THE PRICING OF CORPORATE DEBT AND RELATED ISSUES

By

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DEDICATION

To my mum and brother for their unconditional support, and to the memory of my dad.

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Abstract

The purpose of this thesis is to study the pricing and credit risk of corporate debt using structural and reduced-form approaches. We discuss the theoretical aspects of three important topics in pricing risky debt: (i) the impact of stochastic interest rates, and hence the interaction between market risk and credit risk; (ii) the impact of diversifiable and non-diversifiable jump risks on pricing and default mechanisms; and (iii) a reduced-form model with a firm's fundamental variables.

To investigate the relationships between market risk and credit risk, we develop a flexible binomial framework for valuing credit-sensitive instruments by generalizing the valuation model of Geske [1977]. We price a defaultable coupon bond when interest rates and a firm's asset value are stochastic. Our results confirm our belief that firms with low credit quality should have more market risk than firms with high credit quality. We discuss the implications of the results for capital adequacy. In addition to providing conceptual insights into default behaviour, the flexibility of our method allows for efficient pricing of other credit-sensitive instruments.

To improve the short-end properties of credit spreads, we model a firm's asset value as a jump-diffusion process. We show several significant implications of the jump process for the term structure of credit spreads. We also discuss the effects of the diversifiability of jumps on corporate debt pricing. We prove that without considering systematic jump risk, theoretical models tend to underestimate credit spreads. Another contribution of this thesis is the incorporation of taxes into our model to show that taxes do have significant effects on levels of credit spread. Interestingly, the model implies that a decrease in the federal tax rate may precipitate an earlier default of low-grade bonds.

Finally, we investigate a reduced-form model of corporate debt, by taking into account stochastic interest rates, a firm's equity values, and hazard rates of default. Through a moving average of a log-transformation of equity prices, we introduce structural characteristics of the firm into the model. This is an innovation. We investigate the properties and flexibility of the model for pricing corporate debt. Distinguishing features of the model are fourfold. (i) As with structural models, the model exhibits structural properties in the credit spreads. (ii) As a reduced-form model, it preserves a high degree of flexibility in generating credit spreads. (iii) The analytical and tractable form of the model enables researchers to undertake comparative statics and enhances the empirical applicability of the model. (iv) The model can easily be generalized to deal with counterparty default risk.

CHAPTER 1
Introduction

One of the most dramatic economic events of the past three decades has been the development, evolution and growth of derivatives securities. For example, from 1986 through to 1991, the open interest in exchange-traded derivatives grew by 36% per year, reaching $3.5 trillion at the end of 1991 (Fortune, 1995). Barely five years after this, in 1996, this number almost tripled to $9.9 trillion. Furthermore, the notional principal on which over-the-counter derivatives (interest rate products like swaps, forward rate agreements, caps, collars and floors) traded have enjoyed an annual growth rate of 40%.

This astronomical growth rate has called attention to the possibility of risks that could have a significantly negative impact on the overall financial system. The default and bankruptcy experience of major firms in the United States and Britain, has further led many groups particularly the media and lawmakers to call for caution and regulation of the derivatives industry.

In Europe, the introduction of the single currency is driving the demand for the general awareness of credit issues. With the elimination of foreign exchange exposure, the European fixed income markets have become more aware that the forex elements of many European bonds are disappearing and credit premium is becoming the chief arbiter of returns.

Nevertheless, the most substantial momentum comes from the capital measures for bank regulation by the Banks for International Settlements (BIS). At the time of writing this thesis, the BIS guidelines of 1988 are still in force. Risk capital under the current BIS Accord has been viewed as conservative. Excessive capital may be inappropriately required. In financial industries, a more appropriate risk measure than that specified under the current regulatory regime is generally required. In June 1999 the Committee released a new proposal to replace the 1988 Accord with a more risk-sensitive framework. Apart from proposing for the first time a measure for operational risk, the new framework requiring capital for credit risk have made many regulated institutions more scrupulous about the measurement, management, and control of their credit exposures. The Committee expects the final version of the new Accord to be published in 2002 and to be implemented in 2005. The Accord will have far-reaching implications for credit derivatives markets by closing the gap between economic and regulatory capital. The determination and management of economic relative to regulatory capital will become unprecedentedly important in risk management.

Credit risk management consists of two main structures. At a portfolio level, institutions are concerned with the aggregate amount of risk inherent in their portfolios, and with setting a buffer against any possible adverse outcomes. This buffer may be called economic or regulatory capital. The analysis of the total amount of risk also helps institutions to figure out the sources of risk. At a finer level, institutions can more rationally set credit risk limits and identify candidates of excess risk for hedging or diversification through credit derivatives. This thesis is mainly concerned with the pricing and analysis of credit risk at the individual instrument level.

One way of mitigating default risk without hampering the business activities of the traders and users of these instruments is by developing *objective* measures of their value and risks. Such measures would more accurately reflect the risk inherent in the instruments. To achieve this, developing appropriate models for these instruments has become an important topic for practitioners, regulators, and academics.

We have learned from economic theory that market and credit risk are intrinsically inter-related and they are not separable. Practitioners and regulators frequently estimate the amount of credit and market risk separately, and take the sum as a total measure of capital for the credit and market risk exposures.[1] This approximation cannot be justified because the two types of risk exposures are not perfectly correlated. For regulatory purposes, it is important to count all the risk inherent in portfolio management, but not to double it. The current regulatory approach to capital adequacy tends to overestimate the total amount of capital for credit and market risk. The lack of separability and understanding in the relationships between both affect the determination of economic capital, which is of immediate importance to regulators. Developing appropriate integrated models capturing the two types of risk appears to be a sensible avenue for further research.

In this thesis, we study the pricing and credit risk of corporate debt using the structural and reduced-form approaches. We discuss the theoretical aspects of three important topics in pricing risky debt:

1. **A model with both market risk and credit risk.** As an attempt to study the relationships between market and credit risk, we develop a structural model which involves both types of risk. Our model generalizes Geske's framework [1977] to a two-factor model in order to take into account both the impact of stochastic interest rates and a firm's asset values on the pricing of risky debt. The implications of the results for capital adequacy in the management of portfolios of risky assets are discussed.

2. **The impact of jump risk on pricing and default mechanisms.** The conventional structural approach to the valuation of risky debt is intuitive, but has been criticized for not being able to generate sufficient credit spreads for small maturities of debt. To overcome this problem, we extend Geske's [1977] model to incorporate a jump component in a firm's asset values. We discuss

[1] See Jarrow and Turnbull [2000].

the impact of diversifiable and non-diversifiable jump risk on pricing corporate bonds and default mechanisms.

3. **A reduced-form model with a firm's fundamental variables.** The reduced-form model of corporate bonds proposed by Duffie and Singleton [1999] has been fashionable both in industry and academia. A major advantage of this model is that the modelling approach is mathematically tractable. However, most reduced-form models provide no guidance of structural interpretation of a firm's fundamental variables. Very few models have sought to address this problem by introducing structural properties into the framework, but do not appear to be successful. We propose a reduced-form model to bridge this gap.

The structure of this thesis is as follows: Chapters 2 and 3 give an overview of estimating and pricing credit risk during the last three decades. We start with a discussion of the statistical properties of credit spread behaviour over time. We then review various quantitative models for assessing a company's creditworthiness and default probabilities. Two primary credit risk pricing approaches, namely the structural and reduced-form models, are discussed. We also discuss the applications of portfolio credit risk models and their relationship with credit risk management.

In Chapter 4, using a structural approach, we develop a flexible binomial framework for valuing credit-sensitive instruments, which involves both credit and market risk. By extending the valuation models of Geske [1977], Selby [1983], and Shimko, Tjima, and Van Deventer [1993], we price a defaultable coupon bond when interest rates and a firm's asset value are stochastic. We propose an efficient computation algorithm for the pricing of general risky coupon bonds. The default boundary is determined endogenously by requiring the value of equity to be at least the amount of the coupon just paid, in order to avoid bankruptcy. The properties of defaultable bonds, interactions of market and credit risk are discussed. Our results suggest some implications for capital adequacy.

Further applications of the framework are discussed in Chapter 5. As a binomial method, another characteristic of this algorithm is its flexibility in handling a feature that is peculiar to a specific pricing problem. We show how the algorithm for the computations of defaultable coupon bonds, after some modifications, immediately lends itself to efficient pricing of other credit risk related instruments.

To improve the properties of credit spreads for short maturities, we model a firm's asset value as a jump-diffusion process in Chapter 6. We employ a structural approach to analyse term structures of credit risk and yield spreads for the corporate debt of firms when the value of underlying assets follows a jump-diffusion process. Using a discrete time method for valuing general coupon bonds, we show several significant implications of the jump process for the term structure of credit spreads when systematic jumps are present in the firm's asset value. We also discuss the effects of the diversifiability of jumps on corporate debt pricing. Other important factors include taxes and dividends. The main results are as follows. (i) The presence of jumps in asset values eliminates the undesirable qualitative feature of credit spreads decreasing to zero at the short end. The effects on credit spreads become more persistent when downward jumps are of higher volatility, while the total variance of the firm's asset value remains the same. (ii) Without considering systematic jump risk, theoretical models tend to underestimate the credit spreads. (iii) Taxes do have significant effects on levels of credit spread. Interestingly, the model implies that a decrease in the federal tax rate may precipitate an earlier default of low-grade bonds.

In Chapter 7, we propose a reduced-form model of corporate debt by taking into account stochastic interest rates, a firm's equity values, and hazard rates of default. Through a moving average of a log-transformation of equity prices, we introduce structural characteristics of the firm into the model. This is an innovation that provides a compromise between structural and reduced-form approaches. We investigate the properties and flexibility of the model for pricing corporate debt. Distinguishing features of the model are fourfold. (i) As with structural models, the model is able to capture the effects of economic fundamentals on properties of credit spreads. (ii) As a reduced-form model, it preserves a high degree of flexibility in generating credit spreads. (iii) The analytical and tractable form of the model enables researchers to undertake comparative statics and enhances the empirical applicability of the model. (iv) The model can easily be generalized to deal with counterparty default risk.

Finally, we conclude in Chapter 8. We became aware late in the research that although this thesis places special emphasis on the pricing of corporate bonds, the work is fundamental in nature for the pricing and the analysis of credit derivatives and other credit-sensitive instruments. As an important extension, further research projects relating to the modelling of counterparty default and credit risk are suggested and discussed.

CHAPTER 2
Review of Credit Risk Modelling I: Empirical Properties

2.1 Introduction

In this chapter, we give an overview of the statistical properties of credit spread behaviour over time. We describe how credit spreads are related to other important market variables, including ratings and market indices. Although this section is mainly based on the work of Kao [2000], other empirical studies, for example those of Duffie [1998] and Longstaff and Schwartz [1995a] provide similar results for reference.

2.2 Empirical Properties of U.S. Corporate Credit Spreads

This section focuses on the relationships of credit spreads in financial market information. The credit spread is defined as the spread between the yields in Treasury securities and non-Treasury securities that are identical in all respect except for quality. To estimate and price credit risk, one must understand the underlying factors that drive the changes in credit spreads. Changes in credit spread may be related to some risk factors, such as macroeconomic variables, company specific financial fundamentals, traded assets in other financial market segments, liquidity, or tax effects.

The credit spread increases as credit quality rating declines. As Table 2.1 shows, the differences in spread among ratings exhibit an increasing trend. For example, the spread between A rated and BBB rated bonds had an average of 46 basis points in 1990-1998; in the same period average BBB and BB spreads were 170 basis points. The volatility of credit spreads is higher for lower quality bonds, except for AA rated bonds. Table 2.1 also shows that average changes in credit spreads

Table 2.1: **Corporate Credit Spreads and Spread Changes.** *January 1990-December 1998.*(Adopted from Kao [2000])

	Yield Spread		Spread Change	
Rating	Mean	Standard Deviation	Mean	Standard Deviation
AAA	61.0	20.3	0.1	6.4
AA	61.5	15.0	0.0	6.3
A	85.8	23.7	0.1	7.7
BBB	132.1	36.4	0.3	13.0
BB	301.6	92.2	-0.2	34.2
B	535.2	166.9	-0.8	47.5
Investment grade	89.9	23.1	0.2	8.2
High yield	468.2	189.5	-1.5	44.5

have not been significantly different from zero over time. Spreads have a tendency to revert to the mean with the speed of reversion ranging from one to four years.[1]

Table 2.2 shows correlations of credit spreads with other financial market information. Credit spread changes have positive relationships with changes in the Treasury curve slope, interest rate option volatility (3m-Vol) and swap spreads. They are negatively correlated with changes in LIBOR, interest rate levels, and equity returns. These relationships are described in detail in the following subsections.

The term premium is defined as the difference between credit spreads of long term and short term corporate bonds from the same or similarly rated issuer. It has an important property in credit-risk pricing, because it describes how credit risk evolves over time. Table 2.3 shows that while investment-grade bonds have upward-sloping term premiums of credit spreads, high-yield bonds tend to have negative term premiums. Short-term corporate bonds have higher spread volatilities than long-term corporate bonds.

Table 2.4 shows that term premiums of corporate bonds vary over time. Investment-grade bonds in recent years (1993-1998) have had positive term premiums; and high-yield markets consistently placed higher term premiums.

[1]See Longstaff and Schwartz [1995b].

Table 2.2: **Correlations of Credit Spread Changes with Other Financial Market Information.** *February 1990 - December 1998.*(Adopted from Kao [2000])
Note: IG = Investment-grade bonds; HY = high-yield bonds; IG-S = investment-grade short-maturity bonds (1-10 years); IG-L = investment-grade long-maturity bonds (10+ years); HY-S high-yield short-maturity bonds (1-7 years); HY-L = high-yield long-maturity bonds (7+ years); LIBOR = three-month LIBOR; Level = 10-year Treasury rates; Slope = spread between 2- and 30-year U.S. Treasury rates; 3m-Vol = implied volatility of the three-month OTC option on a 10-year rate (correlations are for March 1991 to December 1998); SwapSpd = 10-year swap spreads; Russell1 = the Frank Russell Company's Russell 1000 Index returns; and Russell2 = Russell 2000 Index returns.

Market Measure	AAA	AA	A	BBB	BB	B	IG	HY	IG-S	IG-L	HY-S	HY-L
LIBOR	-0.18	-0.20	-0.12	-0.10	-0.25	-0.06	-0.14	-0.01	-0.11	-0.17	-0.03	-0.04
Level	-0.24	-0.30	-0.31	-0.33	-0.45	-0.42	-0.33	-0.37	-0.25	-0.40	-0.36	-0.36
Slope	0.34	0.39	0.32	0.27	0.20	0.23	0.33	0.19	0.26	0.39	0.22	0.17
3m-Vol	0.61	0.67	0.65	0.61	0.43	0.61	0.67	0.52	0.67	0.61	0.52	0.52
SwapSpd	0.34	0.38	0.37	0.39	0.35	0.31	0.40	0.33	0.42	0.34	0.34	0.29
Russell1	-0.23	-0.37	-0.30	-0.31	-0.15	-0.28	-0.32	-0.29	-0.32	-0.30	-0.28	-0.28
Russell2	-0.35	-0.50	-0.46	-0.48	-0.33	-0.46	-0.48	-0.49	-0.45	-0.47	-0.47	-0.48

Table 2.3: **Term Premiums of Corporate Credit Spreads.** *February 1990 - December 1998.* (Adopted from Kao [2000])

	Credit Spread				Spread Change			
	Investment Grade		High Yield		Investment Grade		High Yield	
Measure	1-10 Years	10+ Years	1-7 Years	7+ Years	1-10 Years	10+ Years	1-7 Years	7+ Years
Mean	88.8	92.8	520.3	463.1	0.1	0.5	-0.4	-1.7
Standard deviation	29.5	17.9	207.2	176.1	8.5	8.9	55.3	41.3

Table 2.4: **Term Premiums in Various Time Periods.** (Adopted from Kao [2000])

	Mean			Standard Deviation		
Rating	1984-89	1990-92	1993-98	1984-89	1990-92	1993-98
Investment grade	-19.7	-25.8	19.1	20.0	11.6	8.5
High yield	NA	-96.8	-37.4	NA	81.2	33.3

Table 2.5: **Regression relationships of the Term Premium with Interest Rate Levels and Slopes (Independent variables are defined in Table 2.2).**

	Coefficient			t-Statistic			
Rating by Period	Constant	Level	Slope	Constant	Level	Slope	R^2
Investment grade 1990-98	125.0	-15.8	-10.9	14.0	-12.5	-7.6	0.66
High yield 1990-98	23.4	-3.4	-44.5	0.8	-0.8	-9.4	0.46

The time dependence of the term premium is also related to changes in the level and slope of the Treasury yield curve. Table 2.5 shows that term premiums in investment-grade bonds are negatively correlated with both interest rate levels and slopes. Thus, a market environment with the combination of a low interest rate and a flat yield curve is often accompanied by a positive term premium. On the contrary, term premiums in high-yield markets are more negatively correlated with the slope of the Treasury curve, implying that the sensitivity of high-yield bonds' term premiums to the slope of the Treasury curve is substantially higher than that of investment-grade bonds.

2.2.1 Relationship between Credit Spread Changes and Interest Rates

Panel A of Table 2.6 indicates that credit spreads narrow as Treasury rates rise. This relationship is stronger as credit quality declines. It also shows that the change in credit spreads is positively correlated with the slope of the Treasury curve. Credit spreads in high-yield bonds are more sensitive than spreads in investment-grade bonds to changes in Treasury slope. Panel B of Table 2.6 shows that changes in credit spreads exhibit a strong positive relationship with changes in interest rate volatility. In fact, volatility may be the most important risk factor in determining credit spread changes. The importance of interest rate volatility is most apparent in

Table 2.6: **Regression relationships of Credit Spread Changes with Interest Rate Parameters and interest rate volatility (as measured by 3m-Vol).** Panel A: Relationship with level and slope, January 1990-December 1998; Panel B: Relationship with level and volatility, March 1991-December 1998 (Independent variables are defined in Table 2.2)

	Coefficient				t-Statistic				
Rating	Constant	Level	Slope	Volatility	Constant	Level	Slope	Volatility	R^2
Panel A:									
AAA	-0.011	-0.038	0.108		-0.02	-1.77	3.32		0.14
AA	-0.175	-0.057	0.113		-0.32	-2.78	3.65		0.20
A	-0.212	-0.076	0.107		-0.31	-2.94	2.75		0.17
BBB	-0.173	-0.141	0.141		-0.15	-3.21	2.14		0.15
BB	-1.817	-0.523	0.206		-0.61	-4.69	1.23		0.21
B	-2.842	-0.613	0.408		-0.67	-3.87	1.71		0.17
Panel B:									
AAA	-0.469	-0.009		2.085%	-0.90	-0.46		6.84%	0.38
AA	-0.376	-0.031		2.006	-0.84	-1.76		7.69	0.47
A	-0.677	-0.033		2.450	-1.18	-1.48		7.34	0.44
BBB	-0.725	-0.089		3.168	-0.85	-2.66		6.36	0.42
BB	-4.547	-0.446		5.093	-1.68	-4.22		3.22	0.32
B	-6.264	-0.564		11.092	-2.01	-4.62		6.09	0.49

the high-yield bond markets. This result is consistent with Table 2.2, which shows strong correlations between spread changes and changes in interest rate volatility over the past nine years.

2.2.2 Relationship between Credit Spread Changes and Equity Markets

Credit spreads are also related to risk factors common to equity return premiums. As is shown in Table 2.7, changes in credit spreads have strong relationships with returns in equity markets. Like interest rate volatility, equity volatility has a significant positive impact on credit spread changes. This relationship is most apparent for lower rated bonds.

2.3 Other Empirical Studies

One important point of interest is the correlation between the returns on individual stocks and returns on stock markets, and the yield changes of individual bonds. There have been some empirical studies on common risk factors in the returns

Table 2.7: **Regression relationships of Credit Spread Changes with Equity Market Returns, January 1990-December 1998.** Panel A: Equity returns measured by the Russell 1000 Index; Panel B: Equity returns measured by the Russell 2000 Index (Other independent variables are defined in Table 2.2)

	Coefficient				t-Statistic				
Rating	Constant	Level	Slope	Russell Index	Constant	Level	Slope	Russell Index	R^2
Panel A:									
AAA	0.656	-0.070	0.087	-0.005	1.12	-3.08	2.71	-3.35	0.23
AA	0.814	-0.105	0.081	-0.008	1.59	-5.25	2.90	-5.68	0.39
A	0.849	-0.127	0.072	-0.008	1.26	-4.87	1.99	-4.65	0.31
BBB	1.816	-0.236	0.076	-0.015	1.63	-5.47	1.26	-5.27	0.33
BB	2.145	-0.713	0.076	-0.031	0.72	-6.19	0.47	-3.94	0.31
B	4.943	-0.986	0.152	-0.061	1.26	-6.50	0.72	-5.87	0.38
Panel B:									
AAA	0.442	-0.054	0.081	-0.004	0.79	-2.62	2.57	-3.87	0.25
AA	0.446	-0.079	0.076	-0.006	0.92	-4.37	2.76	-6.09	0.41
A	0.509	-0.102	0.064	-0.007	0.18	-4.35	1.80	-5.46	0.35
BBB	1.182	-0.189	0.059	-0.013	1.15	-4.93	1.02	-6.27	0.38
BB	0.955	-0.622	0.039	-0.027	0.34	-5.99	0.25	-4.75	0.35
B	2.419	-0.800	0.090	-0.051	0.68	-5.97	0.44	-6.98	0.44

on stocks and bonds. Kwan [1996] examines the correlation between the returns on individual stocks and the yield changes of individual bonds issued by the same firms. He finds that current bond yield changes are negatively correlated with the issuing firm's current and lagged stock returns. This suggests that firm-specific information tends to be embedded first into individual stock prices and then reflected in individual bond prices. Some interesting properties are discovered. For example, AAA-rated bonds are driven mainly by riskless interest rates and are uncorrelated with their issuing firms' stocks. On the other hand, speculative-grade bonds are highly correlated with their issuing firms' stocks, but insensitive to riskless interest rates. This implies that AAA-rated bonds resemble riskless bonds more than risky bonds, and speculative-grade bonds resemble equity securities more than fixed income securities.

Relationships of common risk factors between the returns on stock market and bonds have been investigated in Fama and French [1993]. They identify five common risk factors in the returns on stocks and bonds. There are three stock-market factors: (i) an overall market factors, (ii) factors related to firm size, and (iii) book-to-market equity. There are two bond-market factors, related to (i) maturity and (ii) default risks. They show that stock returns are linked to bond returns through shared variation in the bond market returns. Most importantly, the five factors seem to explain average returns on stocks and bonds.

Based on the Fama-French three stock-market factor model [1993], Elton, Gruber, Agrawal and Mann [2000] find that expected default accounts for a surprisingly small fraction of the premium in credit spreads of corporate bonds. For example, for a 10-year A-rated industrials expected loss from default accounts for only 17.8% of credit spreads. They conclude that while state taxes explain a substantial portion of the discrepancy, the remaining portion of the spread is closely related to the factors that we commonly accept as explaining risk premiums for common stocks. Up to 85% of the spread that is not accounted for by taxes and expected default can be explained as a reward for bearing systematic risk. They show that a significant portion of the spread is compensation for systematic risk that is affected by the same influences of systematic risks in the stock market.

2.4 Summary

We have reviewed the empirical properties of the credit spreads in the U.S. corporate bond market. Their relationships to ratings, stock market indices, and Treasury yield curves have been described. Other empirical studies have also shown that corporate bond prices contain both firm-specific and market-specific information. Other important factors include state taxes. In the next chapter, we give a review of theoretical modelling of credit risk.

CHAPTER 3
Review of Credit Risk Modelling II: Theoretical Models

3.1 Introduction

In this chapter, we provide a review of pricing credit risk during the last three decades. We start with a discussion of three basic building blocks in modelling corporate bond models. We then review various quantitative models for assessing a company's creditworthiness and default probabilities. We focus on how credit risk should be priced. Two primary credit risk pricing approaches, namely the structural and the reduced-form models, are discussed. We then discuss the applications of credit risk modelling both at individual asset level and portfolio level, with emphasis on their limitations and outstanding issues. We conclude with a discussion of possible future research directions.

3.2 Theoretical Modelling of Credit Risk

The pricing of credit risk has developed into one of the most exciting and promising areas in finance. The subject has extended into credit research, term structure of bond and capital structure theory. A good pricing model has to be internally free of any arbitrage conditions. Whilst it needs to be able to generate a term structure of credit spreads consistent with empirical properties, it also needs to explain the behaviour of credit risk over time, and not only the initial term structure of credit spreads. However, because of lack of empirical data, particularly on bond prices, most of the pricing models are left untested. Models are still subject to empirical justification as regards their applicability.

Many variations of modelling approaches have been proposed in the literature, but all the models lie in three basic building blocks: (i) interest rate process, (ii) process of default mechanism, and (iii) recovery rate process.

3.2.1 Interest Rate Process

As discussed in Chapter 2, understanding the interaction of interest rate risk with credit risk is fundamental in the valuation of corporate bonds and financial derivatives written on them. The specification of a proper interest rate process has been found to be important for credit risk modelling. Some commonly used models are Vasicek [1977], Cox, Ingersoll, and Ross [1985], Ho and Lee [1986], Hull and White [1990], and Heath, Jarrow, and Morton [1992].

The Cox, Ingersoll, and Ross [1985] process is perhaps the most popular interest rate model. A jump component can be appended to the process to take into consideration potential shocks and information surprises. Das [1999] shows that the jump-diffusion process is effective in describing interest rate dynamics.

3.2.2 Default Process

Default is usually defined as a point at which a credit event, such as distress and rating migration, happens. The default trigger point is the most essential part of a default process. It defines when and how the default is deemed to have occurred. The trigger point is usually described in one of the following form:

1. an endogenously determined boundary at which the levels of firm value or stock prices satisfy a certain condition;[1]

2. an exogenously determined boundary;[2] and

3. the hitting time of a stochastic jump process.[3]

[1]See Geske [1977].

[2]See Nielsen, Saá-Requejo, and Santa-Clara [1993].

[3]See Duffie and Singleton [1999].

The default boundary characterizes a set of points at which default happens. In early models, default can only occur at maturity, whereas most recent models allow for premature default.

Cross-default clauses in debt contracts usually ensure that the default probabilities for each of the classes of debt for a firm are the same. This implies that the default probability of the firm determines the default probability for all of the firm's debt or counterparty obligations. However, the loss in the event of default for each of the classes of obligations can vary widely depending on their nature of security and seniority as well as the complicated process of bankruptcy.

3.2.3 Recovery Rate Process

In practice, the recovery process is complex because of the nature of uncertainties in the bankruptcy process. We give a brief description of bankruptcy process as follows.[4] In the event of default, the outcome may be liquidation or reorganization. The underlying legal procedures are referred to respectively as Chapter 7 and Chapter 11 of the 1978 Bankruptcy Reform Act. When the firm files for bankruptcy liquidation, a trustee is appointed to liquidate its assets. The proceeds net of transactions costs are distributed to creditor in order of priority. The priority rule in liquidation is called the *Absolute Priority Rule.*

However, most troubled companies seek protection under Chapter 11. One important feature of this provision is that when the firm enters reorganization, all repayments of capital and interest are postponed until the reorganization is complete. The delay in repayment may be viewed as the exercise of an option purchased by the borrower from the creditor when the bond contract is originally completed. Another feature of Chapter 11 is that prior to court approval, any reorganization plan must be agreed to by a majority of creditors (including the stockholders). This gives rise to a protracted bargaining process. In some circumstances, stockholders may

[4]For details, see White [1983], and Franks and Torous [1989].

exercise an important influence on the reorganization plan. This may largely stem from their managerial representatives remaining in control of the business and their exclusive right to propose a reorganization plan. As a result, senior claimholders may be encouraged to give up some of the value of their claims to stockholders. Such a reduction in claims is referred to as a *deviation from absolute priority.*

Deviations from absolute priority are not unusual in practice. The uncertainty and the extent of the deviation create significant difficulties in specifying a realistic recovery process. Bonds with similar issuers can have more than one recovery value, which reflects the differences in the violation of absolute priority:

1. The bargaining power of bondholders against competitive claims to the remaining assets;

2. the debt issuers; and

3. institutional features of the asset-recovery process.

3.3 Pricing of Corporate Bonds

The pricing of corporate debt is a process that consists of specifying and integrating the three processes into a mathematical framework. Although any bond pricing model can be decomposed into the three building blocks, there are many different approaches to modelling. Models differ in the following: (1) Modelling of default risk, (2) modelling of recovery rate, and (3) integration of the variables.

1. **Modelling of default risk:** There are three main approaches to modelling default risk. Based on the nature of default mechanisms, the three approaches are categorized as follows:-

   (i) default events occur expectedly;

   (ii) default events occur unexpectedly; and

(iii) default events can occur expectedly or unexpectedly.

Pioneered by Merton [1974], the first approach assumes that a firm goes into bankruptcy *expectedly* when its asset level hits a certain lower barrier through a continuous diffusion crossing. Such an approach provides a clear link between the firm's economic fundamentals and default risk. Default risk is determined by the relative positions of the firm's asset value and the face value of debt. In the event of default, the assets of the firm are worth less than the amount of debt. In this case, the default boundary is said to be determined endogenously. It can also be given exogenously. For example, Nielsen, Saá-Requejo, and Santa-Clara [1993] model a defaultable discount bond with a given default boundary.

Contrary to the first approach, the second approach assumes that default events are *surprises*. For example, Duffie and Singleton [1999] model a risky discount bond based on the assumption that a default event occurs at the hitting time of a stochastic jump process. Bond prices can be employed to calibrate his model. Jarrow, Lando, and Turnbull [1997] propose an alternative model that relates default probabilities to the transitions of credit ratings.

Models which have the characteristics of the two approaches have been suggested in the literature. For example, Zhou [1997] assumes that a firm's asset value follows a jump-diffusion process.

2. **Modelling of recovery rate:**

The recovery rate can be generated endogenously by a pricing model. For example, Merton [1974] and Geske's models [1977] generate stochastic recovery rates as the remaining asset value of a firm when a default event occurs. In the case of an exogenously specified recovery rate, it is necessary to make an assumption about the claim made by bondholders in the event of default. Four different formulations of recovery rate are suggested in the literature. The default pay-off is either a fraction of:-

(i) par;

(ii) a risk-free bond with the same structure of cash flows;

(iii) the market value of the security just prior to default; and

(iv) par plus accrued interest.

Madan and Unal [1998] and Madan and Unal [2000] assume fractional loss of par in the event of default. Jarrow and Turnbull [1995] and Hull and White [1995], meanwhile, assume that the claim equals the value of a risk-free bond with the same payment schedule. Duffie and Singleton [1999] assume that the claim is equal to the market value of the bond immediately prior to default. As pointed out by J.P. Morgan [1999] and Jarrow and Turnbull [2000], these assumptions do not correspond to the general practice of bankrucptcy laws in most countries. A realistic assumption is that the claim equals the face value of the bond plus accrued interest.

3. **Integration of the variables:** Credit risk models of risky debt can be categorized in many ways. We choose to distinguish between structural models and reduced-form models. Structural models use the approach developed by Merton [1974], in which credit risk is considered to be a put option on the value of the firm's assets. The first passage time models also use the concept of firm value, the firm value is used to determine the time of default, and recovery rate is often specified exogenously. The modelling types (i) and (iii) of default risk correspond to structural models.

   Reduced-form models are sometimes known as intensity models. These models are structured to be close to data, where the intensity process is calibrated to fit market data. The modelling type (ii) of default risk corresponds to reduced-form models.

In the following sections, we give an overview of the literature in valuing risky bonds, both at individual asset and portfolio levels. Comments are given with emphasis on the models' limitations and outstanding issues.

3.4 Firm Value Models

Firm value models derive the price of default risk by modelling the value of firm's assets relative to its liabilities. Because they model the evolution of the firm's capital structure, they are sometimes called structural models.

3.4.1 Merton's Models

One of the first models for pricing defaultable bonds is developed by Merton [1974]. Based on Black and Scholes [1973], Merton provides an analytical framework, which gives the intuition of interpreting capital structure in terms of option contracts.

It is assumed that the ability of the firm to redeem its debt is determined by the total value of its assets V_t. Consider the firm having a single liability with promised terminal payoff K. This claim is interpreted as a zero-coupon bond. The discount bond is modelled in such a way that equity holders own the firm's asset and buy a put option from the bond holders. If at the maturity date, T, the assets of the firm are worth less than the amount owed to the debt holders, the equity holders can balance the debt due to be redeemed with their payoff from a put option on firm's assets. Thus, a corporate bond can be viewed as a default-free bond minus a put option with strike price K written on the assets of the firm. In this case, the payoff of the bond with promised payoff of K is

$$Min(V_T, K) = K - Max(K - V_T, 0).$$

The dynamics of the firm value under the risk-neutral measure Q are specified as a Geometric Brownian motion of the form

$$\frac{dV_t}{V_t} = rdt + \sigma_v dB_t,$$

where r is the constant riskless interest rate and σ_v is the instantaneous standard deviation of the firm value. Although V_t itself is not a traded asset, the stock of the firm is. Merton [1974] shows that the valuation of derivative of V_t is independent of investors' risk preferences. We can therefore assume risk-neutrality without loss of generality. Thus, the price, $p(t,T)$, of the put option with payoff $max(K - V_T, 0)$ is given by the Black-Scholes option pricing formula.

Given these dynamics, the price of the defaultable bond is obtained by deducting a standard Black-Scholes option from the risk-free value of the bond. The price of the risky bond at time t can be expressed as

$$D(V_t, t, T) = B(t, T) - p(t, T),$$

where $B(t,T)$ is the price of a riskless bond at time t with the same face value K and maturity T. We can write

$$D(V_t, t, T) = B(t, T)N(d - \sigma_v\sqrt{T-t}) + V_t N(-d),$$

where

$$d = \frac{\log \frac{V_t}{K} + (r + \frac{1}{2}\sigma_v^2)(T-t)}{\sigma_v\sqrt{T-t}}.$$

A number of stringent assumptions are made in this model. It assumes that default cannot occur before maturity of the debt. The firm has only a single class of debt; different seniority levels are excluded. Deviations from absolute priority are not considered. Moreover, coupons cannot be handled. Bankruptcy occurs when the firm value is below the face value of the debt; default induced by a liquidity crunch is excluded. Bankruptcy costs are ignored.

3.4.2 Extensions of Merton's Model

Merton's credit risk model for zero-coupon bonds has been extended in several ways. These include the pricing of different types of securities such as coupon bonds, callable bonds, convertible bonds, and variable rate bonds. Other extensions treat the valuation of claims with different maturity dates, seniority classes, and indenture provisions.

For example, Geske [1977] derives closed form solutions for defaultable coupon bonds. He assumes that equity holders make the coupon payment and therefore effectively own a compound option. By paying the coupon, they exercise the compound option and receive the option on the value of the firm for the value of the coupon payment. The price of this compound option can be expressed in terms of multivariate normal integrals with the highest dimension equal to the number of remaining coupon payments.

An important assumption underlying the analysis is that the firm finances each coupon payment with new equity only taken up by shareholders, and that bankruptcy occurs when the firm fails to make a coupon payment because it is unable to raise enough money to fund the payment. Black and Cox [1976] argue that this situation will happen whenever the value of the equity, after payment is made, is less than the value of the payment. This argument is intuitive, in that the firm will find no takers for its stock if they know that the stock will become less valuable than the total value they need to contribute to the promised payment. Geske [1977] extends this argument by assuming that the condition is also necessary for bankruptcy to happen. Given this, let S_{t^+} be the value of the stock immediately after time t, Black and Cox's argument is equivalent to saying that the firm will be able to finance the coupon payment by rights issues if

$$S_{t^+} > C_t,$$

where C_t is the promised coupon payment at time t. Default happens when the above inequality becomes an equality. The corresponding value for V_t is known as the default barrier at time t. In Geske's model [1977], a default can only happen on the day of a coupon payment. In this case, a typical default barrier is a set of points corresponding to the firm's values at which bankruptcy happens on a coupon day.

Selby [1983] generalizes Geske [1977] for valuing senior and subordinate bonds with discrete coupon payments, when two alternative default clauses: liquidation, and reorganization in the event of a default, are considered.

Ho and Singer [1982] analyze the effect on different indenture provisions such as time to maturity, financing restriction on the firm, priority rules, and payment schedules on the credit risk of bonds within the Merton framework. Furthermore, Ho and Singer analyze the effect of sinking fund provisions on the price of risky debt.

Chance [1990] examines the duration of defaultable zero-coupon bonds with the framework of Merton's model. He shows that defaultable bonds have lower duration than their riskless counterparts with the same maturity. This result is consistent with empirical observations that credit risky bonds are less sensitive to interest rate changes.

Shimko, Tejima, and Van Deventer [1993] derive closed-form solution for defaultable discount bonds when interest rates follow a Vasicek [1977] process. They show that the correlation between interest rate movements and the returns on the underlying asset is important in determining the credit spread on risky debt.

3.4.3 First Passage Time Models

Another line of extension of Merton's model follows the first passage time approach. Unlike the model of Merton that does not allow bankruptcy to occur before maturity of the bond, the first passage time models assume that bankruptcy occurs if the firm value crosses a specified boundary. The default boundary is often time-dependent. First passage time models allow for premature default. On the other hand, the recovery rate is usually exogenously specified, and the modelling of which is difficult.

The first passage time model was introduced by Black and Cox [1976]. They modify Merton's firm value model to facilitate the modelling of safety covenants in the indenture provisions. A safety covenant allows the bondholders to force bankruptcy if certain conditions are met. Some safety covenants entitle bondholders the right to force the entire amount of the debt issue due to be redeemed if the debtor is unable to meet interest obligations. In the event of bankruptcy, the entire amount of debt becomes due immediately and forces restructuring or liquidating the firm's assets. The aim of such provisions is to protect debt holders from further devaluations of the firm value.

Black and Cox [1976] modelled such a safety covenant as an exogenous, time-dependent boundary. They set the boundary to be an exponential function in the form of

$$V^d(t) = ke^{-\gamma(T-t)},$$

where k and γ are constants. As soon as the firm value reaches V^d, the firm is forced into bankruptcy, in which case the bondholders take over the firm's assets V^d. Such safety covenants are shown to reduce credit risk by a potentially substantial amount, depending on the specification of the boundary.

Longstaff and Schwartz [1995a] adapt Black and Cox's model to a more realistic setting. Firstly, they allow interest rates to be stochastic, with dynamics as proposed by Vasicek [1977]. Secondly, they do not require the recovery rate to be equal to the boundary value upon first passage, but assume an exogenously given rate w of the face value K. Thirdly, the default boundary is assumed to be K. This approach explicitly allows for deviations from absolute priority.

Nielsen, Saá-Requejo, and Santa-Clara [1993] further generalize the first passage time approach by allowing for a stochastic boundary. In the case of default, recovery rate is specified as an exogenously determined fraction of the face value. In this respect, the model corresponds to Black and Cox [1976]. Nielsen, Saá-Requejo, and Santa-Clara [1993] also assume interest rates to follow Vasicek [1977] or the extended Vasicek model as in Hull and White [1990]. They obtain a closed form solution for the prices of discount bonds if interest rates follow Vasicek [1977].

Mason and Bhattacharya [1981] extend the first passage time model of Black and Cox [1976] by using a jump process to model firm value. The jump process results in a random default time. They show that discontinuous dynamics of firm value can have a significant impact on the value of a safety covenant, as conjectured by Black and Cox [1976]. Zhou [1997] generalizes the model of Mason and Bhattacharya [1981] using a jump-diffusion process. Default can happen expectedly when the asset value hits a lower barrier through a continuous diffusion crossing, or unexpectedly when a downward jump of large magnitude occurs.

Although first time models allow for premature default, they are not without shortcomings. Firstly, it is difficult to estimate a realistic default threshold for the value of the firm. Some models merely assume a constant threshold. Furthermore, with the exception of Black and Cox [1976], most models separate firm value and recovery rate. Usually, the firm value determines the time of default while the recovery rate is an exogenously specified function which may depend on the firm value. The exogenous specification of recovery rate in first passage time models can be seen as a limitation of those models. As is shown in Altman and Kishore [1996], standard deviations of recovery rate are generally high. Also the recovery rate may be correlated with other variables such as interest rates.

3.4.4 First Passage Time Models and Capital Structure

The value of corporate debt and capital structure are interlinked variables. Their inter-relationships are complex, and one cannot be optimized without knowing the value of another one. Leland [1994] examines corporate debt values and optimal capital structure in a unified analytical framework. He relates the value of corporate debt and optimal capital structure to firm risk, taxes, bankruptcy costs and bond covenants. By assuming an infinite life debt structure with time-independent payouts, closed-form solutions are derived for the value of debt, and for optimal capital structure. Two types of debt are considered, namely unprotected debt and protected debt. The first one has the property that bankruptcy is triggered by the inability of the firm to raise sufficient equity to meet its current debt obligations. This condition is consistent with Black and Cox [1976], and Geske [1977]. The second one is the Brennan and Schwartz case with a positive net-worth covenant, which stipulates that the asset value of the firm always exceeds the principal value of the debt.

Another article by Leland and Toft [1996] examines the optimal capital structure between a choice of amount and maturity of its debt. Bankruptcy is determined endogenously. The model predicts that short term debt reduces asset substitution agency costs. This implies that short term debt is more likely to provide incentive compatibility between debt holders and equity holders, although it does not exploit tax benefits as completely as long term debt. Leland [1998] contains a lucid treatment of the development of key research issues in default risk. Callable debt is considered in a framework of dynamic capital structure.

Furthermore, issues in corporate finance and strategic behaviour of shareholders have been taken into account. Examples that consider endogenous capital structure, liquidation policy, re-capitalization and re-organization of debt include Brennan and Schwartz [1984], Leland [1994, 1998], Leland and Toft [1996], Anderson and Sundaresan [1996], and Mella-Baral and Perruadin [1997]. These models allow for endogenous default, optimally determined by equity holders when asset levels fall to a sufficiently low level. Anderson and Sundaresan [2000] conduct an empirical analysis of structural models of corporate bond yields. Their results suggest that recent modifications of the contingent claims models to allow for endogenous default barriers have improved the performance of the models in tracking observed yield spreads.

3.4.5 Comments

The classical Merton model has become an indispensable tool for discussing the distribution of the firm's value between shareholders and bondholders. This approach provides us with a clear link between economic fundamentals and defaults. It also helps us to understand losses on default, and possibly the correlation of default of different firms. However, the main problem with the approach is that firm value is somewhat abstract quantity. It may be difficult to estimate the value of the firm's assets with accuracy.[5] For some types of issuers, such as municipal authorities, it is not clear what asset value process to use. Use of some easily observed proxy for firm value may be preferable.

3.5 Reduced-Form Models

The reduced-form approach to pricing credit risk bypasses the complications of handling firm's economic fundamentals, and deals directly with market prices and spreads. Models in this area include Duffie and Singleton [1999], Jarrow and Turnbull [1995], Jarrow, Lando and Turnbull [1997]. These models do not require a process of default risk that is dependent on the firm's parameters, as in the structural approach. The price or spread of a defaultable bond is related to a riskless bond through some exogenously specified process of default and recovery rates. Because models can be calibrated to market data by taking the term structure of credit spreads, this approach is considered mathematically more tractable. But from the viewpoint of economic fundamentals, it may be less intuitive than the firm value approach.

[5] KMV Corporation has suggested using the equity data and equity volatility to "back out" the estimates of the underlying asset value and its volatility. Apart from providing a mathematical means for estimation, it is evident that this method introduces part of the systematic risk inherent in the stock prices into the asset value process, and hence the credit spreads. See Elton, Gruber, Agrawal, and Mann [2000] for empirical explanations of credit spread.

3.5.1 Duffie and Singleton Model

Duffie and Singleton [1999] assume that default events are surprises. Under a risk neutral measure $\mathcal{Q}$ the default occurs at a hazard rate h_t. Let r_t be a short rate process under $\mathcal{Q}$. They consider a contingent claim consisting of a pair (Z,τ) of random variables, where Z is the payment at the stopping time τ. Then the price of the contingent claim at time t is given by

$$U_t = E_t^{\mathcal{Q}}\Big[\exp(-\int_t^{\tau} r_u \, du) \; Z_\tau\Big].$$

They define a defaultable claim to be a pair $((X,T),(X',T'))$ of contingent claims, where X is the promised payoff at date T, and claimholders receive the payment X' at the stopping time T' when the issuer defaults. This means that the actual claim (Z,τ) can be expressed in terms of $((X,T),(X',T'))$ satisfying

$$\tau = Min(T,T'), \;\; Z = X\mathbf{1}_{\{T<T'\}} + X'\mathbf{1}_{\{T\geq T'\}}.$$

Assuming that at the time of default the claim pays a fraction $(1-L_t)$ of the price immediately before default,

$$X' = (1-L_t)U_{t^-},$$

where U_{t^-} is the price of the claim immediately before default, and L_t is a random variable describing the fractional loss of market value of the claim at default. Under some conditions as in Theorem 1 in their paper, the price process can be expressed alternatively as

$$U_t = E_t^{\mathcal{Q}}\Big[\exp(-\int_t^{T} R_u \, du) \; X\Big].$$

where

$$R_t = r_t + h_t L_t.$$

We call $s_t = h_t L_t$ the short spread. This approach is natural. By discounting at the adjusted short rate R_t, the model accounts for both the probability and timing of default, as well as for the loss effects on default.

There are three main features of this reduced-form model. Firstly, defaults are assumed to occur unexpectedly. If their arrival is modelled as a Poisson process, default happens at the intensity rate of h_t. Secondly, another feature of the model is that the processes (h_t, L_t) can be made independent of the value U_t of the contingent claim. This is a typical assumption in most applications of this approach. As a result, defaultable coupon bonds can be valued as simple portfolios of discount bonds. Thirdly, the assumption on the fractional recovery rate of the debt's market value is made for the sake of technical convenience. However, this is not necessary. As discussed in Section 3.3, other assumptions on recovery rates are also suggested in the literature.

Duffee [1999] applies the Duffie and Singleton's idea to fit yields on bonds issued by individual investment-grade firms to a reduced-form model. He considers a three-factor model in which the instantaneous, default-free short rate process r_t is assumed to be a linear combination of two square-root diffusion processes. The short spread s_t is modelled as another linear combination of three square-root diffusions, where two of them are the same as those in the short rate process. Empirical results appear to be encouraging, as the average error in fitting corporate bond yields is less than 10 basis points. However, Duffie and Singleton [1999] argue that the models used by Duffee [1999] are theoretically incapable of capturing the negative correlation between credit spreads and U.S. Treasury yields, while maintaining non-negative default harzard rates.[6] They then come up with an alternative model with more flexible correlation structures for (r_t, s_t), but the system cannot be solved analytically for bond prices.

Another assumption about the processes (h_t, L_t) is to introduce dependence of the credit spread s_t on the value of the contingent claim. Duffie and Huang [1996] presents a model for valuing claims when contracting parties are subject to default

[6]See Duffee [1998].

risk. Using Duffie and Singleton [1999], the promised cash flows are discounted at a switching-type discount rate. The discount rate function takes the discount rate of the counterparty for whom the swap is currently out of money. Duffie, Schroder, and Skiadas [1996] derive the implications of default risk for the valuation of securities when the fractional recovery rate and hazard rate depend on the market values of instruments issued by the same company. In general, it is reasonable to assume some form of dependence of the processes (h_t, L_t) on the value of the contingent claim. In the case of the valuation of defaultable coupon bonds, however, this brings about another question as to the determination of explicit forms of (h_t, L_t) as functions of debt value.

3.5.2 Jarrow, Lando, and Turnbull Model

Jarrow, Lando, and Turnbull [1997] propose a model that relates default probabilities to credit ratings. They specify a time-homogeneous finite state space Markov chain with a generator matrix of K transition states, and assume that each state corresponds to a rating class. Each entry in the matrix represents a transition rate between the corresponding states. The time spent in one rating class is assumed to be exponentially distributed with a parameter in the diagonal of the matrix. Under the independence assumption on interest rates and default process, and the value of a recovery rate, Jarrow, Lando, and Turnbull [1997] express the price of a defaultable discount bond in terms of a recovery rate, default probabilities, and a riskless discount bond.

The historical approach uses transition probabilities, as they are available from credit rating agencies. Recovery rates can be estimated from historical data. Because transition probabilities are empirical, risk premia have to be estimated. Jarrow, Lando and Turnbull [1997] propose a way of estimating risk premia from traded zero coupon bonds such that model prices match market prices.

Das and Tufano [1996] extend Jarrow, Lando, and Turnbull [1997] to allow for correlation between the term structure of interest rates and recovery rates. This structure results in a model in which credit spreads are correlated with interest rates, as is evidenced in practice.

3.5.3 Other Intensity Models

A middle-way approach has been suggested in the literature. Cathcart and El-Jahel [1999] propose a framework situated between the structural and the reduced-form approaches. A default event occurs in an expected or unexpected manner when the value of a signaling process reaches a certain lower barrier or at the first jump time of a hazard-rate process. Although the model can generate strictly positive credit spreads for small maturities, the simple assumption that the firm goes bankrupt immediately when a jump in the asset value occurs for the first time needs empirical justification.

Duffie and Lando [1999] construct a model with imperfect information to build a bridge between the two approaches. They assume that the value of a firm is not precisely observable but only a noisy process may be observed. As a consequence, the default mechanism concurs with Duffie and Singleton [1999] assumption that default event is a surprise. An intensity process is derived from the structural framework.

To capture the effects of capital structure, a hybrid-type model has been suggested in the literature. Madan and Unal [2000] propose a two-factor hazard rate model to price risky debt. Consistent with the hazard rate literature, the probability of sudden default is governed by the hazard rate. They endogenously derive the hazard rate function in terms of the firm's non-interest sensitive asset values and default-free interest rates. Assuming that default follows a Poisson arrival rate and loss in the case of default has a cumulative distribution function, they come up with a structural definition of the hazard rate process as a product of the two quantities. Although the structural approach is appealing, they fail to obtain an exact analytical solution for the bond price. Instead, an analytical approximation is derived after they express the hazard rate function as a first-order approximation of its Taylor expansion. However, a drawback is that the approximated hazard rate can admit negative values with positive probability.

Duffie and Huang [1996] applies Duffie and Singleton's model to price swaps with counterparties of different default risk. A switching-type default-adjusted short rate process depending on whether the swap value is positive or negative is used. Asymmetric default risk of the counterparties and non-linearity of promised cash flows are then explored in their paper.

Huge and Lando [1999] extend the model of Duffie and Huang [1996] to a framework which takes into account the ratings of two parties. The authors use the framework to investigate plain vanilla interest rate swaps and default swaps. The effects of joint rating are considered in their setting. For example, the results confirm that swap spreads are relatively insensitive to credit quality for interest rate swaps. In addition, their model implies that the effect of the quality of the protection seller in a default swap is relatively small. The flexibility of the method also allows us to examine the effects of early termination provisions on swap spreads. They show that a credit trigger reduces the spreads of an interest rate swap.

Duffie and Singleton's model [1999] has also been applied in a recent paper by Jarrow and Yu [2001]. The paper studies the impact of counterparty default risk on the pricing of defaultable securities, where correlated defaults due to an exposure of common risk factors and firm-specific risks are considered. As with Duffie and Huang's model [1996], Jarrow and Yu [2001] specify in their models switching-type hazard rate processes depending on which counterparties have gone bankrupt. In principle, a framework with multiple layers of counterparty relationship can potentially be applied to pricing defaultable securities.

3.5.4 Comments

An important feature of reduced-form models is its reliance on the existence of traded debt instruments, and market prices and spreads. As a result, it is difficult to apply the model to private debt and commercial loans. Moreover, most of the models exclusively use aggregate market data about default rates, the rating transition matrix, and the recovery rate. Thus, firm specific risk and financial fundamentals are not evaluated, or even ignored.

Pricing models based on rating information have been gaining popularity because they consider not only default risk but also the possibility of other credit migrations. However, such information has to be applied scrupulously. A problem with the transition matrix approach is that the models imply the same yield spread for all bonds belonging to the same rating class. Aggregate rating information is an imprecise measure of an issuer's credit and default risk. This fact is empirically evidenced by the large variations in default rates and observed credit spreads for bonds with the same rating. Longstaff and Schwartz [1995a] show that bonds of equal rating exhibit quite different yield spreads depending on industry sector. Spreads of corporate bonds change even if the rating remains unchanged.

There are some other problems with the use of rating information. As to modelling issues, it is normally assumed that the rating transition is Markovian. A problem is that a rating transition is likely to be varying over time. Furthermore, if transitions between credit classes are governed by a Markov chain, then the times spent in different credit classes would have exponential distributions, and there would be no tendency for a firm to continue to fall through credit classes. This appears to be contrary to empirical evidence. Finally, as shown in Kliger and Sarig [2000], bond ratings are largely about diversifiable default risk.[7] This result casts doubts on

[7]Kliger and Sarig [2000] show that bond ratings are largely about diversifiable default risk, although the result could be due to small sample available for tests. Elton, Gruber, Agrawal,

whether it is sensible to make use of aggregate information on many different firms to price individual corporate bonds, as transition matrices consist of aggregate averages of transition rates from one state to another.

3.6 Pricing of Credit derivatives

Credit derivative are derivatives on pure credit risk, where the payoff is dependent on a credit variable. We give a brief review of debt insurance, spread derivatives, and credit default swaps in this section.

3.6.1 Debt Insurance

The simplest form of credit derivatives is a debt insurance contract. Debt insurance or debt guarantees have existed for a long time and are designed to cover the entire financial loss resulting from a default event. Merton [1977] considers a simple model of firm as in Merton [1974]. The firm borrows money by issuing a single homogenous debt issue. The terms of the debt are that the firm promises to pay a total of K on the maturity date. In absence of a guarantor, in the event that the promised payment is not made, the firm defaults to the bondholders all the assets of the firm. This is the case when the asset value V_t is less equal to K. When V_t is at least equal to K, the firm is able to honour the promised payment.

Consider the impact of a third-party guarantee of the payment to the bondholders where there is no uncertainty about the obligations of the payment K to be made. Here, the terms of guarantee are that in the event of default, the guarantor will meet the payment. In effect, the guarantee has value to the bondholders and imposes a cost on the insurer. Merton [1977] shows that the cost of such an insurance contract can be determined as the put option on the asset value as in Merton [1974]. Merton [1978] extends the model to the case where there are surveillance costs.

and Mann [2000] show that the most significant components of credit spreads are resulted from systematic risk in the stock market and taxes.

3.6.2 Spread Derivatives

Recently more complex credit derivatives have been introduced. These derivatives distinguish themselves from traditional credit derivatives in that they do not insure the loss of default, but a change of credit quality. This change of credit quality can be measured in form of a spread of yields, or change of credit ratings. The simple model of credit derivative presented in the last section cannot be used to value spread credit derivatives.

Das [1995] extends the model by Merton [1974] to credit spread derivatives. The idea of the work of Das [1995] is that a spread option is an option on credit risk, which is in turn an option on the firm value. Das [1995] values a put option with payoff

$$\Lambda(T) = Max\Big[D^{d^*}(T,T') - D(T,T'),0\Big],$$

where $D^{d^*}(T,T') = B(T,T')exp(-r^*(T' - T))$ is the spread-adjusted price of a discount bond. $D(T,T')$ is a defaultable discount bond, and $B(T,T')$ is a riskless discount bond of the same face value. r^* is the strike yield spread. Interest rates are assumed to be stochastic. T is the option maturity, and T' is the bond maturity.

The payoff is positive only if the defaultable discount bond $D(T,T')$ has at least an upward spread r^* at the option maturity T. Das [1995] obtains prices for the option by solving an integral equation numerically.

Alternatively, the yield spread can be modelled directly. This is the approach adopted by Longstaff and Schwartz [1995b]. They assume that the dynamics of the logarithm of the spread X are given by the stochastic differential equation

$$dX_t = (a - bX_t)dt + cdB_t,$$

where a, b, and c are parameters, and B is a standard Brownian motion.

To allow for random interest rates in the model, they make a simple assumption that the riskless term structure is determined by the single-factor Vasicek [1977] model. The two processes are correlated with each other. Longstaff and Schwartz [1995b] price a European call option on the level of the credit spread, with the strike price K of the option.

3.6.3 Credit Default Swaps

Credit default swaps have become increasingly popular. Their purpose is to allow credit risks to be traded and managed in much the same way as market risks. A credit default swap is a contract that provides insurance against the risk of a default by a particular company, which is known as the *reference entity.* A default by the company is known as a *credit event.* A reference asset is normally a bond issued by the *reference entity.*

To obtain protection against the default risk of the reference asset, the buyer of the credit default swap makes periodic payments to the seller until the contract expires, or until a credit event occurs. When a credit event occurs, the contract is then settled by either physical delivery or in cash. If the terms of the contract require physical delivery, the protection buyer delivers the bond to the seller in exchange for its face value. When there is a cash settlement, the seller compensates the buyer for an amount of $(100 - Q)\%$ of the notional principal, where Q is the market price of the reference asset some specified number of days after the credit event.

The pricing of credit default swaps is a complicated task, because of the involvement of three parties with default risk. Hull and White [2000] provides a framework for valuing credit default swaps when there is no counterparty default risk. However, the assumption that interest rates, default probabilities, and recovery rates are independent is unlikely to be true in practice.[8]

Hull and White [2001] extend the analysis to provide a framework for valuing credit default swaps that takes counterparty default risk into account. They assume that the creditworthiness of companies can be defined by credit indexes starting at

[8]See also Davis and Mavroidis [1997]. Under an assumption of no counterparty default risk, they propose a simple valuation model of default swaps taking interest rate risk into account.

zero and following correlated Standard Brownian motions. A default barrier of a company is chosen so that the model is exactly consistent with the default probabilities extracted from bond prices or credit default swap spreads. The credit default swap spread can be computed by Monte Carlo simulation.

3.7 Commercially Available Credit Models

In this section, we briefly review some popular models which have been implemented and are available for commercial use. The models vary in terms of the underlying principles on which they are based, the data used to run them, and also on the transparency of the details of their implementation.

3.7.1 CreditMonitor

KMV Corporation has developed a model of default probability, CreditMonitor, that uses equity prices and financial statements.[9] CreditMonitor calculates the probability of default for the forthcoming year, or years. A default event is defined as a default of any scheduled payments, interest or principal. There are three steps in the determination of the default probability of a firm:

1. **Estimate asset value and volatility.** In this step, the asset value and the asset volatility are estimated from the market value and volatility of the firm's equity, and the book value of its liabilities. Here, Merton's model [1974] is assumed for the value and volatility of the equity.

2. **Calculate the distance to default.** The distance to default is calculated from the asset value and asset volatility estimated in the first step, and the book value of liabilities. This measure combines three credit variables, the value of the firm's assets, its business and industrial risk, and its leverage into a single quantity:-

$$Distance\ to\ default = \frac{Market\ value\ of\ assets - Default\ point}{Market\ value\ of\ assets - Asset\ volatility},$$

[9] See Modelling Default Risk [1997].

where the default point is taken as the asset value at which the firm will default. The default point is assumed to take the book value of liabilities, which lies somewhere between total liabilities and current liabilities. For example, the default point can be defined as short-term liabilities plus 50% of long-term liabilities. Therefore, when default occurs, the distance to default becomes zero as the market value of the firm's assets hits the default point.

3. **Calculate the default probability.** The default probability is determined from the distance to default and the default rate for a given level of the distance to default. If the probability distribution of the asset value is known, the default probability can be computed directly from the distance to default.

Delianedis and Geske [1998] employ a similar approach to modelling default probabilities. However, instead of using the distance to default as a determinant of the default probability, they assume Merton [1974] and Geske's bond pricing models [1977] and derive the corresponding risk-neutral default probabilities. Their empirical study suggests that these risk neutral default probabilities possess significant information about credit rating migrations and default, often more than one year before the event.

3.7.2 CreditMetrics

The evolution of credit derivatives has not been an isolated event. The facility offered by credit derivatives to manage credit risk more actively is certainly noticeable in its own context. However, without methods to identify determinants of portfolio risk, this facility to manage risk would be incomplete.

The advent of public portfolio models has coincided with the growth of credit derivatives markets. There is an important implication here. With these portfolio models, market participants have become more capable of recognizing sources of risk, while credit derivatives provide flexibility to manage these sources. In this and the following subsections, we review two commonly used portfolio credit risk models: CreditMetrics and CreditRisk+.[10]

CreditMetrics models the changes in portfolio value that results from significant credit quality moves, including defaults or rating changes. The model takes information on the individual obligors in the portfolio as inputs, and produces the distribution of portfolio values at some fixed horizon in the future. The model can be described in three parts:

1. **States of the world.** The definition of the possible states for each obligor's credit quality, and a description of how likely obligors are to be in any of these states at the horizon date. We need a rating system. The crucial element here is that we know the probabilities that the obligor migrates to any of the states between now and the horizon. A transition matrix characterizes a rating system by providing the probabilities of migration for all of the system's states.

2. **Revaluation of exposures.** In the Step 1, we are concerned about the migrations of individual credits. Now we need to know the impact of these moves on the portfolio value. In particular, we assume a particular instrument's value is known today, and wish to estimate its value at the risk horizon, conditional on any of the possible credit migrations that the instrument's issuer might undergo. In the case of a bond, the revaluation step consists of estimating the bond's value under each possible transition. For the transition to default, we value the bond through the estimate of a recovery rate. For the non-default states, we obtain an estimate of the bond's horizon value by utilizing the term structure of bond spreads and riskless interest rates. For example, we assume that the forward zero curves for each rating category are available. Other types

[10]See The J. P. Morgan Guide to Credit Derivatives [1999], CreditMetrics - Technical Document [1997], and CreditRisk+ A Credit Risk Management Framework [1997].

of exposures can also be incorporated; this involves defining the values in each possible future rating state of the underlying credit.

3. **Interaction and correlation between credit migrations.** The next step is to construct correlations between exposures. The idea is based on Merton-type frameworks. The intuition behind these models is that default occurs when a firm's asset value drops below the market value of its liabilities. We assume that asset returns follow a normal distribution. We then partition the distribution for the firm according to its transition probabilities.

   In the portfolio framework, once the partitions are defined for every obligor, we take correlations in equity returns as a proxy for the asset value correlations. With the correlations defined, the model is fully specified. The portfolio distribution can then be obtained through a Monte Carlo approach.

   For a single scenario, we draw from a multivariate normal distribution to produce asset value changes, read from the partitions to identify the changes with new rating states and exposure values, and finally aggregate the individual exposures to arrive at a portfolio value of the scenario. Repeating this process over a large number of scenarios, we accumulate a large number of equally likely portfolio values. We are able to estimate the Value-at-Risk (VaR), and other descriptive statistics, of the portfolio.

There are several ways of utilizing the outputs of CreditMetrics to guide credit risk management. At a strategic level, a firm can assess the aggregated amount of risk in terms of VaR calculations. A typical statement for senior management is of the following form: We are X percent certain that we will not lose more than V dollars in a given horizon. At a finer level, concentration analyses can identify pockets of excess risk, which would be a clear candidate for hedging or diversification via credit derivatives.

3.7.3 CreditRisk+

Unlike the simulation-based CreditMetrics, CreditRisk+ uses actuarial techniques to investigate analytically the credit exposures arising from portfolio management. There are three main components in CreditRisk+: CreditRisk+ model, economic capital, and applications.

1. **CreditRisk+ Model.** The model is based on a portfolio approach to modelling credit default risk. It takes into account information relating to size, maturity of an exposure, the credit quality, and systematic risk of an obligor. In order to capture the uncertainty in the level of default rates, volatility of default rates is taken into account. Exposures, default rates, and recovery rates are taken as inputs.

2. **Economic Capital.** Using the derived loss distribution function, the CreditRisk+ model can be used to determine the level of economic capital required to cover the risk of unexpected default losses. Scenario analysis can be carried out in order to identify the financial impact of changes in input data.

3. **Applications.** CreditRisk+ provides the application of provisioning for credit risk. This application reflects the credit losses of the portfolio over several years. Another application is to monitor exposures against limits and provides a trigger mechanism for identifying potentially unwanted exposures.

3.7.4 Comments

CreditMonitor, CreditMetrics, and CreditRisk+ have become the standard methodologies for credit risk management. The CreditMonitor and CreditMetrics methodologies are based on the structural approach, and the CreditRisk+ methodology comes from an actuarial approach to mortality. All these models emphasize the accrual accounting perspective and focus on default risk only. Furthermore, for credit risk management, the time horizon is normally one year or longer. Given that these models are based on the assumption of deterministic interest rates, they are ill-suited to predicting the effects of market risk on the horizon distribution of the value of a portfolio.

CreditMonitor has two main advantages. Firstly, by using the market value of equity to estimate the firm's volatility, it incorporates market information into default probabilities. Secondly, the method is less dependent on underlying distributional assumptions. There are also some disadvantages. Firstly, the value of the firm cannot be directly observed. In the case that the firm does not have traded equity, the estimations of the firm's asset value and return volatility become insurmountable. Secondly, interest rates are assumed to be deterministic. This limits the usefulness of the model when applied to loans and other interest rate sensitive instruments. Thirdly, as with many structural models, one implication of the model is that the default probabilities also tend to zero as the maturities of a credit risky bond tends to zero.

CreditMetrics employs probability transition matrices to develop a portfolio credit risk management framework that measures the marginal impact of individual bonds on the risk and return of the portfolio. This methodology has a number of limitations. Firstly, the model assumes that the term structure of default free interest rates is fixed. CreditMetrics assumes no market risk over a specific period. Secondly, the CreditMetrics default probabilities do not depend upon the state of the economy. This is inconsistent with market practices. Thirdly, the correlation between asset returns is assumed to equal the correlation between equity returns. This is only a crude approximation.

The CreditRisk+ methodology has some advantages, however. Firstly, CreditRisk+ has analytical expressions for the probability distribution of portfolio losses. Thus, the methodology does not require simulation and computation is relatively quick. Secondly, the methodology only requires minimal data inputs of each loan, including the probability of default and the loss rate given default. There are a number of disadvantages. Firstly, CreditRisk+ ignores the stochastic nature of interest rates that affect credit exposure over time. Secondly, the methodology ignores non-linear products such as options.

However, the most noteworthy feature of these models is that their rather different mathematical structures are reconcilable. Gordy [2000] shows that, despite the differences on the surface, the underlying mathematical structures of the CreditMetrics and CreditRisk+ are similar. In fact, if the asset returns have a factor structure, the CreditMetrics and KMV-type models can be written as Bernoulli mixture models, the underlying mathematical framework of CreditRisk+.[11] Based on this concept, Frey and McNeil [2001] further discuss the effects of mixture distributions and underlying mixing distributions on modelling the tail dependence of credit loss distributions of large portfolios. Emphasis is placed on the use of copulas in latent variable framework.[12] Other new approaches to modelling extreme events include Bayesian methods, multivariate Extreme Value Theory, and a random changepoint model as discussed in the work of Smith [2001].

3.8 Outstanding Issues

In the area of corporate debt valuation and credit derivatives, credit risk models are still largely mathematical and theoretical in nature. There are several outstanding issues hinging on the path of extending the research.

The first is lack of data. A large database on corporate debt that is readily available and frequently updated is needed. The information should include bond prices, financial statement information, corporate capital structure, bond covenants,

[11]In Statistics literature, this class of models is referred to as mixture models. See Joe [1997].

[12]See Embrechts, Lindskog, and McNeil [2001], Frey, McNeil and Nyfeler [2001], and Nelsen [1999]. See also Davis and Lo [1999] for an alternative approach to modelling dependence structure in bond porfolios.

and so on. To help understand the empirical and dynamic multivariate nature of credit spreads, the actual behaviour needs to be studied at the level of aggregate bond markets, the sectors and individual bonds.

On the theoretical side, most contingent claims pricing models based on Merton [1974] are not able to deliver the levels of spreads between corporate debt yields and otherwise identical Treasury yields. Even very short-term default-risky securities appear to have significant spreads over their Treasury counterparts. In practice, even for small maturities, the market does not neglect the possibility that some disaster may happen. A portion of the spread may also be due to *taxes* and *liquidity* that need to be incorporated into pricing models. Secondly, there seems to be a persistent negative correlation between the changes in default-free interest rates and the changes in credit spreads. Thirdly, neither the structural approach nor the reduced-form approach is designed to address the empirical regularities in the financial distress literature. Empirical facts such as renegotiations, debt rescheduling, and forgiveness and sometimes costly liquidations need to be reconciled in either the structural or in the reduced-form approach. A realistic pricing model should also consider the implications of managerial actions, as pointed out by Garbade [1999]. He argues that neglect or misspecification of managerial actions may cause models to overvalue senior claims and undervalue junior claims including equity. These actions include

1. early redemption of callable bonds to undertake risky projects and projects perceived to be profitable;

2. dividend policy as a signaling device; and

3. strategic options exercised in the interests of shareholders, for example exchange offers, tender-offer financing, special dividends, and so on.

Fourthly, the key role played by the bankruptcy code in the allocation of residual asset value upon financial distress is yet to be modelled in the valuation models. The code is in fact the key in satisfactorily distinguishing sovereign debts from corporate debts.

With regard to commercially available credit models, including CreditMonitor, CreditMetrics, and CreditRisk+, a main weakness of these models is that they ignore the stochastic nature of interest rates that affect credit exposure over time. Practitioners and regulators often estimate the VaR measures for credit and market risk separately, and take the sum of the measures as a total measure of capital for the credit and market risk exposures.[13] This approach could only be justified if the two types of risk exposure were perfectly correlated. Given that a more appropriate risk measure than that specified under the current regulatory regime is generally required, an improvement of this conservative assumption is a necessary avenue for further research.

3.9 Summary

Each of the two main classes of approach has its strengths and weaknesses. In the structural approach, firm value models have become an indispensable tool for discussing the distribution of the firm's value between shareholders and bondholders. The approach also provides us with a clear link between economic fundamentals and defaults. It helps us to understand losses on default, and possibly the correlation of default of different firms. However, the main problem with the approach is that firm value is somewhat abstract quantity. Use of some easily observed proxy for firm value may be preferable.

The reduced-form models are sufficiently close to data. Although it is always possible to fit some version of the model, but fitted model may not perform well out of sample. As a consequence of its reliance on the existence of traded debt instruments, it is difficult to apply the model to private debt and commercial loans. Moreover, most of the models exclusively use aggregate market data about default

[13] See Jarrow and Turnbull [2000].

rates, the rating transition matrix, and the recovery rate. Thus, firm specific risk and financial fundamentals are not evaluated, or even ignored. Use of matrices of transition ratings as a means to model default process of an individual firm may also brings about another problem.

There is a basic incompatibility in default mechanisms between the Duffie and Singleton [1999] model and the traditional Merton-type models. The "reduced-form" of a structural model is usually taken to mean a version of the model in which endogenous variables are expressed as a function of predetermined variables only.[14] In this definition, the reduced-form model must be equivalent to the original structural one. However, this is not true of the Duffie and Singleton [1999] model. In the Merton-type framework, in which a firm's asset value follows a pure diffusion process, default can only happen expectedly. Given that default has not happened up to the present moment, there is a zero probability that the firm will go into bankruptcy at the next instant. On the contrary, the reduced-form approach typically assumes that default events are surprises. For example, Duffie and Singleton [1999] assume that given a positive hazard rate process h_t, default occurs at a rate of h_t. Neither the pure diffusion nor the reduced-form approach appears to concur completely with empirical evidence that default can happen in both an expected and an unexpected manner. Modelling the firm's asset value as a jump-diffusion process may provide an alternative to the problem. In Chapter 6 we shall show that it also exhibits the interesting properties of leptokurtic feature that are empirically observable in asset returns.

Outstanding issues are summarized as follows. The first problem is lack of data. A large database on corporate debt that is readily available and updated frequently is needed. This information should include bond prices, financial statement information, corporate capital structure, bond covenants, and so on. Secondly, most

[14] For an econometric definition, see Koutsoyiannis [1973].

contingent claims pricing models based on Merton [1974] are not able to deliver the levels of spreads between corporate debt yields and otherwise identical Treasury yields. A portion of the spread may be due to taxes and liquidity that need to be incorporated into pricing models. Thirdly, neither the structural approach nor the reduced-form approach is designed to address empirical regularities in the financial distress literature. Empirical facts such as renegotiations, debt rescheduling, and forgiveness and sometimes costly liquidations need to be reconciled in either the structural or in the reduced-form approach. A realistic pricing model should also consider the implications of managerial actions. Fourthly, the key role played by the bankruptcy code in the allocation of residual asset value upon financial distress must not be ignored. Finally, a more pragmatic approach to modelling the interaction between credit risk and market risk in portfolio models is a necessary avenue for further research.

CHAPTER 4
A Two-Factor Generalization of Geske's model

4.1 Introduction

This chapter develops a flexible binomial framework for valuing credit sensitive instruments in a structural framework. As an application, we price a defaultable coupon bond when interest rates and firm's asset value are stochastic.

The literature on the valuation of defaultable term-structure began with Merton [1974]. In the paper, Merton adopts the Black and Scholes option pricing model to the pricing of risky discount bonds. Under the assumptions of a process of firm's asset value and a constant interest rates economy, Merton's model provides an important insight into the determinants of the risk structure, and shows how the default risk premium is affected by changes in the firm's business risk, debt maturity and the prevailing interest rate. As a generalization of Merton's model of defaultable discount bond, Geske [1977] applies the technique for valuing compound options to the problem of risky coupon bonds. He derives an analytical formula, which consists of multivariate normal integrals with dimensions up to the total number of contractual payments. It is shown that with a special auto-correlation structure, an application of an integral reduction may simplify the numerical computations. In a further paper, Geske [1976] develops a general theory for pricing compound options in terms of multivariate normal integrals. In practice, a wide variety of important problems have turned out to be very closely related to the valuation of compound options.

Selby [1983] generalizes Geske's [1977] work on pricing risky discrete coupon bonds in three ways. Firstly, a continuous dividend, as a known proportion of the firm value, is paid to the equity holders. Secondly, he derives a general valuation formula for valuing individual risky discrete coupon bonds, or tranches of such bonds with equal seniority. Thirdly, he derives general formulae for valuing senior and subordinate bonds with discrete coupons when two alternative default clauses: liquidation, and reorganization in the event of a default, are considered. By using the preference-independent approach first suggested by Cox and Ross [1976], the derivations become much simpler than in the alternative backwards recursion techniques employed by Geske [1977].

Shimko, Tejima, and Van Deventer [1993] generalize Merton's risky debt pricing model to allow for stochastic interest rates as in Vasicek [1977]. The method of analysis is based upon Merton's [1973] earlier work on the valuation of options with stochastic interest rates and time-varying volatility. They obtain a risky discount bond pricing formula, which yields comparative static results consistent with Merton's [1974]. They also show that the combined effect of the term structure of interest rates and credit variables is very important for risky bond pricing.

Each of the valuation formulae for risky coupon bonds developed has a number of common features. From the viewpoint of economic and computational implementation, one common point of paramount importance is that each valuation formula gives rise to a sum of multinormal distribution functions. Moreover, the multinormal distributions are nested, in the sense that the integration region at any stage is dependent on those at the stage of higher dimension. Selby and Hodges [1987] prove a general identity relating sums of nested multinormal distributions. By reducing the number of integrals to be evaluated, the application of the identity significantly improves the computational aspects of both the Geske and Johnson's [1984] analytical American put and the Roll's [1977] formula for American call option. However, these results are more of theoretical interest than practical use. The computation of high dimensional normal integrals has remained a very challenging problem. On the numerical solution of Geske's [1977] formula, however, the computation is not as onerous as would first appear. In the one-factor case where a firm's value is modelled as a geometric Brownian motion and interest rate is taken as a constant, the prices of a defaultable coupon bond can be computed efficiently and accurately by building a binomial tree for the firm value.

Instead of modelling a recovery rate as endogenously given, Longstaff and Schwartz [1995a] adopt a new approach to valuing risky debt by extending Merton's [1973] and Black and Cox's [1976] models in two ways. Firstly, their model incorporates both default risk and interest rate risk. Secondly, they derive the model in such a way that allows for deviations from strict absolute priority. Given the recovery rate as a constant, analytical formulae for fixed rate debt and floating rate debt are derived. By construction the ratio of the firm value to the face amount of the debt is a sufficient statistic for default risk in this model; in order to value a coupon bond it is not necessary to condition on the pattern of cash payments to be made before maturity. As a consequence, coupon bonds can be valued as simple portfolios of discount bonds. This result provides much of the tractability of the model. However, the implication that a firm has a constant value upon default in the typical diffusion approach is problematic. On the one hand, this approach emphasizes the central role of firm value in the determination of default. On the other hand, the approach cannot allow the variation in the recovery rate of a risky bond to depend on the firm's remaining value at default.

Longstaff and Schwartz [1995a] also provide empirical results suggesting that the implications of this valuation model are consistent with the properties of credit spreads implicit in Moody's corporate bond yield averages. This in turn provides strong evidence that both default risk and interest rate risk are necessary components for a valuation model for corporate debt.

In recent years, a new framework, namely the reduced form approach, has been developed. This new approach to pricing credit risk bypasses the complications of handling firm's economic fundamentals, and deals directly with market prices and spreads. Models in this area include Duffie and Singleton [1999], Jarrow and Turnbull [1995], Jarrow, Lando and Turnbull [1997]. These models do not require a process of default risk that is dependent on the firm's parameters, as in the structural approach. The price or spread of a defaultable bond is related to a riskless bond through some exogenously specified process of default and recovery rates. Because models can be calibrated to market data by taking the term structure of credit spreads, this approach is considered mathematically more tractable. But from the viewpoint of economic fundamentals, it may be less intuitive than the structural approach.

In this chapter, we develop a flexible binomial framework for valuing credit sensitive instruments in the structural framework. By extending the valuation models of Geske [1977], Selby [1983], and Shimko, Tejima, and Van Deventer [1993], we price a defaultable coupon bond when interest rates and firm's asset value are stochastic. We propose an efficient computation algorithm for the pricing of general risky coupon bonds. The default boundary is determined endogenously by requiring the value of equity to be at least the amount of coupon just paid in order to avoid bankruptcy. The properties of defaultable bonds, the interaction between market risk and credit risk, and examples of further applications are discussed.

This chapter is structured as follows. Section 4.2 presents a set of assumptions under which risky discount bonds are priced. By using a hypothetical asset as a numeraire, we show that not only does the use of this numeraire significantly simplify the analytical valuation of risky discount bonds, but that it also gives an implication that the computations of a two-factor model can be implemented easily. Section 4.3 derives some basic properties of two underlying processes. In Section 4.4, we propose an algorithm for constructing the binomial processes in Section 4.3 by extending a method suggested by Ho, Stapleton and Subrahmanyam [1995]. The method is to approximate a bivariate lognormal distribution by a bivariate binomial process. Based on Geske's idea, we can price a defaultable coupon bond efficiently by using a three dimensional lattice. A brief discussion on Geske's method of evaluating defaultable coupon bonds is given in Section 4.5. In Section 4.6 we show that the use of the numeraire is supported by computational efficiency, and present an efficient computation algorithm for the pricing of general risky coupon bonds. We generalize the valuation models of Geske [1977], Selby [1983], and Shimko, Tejima, and Van Deventer [1993]. Section 4.7 illustrates the efficiency of our computation algorithm. We investigate the properties of defaultable bonds in an economy of stochastic interest rates. Discussions on the interaction between market risk and credit risk, and implications for capital adequacy in credit risk management are also given in this section. Section 4.8 briefly discusses further applications of the framework. Section 4.9 concludes.

4.2 Valuation of Risky Discount Bonds

4.2.1 Assumptions and Notation

In this section we shall consider a firm which has one discount bond and which has no other form of loan. We make the standard assumptions as follows:

$A1$: Frictionless Markets

- Trading in assets takes place continuously in time,
- There are no taxes, bankruptcy, agency or transaction costs, nor are there problems of indivisibility of assets,
- Every individual acts as though the market price is independent of the amount bought or sold,
- Borrowing and lending are at the same cost, and
- Short sales are permitted, as is full use of the proceeds.

$A2$: The Term Structure

- The term structure of interest rates r is assumed to follow Vasicek's interest rate model.

$A3$: Firm Value Process

- The dynamics for the value of the firm V follow a Geometric Brownian Motion with instantaneous standard derivation σV where σ is non-stochastic and is known. The firm value is assumed to be a traded asset, and the correlation coefficient between the interest rate process and the firm value process is a constant denoted by ρ .

$A4$: Maturity Payment

- Maturity payment K is financed by rights issues only taken up by existing shareholders.

$A5$: Dividends

- Shareholders are entitled to receive a continuous dividend, which is a constant proportion of the value of the assets of the firm.

4.2.2 The Model

Merton [1974] first derives the value of a pure discount corporate bond by employing assumption $A3$. By usual no-arbitrage arguments, a parabolic partial differential equation for the corporate bond is derived. An alternative to Merton's [1973] method of analysis is the use of a probabilistic approach to pricing corporate bonds. We discuss this approach as follows: By assumption $A2$ and $A3$, we can express the stochastic processes of interest rates and the firm values in the following forms:

$$dr_t = (\zeta - \beta r_t)dt + \eta dB_t^r, \tag{4.1}$$

$$dV_t = (r_t - a)V_t dt + \sigma V_t dB_t \tag{4.2}$$

where B_t^r, B_t are two standard Brownian motions under an equivalent martingale measure Q, and satisfy $cor(B_t^r, B_t) = \rho$.[1] [2] ζ is a constant depending on the market price of risk of interest rate risk $-\lambda$.[3]

Let $n_t = \exp(\int_0^t r_t du)$ be the money market account at time t by starting 1 unit of cash at time 0. If $D(V_t, r_t, t)$ represents the price of the risky discount bond at time t, then the relative price of $D(V_t, r_t, t)$ to n_t is a Q-martingale.[4] We write

$$\frac{D(V_s, r_s, s)}{n_s} = E^Q[\frac{D(V_t, r_t, t)}{n_t}|\Im_s]$$

where the expectation is under Q conditional on the information set $\Im_s$, for any $t \geq s \geq 0$. If the risky discount bond matures at time T, then the price of the bond at time 0 is given by

$$D(V_0, r_0, 0) = E^Q[\frac{Min(V_T, K)}{n_T}|\Im_0].$$

This means that the probabilistic approach to pricing a corporate bond is essentially a problem of computing an expectation under Q, which is in turn a problem of computing multivariate normal integrals. The practicality of this approach lies in whether we can express V_T and n_T in the form of simple expressions.[5] In the case of Merton [1974], where the interest rate is assumed to be constant, the expectation is a univariate normal integral, the computation of which is trivial. In our case of a two-factor model, the problem becomes handling of bivariate normal integrals. In general, although it is known that when the integration regions are of some particular forms

[1]Mathematically $B_t^r = \rho B_t + \sqrt{1-\rho^2} B_t'$, where B_t, B_t' are two independent standard Brownian motions.

[2]A measure that is equivalent to the original objective measure. It is also known as a risk neutral measure. The existence of this measure is guaranteed by no-arbitrage arguments. See Harrison and Kreps [1979], and Harrison and Pliska [1981].

[3]If $\alpha - \beta r_t$ is the drift term under the original objective measure, then $\zeta = \alpha + \lambda\eta$ where $-\lambda$ is the market price of risk of interest rates. In general two market prices of risk are needed in our setting, however, by the assumption that the firm value is a traded asset, only the market price of interest rate risk appears.

[4]$D(V_t, r_t, t)$ must be a C^2 function with respect to the first and second coordinates.

[5]Note that V_T and n_T are log-normally distributed under Q.

high dimensional integrals can be reduced to ones with lower dimensions, introduction of stochastic interest rates necessarily complicates the computation issues.

4.2.3 Change of Numeraire

Most of the bond pricing models we mentioned before are based on the approach of using the money market account as a numeraire, associated with which we have the risk-neutral measure $\mathcal{Q}$. While this convenient choice of numeraire leads to an intuitive idea of risk-neutral pricing for asset pricing problems, it is by no means necessary. For example, Geman, Karoui and Rochet [1995] show that many other probability measures can be defined in a similar way to solve option pricing problems. In this section, we shall discuss the use of an alternative numeraire which provides computational convenience for bond pricing problems.

By assumption $A5$ we assume that continuous dividends are paid at a constant rate of per unit of the firm value. The solution of equation (4.2) is expressed as follows.

$$V_t = V_0 n_t \exp(-at - \frac{1}{2}\sigma^2 t + \sigma B_t).$$

Because of the presence of a, it is not difficult to see that the relative price of V_t to the money market account is not a $\mathcal{Q}$-martingale. However, by assuming the existence of a hypothetical asset $V_t^H = V_t \exp(at)$, we see that

$$\frac{V_t^H}{n_t} = \exp(-\frac{1}{2}\sigma^2 t + \sigma B_t)$$

is a $\mathcal{Q}$-martingale.[6] Now let us quote without proof a well-known result in probability theory, which is central to the analysis of changes of measure.[7]

Lemma 4.2.1 *Given two equivalent measures P_1 and P_2. Let $\xi = \frac{dP_2}{dP_1}$ be the Radon-Nikody'm derivative of P_2 relative to P_1. Then for any random variable X integrable*

[6] V_t^H can be regarded as the value of an identical firm that pays no dividends to its shareholders. This assumption is in fact not important, for we merely want to find out a stochastic process, which has nice mathematical properties such that the relative price of the bond to this process is a martingale under another measure.

[7] See Musiela and Rutkowski [1997], p458.

with respect to $(\aleph, P_2)$, *the following abstraction of Bayes theorem holds*

$$E^{P_1}[X\xi|\aleph] = E^{P_1}[\xi|\aleph]E^{P_2}[X|\aleph].$$

This lemma suggests that under some regularity conditions if there exists an equivalent measure Q^V such that the following holds

$$E^{Q}[\frac{D(V_t, r_t, t)}{n_t}|\Im_s] = E^{Q}[\frac{V_t^H}{n_t}|\Im_s]E^{Q^V}[\frac{D(V_t, r_t, t)}{V_t^H}|\Im_s]$$

for any $t \geq s \geq 0$, then the relative price $\frac{D(V_t, r_t, t)}{V_t^H}$ is a Q^V-martingale. The pricing problem is now turned into finding the new measure Q^V. The following theorem guarantees the existence of such a measure, which can be obtained by a simple transformation of standard Brownian motions.

Theorem 4.2.2 *Under the above assumptions on the processes of the interest rates and the firm value, there exists a measure* Q^V *equivalent to* Q *such that under the transformation* $B_t^V = -\sigma t + B_t$, $\frac{n_t}{V_t^H}$ *is a* Q^V*-martingale.*

Proof: $\frac{n_t}{V_t^H} = \frac{1}{V_0}\exp(\frac{1}{2}\sigma^2 t - \sigma B_t)$. By Ito lemma we have

$$d\frac{n_t}{V_t^H} = -\sigma\frac{n_t}{V_t^H}(-\sigma dt + dB_t).$$

Since σ is a constant, then by virtue of Girsanov's theorem there exists an equivalent martingale measure Q^V such that $B_t^V = -\sigma t + B_t$ is a Q^V-martingale. Since $d\frac{n_t}{V_t^H} = -\sigma\frac{n_t}{V_t^H}dB_t^V$ is driftless, this implies that $\frac{n_t}{V_t^H}$ is an exponential martingale. Hence the result follows. ■

4.2.4 Valuation of Risky Discount Bonds

With the results in the last section, the valuation of the risky discount bond becomes fairly straightforward. By applying Lemma 4.2.1, we have

$$E^{Q^V}[\frac{D(V_t, r_t, t)}{V_t^H}|\Im_s] = E^{Q^V}[\frac{n_t}{V_t^H}|\Im_s]E^{Q}[\frac{D(V_t, r_t, t)}{n_t}|\Im_s],$$

for $t \geq s \geq 0$.

Define a normal random variable $Z_T = -\int_0^T r_u du - \frac{1}{2}\sigma^2 T - \sigma B_T^V$. Since by Theorem 4.2.2 $\frac{n_t}{V_t^H}$ is a $\mathcal{Q}^v$-martingale and the relative price of $D(V_t, r_t, t)$ to n_t is a $\mathcal{Q}$-martingale, $\frac{D(V_t,r_t,t)}{V_t^H}$ is a $\mathcal{Q}^V$-martingale, which implies that

$$\begin{aligned} D(V_0, r_0, 0) &= V_0 E^{\mathcal{Q}^V}\Big[\frac{Min(V_T, K)}{V_T^H}|\Im_0\Big] \\ &= V_0 E^{\mathcal{Q}^V}\Big[Min[\exp(-aT), \frac{K}{V_0}\exp(Z_T)]|\Im_0\Big] \end{aligned} \tag{4.3}$$

Suppose that Z_T has mean μ_T and variance σ_T^2, then the price of the defaultable discount bond at time 0 is given by

$$\begin{aligned} D(V_0, r_0, 0) &= V_0 e^{-aT} N\Big[-\frac{\log(V_0/K) - aT - \mu_T}{\sigma_T}\Big] \\ &+ KP(T)N\Big[\frac{\log(V_0/K) - aT - \mu_T}{\sigma_T} - \sigma_T\Big] \end{aligned} \tag{4.4}$$

where $N[\cdot]$ is the cumulative normal distribution function.[8] If $P(T)$ is the price of a default-free zero coupon bond with the face amount of unity, then[9]

$$\begin{aligned} &P(T) = \exp[\tfrac{1-e^{-\beta T}}{\beta}(R(\infty) - r_0) - TR(\infty) - \tfrac{\eta^2}{4\beta^3}(1 - e^{-\beta T})^2], \\ &R(\infty) = \tfrac{\zeta}{\beta} - \tfrac{1}{2}\tfrac{\eta^2}{\beta^2}, \\ &a_T = -\tfrac{1}{\beta}(1 - e^{-\beta T})[r_0 + \tfrac{\zeta}{\beta}(e^{\beta T} - 1)] + \tfrac{\zeta}{\beta}(\tfrac{e^{\beta T}-1}{\beta} - T), \\ &b_T^2 = \tfrac{\eta^2}{\beta^2}T + \tfrac{\eta^2}{2\beta^3}(1 - e^{-2\beta T}) - \tfrac{2\eta^2}{\beta^3}(1 - e^{-\beta T}), \\ &\mu_T = a_T - \tfrac{1}{2}\sigma^2 T + \tfrac{\eta\rho\sigma}{\beta}(\tfrac{1-e^{-\beta T}}{\beta} - T), and \\ &\sigma_T^2 = b_T^2 + \sigma^2 T - 2\tfrac{\eta\rho\sigma}{\beta}(\tfrac{1-e^{-\beta T}}{\beta} - T). \end{aligned}$$

(4.4) is a generalized version of the bond pricing formula with stochastic interest rates given in Shimko, Tejima, and Van Deventer [1993]. We can consider σ_T^2 as the integrated instantaneous variance of Z_T over the life of the risky debt. It is interesting to note that σ_T^2 is a quadratic function of η. For typical values of the parameters β and T, this function has a positive coefficient in the leading term. This

[8]The computations are essentially the same as those required for a single factor model.

[9]The mean and variance are evaluated under the equivalent martingale measure $\mathcal{Q}^V$. See Appendix A for a proof.

implies that σ_T^2 is an increasing function of η when ρ is positive, and σ_T^2 attains its minimum value at some positive value of η when ρ is negative. For small values of η, an important implication on the bond prices is that a decreasing trend is expected when the firm value has a positive correlation with the interest rates.[10] On the contrary when the firm value has a negative correlation with the interest rates, an increasing trend in bond prices is expected.

Let $B(T) = KP(T)$ be the price of a risk-free discount bond, which pays K at maturity time T. Then we can rewrite equation (4.3) as follows:

$$D(V_0, r_0, 0) = V_0 e^{-aT} N[-h_{1T}] + B(T) N[h_{2T}] \tag{4.5}$$

where

$$\begin{aligned} h_{1T} &= \frac{\log(\frac{V_0}{B(T)}) - aT + \frac{1}{2}\sigma_T^2}{\sigma_T}, \\ h &= h_{1T} - \sigma_T. \end{aligned}$$

Note that equation (4.5) resembles the solution derived by Merton [1974]. The firm will be able to make its promised payment K if and only if its value at the bond maturity is at least K. The first term in equation (4.5) represents the expected present value of the firm if default happens at maturity, and the second term is the expected present value of the promised payment if default does not occur at maturity.

4.3 Basic Properties of r_t and Z_t

In the last section, we derived the pricing formula for a defaultable discount bond. The method is appealing in its simplicity by assuming Z_t as the only source of variability. Before going further into the pricing of general coupon bonds, it is tempting to conjecture that such nice properties of Z_t is preserved in a multi-period case so that analytic simplicity can be followed in the same line as the single period case. However, it turns out that this conjecture is wrong. The reason is as follows:

[10]To avoid a high probability that the interest rates go negative, we have to choose small values of η to use.

Given the definition $Z_t = -\int_0^t r_u du - \frac{1}{2}\sigma^2 t - \sigma B_t^V$, we can easily see that Z_t satisfies the following stochastic differential equation

$$dZ_t = -(r_t + \frac{1}{2}\sigma^2)dt - \sigma dB_t^V.$$

This implies that Z_t is a Markovian process of the two state variables r_t and Z_t. Theorem 4.3.1 gives some more properties of Z_t. From now on, we implicitly assume that all expectations and variances are computed under the martingale measure Q^V.

Theorem 4.3.1 *For any time $0 \leq s \leq t$, under the martingale measure Q^V we have the following results:*

$$\begin{aligned} E[r_t|\Im_s] &= a_{s,t} + e^{-\beta(t-s)} r_s, \\ E[Z_t|\Im_s] &= b_{s,t} + Z_s - \frac{1 - e^{-\beta(t-s)}}{\beta} r_s, \end{aligned} \tag{4.6}$$

where

$$\begin{aligned} a_{s,t} &= \frac{\zeta + \eta\rho\sigma}{\beta}\left(1 - e^{-\beta(t-s)}\right) \text{ and} \\ b_{s,t} &= -\frac{1}{2}\sigma^2(t-s) - \frac{\zeta + \eta\rho\sigma}{\beta}\left(t - s - \frac{1 - e^{-\beta(t-s)}}{\beta}\right) \end{aligned}$$

are deterministic functions. Furthermore, if we express the above two equations in the following forms:

$$\begin{aligned} r_t &= a_{s,t} + e^{-\beta(t-s)} r_s + \varepsilon_{s,t,r}, && (4.7) \\ Z_t &= b_{s,t} + Z_s - \frac{1 - e^{-\beta(t-s)}}{\beta} r_s + \varepsilon_{s,t,Z}, \\ &= b_{s,t} + Z_s - \frac{1 - e^{-\beta(t-s)}}{\beta} r_s + \rho_{s,t}\frac{\sigma_{s,t,Z}}{\sigma_{s,t,r}}\varepsilon_{s,t,r} + \sqrt{1 - \rho_{s,t}^2}\sigma_{s,t,Z}\, \varepsilon_{s,t}, && (4.8) \end{aligned}$$

where $\rho_{s,t}$ is the conditional correlation coefficient $Cor(\varepsilon_{s,t,r}, \varepsilon_{s,t,Z}|\Im_s)$. $\varepsilon_{s,t}$ is a standard normal random variable independent of $\varepsilon_{s,t,r}$. $\varepsilon_{s,t,r}$ and $\varepsilon_{s,t,Z}$ are two normal random variables independent of $\Im_s$ satisfying $E[\varepsilon_{s,t,r}|\Im_s] = 0$ and $E[\varepsilon_{s,t,Z}|\Im_s] = 0$, *with*

$$\begin{aligned}
\varepsilon_{s,t,r} &= \eta e^{-\beta t}\int_s^t e^{\beta u} d[\rho B_u^V + \sqrt{1-\rho^2}B_u'], \text{ and} \\
\varepsilon_{s,t,Z} &= -\int_s^t \frac{\eta}{\beta}(1-e^{-\beta(t-u)})d[\rho B_u^V + \sqrt{1-\rho^2}B_u'] - \sigma(B_t^V - B_s^V),
\end{aligned}$$

then

$$\begin{aligned}
Var[r_t] &= e^{-2\beta(t-s)}Var[r_s] + \sigma_{s,t,r}^2, \\
Var[Z_t] &= Var[Z_s] + \Big(\frac{1-e^{-\beta(t-s)}}{\beta}\Big)^2 Var[r_s] - 2\Big(\frac{1-e^{-\beta(t-s)}}{\beta}\Big)Cov[r_s, Z_s] + \sigma_{s,t,Z}^2, \\
Cov[r_t, Z_t] &= e^{-\beta(t-s)}Cov[r_s, Z_s] - e^{-\beta(t-s)}\Big(\frac{1-e^{-\beta(t-s)}}{\beta}\Big)Var[r_s] + \sigma_{s,t,r,Z},
\end{aligned}$$

where the conditional variances and covariance are defined as:

$$\begin{aligned}
\sigma_{s,t,r}^2 &= Var[\varepsilon_{s,t,r}|\Im_s] \\
&= \eta^2\frac{1-e^{-2\beta(t-s)}}{2\beta}, \\
\sigma_{s,t,Z}^2 &= Var[\varepsilon_{s,t,Z}|\Im_s] \\
&= \frac{\eta^2}{\beta^2}\Big[t-s-2\frac{1-e^{-\beta(t-s)}}{\beta} + \frac{1-e^{-2\beta(t-s)}}{2\beta}\Big] + \sigma^2(t-s) \\
&\quad + \frac{2\eta\rho\sigma}{\beta}[t-s-\frac{1-e^{-\beta(t-s)}}{\beta}], \text{ and} \\
\sigma_{s,t,r,Z} &= E[\varepsilon_{s,t,r}\ \varepsilon_{s,t,Z}|\Im_s] \\
&= \frac{\eta^2\rho^2}{2\beta^2}(1-e^{\beta(u-t)})^2 + \frac{\eta\rho\sigma}{\beta}(1-e^{\beta(u-t)}) + \frac{1}{2}\sigma^2.
\end{aligned}$$

Proof: See Appendix B. ∎

This theorem shows that the value of Z_t at time t can be predicted by using the knowledge of r_s and Z_s at time s. Although Z_t is not a Markovian process on its own, its simple decomposition into r_s and Z_s suggests that it is possible to solve the problem of pricing defaultable coupon bonds by using binomial trees. In the following section, we shall extend the method employed by Ho et al. [1995] to build binomial trees for r_s and Z_s.

4.4 A Method for Constructing Approximating Processes

4.4.1 One-Period Case

Ho et al. [1995] show how to construct a multivariate-binomial approximation to a joint lognormal distribution of variables with a recombining binomial lattice. Each variable is lognormally distributed and Markovian on its own.[11] In the present case, we need to modify the procedure, allowing the value of Z_t to depend on the values of r_s and Z_s, for $s \leq t$. Convergence of constructed processes based on the work of Ho, et al [1995] to the real ones will be proved.

As interest rates are assumed to follow normal distributions, the construction of the interest rate tree is the same as they propose. We first consider a one-period case from time 0 to time s. For detailed construction of two-period case, see Appendix E. Let $X_s = \exp(r_s)$. Our method involves the construction of one binomial distribution $X_{s,i}$ for X_s.

We approximate X_s by a vector of $n_s + 1$ numbers:[12]

$$X_{s,i} = X_0 u_s^i d_s^{n_s - i},$$

for $i = 0, \ldots, n_s$, where

$$\begin{aligned} d_s &= \frac{2(E[X_s]/X_0)^{1/n_s}}{1 + \exp(2\sigma_{0,s,r}/\sqrt{n_s})}, \\ u_s &= 2(E[X_s]/X_0)^{1/n_s} - d_s. \end{aligned}$$

On the time-interval $[0, s]$, we choose a transition probability q_s of an up-movement at the initial node such that Property 4.6 holds, that is

$$r_0 + n_s[q_s \log(u_s) + (1 - q_s)\log(d_s)] = a_{0,s} + e^{-\beta s} r_0,$$

hence,

$$q_s = \frac{a_{0,s} + e^{-\beta s} r_0 - r_0 - n_s \log(d_s)}{n_s[\log(u_s) - \log(d_s)]}.$$

[11]For a review of Ho et al. [1995], see Appendix C.

[12]See Appendix C .

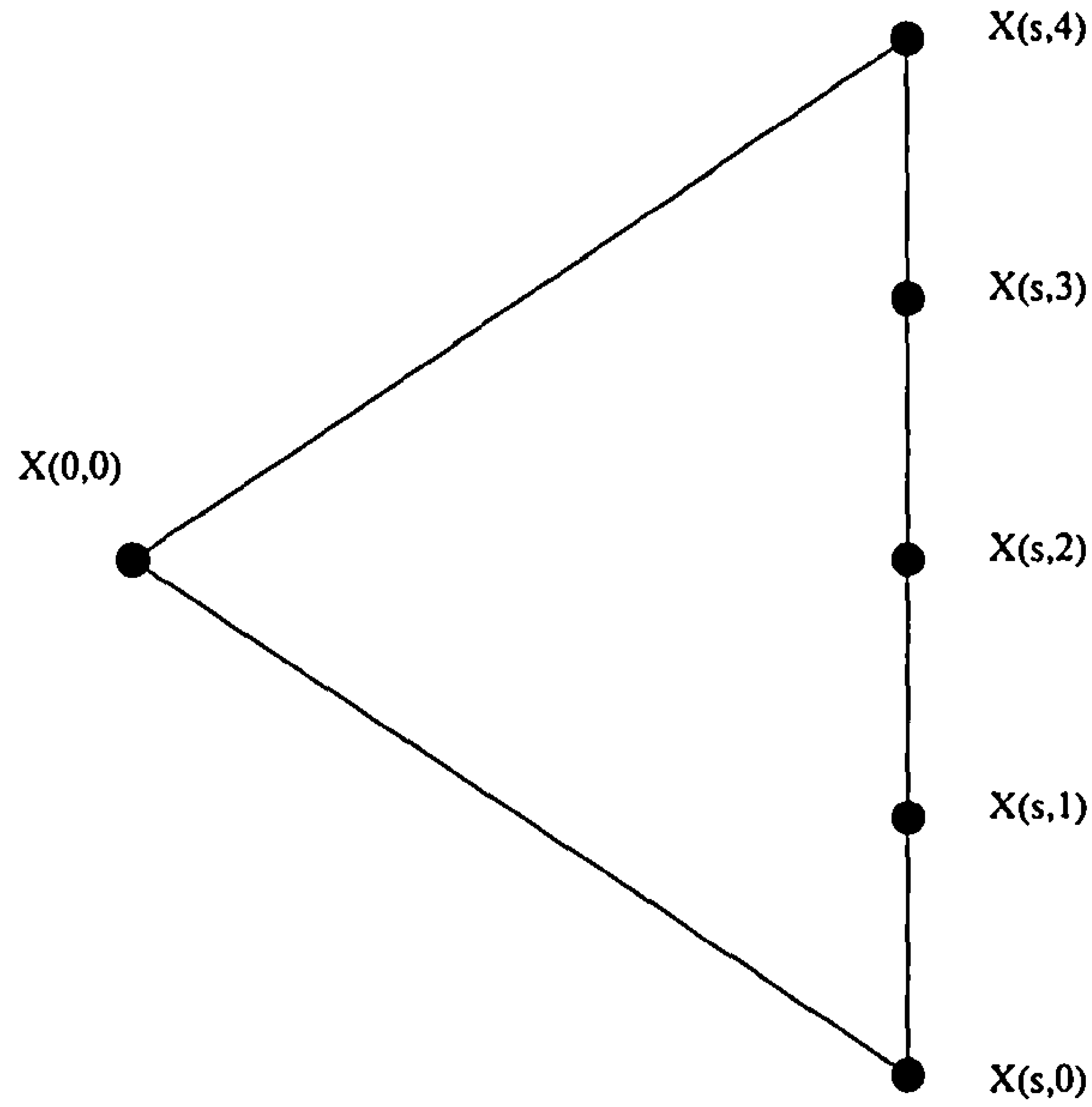

Figure 4.1: **An example of a one-period discrete process for X_s.** There are (n_s+1) nodes at time s numbered $r = 0, 1, \cdots, 4$, where $n_s = 4$. X_0 is the starting point. Note that X_0 can move to any points at time s. Intermediate values on the open interval $(0, s)$ are not defined.

X_0 moves to a point $X_{s,i}$ with the probability $P[X_{s,i}|X_0] = \binom{n_s}{i} q_s^i (1 - q_s)^{n_s - i}$, for $i = 0, 1, \cdots, n_s$.

Given the above interest rate tree, we are now ready to construct a second tree for $Y_s = \exp(Z_s)$ conditional on r_s. At time s, a tree of similar structure Y_s is approximated by a vector of $(m_s + 1)$ numbers:[13]

$$Y_{s,j} = \overline{u}_s^j \overline{d}_s^{m_s - j};\ Y_0 = 1,$$

for $j = 0, \ldots, m_s$, where

$$\begin{aligned}
\overline{d}_s &= \frac{2(E[Y_s])^{1/m_s}}{1 + \exp(2\sigma_{0,s,Z|r_s}/\sqrt{m_s})}, \\
\overline{u}_s &= 2(E[Y_s])^{1/m_s} - \overline{d}_s,
\end{aligned}$$

where $\sigma_{0,s,Z|r_s} = \sqrt{(1 - \rho_{0,s}^2)\ \sigma_{0,s,Z}^2}$ is the volatility of Z_s conditional on r_s.

[13]See Appendix C.

In order to incorporate the correlation structure into the trees, we relate the noise terms in Property 4.7 and 4.8 through their correlation $\rho_{0,s}$ as follows

$$\frac{\varepsilon_{0,s,Z}}{\sigma_{0,s,Z}} = \rho_{0,s}\frac{\varepsilon_{0,s,r}}{\sigma_{0,s,r}} + \sqrt{1-\rho_{0,s}^2}\varepsilon_{0,s},$$

where $\varepsilon_{0,s}$ is a standard normal random variable which is independent of $\varepsilon_{0,s,r}$.

By Property 4.8 we have

$$E[Z_s|r_s] = b_{0,s} - \frac{1-e^{-\beta s}}{\beta}r_0 + \rho_{0,s}\frac{\sigma_{0,s,Z}}{\sigma_{0,s,r}}\varepsilon_{0,s,r}.$$

We choose a transition probability p_s of an up-movement at initial node such that the above equation holds, that is[14]

$$\begin{aligned} p_s(r_{s,i}) &= \frac{b_{0,s} - \frac{1-e^{-\beta s}}{\beta}r_0 + \rho_{0,s}\frac{\sigma_{0,s,Z}}{\sigma_{0,s,r}}\varepsilon_{0,s,r,i} - m_s\log(\overline{d}_s)}{m_s[\log(\overline{u}_s) - \log(\overline{d}_s)]}, \\ r_{s,i} &= a_{0,s} + e^{-\beta s}r_0 + \varepsilon_{0,s,r,i}, \end{aligned}$$

for $i = 0, \cdots, n_s$. Given the value of $r_{s,i}$, Y_0 moves to a point $Y_{s,j}$ with the conditional probability $P[Y_{s,j}|r_{s,i}] = \binom{m_s}{j}p_s^j(1-p_s)^{m_s-j}$, for $j = 0, 1, \cdots, m_s$, where $p_s = p_s(r_{s,i})$.

4.4.2 Multiple-Period Case

In general, both J-period trees can be constructed similarly. Suppose that there are J future dates $t_1, t_2, \ldots, t_J$ on which we are interested in the asset prices. In the case of pricing coupon bonds, $t_1, t_2, \ldots, t_{J-1}$ represent the coupon dates and t_J represents the final maturity of a bond. We are interested in the joint distributions of X_t and Y_t on these dates. Over each of the time interval $[t_{j-1}, t_j]$, we assume that there are exogenously given number of binomial steps n_{t_j} and m_{t_j} for X_t and Y_t respectively. The construction of both trees $X_{t_j,i}$ and $Y_{t_j,i}$ follows from Ho et al. [1995], and are summarized as follows. For $j = 1, 2, \ldots, J$, we approximate X_{t_j} by a

[14] At time s, a discrete realization $r_{s,i}$ of r_s is given by

$$r_{s,i} = r_0 + i\log(u_s) + (n_s - i)\log(d_s),$$

where i is a binomial random variable with parameters (n_s, q_s). Let $r_{s,i} = a_{0,s} + e^{-\beta s}r_0 + \varepsilon_{0,s,r,i}$ for $i = 0, \cdots, n_s$.

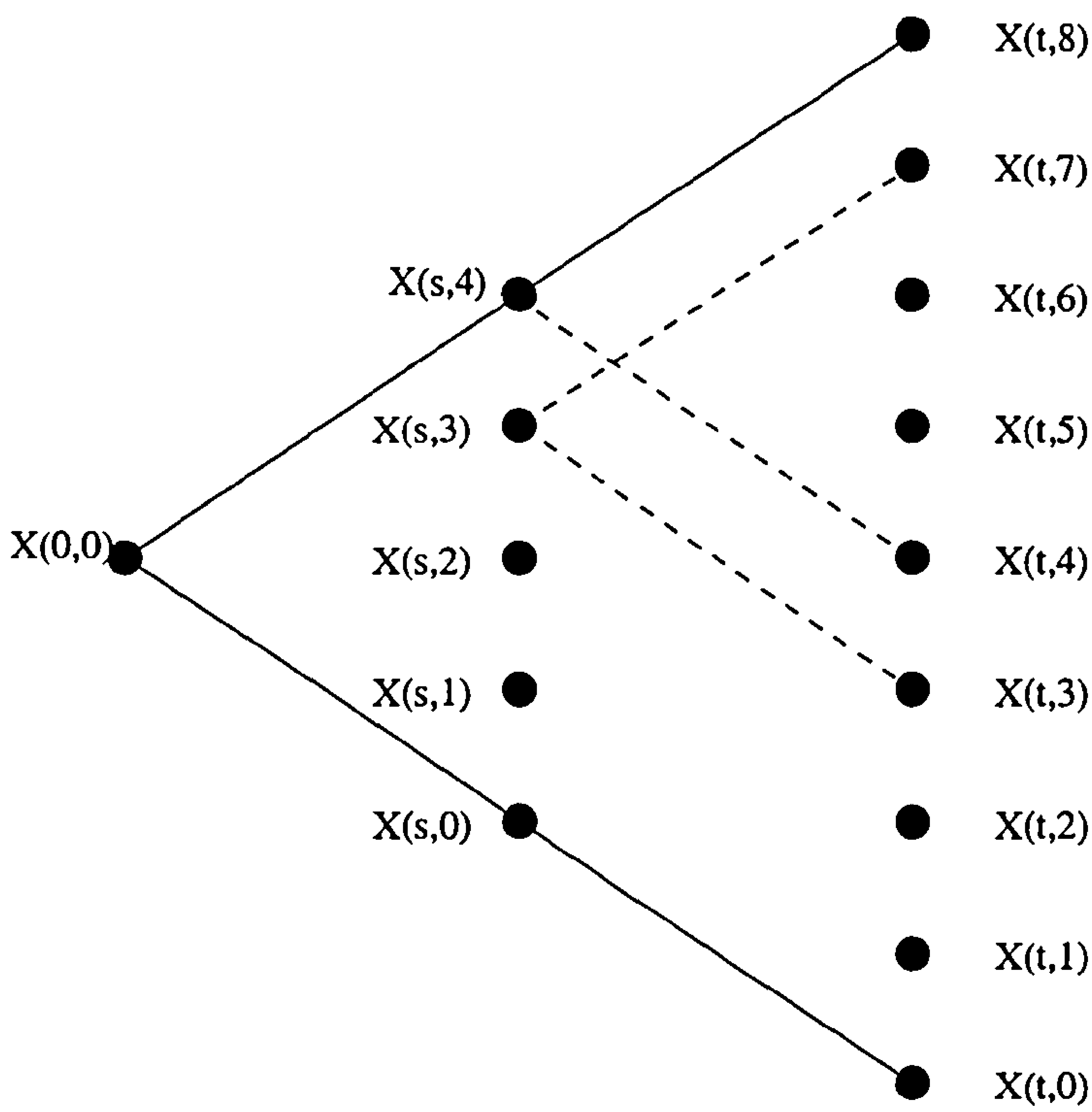

Figure 4.2: **An example of a two-period discrete process for X_s and X_t.** There are (n_s+1) nodes at time s numbered $r = 0, 1, \cdots, 4$, where $n_s = 4$. There are (n_s+n_t+1) nodes at time t numbered $r = 0, 1, \cdots, 8$, where $n_t = 4$. X_0 is the starting point. Note that X_0 can move to any points at time s. Similarly, $X_{s,4}$ can move to any positions corresponding to the 5 points on the top at time t, and $X_{s,3}$ can move to any positions corresponding to those points numbered $r = 3, \cdots, 7$ at time t. Intermediate values on the open intervals $(0, s)$ and (s, t) are not defined.

vector of $N_j + 1$ numbers:

$$X_{t_j,i} = X_0 u_j^i d_j^{N_j - i},$$

for $i = 0, \ldots, N_j$, where

$$\begin{aligned} d_j &= \frac{2(E[X_{t_j}]/X_0)^{1/N_j}}{1 + \exp(2\sigma_{t_{j-1},t_j,r}/\sqrt{n_{t_j}})}, \\ u_j &= 2(E[X_{t_j}]/X_0)^{1/N_j} - d_j, \\ N_j &= \sum_{l=1}^{j} n_{t_l}. \end{aligned}$$

On the time interval $[t_{j-1}, t_j]$, we choose a transition probability q_{t_j} of an up-movement at node i at time t_{j-1} such that Property 4.6 holds,

$$q_{t_j}(r_{t_{j-1},i}) = \frac{a_{t_{j-1},t_j} + e^{-\beta(t_j - t_{j-1})} r_{t_{j-1},i} - r_0 - i\log(u_j) - (N_{j-1} - i)\log(d_j) - n_{t_j}\log(d_j)}{n_j[\log(u_j) - \log(d_j)]},$$

for $i = 0, \cdots, N_{j-1}$. Given the value of $r_{t_{j-1},i}$, $X_{t_{j-1},i}$ moves to a point $X_{t_j,k}$ with the conditional probability $P[X_{t_j,k}|r_{t_{j-1},i}] = \binom{n_{t_j}}{k-i} q_{t_j}^{k-i}(1 - q_{t_j})^{n_{t_j}-k+i}$, for $k = i, i + 1, \cdots, i + n_{t_j}$, where $q_{t_j} = q_{t_j}(r_{t_{j-1},i})$.

At time t_j, Y_{t_j} is approximated by a vector of $M_j + 1$ numbers:

$$Y_{t_j,i} = \overline{u}_j^i \overline{d}_j^{M_j - i},$$

for $i = 0, \ldots, M_j$, where

$$\begin{aligned} \overline{d}_j &= \frac{2E[Y_{t_j}]^{1/M_j}}{1 + \exp(2\sigma_{t_{j-1},t_j,Z|r_{t_j}}/\sqrt{m_{t_j}})}, \\ \overline{u}_j &= 2E[Y_{t_j}]^{1/M_j} - \overline{d}_j, \\ M_j &= \sum_{l=1}^{j} m_{t_l} \end{aligned}$$

where $\sigma_{t_{j-1},t_j,Z|r_{t_j}} = \sqrt{(1 - \rho_{t_{j-1},t_j}^2)\,\sigma_{t_{j-1},t_j,Z}^2}$ is the volatility of Z_{t_j} conditional on $\Im_{t_{j-1}}$ and r_{t_j}. We choose a transition probability p_{t_j} of an up-movement at node i at time t_{j-1} as follows

$$p_{t_j}(r_{t_{j-1},i}, Z_{t_{j-1},l}, r_{t_j,k}) = \frac{b_{t_{j-1},t_j} + Z_{t_{j-1},l} - \frac{1-e^{-\beta(t_j - t_{j-1})}}{\beta} r_{t_{j-1},i} + \rho_{t_{j-1},t_j} \frac{\sigma_{t_{j-1},t_j,Z}}{\sigma_{t_{j-1},t_j,r}} \varepsilon_{t_{j-1},t_j,r,i,k} - \phi}{m_{t_j}[\log(\overline{u}_j) - \log(\overline{d}_j)]},$$

$$\begin{aligned}\phi &= i\log(\overline{u}_j)+(M_{j-1}-i)\log(\overline{d}_j)+m_{t_j}\log(\overline{d}_j),\\ r_{t_j,k} &= a_{t_{j-1},t_j}+e^{-\beta(t_j-t_{j-1})}r_{t_{j-1},i}+\varepsilon_{t_{j-1},t_j,r,i,k},\end{aligned}$$

for $i = 0,\cdots,N_{j-1}$, $k = i,\cdots,i+n_{t_j}$, and $l = 0,\cdots,M_{j-1}$. Given the values of $r_{t_{j-1},i}, Z_{t_{j-1},l}, r_{t_j,k}$, $Y_{t_{j-1},l}$ moves to a point $Y_{t_j,r}$ with the conditional probability $P[Y_{t_j,r}|r_{t_{j-1},i}, Z_{t_{j-1},l}, r_{t_j,k}] = \binom{m_{t_j}}{r-l}p_{t_j}^{r-l}(1-p_{t_j})^{m_{t_j}-r+l}$, for $r = l, l+1,\cdots,l+m_{t_j}$, where $p_{t_j} = p_{t_j}(r_{t_{j-1},i}, Z_{t_{j-1},l}, r_{t_j,k})$.

Figure 4.2 shows an example of a two-period discrete process for X_s and X_t. There are (n_s+1) nodes at time s numbered $r = 0,1,\cdots,4.$, where $n_s = 4$. There are (n_s+n_t+1) nodes at time t numbered $r = 0,1,\cdots,8$, where $n_t = 4$. X_0 is the starting point. Note that X_0 can move to any points at time s. Similarly, $X_{s,4}$ can move to any positions corresponding to the 5 points on the top at time t, and $X_{s,3}$ can move to any positions corresponding to those points numbered $r = 3,\cdots,7$ at time t. Intermediate values on the open intervals $(0,s)$ and (s,t) are not defined. The tree is recombining. In general, it is clear from the construction that the trees $\hat{X}_{t_j}$ and $\hat{Y}_{t_j}$ are recombining.[15] Also note that q_{t_j} and p_{t_j} are well-defined for large values of n_{t_j} and m_{t_j} as they tend to $\frac{1}{2}$ when $n_{t_j}\to\infty$ and $m_{t_j}\to\infty$ respectively.[16]

The following theorem shows the basic properties of the constructed $\hat{r}_t$ and $\hat{Z}_t$ processes that the estimated means, variances and covariances converge to their true values in the limit.

Theorem 4.4.1 *Suppose that the X_t and Y_t trees are constructed as above. Then for $j = 1,2,\ldots,J$, we have*

$$E[\hat{r}_{t_j}] = E[r_{t_j}], \tag{4.9}$$

[15]Here we define $\hat{X}_{t_j}$ as a constructed process of X_{t_j} without reference to node locations. Similarly, we define $\hat{Y}_{t_j}$ as a constructed process of Y_{t_j} without reference to node locations. For the $\hat{X}_{t_j}$ tree, $N_j = \sum_{l=1}^{j} n_{t_l}$. Let $k = max\{n_{t_1},\cdots,n_{t_m}\}$. Since $N_{j+1}\le N_j+k$, for any $j = 1,\cdots,m-1$, the tree is recombining. Similar arguments can be applied to the $\hat{Y}_{t_j}$ tree. For details, see James and Webber [2000].

[16]See Ho, et al. [1995].

$$E[\hat{Z}_{t_j}] = E[Z_{t_j}], \tag{4.10}$$

$$\hat{\sigma}^2_{t_{j-1},t_j,r} \to \sigma^2_{t_{j-1},t_j,r} \; as \; n_{t_j} \to \infty, \tag{4.11}$$

$$\hat{\sigma}^2_{t_{j-1},t_j,Z|r_{t_j}} \to (1-\rho^2_{t_{j-1},t_j})\sigma^2_{t_{j-1},t_j,Z} \; as \; m_{t_j} \to \infty, \tag{4.12}$$

$$\hat{\sigma}^2_{t_{j-1},t_j,Z} \to \sigma^2_{t_{j-1},t_j,Z} \; as \; m_{t_j}, n_{t_j} \to \infty, and \tag{4.13}$$

$$\hat{\sigma}_{t_{j-1},t_j,r,Z} \to \sigma_{t_{j-1},t_j,r,Z} \; as \; m_{t_j}, n_{t_j} \to \infty. \tag{4.14}$$

Hence,[17]

$$Var[\hat{r}_{t_j}] \to Var[r_{t_j}] \; as \; n_{t_j}, \ldots, n_{t_1} \to \infty, \tag{4.15}$$

$$Var[\hat{Z}_{t_j}] \to Var[Z_{t_j}] \; as \; m_{t_j}, n_{t_j}, \ldots, m_{t_1}, n_{t_1} \to \infty, \tag{4.16}$$

$$Cov(\hat{r}_{t_j}, \hat{Z}_{t_j}) \to Cov(r_{t_j}, Z_{t_j}) \; as \; m_{t_j}, n_{t_j}, \ldots, m_{t_1}, n_{t_1} \to \infty. \tag{4.17}$$

Furthermore,

$$E[\hat{X}_{t_j}] \to E[X_{t_j}] \; as \; n_{t_j}, \ldots, n_{t_1} \to \infty, \tag{4.18}$$

$$E[\hat{Y}_{t_j}] \to E[Y_{t_j}] \; as \; m_{t_j}, n_{t_j}, \ldots, m_{t_1}, n_{t_1} \to \infty. \tag{4.19}$$

Proof: See Appendix D ∎

Corollary 4.4.2 *Suppose that the X_t and Y_t trees are constructed as above. Then for $j = 1, 2, \ldots, J$, convergence in distribution is guaranteed*

$$\hat{r}_{t_j}|\Im_{t_{j-1}} \Rightarrow r_{t_j}|\Im_{t_{j-1}} \; as \; n_{t_j} \to \infty,$$

$$\hat{Z}_{t_j}|(\Im_{t_{j-1}}, r_{t_j}) \Rightarrow Z_{t_j}|(\Im_{t_{j-1}}, r_{t_j}) \; as \; m_{t_j} \to \infty.$$

Proof: The proof is followed by the constructions of q_{t_j} and p_{t_j}, and the properties of conditional variances. ∎

This Corollary guarantees that the constructed processes behave entirely the same as the underlying ones in the limit. These results will be applied to the proof of Theorem 4.6.1 in Section 4.6.

[17] Here the order of operations is important. The limits are taken over m_{t_j}, n_{t_j} respectively.

4.5 Geske's Method of Evaluating Defaultable Coupon Bonds

For ease of exposition, we first give a review on the methods of analysis given by Geske [1977] and Selby [1977]. An important assumption that underlies the analysis by them is that the firm finances each coupon payment with equity only taken up by shareholders, and that bankruptcy occurs when the firm fails to make a coupon payment because it is unable to raise enough money to fund the payment. Black and Cox [1976] argue that this will happen whenever the value of the equity, after payment is made, is less than the value of the payment. Black and Cox's [1976] argument is intuitive, in that the firm will find no takers for its stock if they know that the stock will become less valuable than the total value that they need to contribute to the promised payment. Suppose that the firm has an obligation to meet a coupon payment C_t at time t. Let S_{t+} be the value of the stock immediately after time t. Using Black and Cox's [1976] argument, the firm will be able to finance the coupon payment by rights issues if

$$S_{t+} > C_t \tag{4.20}$$

By the Modigliani and Miller theorem, the value of the firm is independent of its capital structure. Therefore the above inequality can be rewritten as

$$V_t > C_t + D_{t+} \tag{4.21}$$

where $V_t = V_{t+} = V_{t-}$ is the firm value at time t.[18] This implies that the value of the firm at time t should be greater than the total value of the coupon to be honored and the debt immediately after the coupon date. Inequality 4.21 is of particular importance when we are working on numerical computations of defaultable bonds.

In the case of Geske's [1977] model, it is fairly straightforward to imply from Inequality 4.20 the existence of a critical value of the firm $\overline{V}$ (below which a default happens). The reason is that when the interest rate is assumed to be constant, there

[18] The money raised by rights issues is used to finance the coupon payment. The firm value remains unchanged before and after the payment.

is only one stochastic variable V_t in the function of the stock price. Moreover, the monotonic increasing property of the stock prices on the firm value always guarantees the existence of such a point $\overline{V}$. Mathematically we write

$$S_{t+}(\overline{V}) = C_t. \tag{4.22}$$

Assuming the existence of such critical values of the firm allows us to price risky coupon bonds analytically.[19] A formula containing multi-variate normal integrals can be derived. However, the result is more of theoretical interest than practical use. Even with the aid of fast computers in existence today, the computation of high dimensional normal integrals is still a very challenging problem.[20] Alternatively Geske's [1977] formula can be computed numerically. In the one-factor case where a firm's value is modelled as a geometric Brownian motion and the interest rate is taken as a constant, the prices of a defaultable coupon bond can be computed efficiently and accurately by building a binomial tree for the firm value.

In a multi-factor framework, the method of solution for the pricing of defaultable coupon bonds is in general more complicated. One complication is the introduction of the stochastic interest rates that makes the determination of an analytical solution for coupon bond prices difficult, if not impossible. In our two-factor case, the stock price is a function of two variables, namely the firm value and the interest rate, and so the solution of equation (4.21) is a function of future interest rates on a coupon date, which are unknown at the time of issue. In other words, $\overline{V}$ is a moving default boundary. In the following section, instead of looking for an analytical solution, we shall propose an efficient numerical technique to price risky coupon bonds by building two binomial trees.

[19] See Cox and Ross [1976], and Selby [1977].
[20] See Papageorgiou and Traub [1997], Paskov and Traub [1997].

4.6 Applications To the Pricing of Defaultable Bonds

In this section, we present the theoretical grounds for computation. In principle computation is independent of the choice of a numeraire. However, an appropriate choice of it does provide much of the computational simplicity. It is both important and interesting to investigate the differences in computational efficiency between using the money market account n_t and the hypothetical asset V_t^H as numeraires in bond pricing problems. We consider the following cases of a defaultable discount bond with maturity T. Suppose we use V_T^H as a numeraire to price the bond, the initial price is estimated by taking the average of the following expression:[21]

$$V_0 \frac{Min(V_T, K)}{V_T^H} = Min\Big(V_0 \exp(-aT), K \exp(Z_T)\Big).$$

With the results in Section 4.4, the bond price can be approximated by building two binomial trees r_T and Z_T. On the other hand, if the money market account n_T is used as a numeraire, the bond price is estimated by taking the average of[22]

$$\frac{Min(V_T, K)}{n_T} = Min\Big(V_0 \exp(-aT - \frac{1}{2}\sigma^2 T + \sigma B_T), K \exp(-\int_0^T r_u du)\Big).$$

Note that in the above expression, the second argument $\exp(-\int_0^T r_u du)$ is a Markovian process on its own and the interest rate. By using the similar techniques as in Section 4.4, $\exp(-\int_0^T r_u du)$ can be mimicked by two binomial trees. Furthermore, we need another binomial tree for the first argument. This result implies that with this choice of numeraire the construction of three binomial trees is necessary to price the bond. Therefore the use of the numeraire V_T^H is supported by the computational efficiency of saving one binomial tree. The following theorem establishes the theoretical justification for numerical computation of discount bonds.

Theorem 4.6.1 *With the construction of two one-period binomial trees r_T and Z_T with binomial steps n and m respectively, the price of a defaultable discount bond of*

[21] The average is taken under Q^V.
[22] Under Q.

maturity T can be estimated by the following two steps:

$$(i)\ E\Big[Min\Big(V_0\exp(-aT), K\exp(\hat{Z}_T)\Big)\Big|\hat{r}_T\Big] \to E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|\hat{r}_T\Big]$$

as $m \to \infty$.

$$(ii)\ E\Big[E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|\hat{r}_T\Big]\Big] \to E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big]$$

as $n \to \infty$,

where the limit is the price of the defaultable discount bond.

Proof: See Appendix F ■

It is obvious to see that the underlying principle of the computations of a discount bond price is the *Tower Property* of expectations:

$$E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big] = E\Big[E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|r_T\Big]\Big].$$

Hence, this theorem shows that the computations consists of three main steps:

(I): We compute $Min\Big(V_0\exp(-aT), K\exp(Z_{T,j})\Big)$ at each node $Z_{T,j}$ of the Z_T tree.

(II): Given each $r_{T,i}$ on the r_T tree, we approximate the conditional mean of the above expression in (I) by the following binomial sum:

$$mean(r_{T,i}) = \sum_{j=0}^{m_T} \binom{m_T}{j} p_T^j (1-p_T)^{m_T-j} Min\Big(V_0\exp(-aT), K\exp(Z_{T,j})\Big).$$

(III): The bond price estimated by taking the mean of the results in step (II) over all $r_{T,i}$ is:

$$\sum_{i=0}^{n_T} \binom{n_T}{i} q_T^i (1-q_T)^{n_T-i} mean(r_{T,i}).$$

In sum, the computational algorithm is to take weighted mean of the points in step (I). Figure 4.3 gives a pictorial representation of the idea for a single-period lattice structure in which $n_T = m_T = 2$. In the figure, we firstly compute the values of the expression in step (I) corresponding to those 9 points at time T. The price of a discount bond is computed by using the steps (II) and (III).

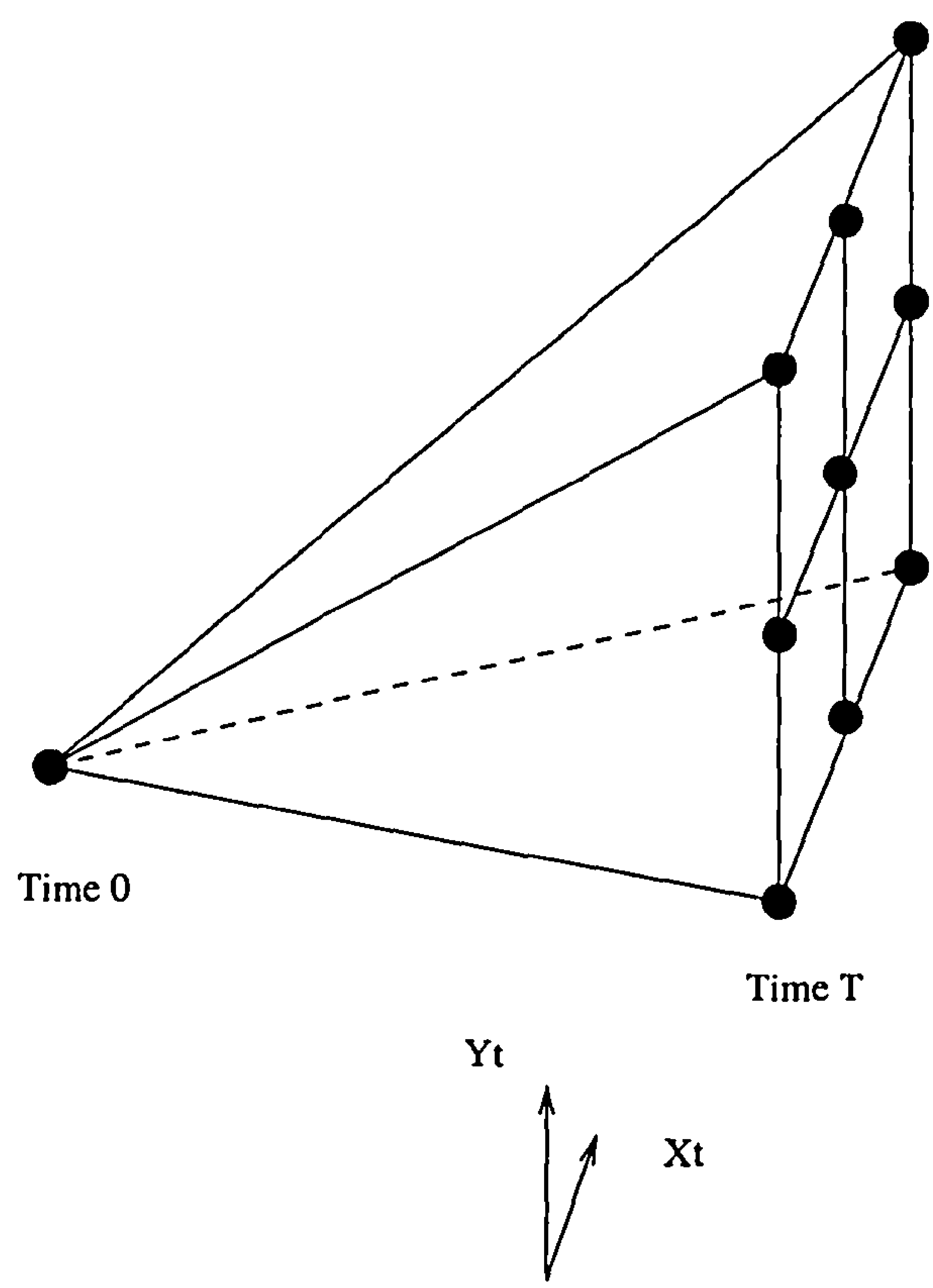

Figure 4.3: **An example of a single-period lattice structure for (X_T, Y_T).** In the horizontal direction, there are (n_T+1) nodes at time T numbered $i = 0, 1, 2$, where $n_T = 2$. In the vertical direction, there are (m_T+1) nodes at time T numbered $j = 0, 1, 2.$, where $m_T = 2$. Note that the movement of the initial point (X_0, Y_0) to any points at time T is governed by the probabilities q_T and p_T, where T represents final maturity of a discount bond.

We can estimate the price of a $c\%$ coupon bond using binomial methods in a similar fashion. With the same notation used as before we firstly consider a two-period case where t_1 and t_2 are the coupon and maturity dates respectively. We can prove in a similar way to Theorem 4.6.1, that

$$(i)\ E\Big[Min\Big(V_0\exp(-at_2), K(1+\frac{1}{2}c\%)\exp(\hat{Z}_{t_2})\Big)\Big|\Im_{t_1}, \hat{r}_{t_2}\Big]$$
$$\to E\Big[Min\Big(V_0\exp(-at_2), K(1+\frac{1}{2}c\%)\exp(Z_{t_2})\Big)\Big|\Im_{t_1}, \hat{r}_{t_2}\Big]\ as\ m_2 \to \infty,$$

and

$$(ii)\ E\Big[E\Big[Min\Big(V_0\exp(-at_2), K(1+\frac{1}{2}c\%)\exp(Z_{t_2})\Big)\Big|\Im_{t_1}, \hat{r}_{t_2}\Big]\Big|\Im_{t_1}\Big]$$
$$\to E\Big[Min\Big(V_0\exp(-at_2), K(1+\frac{1}{2}c\%)\exp(Z_{t_2})\Big)\Big|\Im_{t_1}\Big]\ as\ n_2 \to \infty.$$

Let $B(r_{t_1}, Z_{t_1}) = E\Big[Min\Big(V_0\exp(-at_2), K(1+\frac{1}{2}c\%)\exp(Z_{t_2})\Big)\Big|\Im_{t_1}\Big]$. Then $V_{t_1}^H B(r_{t_1}, Z_{t_1})/V_0$ is the bond price at time t_1. In other words, the bond price at time t_1 is approximated by (i) and (ii) in Theorem 4.6.1. By Condition 4.21, the firm will be able to finance the coupon payment by a rights issue if $V_{t_1} > V_{t_1}^H B(r_{t_1}, Z_{t_1})/V_0 + \frac{K}{2}c\%$, and so the bond price immediately before the coupon date is

$$Min\Big(V_{t_1}, V_{t_1}^H B(r_{t_1}, Z_{t_1})/V_0 + \frac{K}{2}c\%\Big). \tag{4.23}$$

By the martingale property, its initial price is given by

$$V_0 E\Big[Min\Big(V_{t_1}, V_{t_1}^H B(r_{t_1}, Z_{t_1})/V_0 + \frac{K}{2}c\%\Big)/V_{t_1}^H\Big]$$
$$= E\Big[Min\Big(V_0\exp(-at_1), B(r_{t_1}, Z_{t_1}) + \frac{K}{2}c\%\ \exp(Z_{t_1})\Big)\Big].$$

As with Theorem 4.6.1, we can estimate the bond price using the following two steps:

$$(iii)\ E\Big[Min\Big(V_0\exp(-at_1), B(\hat{r}_{t_1}, \hat{Z}_{t_1}) + \frac{K}{2}c\%\ \exp(\hat{Z}_{t_1})\Big)\Big|\hat{r}_{t_1}\Big]$$
$$\to E\Big[Min\Big(V_0\exp(-at_1), B(\hat{r}_{t_1}, Z_{t_1}) + \frac{K}{2}c\%\ \exp(Z_{t_1})\Big)\Big|\hat{r}_{t_1}\Big]\ as\ m_1 \to \infty,$$

and

$$(iv)\ E\Big[E\Big[Min\Big(V_0\exp(-at_1), B(\hat{r}_{t_1}, Z_{t_1}) + \frac{K}{2}c\%\exp(Z_{t_1})\Big)\Big|\hat{r}_{t_1}\Big]\Big]$$
$$\to E\Big[Min\Big(V_0\exp(-at_1), B(r_{t_1}, Z_{t_1}) + \frac{K}{2}c\%\exp(Z_{t_1})\Big)\Big]\ as\ n_1 \to \infty.$$

The idea behind the computations for coupon bonds is based on the computational algorithm of zero-coupon bonds. Figure 4.4 shows an example of a two-period lattice structure for (X_t, Y_t) in which $n_{t_1} = n_{t_2} = 1$ and $m_{t_1} = m_{t_2} = 2$, where t_1 and t_2 represent a coupon date and final maturity of a coupon bond respectively. To compute the price of the coupon bond, we proceed as follows:

(I'): Using the discount bond algorithm, we compute the price of the coupon bond at each point at time t_1. For example, at the point P, the value of the bond is approximated by taking weighted mean of those 6 points on the shaded rectangle.

(II'): Using the bankruptcy condition (4.23), we determine the value of the coupon bond at each point at time t_1 immediately before coupon date.

(III'): Using the discount bond algorithm again, the initial bond price is approximated by taking weighted mean of the results in step (II').

With the one- and two-period cases, we have depicted the essence of numerical techniques for computations of general defaultable coupon bonds. The general method is also based on backwards recursion techniques as above.

4.7 Numerical Computations

4.7.1 Convergence of Bond Prices

For a J-period model, our method of constructing binomial trees r_t and Z_t requires the choice of J-binomial steps $n_{t_1}, n_{t_2}, \ldots, n_{t_J}$ and $m_{t_1}, m_{t_2}, \ldots, m_{t_J}$ respectively. As discussed before, the order of operations is important. To guarantee the convergence of the estimated bond prices with the true value, the limits should be taken in the same order by allowing them to tend to infinity. In general, the estimation of the true value by a choice of realizations of $m_{t_J}, n_{t_J}, \ldots, m_{t_1}, n_{t_1}$ is a difficult task, and depends on the essence of the problem. Improper choice of binomial steps

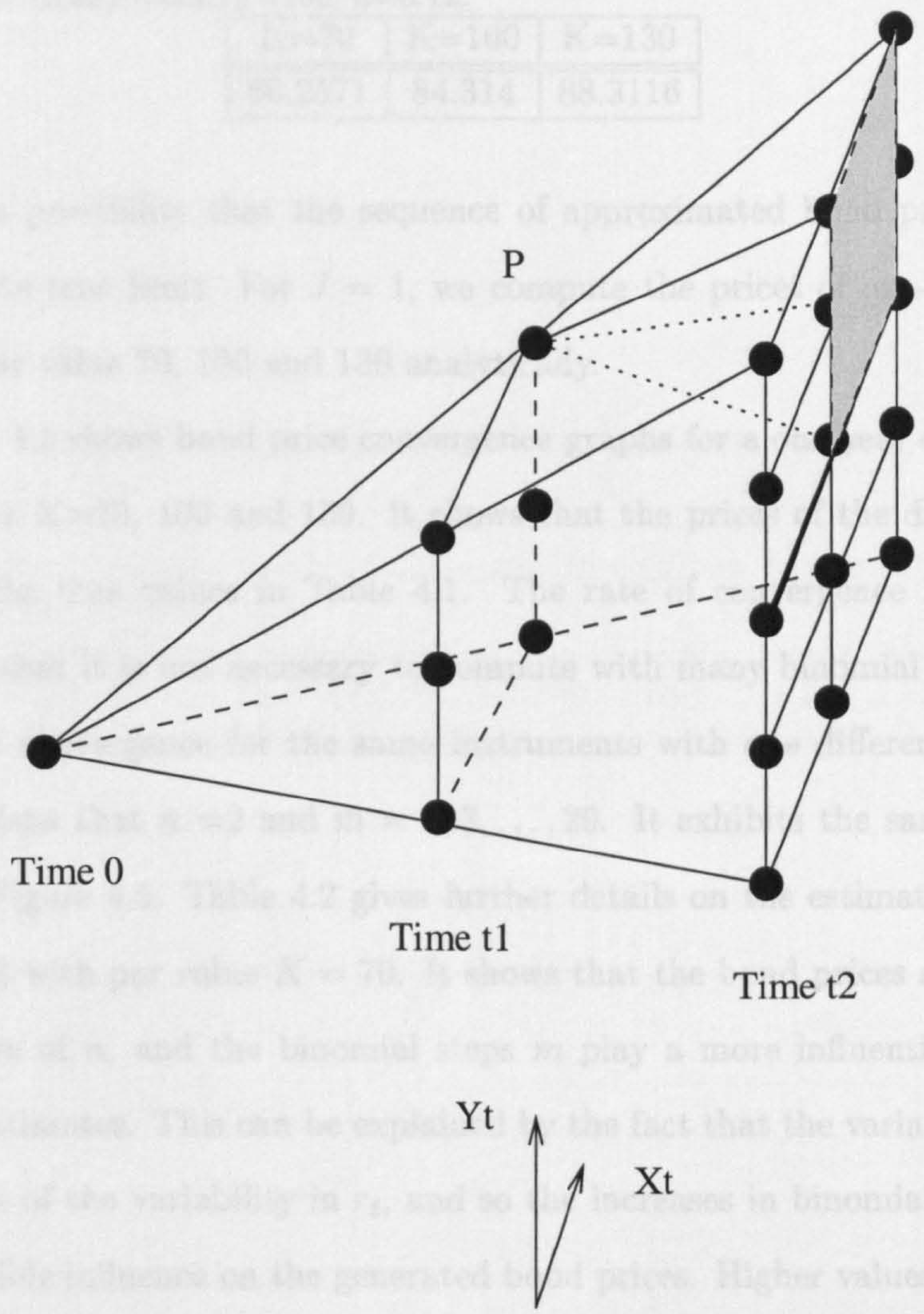

Figure 4.4: **An example of a two-period lattice structure for (X_t, Y_t).** In the horizontal direction, there are $(n_{t_1}+1)$ nodes at time t_1 numbered $i = 0, 1$, where $n_{t_1} = 1$, and $(n_{t_1}+n_{t_2}+1)$ nodes at time t_2 numbered $i = 0, 1, 2$, where $n_{t_2} = 1$. In the vertical direction, there are $(m_{t_1}+1)$ nodes at time t_1 numbered $j = 0, 1, 2$, where $m_{t_1} = 2$, and $(m_{t_1}+m_{t_2}+1)$ nodes at time t_2 numbered $i = 0, 1, \cdots, 4$, where $m_{t_2} = 2$. Note that the movement of the initial point (X_0, Y_0) to any points at time t_1 is governed by the probabilities q_{t_1} and p_{t_1}. The movement of the points at time t_1 is similar. For example, the point P at time t_1 can move to any 6 points at time t_2 on the shaded rectangle, where t_1 and t_2 represent a coupon date and final maturity of a coupon bond respectively.

Table 4.1: **Example of discount bond prices with par value 70, 100 and 130.** The prices are computed analytically using parameter values r_0=0.04, ζ=0.06, β=1,η=0.031,ρ=-0.25,σ=0.2,V_0=100, a=0.12.

K=70	K=100	K=130
66.2571	84.314	88.3116

may lead to a possibility that the sequence of approximated bond prices does not converge to the true limit. For $J = 1$, we compute the prices of one-year discount bonds with par value 70, 100 and 130 analytically.

Figure 4.5 shows bond price convergence graphs for a one-year discount bond with par value K=70, 100 and 130. It shows that the prices of the discount bonds converge to the true values in Table 4.1. The rate of convergence is fairly high, and suggests that it is not necessary to compute with many binomial steps. Figure 4.6 shows the convergence for the same instruments with one difference in the use of binomial steps that n =2 and $m = 2, 3, \ldots, 20$. It exhibits the same pattern as indicated in Figure 4.5. Table 4.2 gives further details on the estimated prices of a discount bond with par value $K = 70$. It shows that the bond prices are insensitive to the changes of n, and the binomial steps m play a more influential role in the precision of estimates. This can be explained by the fact that the variable Z_t already captures most of the variability in r_t, and so the increases in binomial steps n have merely negligible influence on the generated bond prices. Higher values of m tend to give a more accurate approximation of the bond prices. For this reason we shall keep using small values of n in the subsequent computations.

It is noteworthy that the estimated discount bond prices converge to the true limit at a fairly fast rate. A high rate of convergence is also expected in the computations of coupon bond prices as long as the binomial steps m are chosen to be sufficiently large. Therefore to resolve the problem we discussed above, in the computations of coupon bonds we choose $m_{t_1} = m_{t_2} = \ldots = m_{t_J}$.

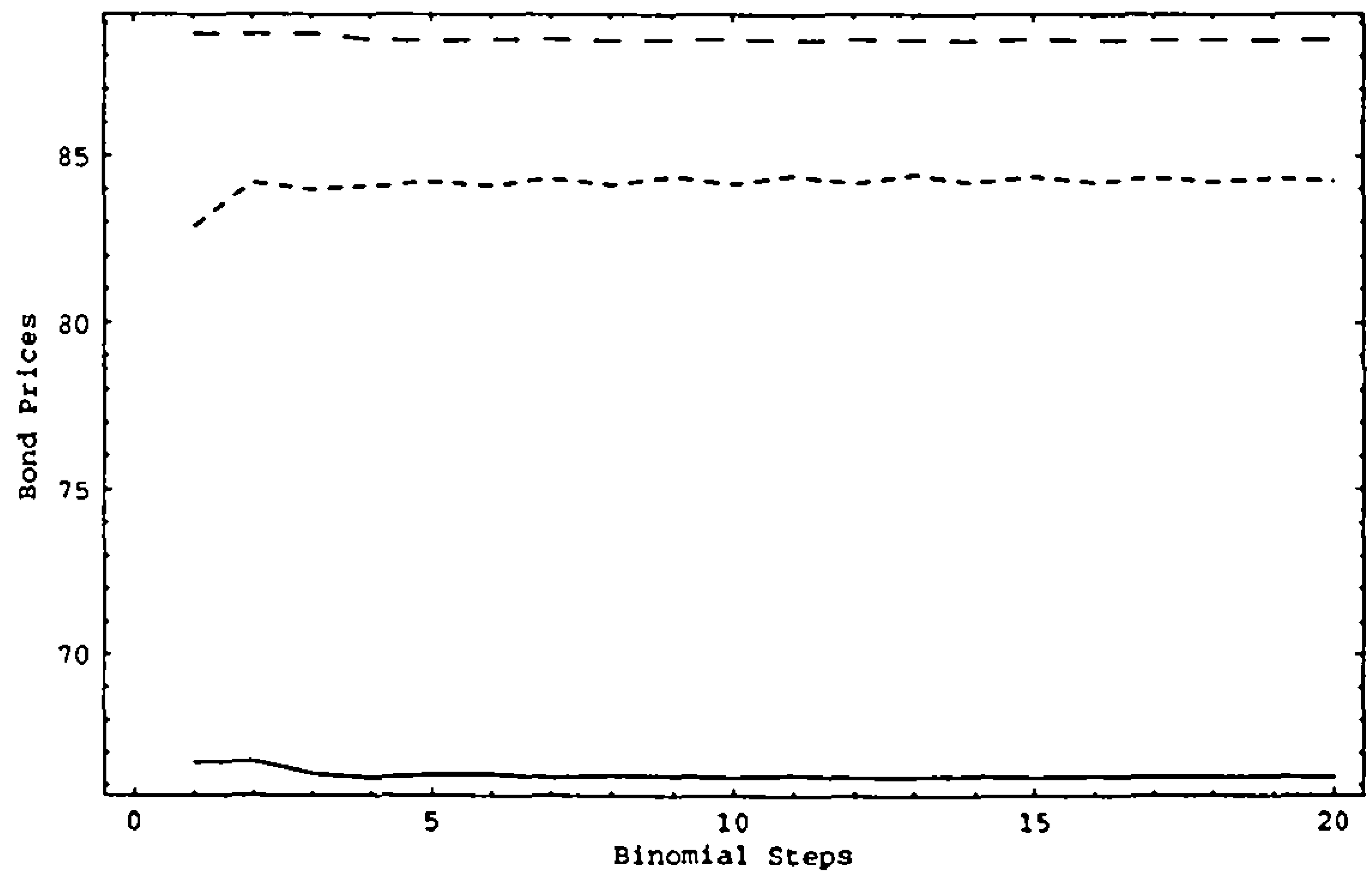

Figure 4.5: **Bond price convergence graphs against binomial steps for a one-year discount bond.**
The bond prices are computed numerically assuming stochastic interest rates using parameter values r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2, V_0=100, a=0.12, $n = m = 2, 3, \ldots, 20$. (i)$K = 70$(solid line),(ii)$K = 100$(short dashed line),(iii)$K = 130$(long dashed line).

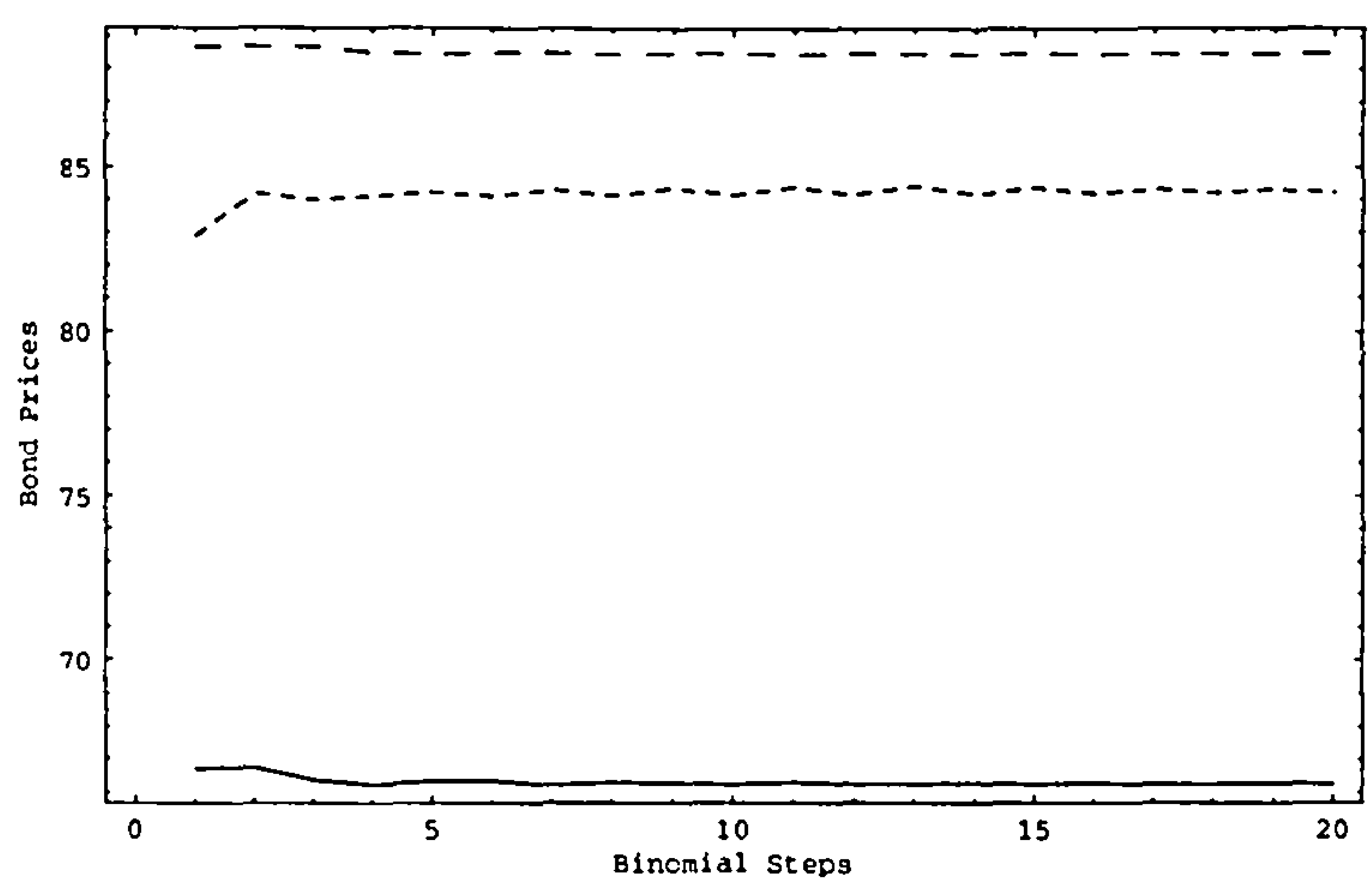

Figure 4.6: **Bond price convergence graphs against binomial steps for a one-year discount bond.**
The bond prices are computed numerically assuming stochastic interest rates using parameter values r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2, V_0=100, a=0.12, $n = 2, m = 2, 3, \ldots, 20$. (i)$K = 70$(solid line),(ii)$K = 100$(short dashed line),(iii)$K = 130$(long dashed line).

Table 4.2: **Example of discount bond prices with par value 70 for the cases (i) n=2,$m = 2, 3, \ldots, 20$, and (ii) $n = m = 2, 3, \ldots, 20$.** The prices are computed numerically using parameter values r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2, V_0=100, a=0.12.

	Case(i)	Case(ii)
$m = 2$	66.7393	66.7393
$m = 3$	66.3665	66.3665
$m = 4$	66.2381	66.2381
$m = 5$	66.3429	66.3429
$m = 10$	66.257	66.2568
$m = 20$	66.256	66.2558
$m = 30$	66.2651	66.265
$m = 40$	66.276	66.2758
$m = 50$	66.2566	66.2564
$m = 60$	66.261	66.2608
$m = 70$	66.2568	66.2566
$m = 80$	66.2632	66.263
$m = 90$	66.2575	66.2573
$m = 100$	66.2607	66.2605

In Table 4.3, we illustrate the efficiency of our method by showing the time taken to compute a one-year discount bond, a one-year 8% coupon bond and a two-year 8% coupon bond. It shows that the computation time increases approximately linearly with the binomial step m.

Table 4.3: **Example of bond prices and their computation times with par value 70 for n=2, $m = 2, 4, \ldots, 10$ (All computations are performed in Mathematica 4.1 for Sun Solaris).** Parameter values r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2,V_0=100, a=0.12.

	1-year discount	2-year 8%	3-year 8%
m=2	66.7393 (0.16s)	71.395 (1.75s)	70.765 (13.79s)
m=4	66.2381 (0.21s)	71.4713 (2.75s)	70.576 (24.02s)
m=6	66.2381 (0.21s)	71.4713 (2.75s)	70.576 (24.02s)
m=8	66.3159 (0.34s)	71.3782 (5.05s)	70.6299 (47.56s)
m=10	66.257 (0.38s)	71.4299 (6.04s)	70.6068 (57.77s)

4.7.2 Credit Spreads of Defaultable Bonds

Credit spread is defined as a spread level over the yield of a default-free bond with the same promised payments and maturity. The plots of credit spreads of a 5-year 8% coupon bond against different parameters are shown in Figure 4.8.

As expected, in most of the plots credit spreads move in a direction that is opposite to the bond prices. However, in Figures 4.7(IV) and 4.8(IV) bond prices and credit spreads are both decreasing functions of the interest rates. The reason for this phenomenon is that in the risk neutral world the firm value tends to increase in response to the increasing interest rates. Bankruptcy is less likely to happen as a result of higher stock prices, therefore credit spreads decrease with interest rates. This result is compatible with the empirical findings in Longstaff and Schwartz [1995a].

4.7.3 Interaction of Market Risk with Credit Risk

In our model, market risk refers to changes in bond prices as a result of changes in interest rates. Credit risk refers to the risk that the issuer of a bond may default. As market events have shown, there is an important interplay between both concepts. Our model integrates market and credit risk together to allow for a more complete picture of the underlying risk. Figures 4.9, 4.10, 4.11 and 4.12 show credit spread sensitivity to parameters.

Figure 4.9 shows that when the firm value is low, credit spread is more sensitive to the changes in interest rates. This confirms our belief that firms with low credit quality should have more market risk than firms with high credit quality. On the contrary, firms with high credit quality are those, which we expect, have only a base level of interest rate exposure.

Furthermore, it is clear from Figure 4.10 and 4.11 that when dividend and coupon rates are higher, credit spread is more sensitive to firm value volatility. This implies that under the assumption that coupons are financed by rights issues, the bond is of higher default risk as coupon rate increases. A similar trend holds for the

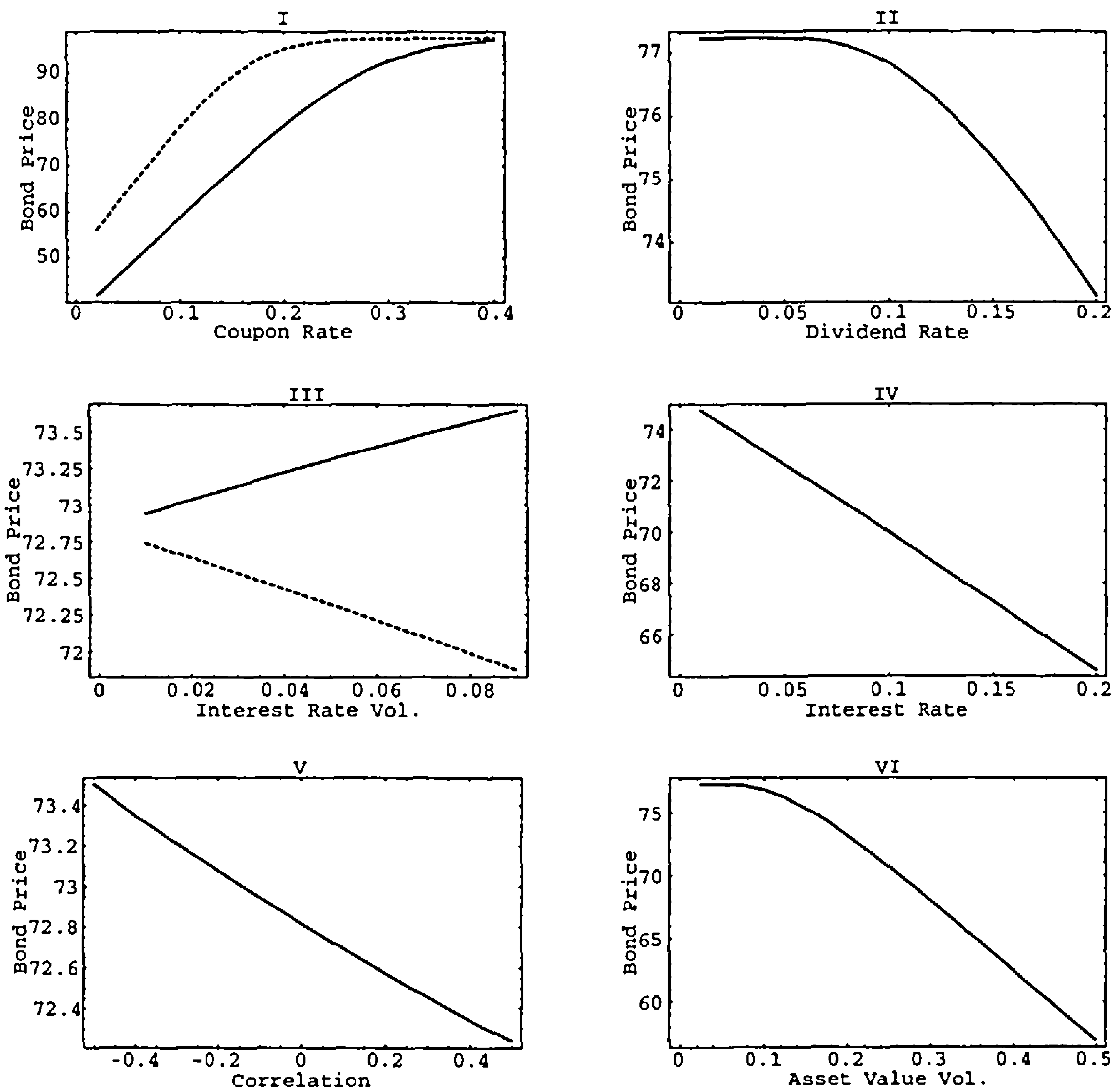

Figure 4.7: **Five-year 8% risky coupon bond prices with half-yearly interest payments and stochastic interest rates:**
r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2,V_0=100,K=70, a=0.05, c=0.08, unless otherwise stated. In (I): K=70 (solid line), K=50 (dashed line). In (III): ρ=-0.25 (solid line), ρ=0.25 (dashed line), $n = 5, m = 50$..

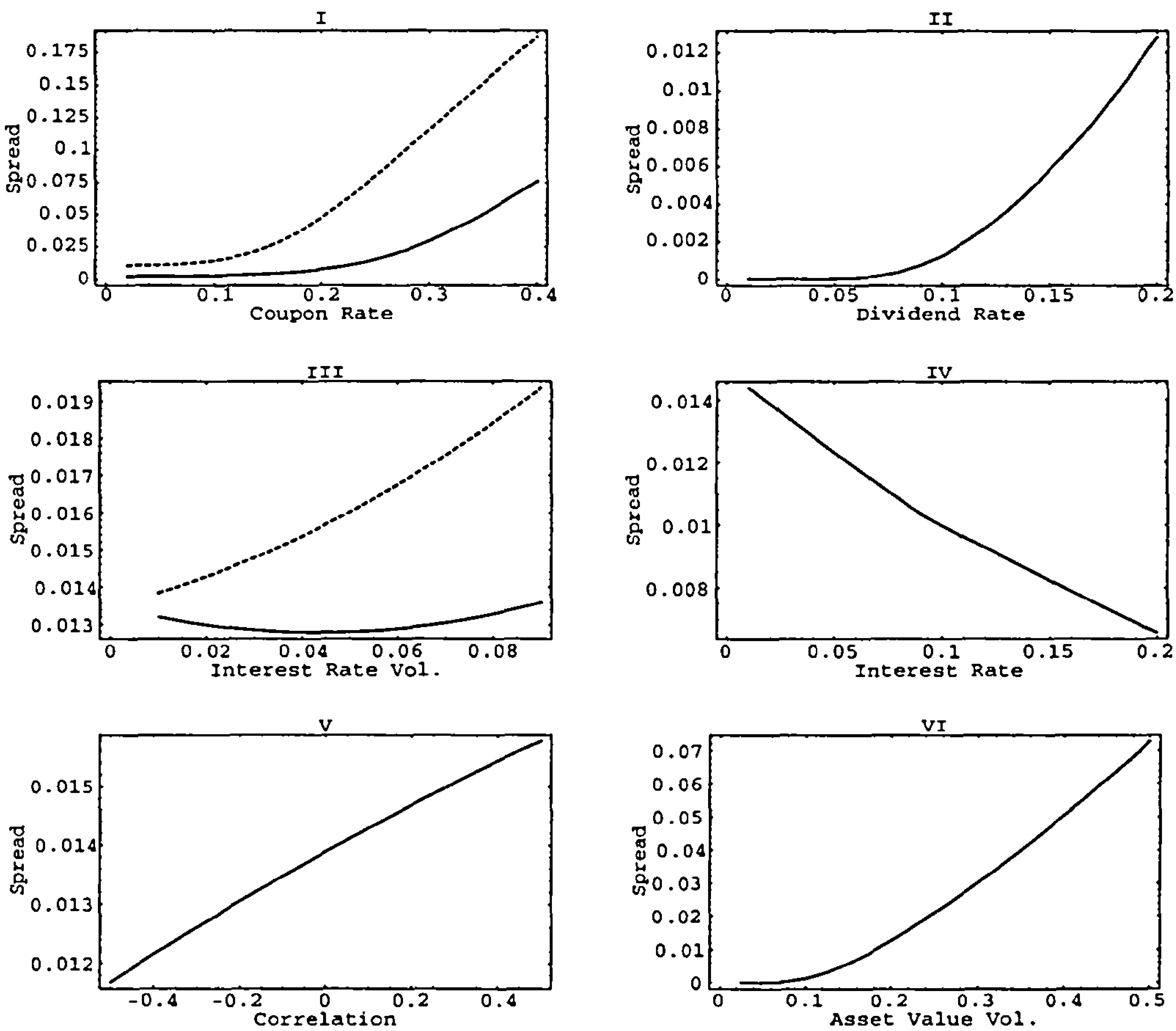

Figure 4.8: **Credit spreads of a five-year 8% risky coupon bond with half-yearly interest payments and stochastic interest rates:**
r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2,V_0=100,K=70, a=0.05, c=0.08, unless otherwise stated. In (I): K=70 (solid line), K=50 (dashed line). In (III): ρ=-0.25 (solid line), ρ=0.25 (dashed line), $n = 5, m = 50$.

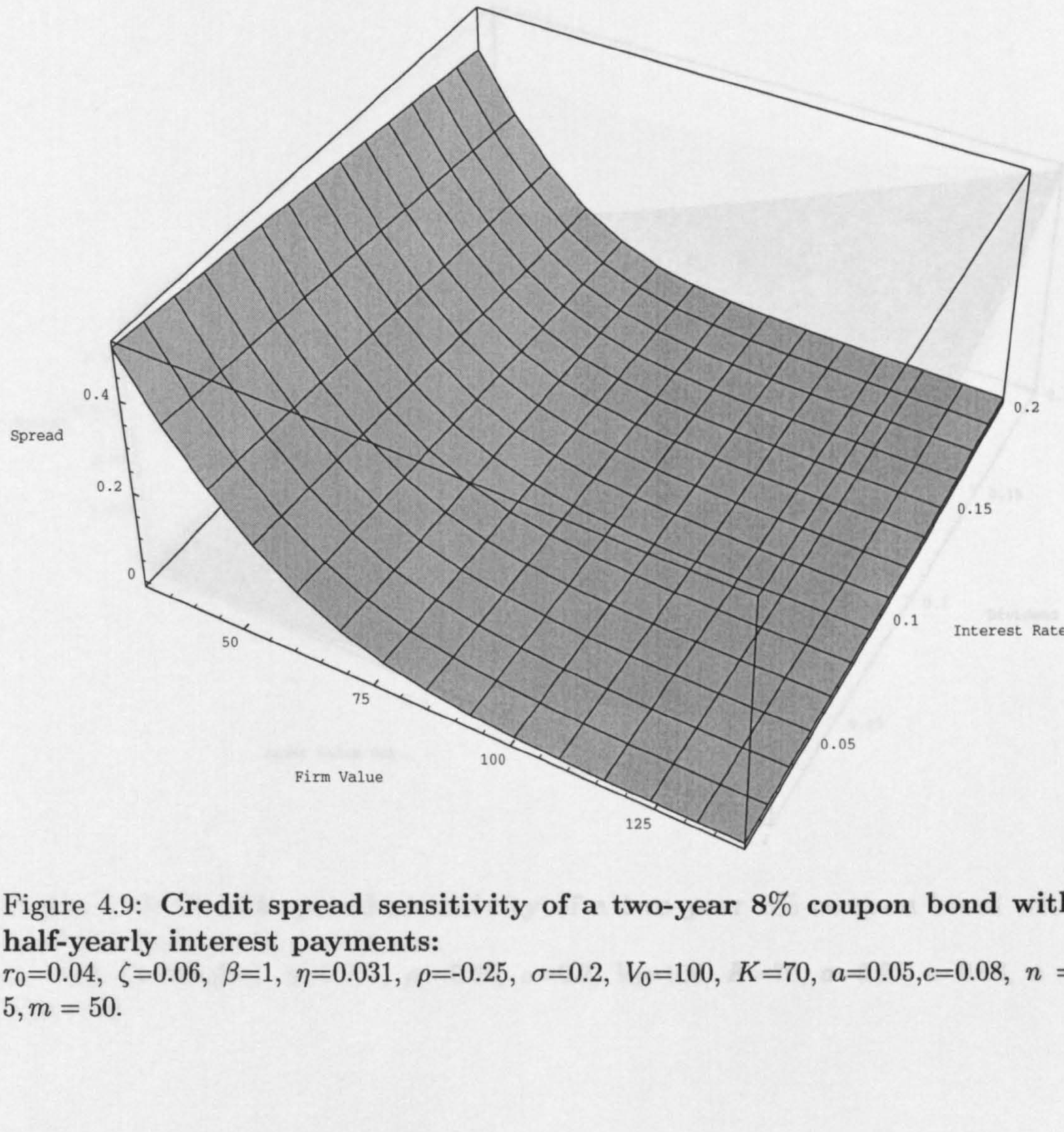

Figure 4.9: **Credit spread sensitivity of a two-year 8% coupon bond with half-yearly interest payments:**
r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2, V_0=100, K=70, a=0.05,c=0.08, $n = 5, m = 50$.

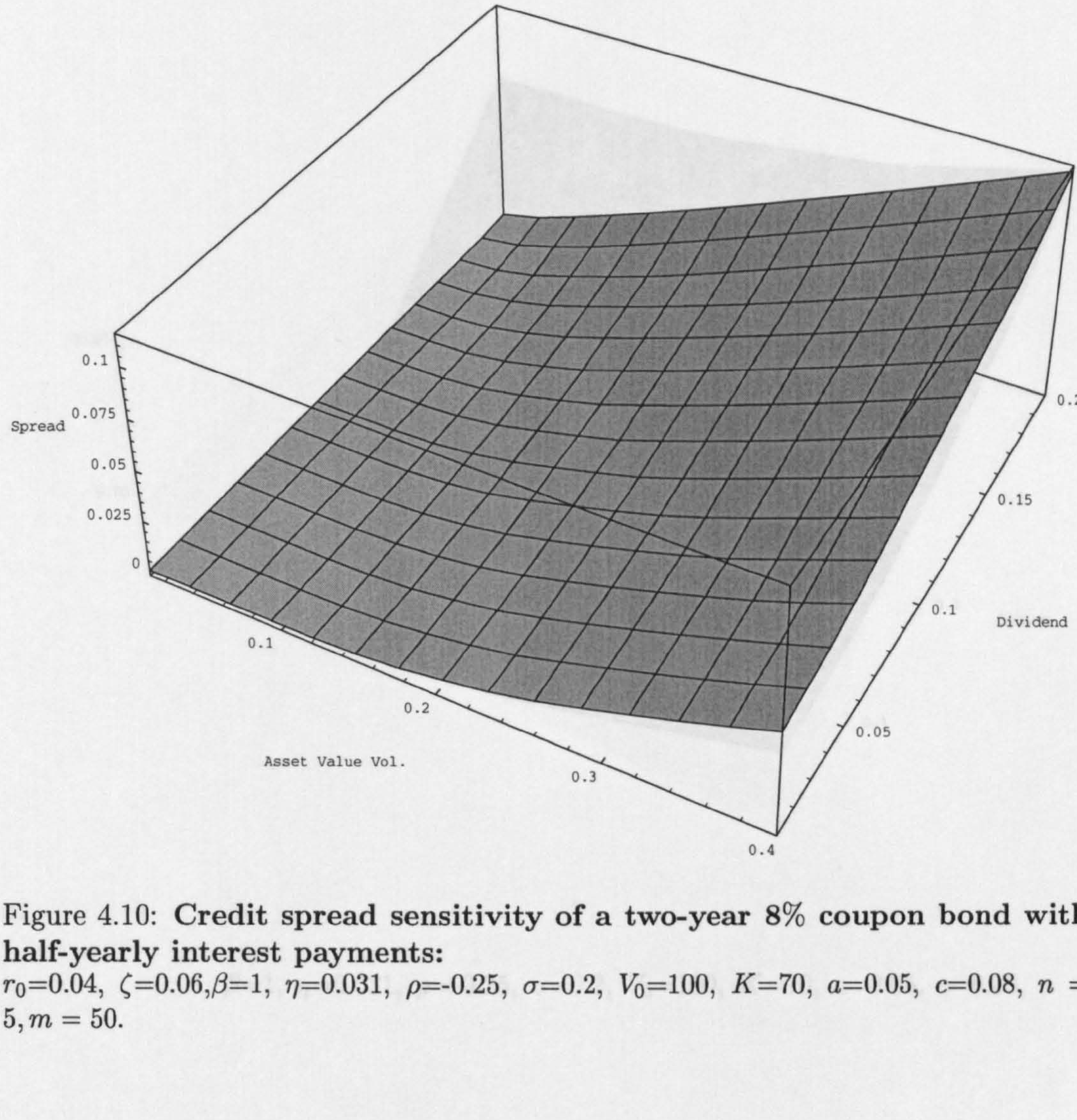

Figure 4.10: **Credit spread sensitivity of a two-year 8% coupon bond with half-yearly interest payments:**
r_0=0.04, ζ=0.06,β=1, η=0.031, ρ=-0.25, σ=0.2, V_0=100, K=70, a=0.05, c=0.08, n = $5, m = 50$.

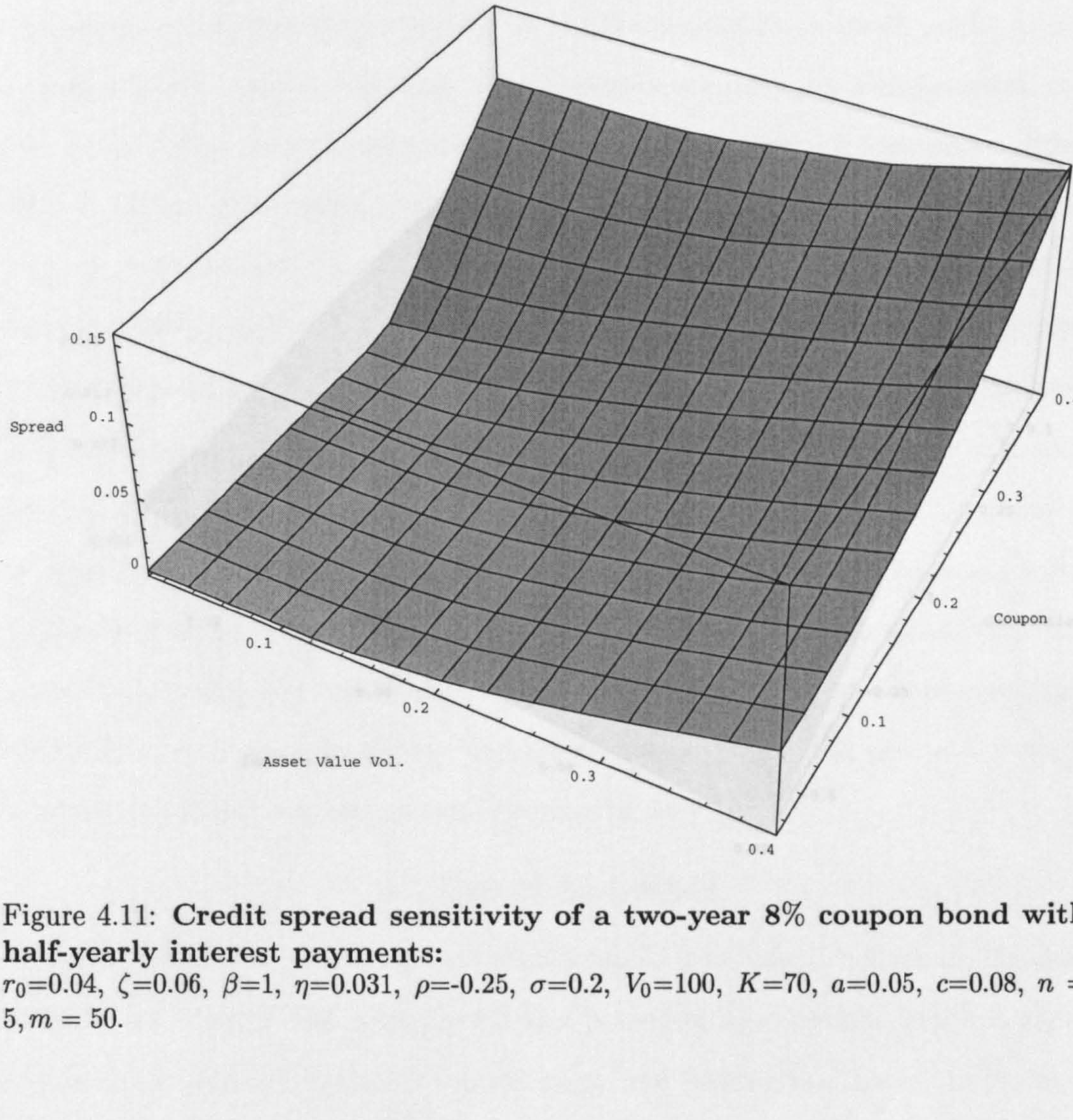

Figure 4.11: **Credit spread sensitivity of a two-year 8% coupon bond with half-yearly interest payments:**
r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2, V_0=100, K=70, a=0.05, c=0.08, $n = 5, m = 50$.

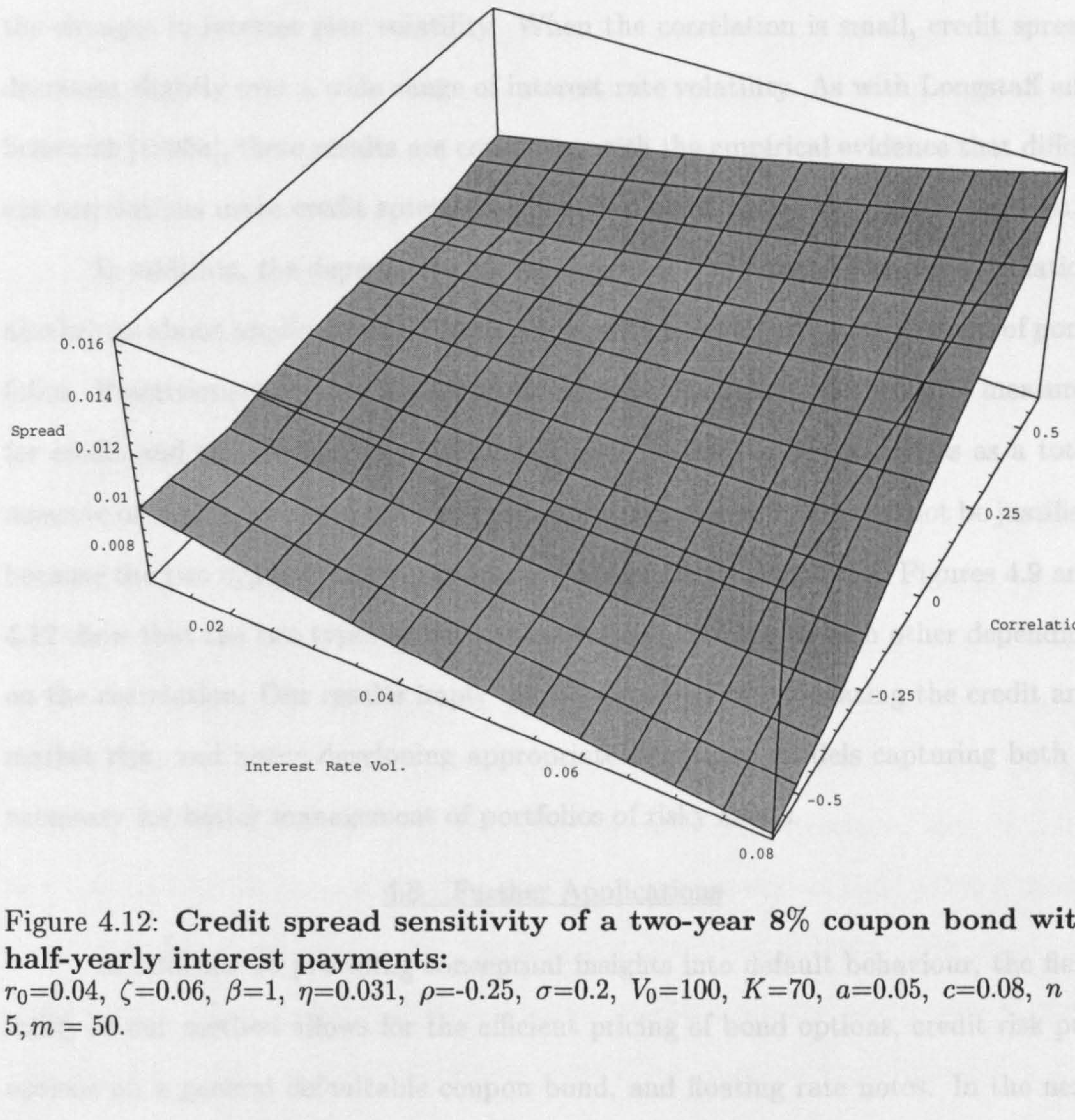

Figure 4.12: **Credit spread sensitivity of a two-year 8% coupon bond with half-yearly interest payments:** r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2, V_0=100, K=70, a=0.05, c=0.08, $n = 5, m = 50$.

case where shareholders are entitled to receive a higher dividend rate as a proportion of firm value.

Figure 4.12 plots the relation of credit spreads with respect to interest rate volatility η and correlation ρ. As shown, the effect of the correlation can be very significant. When the correlation is high, credit spread appears to be sensitive to the changes in interest rate volatility. When the correlation is small, credit spread decreases slightly over a wide range of interest rate volatility. As with Longstaff and Schwartz [1995a], these results are consistent with the empirical evidence that different correlations make credit spreads for bonds of equal rating vary across sections.

In addition, the dependence of both credit and market risk on the correlation also brings about implications for capital adequacy in credit risk management of portfolios. Practitioners and regulators often estimate the value-at-risk (VaR) measures for credit and market risk separately, and take the sum of the measures as a total measure of capital for the credit and market risk exposures.[23] This cannot be justified because the two types of risk exposures are not perfectly correlated. Figures 4.9 and 4.12 show that the two types of risk are intrinsically related to each other depending on the correlation. Our results imply the impossibility of segregating the credit and market risk, and hence developing appropriate integrated models capturing both is necessary for better management of portfolios of risky assets.

4.8 Further Applications

In addition to providing conceptual insights into default behaviour, the flexibility of our method allows for the efficient pricing of bond options, credit risk put options on a general defaultable coupon bond, and floating rate notes. In the next chapter, we will show that the algorithm for computations of defaultable coupon bonds, after some modifications, immediately lends itself to efficient pricing of these instruments.

[23] See Jarrow and Turnbull [2000].

4.9 Summary

In this chapter we have generalized, in computational aspects, Geske's [1977] and Selby's [1977] valuation models of risky coupon bonds to allow for stochastic firm values and interest rates. By using the hypothetical asset V_t^H as a numeraire, it has been shown that not only does the use of this numeraire significantly simplify the analytic valuation of risky discount bonds, but also gives an implication that the two-factor model can be implemented easily. We have discussed computational efficiency when the hypothetical asset V_t^H is used as a numeraire, and showed that this is an appropriate choice of numeraire. In addition, we have suggested an efficient computation algorithm for the pricing of general risky coupon bonds by generalizing the models proposed by Ho et al. [1995]. Much of the simplicity of this method lies in the fact that the two sources of variability, namely interest rate risk and asset value risk, are combined together to form a single stochastic process Z_t.

This analysis can be extended in several ways. The Vasicek model suffers from its implicit assumption that interest rates can become negative with a positive probability at any given time. Whilst it has been shown that this probability can usually be reduced by properly adjusting the process parameters, the weakness of negative interest rates is perhaps offset by Hull and White's [1990] observation that the extended Vasicek model can be used to fit any observable term structure. Moreover, the fact that the Vasicek process can be embedded in a framework of HJM's [1992] model leaves us with an implication that the methods developed in this chapter would be readily generalizable to incorporate a more general term structure consistent interest rate process. Despite this, it still retains much of the computational tractability. For example in the extended Vasicek model, apart from being consistent with initial term structure, any volatility term that is a deterministic function of time can readily be fitted into our framework. In addition, our method of construction of binomial trees can be easily generalized to cope with more general Markovian processes dependent on several stochastic variables. This should be useful in the pricing of some more complicated instruments.

Another drawback is the assumption that asset values are log-normally distributed. The Merton's traditional approach to pricing risky debt has been criticized as being incapable of generating credit spreads consistent with those observed in corporate debt markets. One of the reasons is that the model lacks the fat-tailness properties that we normally observe in asset returns. The introduction of jump processes would be able to resolve some of the issues. Another reason is the absence of a mechanism that allows for costly liquidation in the event of bankruptcy.

Traditional approach to modelling risky debt has been an indispensable tool for discussing the distribution of the firm's value between shareholders and bondholders. In addition to providing conceptual insights into default behaviour, the flexibility of our method allows for the efficient pricing of bond options, credit risk put options on a general defaultable coupon bond, and floating rate notes. This structural approach also paves the way for a further analysis of more complicated debt structures. Incorporation of bankruptcy costs in the model is an important avenue that can be explored in our framework. Efficient numerical valuation of general risky debts when interest rates and firms' values are stochastic should be a crucial step forward in understanding the full complexity of credit analysis.

CHAPTER 5
Applications of the Two-Factor Model

5.1 Introduction

In the last chapter, we developed an efficient algorithm for the pricing of defaultable coupon bonds. By extending the method of Ho, Stapleton and Subrahmanyam [1995], we generalized these author's construction of multivariate binomial trees to deal with a Markovian process $Z_t = -\int_0^t r_u du - \frac{1}{2}\sigma^2 t - \sigma B_t^V$. Because this is a Markovian process of two state variables Z_t and r_t, the tree construction of Z_t consists of two binomial lattices, each representing the evolution of an underlying state variable. As with other binomial methods, another characteristic of this algorithm is its flexibility in handling a feature that is peculiar to a specific pricing problem. For example, the algorithm enables simple handling of credit sensitive instruments with American features.

In this chapter, we will show that the algorithm for the computation of defaultable coupon bond prices, after some modifications, immediately lends itself to efficient pricing of other credit risk related instruments. These instruments include bond options, credit risk put options on a general defaultable coupon bond, and floating rate notes.

5.2 Bond Options

Many papers on financial literature have addressed the important topic of bond option valuation. Of these papers, Jamshidian [1989] and Longstaff [1993] provide analytical formulae for the value of an option on a coupon bond with stochastic interest rates, assuming that the underlying coupon bonds are non-defaultable. In this section, we show that the pricing algorithm for risky coupon bonds can be modified to price options on a defaultable bond.

Consider a European option on a defaultable coupon bond $D(V_t, r_t, t)$ with a fixed exercise price. In this case, we are focusing upon the option with maturity date τ and the underlying bond with final maturity date $\tau + T$. For the sake of simplicity, we only consider European type options for which no compensation will be made to option holders in the case of a default of the underlying instruments before the option maturity date τ. Since the bond option is a derivative of the defaultable coupon bond $D(V_t, r_t, t)$, we can express its value at time s as $C(V_s, r_s, s)$, where $0 \le s \le \tau$.[1] Note that $\frac{C(V_s, r_s, s)}{n_s}$ is a Q-martingale. By Lemma 4.2.1 and Theorem 4.2.2, it is easy to prove that under the transformation $B_s^V = -\sigma s + B_s$, $\frac{C(V_s, r_s, s)}{V_s^H}$ is a Q^V-martingale. Now can we state this result in the following lemma.

Lemma 5.2.1 *The relative price of $C(V_s, r_s, s)$ to V_s^H is a Q^V-martingale for $0 \le s \le \tau$. The initial price of the bond option is given by*

$$C(V_0, r_0, 0) = V_0 E^{Q^V}[\frac{C(V_\tau, r_\tau, \tau)}{V_\tau^H} | \Im_0],$$

where $C(V_\tau, r_\tau, \tau)$ is the payoff of the option at its maturity.

With this lemma, the numerical valuation of European bond options becomes straightforward. This is because we are pricing the bond option $C(V_s, r_s, s)$ under the same numeraire V_s^H of the underlying asset, and so the numerical algorithm developed in the last chapter immediately lends to efficient pricing of bond options. The numerical valuation algorithm for defaultable coupon bonds can be easily modified to price European and American bond calls and bond puts. Table 5.1 shows bond option prices for different levels of initial interest rates. The first six columns represent call and put option prices on a risky coupon bond, and the last three columns are the prices of a call option on a default-free coupon bond with the same payment

[1] It has to be a C^2 function with respect to the first and second coordinates.

schedule as the risky one. The corresponding put option prices are not shown in Table 5.1 because of their extremely small values.[2]

As is shown, the prices of the call option on a non-defaultable bond are uniformly decreasing with interest rate levels, and with exercise prices. This result concurs with the findings in Longstaff [1993] on the valuation of options on default-free coupon bonds. Two features are revealed. Firstly, when the underlying bond is defaultable, the corresponding calls (puts) become less (more) valuable because of the lower values of the underlying asset. Secondly, the risky call prices move in a direction that is opposite to the riskless call prices. The reason for this is similar to the explanation given at the end of Section 4.7.3. The gradually increasing feature of the bond call prices can be explained by the fact that, in the risk-neutral world, the firm's value tends to rise in response to the increasing interest rates. As a result, the bond calls become more valuable as interest rates increase. This shows that default risk of the underlying has a significant effect on the pricing of bond options.

5.3 Credit Risk Options

A credit risk option is a derivative contract on a credit sensitive debt instrument. The writer of the option agrees to compensate the buyer for a predetermined fall in credit standing of the issuer of the underlying instrument. Therefore, a credit risk option protects the buyer against the deterioration in value of the bond if its yield rises above a specified exercise level, because of any changes in creditworthiness of the issuer. Credit risk options are different from debt options, in that the latter usually price the interest rate risk of the bond, whereas credit risk options price only the credit risk of the underlying. With credit risk options, we can efficiently strip off credit risk from corporate debt.

[2]This can be explained by the fact that when the underlying bond is non-defaultable, its price at maturity is likely to be greater than the exercise level, and so the put option becomes deep out-of-the money.

Table 5.1: **Examples of prices for 2-year call and put options on a 5-year 8% coupon bond with face value 70 for different initial interest rate levels r, and exercise price E.** The option values are computed using parameters ζ=0.06,β=1,η=0.031,ρ=-0.25,σ=0.2,V_0=100,K=70, a=0.12, c=0.08, $n=5, m=50$.

	Call Prices(Risky)			Put Prices(Risky)			Call Prices(Riskless)		
r_0	E=60	E=65	E=70	E=60	E=65	E=70	E=60	E=65	E=70
0.01	3.411	1.340	0.317	2.716	5.281	8.893	15.01	10.37	5.739
0.02	3.472	1.368	0.325	2.590	5.081	8.633	14.79	10.20	5.601
0.03	3.354	1.400	0.332	2.468	4.887	8.376	14.57	10.02	5.464
0.04	3.586	1.424	0.337	2.359	4.714	8.142	14.36	9.846	5.329
0.05	3.623	1.441	0.339	2.256	4.551	7.926	14.15	9.674	5.197
0.06	3.659	1.458	0.340	2.154	4.392	7.713	13.94	9.505	5.066
0.07	3.697	1.475	0.344	2.055	4.234	7.504	13.74	9.338	4.938
0.08	3.735	1.491	0.347	1.961	4.079	7.298	13.54	9.174	4.812
0.09	3.774	1.514	0.352	1.867	3.933	7.095	13.34	9.011	4.687
0.10	3.813	1.537	0.357	1.775	3.788	6.900	13.13	8.851	4.565
0.11	3.852	1.560	0.363	1.686	3.645	6.699	12.94	8.694	4.444
0.12	3.894	1.583	0.368	1.600	3.504	6.503	12.75	8.538	4.326
0.13	3.935	1.606	0.374	1.515	3.364	6.310	12.56	8.384	4.209
0.14	3.981	1.629	0.380	1.436	3.227	6.119	12.38	8.233	4.093
0.15	4.027	1.657	0.385	1.358	3.095	5.929	12.19	8.084	3.980

Credit risk options can be structured as put options on a bond price or call options on a bond spread, the modelling of which was first suggested by Das [1995], and Longstaff and Schwartz [1995b] respectively. Although these two instruments are of different forms, they have the same function of stripping off the credit risk of the underlying asset.

In this section, we shall follow the approach of Das [1995] with regard to the pricing of credit risk put options on defaultable coupon bonds. In the case of a European credit risk put option $CRO(V_t, r_t, t)$ on a defaultable coupon bond $D(V_t, r_t, t)$, we focus on the option with maturity τ and the underlying bond with final maturity date $\tau+T$. To allow for the credit risk stripping features, we adjust the exercise level in such a way that compensation from the writer will be made whenever the bond $D(V_t, r_t, t)$ at time τ has a credit spread over the default-free yield greater than a specified spread level r^*. For simplicity, it is assumed that no compensation will be made to option holders in the case of a default of the underlying instruments before the option maturity date τ. Suppose that $B(r_t, r^*, t)$ represents the price of a yield-adjusted riskless coupon bond at time t, which has the same promised payments as the risky bond, and has the yield adjusted upward by the spread level r^*. At the maturity date τ, the payoff of the option is

$$CRO(V_\tau, r_\tau, \tau) = Max\Big(B(r_\tau, r^*, \tau) - D(V_\tau, r_\tau, \tau), 0\Big).$$

By turning the relative price of the credit risk option to the money market account into a Q-martingale, the option price at time 0 is given by[3]

$$CRO(V_0, r_0, 0) = E^Q\Big[\frac{Max\Big(B(r_\tau, r^*, \tau) - D(V_\tau, r_\tau, \tau), 0\Big)}{n_\tau}\Big|\Im_0\Big].$$

The computation of this expectation is difficult. There are two main sources of uncertainty in the above formula: one is the bond price $D(V_\tau, r_\tau, \tau)$, and the other one is the value of the money market account n_τ. By Lemma 4.2.1, we can fit the model into the framework consistent with our numerical algorithm.

Lemma 5.3.1 *The relative price of $CRO(V_s, r_s, s)$ to V_s^H is a Q^V-martingale for $0 \leq s \leq \tau$. The initial price of the credit risk put option is given by*

$$CRO(V_0, r_0, 0) = V_0 E^{Q^V}\Big[\frac{Max\Big(B(r_\tau, r^*, \tau) - D(V_\tau, r_\tau, \tau), 0\Big)}{V_\tau^H}\Big|\Im_0\Big]. \tag{5.1}$$

The computation proceeds as follows. We first build two binomial trees to price the coupon bond. At the option maturity date τ, a set of numbers is computed to mimic the distribution of the defaultable coupon bond $D(V_\tau, r_\tau, \tau)$. It is important to note that the risky bond price $D(V_\tau, r_\tau, \tau)$ tends to the riskless bond price $B(r_\tau, 0, \tau)$, as V_0 approaches infinity. This implies that when V_0 approaches infinity, the prices of the riskless bond can be considered driven by the same stochastic process Z_t, which underlies the valuation of the risky debt. Given the spread level r^*, another

[3] The method can be extended to price American type options.

two binomial trees representing the yield-adjusted riskless coupon bond $B(r_t, r^*, t)$ can then be constructed in a similar way, as in the case of the risky coupon bond above. Not only do the trees have the same structure and shape, but the nodes representing $D(V_T, r_T, T)$ are one-to-one corresponding to those representing $B(r_T, r^*, T)$. The computations inside the expectation E^{Q^V} in equation (5.1) are simply carried out on a one-to-one basis between points on the trees. Therefore, we need not worry about the complicated correlation structure that usually arises from multiple tree constructions for $B(r_T, r^*, T)$ and $D(V_T, r_T, T)$.

Figure 5.1 shows a two-year credit risk put option on a two-year 8% risky coupon bond with usual parameters.[4] As expected, the values of the credit risk option mirror those of the risky debt. Interestingly, the values of the credit risk option decrease with interest rates. The intuition behind this is that the risky bond $D(V_t, r_t, t)$ increases while the yield-adjusted bond $B(r_t, r^*, t)$ decreases, as the firm value tends to increase with interest rates.

5.4 Floating Rate Bonds

We now consider the pricing of risky floating rate bonds. Floating rate bonds are long dated bonds with interest rates linked to short term money market indices. Unlike the fixed rate bonds that we have discussed, floating rate bonds provide interest payments that float with short term interest rates. Suppose that the face amount of a floating rate bond is K, then its actual payments due at time t are structured as:

$$\left(\frac{1}{P(s,t)} - 1 + r^s\right)K,$$

if the bond does not mature at time t, or

$$\left(\frac{1}{P(s,t)} + r^s\right)K,$$

[4]In this case, the underlying bond has a final maturity of 4 years.

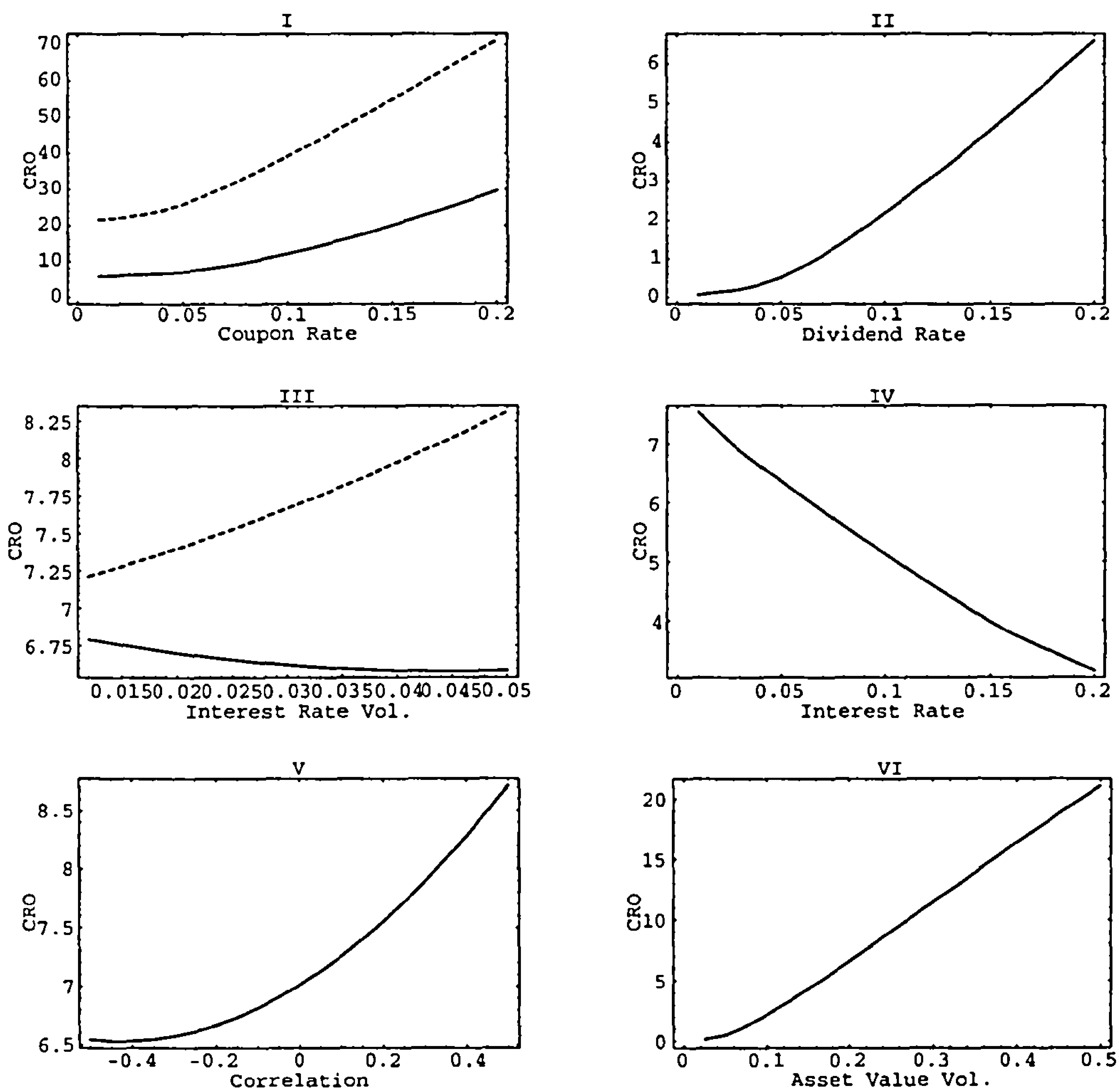

Figure 5.1: **Two-year credit risk put option on a two-year 8% risky coupon bond with stochastic interest rates.**
r_0=0.04, r^*=0.01, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2,V_0=100,K=70, a=0.12, stated. In (I): K=70(solid line), K=100(dashed line). In (III): ρ=-0.25(solid line), ρ=0.25(dashed line), $n = 5, m = 50$.

if the bond matures at time t, where s and t are two consecutive dates for interest payments, and $P(s,t)$ represents the price of a riskless discount bond at time s whose maturity payment of a unity is to be due at time t, and r^s represents a coupon rate spread above the riskless yield to reflect the underlying credit risk.[5] There is a subtle relationship between r^s and risk premium demanded in the market.[6] Instead of modelling the complicated relationship, we simply assume that the markup is a fixed given number.

The interest payments are stochastic, and can only be determined on the last coupon date.[7] Thus in the absence of default risk when r^s=0, investing in a floating rate debt of maturity T is equivalent to depositing an amount of K in a money market account for T years. The current price is obviously K.

In the presence of default risk, the pricing of floating rate bonds can be fitted in our framework in the same way as the pricing of defaultable coupon bonds. By Lemma 4.2.1, it is trivial to see that if a risky floating rate debt has only a single payment, then its relative price to V_T^H is a Q^V-martingale. This fact in turn suggests that risky floating rate bonds can also be priced by the numerical approach that we have developed for the pricing of risky fixed rate bonds. To make the price $P(s,t)$ of the discount bond available for the computations, we use the following formula:

$$\begin{aligned} P(s,t) &= \exp[\frac{1-e^{-\beta(t-s)}}{\beta}(R(\infty)-\hat{r}_s)-(t-s)R(\infty)-\frac{\eta^2}{4\beta^3}(1-e^{-\beta(t-s)})^2], \\ R(\infty) &= \frac{\zeta}{\beta}-\frac{1}{2}\frac{\eta^2}{\beta^2}, \end{aligned}$$

where $\hat{r}_s$ can be approximated by the interest rate lattice developed in Chapter 4.

In Figure 5.2, we plot the prices of a two-year defaultable floating rate bond with different parameters. It is interesting to note that the floating rate bond prices increase with interest rate. The reason for this is that an increase in initial interest

[5]Usually the reference coupon rate used to compute each interest payment is set at a certain percentage point above the contemporaneous yield on Treasury bills, or Libor.

[6]See Ramaswamy and Sundaresan [1986].

[7]Except for the first interest payment,all subsequent ones are stochastic.

rates has an immediate positive effect on the first interest payment, and also tends to raise any subsequent floating rate payments to a higher level. This result implies that the effect of discounting is not sufficiently strong to offset the effect of an increase in interest payment. As a result, the floating rate bond becomes more valuable at higher interest rates.

5.5 Summary

We have shown that the algorithm for the computations of defaultable coupon bonds, after some modifications, immediately lends itself to the efficient pricing of three credit-sensitive instruments. We have considered the pricing of (i) bond options on a defaultable coupon bond, (ii) a credit risk put option on a defaultable coupon bond, and (iii) a defaultable floating rate note.

In the case of bond options, two features are revealed when comparing the corresponding instruments assuming no default risk of the underlying. Firstly, when the underlying bond is defaultable, the call (put) becomes less (more) valuable because of the lower values of the underlying. Secondly, when interest rates rise, call prices move in a direction opposite to the call prices on a riskless underlying. As a consequence, we have shown that the default risk of the underlying asset has a significant effect on the pricing of bond options.

For the credit risk option, we structure the instrument as a put option on a defaultable bond. The exercise level is a yield-adjusted riskless coupon bond that has the same payment schedule as the underlying asset. As in the pricing of the defaultable bond, we can apply the same method to price the exercise level. The pricing of the credit risk put option is based on the fact that the lattice structures representing the underlying asset and the exercise level are one-to-one corresponding to each other. Therefore, the computations become straightforward, as we need not worry about the complicated correlation structure that usually arises from multiple tree constructions.

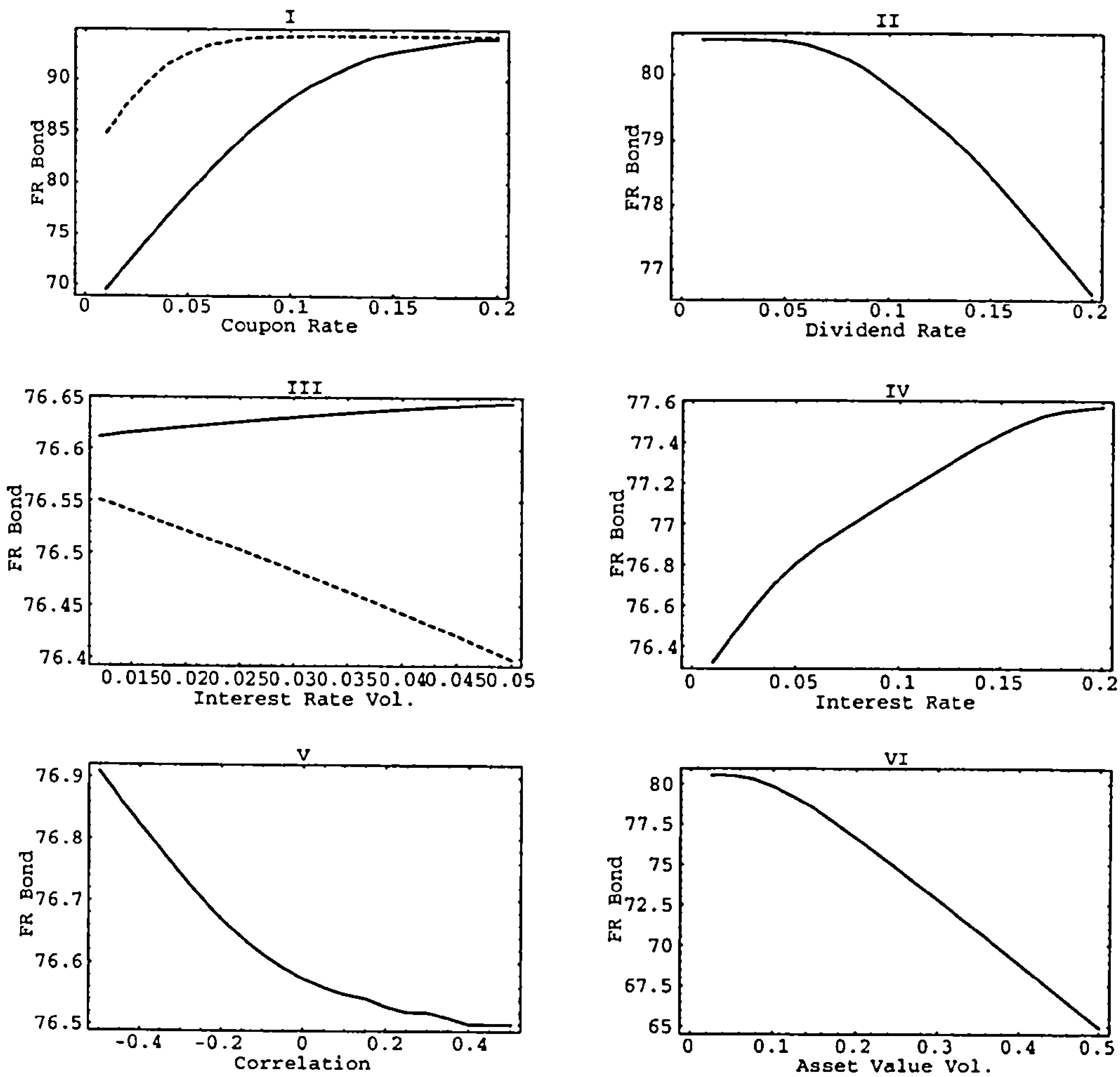

Figure 5.2: **Two-year defaultable floating rate bond prices with stochastic interest rates.**
r_0=0.04, ζ=0.06, β=1, η=0.031, ρ=-0.25, σ=0.2, V_0=100, K=70, a=0.12, r^s=0.04, unless otherwise stated. In (I): K=70 (solid line), K=100 (dashed line). In (III): ρ=-0.25 (solid line), ρ=0.25 (dashed line), $n = 5, m = 50$.

In the case of pricing the floating rate note, we employ a similar method of solution used for the valuation of credit put option. Here, we manage to approximate a time-series of six-month riskless discount bond prices from the time of issue up to the second last coupon date. Numerical computations have shown that unlike fixed rate bonds, the floating rate bond prices increase with interest rate.

In times of stringent economic situations and tightening monetary policies, increases in interest rates normally bring about difficulties in financing the coupon with variable interest payments. This is an interesting issue. With regard to further research, the model can be employed to investigate the impact of interest rate changes on the default barrier of the model. Here, we can adopt the same default condition as in the pricing of fixed rate bonds to investigate the properties of the bankruptcy barrier.

CHAPTER 6
A Jump-Diffusion Model

6.1 Introduction

This chapter employs a structural approach to analyse term structures of credit risk and yield spreads for the corporate debt of firms when the value of underlying assets follows a jump-diffusion process. Using a discrete time method for valuing general coupon bonds, we show several significant implications of the jump process for the term structure of credit spreads when systematic jumps are present in the firm's asset value. We also discuss the effects of diversifiability of jumps on corporate debt pricing. Other important factors include taxes and dividends. The main results are as follows. Firstly, the presence of jumps in asset values eliminates the undesirable qualitative feature of credit spreads decreasing to zero at the short end. The effects on credit spreads become more persistent when downward jumps are of higher variance while the total variance of the firm's asset value remains the same. Secondly, without considering systematic jump risk, theoretical models tend to underestimate the credit spreads. Thirdly, taxes do have significant effects on levels of credit spread. Interestingly, the model implies that a decrease in the federal tax rate may affect earlier default of low-grade bonds.

The valuation of corporate debt is central to theoretical and empirical work in corporate finance. The literature on pricing risky debt has evolved in two main directions: the structural approach and the reduced-form approach. As in Merton [1974], the structural approach takes the dynamics of the assets of the issuing firm as given, and priced corporate bonds as contingent claims on the assets. Black and Cox [1976] and Geske [1977] provide generalizations that take into account the effects of coupon and bond indenture provisions. The application of stochastic interest rates in the valuation of corporate debt using Merton's [1974] framework is discussed in Shimko, Tejima, and Van Deventer [1993]. Longstaff and Schwartz [1995a] adapt Black and Cox's model to a more realistic setting.

By introducing bankruptcy costs and tax effects, the framework has been extended to a richer extent taking into account issues in corporate finance. Examples that consider endogenous capital structure, liquidation policy, re-capitalization and re-organization of debt include Brennan and Schwartz [1984], Leland [1994, 1998], Leland and Toft [1996], Anderson and Sundaresan [1996], and Mella-Baral and Perruadin [1997]. These models allow for endogenous default, optimally determined by equity holders when asset levels fall to a sufficiently low level. Anderson and Sundaresan [2000] conduct an empirical analysis of structural models of corporate bond yields. Their results suggest that recent modifications of the contingent claims models to allow for endogenous default barriers have improved the performance of the models in tracking observed yield spreads.

Nevertheless, the structural approach to the valuation of risky debt has been criticized for not being able to generate sufficient credit spreads for small maturities of debt. In practice, even for small maturities, the market does not neglect the possibility that some disaster may happen.

As opposed to the structural models, more recent literature has adopted an alternative approach that offers a high degree of tractability for credit risky bonds. This reduced-form approach relates default time to the stopping time of an exogenously given hazard rate process, and derived formulae are calibrated to market data. This is illustrated by Jarrow and Turnbull [1995], Jarrow, Lando, and Turnbull [1997], Lando [1995], Madan and Unal [2000], Duffee [1999], and Duffie and Singleton [1999]. This approach provides us with a model that is close to the data, and it is always possible to fit some version of the model. However, the fitted model may not perform well on "out of sample" analysis.

A middle-way approach has been suggested in the literature. Cathcart and El-Jahel [1999] propose a framework situated between structural and reduced-form approaches, within which a default event occurs in an expected or unexpected manner when the value of a signalling process reaches a certain lower barrier or at the first jump time of a hazard-rate process. Although the model can generate strictly positive credit spreads for small maturities, the simple assumption that the firm goes bankrupt immediately when a jump in the asset value occurs for the first time needs empirical justification. Zhou [1997] proposes a numerical model for pricing discount bonds in much the same spirit as in Longstaff and Schwartz [1995a], when the underlying asset value follows a jump-diffusion process which is similar to the stock price process in Merton [1976]. Coupon bonds are not considered in the paper. Instead of employing a jump process as a determinant of default mechanism, Duffie and Lando [1999] study the implications of imperfect accounting information for modelling corporate bonds. They suppose that bond investors cannot directly observe the issuer's assets directly, and receive only periodic and imperfect accounting reports. As a consequence of the uncertainty in asset values, bounding short spreads away from zero can be obtained in their model.

There is a basic incompatibility in default mechanisms between the Duffie and Singleton [1999] model and the traditional Merton-type models. The "reduced-form" of a structural model is usually taken to mean a version of the model in which endogenous variables are expressed as a function of predetermined variables only.[1] In this definition, the reduced-form model must be equivalent to the original structural one. However, this is not true of the Duffie and Singleton [1999] model. In the Merton-type framework, in which a firm's asset value follows a pure diffusion

[1] For an econometric definition, see Koutsoyiannis [1973].

process, default can only happen expectedly. Given that default has not happened up to the present moment, there is a zero probability that the firm will become bankrupt at the next instant. On the contrary, the reduced-form approach typically assumes that default events are surprises. For example, Duffie and Singleton [1999] consider that given a positive hazard rate process h_t, default occurs at a rate of h_t.

Neither the pure diffusion nor the reduced-form approach appears to concur completely with empirical evidence that default can happen in both an expected and an unexpected manner.[2] Motivated by this observation, we provide an alternative to the problem by modelling the firm's asset value as a jump-diffusion process. While a model based on jump-diffusion captures both expected and unexpected defaults, we show that it also exhibits the interesting properties of the leptokurtic feature that are empirically observable in asset returns. We present a tractable, discrete time model for valuing general coupon bonds under a jump-diffusion process in asset value. For simplicity, we assume that a default event can only happen on payment dates.[3] The firm goes into bankruptcy *expectedly* when the asset level hits a certain lower barrier through a continuous diffusion crossing, or *unexpectedly* when its value drops precipitously below the barrier.[4] Consistent with Geske [1977], Leland and Toft [1996], and Leland [1994, 1998], the default boundary is determined endogenously by requiring the value of equity to be at least the amount of the coupon just paid, in order to avoid bankruptcy. As investors in corporate bonds are subject to state and local taxes, we also consider the effects of tax premiums in an economy where jump risk is correlated to a market portfolio.

[2]See Delianedis and Geske [1998]. Their empirical study indicates that risk neutral default probabilities using Merton [1974] and Geske's [1977] diffusion models appear to possess significant information about credit migrations and default.

[3]The method is flexible enough to be modified and allow bankruptcy events to happen in any between-payment dates.

[4]Hence one would expect to see a marked increase in volatility of bond returns and a sudden drop in equity prices.

The objectives of this chapter are as follows. We show several significant implications of the jump process for the level and the term structure of credit spreads. For example, it is interesting to note that while the presence of jumps in asset values eliminates the undesirable qualitative feature of credit spreads decreasing to zero at the short end, negative jumps can have significant and persistent effects on spread levels. The jump effects on spread levels are conspicuous for short maturities. For long maturities, credit spreads are indistinguishable from those generated by a pure diffusion model. However, when downward jumps are of higher volatility while the total variance of the firm's asset value remains the same, the effects on credit spreads become more persistent. Other important factors include taxes and dividends. As suggested by Elton, Gruber, Agrawal, and Mann [2000], we show that taxes do have significant effects on the levels of credit spread. A further contribution of this chapter is that we provide characterizations of the default boundary, and show how the endogenous default mechanism is affected in the presence of jumps and taxes. Our results suggest that a decrease in the federal tax may affect earlier default of low-grade bonds.

The chapter is divided into six sections. In the next section, we review a continuous jump-diffusion model as presented by Zhou [1997]. We show a generalization of the model to incorporate systematic jump risk. As far as parameter estimation is concerned, emphasis is placed on the structural similarities between the two models. In order to facilitate efficient computation, we adopt Amin's [1993] discrete time approach to approximate the continuous models in Section 6.3. In Section 6.4, we extend Geske's [1977] model to incorporate a jump component by the discrete time approximation, as in Amin [1993], and price a defaultable coupon bond with general jump risk in total firm value. We investigate some properties of credit spreads, and show how term structures of spread levels under the jump-diffusion process differ from those generated by a pure diffusion approach. We present some noteworthy results about the effects of non-diversifiable jumps on debt pricing. Section 6.5 discusses a more practical approach to pricing corporate bonds by introducing state and federal taxes into our model. The effects of state taxes and a dividend payout rate on spread levels and bankruptcy mechanisms are investigated. In Section 6.6, we conclude and provide suggestions for further research.

6.2 Theoretical Models

6.2.1 A Continuous Time Model with Non-Systematic Jump Risk

Zhou's [1997] model is in much the same spirit as that of Longstaff and Schwartz [1995a]. The underlying process of the firm's asset value is modelled as a jump-diffusion process where the jump risk is assumed to be diversifiable. Such an approach is analogous to the modelling of the stock price process in Merton [1976].

Let V_t be the total market value of the assets of the firm at time t. Under an equivalent martingale measure, Zhou [1997] assumes that the dynamics of the firm's asset value process V_t follows the following jump-diffusion process:

$$dV_t/V_t = (r - \lambda_J \overline{m})\ dt + \sigma\ dZ_t + m\ dJ, \tag{6.1}$$

where

r is the contant spot rate of interest based on continuous compounding;

σ is the instantanous volatility conditional on no jumps;

Z_t is a standard Brownian motion under the risk-neutral measure;

m is the random percentage change in firm value if a Poisson jump occurs: $1+m$ is log-normally distributed, $\log(1+m) \sim N[\gamma_m - \frac{1}{2}\sigma_m^2, \sigma_m^2]$, $E(m) = \overline{m} = e^{\gamma_m} - 1$; and λ_J is the intensity of the Poisson jump process $J : P[dJ = 1] = \lambda_J\ dt$.

The process most often resembles geometric Brownian motion, but on average λ_J times per year, the price jumps discretely by a random amount. Thus the total change in asset value of the firm is posited to be the composition of two types of changes. The first is the normal vibrations in value due to changes in the economic outlook that causes marginal changes in the asset value. This component is modelled by the standard geometric Brownian motion, Z_t, with the constant variance per unit time. The second is the abnormal vibrations in asset value due to the arrival of important new information about the firm. This has more than a marginal effect on price, and is modelled by the jump process, J, reflecting that such information arrives only at discrete points in time. Under the assumptions of Merton [1976] and Zhou [1997] that the two sources of uncertainties are independent of each other and the information is specific to the firm, the jump risk is uncorrelated to the market and is not priced in equilibrium.

Zhou [1997] also assumes the presence of a positive threshold value, K, for the firm at which financial distress occurs. If the firm value, V_t, falls to or below the threshold level, K, the firm defaults on all of its obligations immediately and some form of corporate restructuring takes place. The bondholder receives $1-w(X_t)$ times the face value of the security at maturity T, where $X_t = V_t/K$ is the ratio of the firm value V_t to K. Under these assumptions, the bond price $B(X_0, 0)$, with a promised final payment of 1 at time T is given by

$$B(X_0, 0) = \exp(-rT)E[I_{V_T>K} + (1 - w(X_T))I_{V_T\leq K}].$$

Zhou [1997] proposes a numerical algorithm for computations of the bond price. He also considers the case where interest rates follow a Vasicek [1977] process. Neither coupon bonds nor the effects of change of measures on the firm's asset value process are discussed in his paper. Unlike Zhou's [1997] framework, our model deals with endogenous default mechanisms by employing Geske's [1977] idea. In this chapter, we extend Zhou's model in three ways: (i) we take coupons into account; (ii) we assume that default barrier is determined endogenously; and (iii) we consider the effect of systematic jumps.

6.2.2 A Continuous Time Model with General Jump Risk

The jump-diffusion processes as described by Merton [1976] and Zhou [1997] are perhaps the simplest type of models to include jumps in asset prices. The crucial assumption is that the jump risk is diversifiable and non-systematic. This assumption is questionable as asset prices appear to be correlated with market movements.

In an empirical study of an economy where stock prices are assumed to follow a jump-diffusion process, Jarrow and Rosenfeld [1984] investigate the satisfaction of assuming jumps to be diversifiable. Evidence has been found to show that the jump component of stock's returns has a strong correlation with the market portfolio, that is, the market portfolio appears to contain a jump component. A similar conclusion is drawn by Kim, Oh, and Brooks [1994] who study 20 component stocks of the Major Market Index. They find that Poisson-type jumps observed from both the index and its component stocks constitute non-diversifiable risk. This implies that the standard assumption in option pricing as in Merton [1976] that those jumps are not priced may be invalid.

Relationships of common risk factors between the returns on stocks and bonds have been investigated in Fama and French [1993], and Elton et al. [2000]. Based on the Fama-French three-factor model [1993], Elton et al. [2000] find that expected default accounts for a surprisingly small fraction of the premium in credit spreads of corporate bonds. They conclude that, while state taxes explain a substantial portion of the discrepancy, the remaining portion of the spread is closely related to factors commonly accepted as explaining risk premiums for common stocks. They show that a significant portion of the spread is compensation for systematic risk that is affected by the same influences of systematic risks in the stock market. These results imply that it is hardly plausible to maintain Merton and Zhou's simplifying assumptions that jump risk is non-systematic and diversifiable. To incorporate systematic jump risk into process (6.1), we adopt a jump-diffusion process in Bates [1991] as a generalization of the model. Firstly, we assume that under the objective measure, the dynamics of the firm's asset value are as follows:

$$dV_t/V_t = (\mu - \lambda\bar{k} - \delta)\ dt + \sigma\ dB_t + k\ dq, \tag{6.2}$$

where

μ is the instantaneous cum-dividend expected return on the asset;

δ is a constant payout rate as a fraction of firm value;

σ is the instantanous volatility conditional on no jumps;

B_t is a standard Brownian motion under the objective measure; and

k is the random percentage change in firm value if a Poisson jump occurs: $1+k$ is log-normally distributed, $\log(1+k) \sim N[\gamma - \frac{1}{2}\sigma_k^2, \sigma_k^2], E(k) = \bar{k} = e^\gamma - 1$; λ is the intensity of the Poisson jump process $q : P[dq = 1] = \lambda\ dt$.

Secondly, the following restrictions on utility preferences are imposed:

$A1$: Frictionless Markets.

$A2$: Optimally invested wealth W_t follows a jump-diffusion,

$$dW_t/W_t = (\mu_w - \lambda\bar{k}_w - C/W)\ dt + \sigma_w\ dB_t^{'} + k_w\ dq,$$

where μ_w is constant and k_w is the percentage change in wealth when the Poisson jump happens. $1+k_w$ is log-normally distributed, $\log(1+k_w) \sim N[\gamma_w - \frac{1}{2}\sigma_{k_w}^2, \sigma_{k_w}^2], E(k_w) = \bar{k}_w = e^{\gamma_w} - 1$, and $Cov[\log(1+k), \log(1+k_w)] = \sigma_{vw}$.

$A3$: The representative consumer has time-separable power utility,

$$E_\tau \int_\tau^\infty e^{-\rho t} U(C_\tau)\ dt,\ U(C) = (C^{1-R} - 1)/(1-R).$$

Assuming that jump risk is systematic, all asset prices and wealth jump simultaneously, possibly by different amounts. Analogous to Bates' model [1991] of systematic jump risk, we assume that under a risk-neutral measure, the jump-diffusion model (6.2) takes the following form:

$$dV_t/V_t = (r - \lambda^* \overline{k}^* - \delta)\ dt + \sigma\ dB_t^* + k^*\ dq^*, \tag{6.3}$$

where

σ and δ are as before;

$\lambda^* = \lambda \exp[-R\gamma_w + \frac{1}{2}R(1+R)\sigma_{k_w}^2]$;

q^* is a Poisson counter with intensity λ^*; and

k^* is the random percentage change in firm value if a Poisson jump occurs: $1+k^*$ is log-normally distributed, $\log(1+k^*) \sim N[\gamma^* - \frac{1}{2}\sigma_k^2, \sigma_k^2]$, $E(k^*) = \overline{k}^* = e^{\gamma^*} - 1$, and $\gamma^* = \gamma - R\sigma_{vw}$.

It is noteworthy that process (6.3) is a generalization of (6.1). The process (6.3) reduces to (6.1) when the dividend payout rate $\delta = 0$ and the coefficient of relative risk aversion $R = 0$, meaning that investors are risk neutral. Furthermore, in the event that jump risk is "firm-specific", $\gamma = \sigma_{k_w} = \sigma_{vw} = 0$ and the model reduces to the Merton model with $\overline{k}^* = \overline{k}$, $\overline{\lambda}^* = \lambda$.

A noticeable feature of the two processes is that they share the same modelling structure. The implicit nature of the relative risk aversion coefficient, R, lends much to their structural similarity and to their convenience for pricing on the basis of process (6.3). For example, pricing European options from process (6.3) is straightforward (see Merton [1976]). This is also the case for parameter estimation.[5] Parameters estimated from a pricing model based on the underlying risk-neutral jump-diffusion

[5]See Bates [1991].

process (6.3) are of course those of the same process. Inferring the true parameters[6] in process (6.2) requires additional assumptions about the coefficient of relative risk aversion and about the degree to which jumps in process (6.3) are related to jumps in wealth. To implement the model, we need a discrete time formulation that provides us with a computational tool for pricing derivative instruments.

6.3 A Discrete Time Approximation

In this section, we apply the technique of a discrete time approach to the solution of process (6.3). As in Merton [1976] and Zhou [1997], the asset value of the firm under the risk neutral measure is of the form:

$$V_t = V_0 \exp\left([r - \frac{1}{2}\sigma^2 - \lambda^* \overline{k}^* - \delta]t + \sigma B_t^* + \sum_{j=0}^{NJ} \log(1 + k_j^*)\right), \tag{6.4}$$

where NJ is the total number of Poisson jumps over time t, and the Poisson jump sizes $(1 + k_j)$'s are independent and identically log-normally distributed random variables with parameters $N[\gamma^* - \frac{1}{2}\sigma_k^2, \sigma_k^2]$.

For computational convenience, we turn V_t into a logarithmic scale and define the drift of the logarithm of the asset value in equation (6.4) by:

$$\alpha = r - \delta - \frac{1}{2}\sigma^2 - \lambda^* \overline{k}^*.$$

The process (6.3) can be rewritten in the following form:

$$X_t = \log(V_t/V_0) = \alpha t + \sigma B_t^* + \sum_{j=0}^{NJ} \log(1 + k_j^*).$$

To approximate X_t, we adopt a method used by Amin [1993] to discretize the process. The discrete time formulation is based on the work of Cox, Ross, and Rubinstien [1976] as the starting point. Multivariate jumps are superimposed on the model to obtain the model with a limiting jump-diffusion process. Let T be the

[6] It is not necessary to know the true parameters in process (6.2) as long as pricing issues are concerned.

Figure 6.1: **State space of the discrete approximation of firm value V_t for a fixed parameter n.** This figure shows the state space for the discrete model at dates 0, h_n and at some arbitrary date ih_n.

$$
\begin{array}{cccc}
\text{Time}=0 & \text{Time}=h_n & \dots & \text{Time}=ih_n \\
 & \dots & \dots & \dots \\
 & \log\frac{V_4(h_n)}{V_0} = \alpha h_n + 4\sigma\sqrt{h_n} & \dots & \log\frac{V_4(ih_n)}{V_0} = \alpha ih_n + 4\sigma\sqrt{h_n} \\
 & \log\frac{V_3(h_n)}{V_0} = \alpha h_n + 3\sigma\sqrt{h_n} & \dots & \log\frac{V_3(ih_n)}{V_0} = \alpha ih_n + 3\sigma\sqrt{h_n} \\
 & \log\frac{V_2(h_n)}{V_0} = \alpha h_n + 2\sigma\sqrt{h_n} & \dots & \log\frac{V_2(ih_n)}{V_0} = \alpha ih_n + 2\sigma\sqrt{h_n} \\
 & \log\frac{V_1(h_n)}{V_0} = \alpha h_n + 1\sigma\sqrt{h_n} & \dots & \log\frac{V_1(ih_n)}{V_0} = \alpha ih_n + 1\sigma\sqrt{h_n} \\
0 & \log\frac{V_0(h_n)}{V_0} = \alpha h_n & \dots & \log\frac{V_0(ih_n)}{V_0} = \alpha ih_n \\
 & \log\frac{V_{-1}(h_n)}{V_0} = \alpha h_n - 1\sigma\sqrt{h_n} & \dots & \log\frac{V_{-1}(ih_n)}{V_0} = \alpha ih_n - 1\sigma\sqrt{h_n} \\
 & \log\frac{V_{-2}(h_n)}{V_0} = \alpha h_n - 2\sigma\sqrt{h_n} & \dots & \log\frac{V_{-2}(ih_n)}{V_0} = \alpha ih_n - 2\sigma\sqrt{h_n} \\
 & \log\frac{V_{-3}(h_n)}{V_0} = \alpha h_n - 3\sigma\sqrt{h_n} & \dots & \log\frac{V_{-3}(ih_n)}{V_0} = \alpha ih_n - 3\sigma\sqrt{h_n} \\
 & \log\frac{V_{-4}(h_n)}{V_0} = \alpha h_n - 4\sigma\sqrt{h_n} & \dots & \log\frac{V_{-4}(ih_n)}{V_0} = \alpha ih_n - 4\sigma\sqrt{h_n} \\
 & \dots & \dots & \dots
\end{array}
$$

maturity of a coupon bond. For a fixed positive integer, n, we divide the interval $[0, T]$ into n subintervals of width $h_n = T/n$. The state space of the discrete approximation of firm value V_t is depicted in Figure 6.1.

At each date, ih_n, the value of the approximate process, X_i is shifted upward by αh_n relative to the grid at time $(i-1)h_n$. Therefore, the asset value at time ih_n and in state j relative to date 0 is given by $V_i = V_0 \exp(\alpha ih_n + j\sigma\sqrt{h_n})$. Any point at time $(i-1)h_n$ can move to any other points at time ih_n. As discussed in Amin [1993], there are two types of movements as to the dynamics of changes in the asset values over time, namely the local and non-local change. The asset price undergoes either of the two different types of mutually exclusive price changes. In most periods, the asset prices undergo only local changes. This is as in Cox, Ross, and Rubinstein [1976] where the asset price moves up or down by one step. This price change is due to the diffusion component of the continuous time case. Given that a local change of the asset prices has occurred, for a sufficiently large integer, n, the probabilities of an upward movement and of a downward movement are taken to be $p = \frac{1}{2}$.

If the change of asset price corresponds to any other number of steps, then a jump has occurred. This event has a low probability of occurring in any given period. In this case, determining of the corresponding transition probabilities is more tedious. At each discrete date, we approximate the jump distribution on the entire real line by breaking it down into non-overlapping intervals of equal width. The entire probability mass over each of these intervals is assigned to the state contained in one of these intervals. Let the cumulative density function of the jump size be $F(.)$. For any state, l, not equal to -1, 0, and 1, the jump probability P_j is given by:

$$P_j = F(\alpha h_n + (l + \frac{1}{2})\sigma\sqrt{h_n}) - F(\alpha h_n + (l - \frac{1}{2})\sigma\sqrt{h_n})$$

When l =0, we take

$$P_j = F(\alpha h_n + (1 + \frac{1}{2})\sigma\sqrt{h_n}) - F(\alpha h_n - (1 + \frac{1}{2})\sigma\sqrt{h_n})$$

Finally, when l = -1 or +1,

$$P_j = 0$$

This completes the specification of the risk neutral measure for the discrete time framework.[7]

Let $B_s(i)$ be the value of the risky bond at time ih_n and in state s. Then for any state k, the bond price between two consecutive payment dates is given by the

[7]Amin [1993] shows that the discrete time process converges weakly to the continuous time process. This guarantees that the prices of European options computed from the discrete time model will converge to their corresponding continuous time values under fairly mild regularity conditions. For example, the option payoff must be uniformly integrable in h_n.

iterative formula:

$$B_k(i) = e^{-rh_n}\Big(\lambda^* h_n E_Y\Big[B_{k+y}(i+1)\Big] + \frac{1}{2}(1-\lambda^* h_n)\Big(B_{k-1}(i+1) + B_{k+1}(i+1)\Big)\Big).$$

The first term on the right hand side represents a fraction of the bond price as a consequence of non-local changes in the asset prices, whereas the second term is the expected value of two bond prices resulting from local movements of the asset value. Here, we assume that the probability of a jump in the discrete model at any time is equal to $\lambda^* h_n$. We also assume that h_n is so small that multiple jumps cannot occur within the same period. At each coupon date $t = ih_n$, the bond prices immediately before coupon payment is given by:

$$B_k(i^-) = Min\Big(V_t, coupon + B_k(i^+)\Big).$$

We assume Geske's [1977] condition that coupon payments are financed by issues of new equity. The firm goes bankrupt only when its stock value immediately after a coupon date is less than the total coupon payment. Black and Cox [1976] argue that this situation will happen whenever the value of the equity, after payment is made, is less than the value of the payment. The argument is intuitive in that the firm will find no takers for its stock if they know that the stock will become less valuable than the total value they need to contribute to the promised payment. This condition endogenously determines the asset level position of default barriers that is consistent with Leland and Toft [1996] and Leland [1994, 1998]. However, this condition is not completely necessary; other bankruptcy criteria can also be used in our model. For example, Kim, Ramaswamy and Sundaresan [1993] assume a lower reorganization boundary for the firm's value, at which the total cash flow per unit time will be just sufficient to pay the contractual coupon. Furthermore, we also assume that bankruptcy costs[8] are not a significant determinant of firm value. With

[8]See Kliger and Sarig [2000] for an empirical justification.

this discrete time formulation, we can compute bond prices by dynamic programming using a backward recursion on the state space that we have developed.[9]

6.4 Pricing With General Jump Risk

In this section, we illustrate the model by numerical results. We consider a corporate bond with half-yearly coupon payments, $c\,K/2$, where K is the face value. In order to study the properties of credit spreads, we consider a base case environment in the risk-neutral world with the following parameters: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, $\sigma_k = 0.25$ and the number of subintervals is 50 per year. Based on an observation that downward jumps are more likely to happen than upward jumps, we also assume $\gamma^* = -0.1$. The parameter $\overline{k}^*$ in process (6.3) becomes -9.5%, implying a negative average jump size in asset value.

We first consider the dynamics of bond prices and credit spreads under the influence of jumps. The graphs presenting the term structure of bond prices and credit spreads under a jump-diffusion process are depicted in Figures 6.2, 6.3, and 6.4.

Figure 6.2 shows the term structure of a 5% coupon bond with maturities from one up to 20 years. Figure 6.3 shows the corresponding term structures of credit spreads for three jump-diffusion cases and a pure diffusion case. In the jump-diffusion cases, the total variances under the risk-neutral measure are kept the same, and the contributions of jump components to total variance are (i) 55% and (ii) 9% respectively.[10] We assume that the pure diffusion process has a constant variance equal to the total variance of the jump-diffusion case. We observe that while the term structures are similar in shape for both jump-diffusion and pure diffusion cases,

[9] The lattice method can be benchmarked with Monte-Carlo simulations.

[10] The total variance is $\sigma^2 + \lambda^*\left(\sigma_k^2 + (\gamma^* - \frac{1}{2}\sigma_k^2)^2\right)$, where the first and second terms are due to the diffusion and jump components respectively.

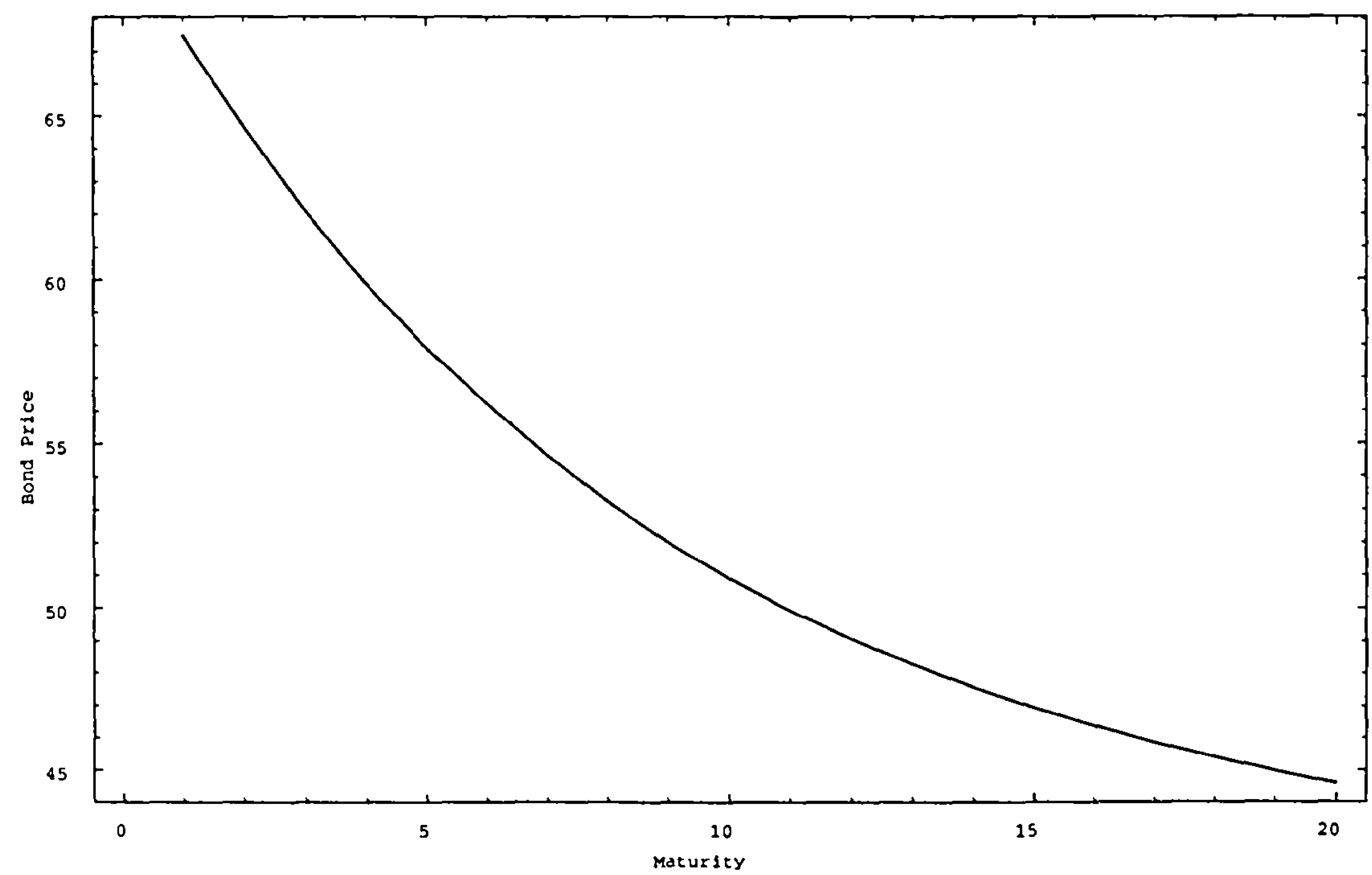

Figure 6.2: **Term structures of bond prices under a jump-diffusion process.** This figure shows the term structure of bond prices of a 5% coupon bond when the underlying process follows a jump-diffusion process. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, $\sigma_k = 0.25$, $\gamma^* = -0.1$ and $\overline{k}^* = -9.5\%$.

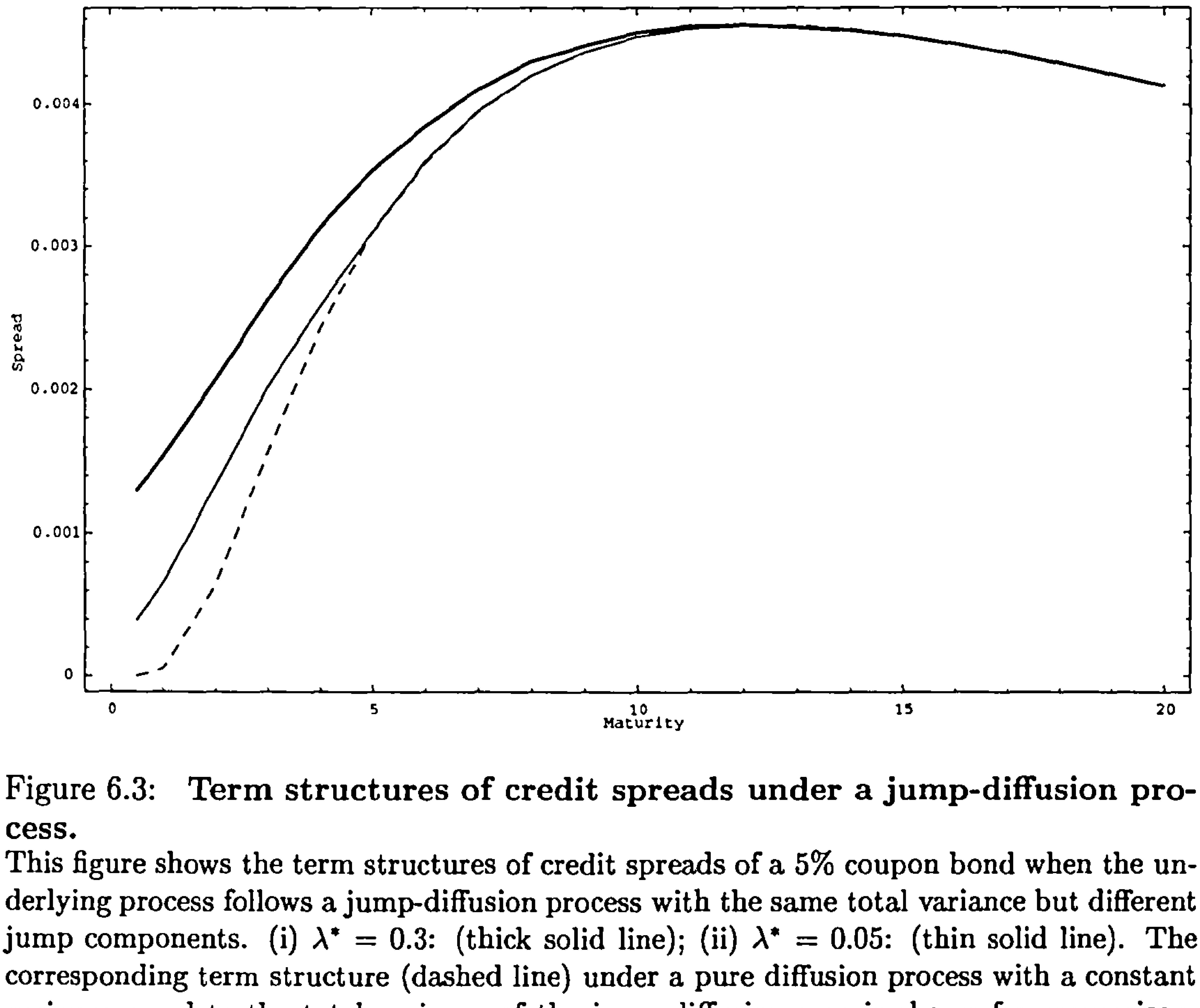

Figure 6.3: **Term structures of credit spreads under a jump-diffusion process.**
This figure shows the term structures of credit spreads of a 5% coupon bond when the underlying process follows a jump-diffusion process with the same total variance but different jump components. (i) $\lambda^* = 0.3$: (thick solid line); (ii) $\lambda^* = 0.05$: (thin solid line). The corresponding term structure (dashed line) under a pure diffusion process with a constant variance equal to the total variance of the jump-diffusion case is shown for comparison. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $K = 50$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\overline{k}^* = -9.5\%$, unless stated otherwise. Total variance per unit time remains constant in all cases.

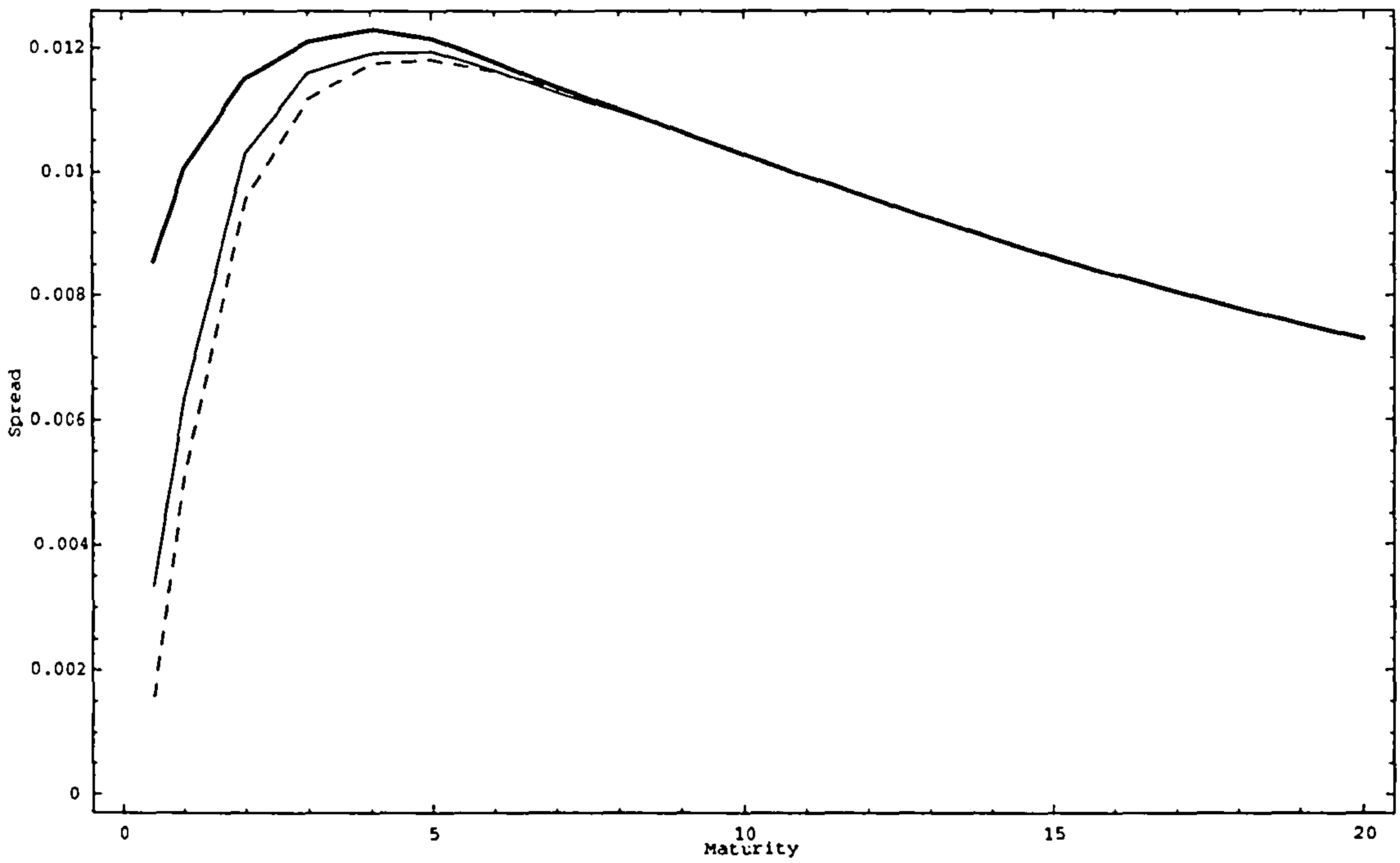

Figure 6.4: **Term structures of credit spreads under a jump-diffusion process.**
This figure shows the term structures of credit spreads of a 5% coupon bond when the underlying process follows a jump-diffusion process with the same total variance but different jump components. (i) $\lambda^* = 0.3$: (thick solid line); (ii) $\lambda^* = 0.05$: (thin solid line). The corresponding term structure (dashed line) under a pure diffusion process with a constant variance equal to the total variance of the jump-diffusion case is shown for comparison. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $K = 70$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\overline{k}^* = -9.5\%$, unless stated otherwise. Total variance per unit time remains constant in all cases.

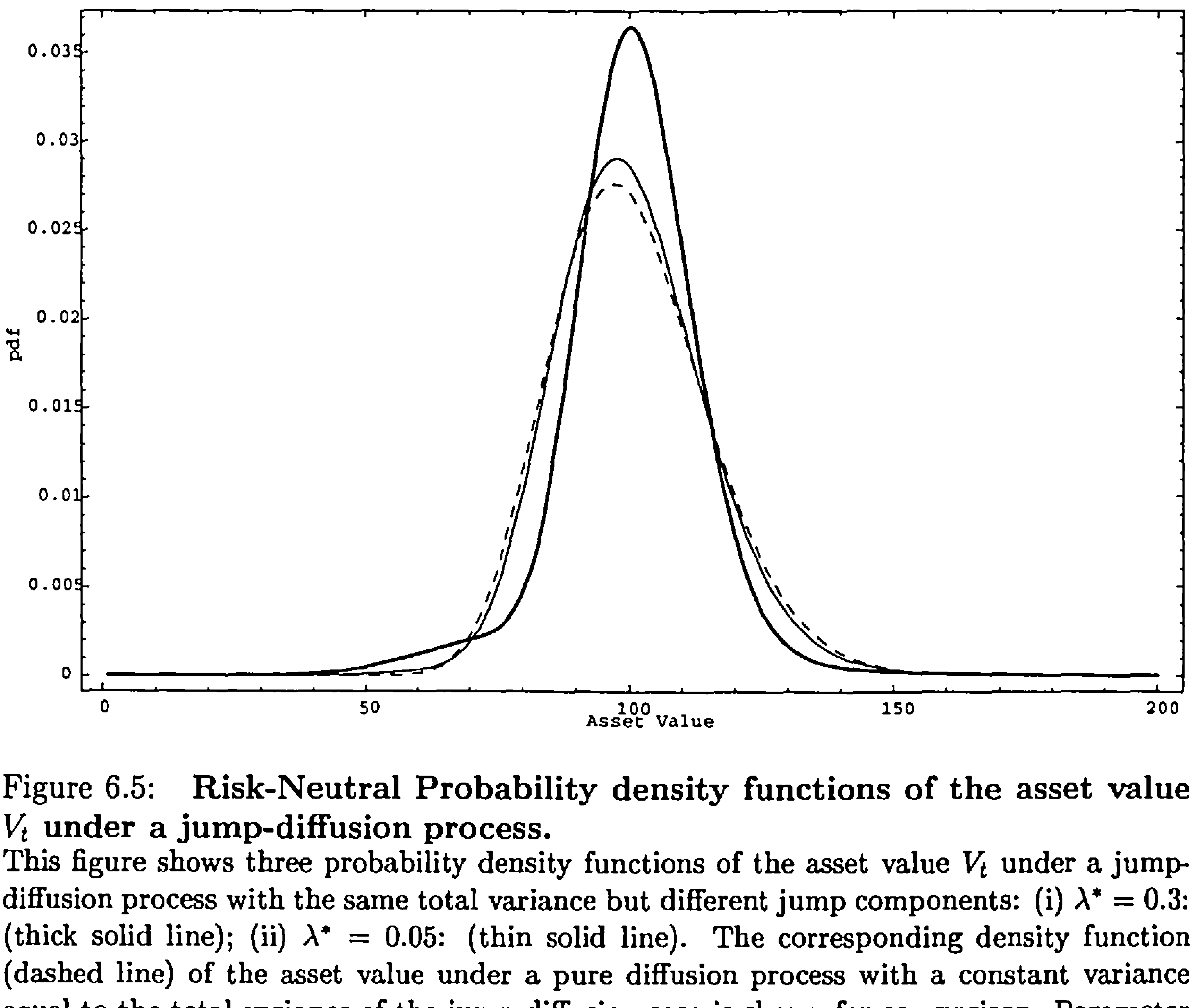

Figure 6.5: **Risk-Neutral Probability density functions of the asset value V_t under a jump-diffusion process.**
This figure shows three probability density functions of the asset value V_t under a jump-diffusion process with the same total variance but different jump components: (i) $\lambda^* = 0.3$: (thick solid line); (ii) $\lambda^* = 0.05$: (thin solid line). The corresponding density function (dashed line) of the asset value under a pure diffusion process with a constant variance equal to the total variance of the jump-diffusion case is shown for comparison. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, $c = 0.05$, $V_0 = 100$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\overline{k}^* = -9.5\%$, unless stated otherwise. Total variance per unit time remains constant in all cases.

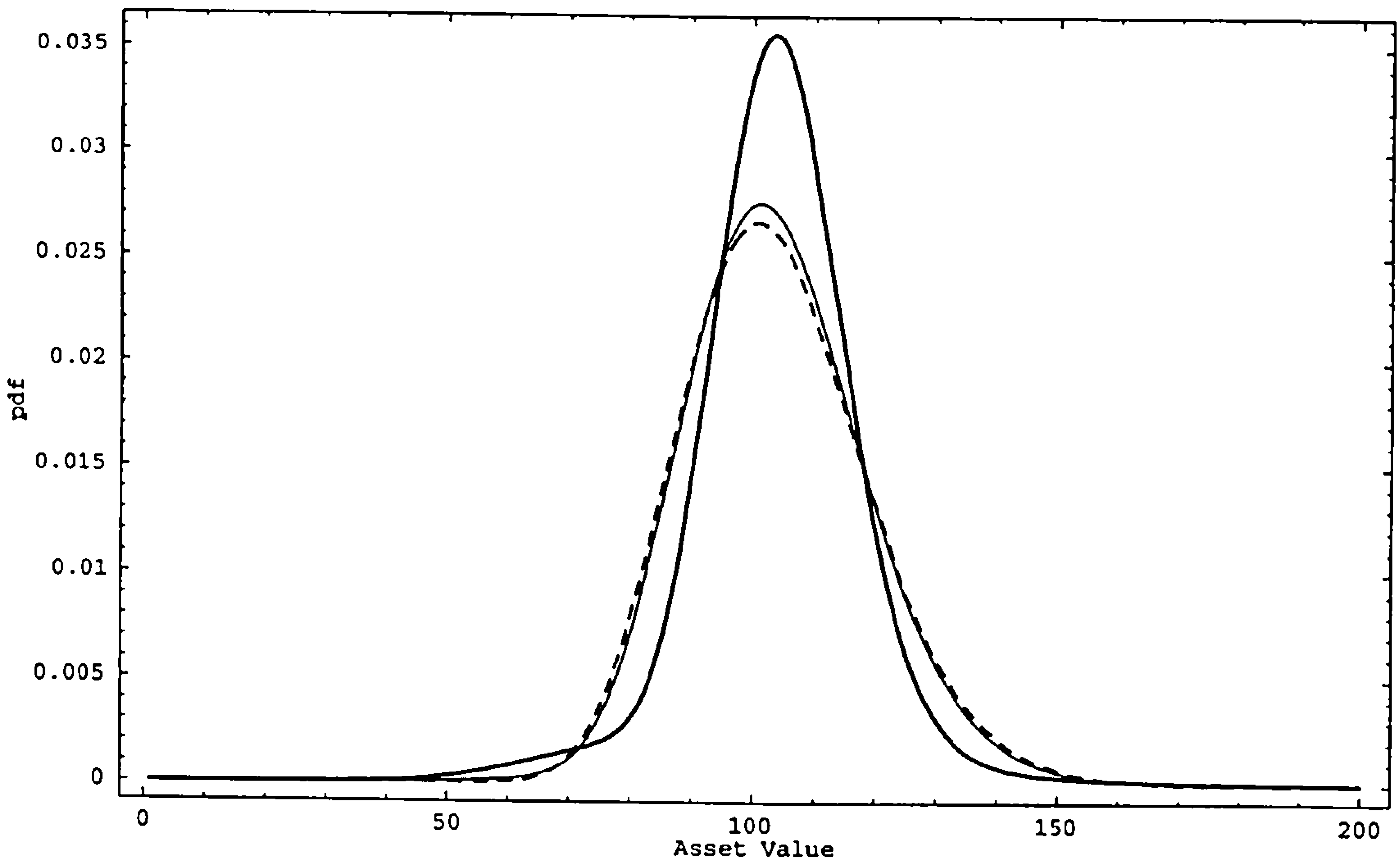

Figure 6.6: **Objective Probability density functions of the asset value V_t under a jump-diffusion process.**
This figure shows the objective probability density functions of the asset value V_t under a jump-diffusion process corresponding to the risk-neutral ones in Figure 6.5. The density functions are computed under the assumptions:$\mu = 15\%$, $R = 0.5$, $\gamma_w = \gamma$, and $\sigma_{vw} = \sigma_k^2 = \sigma_{k_w}^2$

the gap between credit spreads in the two cases narrows with maturities. In the presence of jumps, the possibility of sudden default only raises the levels of credit spread for short maturities.[11] This effect becomes more evident for bonds with shorter maturities when jumps occur more frequently and have higher variability. As maturity increases, the differences in credit spreads dwindle, implying that the effect of a jump becomes less prominent for bonds with long maturities. Similar results are shown in Figure 6.4, where the term structure of credit spreads for a different face value $K = 70$ is plotted. The rationale behind these results is as follows.

Recall that the total variance per unit time remains constant in all cases. When the maturity is small, the diffusion volatility in the pure diffusion model can only cause relatively small changes in the asset value. On the contrary, a single jump can cause a relatively large change in the asset value.

Figure 6.5 shows the risk-neutral probability density functions for the jump-diffusion and pure diffusion cases. It is evident that there is an empirical property[12] of fat-tailness when jumps can happen. Here, we choose the time horizon of half a year. Numerical computations show that the differences in shape gradually disappear as the time horizon increases. According to the Central Limit Theorem, one insight into the nature of these results is that the horizon distribution of a jump-diffusion process converges to the pure diffusion one as maturity increases. This shows that jumps bring about additional risk for short-term bonds only. Similar results can be obtained when we increase the jump variance, σ_k, and decrease the jump frequency, λ^*, while keeping its total contribution and total variance constant.

It is interesting to compare the differences between risk-neutral and the corresponding objective density function of the firm's asset return. However, as the

[11] Analytically, we can prove that there is a positive instantaneous default hazard at time 0, which is equal to λ^* times probability of default in the case of a jump.

[12] The leptokurtic properties of fat-tailness, especially in the left tail, and high peak that are evident in Figures 6.5 and 6.6, are normally observable in empirical densities of market indices, for example, the S&P 500 Index.

determination of the objective density function requires a knowledge of estimation of an instantaneous expected return on the asset in the real world, which lies beyond the scope of this chapter, here we simply assume that the asset return in the real world is $\mu = 15\%$, $R = 0.5$, $\gamma_w = \gamma$, and $\sigma_{vw} = \sigma_k^2 = \sigma_{k_w}^2$. Figure 6.6 plots the objective density functions. Two points are revealed. Firstly, the risk-neutral density function tends to give a higher estimate of default probability than the objective one, as a consequence of the lower instantaneous asset return in the risk-neutral world. Secondly, the close resemblance in shape to the corresponding risk-neutral ones shows that the empirical property of fat-tailness in asset return can also be observed in the real world.

Note that our comparison analysis is based on the assumption of constant total variance, measured in the risk-neutral world. There is a point of paramount importance in our credit-spread analysis. In the case of non-systematic jumps, the total variances of the jump-diffusion process in the real and risk-neutral world are the same. This implies that the theoretical levels of credit spread due to default risk can be estimated by the observable parameters in the real world. However, if the jump risk is systematic, estimation of spread levels becomes subtle. This is particularly the case if the average jump sizes in the asset value process and market portfolio are negative, that is, γ, $\gamma_w < 0$.[13] Under this assumption, it is trivial to see that:

$$\begin{aligned} \lambda^* &= \lambda \exp[-R\gamma_w + \frac{1}{2}R(1+R)\sigma_{k_w}^2] > \lambda, \\ \gamma^* &= \gamma - R\sigma_{vw} < \gamma < 0. \end{aligned}$$

The implication is that the total variance measured in the risk-neutral world is higher than that observed in the real world. Jumps in the risk-neutral world tend to be more influential as they become more negative. This is also true in times of economic recession where investors become more averse to risk. In the presence of systematic jumps, the true levels of spread cannot be approximated accurately by the theoretical

[13]Downward jumps are more likely than upward jumps.

ones based on the observable parameters without making specific assumptions about risk aversion and market portfolio. In fact, without knowledge of the risk aversion and market portfolio, the jumps would not be priced correctly, and so there is a tendency to underestimate the spread levels. Consistent with the findings in Elton et al. [2000], this result implies that without taking the systematic jump risk into account, Merton-type models tend to underestimate the credit spreads.

By comparing with empirical properties of credit spread, it is evident that the spread levels generated in Figures 6.3 and 6.4 do not quite resemble those observed in markets. The empirical findings in Kim, Ramaswamy, and Sundaresan [1993] indicate that over the 1926-1986 period, the yield spreads on high-grade corporates (AAA-rated) ranged from 15 to 215 basis points and averaged 77 basis points; and the yield spreads on BAA ranged from 51 to 787 basis points and averaged 198 basis points. To improve spread levels, we shall now incorporate state taxes into our model in the next section.

6.5 Pricing With General Jump Risk, Tax, and Dividend Effects

State taxes have been ignored in almost all modelling of defaultable bonds (see, for example, Jarrow, Lando, and Turnbull [1997] and Duffee [1999]). In this section, we introduce the two important factors of taxes and dividends into the model. We then investigate the effects of state taxes on spread levels and default mechanisms. Tax effects are important because investors in corporate bonds are subject to state and local taxes on interest payment while government bonds are not subject to these taxes. Thus, corporate bonds have to offer a higher pre-tax return for investors to compensate for tax expenses.

Elton, Gruber, Agrawal, and Mann [2000] show that expected default accounts for a small fraction of the premium in corporate yields over treasuries. State taxes explain a substantial portion of the difference. Taxes account for a significantly larger portion of the spreads than do expected losses. They find that for 10-year A-rated bonds, taxes accounted for 36.1% of the spreads, compared to the 17.8% accounted for by expected losses.

Since state tax is deductible from income for the purpose of federal tax, state tax is reduced by the federal tax rate. Hence, the effective tax rate, as a measure of marginal impact of state taxes, is of the form:

$$\tau = \tau_s(1 - \tau_g),$$

where τ_s is a state tax, and τ_g is the federal tax rate. We assume $\tau = 4.875\%$ by following the arguments in Elton et al. [2000] that we choose $\tau_s = 7.5\%$ as the mid-point of maximum marginal state taxes, and $\tau_g = 35\%$ as the maximum federal tax rate. It is easy to modify the model in the last section to fit with this tax factor. We assume that default can only happen on coupon payment dates. There are two cases where bond price will be affected. On each coupon payment date, if default does not happen, then the actual bond value is the original bond value less the total amount of tax on the interest payment. If the bond defaults, then the bond price becomes the residual asset value plus the tax refund due to a capital loss.[14] We take the after-tax coupon rate to be 5%.

The tax effects on spread levels are shown in Figures 6.7 and 6.8. Figure 6.8 shows the term structures of credit spread of two risky bonds with different face values and their spread component generated by pure tax effects.[15] Note that the pure tax level is the same in both cases.[16] As expected, tax effects contribute to considerable portion of total spread levels. The persistent difference in spread levels between the spread curves indicates that taxes are a more important influence on spreads. There are two important points worth mentioning. Firstly, pure tax effects appear to be proportionally more significant when the face value becomes smaller.

[14] When a bond defaults at time t, the amount $K - V_{t+}$ lost in default is a capital loss and taxes $\tau(K - V_{t+})$ are recovered. See Elton et al. [2000].

[15] We assume that they are issued by two identical firms.

[16] Spread levels are computed when the firm's asset value is large relative to the amount of debt.

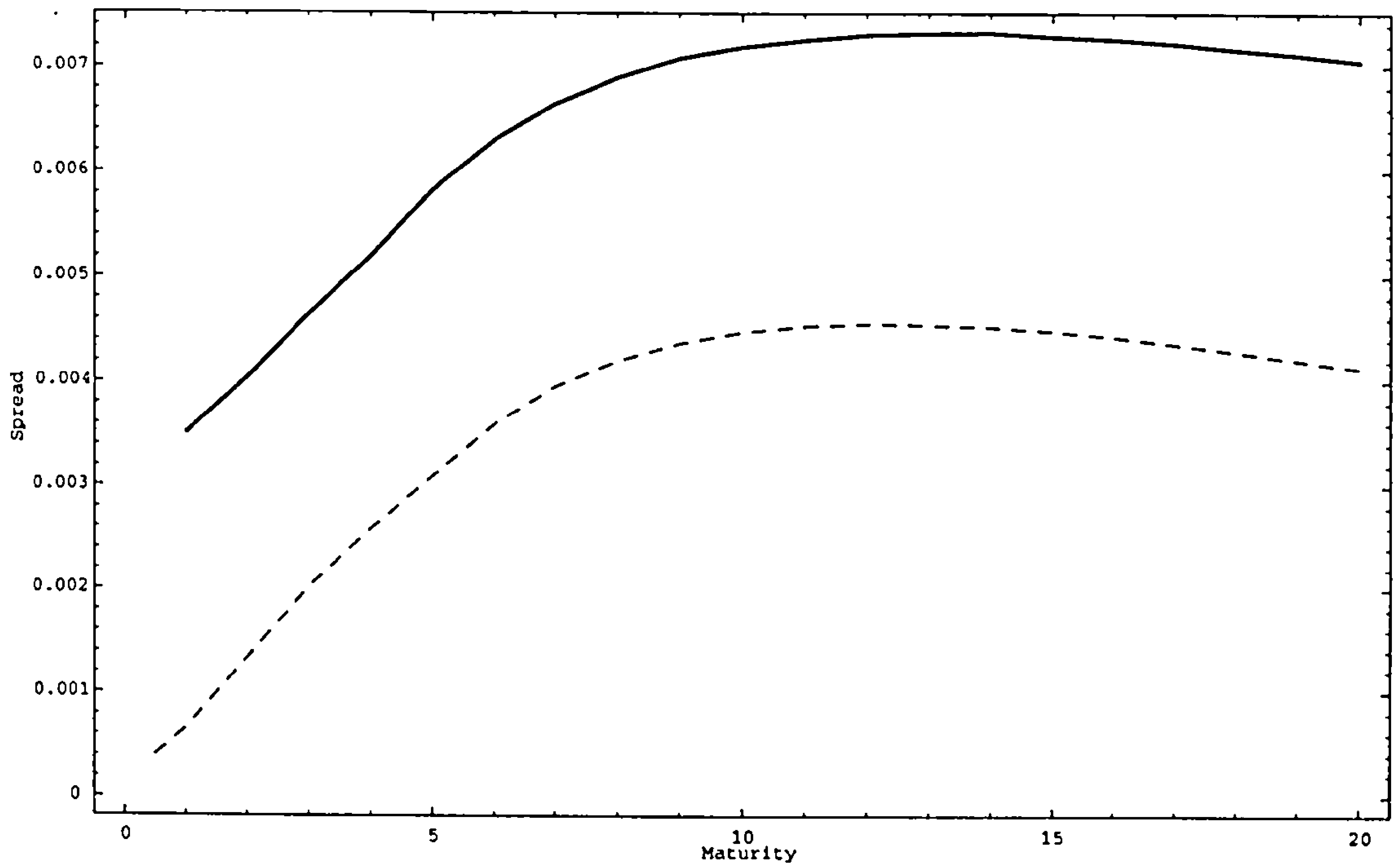

Figure 6.7: **Term structure of credit spread under a jump-diffusion process for face values K =50 with tax effects.**
This figure shows the term structure of credit spreads (solid line) of a 5% (*after-tax rate*) coupon bond when the underlying process follows a jump-diffusion process. The level of credit spread without taking tax effects into account is shown in dashed line. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, after-tax $c = 0.05$, $V_0 = 100$, $\lambda^* = 0.05$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\overline{k}^* = -9.5\%$. Total variance per unit time remains constant in all cases.

This is because the spread levels remain the same while the fraction of premium due to default risk becomes smaller when $K = 50$. Secondly, the spread levels are closer[17] to the empirical averages of yield spreads on high-grade and on medium-grade bonds as found in Kim et al. [1993]. The model with tax effects is more capable of producing realistic spread levels.

Figure 6.9 shows that the effects of downward jumps on default boundaries are fractional. The term structure of default barrier remains nearly stationary, even when an average jump size in asset value is $\overline{k}^* = -9.5\%$ under the influence of downward jumps. The implication is that spread levels rise, as downward jumps accelerate default mechanism by increasing the probability of bankruptcy, rather than raising

[17]The remaining discrepancies may be due to the illiquidity of corporate bonds.

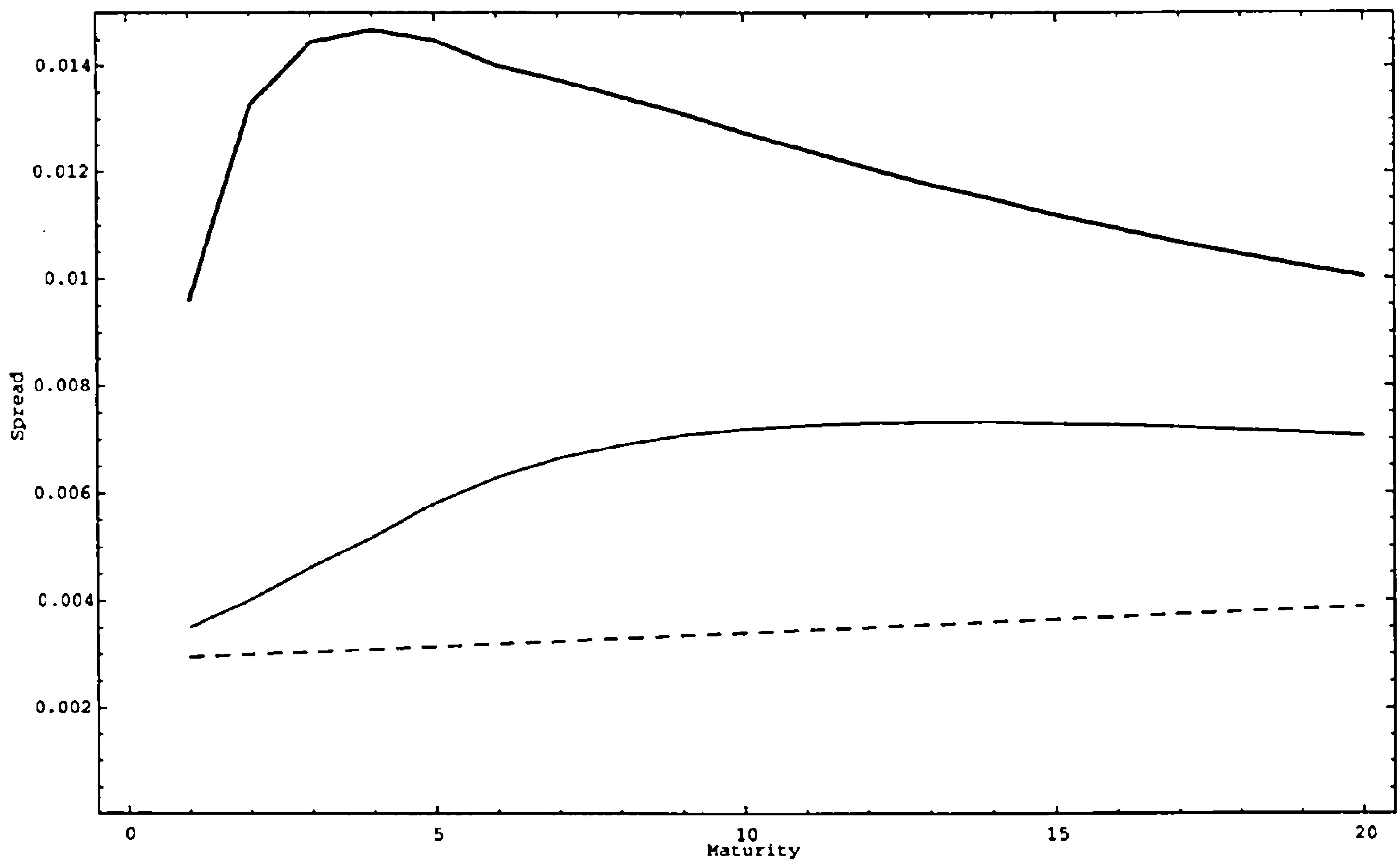

Figure 6.8: **Term structures of credit spread under a jump-diffusion process for different face values of K =70 and K =50 with tax effects.**
This figure shows the term structures of credit spreads of a 5% (*after-tax rate*) coupon bonds with different face values: (i) K =70 (thick solid line) and (ii) K =50 (thin solid line), when the underlying process follows a jump-diffusion process. The level of credit spread generated by pure tax effects is shown in dashed line. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, after-tax $c = 0.05$, $\tau = 4.875\%$, $V_0 = 100$, $\lambda^* = 0.05$, and $\sigma_k = 0.25$, unless stated otherwise. Total variance per unit time remains constant in all cases.

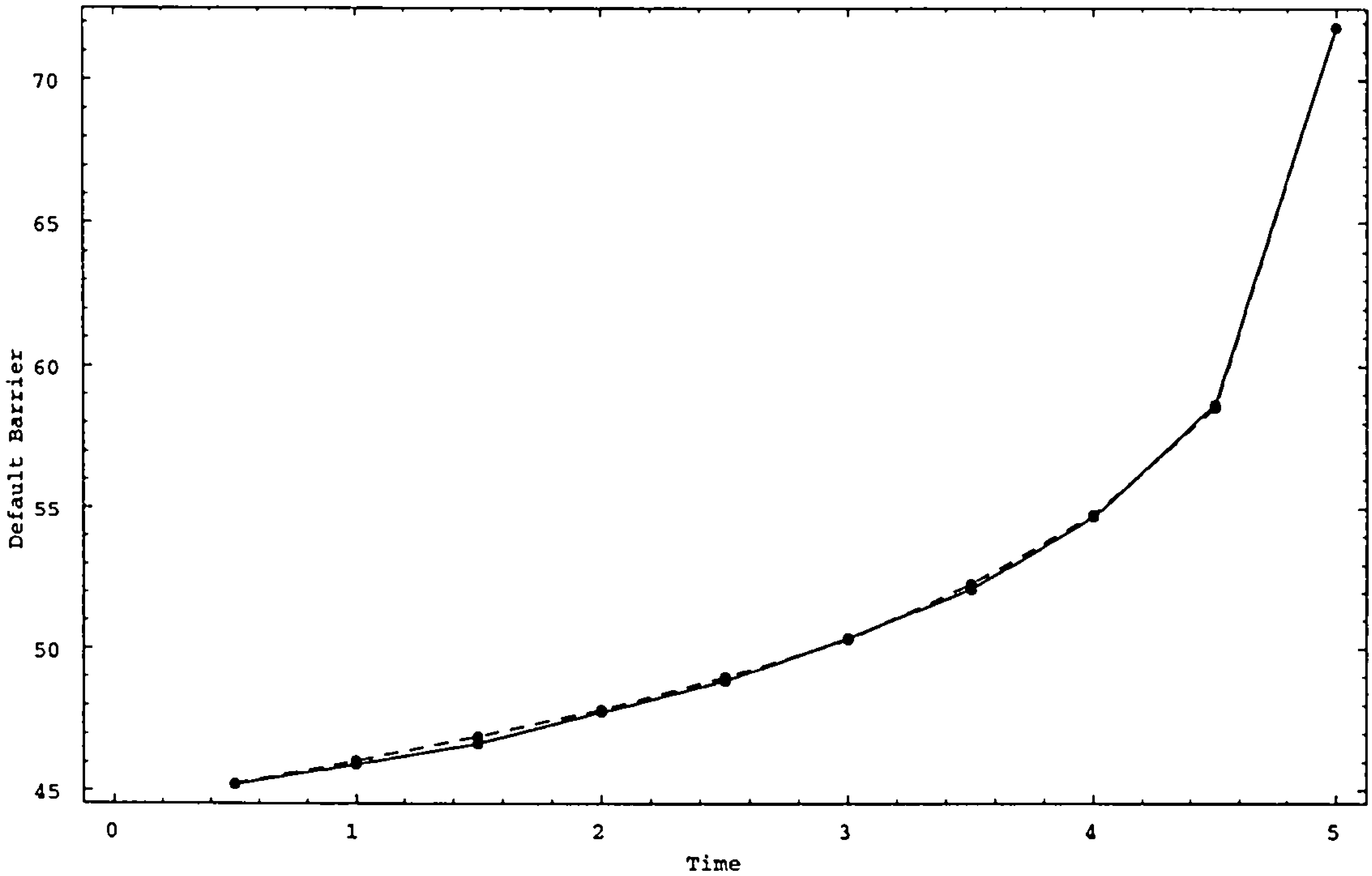

Figure 6.9: **Default boundaries under a jump-diffusion process for different values of $\overline{k}^*$ =0 and -9.5%.**
This figure shows the default boundaries of a 5-year 5% (*after-tax rate*) coupon bonds with different values of $\overline{k}^*$:(i) $\overline{k}^*$ =0 (dashed line), (ii) $\overline{k}^* = -9.5\%$ (solid line), when the underlying process follows a jump-diffusion process. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, after-tax rate $c = 0.05$, $\tau = 4.875\%$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, and $\sigma_k = 0.25$. Total variance per unit time remains constant in all cases.

the barrier to a higher level. The same result holds for cases where different face values K are used.

We investigate the properties of spread levels and default barriers when there is a change in effective tax rate. We assume that the distribution of state taxes remains unchanged. When we reduce the federal tax rate by 5% to 30%, there is an increase in effective tax rate to $\tau = 5.25\%$. We plot Figures 6.10 and 6.11 to indicate the effects of a change in effective tax rate. It is observed that the spread level becomes higher as a result of the increase in τ. Furthermore, the default barrier is raised to a higher level. This can be explained as follows. In order to avoid bankruptcy, the asset value has to be maintained at such a level that the stock price immediately after a coupon date is at least worth the coupon payment. The bankruptcy boundary is determined at the asset level where stock price is equal to the coupon payment. In case such a default is imminent, total debt value increases slightly because of the refund of capital loss in the event of bankruptcy. As a consequence, the rise in effective tax rate raises the bankruptcy barrier. Furthermore, as the firm's value drops to a low level, tax shelter for coupon payments will not be fully realized.[18] While how firms consider taxes in making default decisions is a complicated issue, our result suggests that a change in the federal tax rate may be a factor for earlier default of low-grade bonds.

Finally, the effect of dividends is shown in Figure 6.12, where we plot the term structures of credit spread under a jump-diffusion process for different values of dividend payout rate $\delta = 7\%$ and 5%. It is evident that a small change in dividend rates can have significant and persistent effects on spread levels.

6.6 Summary

This chapter has compared the structural framework of bond pricing models under a jump-diffusion process with those under a pure diffusion process. We have employed a tractable, discrete time model for the valuation of defaultable coupon bonds when the underlying firm value process follows a jump-diffusion process. The method yields a framework which adopts only simple mathematics. It appears that the modelling of a firm's total asset value as a jump-diffusion process can provide a more realistic model of spread levels which, unlike diffusion based models, does not go to zero for short maturities. This is because the jump-diffusion model enables us to generate leptokurtic (fat-tailed) distribution for firm's asset values. We have also found that negative jumps can have significant and persistent effects on spread levels.

[18]Under U.S. tax codes, to benefit fully with tax shelter, the firm must have earnings before interest and taxes that are not less than total coupon payments. When default is imminent, it is quite possible that profits will be less than the coupon payout and tax savings will not be fully realized. See Leland [1994].

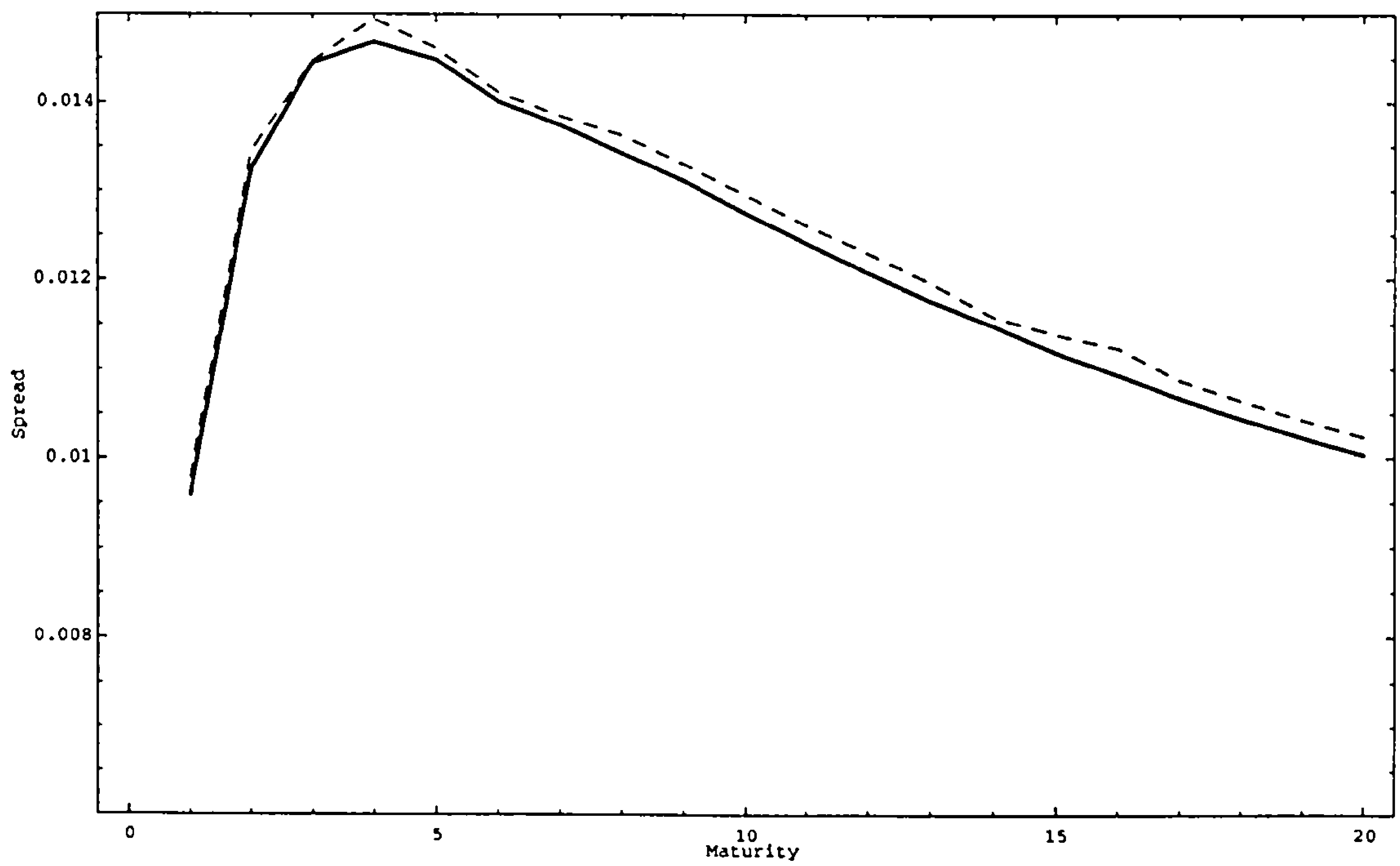

Figure 6.10: **Term structures of credit spread under a jump-diffusion process for different values of $\tau = 4.875\%$ and 5.25%.**

This figure shows the term structures of credit spreads of a 5% (*after-tax rate*) coupon bonds with different values of τ:(i) $\tau = 4.875\%$ (solid line), (ii) $\tau = 5.25\%$ (dashed line), when the underlying process follows a jump-diffusion process. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, after-tax $c = 0.05$, $\tau = 4.875\%$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, $\sigma_k = 0.25$, $\gamma^* = -0.1$, and $\overline{k}^* = -9.5\%$.

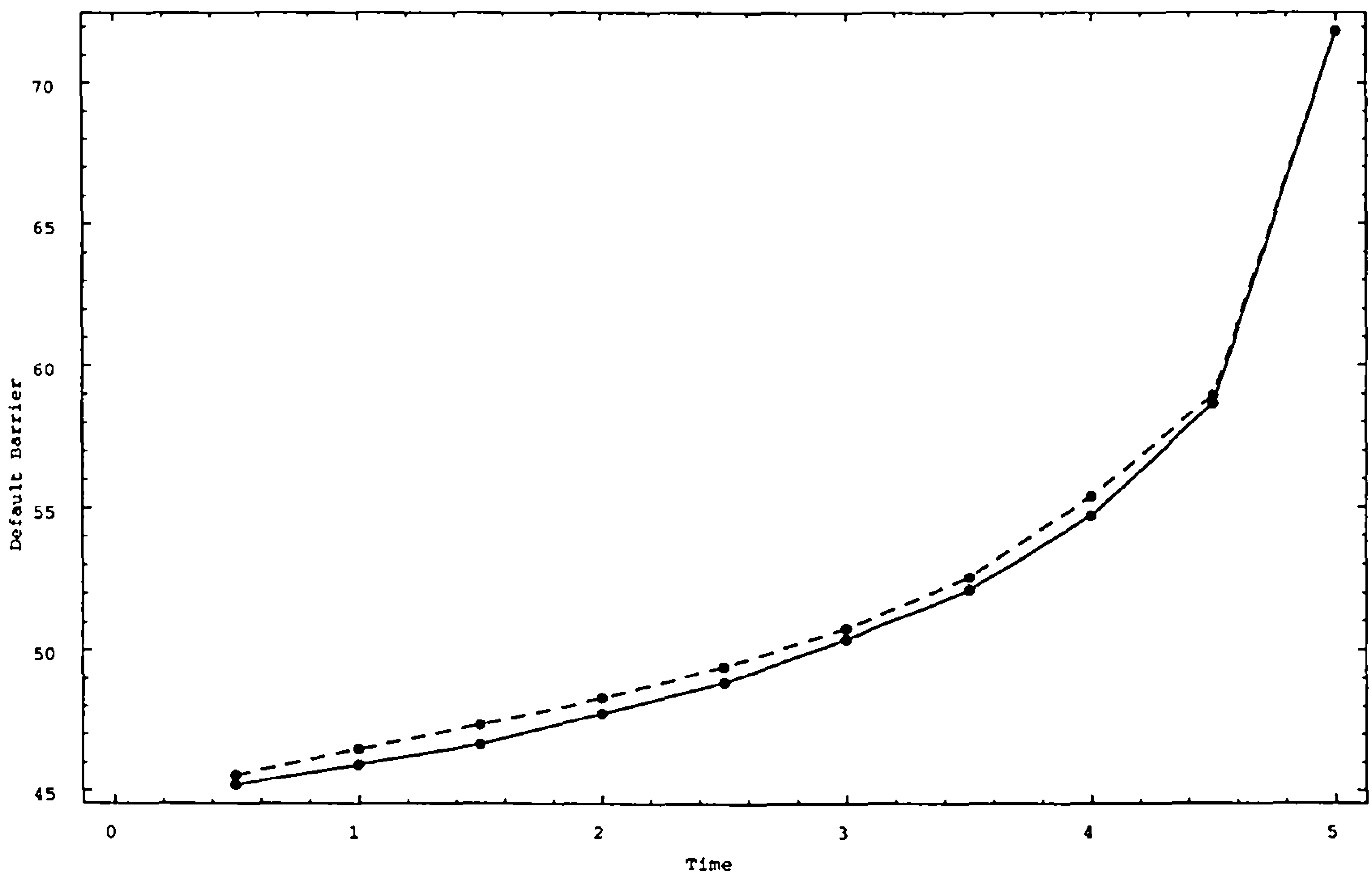

Figure 6.11: **Default boundaries under a jump-diffusion process for different values of $\tau = 4.875\%$ and 5.25%.**

This figure shows the default boundaries of a 5-year 5% (*after-tax rate*) coupon bonds with different values of τ:(i) $\tau = 4.875\%$ (solid line), (ii) $\tau = 5.25\%$ (dashed line), when the underlying process follows a jump-diffusion process. Parameter values: $r_0 = 0.08$, $\delta = 0.07$, $\sigma = 0.2$, after-tax rate $c = 0.05$, $\tau = 4.875\%$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, and $\sigma_k = 0.25$.

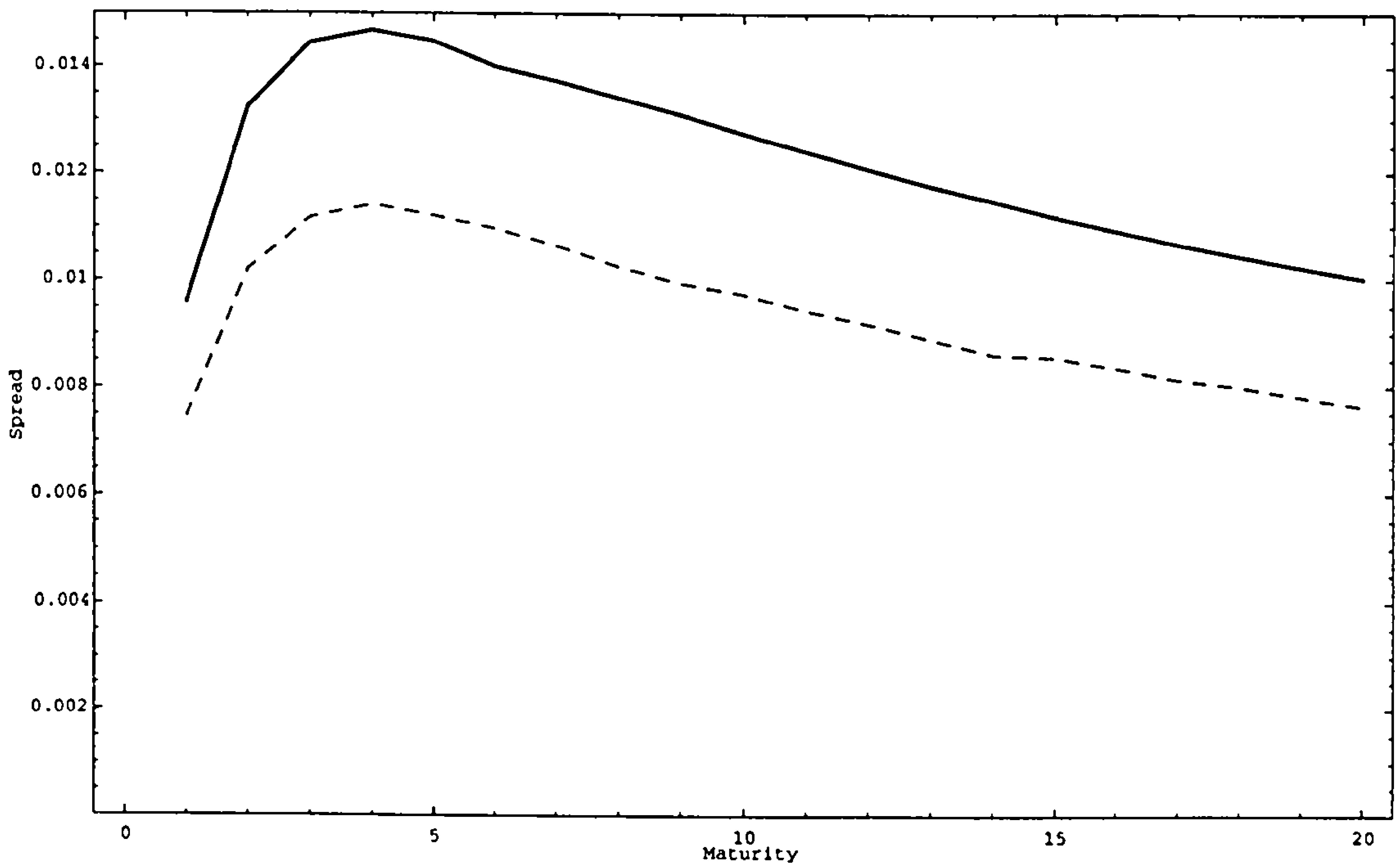

Figure 6.12: **Term structures of credit spread under a jump-diffusion process for different values of dividend payout rate $\delta = 7\%$ and 5%.**
This figure shows the term structures of credit spreads of a 5% (*after-tax rate*) coupon bonds with different values of delta:(i) $\delta = 7\%$ (solid line), (ii) $\delta = 5\%$ (dashed line), when the underlying process follows a jump-diffusion process. Parameter values: $r_0 = 0.08$, $\sigma = 0.2$, after-tax rate $c = 0.05$, $\tau = 4.875\%$, $V_0 = 100$, $K = 70$, $\lambda^* = 0.05$, and $\sigma_k = 0.25$.

As in Geske [1977], we assume that a default event can only happen on payment dates. Bankruptcy can happen *expectedly* when the asset level hits a certain lower barrier through a continuous diffusion crossing, or *unexpectedly* when its value drops precipitously below the barrier. Consistent with Leland and Toft [1996], and Leland [1994, 1998], the default boundary is determined endogenously by requiring the value of equity to be at least the amount of the coupon just paid, in order to avoid bankruptcy. However, alternative bankruptcy criteria can also be used in our model. For example, our model is flexible enough to accommodate a given lower reorganization boundary, below which the firm will be unable to pay contractual coupons.

The effects of jumps on the levels of credit spreads can be significant and persistent over time. The effects on spread levels are conspicuous for short maturities. The higher the jump frequency and variability, the higher the short-term spreads. For long maturities, credit spreads are not distinguishable from those generated by a pure diffusion model. However, when downward jumps are of higher volatility, the effects on credit spreads become more persistent. Furthermore, if the downward jumps are systematic, there is a tendency to underestimate the spread levels. This result may partly explain why without taking the systematic jump risk into account, Merton-type models tend to underestimate the credit spreads.

Other important factors include taxes and dividends. State taxes have been ignored in almost all modelling of defaultable bonds. As a further contribution, this chapter has introduced the important factor of tax into the model. As motivated by Elton et al. [2000], we have shown that taxes do have significant and persistent effects for bonds with long maturities. In fact, credit spread increases with effective tax rate on coupon payments. Tax effects appear to be the second most important factor for spread levels, as documented in Elton et al. [2000]. We have also found that while downward jumps in firm value increase the probability of default, the bankruptcy boundary does not seem to be affected.

We have also investigated the effects of state and federal taxes on default mechanisms. Assuming the distribution of state taxes remains unchanged, we have shown that a change in the federal tax rate may be a factor for earlier default of low-grade bonds. Finally, we have found that dividend payout rates can have significant and persistent effects on spread levels. With deployment of the additional factors of taxes and dividends, the jump-diffusion model has been shown to be more flexible than pure diffusion ones in fitting empirical spreads. It remains to be seen whether it is sufficiently flexible and sufficiently easy to fit for it to be useful in empirical work.

CHAPTER 7
A Reduced-form Model incorporating Fundamental Variables

7.1 Introduction

This chapter proposes a reduced-form model of corporate debt, by taking into account stochastic interest rates, a firm's equity values, and hazard rates of default. Through a moving average of a log-transformation of equity prices, we introduce structural characteristics of the firm into the model. This is an innovation that provides a compromise between the structural and the reduced-form approaches. We investigate the properties and flexibility of the model for pricing corporate debt. Distinguishing features of the model are fourfold. Firstly, as with structural models, the model is able to capture the effects of economic fundamentals on properties of credit spreads. Secondly, as a reduced-form model, it preserves a high degree of flexibility in generating credit spreads. Thirdly, the analytical and tractable form of the model enables researchers to undertake comparative statics and enhance its empirical applicability. Finally, the model can easily be generalized to deal with counterparty default risk.

The literature on pricing risky debt has evolved in two main directions: the structural approach and the reduced-form approach. Pioneered by Merton [1974], the structural approach has taken the dynamics of the assets of the issuing firm as given, and priced corporate bonds as contingent claims on the assets. A vast literature that used and extended Merton's [1974] model includes Black and Cox [1976], Geske [1977], Shimko, Tejima, and Van Deventer [1993], and Longstaff and Schwartz [1995]. Other examples that consider endogenous capital structure, liquidation policy, re-capitalization, and re-organization of debt include Brennan and Schwartz [1984], Leland [1994, 1998], Leland and Toft [1996], Anderson and Sundaresan [1996], and Mella-Baral and Perruadin [1997].

A recent paper by Collin-Dufresne and Goldstein [2001] employs a structural approach to investigate the effects of a firm's capital structure on debt pricing. They propose a structural model of default with stochastic interest rates and the firm's asset values that captures mean-reverting feature of leverage ratios. Effectively, their model allows for an amount of debt to be issued in the future when the firm's asset values increase. They derive the value of a risky discount bond in the form of an infinite series in line with Longstaff and Schwartz's [1995] model. The levels of credit spread generated appear to be more consistent with empirical findings.

The structural approach to the valuation of risky debt has been criticized for not being able to generate sufficient credit spreads for small maturities of debt. Although these structural models can answer questions about the implications for debt pricing in changes of firm-specific variables such as debt restructuring, this important feature is compromised by their inability to generate realistic credit spreads for short maturity bonds. In practice, even for small maturities, the market does not neglect the possibility that some disaster may happen. As noted in Kim, Ramaswamy, and Sundaresan [1993], realistic values of leverage and the volatility of the value of firm asset seem incapable of producing the credit spreads that are actually observed in the market.

In contrast to the structural models, the literature has adopted an alternative approach that offers a high degree of tractability for credit risky bonds. This reduced-form approach bypasses the complications of handling a firm's economic fundamentals, and deals directly with market prices and spreads. The method involves relating default time to the stopping time of an exogenously given hazard rate process. Models in this area include those of Jarrow and Turnbull [1995], Jarrow, Lando, and Turnbull [1994], Lando [1995], Madan and Unal [1998], Duffee [1999], and Duffie and Singleton [1999].

There have been many applications of Duffie and Singleton's framework [1999] in the literature. Lando [1998] illustrates how doubly stochastic Poisson processes, also known as Cox processes, can be applied to model prices of financial instruments in which credit risk is a significant factor. The idea is based on Duffie and Singleton's [1999] model with the specification of a hazard rate process as a Cox process. Because of the general nature of Cox processes, Lando's [1998] approach allows default characteristics of firms, such as rating transitions, to be captured into his model.

Duffie and Huang [1996] apply Duffie and Singleton's model to price swaps with counterparties of different default risks. A switching-type, default-adjusted short rate process is used depending on whether the swap value is positive or negative. Asymmetric default risk of the counterparties and non-linearity of promised cash flows are then explored. Another application of Duffie and Singleton's model [1999] is a recent paper by Jarrow and Yu [2001]. The paper studies the impact of counterparty default risk on the pricing of defaultable securities, where correlated defaults due to an exposure of common risk factors and firm-specific risks are considered. As with Duffie and Huang's [1996] model, Jarrow and Yu [2001] specify in their models switching-type hazard rate processes depending on which counterparties have gone bankrupt. In principle, a framework with multiple layers of counterparty relationship can potentially be applied to pricing defaultable securities.

A major advantage of reduced-form models is that they provide us with a model that is close to the data, and it is always possible to fit some version of the model. However, the fitted model may not perform well on "out of sample" analysis. Another potential drawback in the construction of an underlying hazard rate process is that these models lack a connection of a firm's economic fundamentals to default events. As a consequence, they provide no guidance of structural interpretation in the changes of firm-specific variables. Firm-specific risk and financial fundamentals are not evaluated and may even be ignored.

In addition to the basic incompatibility in the default mechanisms of the two approaches that we have discussed in the Chapter 6, Section 6.1, there is another key theoretical difference between them. A structural model completely rules out the use of a hazard rate process that is common in the reduced-form approach, and such a structural model implies a hazard rate that would be zero before default and infinite at default. Madan and Unal [1998] have come up with a reduced-form model whose hazard rate process concurs with the diffusion-based structural approach in this respect. However, the model still lacks an interpretation of a firm's structural characteristics.

To capture the effects of capital structure, a hybrid-type model has been suggested in the literature. Madan and Unal [2000] propose a two-factor hazard rate model to price risky debt. Consistent with the hazard rate literature, the probability of sudden default is governed by the hazard rate. They derive the hazard rate function endogenously in terms of the firm's non-interest sensitive asset values and default-free interest rates. Assuming that default follows a Poisson arrival rate and loss in the case of default has a cumulative distribution function, they come up with a structural definition of the hazard rate process as a product of the two quantities. Although the structural approach is appealing, they fail to obtain an exact analytical solution for the bond price. Instead, an analytical approximation is derived after they express the hazard rate function as a first-order approximation of its Taylor expansion. Other attempts to introduce structural properties into the reduced-form framework include Cathcart and El-Jahel [1998], Jarrow [2001], Jarrow and Turnbull [2000], and Hübner [2001]. In this chapter, we extend their results by incorporating current and lagged effects of individual stocks into the pricing of corporate bonds.

This chapter proposes a reduced-form model of corporate debt by taking into account stochastic interest rates, a firm's asset values, and hazard rates of default. Consistent with the literature of the reduced-form models, we assume that default can only happen unexpectedly. As in Duffie and Singleton [1999] and Duffee's [1999] work, we take a hazard rate process as exogenously given. Unlike those models, there is a crucial distinction in the specification of the process in our model. We introduce structural characteristics of the firm into the hazard rate process, through a factor providing a measure of a firm's performance in equity. The use of such a measure has two important features. Firstly, instead of solely using a firm's current value as conventional Merton-type models do, we take the past performance of the firm into account. Unlike Madan and Unal [2000], we employ relative values of *observable* equity prices to measure a firm's performance as well as leverage effect. Secondly, having high equity values alone may not necessarily be a good indicator of a firm's creditworthiness. In our model, we take a broader view of the financial health of a firm by considering the current asset level relative to its past positions. The debt becomes more risky when the relative levels are lower; when the relative levels are higher, the debt becomes safer. As a consequence, a peculiar feature of financial markets that news on corporate earnings is normally reflected in equity prices first, and then bond prices, can be captured in our model.

The objectives of this chapter are as follows. We seek to propose a flexible model of corporate debt in analytical form. The structural characteristics of a firm and stochastic interest rates are taken into account. Our crucial assumption is that the default hazard of the firm, unlike the structural approach, depends on the current relative price of equity to its recent past levels. We consider a moving average of logarithm of recent stock prices. The use of this measure is an innovative idea that allows economic fundamentals of the firm to be captured in the hazard rate process, and hence bond prices. Three features are noteworthy. Firstly, as with other structural models, we show the structural impact of interest rate movements and their correlation with equity returns on the pricing of risky debt. For example, we demonstrate that the levels of spread increase with interest rate volatility, equity return volatility, and the correlation. Secondly, as a reduced-form model, the model preserves a high degree of flexibility in generating credit spreads. Numerical computations show that the model is flexible enough to generate many different term structures of credit spreads by using appropriately chosen parameters. We investigate analytically how parameter values affect the shape of credit spread curve in terms of its intercept, slope at zero maturity, and spread level for long maturity. Finally, the analytical and tractable form of the model enables researchers to undertake comparative statics and enhance its empirical applicability.

The chapter is divided into eight sections. In the next section, we state in advance a main result of this chapter. We postpone detailed discussion of economic implications and construction of underlying processes to Section 7.3 and 7.4. Section 7.5 shows the short- and long-term behaviour of credit spreads, and their relationships to the structural characteristics of the firm. Emphasis is placed on the flexibility of the model in generating credit spreads in relation to model parameters. A method of empirical calibration of the model is discussed in Section 7.6. Section 7.7 shows how we can extend the model to deal with counterparty default risk, whose impact on credit spreads is also presented. Finally, we conclude in Section 7.8 with a summary and a discussion of further research.

7.2 The Model

For ease of exposition, we state in advance the solution of our model in this section, and postpone detailed discussion of economic implications and construction of underlying processes to Section 7.3 and 7.4. We consider a risky zero-coupon bond of unit face value and maturity date T. The default-free interest rate process is r_t. Consistent with Duffie and Singleton [1999], we suppose that default occurs randomly, and that the risky debt has a risk-neutral short spread process s_t. We also allow economic fundamentals of the firm to be captured in the short spread, through a process Y_t. For construction and interpretations of processes s_t and Y_t, please refer to Section 7.4.

Given that these processes s_t, Y_t, and r_t have the following affine representations $(*)$:

$$(*)\begin{cases} dr_t = k_r(\theta_r - r_t)\ dt + \sigma_r\ dB^r_t, \\ dY_t = (r_t - \alpha Y_t - \sigma_s^2/2 - a)\ dt + \sigma_s\rho\ dB^r_t + \sigma_s\sqrt{1-\rho^2}\ dB^s_t, \textit{and} \\ ds_t = (\delta\theta_h + k_h s_t + \delta k_{hy}Y_t + \delta k_{hr}r_t)dt + \delta\sigma_{hr}dB^r_t + \delta\sigma_{hs}dB^s_t + \sigma_h\sqrt{\delta}\sqrt{s_t}dB^h_t, \end{cases}$$

where B^r_t, B^s_t, and B^h_t are independent standard Brownian motions, then as in Duffie and Kan [1996], we assume that the bond price can be expressed in exponential-affine form in terms of the three factors. The time-t_0 price, $D(t_0,T)$, of the risky bond is of the form:

$$D(t_0,T) = \exp\left(A(t_0,T) + B_1(t_0,T)s_{t_0} + B_2(t_0,T)Y_{t_0} + B_3(t_0,T)r_{t_0}\right). \qquad (7.1)$$

Now we state a main result in this chapter as follows:

Proposition 7.2.1 *Suppose that the bond price satisfies equation (7.1).*[1] *Then*

$$\begin{aligned} B_1(t_0,T) &= -\frac{2[1-e^{-\sqrt{k_h^2+2\delta\sigma_h^2}(T-t_0)}]}{(\sqrt{k_h^2+2\delta\sigma_h^2}-k_h) + (\sqrt{k_h^2+2\delta\sigma_h^2}+k_h)e^{-\sqrt{k_h^2+2\delta\sigma_h^2}(T-t_0)}}, \\ B_2(t_0,T) &= \delta k_{hy}\int_{t_0}^T e^{-\alpha(u-t_0)}B_1(u,T)du, \\ B_3(t_0,T) &= -\int_{t_0}^T e^{-k_r(u-t_0)}\Big[1 - \delta k_{hr}B_1(u,T) - B_2(u,T)\Big]du, \ \textit{and} \\ A(t_0,T) &= \int_{t_0}^T \Big[\delta\theta_h B_1(u,T) - (\sigma_s^2/2+a)B_2(u,T) + k_r\theta_r B_3(u,T) + \frac{1}{2}\sigma_s^2 B_2(u,T)^2 \\ &\quad + \frac{1}{2}\sigma_r^2 B_3(u,T)^2 + \frac{\delta^2}{2}(\sigma_{hr}^2+\sigma_{hs}^2)B_1(u,T)^2 + \sigma_r\delta\sigma_{hr}B_1(u,T)B_3(u,T) \end{aligned}$$

[1] $B_2(t_0,T)$ and $B_3(t_0,T)$ are expressible in terms of a hypergeometric function and its integrals. A hypergeometric function ${}_2F_1$ can be written as an integral:

$${}_2F_1(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\ dt.$$

$$+ \sigma_r \sigma_s \rho B_2(u,T) B_3(u,T) + \delta(\sigma_s \sigma_{hr} \rho + \sigma_s \sigma_{hs} \sqrt{1-\rho^2}) B_1(u,T) \times B_2(u,T) \Big] du.$$

Proof: See Appendix G. ■

Proposition 7.2.1 shows that the bond price has an analytical form which can be expressed in terms of hypergeometric functions of the type ${}_2F_1(\cdot,\cdot,\cdot,\cdot)$. In addition to tractability, the analytical form of the model also enables us to undertake comparative statics analysis and empirical research. Furthermore, the influences of interest rates, a firm's economic fundamentals, as well as the probabilities of default are synthesized into the price of the risky debt in equation (7.1).

7.3 The Framework

In this section, we set out an overall structure of the model. We assume a frictionless economy with a prevailing default-free interest rate r_t. Under a risk-neutral measure, the evolution of the short rate follows a Vasicek process [1977]:

$$dr_t = k_r(\theta_r - r_t)\ dt + \sigma_r\ dB_t^r, \tag{7.2}$$

where B_t^r is a standard Brownian motion under the risk-neutral measure, k_r is the speed at which the interest rate r_t tends to its long term mean θ_r, and σ_r is the volatility of changes in the instantaneous interest rate.

We consider a firm whose equity has value S_t, which follows a diffusion process with constant volatility of rate of return:

$$dS_t/S_t = (r_t - a)\ dt + \sigma_s \rho\ dB_t^r + \sigma_s \sqrt{1-\rho^2}\ dB_t^s, \tag{7.3}$$

where B_t^s is another standard Brownian motion under the same measure, and is independent of B_t^r, a is the total dividend rate to shareholders, σ_s is the volatility of equity returns, and ρ is the correlation between the increments of r_t and the instantaneous returns of equity.

We consider a risky zero-coupon bond, issued by the firm, of unit face value and maturity date T. Consistent with Duffie and Singleton [1999], we suppose that default occurs at a random time τ, $\tau < T$, and that the corresponding risk-neutral hazard rate process is h_t. Assuming that in the event of default, the mean fractional loss of the market value of the claim is a constant δ, where $0 < \delta < 1$, then the short spread can be expressed as:[2] [3]

$$s_t = \delta h_t. \tag{7.4}$$

According to Duffie and Singleton [1999], the time-t_0 price, $D(t_0, T)$, of the risky bond is of the following form:

$$D(t_0, T) = E_{t_0}\Big[\exp(-\int_{t_0}^{T} R_u \, du)\Big], \tag{7.5}$$

where $R_t = r_t + s_t$. The intuition behind the model is as follows. By discounting at the adjusted short rate R_t, the model accounts for both the probability and timing of default, as well as for the loss effects on default. Furthermore, the bond corresponds to having a thinned default intensity δh_t, and a recovery rate of *zero* in the event of default. Hence the bond can be valued alternatively as in the Cox process case in Lando [1998].

Before we prove Proposition 7.2.1, we need to specify other processes through which the firm's economic fundamentals are incorporated into the model. The next section achieves this by relating the equity prices to the instantaneous hazard rate.

7.4 Modelling of the Hazard Rate Process

Having high equity values alone may not necessarily provide a good indicator of firm's creditworthiness. Directors of a firm may consider issuing more debt after

[2] There are four different formulations of the loss function are suggested in the literature. The default pay-off is either a fraction of (i) par (Madan and Unal [1998]), (ii) par plus accrued interest (J.P. Morgan [1999]; Jarrow and Turnbull [2000]), (iii) a risk-free bond with the same structure of cash flows (Jarrow and Turnbull [1995]), and (iv) the market value of the security just prior to default (Duffie and Singleton [1999]). In this chapter, we adopt Duffie and Singleton's [1999] approach and assume that the loss rate is a constant fraction of the bond price immediately before default.

[3] The mean loss rate is heavily dependent on the seniority of the bond.

realizing an increase in the firm's asset values, but the increase in debt levels would bring extra risk into the firm's capital structure. On the contrary, they may consider reducing debt levels by issuing new equity when realising a continual decline in the equity prices, the firm's debt-equity ratio would subsequently be lowered. This observation is consistent with Malitz's [1994] findings that bond covenants typically allow directors to have a degree of flexibility in changing the debt levels in the future.

In our model, we take a broader view of the financial health of the firm to allow for such a dynamic restructuring of capital structure, by considering the current equity level relative to its past positions. The debt becomes more risky when the relative levels are lower; when the relative levels are higher, the debt becomes safer. With this motivation, we consider a continuous moving average M_t of $\log(S_t)$:

$$dM_t = \alpha\Big(\log(S_t) - M_t\Big)\ dt, \tag{7.6}$$

where $\alpha > 0$ is a smoothing parameter. Note that instead of taking averages of the equity prices, we define M_t as the continuous moving average of $\log(S_t)$ for the sake of tractability. To solve equation (7.6) for $t \geq s \geq 0$, we have the following expression:

$$M_t = e^{-\alpha(t-s)}M_s + \int_s^t \alpha e^{-\alpha(t-u)} \log(S_u)\ du. \tag{7.7}$$

This variable has been employed in the literature of bond pricing and stochastic volatility models. Instead of using equity prices, Collin-Dufresne and Goldstein [2001] structure a log-default threshold in a similar way by considering a firm's asset values. Tompkins [2000] has shown that the exponentially-weighted return series of futures prices on a stock index is significantly related to the volatility of the futures prices, and hence leverage effect.[4] Equation (7.7) is a straightforward generalization of exponential moving average models in discrete case. This expression shows that

[4]Tompkins [2000] uses this variable as an attribute to measure leverage effect. He found that recent relative prices are negatively correlated to the series of 20-day unconditional volatility of stock index futures. This result is consistent with the negative leverage effects that Christie [1982] has pointed out for individual stocks.

the moving average M_t depends on the equity in two manners: (i) the equity prices before time s, and (ii) those entering the system from time s to t. More precisely, M_t is a continuous exponentially-weighted mean of its value at time s and all values of $\log(S_u)$ between time s and t, for which the weights are $e^{-\alpha(t-s)}$ and $1 - e^{-\alpha(t-s)}$ respectively. It is evident that the higher the value of α, the more the moving average is dependent on the recent values of equity price. The value of α must be chosen to ensure that the current value of M_t does not depend overwhelmingly on those in the past. We will show later in this chapter how the choice of α affects the term structure of credit spreads.

We define a measure of the relative levels of equity price as:

$$Y_t = \log(S_t) - M_t.$$

This variable measures how far $\log(S_t)$ is from its recent mean level, and provides an attribute to indicate the firm's business outlook. Since, as documented in Kwan [1996], both current and lagged values of equity return have been shown to have impact on changes in bond yields, we incorporate these empirical properties into our model and postulate that the structural characteristics of the firm enter into the prices of risky bond through the process Y_t in the following way:

$$dh_t = (\theta_h + k_h h_t + k_{hy} Y_t + k_{hr} r_t)dt + \sigma_{hr} dB_t^r + \sigma_{hs} dB_t^s + \sigma_h \sqrt{h_t} dB_t^h, \qquad (7.8)$$

where B_t^h is a standard Brownian motion independent of B_t^r and B_t^s. Using equation (7.4), the short spread s_t follows the following process:

$$ds_t = (\delta\theta_h + k_h s_t + \delta k_{hy} Y_t + \delta k_{hr} r_t)dt + \delta\sigma_{hr} dB_t^r + \delta\sigma_{hs} dB_t^s + \sigma_h \sqrt{\delta}\sqrt{s_t} dB_t^h. \quad (7.9)$$

It is interesting to note the role of Y_t in the processes (7.8) and (7.9). The presence of the process Y_t is a structural difference between the short spread process (7.9) and many others that have been suggested in the literature. For example, Duffee [1999] applies Duffie and Singleton's [1999] idea to fit yields on bonds issued by individual investment-grade firms to a reduced-form model, in which no factors of firm's economic fundamentals are taken into account.[5] Four important features are captured in our setting:

(i) Both the hazard rate and the short spread are modelled as square-root processes.[6] We know from the work of Longstaff and Schwartz [1995b] that credit spread displays a significant amount of stability. To be consistent with this property, mean reverting feature of credit spreads is incorporated into the model by specifying $k_h < 0$;

(ii) As documented in Duffee [1998], yield spreads for high-quality firms are positive, even at the short end of spread curve. This suggests that there is a positive spread at zero maturity, which is $s_{t_0} = \delta h_{t_0}$ in the model;

(iii) The short spreads are stochastic, fluctuating with the firm's structural characteristics, captured by Y_t. It is interesting to note that any latent variable with the same structural and mathematical properties can be employed in the place of Y_t; and

[5]Duffee [1999] considers a three-factor model in which the instantaneous, default-free short rate process r_t is assumed to be a linear combination of two square-root diffusion processes. Short spread s_t is modelled as another linear combination of three square-root diffusions, where two of them are the same as those in the short rate process. No factors of firm's economic fundamentals are taken into account. An analytical form of solution for bond prices is obtained. Although empirical results appear to be encouraging as the average error in fitting corporate bond yields is less than 10 basis points, Duffie and Singleton [1999] argue that the models used by Duffee [1999] are theoretically incapable of capturing the negative correlation between credit spreads and U.S. Treasury yields while maintaining non-negative default hazard rates. They succeed in coming up with an alternative model with more flexible correlation structures for $(r_t,\ s_t)$, but the system cannot be solved analytically for bond prices. We discuss a method of solution for our model in Appendix G.

[6]In this formulation, the risk-neutral hazard-rate and short spread processes can become negative. However, it can be shown by Monte-Carlo simulations that when θ_h is sufficiently large, it is unlikely for the processes to hit 0. In particular, for the numerical examples in this chapter, Monte-Carlo simulations show that if we assume that the true hazard rate process is of the form: $h_t^* = max\{h_t, 0\}$, then our model tends to underestimate the true levels of credit spreads by no more than 10 basis points. Therefore, given the tractability of the subsequent expressions, this is an acceptable approximation.

(iv) The short spreads can be structured to be systematically related to variations in the default-free term structure, as documented in empirical literature.[7]

It is important to investigate the properties of processes (7.8) and (7.9) in relation to the process Y_t, α, and other parameters. Recall that M_t is defined as a mean of M_s and $\log(S_u)$ from time s to t, weighted by $e^{-\alpha(t-s)}$ and $1 - e^{-\alpha(t-s)}$ respectively. A noticeable feature is that the contribution of M_s becomes negligible when $\alpha(t-s)$ is large. The larger the value of α, the more significant the contribution of the second term to the overall average M_t. As a consequence, the averaging is mainly performed on $\log(S_u)$ from time s to t. An intuition is that when equity prices are continually rising, Y_t tends to be positive. However, when equity prices are continually declining, Y_t tends to be negative. Such a property of Y_t provides us with a clue as to the appropriate signs of k_{hy} and k_{hr}. To capture the property of negative correlation between the interest rate r_t and the short spread s_t, we specify that k_{hy}, $k_{hr} \leq 0$. On the other hand, a positive value of σ_{hr} induces positive correlation between the increments of r_t and s_t. By construction, this model also has a fairly high degree of flexibility in correlation structures. We will discuss the flexibility of the model further in Section 7.5.

Before we finish this section, we state in the following proposition that the specification of Y_t is affine. Hence, together with previous results, we have structured the framework in terms of the three main processes, r_t, s_t, and Y_t, in affine representations. We are now ready to present a proof of Proposition 7.2.1, see Appendix G.

Proposition 7.4.1 *The process Y_t satisfies the following stochastic differential equation:*

$$dY_t = (r_t - \alpha Y_t - \sigma_s^2/2 - a)\ dt + \sigma_s \rho\ dB_t^r + \sigma_s \sqrt{1-\rho^2}\ dB_t^s. \tag{7.10}$$

[7]See Duffee [1998], and Longstaff and Schwartz [1995a] for empirical justifications.

Proof: Rewrite process (7.3) as

$$d\log(S_t) = (r_t - \sigma_s^2/2 - a)\ dt + \sigma_s\rho\ dB_t^r + \sigma_s\sqrt{1-\rho^2}\ dB_t^s.$$

By definitions of Y_t and M_t, the result follows. ∎

7.5 Properties of Credit Spreads

To better understand the impacts of the underlying processes on risky debt, we conduct an analysis of credit spreads as follows. Let the credit spread $s(t_0, T)$ be the difference in yields between the risky bond $D(t_0, T)$ and default free bond $B(t_0, T)$. Then

$$s(t_0, T) = \frac{\log\Big(B(t_0,T)/D(t_0,T)\Big)}{T - t_0},$$

where

$$\begin{aligned} B(t_0,T) &= \exp\Big(- B_0(t_0,T)r_{t_0} + \frac{(B_0(t_0,T) - T + t_0)(k_r^2\theta_r - \sigma_r^2/2)}{k_r^2} - \frac{\sigma_r^2 B_0(t_0,T)^2}{4k_r}\Big), \\ B_0(t_0,T) &= \frac{1 - e^{-k_r(T-t_0)}}{k_r}. \end{aligned}$$

By proposition 7.2.1, credit spread $s(t_0, T)$ is of the following form:

$$\begin{aligned} s(t_0,T) &= -\frac{A(t_0,T)}{T-t_0} - \frac{B_1(t_0,T)}{T-t_0}s_{t_0} - \frac{B_2(t_0,T)}{T-t_0}Y_{t_0} - \frac{B_3(t_0,T)}{T-t_0}r_{t_0} \\ &\quad -\frac{B_0(t_0,T)}{T-t_0}r_{t_0} + \frac{(B_0(t_0,T)/(T-t_0) - 1)(k_r^2\theta_r - \sigma_r^2/2)}{k_r^2} \\ &\quad -\frac{\sigma_r^2 B_0(t_0,T)^2}{4k_r(T-t_0)}. \end{aligned} \tag{7.11}$$

It is evident that the spread $s(t_0, T)$ is a linear function depending on the current states of economy, r_{t_0}, s_{t_0}, and Y_{t_0}. The role of Y_{t_0} in the spread function appears to concur with a finding in the work of Kwan [1996] that current bond yield changes are negatively correlated with the issuing firm's current and lagged stock returns, and so firm-specific information tends to be embedded first into individual stock prices and then reflected in individual bond prices. Furthermore, the linear relationship of our model can potentially capture both the specific and systematic risks of the firm. This is an important point, as shown in Elton, Gruber, Agrawal, and Mann [2000], that the most significant components of credit spreads result from expected default risk, taxes, and systematic risk in the stock market.[8]

To evaluate the effects of the three factors on the yield spreads, we state the short and long-term properties of the spread function $s(t_0, T)$ in the following proposition:

Proposition 7.5.1 *The spread function $s(t_0, T)$ has the following properties:*

(i) Short-term level of spreads $s(t_0, t_0) = \delta h_{t_0}$,

(ii) Short-term slope of the spread curve $= \delta\theta_h + k_h s_{t_0} + \delta k_{hy} Y_{t_0} + \delta k_{hr} r_{t_0}$,

(iii) Long-term properties of $s(t_0, T)$ are stationary, and

$$\begin{aligned}\lim_{T\to\infty} s(t_0, T) &= -\delta\theta_h l_1 + (\sigma_s^2/2 + a)l_2 - \frac{1}{2}\sigma_s^2 l_2^2 - \frac{1}{2}\sigma_r^2 l_3^2 - \frac{\delta^2}{2}(\sigma_{hr}^2 + \sigma_{hs}^2)l_1^2 \\ &\quad -\sigma_r \delta \sigma_{hr} l_1 l_3 - \sigma_r \sigma_s \rho l_2 l_3 - \delta(\sigma_s \sigma_{hr}\rho + \sigma_s \sigma_{hs}\sqrt{1-\rho^2})l_1 l_2 \\ &\quad -k_r \theta_r l_3 - \frac{k_r^2\theta_r - \sigma_r^2/2}{k_r^2},\end{aligned}$$

where

$l_1 = \frac{-2}{\sqrt{k_h^2 + 2\delta\sigma_h^2} - k_h}$,

$l_2 = \frac{\delta k_{hy} l_1}{a}$, *and*

$l_3 = -\frac{1}{k_r} + \frac{\delta k_{hr} l_1}{k_r} + \frac{l_2}{k_r}$.

Proof: See Appendix H. ■

As discussed in Section 7.4, our model preserves the property that the short-term spreads are positive. This is the case if the firm has a non-zero loss rate and

[8]Elton et al. [2000] show that almost all of the differences between government and corporate credit yields are explained by expected default risk, taxes, and systematic risk. We neglect tax effects in our discount bond model.

positive default hazard for short maturity debt. More interestingly, the slope of short-term spread is a linear combination of *initial* values of zero maturity spread, interest rate, and the relative position of the equity price to its average of past equity levels. The dependence of the *current* state of economy allows us to generate richer term structures of credit spread. Assuming that $\theta_h > 0$, k_h, k_{hy}, and $k_{hr} \leq 0$, the spread curve tends to be upward (or download) sloping when $Y_{t_0} < 0$ (or $Y_{t_0} > 0$). This concurs with our intuition that there is a tendency for the spreads to rise when a firm's equity is continually declining. On the contrary, when the firm's equity is continually rising, the spreads tend to be sloping downward. On the other hand, the current interest rates have similar effects on the slope of short term spreads. Such empirical properties have been documented by Duffee [1998], who demonstrates that non-callable bond yield spreads fall when the levels of Treasury term structure rises. Furthermore, the extent of the decline depends on the initial credit quality of the bond. Duffee [1998] shows that the decline is small for high-grade bonds and large for low-grade bonds. By appropriate choices of k_h, k_{hy}, k_{hr}, and Y_{t_0}, the model appears to have a high degree of flexibility in reconciling these empirical results. For long-term debt, the levels of spread are stationary and independent of the current states of economy. The spread levels depend only on the present estimates of parameters. For example, it is trivial to observe that the levels of long-term spread increase with the value of θ_h.

In the following, we illustrate the model by numerical results. In order to study the properties of credit spreads, we consider a particular case of the model with a base case environment in which the parameters take the following values: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\sigma_s = 0.2$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = 0$, and $r_{t_0} = 0.05$. In this case, we are assuming $Y_{t_0} = 0$, that there are no particular substantial upward or downward movements in recent equity prices. Also we specify the model in such a way that while it becomes simpler as $k_{hr} = 0$, $\sigma_{hr} = 0$, and $\sigma_{hs} = 0$, it is rich enough to capture a negative correlation between the interest rate and the spread movements.

Figures 7.1, 7.2, and 7.3 illustrate structural properties of the model. The plots show that the movements of spread are similar to those of Merton-type frameworks, as documented, for example, in Shimko, Tejima, and Van Deventer [1993]. Figure 7.1 shows that the levels of spread tend to increase with the correlation. Figure 7.3 shows that equity return volatility has a significant impact on the levels of credit spread. The effects of equity volatility tend to increase the spreads through the Y_{t_0} term. Furthermore, interest rate volatility has a similar effect on spread levels.

The effects of α are demonstrated in Figures 7.4 and 7.5. Figure 7.4 shows that in the case where there is a recent decline in equity prices and α is increasing, the spreads tend to move toward to the level of spread (dashed line with dots) generated in the case where $k_{hy} = 0$. Similar results are shown in Figure 7.5, where equity prices are assumed to be continually rising. The rationale behind this is as follows. Recall that the higher the value of α, the greater the dependence of the moving average on the recent values of equity price. When α is increasing, the value of Y_t tends to move to zero, and hence the effects of Y_t in the hazard rate process and short rate process vanish.

Figures 7.4 and 7.5 also illustrate that the spread curves tend to be upward (or downward) sloping when the equity prices are continually declining (or rising). The result is trivial as, by Proposition 7.5.1, the slope of a spread curve at short maturity is negatively related to Y_{t_0}. Furthermore, it appears that the terms Y_t in processes (7.8) and (7.9) have another importance in generating spread curves with a humped shape. Numerical computations show that this is likely to be the case, as the slope at short maturity tends to be positive when equity prices are recently declining.

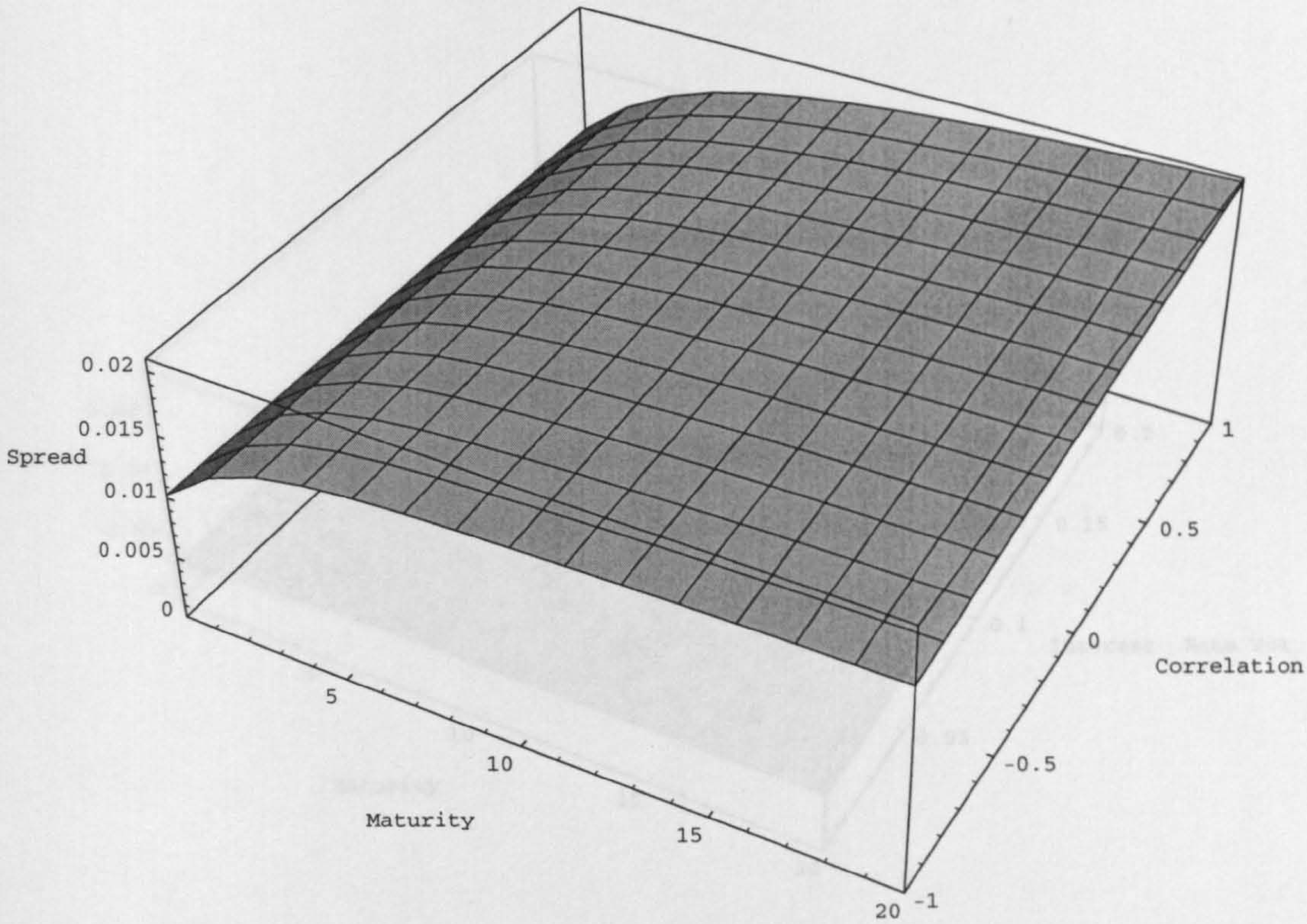

Figure 7.1: **Credit spread surface as a function of maturity T and correlation ρ.**

This figure shows the term structures of credit spreads of a risky discount bond with different values of correlation ρ: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\sigma_s = 0.2$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = 0$, and $r_{t_0} = 0.05$, unless stated otherwise.

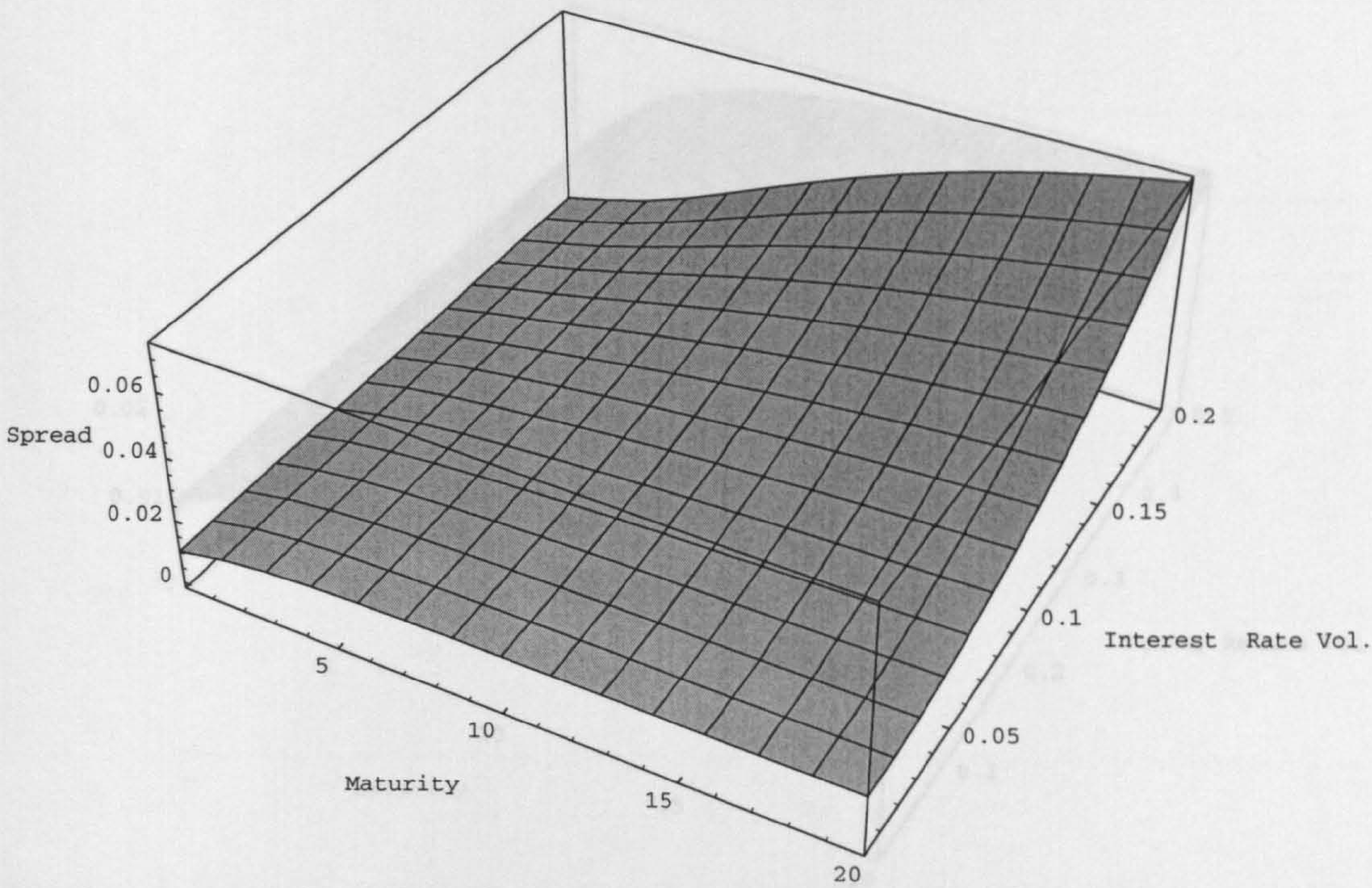

Figure 7.2: **Credit spread surface as a function of maturity T and interest rate volatility σ_r.**
This figure shows the term structures of credit spreads of a risky discount bond with different values of interest rate volatility σ_r: $k_r = 0.2$, $\theta_r = 0.06$, $a = 0.07$, $\sigma_s = 0.2$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = 0$, and $r_{t_0} = 0.05$, unless stated otherwise.

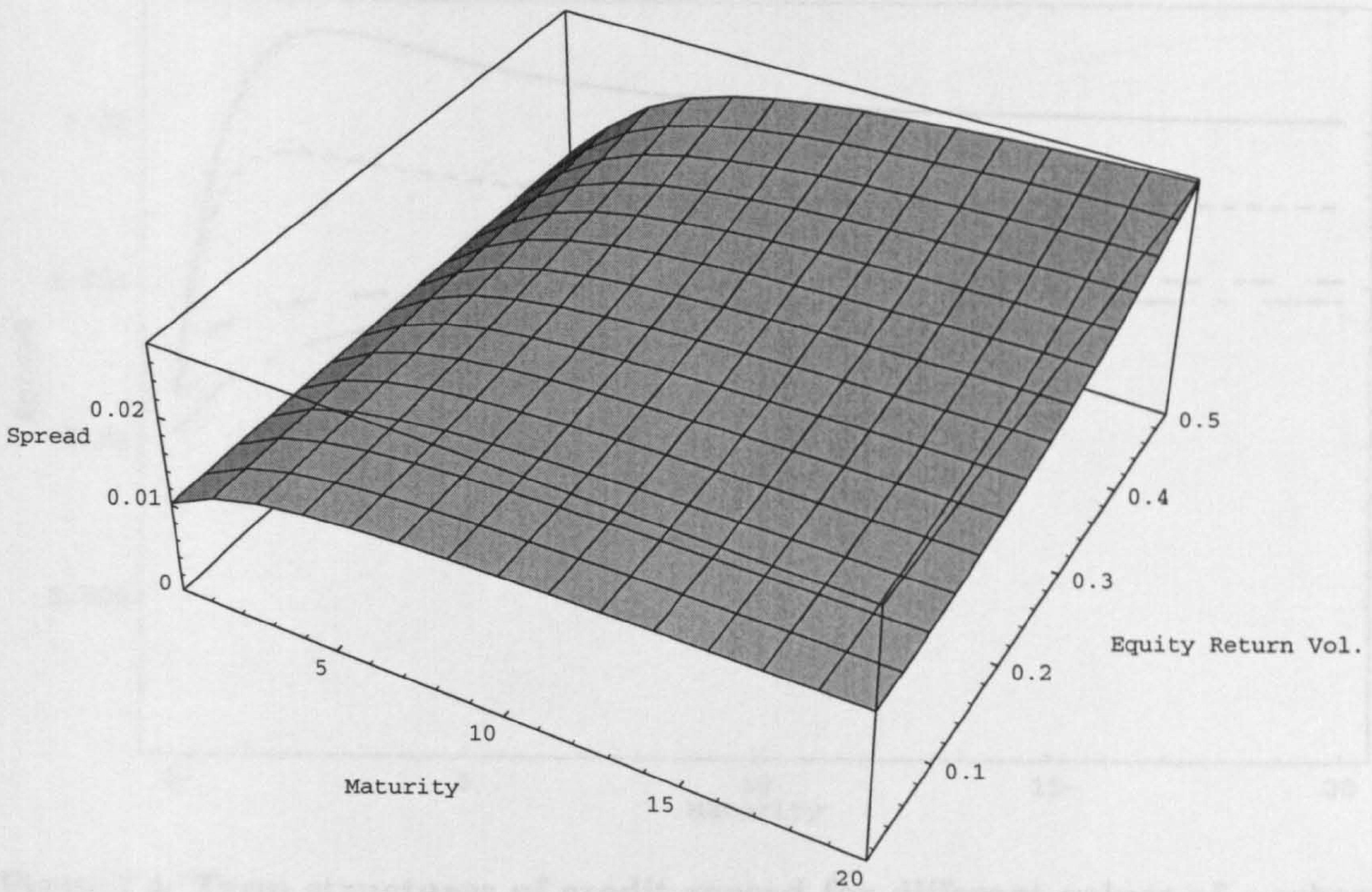

Figure 7.3: **Credit spread surface as a function of maturity T and equity return volatility σ_s.**
This figure shows the term structures of credit spreads of a risky discount bond with different values of equity return volatility σ_s: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = 0$, and $r_{t_0} = 0.05$, unless stated otherwise.

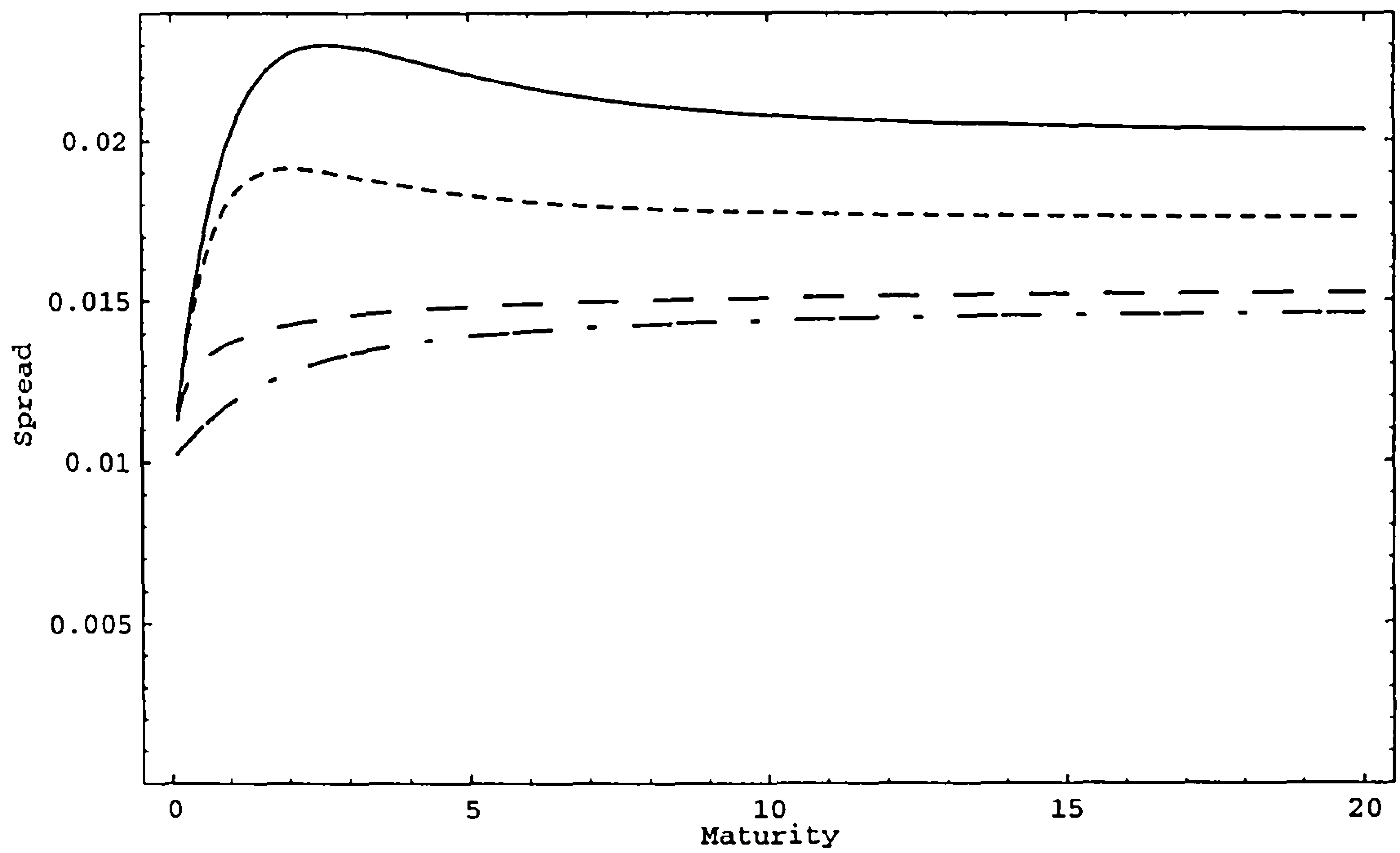

Figure 7.4: **Term structures of credit spread for different values of α when $Y_{t_0} < 0$.**

This figure shows the term structures of credit spreads of a risky discount bond with different values of α: (i) $\alpha = 1$ (solid line), (ii) $\alpha = 2$ (short dashed line), (iii) $\alpha = 10$ (long dashed line), and (iv) $k_{hy} = 0$ (dashed line with dots). Parameter values: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\sigma_s = 0.2$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = -0.3$, and $r_{t_0} = 0.05$, unless stated otherwise.

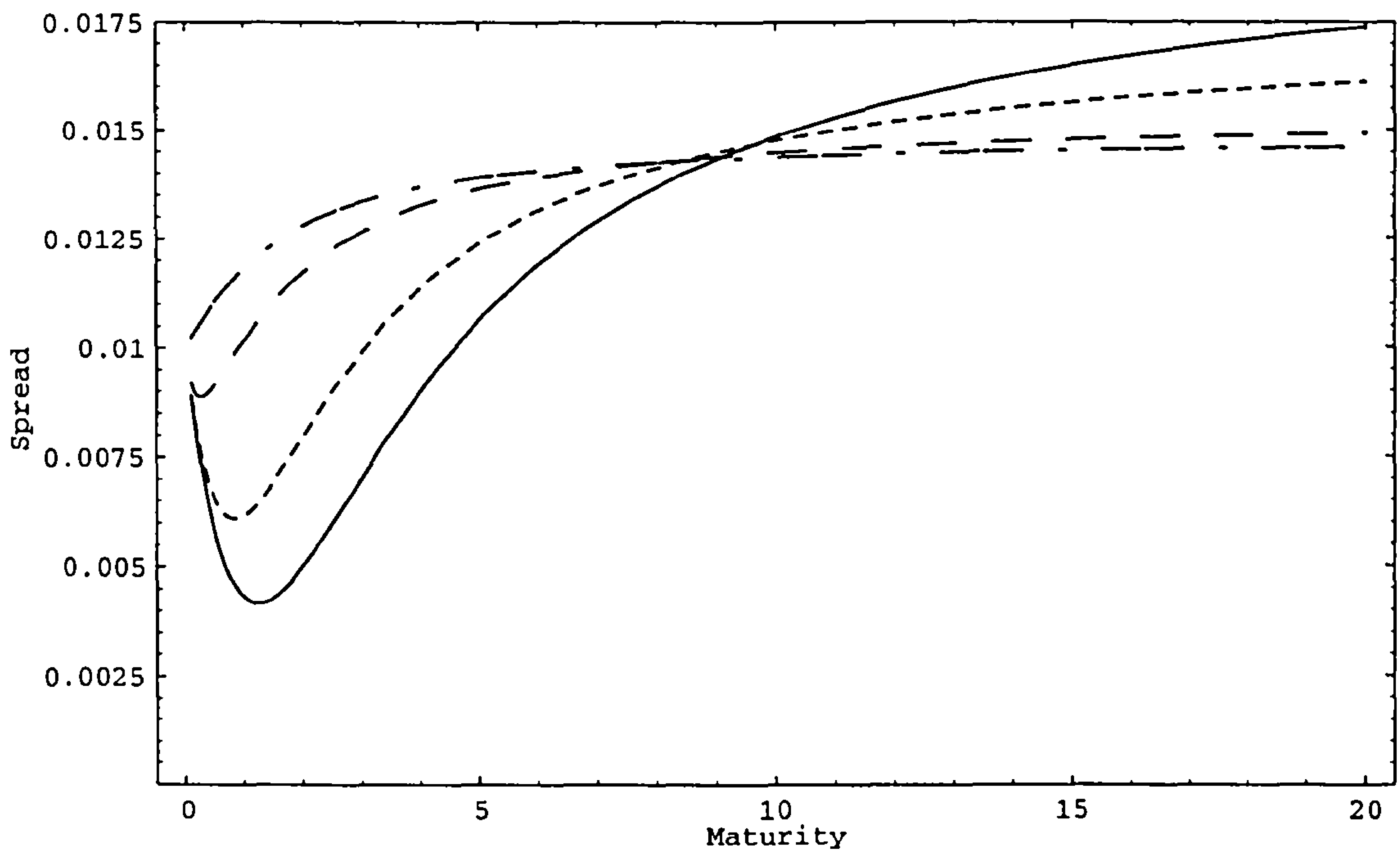

Figure 7.5: **Term structures of credit spread for different values of α when $Y_{t_0} > 0$.**
This figure shows the term structures of credit spreads of a risky discount bond with different values of α: (i) $\alpha = 1$ (solid line), (ii) $\alpha = 2$ (short dashed line), (iii) $\alpha = 10$ (long dashed line), and (iv) $k_{hy} = 0$ (dashed line with dots). Parameter values: $k_r = 0.2$, $\theta_r = 0.06$, $\sigma_r = 0.031$, $a = 0.07$, $\sigma_s = 0.2$, $\rho = 0.1$, $\theta_h = 0.03$, $k_h = -1$, $k_{hy} = -0.2$, $k_{hr} = 0$, $\sigma_{hr} = 0$, $\sigma_{hs} = 0$, $\sigma_h = 0.2$, $\alpha = 1$, $\delta = 0.5$, $t_0 = 0$, $h_{t_0} = 0.02$, $Y_{t_0} = +0.3$, and $r_{t_0} = 0.05$, unless stated otherwise.

7.6 A Method of Empirical Calibration

A noticeable feature of the discount bond model is its analytical tractability. This enables researchers to undertake comparative statics and enhance the empirical applicability of the model. In this section, we propose a method of empirical calibration of the model. For details of the method, refer to Chen and Scott [1995], Duan and Simonato [1995], Harvey [1989], Lund [1997a], and Lund [1997b].

The estimation procedure consists of three main steps: estimation of (i) the default-free process parameters, (ii) the equity process parameters, and (iii) the hazard rate process parameters.

For step (i), we suggest using the method of linear Kalman Filtering. The data set is assumed to contain time-series yields of Treasury zero-coupon bonds of different maturities. This method is feasible as the short rate process (7.2) is Gaussian and the bond yields are linear in the state variable of short rate.[9]

Given time-series data of a firm's common stock prices, we can estimate the average dividend rate a and volatility of stock returns σ_s. The correlation ρ between the short rates (7.2) and equity prices (7.3) can be estimated by considering the time-series of the Treasury yields and the stock prices.

For step (iii), we consider two cases depending on whether or not in the same seniority of risky debt there are plenty of coupon bonds of different maturities. If so, a procedure for stripping risky zero-coupon bond prices can be employed to strip out the zeros, whose yields can be computed.[10] In this case, after some modifications of the state variable distributions, the estimation procedure continues as described in step (i).(See Appendix I.1).

[9] For details, see Harvey [1989] and Lund [1997a].
[10] See Jarrow, Lando, and Turnbull [1997].

However, when there are only few coupon bonds available in the same class of seniority, the estimation procedure becomes more complicated. Instead, an extended Kalman filter procedure is used to extract information about the hazard rate process. We assume that time-series prices of a corporate bond issued by the same firm are available. Simple no-arbitrage arguments show that the price of the risky coupon bond is the sum of the prices of the individual coupon payments and principle payments. Each of these prices is a discounted value of the corresponding promised payment, which is given by our model in equation (7.1) and Proposition 7.2.1. Unlike step (i), a non-linear procedure is required for parameter estimation. As the model is exponential-affine, we suggest using the method of iterative extended Kalman filtering employed by Lund [1997b]. Here, the analytical tractability of our model becomes crucial because it allows derivatives to be derived analytically for the efficient iterative scheme. (See Appendix I for details).

There is a point of importance as to the above estimation procedure. Given a value of the market loss rate δ, we are able to estimate the hazard rate process from corporate bond prices.[11] Hence we can compute analytically survival probabilities of the firm for a time horizon. However, as discussed in Duffie and Singleton [1999], if δ is unknown and has to be estimated from data, we also have the same identification problem of the market loss rate δ and the hazard rate process h_t from our model. The reason for this is evident from the structure of equation (7.9). For any positive value of $\delta < 1$, it is always possible to choose other parameter values so as to make the whole process (7.9) remain unchanged. Trivially, the problem may be resolved by imposing further restrictions on the parameters of the hazard rate process (7.8), as, for example, suggested by Jarrow [2001] where equity and bond prices are used to segregate the two variables. While it remains to be seen whether Jarrow's method can be justified

[11] Recovery rates are usually recorded as fractions of face value. Here, as an approximation, we can take δ as a recovery rate at a fraction of the face value. Duffie and Singleton [1999] have shown numerically that differences between the two formulations of recovery rates have little effect on bond prices.

by empirical work, the analytical computations of survival probability of the firm, especially with the hazard rate process linked to underlying fundamental variables, may provide us with deeper insights into understanding the default mechanism.

7.7 An Extension: A Model with Counterparty Default Risk

It is interesting to see how we can extend our model to deal with the default risk of firm's counterparty. In this section, we introduce this element of risk into the model by employing the ideas in Jarrow and Yu [2001]. We consider a simple primary-secondary framework of two firms, A and B. Firm A is a primary firm whose default process depends only on macro-variables. Firm B is a secondary firm having default process dependent on the macro-variables and the default probability of firm A. This assumption can be taken to mean that firm B is holding a significant amount of long (or short) positions of assets issued by firm A, and firm A is not holding any firm B's equity or debt. In principle, default processes of the two firms should be correlated. However, we intend to weaken this assumption, as we are only concerned with the impact of firm A's default risk on the credit spread of a bond issued by firm B. For the sake of technical simplicity, we assume that firm A has a constant default rate process,

$$h_t^A = h^A \ > 0,$$

and the default rate process h_t^B of firm B consists of two parts relating to: (i) its own economic fundamentals, and (ii) a default hazard induced by the default hazard of firm A. We define h_t^B as

$$h_t^B = h_t + p\mathbf{1}_{\{\tau^A \leq t\}},$$

where τ^A is the default time of firm A, and h_t is defined by equation (7.8) and $p > 0$ (or $p < 0$) is a constant. The interpretation of this equation is that firm B is holding some assets issued by firm A, and the default of firm A increases (or decreases) the instantaneous hazard rate of firm B. Furthermore, we can relate the value of p to the nature of underlying assets, since a portfolio of holding a significant amount of long (or short) positions of firm A's assets is normally associated with a large positive (or negative) value of p. Assuming that debt issued by firm B has a loss rate δ of its market value in the event of default,[12] then the short spread process of firm B's debt is of the following form:

$$s_t^B = s_t + \delta p \mathbf{1}_{\{\tau^A \leq t\}},$$

where s_t is defined by equation (7.9). This equation means that the short spread increases (or decreases) by an amount of δp after firm A have gone bankrupt. The price of a discount bond with a unit face value issued by firm B is then:

$$D^B(t_0,T) = D(t_0,T)E_{t_0}\Big[\exp(-\int_{t_0}^{T} \delta p \mathbf{1}_{\{\tau^A \leq u\}}\, du)\Big], \tag{7.12}$$

if firm B has not defaulted by time t_0. By construction, as h^A is assumed to be constant, the bond price can be separated into a product of two parts. The first part is exactly the same as the solution given in equation (7.1) and Proposition 7.2.1. The second part of the price is entirely due to the risk of holding the portfolio of firm A's assets. The following proposition shows how the default risks of firm A affect the credit spread of firm B's debt.

Proposition 7.7.1 *Assuming that the above conditions hold, and that both firm A and B have not defaulted by time t_0, then*

(i) the bond price $D^B(t_0,T)$ of firm B is given by equation (7.12), where

$$E_{t_0}\Big[\exp(-\int_{t_0}^{T} \delta p \mathbf{1}_{\{\tau^A \leq u\}}\, du)\Big] = \begin{cases} \frac{\delta p e^{-h^A(T-t_0)} - h^A e^{-\delta p(T-t_0)}}{\delta p - h^A} & \text{if } h^A \neq \delta p \\ e^{-h^A(T-t_0)}[h^A(T-t_0)+1] & \text{if } h^A = \delta p. \end{cases}$$

(ii) The credit spread $s^B(t_0,T)$ of the risky bond is of the form:

$$s^B(t_0,T) = s(t_0,T) + CS^A(t_0,T),$$

[12] This assumption on the loss of market value is different from Jarrow and Yu's [2001] approach. They assume that the loss rate is a fraction of face value and final payoff is always made at maturity.

where $s(t_0, T)$ is given by equation (7.11), $CS^A(t_0, T)$ is a component of the total spread, due to the default risk of firm A only, and

$$CS^A(t_0,T) = \begin{cases} \frac{-1}{T-t_0}\log\left(\frac{\delta p e^{-h^A(T-t_0)} - h^A e^{-\delta p(T-t_0)}}{\delta p - h^A}\right) & \text{if } h^A \neq \delta p \\ \frac{-1}{T-t_0}\log\left(e^{-h^A(T-t_0)}[h^A(T-t_0)+1]\right) & \text{if } h^A = \delta p. \end{cases}$$

Furthermore, $CS^A(t_0, T)$ has the following properties:

$$\begin{aligned} \lim_{T \to t_0^+} CS^A(t_0,T) &= 0, \text{ and} \\ \lim_{T \to \infty} CS^A(t_0,T) &= Min\{h^A, \delta p\}. \end{aligned}$$

Proof: See Appendix J. ■

Proposition 7.7.1 shows that the default risk of counterparty A has an impact on the credit spread of firm B's debt. Interestingly, the effect is small when the maturity is short; when the maturity is long, an additional amount of $Min\{h^A, \delta p\}$ adds to the level $s(t_0, T)$ of the spreads. The short-end property of the credit spreads is due to the assumption that firm A has not yet defaulted at the issue time t_0. This implies that the zero maturity level of spread curve remains the same as $s_{t_0} = \delta h_{t_0}$. The effect of default risk of the counterparty becomes prominent only for debt of longer maturities. At the long end of the spread curve, the spread level $CS^A(t_0, T)$ is positive if $p > 0$ when firm B is holding the portfolio of long positions of firm A's assets. The spread level $CS^A(t_0, T)$ is negative if $p < 0$ when the holding is largely short positions of firm A's assets.

The long-term properties of the counterparty default risk have an economic implication that can be visualised in the following situations. When the counterparty A has very high hazard of default, firm B would prefer to choose a portfolio in such a way that the value of p is minimized, for example, by maintaining an optimal composition of long and short positions of firm A's assets in the portfolio. On the contrary, when the counterparty A has a very small hazard rate, it would be safe for firm B to hold a large portfolio of firm A's assets. Figure 7.6 shows the term

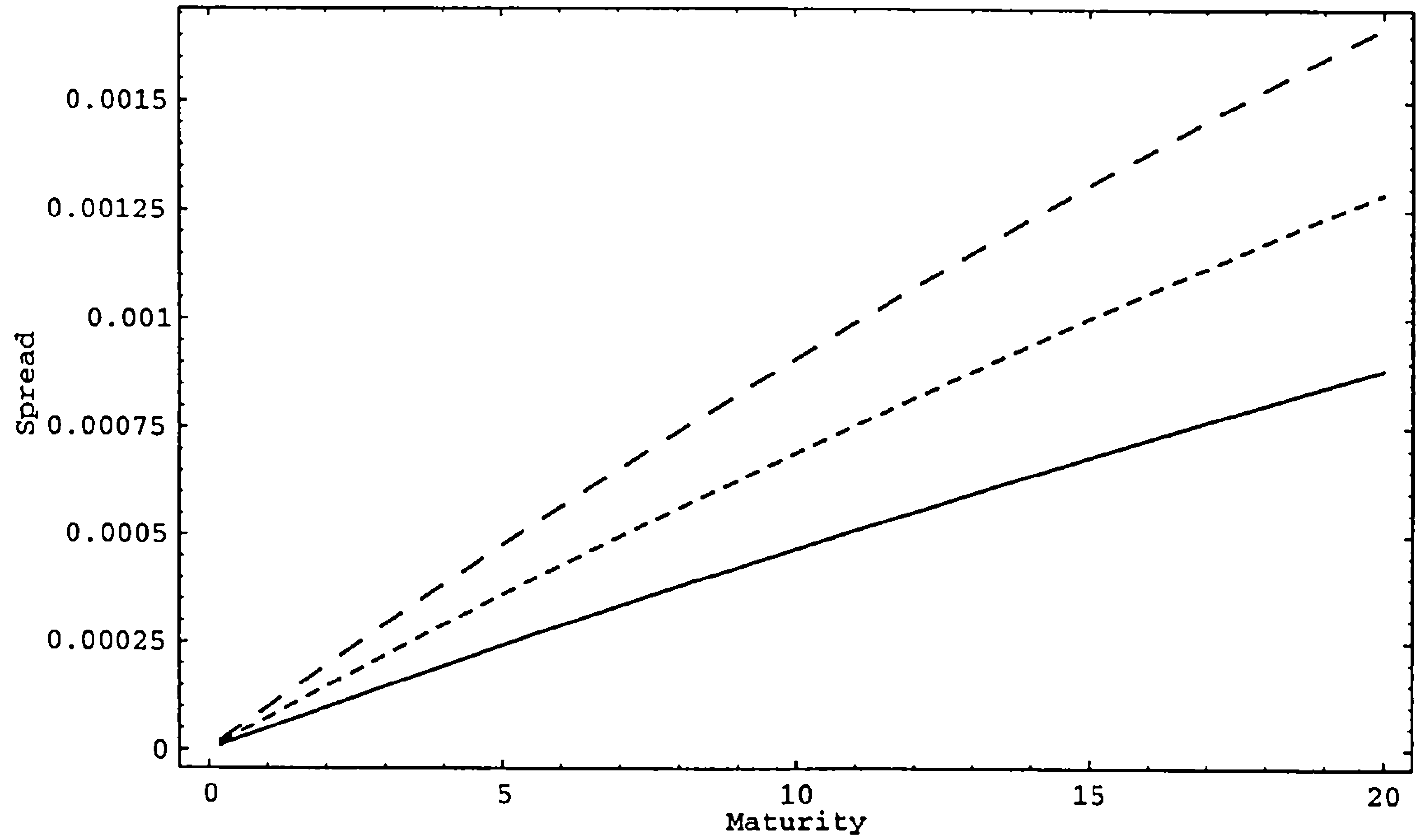

Figure 7.6: **Term structures of credit spread $CS^A(t_0,T)$ for different values of p.**
This figure shows the term structures of credit spreads due to the default risk of firm A for different values of p: (i) $p = 0.02$ (solid line), (ii) $p = 0.03$ (short dashed line), and (iii) $p = 0.04$ (long dashed line). Parameter values: $h^A = 0.01$, $\delta = 0.5$, and $t_0 = 0$.

structures of the spread level CS^A for different values of p. Note that the spread curves $CS^A(t_0,T)$ increase more steeply for larger values of p.

7.8 Summary and Suggestions for Further Research

The literature on pricing risky debt has evolved in two main directions: the structural approach and the reduced-form approach. The two approaches have pros and cons. Although appealing, structural models have been criticized for not being able to generate sufficient credit spreads for small maturities of debt. The models' reliance on economic fundamentals and the value of a firm's asset make them hard to estimate in practice. On the contrary, the reduced-form approach has a major advantage in that it provides us with a model of very high tractability and ease of calibration. However, most reduced-form models have a structural drawback that lacks a connection between a firm's economic fundamentals and default events, although some suggestions for improvement have been put forward in the literature. This motivates us to propose in this chapter the flexible analytical model which provides a compromise between the two approaches.

Our model of corporate debt has taken into account stochastic interest rates, a firm's asset values, and hazard rates of default. Consistent with the literature of the reduced-form models, we have assumed that default can only happen unexpectedly. As in Duffie and Singleton [1999] and Duffee [1999], we take a hazard rate process as exogenously given. Unlike those models, there is a crucial and innovative distinction in the specification of the process in our model. We have introduced structural characteristics of the firm into the hazard rate process, through a moving average providing a measure of the firm's performance in equity as well as its leverage effect. Furthermore, our model also has a fairly high degree of flexibility in correlation structures.

The model has another four important features. Firstly, instead of solely using a firm's current value as the conventional Merton-type and recent Madan and Unal's [2000] models do, we take a broader view of the financial health of the firm by considering the current asset level relative to its past positions. The implication of this is that the debt becomes more risky when the relative levels are lower. When the relative levels are higher, the debt becomes safer. As with other structural models, we have shown that our model is able to capture the effects of economic fundamentals on properties of credit spreads. For example, in a simplified version of the model, we have demonstrated that the levels of spread increase with interest rate volatility, equity return volatility, and the correlation.

Secondly, our model preserves a high degree of flexibility in generating credit spreads. Numerical computations have shown that the model is flexible enough to generate many different term structures of credit spreads by using appropriately chosen parameters. We have investigated analytically how parameter values affect the shape of the credit spread curve, in terms of its intercept, slope at zero maturity, and spread level for long maturity. Thirdly, the analytical and tractable form of the model enables researchers to undertake comparative statics and enhance its empirical applicability.

Fourthly, as an interesting extension, we have demonstrated how we can generalize our model to deal with the default risk of a counterparty. Although in a simple setting, we believe that the extended model has captured essential features of counterparty default risk, whose properties have been shown to have an economic implication in holding a portfolio of assets issued by the counterparty.

The analytical tractability of the model has another advantage. Given the value of the market loss rate δ, we are able to estimate other model parameters, and hence compute analytically survival probabilities of firms. However, as shown in Duffie and Singleton [1999], if δ is unknown, we also have the same identification problem of the market loss rate δ and the hazard rate process h_t from our model. Jarrow [2001] develops a procedure for segregating the two variables by using equity and bond prices. Two recent papers by Janosi, Jarrow, and Yildirim [2001a, 2001b] apply Jarrow's model [2001] to estimate default probabilities implicit in equity prices, and to estimate expected losses and liquidity discounts implicit in debt prices respectively. Liquidity effects can also be incorporated into our reduced-form model similarly as in the work of Janosi, Jarrow, and Yildirim [2001b].

While it remains to be seen whether Jarrow's idea segregating of the two variables can be justified by empirical work, the modelling of survival probabilities of firms in terms of hazard rate processes is an important area of research in credit risk analysis. More importantly, by linking the hazard rate processes to underlying fundamental variables, the result may provide us with deeper insights into understanding the default mechanism.

Finally, the flexibility of our model also paves the way for further generalizations. For example, the modelling of a framework with several counterparties and the pricing of credit default swaps are attractive avenues for further work.

CHAPTER 8

Conclusions and Suggestions for Further Research

8.1 Conclusions

This thesis studies the pricing and credit risk of corporate debt using the structural and the reduced-form approaches. We have discussed the theoretical aspects of three important topics on pricing risky debt: (i) the impact of stochastic interest rates, and hence the interaction between market risk and credit risk; (ii) the impact of diversifiable and non-diversifiable jump risks on pricing and default mechanisms; and (iii) an analysis of a reduced-form model with fundamental variables.

In Chapter 4, we have generalized, in computational aspects, Geske's [1977] and Selby's [1977] valuation models of risky coupon bonds to allow for stochastic firm values and interest rates. By using the hypothetical asset V_t^H as a numeraire, it has been shown that not only does the use of this numeraire significantly simplify the analytic valuation of risky discount bonds, but also gives an implication that the two-factor model can be implemented easily. We have discussed computational efficiency when the hypothetical asset V_t^H is used as a numeraire, and showed that this is an appropriate choice of numeraire. In addition, we have suggested an efficient computation algorithm for the pricing of general risky coupon bonds by generalizing the models proposed by Ho, Stapleton and Subrahmanyam [1995]. Much of the simplicity of this method lies in the fact that the two sources of variability, namely interest rate risk and asset value risk, are combined together to form a single stochastic process Z_t.

Numerical computations have confirmed our belief that firms with low credit quality should have more market risk than firms with high credit quality. We have shown that when the firm's value is low, credit spread is more sensitive to the changes in interest rates. On the contrary, firms with high credit quality are those which, as we expect, have only a base level of interest rate exposure. Also when dividend and coupon rates are higher, credit spread is more sensitive to the firm value volatility. Furthermore, the effect of the correlation between the increments of interest rates and instantaneous returns of asset value can be very significant. When correlation is high, credit spread appears to be sensitive to the changes in interest rate volatility. When correlation is small, credit spread decreases slightly over a wide range of interest rate volatility. These results show that credit risk and market risk are intrinsically interrelated. The impossibility of segregating the two types of risk implies that developing appropriate integrated models capturing both is necessary for better management of portfolio credit risk.

Chapter 5 has demonstrated that the algorithm for the computations of defaultable coupon bonds, after some modifications, immediately lends itself to efficient pricing of other credit risk related instruments. In the case of a bond option, we have shown that default risk of the underlying has a significant effect on the pricing of bond options.

To improve the properties of credit spreads for short maturities, we model a firm's asset value as a jump-diffusion process. Chapter 6 has compared the structural framework of bond pricing models under a jump-diffusion process with those under a pure diffusion process. We have employed a tractable, discrete time model for the valuation of defaultable coupon bonds when the underlying firm value process follows a jump-diffusion process. The method yields a framework which adopts only simple mathematics. It appears that the modelling of a firm's total asset value as a jump-diffusion process can provide a more realistic model of spread levels which, unlike diffusion based models, does not go to zero for short maturities. This is because the jump-diffusion model enables us to generate leptokurtic (fat-tailed) distribution for firm's asset values. We have also found that negative jumps can have significant and persistent effects on spread levels.

Furthermore, if the downward jumps are systematic, there is a tendency to underestimate the spread levels. This result may partly explain why without taking the systematic jump risk into account, Merton-type models tend to underestimate the credit spreads.

Other important factors include taxes and dividends. State taxes have been ignored in almost all modelling of defaultable bonds. As a further contribution, this chapter has introduced the important factor of tax into the model. As motivated by Elton, Gruber, Agrawal, and Mann [2000], we have shown that taxes do have significant and persistent effects on bonds with long maturities. Interestingly, assuming that the distribution of state taxes remains unchanged, we have shown that a decrease in the federal tax rate may precipitate earlier default of low-grade bonds.

Both the structural and the reduced-form approaches have pros and cons. Although structurally appealing, the reliance of structural models on economic fundamentals and the value of a firm's asset make them hard to estimate in practice. On the contrary, the reduced-form approach has the major advantage of providing a model of very high tractability and ease of calibration. However, most reduced-form models have a structural drawback that lack a connection of a firm's economic fundamentals to default events, although some suggestions for improvement have been put forward in the literature. This provides the motivation for us to propose in Chapter 7 the flexible analytical model, which provides a compromise between the two approaches.

Our model of corporate debt takes into account stochastic interest rates, a firm's asset values, and hazard rates of default. Consistent with the literature of the reduced-form models, we assume that default can only happen unexpectedly. As in Duffie and Singleton [1999] and Duffee [1999], we take a hazard rate process as exogenously given. Unlike these models however, there is a crucial and innovative distinction in specification of the process in our model. We introduce structural characteristics of the firm into the hazard rate process through a moving average so as to provide a measure of the firm's performance in equity as well as its leverage effect. Furthermore, our model also has a fairly high degree of flexibility in correlation structures.

The model has another four important features. Firstly, instead of solely using current firm's value as do the conventional Merton-type and the recent Madan and Unal's [2000] models, we take the broader view of the financial health of the firm into account by considering the current asset level relative to its past positions. Secondly, our model preserves a high degree of flexibility in generating credit spreads. Thirdly, the analytical and tractable form of the model enables researchers to undertake comparative statics and enhances the empirical applicability of the model. Fourthly, as an interesting extension, we have demonstrated how we can generalize our model to deal with default risk of a counterparty. Although situated in a simple setting, we believe that the extended model has captured essential features of counterparty default risk, whose properties have been shown to have an economic implication in holding a portfolio of assets issued by the counterparty.

8.2 Suggestions for Further Research

There are some outstanding empirical issues. In the case of the jump-diffusion model, we have found that with the deployment of the additional factors of taxes and dividends, the jump-diffusion model has been shown to be more flexible than pure diffusion ones in fitting empirical spreads. It remains to be seen whether it is sufficiently flexible and sufficiently easy to fit for it to be useful in empirical work.

In the case of the reduced-form model, the analytical tractability has a major advantage for ease of calibration. Given the value of the market loss rate δ, we are able to estimate other model parameters, and hence compute analytically the survival probabilities of firms. However, as shown in Duffie and Singleton [1999], if δ is unknown, we also have the same identification problem of the market loss rate δ and the hazard rate process h_t from our model. Jarrow [2001] develops a procedure for segregating the two variables by using equity and bond prices. Two recent papers by Janosi, Jarrow, and Yildirim [2001a, 2001b] apply Jarrow's model [2001] to estimate default probabilities implicit in equity prices, and to estimate expected losses and liquidity discounts implicit in debt prices respectively. Liquidity effects can also be incorporated into our reduced-form model similarly as in the work of Janosi, Jarrow, and Yildirim [2001b].

While it remains to be seen whether Jarrow's idea of segregating the two variables can be justified by empirical work, the modelling of survival probabilities of firms in terms of hazard rate processes is an important area of research in credit risk analysis. More importantly, by linking the hazard rate processes to underlying fundamental variables, the result may provide us with deeper insights into understanding the default mechanism.

On the theoretical side, we note that neither the structural approach nor the reduced-form approach is designed to address the empirical regularities in the financial distress literature. Empirical facts such as renegotiation, debt rescheduling, and forgiveness and sometimes costly liquidations need to be reconciled in either the structural or in the reduced-form approach. A realistic pricing model should also consider the implications of managerial actions, as pointed out by Garbade [1999]. Furthermore, the key role played by the bankruptcy code in the allocation of residual asset value upon financial distress must not be ignored.

The issues of correlated defaults have become a focus of attention for practitioners, regulators, and academics alike. In studying the impact of counterparty default risk on pricing defaultable securities, Jarrow and Yu [2001] specify in their models switching-type hazard rate processes depending on which counterparties have gone bankrupt. In a simple case of having two firms A and B, Jarrow and Yu [2001] define the firms' default intensities to be:

$$\begin{aligned} \lambda_t^A &= a_1 + a_2 \mathbf{1}_{\{\tau^B \leq t\}}, \textit{ and} \\ \lambda_t^B &= b_1 + b_2 \mathbf{1}_{\{\tau^A \leq t\}}. \end{aligned}$$

where a_1, a_2, b_1, and b_2 are constants. Therefore, the pricing of bonds issued by these firms requires knowledge of the distribution of the first jump times τ^A and τ^B. Although Jarrow and Yu [2001] claim that this setting can be generalized similarly to deal with a situation of more than two firms, working out the joint distributions is difficult. They show that even in the simple case of two firms, the corresponding distribution functions for τ^A and τ^B must be solved numerically unless the firms have identical default intensities.

Instead of solving the bond pricing problems numerically in this sitting, Jarrow and Yu [2001] impose further restrictions on the setting by considering a primary-secondary framework, in which $a_2 = 0$. In this case, correlated defaults due to an exposure of common risk factors and firm-specific risks are considered analytically.

There are some problems with Jarrow and Yu's [2001] framework. Firstly, the assumption that counterparty default risk only enters a hazard rate process through the corresponding first jump time is simplistic. Strictly speaking, the creditworthiness of an individual firm can affect the financial strength of many other firms, including its business partners, and vice versa. This implies that counterparty default risk enters into the system in a far more complicated way. Secondly, in order to obtain analytical solutions, Jarrow and Yu [2001] ignore the modelling of the firms' fundamental variables. We believe that the firms' equity prices and some macro-economic variables should be taken into account in a general situation. Thirdly, in an attempt to introduce common market risk factors into their models, Jarrow and Yu [2001] model the hazard rate process as having an additive factor of interest rates. However, this allows the hazard rate to become negative with positive probability because the interest rates are assumed to be Gaussian.

We have become aware late in the research that the main model in Chapter 7 can be taken as a starting point for handling a more general framework of counterparty default and credit risk. In particular, the model can be accommodated to solving the above problems. In the case of two firms, we can postulate that the hazard rate process of an individual firm takes a typical form of:

$$\begin{aligned} dh_t &= (\theta_h + k_h h_t + k_{hy_1} Y_{1,t} + k_{hy_2} Y_{2,t} + k_{hr} r_t)dt + \sigma_{hr} dB_t^r \\ &\quad + \sigma_{hs_1} dB_t^{s_1} + \sigma_{hs_2} dB_t^{s_2} + \sigma_h \sqrt{h_t} dB_t^h, \end{aligned}$$

where $Y_{1,t}$ and $Y_{2,t}$ are two measures of relative levels of equity prices for the two firms. Note that the process assumes a similar form as the one defined in Chapter 7. However, there is a main structural difference in that the above process is capable of capturing the interaction of counterparty effects through the measures $Y_{1,t}$, $Y_{2,t}$, and the covariance terms.

As with Chapter 7, this process can be applied to pricing a risky discount bond. There are four noticeable features:

(i) The nature of the counterparty effects is captured by the coefficients k_{hy_1}, k_{hy_2}, and the covariance terms. The financial strength of a firm and the counterparty effects are first reflected in the hazard rate process through the firms' equity prices and then the bond price;

(ii) A method of solution similar to that in Chapter 7 is applicable. We can show that the model of the corporate bond is analytical. Its structural resemblance to the main model in Chapter 7 implies that the model has the same desirable features as we discussed before;

(iii) Macro-economic variables, for example a stock market index, can be incorporated into the hazard rate process in the same way as the measures $Y_{1,t}$ and $Y_{2,t}$ do; and

(iv) The structure of the framework may pave the way for the analysis of a bond portfolio. Other macro-economic variables, including GDP growth rate, overall unemployment and so on, can be introduced into the hazard rate processes similarly. As we know form the work of Altman [1983, 1990] and Wilson [1997a, 1997b] that these macro-economic factors have explanatory power in predicting the number of defaults. This is an important point when we are investigating the structural properties of a portfolio of corporate bonds.

Although this thesis has placed particular emphasis on the pricing of corporate bonds, the work is fundamental in nature as regards the pricing and the analysis of credit derivatives and other credit-sensitive instruments. Further research projects relating to the modelling of counterparty default and credit risk are ongoing. With the setting discussed above, the modelling of a framework with several counterparties is our main focus of further research.

APPENDIX A
Proof

Proof: For reference, see Musiela and Rutkowski [1997]. Since r_t follows a Vasicek process, it is not difficult to prove that under the equivalent measure Q,

$$-\int_0^T r_s ds = a_T + \frac{\eta}{\beta} e^{-\beta T} \int_0^T e^{\beta s} dB_s^r - \frac{\eta}{\beta} B_T^r,$$

where

$$a_T = -\frac{1}{\beta}(1 - e^{-\beta T}) \left[r_0 + \frac{\zeta}{\beta}(e^{\beta T} - 1) \right] + \frac{\zeta}{\beta}(\frac{e^{\beta T} - 1}{\beta} - T),$$
$$B_s^r = \rho B_s + \sqrt{1 - \rho^2} B_s'.$$

By Theorem 4.2.2, under the equivalent martingale measure Q^V and the transformation $B_t^V = -\sigma t + B_t$, we have

$$\begin{aligned} -\int_0^T r_s ds &= a_T + \frac{\eta}{\beta} e^{-\beta T} \int_0^T e^{\beta s} d[\rho B_s^V + \sqrt{1 - \rho^2} B_s'] \\ &\quad - \frac{\eta}{\beta}[\rho B_T^V + \sqrt{1 - \rho^2} B_T'] + \frac{\rho \eta \sigma}{\beta}[\frac{1 - e^{-\beta T}}{\beta} - T], \end{aligned}$$

which is a normal random variable.

By definition $Z_T = -\int_0^T r_u du - \frac{1}{2}\sigma^2 T - \sigma B_T^V$,

$$\sigma_T^2 = Var[Z_T] = b_T^2 + \sigma^2 T - 2\frac{\eta \rho \sigma}{\beta}(\frac{1 - e^{-\beta T}}{\beta} - T),$$

where

$$b_T^2 = \frac{\eta^2}{\beta^2} T + \frac{\eta^2}{2\beta^3}(1 - e^{-2\beta T}) - \frac{2\eta^2}{\beta^3}(1 - e^{-\beta T}).$$

■

APPENDIX B
Proof of Theorem 4.3.1

Proof: Under the assumption that r_t follows a Vasicek process as described by equation (4.1), for $t \geq s \geq 0$, under the measure Q, we have

$$r_t = r_s e^{-\beta(t-s)} + \frac{\zeta}{\beta}(1 - e^{-\beta(t-s)}) + \eta e^{-\beta t} \int_s^t e^{\beta u} dB_u^r,$$

where $B_t^r = \rho B_t + \sqrt{1-\rho^2} B_t'$. By Theorem 4.2.2, under the equivalent martingale measure Q^V and the transformation $B_t^V = -\sigma t + B_t$, the above equation can be expressed as

$$\begin{aligned} r_t &= r_s e^{-\beta(t-s)} + \frac{\zeta + \eta\rho\sigma}{\beta}(1 - e^{-\beta(t-s)}) \\ &\quad + \eta e^{-\beta t} \int_s^t e^{\beta u} d[\rho B_u^V + \sqrt{1-\rho^2} B_u']. \end{aligned} \tag{B.1}$$

We can prove that

$$\begin{aligned} E[r_t|\Im_s] &= r_s e^{-\beta(t-s)} + \frac{\zeta + \eta\rho\sigma}{\beta}(1 - e^{-\beta(t-s)}), \\ \varepsilon_{s,t,r} &= \eta e^{-\beta t} \int_s^t e^{\beta u} d[\rho B_u^V + \sqrt{1-\rho^2} B_u'] \\ &= \eta\rho \int_s^t e^{\beta(u-t)} dB_u^V + \eta\sqrt{1-\rho^2} \int_s^t e^{\beta(u-t)} dB_u'. \end{aligned}$$

The result (4.7) follows immediately using equation (B.1).

By definition $Z_t = -\int_0^t r_u du - \frac{1}{2}\sigma^2 t - \sigma B_t^V$, we have

$$Z_t = Z_s - \frac{1}{2}\sigma^2(t-s) - \int_s^t r_u du - \sigma(B_t^V - B_s^V). \tag{B.2}$$

After integrating equation (B.1), we have

$$\begin{aligned} -\int_s^t r_u du &= -\frac{1 - e^{-\beta(t-s)}}{\beta} r_s - \frac{\zeta + \eta\rho\sigma}{\beta}(t - s - \frac{1 - e^{-\beta(t-s)}}{\beta}) \\ &\quad - \int_s^t \eta e^{-\beta v} \int_s^v e^{\beta u} d[\rho B_u^V + \sqrt{1-\rho^2} B_u']\, dv. \end{aligned}$$

Equation (B.2) becomes

$$
\begin{aligned}
Z_t &= Z_s - \frac{1}{2}\sigma^2(t-s) \\
&\quad -\frac{1-e^{-\beta(t-s)}}{\beta}r_s - \frac{\zeta+\eta\rho\sigma}{\beta}(t-s-\frac{1-e^{-\beta(t-s)}}{\beta}) + \varepsilon_{s,t,Z} \\
&= b_{s,t} + Z_s - \frac{1-e^{-\beta(t-s)}}{\beta}r_s + \varepsilon_{s,t,Z},
\end{aligned}
$$

where

$$
\begin{aligned}
\varepsilon_{s,t,Z} &= -\int_s^t \eta e^{-\beta v}\int_s^v e^{\beta u} d[\rho B_u^V + \sqrt{1-\rho^2}B_u^{'}]\, dv - \sigma(B_t^V - B_s^V) \\
&= -\int_s^t \frac{\eta}{\beta}(1-e^{-\beta(t-u)})d[\rho B_u^V + \sqrt{1-\rho^2}B_u^{'}] - \sigma(B_t^V - B_s^V) \\
&= -\int_s^t \left[\frac{\eta\rho}{\beta}(1-e^{-\beta(t-u)}) + \sigma\right]dB_u^V - \frac{\eta\sqrt{1-\rho^2}}{\beta}\int_s^t (1-e^{-\beta(t-u)})dB_u^{'}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\sigma_{s,t,r}^2 &= Var[\varepsilon_{s,t,r}|\Im_s] \\
&= \eta^2 e^{-2\beta t}\int_s^t e^{2\beta u}du \\
&= \eta^2\frac{1-e^{-2\beta(t-s)}}{2\beta}, \\
\sigma_{s,t,Z}^2 &= Var[\varepsilon_{s,t,Z}|\Im_s] \\
&= \int_s^t \left[\frac{\eta\rho}{\beta}(1-e^{-\beta(t-u)}) + \sigma\right]^2 du + \frac{\eta^2(1-\rho^2)}{\beta^2}\int_s^t (1-e^{-\beta(t-u)})^2 du \\
&= \frac{\eta^2}{\beta^2}\left[t-s-2\frac{1-e^{-\beta(t-s)}}{\beta} + \frac{1-e^{-2\beta(t-s)}}{2\beta}\right] + \sigma^2(t-s) \\
&\quad + \frac{2\eta\rho\sigma}{\beta}[t-s-\frac{1-e^{-\beta(t-s)}}{\beta}], \\
\sigma_{s,t,r,Z} &= E[\varepsilon_{s,t,r}\, \varepsilon_{s,t,Z}|\Im_s] \\
&= -\eta\rho\int_s^t \left[\frac{\eta\rho}{\beta}(1-e^{-\beta(t-u)}) + \sigma\right]e^{\beta(u-t)}du - \frac{\eta^2(1-\rho^2)}{\beta}\int_s^t (1-e^{-\beta(t-u)})e^{\beta(u-t)}du \\
&= \frac{\eta^2\rho^2}{2\beta^2}(1-e^{\beta(u-t)})^2 + \frac{\eta\rho\sigma}{\beta}(1-e^{\beta(u-t)}) + \frac{1}{2}\sigma^2.
\end{aligned}
$$

■

APPENDIX C
A Review of Ho, Stapleton and Subrahmanyam's Model [1995]

The method Ho, Stapleton and Subrahmanyam [1995] use to approximate the multivariate process is closely related to previous contributions by Amin [1990, 1991], and Nelson and Ramaswamy [1990]. Nelson and Ramaswamy [1990] approximate a given univariate process for the price for the underlying asset by a simple binomial process.[1] In order to ensure that the process has the desired variance characteristics, while remaining simple, Nelson and Ramaswamy [1990] suggest an adjustment in the conditional probabilities of the binomial process over time.

Ho, et al. [1995] also construct simple binomial processes. However, in contrast to Nelson and Ramaswamy [1990], they allow the number of binomial steps n_{t_i} between any two points t_{i-1} and t_i to be greater than 1. This means that, in a univariate case, the method of Ho, et al. [1995] can be regarded as a generalization of Nelson and Ramaswamy [1990]. In a multivariate case, a multivariate process can also be modelled by changing the conditional probabilities associated with nodes. A general multivariate process, where the individual assets have different rates of change of variance and mean reversion, is modelled.

C.1 Notation

We assume that the prices of each of the underlying assets, $X_1, X_2, \cdots, X_J$ follows a lognormal diffusion process:

$$d\ln(X_j) = \mu_j(X_j, t)dt + \sigma_j(t)dZ_j \tag{C.1}$$

[1] In the context of Nelson and Ramaswamy [1990], "simple" means that the number of nodes of the binomial process increases linearly with time.

for $j = 1, 2, \cdots, J$, where μ_j and σ_j are the instantaneous drift and volatility of $\ln(X_j)$, and dZ_j is a standard Brownian motion. The instantaneous correlation between the Brownian motions dZ_j and dZ_k is $\rho_{j,k}(t)$. The instantaneous drift in the above equation is a function of X_j and t, which allows for mean reversion that may change over time. We assume that $\mu_j(X_j, t)$ is linear in X_j and the instantaneous variances and covariances are non-stochastic functions of time. Hence, the asset prices are lognormally distributed at any time t.

There are a finite number (m) of future dates in the time interval $[0, T]$ at which we are interested in the asset prices. The dates are numbered $t_1, t_2, \cdots, t_m = T$. We are interested in the joint distribution of the prices of the assets on these dates. We denote the unconditional mean (at time 0) of the logarithmic j-th asset return at time t_i as $\mu_{t_i,j} = E_0[\ln(X_{t_i,j})]$. The conditional volatility over the period t_{i-1} and t_i is denoted $\sigma_{t_{i-1},t_i,j} = \sqrt{Var[\ln(X_{t_i})|\Im_{t_{i-1}}]}$, and the unconditional volatility is $\sigma_{0,t_i,j} = \sqrt{Var[\ln(X_{t_i})|\Im_0]}$.

C.2 A Method for Constructing a Univariate Binomial Process with Specified Variances

In the univariate case, we drop the subscript j in this section. The problem is to approximate with a binomial process the true process for X_{t_i}, given the means μ_{t_i}, conditional volatilities σ_{t_{i-1},t_i}, and unconditional volatilities σ_{0,t_i}. The conditional volatilities of the approximating binomial process will be denoted $\hat{\sigma}_{t_{i-1},t_i}(n_{t_i})$ since they will be a function of n_{t_i}, the number of binomial steps between times t_{i-1} and t_i. We require that

$$\lim_{n_{t_i} \to \infty} \hat{\sigma}_{t_{i-1},t_i}(n_{t_i}) = \sigma_{t_{i-1},t_i}, \tag{C.2}$$

for all $i = 1, 2, \cdots, m$.

The unconditional volatility of the approximating process over the period $(0, t_i)$ is similarly denoted $\hat{\sigma}_{0,t_i}(n_{t_1}, n_{t_2}, \cdots, n_{t_i})$, since it is, in general, a function of the number of binomial steps over each of the subperiods $t_1, t_2, \cdots, t_i$. Here we require

$$\lim_{n_{t_l}\to\infty} \hat{\sigma}_{0,t_i}(n_{t_1}, n_{t_2}, \cdots, n_{t_i}) = \sigma_{0,t_i}, \tag{C.3}$$

for all $i = 1, 2, \cdots, m$, $l = 1, \cdots, i$.

In addition, we constrain the mean of the approximating process to be equal to μ_{t_i} for all i.

$$\lim_{n_i\to\infty} \hat{\mu}_{t_i} = \mu_{t_i}, \tag{C.4}$$

for all $i = 1, 2, \cdots, m$.

The work of Ho, et al. [1995] involves the construction of m separate binomial distributions, for the time periods $[t_{i-1}, t_i]$, $i = 1, 2, \cdots, m$. The set of these distributions forms a discrete stochastic process for X_{t_i}:

$$(\hat{X}_{t_1}, \hat{X}_{t_2}, \cdots, \hat{X}_{t_m}),$$

where $\hat{X}_{t_i}$ is only defined at the time t_i.[2]

$\hat{X}_{t_1}$ takes values in an $(N_1 + 1)$ vector with k-th element

$$X_{t_1,k} = X_0 u_1^k d_1^{N_1-k},$$

where $N_1 = n_{t_1}$, $k = 0, \cdots, N_1$. $\hat{X}_{t_2}$ takes values in an $(N_2 + 1)$ vector with k-th element

$$X_{t_2,k} = X_0 u_2^k d_2^{N_2-k},$$

where $N_2 = n_{t_1} + n_{t_2}$, $k = 0, \cdots, N_2$. In general, $\hat{X}_{t_i}$ takes values in an $(N_i + 1)$ vector with k-th element

$$X_{t_i,k} = X_0 u_i^k d_i^{N_i-k},$$

where $N_i = \sum_{r=1}^{i} n_{t_r}$, $k = 0, \cdots, N_i$.

Ho, et al. [1995] choose the up and down movements $u_1, \cdots, u_m$, $d_1, \cdots, d_m$ and the conditional probabilities of an up movement for each time interval $[t_{i-1}, t_i]$,

[2] Here we define $\hat{X}_{t_i}$ as a constructed process of X_{t_i} without reference to node locations.

given the location of a node at time t_{i-1}, such that the convergence Equations (C.2), (C.3) and (C.4) are satisfied. We denote

$$x_{t_i} = \ln(X_{t_i}/X_0).$$

On the lattice, we define

$$x_{t_i,k} = \ln(X_{t_i,k}/X_0),$$

and the conditional probability of an up movement of a single binomial step at a node k at t_{i-1} as

$$q(x_{t_{i-1},k}).$$

Figure C.1 shows an example of a two-period discrete process for X_s and X_t, where $m = 2$ and $s < t$. There are (n_s+1) nodes at time s numbered $i = 0, 1, \cdots, 4$, where $n_s = 4$. There are (n_s+n_t+1) nodes at time t numbered $j = 0, 1, \cdots, 8$, where $n_t = 4$. X_0 is the starting point. Ho, et al [1995] assume that X_0 can move to any point $X_{s,i}$ associated with the probability $\binom{n_s}{i} q_0^i (1 - q_0)^{n_s - i}$, where $q_0 = q(x_0)$, for $i = 0, 1, \cdots, n_s$.

At time s, $X_{s,4}$ can move to any positions corresponding to the 5 points on the top at time t, and $X_{s,3}$ can move to any positions corresponding to those points numbered $j = 3, \cdots, 7$ at time t.

In general, $X_{s,i}$ can move to a point $X_{t,j}$ associated with the probability $\binom{n_t}{j-i} q_{s|i}^{j-i} (1 - q_{s|i})^{n_t - j + i}$, where $q_{s|i} = q(x_{s,i})$, for $j = i, i + 1, \cdots, i + n_t$. Intermediate values on the open intervals $(0, s)$ and (s, t) are not defined. The tree is recombining. It is clear from the construction that the tree $\hat{X}_{t_i}$ is recombining.[3]

Ho, et al [1995] establish the following lemma to guarantee that the conditional volatility and the unconditional mean converge to their values in Equations (C.2) and (C.4).

[3] Here we define $\hat{X}_{t_i}$ as a constructed process of X_{t_i} without reference to node locations. For the $\hat{X}_{t_i}$ tree, $N_i = \sum_{l=1}^{i} n_{t_l}$. Let $k = max\{n_{t_1}, \cdots, n_{t_m}\}$. Since $N_{i+1} \leq N_i + k$, for any $i = 1, \cdots, m-1$, the tree is recombining. For details, see James and Webber [2000].

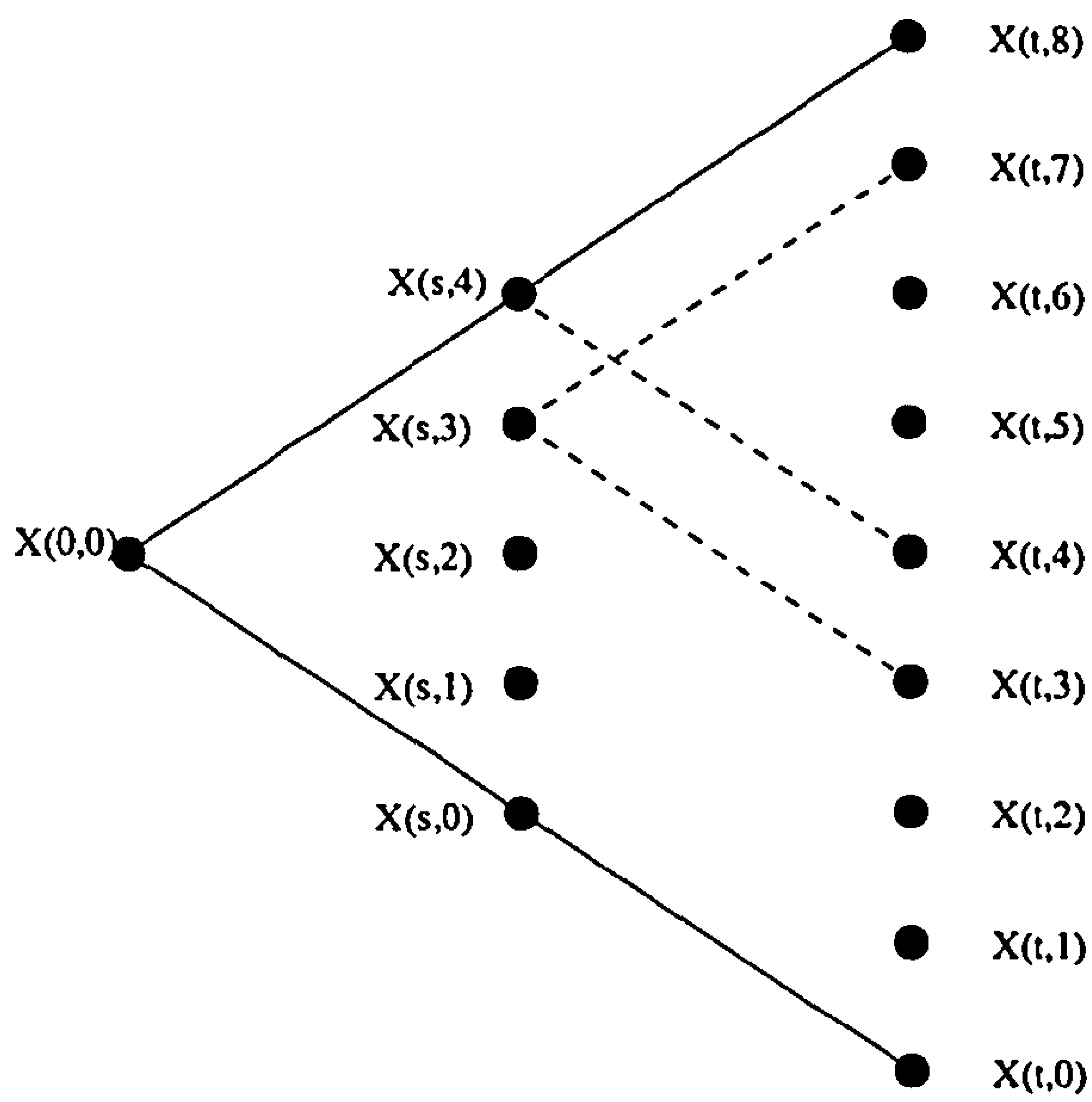

Figure C.1: **An example of a two-period discrete process for X_s and X_t.** There are (n_s+1) nodes at time s numbered $i = 0, 1, \cdots, 4$, where $n_s = 4$. There are (n_s+n_t+1) nodes at time t numbered $j = 0, 1, \cdots, 8$, where $n_t = 4$. X_0 is the starting point. Note that X_0 can move to any points at time s. Similarly, $X_{s,4}$ can move to any positions corresponding to the 5 points on the top at time t, and $X_{s,3}$ can move to any positions corresponding to those points numbered $j = 3, \cdots, 7$ at time t. Intermediate values on the open intervals $(0, s)$ and (s, t) are not defined.

Lemma C.2.1 *Suppose that the up and down movements u_i and d_i are chosen so that*

$$
\begin{aligned}
d_i &= \frac{2(E_0(X_{t_i})/X_0)^{\frac{1}{N_i}}}{1+\exp(2\sigma_{t_{i-1},t_i}/\sqrt{n_{t_i}})}, & \text{(C.5)} \\
u_i &= 2(E_0(X_{t_i})/X_0)^{\frac{1}{N_i}} - d_i, & \text{(C.6)}
\end{aligned}
$$

$i = 1, 2, \cdots, m$, where $N_i = \sum_{r=1}^{i} n_{t_r}$, then if, for all i, the conditional probability $q(x_{t_{i-1},k}) \to 0.5$ as $n_{t_l} \to \infty$, for $l = 1, 2, \cdots, i$, then the unconditional mean and the conditional volatility of the approximating process approach respectively their true values:

$$
\begin{aligned}
\lim_{n_{t_l}\to\infty} E(\hat{X}_{t_i}) &= E(X_{t_i}), \textit{for all } l = 1, 2, \cdots, i, \\
\lim_{n_{t_i}\to\infty} \hat{\sigma}_{t_{i-1},t_i} &= \sigma_{t_{i-1},t_i}.
\end{aligned}
$$

The up and down movements d_i and u_i are chosen to match the true mean and conditional volatility. Furthermore, since the conditional volatilities are allowed to change over time, the u_i and d_i change correspondingly.

The remaining problem is to choose the conditional probabilities $q(x_{t_{i-1},k})$ in such a manner that the the unconditional volatility converges to the true value as in Equation (C.3). Since $x_{t_i} = \ln(X_{t_i}/X_0)$ is a conditionally normally distributed Markovian random variable, it follows that the regression

$$
x_{t_i} = a_{t_i} + b_{t_i} x_{t_{i-1}} + \varepsilon_{t_i},
$$

where $E_{t_{i-1}}(\varepsilon_{t_i}) = 0$, is linear with

$$
\begin{aligned}
b_{t_i} &= \sqrt{\frac{\sigma^2_{0,t_i} - \sigma^2_{t_{i-1},t_i}}{\sigma^2_{0,t_{i-1}}}}, \\
a_{t_i} &= E(x_{t_i}) - b_{t_i} E(x_{t_{i-1}}).
\end{aligned}
$$

Theorem C.2.2 *Suppose that the X_{t_i} are joint lognormally distributed. If the X_{t_i} are approximated with binomial distributions with $N_i = N_{i-1} + n_{t_i}$ steps, and u_i and*

d_i are given by Equations (C.5), (C.6), and if the conditional probability of an up movement at node k at time t_{i-1} is

$$q(x_{t_{i-1},k}) = \frac{a_{t_i} + b_{t_i} x_{t_{i-1},k} - k\ln(u_i) - (N_{i-1} - k)\ln(d_i) - n_{t_i}\ln(d_i)}{n_{t_i}(\ln(u_i) - \ln(d_i))},$$

for all $i = 1, 2, \cdots, m$; $k = 0, 1, \cdots, N_{i-1}$, then $\hat{\mu}_{t_i} \to \mu_{t_i}, \hat{\sigma}_{0,t_i} \to \sigma_{0,t_i}$, and $\hat{\sigma}_{t_{i-1},t_i} \to \sigma_{t_{i-1},t_i}$, as $n_{t_i} \to \infty$, for all i.

Theorem C.2.2 allows us to approximate a process with given mean, variance, and covariance characteristics over the periods $(0, t_i)$ and (t_{i-1}, t_i), where $i = 1, 2, \cdots, m$. We can therefore construct a process using all the dates $t_1, t_2, \cdots, t_m$. Successive application of Theorem C.2.2 guarantees that the volatilities converge to their given values over each time period.

C.3 The Multivariate Case

Ho, et al [1995] only confine to a simple case where there are two relevant random variables (X, Y). Suppose that (X_{t_1}, Y_{t_1}) and (X_{t_2}, Y_{t_2}) are multivariate lognormally distributed with volatilities $(\sigma_{0,t_1,x}, \sigma_{0,t_2,x}, \sigma_{t_1,t_2,x})$ and $(\sigma_{0,t_1,y}, \sigma_{0,t_2,y}, \sigma_{t_1,t_2,y})$. Also, assume that the correlation between x_{t_1} and y_{t_1} is ρ_{0,t_1}. The conditional correlation between x_{t_2} and y_{t_2} is denoted ρ_{t_1,t_2}. Note that, for the joint lognormal distribution, we assume that the conditional correlation is non-stochastic but is allowed to change over time.

The steps in the computation for the general case are:

(I): Compute the node locations for X_{t_1} and Y_{t_1}, and X_{t_2} and Y_{t_2}, independently using the techniques described in Section C.2. Specifically, $\hat{Y}_{t_1}$ is constructed using the conditional volatility of y_{t_1} given x_{t_1}, $\hat{X}_{t_2}$ requires the the conditional volatility $\sigma_{t_1,t_2,x}$ of x_{t_2} given x_{t_1}, and $\hat{Y}_{t_2}$ requires the conditional volatility of y_{t_2} given both y_{t_1} and x_{t_2},

(II): Compute the conditional probability of an up movement at Y_0 given a value of X_{t_1} using the following equation

$$q_Y(x_{t_1,k}) = \frac{\alpha_{t_1} + \beta_{t_1} x_{t_1,k} - m_{t_1} \ln(d_{1,y})}{m_{t_1}(\ln(u_{1,y}) - \ln(d_{1,y}))},$$

where α_{t_1} and β_{t_1} are the coefficients from the simple regression of y_{t_1} on x_{t_1}. β_{t_1} implicitly contains the correlation term ρ_{0,t_1}. m_{t_1} is the number of binomial steps for $\hat{Y}_{t_1}$. Y_0 moves to $Y_{t_1,j}$ with the probability $P[Y_{t_1,j}|x_{t_1,k}] = \binom{m_{t_1}}{j} q_Y(x_{t_1,k})^j \left(1 - q_Y(x_{t_1,k})\right)^{m_{t_1}-j}$, $j = 0, 1, \cdots, m_{t_1}$,

(III): Compute the conditional probability of an up movement at X_{t_1} given a value of X_{t_1} using the univariate techniques discussed in Section C.2 and the equation $q_X(x_{t_1,k})$ in Theorem C.2.2.

(IV): Compute the conditional probability of an up movement at Y_{t_1} given both Y_{t_1} and X_{t_2}. We denote this probability as $q_Y(y_{t_1,k}, x_{t_2,s})$, where

$$q_Y(y_{t_1,k}, x_{t_2,s}) = \frac{a_{t_2} + b_{t_2} y_{t_1,k} + c_{t_2} x_{t_2,s} - k \ln(u_{2,y}) - (M_1 - k)\ln(d_{2,y}) - m_{t_2} \ln(d_{2,y})}{m_{t_2}(\ln(u_{2,y}) - \ln(d_{2,y}))},$$

where a_{t_2}, b_{t_2} and c_{t_2} are the multiple regression coefficients from the regression of y_{t_2} on y_{t_1} and x_{t_2}. c_{t_2} implicitly contains the correlation term ρ_{t_1,t_2}. $M_1 = m_{t_1}$ is the number of binomial steps for $\hat{Y}_{t_1}$ and $M_2 = m_{t_1} + m_{t_2}$ is the number of binomial steps for $\hat{Y}_{t_2}$. $Y_{t_1,k}$ moves to $Y_{t_2,j}$ with the probability $P[Y_{t_2,j}|y_{t_1,k}, x_{t_2,s}] = \binom{m_{t_2}}{j-k} q_Y(y_{t_1,k}, x_{t_2,s})^{j-k} \left(1 - q_Y(y_{t_1,k}, x_{t_2,s})\right)^{m_{t_2}-j+k}$, $j = k, k+1, \cdots, k + m_{t_2}$.

It is evident that the joint probabilities of transition from one point to another are given as follows:

$$\begin{aligned} P[(X_{t_1,k}, Y_{t_1,j})|(X_0, Y_0)] &= P[X_{t_1,k}|x_0] P[Y_{t_1,j}|x_{t_1,k}], \\ P[(X_{t_2,s}, Y_{t_2,i})|(X_{t_1,k}, Y_{t_1,j})] &= P[X_{t_2,s}|x_{t_1,k}] P[Y_{t_2,i}|y_{t_1,j}, x_{t_2,s}]; \end{aligned}$$

where $k = 0, 1, \cdots, n_{t_1}$, $j = 0, 1, \cdots, m_{t_1}$, $s = k, k+1, \cdots, k+n_{t_2}$ and $i = j, j+1, \cdots, j+m_{t_2}$.

Ho, et al [1995] prove that if the up and down movements of the two correlated random variables X and Y, and the conditional probabilities are constructed as above, then the approximated value of the conditional covariance converges to its true value. They also claim without proof that an extension of Theorem C.2.2 can be used to show that when the conditional probabilities are chosen in this manner, both the variances and the correlations of the multivariate processs converge to their given values. In Chapter 4, we consider a similar situation where there are two lognormal processes, one of which is a Markovian random variable on itself and another one is a Markovian random variable on itself and the first variable. Convergence of constructed processes based on the work of Ho, et al [1995] to the real ones is proved.

APPENDIX D
Proof of Theorem 4.4.1

Proof: We can prove equation (4.9) by induction. Note that by the choice of the probabilities q_{t_j}, we have

$$E[\hat{r}_{t_j}|\Im_{t_{j-1}}] = a_{t_{j-1},t_j} + e^{-\beta(t_j - t_{j-1})}\hat{r}_{t_{j-1}}.$$

This implies that

$$E[\hat{r}_{t_j}] = a_{t_{j-1},t_j} + e^{-\beta(t_j - t_{j-1})}E[\hat{r}_{t_{j-1}}].$$

It is easy to check that

$$E[\hat{r}_{t_1}] = E[r_{t_1}].$$

By induction, the result (4.9) follows.

Similarly we can prove equation (4.10).

For property (4.11), we only consider the case when $j = 1$. The proof for a general j is similar. At time t_1, a realization of r_{t_1} can be estimated by

$$\hat{r}_{t_1} = r_0 + i\log(u_1) + (n_{t_1} - i)\log(d_1),$$

where i is a binomial random variable with parameters (n_{t_1}, q_{t_1}). Hence,

$$\hat{\sigma}^2_{0,t_1,r} = n_{t_1}q_{t_1}(1 - q_{t_1})\Big(\log(u_1) - \log(d_1)\Big)^2.$$

Note that

$$\begin{aligned}
\lim_{n_{t_1}\to+\infty} q_{t_1} &= \lim_{n_{t_1}\to+\infty} \frac{a_{0,t_1} + e^{-\beta t_1}r_0 - r_0 - n_{t_1}\log(d_{t_1})}{n_{t_1}[\log(u_{t_1}) - \log(d_{t_1})]} \\
&= \lim_{n_{t_1}\to+\infty} -\frac{\log(2) + \log(E[X_{t_1}]/X_0)/n_{t_1} - \log(1 + e^{2\sigma/\sqrt{n_{t_1}}})}{2\sigma/\sqrt{n_{t_1}}} \\
&= \lim_{h\to 0^+} -\frac{\log(2) + h^2\log(E[X_{t_1}]/X_0) - \log(1 + e^{2\sigma h})}{2\sigma h} \\
&= \frac{1}{2}.
\end{aligned}$$

Since $q_{t_1} \to \frac{1}{2}$ as $n_{t_1} \to \infty$, we have $\hat{\sigma}^2_{0,t_1,r} \to \sigma^2_{0,t_1,r}$ as $n_{t_j} \to \infty$. The result (4.11) obtains.

Similarly we can prove that

$$\hat{\sigma}^2_{t_{j-1},t_j,Z|r_{t_j}} \to \sigma^2_{t_{j-1},t_j,Z|r_{t_j}} \text{ as } m_{t_j} \to \infty.$$

By property (4.8), we have

$$Z_{t_j} = b_{t_{j-1},t_j} + Z_{t_{j-1}} - \frac{1 - e^{\beta(t_j - t_{j-1})}}{\beta} r_{t_{j-1}} + \varepsilon_{t_{j-1},t_j,Z},$$

where $\varepsilon_{t_{j-1},t_j,Z} = \rho_{t_{j-1},t_j} \frac{\sigma_{t_{j-1},t_j,Z}}{\sigma_{t_{j-1},t_j,r}} \varepsilon_{t_{j-1},t_j,r} + \sqrt{1 - \rho^2_{t_{j-1},t_j}} \sigma_{t_{j-1},t_j,Z} \varepsilon_{t_{j-1},t_j}$, for a standard normal random variable $\varepsilon_{t_{j-1},t_j}$ that is independent of $\varepsilon_{t_{j-1},t_j,r}$. This implies that $\sigma^2_{t_{j-1},t_j,Z|r_{t_j}} = (1 - \rho^2_{t_{j-1},t_j})\sigma^2_{t_{j-1},t_j,Z}$. Hence the result (4.12) holds.

By construction, the choice of p_j satisfies

$$E[\hat{Z}_{t_j}|\Im_{t_{j-1}}, \hat{r}_{t_j}] = b_{t_{j-1},t_j} + \hat{Z}_{t_{j-1}} - \frac{1 - e^{-\beta(t_j - t_{j-1})}}{\beta} \hat{r}_{t_{j-1}} + \rho_{t_{j-1},t_j} \frac{\sigma_{t_{j-1},t_j,Z}}{\sigma_{t_{j-1},t_j,r}} \hat{\varepsilon}_{t_{j-1},t_j,r},$$

which implies that

$$Var\Big[E[\hat{Z}_{t_j}|\Im_{t_{j-1}}, \hat{r}_{t_j}]|\Im_{t_{j-1}}\Big] = \rho^2_{t_{j-1},t_j} \frac{\sigma^2_{t_{j-1},t_j,Z}}{\sigma^2_{t_{j-1},t_j,r}} \hat{\sigma}^2_{t_{j-1},t_j,r} \to \rho^2_{t_{j-1},t_j} \sigma^2_{t_{j-1},t_j,Z}$$

as $n_{t_j} \to \infty$ by (4.11). As $\hat{\sigma}^2_{t_{j-1},t_j,Z} = Var\Big[E[\hat{Z}_{t_j}|\Im_{t_{j-1}}, \hat{r}_{t_j}]|\Im_{t_{j-1}}\Big] + E\Big[Var[\hat{Z}_{t_j}|\Im_{t_{j-1}}, \hat{r}_{t_j}]|\Im_{t_{j-1}}\Big]$, the result (4.13) follows.

Proof of (4.14) is similar.

We prove the properties (4.15), (4.16) and (4.17) by induction on j. By (4.11), (4.13) and (4.14), it is obvious that the proposition holds when $j = 1$. Assume that the proposition holds for some $j = i - 1$. Let $\hat{r}_{t_j}$ and $\hat{Z}_{t_j}$ denote the constructed discrete processes $r_{t_j,i}$ and $Z_{t_j,l}$ respectively without reference to node locations. By construction, we have

$$\begin{aligned} \hat{r}_{t_i} &= a_{t_{i-1},t_i} + e^{-\beta(t_i - t_{i-1})} \hat{r}_{t_{i-1}} + \hat{\varepsilon}_{t_{i-1},t_i,r}, \\ \hat{Z}_{t_i} &= b_{t_{i-1},t_i} + \hat{Z}_{t_{i-1}} - \frac{1 - e^{-\beta(t_i - t_{i-1})}}{\beta} \hat{r}_{t_{i-1}} + \hat{\varepsilon}_{t_{i-1},t_i,Z}. \end{aligned}$$

Then,

$$\begin{aligned}
Var[\hat{r}_{t_i}] &= e^{-2\beta(t_i-t_{i-1})}Var[\hat{r}_{t_{i-1}}] + \hat{\sigma}^2_{t_{i-1},t_i,r}, \\
Var[\hat{Z}_{t_i}] &= Var[\hat{Z}_{t_{i-1}}] + \Big(\frac{1-e^{-\beta(t_i-t_{i-1})}}{\beta}\Big)^2 Var[\hat{r}_{t_{i-1}}] \\
&\quad -2\Big(\frac{1-e^{-\beta(t_i-t_{i-1})}}{\beta}\Big)Cov[\hat{r}_{t_{i-1}},\hat{Z}_{t_{i-1}}] + \hat{\sigma}^2_{t_{i-1},t_i,Z}, \\
Cov[\hat{r}_{t_i},\hat{Z}_{t_i}] &= e^{-\beta(t_i-t_{i-1})}Cov[\hat{r}_{t_{i-1}},\hat{Z}_{t_{i-1}}] - e^{-\beta(t_i-t_{i-1})}\Big(\frac{1-e^{-\beta(t_i-t_{i-1})}}{\beta}\Big)Var[\hat{r}_{t_{i-1}}] \\
&\quad + \hat{\sigma}_{t_{i-1},t_i,r,Z}.
\end{aligned}$$

By properties (4.11), (4.13), (4.14) and induction assumption, the proposition also holds for $j = i$.

To justify (4.18), it suffices to prove by induction on j that for any real α number

$$E[\exp(\alpha\hat{r}_{t_j})] \to E[\exp(\alpha r_{t_j})] \text{ as } n_{t_j},\ldots,n_{t_1} \to \infty.$$

For $j = 1$, as $n_{t_1} \to \infty$

$$\begin{aligned}
&E[\exp(\alpha\hat{r}_{t_1})] = X_0^\alpha[q_{t_1}u_{t_1}^\alpha + (1-q_{t_1})d_{t_1}^\alpha]^{n_{t_1}} \\
&\to X_0^\alpha \exp\Big(\alpha(a_{0,t_1} + e^{-\beta t_1}r_0 - r_0) + \frac{1}{2}\alpha^2\sigma^2_{0,t_1,r}\Big) \\
&= \exp\Big(\alpha(a_{0,t_1} + e^{-\beta t_1}r_0) + \frac{1}{2}\alpha^2\sigma^2_{0,t_1,r}\Big) \\
&= E[\exp(\alpha r_{t_1})].
\end{aligned}$$

Suppose that the proposition is true for some j. For any real number α, conditional on the node i at time t_j, we consider the following. When $n_{t_{j+1}} \to \infty$,

$$\begin{aligned}
&E[\exp(\alpha\hat{r}_{t_{j+1}})|\Im_{t_j}] = X_0^\alpha(u_{t_{j+1}}^\alpha)^i(d_{t_{j+1}}^\alpha)^{N_j-i}[q_{t_{j+1}}u_{t_{j+1}}^\alpha + (1-q_{t_{j+1}})d_{t_{j+1}}^\alpha]^{n_{t_{j+1}}} \\
&\to X_0^\alpha \exp[\alpha(a_{t_j,t_{j+1}} + e^{-\beta(t_{j+1}-t_j)}r_{t_j} - r_0) + \frac{1}{2}\alpha^2\sigma^2_{t_j,t_{j+1},r}] \\
&= \exp(\alpha' r_{t_j})\exp(\alpha a_{t_j,t_{j+1}} + \frac{1}{2}\alpha^2\sigma^2_{t_j,t_{j+1},r})
\end{aligned}$$

where $\alpha' = \alpha e^{-\beta(t_{j+1}-t_j)}$. By assumption, we have

$$E[\exp(\alpha\hat{r}_{t_{j+1}})] \to \exp(\alpha' E[r_{t_j}] + \frac{1}{2}\alpha'^2 Var[r_{t_j}])\exp(\alpha a_{t_j,t_{j+1}} + \frac{1}{2}\alpha^2\sigma^2_{t_j,t_{j+1},r})$$

as $n_{t_{j+1}}, \ldots, n_{t_1} \to \infty$. By Theorem 4.3.1, right hand side of the above expression is of the form

$$\exp(\alpha E[r_{t_{j+1}}] + \frac{1}{2}\alpha^2 Var[r_{t_{j+1}}]).$$

Hence the proposition holds for $j+1$.

To prove (4.19), it suffices to prove an alternative proposition by induction on j for any real number α,

$$E[\exp(\hat{Z}_{t_j} + \alpha \hat{r}_{t_j})] \to E[\exp(Z_{t_j} + \alpha r_{t_j})] \text{ as } m_{t_j}, n_{t_j}, \ldots, m_{t_1}, n_{t_1} \to \infty.$$

The method is similar to the proof of (4.18). ∎

APPENDIX E
Construction of A Two-Period Tree

In the text, we have constructed the one-period trees for X_s and Y_s. Now we consider the construction of the trees at a second time period t, with $t > s$, by following the similar fashion to approximate X_t by a vector of $n_s + n_t + 1$ numbers:

$$X_{t,i} = X_0 u_t^i d_t^{n_s+n_t-i},$$

for $i = 0, \ldots, n_s + n_t$, where

$$\begin{aligned} d_t &= \frac{2(E[X_t]/X_0)^{1/(n_s+n_t)}}{1+\exp(2\sigma_{s,t,r}/\sqrt{n_t})}, \\ u_t &= 2(E[X_t]/X_0)^{1/(n_s+n_t)} - d_t. \end{aligned}$$

On the time-interval $[s,t]$, we choose a transition probability q_t of an up-movement at node i at time s such that property (4.6) holds, that is

$$r_0 + i\log(u_t) + (n_s - i)\log(d_t) + n_t\Big(q_t\log(u_t) + (1-q_t)\log(d_t)\Big) = a_{s,t} + e^{-\beta(t-s)}\hat{r}_s,$$

hence,

$$q_t(r_{s,i}) = \frac{a_{s,t} + e^{-\beta(t-s)}r_{s,i} - r_0 - i\log(u_t) - (n_s-i)\log(d_t) - n_t\log(d_t)}{n_t\Big(\log(u_t) - \log(d_t)\Big)},$$

for $i = 0, \cdots, n_s$.

Given the above interest rate tree, we are now ready to construct a second tree for Y_t conditional on r_t. At time t, we create a vector $m_s + m_t + 1$ of numbers

$$Y_{t,i} = \overline{u}_t^i \overline{d}_t^{m_s+m_t-i},$$

for $i = 0, \ldots, m_s + m_t$, where

$$\begin{aligned} \overline{d}_t &= \frac{2(E[Y_t])^{1/(m_s+m_t)}}{1+\exp(2\sigma_{s,t,Z|r_t}/\sqrt{m_t})}, \\ \overline{u}_t &= 2(E[Y_t])^{1/(m_s+m_t)} - \overline{d}_t, \end{aligned}$$

where $\sigma_{s,t,Z|r_t}$ is the volatility of Z_t conditional on $\Im_s$ and r_t. A transition probability p_t of an up-movement at node i at time s is chosen such that the following property holds

$$E[Z_t|\Im_s, r_t] = b_{s,t} + Z_s - \frac{1-e^{-\beta(t-s)}}{\beta} r_s + \rho_{s,t}\frac{\sigma_{s,t,Z}}{\sigma_{s,t,r}}\varepsilon_{s,t,r}$$

that is,

$$\begin{aligned} & i\log(\overline{u}_t) + (m_s - i)\log(\overline{d}_t) + m_t\Big(p_t\log(\overline{u}_t) + (1-p_t)\log(\overline{d}_t)\Big) \\ & = b_{s,t} + Z_{s,l} - \frac{1-e^{-\beta(t-s)}}{\beta} r_{s,i} + \rho_{s,t}\frac{\sigma_{s,t,Z}}{\sigma_{s,t,r}}\varepsilon_{s,t,r,i,k}, \end{aligned}$$

hence

$$\begin{aligned} p_t(r_{s,i}, Z_{s,l}, r_{t,k}) &= \frac{b_{s,t} + Z_{s,l} - \frac{1-e^{-\beta(t-s)}}{\beta} r_{s,i} + \rho_{s,t}\frac{\sigma_{s,t,Z}}{\sigma_{s,t,r}}\varepsilon_{s,t,r,i,k} - \phi}{m_t\Big(\log(\overline{u}_t) - \log(\overline{d}_t)\Big)}, \\ \phi &= i\log(\overline{u}_t) + (m_s - i)\log(\overline{d}_t) + m_t\log(\overline{d}_t), \\ r_{t,k} &= a_{s,t} + e^{-\beta(t-s)} r_{s,i} + \varepsilon_{s,t,r,i,k}, \end{aligned}$$

for $i = 0, \cdots, n_s$, $k = i, \cdots, i + n_t$, and $l = 0, \cdots, m_s$.

APPENDIX F
Proof of Theorem 4.6.1

Proof: (i) By Corollary 4.4.2, $\hat{Z}_T|\hat{r}_T$ converges to $Z_T|\hat{r}_T$ in distribution. Since $Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|\hat{r}_T$ is continuous at Z_T, Billingsley([7], p.334, corollary 1) implies that

$$Min\Big(V_0\exp(-aT), K\exp(\hat{Z}_T)\Big)\Big|\hat{r}_T \Rightarrow Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|\hat{r}_T,$$

as $m \to \infty$. Note that $Min\Big(V_0\exp(-aT), K\exp(\hat{Z}_T)\Big)\Big|\hat{r}_T$ is uniformly integrable as it is bounded above by V_0. By Billingsley([7], p.338, Theorem 25.12), The result (i) follows.

(ii) By Billingsley([7], p.334, corollary 1) again, since $\hat{r}_T$ converges to r_T in distribution and $E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|r_T\Big]$ is continuous at r_T,

$$E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|\hat{r}_T\Big] \Rightarrow E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|r_T\Big],$$

as $n \to \infty$. $E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|\hat{r}_T\Big]$ is uniformly integrable, and so by Billingsley([7], p.338, Theorem 25.12) again, we have

$$\begin{aligned} &E\Big[E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|\hat{r}_T\Big]\Big] \\ &\to E\Big[E\Big[Min\Big(V_0\exp(-aT), K\exp(Z_T)\Big)\Big|r_T\Big]\Big] \end{aligned}$$

as $n \to \infty$. The result (ii) follows. ■

APPENDIX G
Proof of Proposition 7.2.1

Proof: Given the three affine processes s_t, Y_t, and r_t as in ($*$), by Duffie and Kan [1996], we can express the solution in the form of the equation (7.1). For equation (7.5) to satisfy the corresponding backward Kolmogorov partial differential equation,[1] we have the following set of differential equations:[2]

$$
\begin{aligned}
0 &= -1 + B_1'(t,T) + k_h B_1(t,T) + \frac{\delta}{2}\sigma_h^2 B_1(t,T)^2, && \text{(G.1)}\\
0 &= B_2'(t,T) + \delta k_{hy} B_1(t,T) - \alpha B_2(t,T), && \text{(G.2)}\\
0 &= -1 + B_3'(t,T) - k_r B_3(t,T) + \delta k_{hr} B_1(t,T) + B_2(t,T), and && \text{(G.3)}\\
0 &= A'(t,T) + \delta\theta_h B_1(t,T) - (\sigma_s^2/2 + a) B_2(t,T) + k_r\theta_r B_3(t,T) + \frac{1}{2}\sigma_s^2 B_2(t,T)^2 \\
&\quad + \frac{1}{2}\sigma_r^2 B_3(t,T)^2 + \frac{\delta^2}{2}(\sigma_{hr}^2 + \sigma_{hs}^2) B_1(t,T)^2 + \sigma_r\delta\sigma_{hr} B_1(t,T) B_3(t,T) && \text{(G.4)}\\
&\quad + \sigma_r\sigma_s\rho B_2(t,T) B_3(t,T) + \delta(\sigma_s\sigma_{hr}\rho + \sigma_s\sigma_{hs}\sqrt{1-\rho^2}) B_1(t,T) B_2(t,T),
\end{aligned}
$$

with the boundary conditions $A(T,T) = 0$, $B_1(T,T) = 0$, $B_2(T,T) = 0$, and $B_3(T,T) = 0$. From equation (G.1), we have

$$
\begin{aligned}
-B_1'(t,T) &= -1 + k_h B_1(t,T) + \frac{\delta}{2}\sigma_h^2 B_1(t,T)^2 \\
&= \frac{\delta}{2}\sigma_h^2\Big(B_1(t,T) + \beta_1\Big)\Big(B_1(t,T) - \beta_2\Big),
\end{aligned}
$$

where

$$
\begin{aligned}
-\beta_1 &= \frac{-k_h - \sqrt{k_h^2 + 2\delta\sigma_h^2}}{\delta\sigma_h^2}, \\
\beta_2 &= \frac{-k_h + \sqrt{k_h^2 + 2\delta\sigma_h^2}}{\delta\sigma_h^2}.
\end{aligned}
$$

[1] We know from the Feynman-Kac formula that, under some technical conditions, equation (7.5) solves the backward Kolmogorov partial differential equation of the problem.

[2] All derivatives are computed with respect to time t.

The above equation can be expressed in the following form:

$$\int_t^T \frac{\delta}{2}\sigma_h^2 ds = -\int_t^T \frac{dB_1(s,T)}{\Big(B_1(s,T)+\beta_1\Big)\Big(B_1(s,T)-\beta_2\Big)}.$$

Using partial fractions, we have

$$\begin{aligned}
\frac{\delta}{2}\sigma_h^2(T-t) &= -\int_t^T \frac{dB_1(s,T)}{\Big(B_1(s,T)+\beta_1\Big)\Big(B_1(s,T)-\beta_2\Big)} \\
&= \frac{1}{\beta_1+\beta_2}\int_t^T \frac{dB_1(s,T)}{\Big(B_1(s,T)+\beta_1\Big)} - \frac{1}{\beta_1+\beta_2}\int_t^T \frac{dB_1(s,T)}{\Big(B_1(s,T)-\beta_2\Big)} \\
\frac{B_1(t,T)+\beta_1}{B_1(t,T)-\beta_2} &= -\frac{\beta_1}{\beta_2}\exp\Big(-\frac{1}{2}(\beta_1+\beta_2)\delta\sigma_h^2(T-t)\Big).
\end{aligned}$$

Solving for $B_1(t,T)$,

$$B_1(t,T) = -\frac{2[1-e^{-\sqrt{k_h^2+2\delta\sigma_h^2}(T-t)}]}{(\sqrt{k_h^2+2\delta\sigma_h^2}-k_h)+(\sqrt{k_h^2+2\delta\sigma_h^2}+k_h)e^{-\sqrt{k_h^2+2\delta\sigma_h^2}(T-t)}}.$$

To solve equation (G.2), we use the method of integrating factor.

$$\begin{aligned}
\Big[e^{-\alpha t}B_2(t,T)\Big]' &= -\alpha k_{hy}e^{-\alpha t}B_1(t,T), \\
B_2(t_0,T) &= \delta k_{hy}\int_{t_0}^T e^{-\alpha(u-t_0)}B_1(u,T)du.
\end{aligned}$$

Equation (G.3) can be solved similarly. ∎

APPENDIX H
Proof of Proposition 7.5.1

Proof: Part (i) and (ii) are trivial.

For part (iii), we prove the result by using Proposition 7.2.1 and equation (7.11). Note that $B_1(t_0,T) \to l_1$ as $T \to \infty$, where $l_1 = \frac{-2}{\sqrt{k_h^2+2\delta\sigma_h^2}-k_h}$.

Let $B_2(t_0,T) \to l_2$ as $T \to \infty$, where l_2 is independent of time t_0. Then for $t > t_0 \geq 0$,

$$
\begin{aligned}
& \delta k_{hy} \int_{t_0}^{T} e^{-\alpha(s-t_0)} B_1(s,T)\, ds \\
= \;& \delta k_{hy} \int_{t_0}^{t} e^{-\alpha(s-t_0)} B_1(s,T)\, ds + \delta k_{hy} \int_{t}^{T} e^{-\alpha(s-t_0)} B_1(s,T)\, ds \\
= \;& \delta k_{hy} \int_{t_0}^{t} e^{-\alpha(s-t_0)} B_1(s,T)\, ds + \delta k_{hy} e^{-\alpha(t-t_0)} \int_{t}^{T} e^{-\alpha(s-t)} B_1(s,T)\, ds.
\end{aligned}
$$

Taking limits on both sides, as $T \to \infty$,

$$
l_2 = \delta k_{hy} \int_{t_0}^{t} e^{-\alpha(s-t_0)} l_1\, ds + e^{-\alpha(t-t_0)} l_2,
$$

this implies that $l_2 = \frac{\delta k_{hy} l_1}{\alpha}$.

Similarly, we can prove that $l_3 = -\frac{1}{k_r} + \frac{\delta k_{hr} l_1}{k_r} + \frac{l_2}{k_r}$. Hence,

$$
\lim_{T\to\infty} \frac{B_i(t_0,T)}{T-t_0} = 0,
$$

for i = 0,1,2, and 3.

To compute $\lim_{T\to\infty} \frac{A(t_0,T)}{T-t_0}$, we consider

$$
\lim_{T\to\infty} \frac{\int_{t_0}^{T} B_1(s,T)\, ds}{T-t_0}.
$$

Note that $B_1(s,T)$ depends on s and T through their difference $T-s$, and that $\lim_{T\to\infty}\int_{t_0}^{T} B_1(s,T)\,ds \to -\infty$ as $T\to\infty$. Therefore,

$$\begin{aligned}
\lim_{T\to\infty}\frac{\int_{t_0}^{T} B_1(s,T)\,ds}{T-t_0} &= \lim_{T\to\infty}\frac{\partial}{\partial T}\int_{t_0}^{T} B_1(s,T)\,ds \\
&= \lim_{T\to\infty} -\frac{\partial}{\partial t_0}\int_{t_0}^{T} B_1(s,T)\,ds \\
&= \lim_{T\to\infty} B_1(t_0,T) \\
&= l_1
\end{aligned}$$

All remaining terms in $\lim_{T\to\infty}\frac{A(t_0,T)}{T-t_0}$ can be computed similarly. The result follows.

■

APPENDIX I
Iterative Extended Kalman Filter

We provide a general method for the iterative extended Kalman algorithm when underlying state space model is non-Gaussian. An approximate discrete-time distribution of state variables and a description of the iteration extended Kalman filter algorithm are given in the following two sections I.1 and I.2. For details, see Chen and Scott [1995], Lund [1997a], and Lund [1997b].

I.1 Discrete-time distribution of state variables

We consider a general class of term-structure models as in Duffie and Kan [1996]. Under a risk-neutral measure, we assume the state variables, X_t, are governed by the process:

$$dX_t = \mathcal{K}(\Theta - X_t)dt + C\sigma(X_t)dW_t, \tag{I.1}$$

where $\sigma(X_t)$ is a $m \times m$ diagonal matrix with the i-th diagonal element given by

$$[\sigma(X_t)]_{ii} = \sqrt{\alpha_i + \beta_i' X_t}.$$

We assume that the m univariate standard Brownian motions represented by W_t are independent, and the dependence structure between the innovations to X_t is captured by the $m \times m$ matrix C.

Note that the main three processes ($*$) described in Section 7.2 are in this format. Except for some special cases, the exact discrete-time distribution for process I.1 is not available in closed form. Following the ideas presented in Chen and Scott [1995] and Duan and Simonato [1995], we focus on the first and second conditional moments of X_t for which closed-form expressions are straightforward to obtain. We describe the procedure as follows.

By Ito's lemma and properties of matrix exponential functions, process I.1 has the following representation:[1]

$$X_t = e^{-\mathcal{K}(t-s)}X_s + \int_s^t e^{-\mathcal{K}(t-u)}\mathcal{K}\Theta du + u(s,t), \ s \leq t, \tag{I.2}$$

where

$$u(s,t) = \int_t^s e^{-\mathcal{K}(t-u)}C\sigma(X_u)dW_u.$$

Assuming that $\mathcal{K}$ is non-singular, the conditional mean of X_t follows directly from equation (I.2),

$$E[X_t|X_s] = e^{-\mathcal{K}(t-s)}X_s + (I_m - e^{-\mathcal{K}(t-s)})\Theta. \tag{I.3}$$

The conditional covariance matrix is given by:

$$\begin{aligned} Cov[X_t|X_s] &= E[u(s,t)u(s,t)'|X_s] \\ &= \int_s^t e^{-\mathcal{K}(t-u)}CE[\sigma^2(X_u)|X_s]C'e^{-\mathcal{K}'(t-u)}du, \end{aligned} \tag{I.4}$$

where $E[\sigma^2(X_u)|X_s]_{ii} = \alpha_i + \beta_i' E[X_u|X_s]$ and all off-diagonal elements are zero. When $\sigma(\cdot)$ is a constant function, the conditional covariance in (I.4) becomes constant as in the case where the underlying process is Gaussian.

I.2 The iterative extended Kalman filter algorithm

We assume that the state space model has the following form:

$$y_k = Z_k(X_k, \psi) + \epsilon_k, \ \epsilon_k \sim D(0, H_k(\psi)), \tag{I.5}$$

$$X_k = \Phi_{k0}(\psi) + \Phi_{k1}(\psi)X_{k-1} + u_k, \ u_k \sim D(0, V_k(\psi)), \tag{I.6}$$

where $D(0,Q)$ refers to an arbitrary zero-mean distribution with covariance matrix Q, and the vector ψ contains all parameters of the model. Here for simplicity we use X_k to represent X_{t_k}, for $k = 1, \cdots, n$.

[1]Techniques for computing matrix exponential functions and integrals are discussed in Golub and Van Loan [1989], Moler and Van Loan [1978], and Van Loan [1978].

We interpret the state space model as follows. When applying the framework to bond prices, we have a non-linear measurement equation (I.5) with $Z_k(X_k, \psi)$ representing the price of a corporate coupon bond, and y_k the observed market price at time t_k. The dynamics of the state variables X_k are expressed in the linear form as in the system equation (I.6). This is a consequence of the application of equation (I.3) and (I.4) to the state variables (*).

The filtering algorithm of the Iterative Extended Kalman Filter consists of two steps: (i) a prediction step and (ii) an update step. Since, by construction, the transition equation (I.6) is linear, the prediction step is:

$$\hat{X}_{k|k-1} = \Phi_{k0} + \Phi_{k1}\hat{X}_{k-1},$$

with MSE matrix

$$\Sigma_{k|k-1} = \Phi_{k1}\Sigma_{k-1}\Phi_{k1}' + V_k.$$

As the measurement equation (I.5) is non-linear, the update step is less straightforward. Lund [1997b] suggests updating the above estimate by the method of non-linear generalised least square:

$$\hat{X}_k = arg\ min_X F(X),$$

where

$$F(X) = (X - \hat{X}_{k|k-1})'\Sigma_{k|k-1}^{-1}(X - \hat{X}_{k|k-1}) + (y_k - Z_k(X))'H_k^{-1}(y_k - Z_k(X)).$$

The MSE matrix for $\hat{X}_k$ is defined as:

$$\Sigma_k = \left(\Sigma_{k|k-1}^{-1} + \frac{\partial Z_k(\hat{X}_k)'}{\partial X} H_k^{-1} \frac{\partial Z_k(\hat{X}_k)}{\partial X'}\right)^{-1}.$$

The Gauss-Newton algorithm with analytical derivatives can be employed to solve the minimization problem. Finally, we estimate the model parameters by the quasi-maximum likelihood principle. The quasi log-likelihood function is given by

$$\log L(y_1, \cdots, y_n; \psi) = -\frac{1}{2}\sum_{k=1}^{n}\left(\log(|F_k|) + v_k'F_k^{-1}v_k\right),$$

where

$$\begin{aligned} v_k &= y_k - Z_k(\hat{X}_{k|k-1}), \\ F_k &= \frac{\partial Z_k(\hat{X}_{k|k-1})}{\partial X'} \Sigma_{k|k-1} \frac{\partial Z_k(\hat{X}_{k|k-1})'}{\partial X} + H_k. \end{aligned}$$

Because Z_k and its derivatives are of analytical forms, the optimal values of ψ can be computed iteratively by applying the method of steepest descent to the quasi log-likelihood function.

APPENDIX J
Proof of Proposition 7.7.1

Proof: Part (i): Note that for $t_0 \le u \le T$, $\mathbf{1}_{\{\tau^A \le u\}} = \mathbf{1}_{\{\tau^A \le u\}}\mathbf{1}_{\{\tau^A \le T\}}$.

$$
\begin{aligned}
& E_{t_0}\Big[\exp(-\int_{t_0}^{T} \delta p \mathbf{1}_{\{\tau^A \le u\}}\, du)\Big] \\
= \; & E_{t_0}\Big[\exp(-\delta p(T-\tau^A)\mathbf{1}_{\{\tau^A \le T\}})\Big] \\
= \; & \int_{t_0}^{\infty} e^{-\delta p(T-s)\mathbf{1}_{\{s \le T\}}} h^A e^{-h^A(s-t_0)} ds \\
= \; & \int_{t_0}^{T} e^{-\delta p(T-s)} h^A e^{-h^A(s-t_0)} ds + \int_{T}^{\infty} h^A e^{-h^A(s-t_0)} ds \\
= \; & \int_{t_0}^{T} e^{-\delta pT + h^A t_0} h^A e^{-(h^A - \delta p)s} ds + e^{-h^A(T-t_0)}.
\end{aligned}
$$

If $\delta p \ne h^A$, then the above integral becomes $\frac{h^A\left[e^{-h^A(T-t_0)} - e^{-\delta p(T-t_0)}\right]}{\delta p - h^A}$. Otherwise, it becomes $e^{-\delta pT + h^A t_0} h^A (T - t_0)$. The results follow.

Part (ii) is trivial. ∎

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