A note on zero sets of absolutely monotonic functions

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Abstract

A new necessary condition for a set to be the zero set of an absolutely monotonic function is given. If $A \subset \{z : \mathrm{Im}\ z < 0\}$ is the zero set of an absolutely monotonic function then for any $\beta \in (0, \pi/2)$ there exists a nonnegative continuous function h_β, $h_\beta \in L^1(-\infty, -1]$, such that

$$\sum_{a \in A \bigcap \{a : |\arg a - \pi| < \beta\}} \frac{1}{(x-a)^2} \leq h_\beta(x).$$

It is shown that this condition is not a consequence of conditions known before.

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A function $f \in C^\infty(-\infty, 0]$ is said to be absolutely monotonic if

$$f^{(k)}(x) > 0, \quad \forall k \in \mathbb{N} \bigcup \{0\}, \quad \forall x \in (-\infty, 0]. \tag{1}$$

The notion of absolutely monotonic function was introduced by S. Bernstein [1]. By the well-known S. Bernstein's theorem (see [2]) the class of absolutely monotonic functions coincides with the class of functions representable in the form

$$f(x) = \int_0^\infty e^{xu} P(du), \quad x \in (-\infty, 0], \tag{2}$$

where P is a nonnegative finite Borel measure on $[0, \infty)$. (2) shows that any absolutely monotonic function f is analytic in $\mathbb{C}_0 := \{z : \text{Re } z < 0\}$, continuous in $\{z : \text{Re } z \leq 0\}$, and

$$f(z) = \int_0^\infty e^{zu} P(du), \quad \text{Re } z \leq 0, \tag{3}$$

where the integral converges absolutely for each z. Absolutely monotonic functions are bounded in the half-plane:

$$|f(z)| \leq f(0), \quad \text{Re } z \leq 0. \tag{4}$$

There is a natural problem to characterize the class of subsets of $\mathbb{C}_0$ which can serve as zero sets of absolutely monotonic functions. Let us mention some known properties of such zero sets. It is obvious that if $A \subset \mathbb{C}_0$ is the zero set of an absolutely monotonic function then A is at most countable set without accumulation points in $\mathbb{C}_0$ and

$$A \bigcap R = \emptyset, \quad a \in A \Leftrightarrow \bar{a} \in A, \tag{5}$$

(multiplicities of a and $\bar{a}$ are equal). (4) implies that A satisfies the well-known Blaschke condition for a half-plane:

$$\sum_{a \in A} \frac{\text{Re } a}{|a|^2 + 1} < \infty. \tag{6}$$

The following necessary condition not depending on the previous ones was mentioned in [3]:

$$\text{dist}(x, A) \to +\infty, \quad x \to -\infty. \tag{7}$$

I.V. Ostrovskii showed (oral communication) that the following independent condition is also necessary:

$$(\forall \alpha \in (0, \pi/2)) : \sum_{a \in A_\beta} \text{Re } \frac{1}{x - a} \to 0, \quad x \to -\infty, \tag{8}$$

where

$$A_\beta := A \bigcap \{z : |\arg z - \pi| < \alpha\}.$$

In [3] it was proved that any finite set satisfying (5) can serve as the zero set of an absolutely monotonic function (even of an entire absolutely monotonic function). We can show that if there exists $\alpha \in (0, \pi/2)) : A_\alpha = \emptyset$ and A does not have finite accumulation points then (5) and (6) are sufficient conditions for a set A to be the zero set of an entire absolutely monotonic function (to appear). The aim of this note is to obtain one new necessary condition for zero sets of absolutely monotonic functions, to show that this condition is not a consequence of previous ones and to discuss some examples.

Theorem 1. *Let $A \subset \mathbb{C}_0$ without finite accumulation points is the zero set of absolutely monotonic function $f(z)$. Let $B(z)$ be the Blaschke product:*

$$B(z) := \prod_{a \in A} \frac{1 - \frac{z}{a}}{1 + \frac{z}{\bar{a}}}. \tag{9}$$

Then there exists a nonnegative function $g \in C(-\infty, -1] \bigcap L^1(-\infty, -1]$ such that

$$(\log B(x))'' \geq -g(x). \tag{10}$$

Proof. Without loss of generality we can assume $f(0) = 1$. So $f(z)$ is an analytic function in $\mathbb{C}_0$, continuous function in $\{z : \text{Re } z \leq 0\}$ (since A has no finite accumulation points) and (4) shows that $|f(z)| \leq 1$. We shall use the well-known representation for a function bounded in a half-plane (see, for example [4], chapt. 6):

$$f(z) = B(z) \exp(kz + \frac{1}{\pi} \int_{-\infty}^{\infty} \log |f(it)| (\frac{1}{it - z} + \frac{it}{1 + t^2}) dt). \tag{11}$$

(3) implies the ridge condition for $f(z)$:

$$|f(z)| \leq f(\text{Re } z). \tag{12}$$

The ridge condition implies (see [5], chapt. 2) that $\log f(x)$ is a convex function on $(-\infty, 0]$, so $(\log f(x))'' \geq 0$ for $x \leq 0$. From (11) we have for $x \leq 0$:

$$(\log f(x))'' = (\log B(x))'' + \frac{2}{\pi} \int_{-\infty}^{\infty} \log |f(it)| \frac{dt}{(it - x)^3}. \tag{13}$$

Since $|f(it)| = |f(-it)| \leq 1$ we obtain from $(\log f(x))'' \geq 0$

$$(\log B(x))'' \geq \frac{4}{\pi}\int_0^\infty (-\log|f(it)|)\text{Re }(\frac{1}{(it-x)^3})dt. \tag{14}$$

That is why

$$(\log B(x))'' \geq \frac{-12}{\pi}\int_0^\infty (-\log|f(it)|)\frac{(-x)t^2}{(x^2+t^2)^3}dt =: -g(x). \tag{15}$$

It is easy to see that $g(x) \geq 0$, $g \in C(-\infty, 0]$ and

$$\int_{-\infty}^{-1} g(x)dx = \frac{3}{\pi}\int_0^\infty (-\log|f(it)|)\frac{t^2}{(1+t^2)^2}dt < \infty. \qquad \square$$

Theorem 2. *The statement of Theorem 1 can be written in equivalent form: for any $\beta \in (0, \pi/2)$ there exists a nonnegative continuous function h_β , $h_\beta \in L^1(-\infty, -1]$, such that*

$$\sum_{a\in A_\beta} \frac{1}{(x-a)^2} \leq h_\beta(x). \tag{16}$$

Proof. We have

$$(\log B(x))'' = \sum_{a\in A}(\frac{1}{(x+\bar{a})^2} - \frac{1}{(x-a)^2}).$$

Let us show that

$$S_1 := \sum_{a\in A\bigcap\{a:\beta\leq|\arg a-\pi|<\pi/2\}} (\frac{1}{(x+\bar{a})^2} - \frac{1}{(x-a)^2}) \in L^1(-\infty, -1].$$

We obtain by elementary calculations

$$|S_1| \leq \sum_{a\in A\bigcap\{a:\beta\leq|\arg a-\pi|<\pi/2\}} \frac{|-4x\text{Re }a + 4i\text{Re }a\text{ Im }a|}{|x-a|^2|x+\bar{a}|^2} \leq$$

$$\sum_{a\in A\bigcap\{a:\beta\leq|\arg a-\pi|<\pi/2\}} \frac{4|x||\text{Re }a| + 4|\text{Re }a||\text{Im }a|}{|x-a|^2|x+\bar{a}|^2} =$$

$$\sum_{a\in A\bigcap\{a:\beta\leq|\arg a-\pi|<\pi/2\}} \frac{4|x||\text{Re }a| + 4|\text{Re }a||\text{Im }a|}{(x^2-|a|^2)^2 + 4x^2(\text{Im }a)^2} \leq$$

$$C_\beta \sum_{a\in A\bigcap\{a:\beta\leq|\arg a-\pi|<\pi/2\}} \frac{4|x||\text{Re }a| + 4|\text{Re }a||\text{Im }a|}{(x^2+|a|^2)^2} =: g_1(x), \tag{17}$$

where C_β is a positive constant depending only on β. Let us show that $g_1 \in L^1(-\infty, 0]$

$$\int_{-\infty}^{0} g_1(x)dx = \sum_{a\in A\bigcap\{a:\beta\le|\arg a-\pi|<\pi/2\}} \int_0^\infty \frac{4y|\text{Re } a| + 4|\text{Re } a||\text{Im } a|}{(y^2+|a|^2)^2}dy =$$

$$\sum_{a\in A\bigcap\{a:\beta\le|\arg a-\pi|<\pi/2\}} \int_0^\infty \frac{4|a||\text{Re } a|t + 4|\text{Re } a||\text{Im } a|}{|a|^4(1+t^2)^2}|a|dt \le$$

$$\sum_{a\in A\bigcap\{a:\beta\le|\arg a-\pi|<\pi/2\}} \frac{4|\text{Re } a|}{|a|^2}\int_0^\infty \frac{(t+1)}{(1+t^2)^2}dt < \infty \qquad (18)$$

(see (6)). To obtain (16) it remains to show that

$$\sum_{a\in A_\beta} \frac{1}{(x+\bar{a})^2} \in L^1(-\infty,-1].$$

We have

$$\int_{-\infty}^{-1} |\sum_{a\in A_\beta} \frac{1}{(x+\bar{a})^2}|dx \le$$

$$\int_{-\infty}^{-1} \sum_{a\in A_\beta} \frac{1}{|x+\bar{a}|^2}dx$$

$$\le \int_{-\infty}^{-1} \sum_{a\in A_\beta} \frac{1}{x^2+(\text{Re } a)^2}$$

$$\le \sum_{a\in A_\beta} \frac{1}{\text{Re } a}\int_0^\infty \frac{dt}{1+t^2} < \infty \qquad (19)$$

(we use condition (6) for the case $|\arg a - \pi| < \beta$). Taking into account (18) and (19) we conclude that the statement of Theorem 1 is equivalent (16). □

Statement 1. *Let $\{\alpha_k\}_{k=1}^\infty$ and $\{\beta_k\}_{k=1}^\infty$ be two sequences of positive numbers, $\alpha_k \to +\infty$, $\beta_k \to +\infty$, and*

$$\sum_{k=1}^\infty \frac{1}{\alpha_k} < \infty, \quad \beta_k = o(\alpha_k), \quad \sum_{k=1}^\infty \frac{1}{\beta_k} = \infty.$$

Assume that there exists $\lambda > 1$ such that

$$\alpha_k + \lambda\beta_k \le \alpha_{k+1} - \beta_{k+1}, \quad k \in \mathbb{N}. \tag{20}$$

Then $A := \{\alpha_k \pm i\beta_k\}$ does not satisfy condition (16).

Proof. Since A is symmetric with respect to the real axis (16) can be written in the form :there exists a nonnegative continuous function h, $h \in L^1(-\infty, -1]$, such that

$$\sum_{a\in A} \mathrm{Re}\, \frac{1}{(x-a)^2} \le h(x). \tag{21}$$

For any k

$$\mathrm{Re}\, \frac{1}{(x-a)^2} = \frac{(x-\alpha_k)^2 - \beta_k^2}{((x-\alpha_k)^2 + \beta_k^2)^2} > 0, \quad x \in (-\infty, \alpha_k - \beta_k) \bigcup (\alpha_k + \beta_k, \infty).$$

Let us denote $I := \bigcup_{k=1}^{\infty} (\alpha_k + \beta_k, \alpha_{k+1} - \beta_{k+1})$ (by (20) intervals $(\alpha_k + \beta_k, \alpha_{k+1} - \beta_{k+1})$ do not intersect). So

$$\sum_{a\in A} \frac{(x-\alpha_k)^2 - \beta_k^2}{((x-\alpha_k)^2 + \beta_k^2)^2} \ge \frac{(x-\alpha_j)^2 - \beta_j^2}{((x-\alpha_j)^2 + \beta_j^2)^2}, x \in (\alpha_j + \beta_j, \alpha_{j+1} - \beta_{j+1}). \tag{22}$$

The last estimation imply (see also (20))

$$\begin{aligned}
\int_I f(x)dx \quad &\ge \quad \sum_{k=1}^{\infty} \int_{\alpha_k+\beta_k}^{\alpha_{k+1}-\beta_{k+1}} \frac{(x-\alpha_k)^2 - \beta_k^2}{((x-\alpha_k)^2 + \beta_k^2)^2} dx \ge \\
&\sum_{k=1}^{\infty} \int_{\alpha_k+\beta_k}^{\alpha_k+\lambda\beta_k} \frac{(x-\alpha_k)^2 - \beta_k^2}{((x-\alpha_k)^2 + \beta_k^2)^2} dx = \\
&\sum_{k=1}^{\infty} \frac{1}{\beta_k} \int_1^{\lambda} \frac{u^2-1}{(u^2+1)^2} du = C(\lambda) \sum_{k=1}^{\infty} \frac{1}{\beta_k} = \infty. \qquad \square
\end{aligned}$$

Example. Using Statement we obtain that $A_\gamma := \{-k^\gamma \pm ik\}, k \in \mathbb{N}$ can not serve as the zero set of an absolutely monotonic function for $\gamma \ge 2$, but A_γ satisfies (5), (6) and (7). By direct calculation one can see that A_2 satisfies (8).

Remark. Taking into account (8) we can rewrite (16) in the form: for any $\beta \in (0, \pi/2)$

$$\sum_{a \in A \bigcap \{a: |\arg a - \pi| < \beta\}} \frac{1}{(x-a)^2} \in L^1(-\infty, -1].$$

REFERENCES

[1] Bernstein, S.N. Sur les fonctions absolument monotones, *Acta math.*, **52** (1928), 1 - 66.

[2] Akhiezer, N.I. The classical moment problem and some related questions in Analysis, Hafner, New York, 1965.

[3] Ostrovskii, I.V. On zeros of entire characteristic functions of distributions concentrated on a half-line. *Theor. Probab. Applications*, **33** (1988), 167-171.

[4] Koosis, P. Lectures on H_p spaces, Lond. Math. Soc. Lect. Notes Series, No 40, Cambridge Univ. Press, London, 1980.

[5] Linnik, Ju.V. and Ostrovskii, I.V. Decompositions of random variables and vectors, "Nauka", Moscow, 1972, English transl., Amer. Math. Soc., Providence, R.I., 1977.