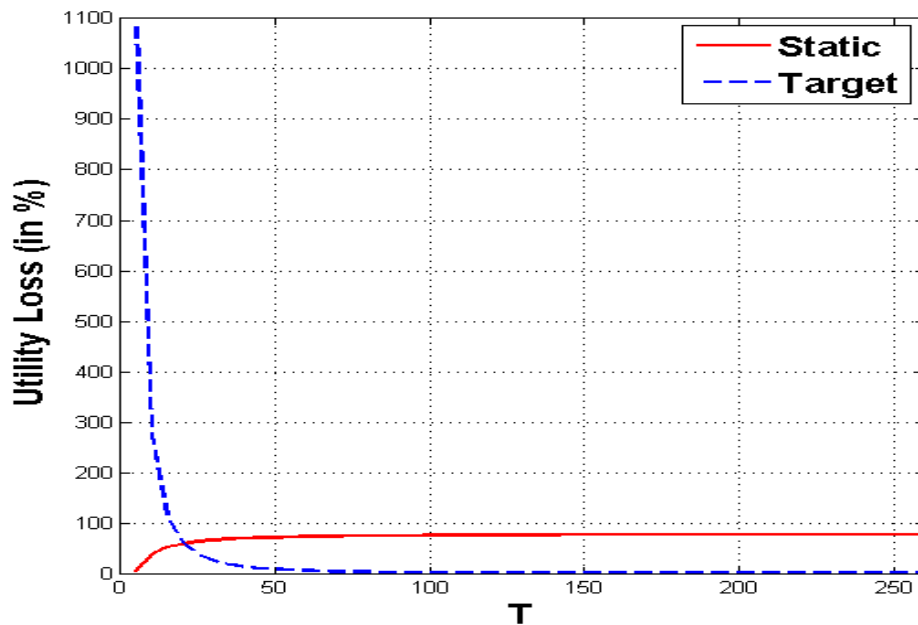
**Figure A.4: Utility losses with proportional transaction costs.**

This figure depicts the utility loss of the static and target portfolios for the dataset with 15 commodity futures as a function of the transaction cost parameter  $\kappa$  (Panel (a)) and the investment horizon  $T$  (Panel (b)). In the base case, we consider proportional transaction costs parameter  $\kappa = 0.0050$ , risk-aversion parameter  $\gamma = 1e - 6$ , annual discount rate  $\rho = 2\%$ , and investment horizon  $T = 22$ . The price-change mean and covariance matrix are set equal to the sample estimators for the dataset that contains 15 commodity price changes.

(a) Utility losses depending on  $\kappa$ .



(b) Utility losses depending on investment horizon  $T$ .



## Appendix B. Proofs of all results

### Proof of Proposition 1

**Part 1.** When  $\rho = 0$ , the change in excess terminal wealth net of transaction costs for a multiperiod investor is

$$W_T = \sum_{t=1}^T [x_t^\top r_{t+1} - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p]. \quad (\text{B.1})$$

From Assumptions 1 and 2, it is straightforward that the expected change in terminal wealth is

$$E_0(W_T) = \sum_{t=1}^T (x_t^\top \mu - \kappa \|\Lambda^{1/p}\|_p^p \|x_t - x_{t-1}\|_p^p). \quad (\text{B.2})$$

Using the law of total variance, the variance of change in terminal wealth can be decomposed as

$$\text{Var}_0(W_T) = E_0 [\text{Var}_s(W_T)] + \text{Var}_0 [E_s(W_T)]. \quad (\text{B.3})$$

Taking into account that

$$\begin{aligned} E_0 [\text{Var}_s(W_T)] &= E_0 \left\{ \text{Var}_s \left[ \sum_{t=1}^T [x_t^\top r_{t+1} - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p] \right] \right\} \\ &= E_0 \left[ \sum_{t=s}^T x_t^\top \Sigma x_t \right] = \sum_{t=s}^T x_t^\top \Sigma x_t, \end{aligned} \quad (\text{B.4})$$

and

$$\begin{aligned} \text{Var}_0 [E_s(W_T)] &= \text{Var}_0 \left\{ E_s \left[ \sum_{t=1}^T [x_t^\top r_{t+1} - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p] \right] \right\} \\ &= \text{Var}_0 \left\{ \sum_{t=1}^{s-1} [x_t^\top r_{t+1} - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p] + \sum_{t=s}^T [x_t^\top \mu - \kappa \|\Lambda^{1/p}(x_t - x_{t-1})\|_p^p] \right\} \\ &= \sum_{t=1}^{s-1} x_t^\top \Sigma x_t, \end{aligned} \quad (\text{B.5})$$

the variance of the change in excess terminal wealth can be rewritten as

$$\text{Var}_0(W_T) = \sum_{t=s}^T x_t^\top \Sigma x_t + \sum_{t=1}^{s-1} x_t^\top \Sigma x_t = \sum_{t=1}^T x_t^\top \Sigma x_t. \quad (\text{B.6})$$

Consequently, the mean-variance objective of the change in excess terminal wealth for a multiperiod investor is

$$\begin{aligned} & \max_{\{x_t\}_{t=1}^T} E(W_T) - \frac{\gamma}{2} \text{Var}(W_T) \\ & \equiv \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T (x_t^\top \mu - \kappa \|\Lambda^{1/p}\| x_t - x_{t-1}\|_p^p) - \frac{\gamma}{2} \sum_{t=1}^T x_t^\top \Sigma x_t \\ & \equiv \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t) - \kappa \|\Lambda^{1/p}\| (x_t - x_{t-1})\|_p^p \right]. \end{aligned} \quad (\text{B.7})$$

This is exactly objective function (1) when the value of discount rate  $\rho = 0$ .

**Part 2.** For the model with proportional transaction costs, the optimal policy is to trade at the first period to the boundary of the no-trade region given by (5), and not to trade for periods  $t = 2, 3, \dots, T$ . For an investor who is now sitting at period  $j$  with  $j > 0$ , the no-trade region for the remaining periods  $t = j + 1, j + 2, \dots, T$  is given by

$$\|\Sigma(x - x^*)\|_\infty \leq \frac{\kappa}{(1 - \rho)\gamma} \frac{\rho}{1 - (1 - \rho)^{T-j}}, \quad (\text{B.8})$$

which defines a region that contains the region defined by constraint (5). We could infer that the “initial position” for stage  $s$ , which is on the boundary of the no-trade region defined by (5), is inside the no-trade region defined in (B.8). Hence for any  $\tau > j$ , the optimal strategy for an investor who is sitting at  $j$  is to stay at the boundary of the no-trade region defined in (5), which is consistent with the optimal policy obtained at time  $t = 0$ .

For the model with temporary market impact costs, for simplicity of exposition we consider  $\Lambda = \Sigma$ . At  $t = 0$ , the optimal trading strategy for period  $\tau$  is on the boundary of the following rebalancing region

$$\frac{\|\sum_{s=\tau}^T (1 - \rho)^{s-\tau} \Sigma^{1/q} (x_{s|0} - x^*)\|_q}{p \|\Sigma^{1/p} (x_{t|0} - x_{t-1|0})\|_p^{p-1}} \leq \frac{\kappa}{(1 - \rho)\gamma}. \quad (\text{B.9})$$

For the investor who is now at  $t = j$  for  $j < \tau$ , the optimal trading strategy is at the

boundary of the rebalancing region given by

$$\frac{\|\sum_{s=\tau}^T (1-\rho)^{s-t} \Sigma^{1/q} (x_{s|j} - x^*)\|_q}{p \|\Sigma^{1/p} (x_{t|j} - x_{t-1|j})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (\text{B.10})$$

Because  $x_{\tau|j} = x_{\tau|0}$  for  $\tau \leq j$  and taking into account that  $x_{j|j} = x_{j|0}$ , we can infer that

$$\frac{\|\sum_{s=j}^T (1-\rho)^{s-t} \Sigma^{1/q} (x_{s|j} - x^*)\|_q}{p \|\Sigma^{1/p} (x_{j|j} - x_{j-1|j})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma} \quad (\text{B.11})$$

defines the same rebalancing region for period  $j$  as the following one

$$\frac{\|\sum_{s=j}^T (1-\rho)^{s-t} \Sigma^{1/q} (x_{s|0} - x^*)\|_q}{p \|\Sigma^{1/p} (x_{j|0} - x_{j-1|0})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (\text{B.12})$$

That is,  $x_{\tau|j} = x_{\tau|0}$  for  $j < \tau$ , and hence the optimal policies for the model with temporary market impact are time-consistent.

For the model with quadratic transaction costs, for simplicity of exposition we consider  $\Lambda = \Sigma$ . For an investor sitting at period  $t = 0$ , the optimal trading strategy for a future period  $t = \tau$  is given by (B.44) (when  $t = T$ , simply let  $\alpha_1 = \beta_1$ ,  $\alpha_2 = \beta_2$ ,  $\alpha_3 = 0$ ), that is,

$$x_{\tau|0} = \alpha_1 x^* + \alpha_2 x_{\tau-1|0} + \alpha_3 x_{\tau+1|0}. \quad (\text{B.13})$$

For the investor who is at  $t = j$ , for  $j < \tau$ , the optimal trading strategy is

$$x_{\tau|j} = \alpha_1 x^* + \alpha_2 x_{\tau-1|j} + \alpha_3 x_{\tau+1|j}. \quad (\text{B.14})$$

Because  $x_{\tau|j} = x_{\tau|0}$  for  $\tau \leq j$  and taking into account that

$$\begin{aligned} x_{j|0} &= \alpha_1 x^* + \alpha_2 x_{j-1|0} + \alpha_3 x_{j+1|0} \\ &= x_{j|j} = \alpha_1 x^* + \alpha_2 x_{j-1|j} + \alpha_3 x_{j+1|j} \\ &= \alpha_1 x^* + \alpha_2 x_{j-1|0} + \alpha_3 x_{j+1|j}, \end{aligned} \quad (\text{B.15})$$

which gives  $x_{j+1|0} = x_{j+1|j}$ , we can use the relation recursively to show that for all  $\tau > j$ , it holds that  $x_{\tau|j} = x_{\tau|0}$ .  $\square$

## Proof of Theorem 1

**Part 1.** Define  $\Omega_t$  as the subdifferential of  $\kappa\|x_t - x_{t-1}\|_1$

$$s_t \in \Omega_t = \{u_t \mid u_t^\top(x_t - x_{t-1}) = \kappa\|x_t - x_{t-1}\|_1, \|u_t\|_\infty \leq \kappa\}, \quad (\text{B.16})$$

where  $s_t$  denotes a subgradient of  $\kappa\|x_t - x_{t-1}\|_1$ ,  $t = 1, 2, \dots, T$ . If we write  $\kappa\|x_t - x_{t-1}\|_1 = \max_{\|s_t\|_\infty \leq \kappa} s_t^\top(x_t - x_{t-1})$ , objective function (2) can be sequentially rewritten as

$$\begin{aligned} & \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \\ &= \max_{\{x_t\}_{t=1}^T} \min_{\|s_t\|_\infty \leq \kappa} \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} s_t^\top(x_t - x_{t-1}) \right] \\ &= \min_{\|s_t\|_\infty \leq \kappa} \max_{\{x_t\}_{t=1}^T} \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} s_t^\top(x_t - x_{t-1}) \right]. \end{aligned} \quad (\text{B.17})$$

The first-order condition for the inner objective function of (B.17) with respect to  $x_t$  is

$$0 = (1-\rho)(\mu - \gamma \Sigma x_t) - s_t + (1-\rho)s_{t+1}, \quad (\text{B.18})$$

and hence

$$x_t = \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{t+1}) - \frac{1}{(1-\rho)\gamma} \Sigma^{-1} s_t, \quad \text{for } s_t \in \Omega_t, s_{t+1} \in \Omega_{t+1}. \quad (\text{B.19})$$

Denote  $x_t^*$  as the optimal solution for stage  $t$ . Then, there exists  $s_t^*$  and  $s_{t+1}^*$  such that

$$x_t^* = \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{t+1}^*) - \frac{1}{(1-\rho)\gamma} \Sigma^{-1} s_t^*, \quad \forall t. \quad (\text{B.20})$$

We now let  $s_t^* = \frac{1-(1-\rho)^{T-t+2}}{\rho} s_T^*$ , for  $t = 1, 2, \dots, T-1$  and  $s_T^* = (1-\rho)(\mu - \gamma \Sigma x_T^*)$ . Rewrite  $x_t^*$  as

$$\begin{aligned} x_t^* &= \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{t+1}^*) - \frac{1}{(1-\rho)\gamma} \Sigma^{-1} s_t^* \\ &= \frac{1}{\gamma} \Sigma^{-1}(\mu + s_{r+1}^*) - \frac{1}{(1-\rho)\gamma} \Sigma^{-1} s_r^* = x_r^*, \quad \forall t, r, \end{aligned} \quad (\text{B.21})$$

where  $\|s_t\|_\infty \leq \kappa$ . By this means, we find the value of  $s_t^*$  such that  $x_t^* = x_r^*$  for all  $t \neq r$ . We

conclude that  $x_1 = x_2 = \dots = x_T$  satisfies the optimality conditions.

**Part 2.** Because  $x_1 = x_2 = \dots = x_T$ , one can rewrite the objective function (2) as

$$\begin{aligned}
 & \max_{\{x_t\}_{t=1}^T} \left\{ \sum_{t=1}^T \left[ (1-\rho)^t \left( x_t^\top \mu - \frac{\gamma}{2} x_t^\top \Sigma x_t \right) - (1-\rho)^{t-1} \kappa \|x_t - x_{t-1}\|_1 \right] \right\} \\
 = & \max_{x_1} \left\{ \sum_{t=1}^T \left[ (1-\rho)^t \left( x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) \right] - \kappa \|x_1 - x_0\|_1 \right\} \\
 = & \max_{x_1} \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \left( x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - \kappa \|x_1 - x_0\|_1. \tag{B.22}
 \end{aligned}$$

Let  $s$  be the subgradient of  $\kappa \|x_1 - x_0\|_1$  and let  $\Omega$  be the subdifferential

$$s \in \Omega = \left\{ u \mid u^\top (x_1 - x_0) = \kappa \|x_1 - x_0\|_1, \|u\|_\infty \leq \kappa \right\}. \tag{B.23}$$

If we write  $\kappa \|x - x_0\|_1 = \max_{\|s\|_\infty \leq \kappa} s^\top (x - x_0)$ , objective function (B.22) can be sequentially rewritten as:

$$\begin{aligned}
 & \max_{x_1} \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \left( x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - \kappa \|x_1 - x_0\|_1 \\
 = & \max_{x_1} \min_{\|s\|_\infty \leq \kappa} \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \left( x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - s^\top (x_1 - x_0) \\
 = & \min_{\|s\|_\infty \leq \kappa} \max_{x_1} \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \left( x_1^\top \mu - \frac{\gamma}{2} x_1^\top \Sigma x_1 \right) - s^\top (x_1 - x_0). \tag{B.24}
 \end{aligned}$$

The first-order condition for the inner objective function in (B.24) is

$$0 = \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} (\mu - \gamma \Sigma x_1) - s, \tag{B.25}$$

and hence  $x_1 = \frac{1}{\gamma} \Sigma^{-1} \left( \mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s \right)$  for  $s \in \Omega$ . Then we plug  $x_1$  into (B.24),

$$\begin{aligned}
 & \min_{\|s\|_\infty \leq \kappa} \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \left\{ \left[ \frac{1}{\gamma} \Sigma^{-1} \left( \mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s \right) \right]^\top \mu \right. \\
 & \quad \left. - \frac{\gamma}{2} \left[ \frac{1}{\gamma} \Sigma^{-1} \left( \mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s \right) \right]^\top \Sigma \left[ \frac{1}{\gamma} \Sigma^{-1} \left( \mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s \right) \right] \right\} \\
 & \quad - s^\top \left[ \frac{1}{\gamma} \Sigma^{-1} \left( \mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s \right) - x_0 \right] = \tag{B.26}
 \end{aligned}$$

$$\begin{aligned} \min_{\|s\|_\infty \leq \kappa} & \frac{(1-\rho) - (1-\rho)^{T+1}}{2\rho\gamma} \left( \mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s \right)^\top + \dots \\ & \Sigma^{-1} \left( \mu - \frac{\rho}{(1-\rho) - (1-\rho)^{T+1}} s \right) + s^\top x_0. \end{aligned} \quad (\text{B.27})$$

Note that from (B.25), we have  $s = \frac{(1-\rho)-(1-\rho)^{T+1}}{\rho}(\mu - \gamma\Sigma x_1) = \frac{(1-\rho)-(1-\rho)^{T+1}}{\rho}[\gamma\Sigma(x^* - x_1)]$  as well as  $\|s\|_\infty \leq \kappa$ , where  $x^* = \frac{1}{\gamma}\Sigma^{-1}\mu$  is the Markowitz portfolio. Replacing  $s$  in (B.27), we conclude that problem (B.27) is equivalent to the following

$$\min_{x_1} \quad \frac{\gamma}{2} x_1^\top \Sigma x_1 - \gamma x_1^\top \Sigma x_0 \quad (\text{B.28})$$

$$s.t. \quad \left\| \frac{(1-\rho) - (1-\rho)^{T+1}}{\rho} \gamma \Sigma (x_1 - x^*) \right\|_\infty \leq \kappa. \quad (\text{B.29})$$

Be aware that the term  $\frac{\gamma}{2} x_0^\top \Sigma x_0$  is a constant term for objective function (B.28). Thus, it can be rewritten as:

$$\begin{aligned} \max_{x_1} & \quad \frac{\gamma}{2} x_1^\top \Sigma x_1 - \gamma x_1^\top \Sigma x_0 \\ \equiv \max_{x_1} & \quad \frac{\gamma}{2} x_1^\top \Sigma x_1 - \gamma x_1^\top \Sigma x_0 + \frac{\gamma}{2} x_0^\top \Sigma x_0. \end{aligned} \quad (\text{B.30})$$

It follows immediately that objective function (B.30) together with constraint (B.29) is equivalent to

$$\min_{x_1} \quad (x_1 - x_0)^\top \Sigma (x_1 - x_0), \quad (\text{B.31})$$

$$s.t. \quad \left\| \Sigma (x_1 - x^*) \right\|_\infty \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T}. \quad (\text{B.32})$$

**Part 3.** Note that constraint (5) is equivalent to

$$-\frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T} e \leq \Sigma(x_1 - x^*) \leq \frac{\kappa}{(1-\rho)\gamma} \frac{\rho}{1 - (1-\rho)^T} e,$$

which is a parallelogram centered at  $x^*$ . To show that constraint (5) defines a no-trade region, note that when the starting portfolio  $x_0$  satisfies constraint (5),  $x_1 = x_0$  is a minimizer of objective function (4) and is feasible with respect to the constraint. On the other hand, when  $x_0$  is not inside the region defined by (5), the optimal solution  $x_1$  must be the point on the boundary of the feasible region that minimizes the objective. We then conclude that constraint (5) defines a no-trade region.  $\square$



## Proof of Proposition 2

Differentiating objective function (7) with respect to  $x_t$  gives

$$(1 - \rho)(\mu - \gamma \Sigma x) - \kappa p \Lambda^{1/p} |\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0)) = 0, \quad (\text{B.33})$$

where  $|a|^{p-1}$  denotes the absolute value to the power of  $p - 1$  for each component

$$|a|^{p-1} = (|a_1|^{p-1}, |a_2|^{p-1}, \dots, |a_N|^{p-1}),$$

and  $\text{sign}(\Lambda^{1/p}(x - x_0))$  is a vector containing the sign of each component for  $\Lambda^{1/p}(x - x_0)$ .

Given that  $\Lambda$  is symmetric, rearranging (B.33) we have

$$(1 - \rho) \Lambda^{-1/p} \gamma \Sigma (x^* - x) = \kappa p |\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0)). \quad (\text{B.34})$$

Note that  $x = x_0$  cannot be the optimal solution unless the initial position  $x_0$  satisfies  $x_0 = x^*$ . Otherwise, take  $q$ -norm on both sides of (B.34):

$$\|\Lambda^{-1/p} \Sigma (x - x^*)\|_q = \frac{\kappa}{(1 - \rho) \gamma} p \|\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0))\|_q, \quad (\text{B.35})$$

where  $q$  is the value such that  $\frac{1}{p} + \frac{1}{q} = 1$ . Noting that  $\|\Lambda^{1/p}(x - x_0)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x - x_0))\|_q = \|x - x_0\|_p^{p-1}$ , we conclude that the optimal trading strategy satisfies

$$\frac{\|\Lambda^{-1/p} \Sigma (x - x^*)\|_q}{p \|\Lambda^{1/p}(x - x_0)\|_p^{p-1}} = \frac{\kappa}{(1 - \rho) \gamma}. \quad (\text{B.36})$$

## Proof of Theorem 2

Differentiating objective function (9) with respect to  $x_t$  gives

$$(1 - \rho)^t (\mu - \gamma \Sigma x_t) - (1 - \rho)^{t-1} p \kappa \Lambda^{1/p} |\Lambda^{1/p}(x_t - x_{t-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_t - x_{t-1})) \\ + (1 - \rho)^t p \kappa \Lambda^{1/p} |\Lambda^{1/p}(x_{t+1} - x_t)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_{t+1} - x_t)) = 0. \quad (\text{B.37})$$

Specifically, given that  $\Lambda$  is symmetric, the optimality condition for stage  $T$  reduces to

$$(1 - \rho) \Lambda^{-1/p} (\mu - \gamma \Sigma x_T) = p \kappa |\Lambda^{1/p}(x_T - x_{T-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_T - x_{T-1})). \quad (\text{B.38})$$

Note that  $x_T = x_{T-1}$  cannot be the optimal solution unless  $x_{T-1} = x^*$ . Otherwise, take  $q$ -norm on both sides of (B.38) and rearrange terms,

$$\frac{\|\Lambda^{-1/p}\Sigma(x_T - x^*)\|_q}{p\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}} \leq \frac{\kappa}{(1-\rho)\gamma}. \quad (\text{B.39})$$

Summing up the optimal conditions recursively gives

$$p\kappa|x_t - x_{t-1}|^{p-1} \cdot \text{sign}(x_t - x_{t-1}) = \sum_{s=t}^T (1-\rho)^{s-t+1} \gamma \Lambda^{-1/p}\Sigma(x^* - x_s). \quad (\text{B.40})$$

Note that  $x_t = x_{t-1}$  cannot be the optimal solution unless  $x_{t-1} = x^*$ . Otherwise, take  $q$ -norm on both sides, and it follows straightforwardly that

$$\frac{\|\sum_{s=t}^T (1-\rho)^{s-t} \Lambda^{-1/p}\Sigma(x_s - x^*)\|_q}{p\|\Lambda^{1/p}(x_t - x_{t-1})\|_p^{p-1}} = \frac{\kappa}{(1-\rho)\gamma}. \quad (\text{B.41})$$

We conclude that the optimal trading strategy for period  $t$  satisfies (B.41) whenever the initial portfolio is not  $x^*$ .  $\square$

### Proof of Proposition 3

**Part 1.** We first define the function  $g(x) = (1-\rho)\Lambda^{-1/p}\Sigma(x - x^*)$ . For period  $T$  it holds that  $\|g(x_T)\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$ . Moreover, for the following last period it holds that  $\|g(x_{T-1}) + (1-\rho)g(x_T)\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}$ . Noting that  $\|A+B\|_q \geq \|A\|_q - \|B\|_q$ , it follows immediately that

$$\begin{aligned} p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1} &\geq \|g(x_{T-1}) + (1-\rho)g(x_T)\|_q \\ &\geq \|g(x_{T-1})\|_q - (1-\rho)\|g(x_T)\|_q \\ &\geq \|g(x_{T-1})\|_q - (1-\rho)p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}, \end{aligned}$$

where the last inequality holds based on the fact  $\|g(x_T)\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$ .

Rearranging terms,

$$\|g(x_{T-1})\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1} + (1-\rho)p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1},$$

which implies that

$$\frac{\|g(x_{T-1})\|_q}{p\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}} \leq \frac{\kappa}{\gamma} + (1 - \rho) \frac{\kappa}{\gamma} \frac{\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}}{\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}},$$

which gives a wider area than the rebalancing region defined for  $x_T$ :  $\frac{\|g(x_T)\|_q}{p\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}} \leq \frac{\kappa}{\gamma}$ . Similarly, where  $\|g(x_{T-2}) + (1 - \rho)g(x_{T-1}) + (1 - \rho)^2g(x_T)\|_q \leq \frac{\kappa}{\gamma}p\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1}$ , it follows that

$$\begin{aligned} \frac{\kappa}{\gamma}p\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1} &\geq \|g(x_{T-2}) + (1 - \rho)g(x_{T-1}) + (1 - \rho)^2g(x_T)\|_q \\ &\geq \|g(x_{T-2}) + (1 - \rho)\|g(x_{T-1})\|_p - (1 - \rho)^2\|g(x_T)\|_q, \\ &\geq \|g(x_{T-2}) + (1 - \rho)\|g(x_{T-1})\|_p - (1 - \rho)^2p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}, \end{aligned}$$

where the last inequality holds because  $\|g(x_T)\|_q \leq p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}$ .

Rearranging terms,

$$\|g(x_{T-2}) + (1 - \rho)\|g(x_{T-1})\|_p \leq \frac{\kappa}{\gamma}p\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1} + (1 - \rho)^2p\frac{\kappa}{\gamma}\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1},$$

which implies that

$$\frac{\|g(x_{T-2}) + (1 - \rho)\|g(x_{T-1})\|_p}{p\|x_{T-2} - x_{T-3}\|_p^{p-1}} \leq \frac{\kappa}{\gamma} + (1 - \rho)^2 \frac{\kappa}{\gamma} \frac{\|\Lambda^{1/p}(x_T - x_{T-1})\|_p^{p-1}}{\|\Lambda^{1/p}(x_{T-2} - x_{T-3})\|_p^{p-1}}.$$

The above inequality defines a region which is wider than the rebalancing region defined by

$$\frac{\|g(x_{T-1}) + (1 - \rho)\|g(x_T)\|_p}{p\|\Lambda^{1/p}(x_{T-1} - x_{T-2})\|_p^{p-1}} \leq \frac{\kappa}{\gamma} \text{ for } x_{T-1}.$$

Recursively, we can deduce that the rebalancing region corresponding to each period shrinks along  $t$ .

**Part 2.** Note that the rebalancing region for period  $t$  relates to the trading strategies thereafter. Moreover, the condition  $x_1 = x_2 = \dots = x_T = x^*$  satisfies inequality (10). We then conclude that the rebalancing region for stage  $t$  contains Markowitz strategy  $x^*$ .

**Part 3.** The optimality condition for period  $T$  satisfies

$$(1 - \rho)(\mu - \gamma\Sigma x_T) - p\kappa\Lambda^{1/p}|\Lambda^{1/p}(x_T - x_{T-1})|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(x_T - x_{T-1})) = 0. \quad (\text{B.42})$$

Let  $\omega$  to be the vector such that  $\lim_{T \rightarrow \infty} x_T = \omega$ . Taking limit on both sides of (B.42),

$$(1 - \rho)(\mu - \gamma \Sigma \omega) - p \kappa \Lambda^{1/p} |\Lambda^{1/p}(\omega - \omega)|^{p-1} \cdot \text{sign}(\Lambda^{1/p}(\omega - \omega)) = 0. \quad (\text{B.43})$$

Noting that  $\lim_{T \rightarrow \infty} x_T = \lim_{T \rightarrow \infty} x_{T-1} = \omega$ , it follows that

$$(1 - \rho)(\mu - \gamma \Sigma \omega) = 0,$$

which gives that  $\omega = \frac{1}{\gamma} \Sigma^{-1} \mu = x^*$ . We conclude that the investor will move to Markowitz strategy  $x^*$  in the limit case.  $\square$

## Proof of Corollary 2

Substituting  $\Lambda$  with  $\Sigma$  we obtain the optimal trading strategy as

$$x_t = \alpha_1 x^* + \alpha_2 x_{t-1} + \alpha_3 x_{t+1}, \quad \text{for } t = 1, 2, \dots, T-1 \quad (\text{B.44})$$

$$x_t = \beta_1 x^* + \beta_2 x_{t-1}, \quad \text{for } t = T. \quad (\text{B.45})$$

where  $\alpha_1 = (1 - \rho)\gamma / ((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$ ,  $\alpha_2 = 2\kappa / ((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$ ,  $\alpha_3 = 2(1 - \rho)\kappa / ((1 - \rho)\gamma + 2\kappa + 2(1 - \rho)\kappa)$  with  $\alpha_1 + \alpha_2 + \alpha_3 = 1$ , and  $\beta_1 = (1 - \rho)\gamma / ((1 - \rho)\gamma + 2\kappa)$ ,  $\beta_2 = 2\kappa / ((1 - \rho)\gamma + 2\kappa)$  with  $\beta_1 + \beta_2 = 1$ . Since  $x_T$  is a linear combination of  $x_{T-1}$  and  $x^*$ , it indicates that  $x_T$ ,  $x_{T-1}$  and  $x^*$  are on a straight line. On the contrary, if we assume that  $x_{T-2}$  is not on the same line, then  $x_{T-2}$  cannot be expressed as a linear combination of  $x_T$ ,  $x_{T-1}$  and  $x^*$ . Recall from equation (B.44) when  $t = T-1$  that

$$x_{T-1} = \alpha_1 x^* + \alpha_2 x_{T-2} + \alpha_3 x_T, \quad (\text{B.46})$$

rearranging terms

$$x_{T-2} = \frac{1}{\alpha_2} x_{T-1} - \frac{\alpha_1}{\alpha_2} x^* - \frac{\alpha_3}{\alpha_2} x_T.$$

Noting that  $\frac{1}{\alpha_2} - \frac{\alpha_1}{\alpha_2} - \frac{\alpha_3}{\alpha_2} = 1$ , it is contradictory with the assumption that  $x_{T-2}$  is not a linear combination of  $x_T$ ,  $x_{T-1}$  and  $x^*$ . By this means, we can show recursively that all the policies corresponding to the model when  $\Lambda = \Sigma$  lie on the same straight line.  $\square$

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