A Continual Analogue of a Theorem by M. Fekete and G. Pólya

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1 Introduction and statement of results

In 1912 M. Fekete and G. Pólya [1] proved the following theorem.

Theorem A *Let $f(u) = p_0 + p_1 u + \ldots + p_n u^n$ be a polynomial of degree n with real coefficients such that*

$$f(u) > 0 \quad \text{for any} \quad u \geq 0. \tag{1}$$

Then there exists a positive number λ_ε such that for any $\lambda \geq \lambda_\varepsilon$ the entire function $\exp(\lambda u) f(u)$ has nonnegative Taylor coefficients.

We are going to give a continual analogue of this theorem. For a function $f : [0; +\infty) \to R$ we denote by $L(x, f)$ its Laplace transform : $L(x, f) := \int_0^\infty \exp(xt) f(t) dt$.

The main result of this paper is the following

Theorem 1 *Let $p : [0; +\infty) \to R$ be a function satisfying the following conditions:*

(a) $p \in \boldsymbol{C}[0; +\infty)$,

(b) $L(x, |p|) < \infty$ *for any* $x \geq 0$,

(c) there exist real numbers a, b, $0 < a \leq b < +\infty$, such that $p(t) \geq 0$ for $t \in [0, a] \cup [b, +\infty)$,

(d) there exist $t_1 \in [0,a]$ *and* $t_2 \in [b,+\infty)$*, such that* $p(t_1) > 0$*,* $p(t_2) > 0$*,*

(e) $L(x,p) > 0$ *for any* $x \in R$*.*

Then $(\exists \varepsilon_0 > 0)\,(\forall \varepsilon \in (0;\varepsilon_0])\,(\exists \lambda_\varepsilon > 0)\,(\forall \lambda \geq \lambda_\varepsilon)$ *the function*

$$F_{\lambda,\varepsilon}(x) = L(x,p)\exp(\lambda e^{\varepsilon x}) \tag{2}$$

is the Laplace transform of a nonnegative function $q_{\lambda,\varepsilon}(t)$ *.*

The following theorem about compact supported functions is an immediate corollary of Theorem 1.

Theorem 2 *Let* $p : [a;b] \to R$ *be a function satisfying the following conditions:*

(a) $p \in \boldsymbol{C}[a;b]$*,*

(b) $p(a) > 0$*,* $p(b) > 0$*,*

(c) $L(x,p) > 0$ *for any* $x \in R$*.*

Then $(\exists \varepsilon_0 > 0)\,(\forall \varepsilon \in (0;\varepsilon_0])\,(\exists \lambda_\varepsilon > 0)\,(\forall \lambda \geq \lambda_\varepsilon)$ *the function*

$$F_{\lambda,\varepsilon}(x) = L(x,p)\exp(\lambda e^{\varepsilon x}) \tag{3}$$

is the Laplace transform of a nonnegative function $q_{\lambda,\varepsilon}(t)$*.*

This theorem can be viewed as a continual analogue of Theorem A by M. Fekete and G. Pólya. In fact, if we interpret coefficients of the polynomial f as values of the discrete measure μ, supported by the set $\{0,1,2,\ldots,n\}$, $\mu(\{k\}) = p_k$, then the Laplace transform of this measure at point $x = \log u$, $u > 0$, will be a polynomial $f(u)$. In this interpretation condition (c) of Theorem 2 transforms into condition (1) of Theorem A. Note that (1) implies $p_0 > 0$ and $p_n > 0$ which is an analogue of condition (b) of Theorem 2.

Condition (a) in Theorems 1, 2 could be weakened, but then the proof would be too cumbersome.

The function $\exp(\lambda e^{\varepsilon x})$ is the Laplace transform of the measure $\mu_{\lambda,\varepsilon}$ supported by $\{\varepsilon k\}_{k=0}^{\infty}$, $\mu_{\lambda,\varepsilon}(\{\varepsilon k\}) = \lambda^k/k!$. Therefore Theorems 1 and 2 mean that the convolution $p * \mu_{\lambda,\varepsilon}$ is nonnegative on $[0,\infty)$ for sufficiently small ε and large λ. Conditions on a function p providing the existence of a non-negative Borel measure μ such that $p * \mu$ is nonnegative were investigated by H. Diamond and M. Essén [2]. One of distinctions from their results is that the measure $\mu_{\lambda,\varepsilon}$ in Theorems 1 and 2 is "standard" i.e. its form does not depend on p. More essential distinction is that the Laplace transform of $\mu_{\lambda,\varepsilon}$ does not vanish in the whole plane and hence the zero-sets of $L(x,p)$ and $L(x, p * \mu_{\lambda,\varepsilon})$ coincide. In a bit different situation, when p and μ are supported by the whole real line R, such a preservation of zero-sets plaid an important role in the solution (given by I.P. Kamynin and I.V. Ostrovskii [4]) to the problem of characterization of zero-sets of entire Fourier transforms of nonnegative measures supported by R. Therefore Theorems 1 and 2 are of interest in connection with the open problem [3] of characterization of zero-sets of entire Fourier transforms of nonnegative measures supported by a half-line or a finite interval.

The rest of this paper is devoted to the proof of Theorems 1 and 2.

2 Equivalence of Theorems 1 and 2

It is obvious that Theorem 2 is a particular case of Theorem 1. Let us show that Theorem 1 follows from Theorem 2. We shall suppose that the statement of Theorem 2 is true and the conditions of Theorem 1 are satisfied. Denote by l and r the left and right endpoints of the interval supporting the function p, $0 \leq l \leq t_1$, $t_2 \leq r \leq \infty$ (the numbers t_1 and t_2 are from condition (d) of Theorem 1). . It is clear that

$$f(x) := \int_l^r \exp(xt)p(t)dt > 0, x \in R. \tag{4}$$

Since the number r is the right boundary point supporting p (maybe $r = \infty$) and the conditions (c) and (d) of Theorem 1 are satisfied, it follows that there exists a sequence of segments $\{[r'_k, r''_k]\}_{k=1}^{\infty}$, $r'_1 < r''_1 < r'_2 < r''_2 < \cdots$, $\lim_{k\to+\infty} r''_k = r$, such that $p(t) > 0$ for any $t \in [r'_k, r''_k]$ and $k \in N$.

Reasoning similarly, we obtain that there exists a sequence of segments $\{[l'_k, l''_k]\}_{k=1}^{\infty}$, $l''_1 > l'_1 > l''_2 > l'_2 > \cdots$, $\lim_{k\to+\infty} l'_k = l$, such that $p(t) > 0$ for any $t \in [l'_k, l''_k]$ and $k \in N$. We suppose $r'_1 \geq b$, $l''_1 \leq a$ (the numbers a and b are from condition (c) of Theorem 1). Set

$$f_k(x) := \int_{l'_k}^{r''_k} \exp(xt)p(t)dt, \qquad k \in N. \tag{5}$$

Let us show that $f_{k_0}(x) > 0$ for some $k_0 \in N$ and any $x \in R$. We have

$$f_k(x) \geq \int_{l'_1}^{r''_1} \exp(xt)p(t)dt = f_1(x), \qquad x \in R. \tag{6}$$

Let $x \geq 0$, then it follows from (6) that

$$f_k(x) \geq \int_{l'_1}^{r''_1} \exp(xt)p(t)dt \geq \int_{(r'_1+r''_1)/2}^{r''_1} \exp(xt)p(t)dt -$$

$$- \int_{l'_1}^{r'_1} \exp(xt)|p(t)|dt \geq K_1 \exp((r'_1 + r''_1)x/2) - K_2 \exp(r'_1 x), \tag{7}$$

where $K_1 > 0$ and $K_2 > 0$ are independent of x and k. Thus there exists a positive number R_1 such that $f_k(x) > 0$ for any $x \geq R_1$ and $k \in N$. Reasoning similarly for $x \leq 0$ we obtain

$$f_k(x) \geq K_3 \exp((l_1^{'} + l_1^{''})x/2) - K_4 \exp(l_1^{''}x), \tag{8}$$

where $K_3 > 0$ and $K_4 > 0$ do not depend on x and k. So there exists a negative number R_2 such that $f_k(x) > 0$ for any $x \leq R_2$ and $k \in N$. Thus $f_k(x) > 0$ for any $k \in N$ and for any $x \in (-\infty; R_2] \cup [R_1; +\infty)$. It remains to show that there exists $k \in N$ such that $f_k(x) > 0$, when $x \in [R_2; R_1]$. It follows from conditions (a), (b) of Theorem 1 that $f_k(x) \to f(x)$, $k \to +\infty$ uniformly on any compact set in R. It follows from condition (e) of Theorem 1 that there exists $k_0 \in N$ such that for any $x \in [R_2; R_1]$ the inequality $f_{k_0}(x) > 0$ holds. Therefore we have $f_{k_0}(x) > 0$ for any $x \in R$.

Denote

$$\begin{aligned} p_1(t) &= p(t)\chi[l, l_{k_0}^{'}](t), & (9)\\ p_2(t) &= p(t)\chi[l_{k_0}^{'}, r_{k_0}^{''}](t), & (10)\\ p_3(t) &= p(t)\chi[r_{k_0}^{''}, r](t), & (11) \end{aligned}$$

where $\chi[m, n](t)$ means the characteristic function of segment $[m, n]$:

$$\chi[m, n](t) = \begin{cases} 1, & t \in [m, n], \\ 0, & t \in (-\infty; m) \cup (n; +\infty). \end{cases} \tag{12}$$

Then

$$p(t) = p_1(t) + p_2(t) + p_3(t), \tag{13}$$

where

$$p_1(t) \geq 0, \qquad p_3(t) \geq 0. \tag{14}$$

It is clear that

$$\exp(\lambda e^{\varepsilon x}) = \sum_{k=0}^{\infty} \frac{\lambda^k \exp(k\varepsilon x)}{k!} = \int_0^{\infty} \exp(xt) d\Phi_{\lambda,\varepsilon}(t) \tag{15}$$

where $\Phi_{\lambda,\varepsilon}(t)$ is the step function with jump $\lambda^k/k!$ at the point εk, $k = 0, 1, 2, \ldots$. Thus the function $F_{\lambda,\varepsilon}(x) = f(x) \exp(\lambda e^{\varepsilon x})$ has the representation

$$F_{\lambda,\varepsilon}(x) = \int_0^{\infty} \exp(xt) q_{\lambda,\varepsilon}(t) dt, \tag{16}$$

where

$$q_{\lambda,\varepsilon}(t) = (p * \Phi_{\lambda,\varepsilon})(t) = (p_1 * \Phi_{\lambda,\varepsilon})(t) + (p_2 * \Phi_{\lambda,\varepsilon})(t) + (p_3 * \Phi_{\lambda,\varepsilon})(t). \tag{17}$$

For $\varepsilon > 0$, $\lambda > 0$ and $t > 0$ we have

$$(p_1 * \Phi_{\lambda,\varepsilon})(t) \geq 0, \qquad (p_3 * \Phi_{\lambda,\varepsilon})(t) \geq 0. \tag{18}$$

Thus we have to prove that $(p_2 * \Phi_{\lambda,\varepsilon})(t) \geq 0$ for $0 < \varepsilon \leq \varepsilon_0$ and $\lambda \geq \lambda_\varepsilon$. This follows from Theorem 2 (because the function p_2 satisfies all conditions of this theorem).

So, Theorem 1 will be proved if we prove Theorem 2.

3 The proof of Theorem 2

3.1

Without loss of generality we can suppose that $[a, b] = [0, 1]$. It follows from conditions (a), (b) that there exists $\Delta \in (0, 1)$ such that

$$p(t) > 0, \qquad \text{when} \qquad t \in [0, \Delta] \cup [1 - \Delta, 1]. \tag{19}$$

We shall assume that p is defined at every point of the real axis by putting $p(t) = 0$, when $t \in (-\infty, 0) \cup (1; +\infty)$.

We have

$$F_{\lambda,\varepsilon}(x) = f(x) \exp(\lambda e^{\varepsilon x}) = \int_0^\infty \exp(xt) q_{\lambda,\varepsilon}(t) dt, \tag{20}$$

where

$$q_{\lambda,\varepsilon}(t) = (p * \Phi_{\lambda,\varepsilon})(t). \tag{21}$$

Let us compute $q_{\lambda,\varepsilon}$. By definition of $\Phi_{\lambda,\varepsilon}(t)$ (see (15)) we have

$$q_{\lambda,\varepsilon}(t) = \int_{t-1}^{t} p(t - u) d\Phi_{\lambda,\varepsilon}(u). \tag{22}$$

By virtue of (22) it is clear that

$$q_{\lambda,\varepsilon}(t) = 0, \qquad \text{when} \qquad t \leq 0 \tag{23}$$

moreover by (19) we have

$$q_{\lambda,\varepsilon}(t) \geq 0, \qquad \text{when} \qquad t \in [0, \Delta]. \tag{24}$$

We are interested in the form of the function $q_{\lambda,\varepsilon}(t)$ for $t \geq \Delta$. Let us fix such t.

We will use the following notations: $m = m(\varepsilon, t) := [t/\varepsilon]$, $\delta = \delta(\varepsilon, t) := t - [t/\varepsilon]\varepsilon$, $n = n(\varepsilon, t) := [(1 - \delta)/\varepsilon]^*$, where $[a]$ denotes the largest integer $< a$, $[a]^*$ denotes the smallest integer $> a$.

Note that

$$n(\varepsilon, t) \leq [\varepsilon^{-1}] + 1. \tag{25}$$

We have from (15) and (22)

$$\begin{aligned} q_{\lambda,\varepsilon}(t) &= \int_{t-1}^{t} p(t-u)d\Phi_{\lambda,\varepsilon}(u) = \\ &= \frac{p(\delta)\lambda^m}{m!} + \frac{p(\delta+\varepsilon)\lambda^{m-1}}{(m-1)!} + \ldots + \\ &\quad + \frac{p(\delta+(n-1)\varepsilon)\lambda^{m-n+1}}{(m-n+1)!} = \\ &= \frac{\lambda^m}{m!}\left(p(\delta) + p(\delta+\varepsilon)\frac{m}{\lambda} + p(\delta+2\varepsilon)\frac{m}{\lambda}\left(\frac{m}{\lambda} - \frac{1}{\lambda}\right) + \ldots + \right. \\ &\quad \left. + p(\delta+(n-1)\varepsilon)\frac{m}{\lambda}\left(\frac{m}{\lambda} - \frac{1}{\lambda}\right)\cdots\left(\frac{m}{\lambda} - \frac{n-2}{\lambda}\right)\right). \end{aligned} \tag{26}$$

3.2

Let us introduce the following system of polynomials on y:

$$\begin{aligned} Q_n(y,\mu,\beta) &= p(\mu) + p(\mu+\varepsilon)y + p(\mu+2\varepsilon)y(y-\beta) + \ldots + \\ &\quad + p(\mu+(n-1)\varepsilon)y(y-\beta)\cdots(y-(n-2)\beta), \end{aligned} \tag{27}$$

where

$$0 \leq \mu \leq \varepsilon, \quad 0 \leq \beta \leq 1, \quad 0 \leq \varepsilon \leq 1, \quad n \leq [\varepsilon^{-1}] + 1, \tag{28}$$

(compare with (25) and (26)).

From (26) we see that

$$q_{\lambda,\varepsilon}(t) = \frac{\lambda^m}{m!} Q_n\left(\frac{m}{\lambda}, \delta, \frac{1}{\lambda}\right), t \geq 0. \tag{29}$$

We shall study the system of the polynomials $Q_n(y,\mu,\beta)$. Let $\varepsilon > 0$ be fixed and sufficiently small. (The value of $\varepsilon > 0$ will be chosen later). By (27) the degrees of $Q_n(y,\mu,\beta)$ are bounded, when ε is fixed. Let us show that under these conditions the polynomials $Q_n(y,\mu,\beta)$ have the common boundary of positive zeros for any $\mu \in [0,\varepsilon]$ and $\beta \in [0,1]$. We shall use the following well-known theorem.

Theorem B *The polynomial* $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ *has no zeros when* $|z| \geq R$, *where*

$$R = 1 + (1/a_n)\max\{|a_j| : 0 \leq j \leq n-1\}. \tag{30}$$

The number $p(\mu + (n-1)\varepsilon)$ is the leading coefficient of the polynomial $Q_n(y, \mu, \beta)$. Let $\varepsilon > 0$ be a fixed number, $\varepsilon < \Delta/4$ (see (19)). If $\mu \in [0, \varepsilon]$, $\beta \in [0, 1]$, $n \leq [\varepsilon^{-1}] + 1$ we have

$$p(\mu + (n-1)\varepsilon) \geq \min \{p(t) : t \in [1 - \Delta, 1]\} > 0. \tag{31}$$

It is clear that the rest of the coefficients of $Q_n(y, \mu, \beta)$ are bounded from above by the constant depending only on $\max \{|p(t)| : 0 \leq t \leq 1\}$ if ε is fixed, $\varepsilon \in (0, \Delta/4]$, $\mu \in [0, \varepsilon]$, $\beta \in [0, 1]$. Thus for fixed $\varepsilon \in (0, \Delta/4]$ there exists a real number $R_0 > 1$ not depending on β such that

$$Q_n(y, \mu, \beta) > 0, \qquad y \geq R_0 \tag{32}$$

3.3

Let $\varepsilon > 0$ be a fixed small number. If it is remembered that p is a continuous function, the number $n = n(\varepsilon, t)$ satisfies (25), we obtain the following :

$$\begin{aligned} \lim_{\beta \to 0} Q_n(y, \mu, \beta) &= p(\mu) + p(\mu + \varepsilon)y + p(\mu + 2\varepsilon)y^2 + \ldots + \\ &\quad + p(\mu + (n-1)\varepsilon)y^{n-1} =: Q_n(y, \mu), \end{aligned} \tag{33}$$

where the limit is uniform with respect to $\mu \in [0, 1]$ and $y \in C$, for every compact $C \subset C$.

Lemma 1 *There exist positive numbers ε_0 and ν_0 such that*

$$Q_n(y, \mu) \geq \nu_0 > 0 \tag{34}$$

for any $\varepsilon \in (0, \varepsilon_0]$, $\mu \in [0, \varepsilon]$ and $y \geq 0$.

Before proving Lemma 1 we show that Theorem 2 follows from Lemma 1.

3.4

Let us assume that Lemma 1 is true. We choose and fix a positive number $\varepsilon \in (0, \varepsilon_0]$ (ε_0 is the same as in Lemma 1). Consider the system of polynomials $Q_n(y, \mu, \beta)$ on the segment $[0, R_0]$, where R_0 is from (32). Since ε_0 is fixed, by (25) we have that the degrees of these polynomials are bounded by the number $[\varepsilon^{-1}] + 1$. By (34) and uniform convergence $Q_n(y, \mu, \beta)$ to $Q_n(y, \mu)$ with respect to $y \in [0, R_0]$ there exists a positive number β_0 such that

$$Q_n(y, \mu, \beta) \geq \nu_0/2 > 0 \tag{35}$$

for any $\beta \in (0, \beta_0]$, $y \in [0, R_0]$ and $\mu \in [0, \varepsilon]$. By (32) and (35) there exists a positive number ε_0 and for any $\varepsilon \in (0, \varepsilon_0]$ we can choose $\beta_0 > 0$ such that for any $\beta \in (0, \beta_0]$, $y \geq 0$ the inequality $Q_n(y, \mu, \beta) > 0$ is valid. Putting $\lambda_\varepsilon = (1/\beta_0)$ and recalling (29) we obtain the statement of Theorem 2.

Thus Theorem 2 will be proved if we prove Lemma 1.

3.5

To prove Lemma 1 we have to find more precise boundary of positive zeros of the polynomial $Q_n(y, \mu, \beta)$ than that given by Theorem B.

Lemma 2 *Let* $F(t) = a_0 + a_1 t + \ldots + a_{n-1} t^{n-1}$ *be a polynomial with real coefficients. Suppose that for some* $q \in N$ *such that* $n/2 \leq q < n$ *the inequalities*

$$a_q > 0, \qquad a_{q+1} > 0, \qquad \ldots, \qquad a_{n-1} > 0 \tag{36}$$

are valid. Let

$$\max\{|a_j| : 0 \leq j \leq q-1\} \leq B, \qquad \min\{a_j : q \leq j \leq n-1\} \geq b. \tag{37}$$

Then $F(t) > 0$ *for*

$$t \geq \max\left\{1, \left(\frac{Bq}{b(n-q)}\right)^{1/(n-q)}\right\}. \tag{38}$$

3.6 Proof of Lemma 2

We have (for $t \geq 0$)

$$\begin{aligned} F(t) &= a_{n-1}t^{n-1} + \ldots + a_q t^q + a_{q-1}t^{q-1} + \ldots + a_0 \geq \\ &\geq b\left(t^{n-1} + t^{n-2} + \ldots + t^q\right) - B\left(t^{q-1} + t^{q-2} + \ldots + 1\right) = \\ &= bt^q \frac{t^{n-q}-1}{t-1} - B\frac{t^q - 1}{t-1} = \\ &= b\frac{t^{n-q}-1}{t-1}\left[t^q - \frac{B}{b}\frac{t^q-1}{t^{n-q}-1}\right]. \end{aligned} \tag{39}$$

It is not hard to see that under the conditions of Lemma 2

$$\frac{t^q - 1}{t^{n-q}-1} \leq \frac{q}{n-q} t^{2q-n} \tag{40}$$

for $t \geq 1$. (39) and (40) yield

$$F(t) \geq b\frac{t^{n-q}-1}{t-1}\left[t^q - Bq\frac{t^{2q-n}}{b(n-q)}\right]. \tag{41}$$

Thus, if $t \geq 1$ and satisfies the condition

$$t^q - Bq\frac{t^{2q-n}}{b(n-q)} > 0, \tag{42}$$

then $F(t) > 0$. It is easily verified that (42) implies the statement of Lemma 2.

3.7 Proof of Lemma 1

We shall find the upper and lower boundaries of the positive zeros of the polynomial $Q_n(y,\mu)$ with the help of Lemma 2. Put

$$B=\max\{|p(t)|:0\le t\le 1\}\,;\quad b=\min\{p(t):t\in[0,\Delta]\cup[1-\Delta,1]\}>0. \quad (43)$$

The number $p(\mu+k\varepsilon)$ (see (33)) is the coefficient of the polynomial $Q_n(y,\mu)$ of y^k. Under the condition we have $p(t)>0$ for $t\in[0,\Delta]\cup[1-\Delta,1]$. We apply Lemma 2 to the polynomials

$$F_1(y)=Q_n(y,\mu)-b/2 \quad (44)$$

and

$$\begin{aligned} F_2(y) &= y^{n-1}\left(Q_n(1/y,\mu)-b/2\right)=\left(p(\mu)-b/2\right)y^{n-1}+ \\ &+p(\mu+\varepsilon)y^{n-2}+p(\mu+2\varepsilon)y^{n-3}+\ldots+p(\mu+(n-1)\varepsilon). \end{aligned} \quad (45)$$

Put

$$\varepsilon_1=\min\{1/4,\Delta/4\}\,. \quad (46)$$

We shall choose $\varepsilon_0<\varepsilon_1$ (recall that $0<\mu\le\varepsilon<\varepsilon_0$). Let us estimate the positive integer numbers n and q from Lemma 2 for polynomials $F_1(y)$ and $F_2(y)$. Note that the number n from Lemma 2 is the number of all coefficients of the polynomial F and the number q is the number of the positive "leading" coefficients. By (19) "leading" coefficients of polynomial $F_1(y)$ will be positive, if the index k satisfies $1-\Delta<\mu+k\varepsilon<1$. So for polynomial $F_1(y)$ the number of positive "leading" coefficients is not less than $[\Delta/\varepsilon]-1$, and by (25) the general number of the coefficients are not exceeding $[1/\varepsilon]+1$. Applying Lemma 2 to the polynomial $F_1(y)$ we obtain

$$F_1(y)>0,\quad \text{when}\quad y\ge\max\left\{1,\left(\frac{2B([(1-\Delta)/\varepsilon]+1)}{b([\Delta/\varepsilon]-1)}\right)^{1/([\Delta/\varepsilon]-1)}\right\}. \quad (47)$$

Thus, there exists a constant $C_1>0$ not depending on μ and ε such that

$$Q_n(y,\mu)>b/2,\qquad \text{when}\qquad y\ge\exp(C_1\varepsilon). \quad (48)$$

Reasoning similarly for polynomial $F_2(y)$ we obtain that there exists a constant $C_2>0$ not depending on μ and ε such that

$$Q_n(y,\mu)>b/2,\qquad \text{when}\qquad 0\le y\le\exp(-C_2\varepsilon). \quad (49)$$

It remains to prove that the polynomials $Q_n(y,\mu)$ are bounded from below by a positive constant, when $\exp(-C_2\varepsilon)\le y\le\exp(C_1\varepsilon)$. We apply the substitution $y=\exp(x\varepsilon)$. This substitution is correct, because we consider the system of polynomials $Q_n(y,\mu)$ for $y>0$. Put

$$\begin{aligned} q_n(x,\mu) &= Q_n(e^{x\varepsilon},\mu)=p(\mu)+p(\mu+\varepsilon)e^{x\varepsilon}+ \\ &+p(\mu+2\varepsilon)e^{2x\varepsilon}+\ldots+p(\mu+(n-1)\varepsilon)e^{(n-1)x\varepsilon}. \end{aligned} \quad (50)$$

By virtue (48) and (49) we have

$$q_n(x,\mu) > b/2, \qquad \text{when} \qquad |x| \geq C = \max\{C_1, C_2\}, \tag{51}$$

where C is independent of μ and ε. We shall consider the functions $q_n(x,\mu)$ for $|x| \leq C$. By conditions (a), (c) of Theorem 2 there exists a positive number a_1 such that

$$f(x) = \int_0^1 \exp(xt)p(t)dt \geq a_1 \tag{52}$$

for any $|x| \leq C$. Note that

$$\begin{aligned}
\varepsilon \exp(x\mu) q_n(x,\mu) &= \varepsilon\Big(p(\mu) + \\
&\quad + p(\mu+\varepsilon)e^{x(\varepsilon+\mu)} + p(\mu+2\varepsilon)e^{x(2\varepsilon+\mu)} + \ldots + \\
&\quad + p(\mu+(n-1)\varepsilon)e^{x(n-1)\varepsilon+\mu)}\Big) \to \\
&\to \int_0^1 \exp(xt)p(t)dt \geq a_1, \qquad \varepsilon \to 0,
\end{aligned} \tag{53}$$

uniformly with respect to $x \in [-C, C]$. Thus, there exists a positive number $\varepsilon_0(C)$ such that for any $\varepsilon \in (0, \varepsilon_0(C)]$ we have

$$\varepsilon \exp(x\mu) q_n(x,\mu) \geq a_1/2, \qquad \text{when} \qquad |x| \leq C, \qquad \mu \in [0, \varepsilon]. \tag{54}$$

From (54) we obtain that

$$q_n(x,\mu) > a_1 \exp(-C\varepsilon_0)/(2\varepsilon_0), \qquad \text{when} \qquad |x| \leq C. \tag{55}$$

Put

$$\varepsilon_0 = \min\{\varepsilon_1, \varepsilon_0(C)\}, \qquad \nu_0 = \min\{b/2, a_1 \exp(-C\varepsilon_0)/(2\varepsilon_0)\}. \tag{56}$$

Then for any $\varepsilon \in (0; \varepsilon_0]$ and $x \in [-C, C]$ the inequality

$$q_n(x,\mu) \geq \nu_0 > 0 \tag{57}$$

holds. By (51) and (57) we obtain $q_n(x,\mu) \geq \nu_0 > 0$ for any $\varepsilon \in (0; \varepsilon_0]$, $\mu \in (0; \varepsilon]$, $x \in R$. Recalling that $Q_n(y,\mu) = q_n(\varepsilon^{-1} \ln y, \mu)$, we obtain

$$Q_n(y,\mu) \geq \nu_0 > 0 \tag{58}$$

for any $\varepsilon \in (0; \varepsilon_0]$, $\mu \in (0; \varepsilon]$, $y \geq 0$. Thus Lemma 1 is proved. This completes the proof of Theorem 2 and Theorem 1.

Remark. The function p in Theorems 1 and 2 can be viewed as a density of some real-valued measure m_p. Theorems 1 and 2 can be reformulated in such a way. Let m_p be a real-valued measure with a continuous density supported by the positive half-axis or a segment. Let the Laplace transform of this measure $L_{m_p}(x) > 0$ for all $x \in R$ and m_p is positive in some neighborhoods of the endpoints of its support. Then for some $\varepsilon > 0$ and $\lambda_\varepsilon > 0$ the convolution $m_p * \mu_{\lambda,\varepsilon} \geq 0$ (where $L_{\mu_{\lambda,\varepsilon}}(x) = \exp(\lambda e^{\varepsilon x})$). A question can be asked: is such statement true for any measure supported by half-axis or a segment which is positive in some neighborhoods of the endpoints of its support and with positive Laplace transform? The answer is negative. Let $\alpha, \beta \in (0,1)$ be irrational non-comeasurable numbers. Let $m := D_0 - \varepsilon D_\alpha - \varepsilon D_\beta + D_1$, where D_x is the Dirac measure at the point x and $\varepsilon > 0$. It is clear that if ε is sufficiently small then $L_m(x) = \int_0^1 \exp(xt)dm(t) > 0$ for all real x. If the convolution $m * \mu_{\lambda,\varepsilon}$ is nonnegative in the point α then $\varepsilon = \frac{\alpha}{k}, k \in N$, and if $m * \mu_{\lambda,\varepsilon}$ is nonnegative in the point β then $\varepsilon = \frac{\beta}{l}, l \in N$. So for any choice of ε, λ the convolution $m * \mu_{\lambda,\varepsilon}$ can not be nonnegative.

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