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STABILITY OF ERYTHROCYTE SEDIMENTATION IN A CONSTANT MAGNETIC FIELD

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The stability of erythrocyte sedimentation in the presence of a transverse component of the ponderomotive force is investigated.

The processes of erythrocyte aggregation lead to the sedimentation being unstable with respect to small variations of the uniform horizontal cell distribution in the sedimentation tube. If an axisymmetric cell distribution is assumed, the system of equations describing erythrocyte sedimentation in blood plasma can be reduced to two-dimensional form and the investigation of this system, both with and without allowance for the viscous components and inertial terms, has shown that it is unstable with respect to small perturbations [1]. The instability may sometimes result in the widely employed ESR test not being exclusively determined by the rheological characteristics of the blood modified, for example, by disease. Accidental shaking of the capillary containing the blood or some other mechanical influence may lead to aggregation of the erythrocytes at the top of the tube and a sharp acceleration of the entire sedimentation process.

In [2] a one-dimensional model of erythrocyte sedimentation with allowance for aggregation and the trapping of some of the fluid inside the aggregations was constructed. In [3] the model was extended to the case of sedimentation with allowance for aggregation in the axisymmetric magnetic field of a solenoid. It was shown that for a fairly long solenoid and a thin capillary, when the capillary was positioned along the axis of the solenoid, the presence of a horizontal component of the magnetic field has almost no effect on the sedimentation. However, shorter solenoids, in which the nonuniformity of the distribution of the horizontal component of the magnetic field along the tube may be important, are often used for setting up experiments. Consequently, if we assume that the only mechanism by which the constant magnetic field acts on the sedimentation is magnetization of the erythrocytes [3], we must take into account the presence of a horizontal force acting on the erythrocytes in the radial direction.

We will consider the sedimentation of an erythrocyte suspension (two-phase medium) in an external magnetic field $\mathbf{H}$ on the assumption that the erythrocytes are only weakly magnetizable. The system of equations describing the motion of the suspension is written in the form [3]:

$$\frac{\partial C}{\partial t}+\operatorname{div}\mathbf{v}^1C=0,\quad \operatorname{div}[\mathbf{v}^1C+\mathbf{v}^2(1-C)]=0$$

$$\rho_s C\left(\frac{\partial\mathbf{v}^1}{\partial t}+(\mathbf{v}^1\nabla)\mathbf{v}^1\right)=-C\nabla p+\rho_s Cg\mathbf{e}_z-DC(\mathbf{v}^1-\mathbf{v}^2)+\frac{1}{2}\chi_s C\nabla H^2 \tag{1}$$

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$$\rho_f(1-C)\left(\frac{\partial \mathbf{v}^2}{\partial t}+(\mathbf{v}^2\nabla)\mathbf{v}^2\right)=-(1-C)\nabla p+\rho_f(1-C)g\mathbf{e}_z+DC(\mathbf{v}^1-\mathbf{v}^2)+\frac{1}{2}\chi_f(1-C)\nabla H^2$$

where C is the volume erythrocyte concentration, $\mathbf{v}^1$ and $\mathbf{v}^2$ are the velocities of the erythrocyte and plasma phases, ρ_s and ρ_f are the true phase densities, $\mathbf{e}_z$ is the unit vector in the vertical direction, χ_s and χ_f are the magnetic susceptibilities of the phases, and D(C) is the phenomenological Stokes drag coefficient of the particles.

We will consider the two-dimensional problem, to which the system (1) can be reduced for certain configurations of the external magnetic field, for example, in a flat cell ($z\in[0; L]$, $x\in[0; R]$, $y\in[0; h]$, $R/h\ll 1$, $R/L\ll 1$), where the z axis is directed vertically downwards. If the sedimentation takes place in a solenoid of circular cross section, then the problem will likewise be two-dimensional.

If we neglect the acceleration of the particles and the inertia as compared with the viscous forces, the undisturbed solution for vertical sedimentation takes the form [3]:

$$w=v_{z_0}{}^1=\frac{(1-C)^2}{D}\left[g(\rho_s-\rho_f)+\frac{\chi_s-\chi_f}{2}\frac{\partial}{\partial z}H_z{}^2\right],\ C=C_0=\text{const} \tag{2}$$

Assuming that the disturbed solution is the sum of the undisturbed solution and a wave $\exp[i(kx+\ell z-\omega t)]$ with corresponding amplitude, for the amplitudes $v^{*i}=(v_x^{*i}, v_z^{*i})$, i = 1, 2, $C=C^*/C_0$, p^* of the disturbed motion we obtain the system

$$(\rho_s\Omega_1-iD)v_x{}^{*1}+iDv_x{}^{*2}+kp^*+i\chi_s\Psi_xC=0,\ \ (\rho_s\Omega_1-iD)v_z{}^{*1}+iDv_z{}^{*2}+lp^*+i(\chi_s\Psi_z-wDF(1+r))C=0$$

$$irDv_x{}^{*1}+(\rho_f\Omega_2-irD)v_x{}^{*2}+kp^*-ir\chi_f\Psi_xC=0$$

$$irDv_z{}^{*1}+(\rho_f\Omega_2-irD)v_z{}^{*2}+lp^*-ir[\chi_f\Psi_z-D(1+r)(1+r+F)w]C=0 \tag{3}$$

$$kv_x{}^{*1}+lv_z{}^{*2}+\Omega_1C=0,\ kv_x{}^{*1}+kv_x{}^{*2}/r+lv_z{}^{*1}+lv_z{}^{*2}/r+lw(1+r)C=0$$

$$\Omega_1=lw-\omega,\ \Omega_2=-rlw-\omega,\ r=\frac{C}{1-C},\quad F=\frac{C}{D}\frac{dD}{dC},\quad \Psi_x=\frac{\partial}{\partial x}\frac{H^2}{2},\quad \Psi_z=\frac{\partial}{\partial z}\frac{H^2}{2}$$

The condition of existence of a solution of system (3) is that the determinant of the coefficients of the corresponding variables be equal to zero. The determinant can be represented in the form of a product of two polynomials; therefore it will be equal to zero if one of the following two conditions is satisfied:

$$\omega^2+[r-1+id(\rho r+1)]\omega_0\omega-r[1+id(\rho-1)]/\omega_0{}^2=0 \tag{4}$$

$$(r+\rho)\omega^2+[2(r^2-\rho)+id\rho(r+1)^2]\omega_0\omega+[r^3+\rho+id\rho(r+1)^2(2r-1+F-\mu)]\omega_0{}^2=0 \tag{5}$$

$$\varkappa=\frac{(\chi_s-\chi_f)\Psi_z}{(\rho_s-\rho_f)g},\quad \omega_0=\frac{lg(\rho_s-\rho_f)(1+\varkappa)}{D(1+r)^2},\quad \rho=\frac{\rho_s}{\rho_f},\ d=\frac{D}{\rho_s\omega_0},\quad \mu=\frac{(\chi_s+r\chi_f)(k\Psi_x+l\Psi_z)}{D\omega_0(1+r)^2}$$

When $\Psi_x=0$, $\Psi_z=0$ conditions (4) and (5) coincide with the corresponding conditions from [1].

The motion defined by system (1) will be stable if Im $\omega<0$. For Eq. (4) the relation Im $\omega<0$ always holds. For Eq. (5) there is always one solution with Im $\omega>0$. We will investigate the positive solution $\omega_a=\mathrm{Im}(\omega/\omega_0)$ of Eq. (5).

An analysis shows that the instability growth rate will decrease relative to that of gravitational sedimentation if

$$\frac{\varkappa}{1+\varkappa}<0,\quad \mu[2\mu^*-\mu]>0,\quad \mu^*=2r-1+F+\frac{\rho^2-r}{\rho+r}$$

Thus, if the particles settle in the longitudinal ponderomotive force field $\Psi_x=0$, $\Psi_z\neq 0$ stabilization will take place when $\varkappa\in]-1;\ 0[$. The action of the transverse ponderomotive force $\Psi_x\neq 0$ will stabilize sedimentation when $\mu\in]0;2\mu^*[$.

We will assume that sedimentation is almost stable if for any values of k and ℓ the

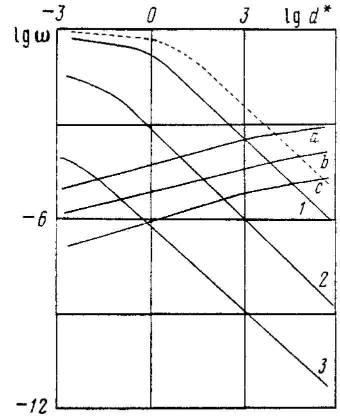

Fig. 1

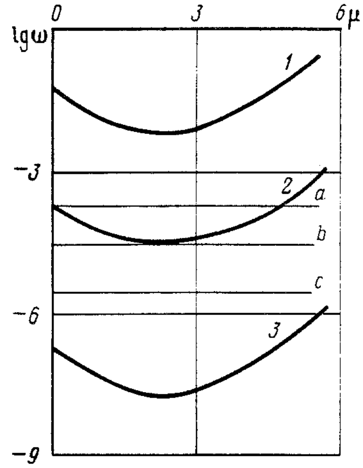

Fig. 2

characteristic sedimentation time $t^* \leq 2\pi/\omega_a$. Since the sedimentation experiments and the numerical calculations were carried out for a quasi-one-dimensional model of sedimentation in a long tube of circular cross section [2, 3], we will also base our estimation of t^* on this model.

We set $t^* = t_b$, the time required to reach the maximum sedimentation rate. When $t < t_b$ there is a zone of quasione-dimensional sedimentation, but when $t > t_b$ the bottom begins to exert a greater influence and compaction of the aggregations and filtration of the trapped fluid, processes which lie outside the context of the present problem, take place in the sedimentation zone.

Since we have in view only a qualitative investigation of the effect of a magnetic field on sedimentation, we will assume that Ψ_x = const, Ψ_z = const. For the time t_b, the average aggregation volume δ and the volume aggregation concentration in the quasi-one-dimensional sedimentation zone we have ($t < t_b$) [3]

$$t_b=\left[\left(\frac{10K}{27(1+\varkappa)(1-C)^{3,5}}+1\right)^{0,6}-1\right]\frac{L}{KCu_0} \tag{6}$$

$$\delta=\delta_0\left[1+\frac{KCu_0t_a}{L}\right], \quad C=C_0 \tag{7}$$

where δ_0 is the average erythrocyte volume, u_0 is the characteristic sedimentation rate of the individual erythrocyte, t_a is the time during which the erythrocytes aggregate, followed by quasione-dimensional sedimentation, and K is a dimensionless parameter characterizing the aggregation rate.

In [2] the following expression was obtained for the phenomenological coefficient D:

$$D=\eta\delta^{-2/3}(1-C)^{-2,5} \tag{8}$$

For the perturbation wave to interact with the system, the conditions $2\pi/\ell \ll L$, $2\pi/k \ll R$ must be satisfied. On the other hand, it is reasonable to limit the values of the wave numbers by imposing the conditions $2\pi/\ell \gg a$, $2\pi/k \gg a$, where a is the hydrodynamic radius of the erythrocyte.

For numerical calculation purposes we will take $C_0 = 0.4$, $\eta = 15\cdot10^{-3}$ km/m·sec, $\rho_s = 1080$ kg/m³, $\rho_f = 1030$ kg/m³, $\chi_s = 10^{-6}$, $\chi_f = 10^{-7}$, $a = 5\cdot10^{-6}$ m, L = 0.1 m, R = 10^{-3} m, $u_0 = 8\cdot10^{-7}$ m/sec, and $\delta_0 = 0.87\cdot10^{-16}$ m³. We also assume that t_a = 5 min, and $K\in[10^3, 10^5]$ [2]. For k and ℓ we obtain $10 \leq \ell \leq 10^3$, $10^2 \leq k \leq 10^3$.

In Fig. 1 we have plotted the dependence of the quantity $\omega=\omega_a(1+\varkappa)$ on the dimensionless parameter $d^*=D(1+\varkappa)/d$ for gravitational sedimentation $\varkappa=0$, $\mu=0$ (broken curve) and sedimentation in magnetic fields $\mu = \mu^* = 2.352$, $\varkappa=0$ (curve 1), $\mu = \mu^*$, $\varkappa = -0.95$ (curve 2), $\mu = \mu^*$, $\varkappa = -0.999$ (curve 3). Curves a, b, and c represent the dependence $\omega_b(d^*) = 2\pi/t_b$ for $\mu = \mu^*$ and $\varkappa$ = 0, −0.95, and −0.999, respectively. Beneath each of curves a, b, and c there is a region of values of ω giving stable sedimentation for the corresponding value of $\varkappa$, and above the curve a region of values giving unstable sedimentation. The values of d^* were determined for the given K from (7) and (8).

Figure 2 shows the parabolic dependence of ω on the parameter μ for $d^* = 52.5$ and various values of $\varkappa$. The straight lines a, b, and c — the values of ω_b for $\varkappa = 0, -0.95$, and -0.999, respectively — divide the entire region into zones of stability and instability. The minimum value of ω for each fixed $\varkappa$ is reached when $\mu = \mu^*$.

All the calculations were made for $\ell = 10^3$. Since for other values of ℓ on the chosen interval the values of ω for all d^* are less than the corresponding values for $\ell = 10^3$, when $d^* > 1.38 \cdot 10^4$ (or $K < 7.2 \cdot 10^4$) gravitational sedimentation is almost stable (Fig. 1). By superimposing an external magnetic field it is possible to extend this interval of values of K. As shown in Fig. 2, for fixed $\varkappa$ the maximum extension can be obtained when $\mu = \mu^*$.

When the ponderomotive force acts only in the transverse direction $\Psi_z = 0$, $\Psi_x \neq 0$ ($\varkappa = 0$, $\mu = \mu^*$) sedimentation will be stable for $d^* > 1.38 \cdot 10^3$ ($K < 4.07 \cdot 10^5$) (Fig. 1). When the ponderomotive force acts only in the longitudinal direction this stabilizing effect is not achieved. In this case the maximum extension of the interval of stable values of K is obtained when $\varkappa = -0.37$, $\mu = -0.7$, and sedimentation will be stable for $d^* > 6.01 \cdot 10^3$, ($K < 1.34 \cdot 10^5$).

As $\varkappa$ approaches the value $\varkappa = -1$, the range of values of μ ensuring stable sedimentation expands (Fig. 2). By an appropriate choice of the parameters $\varkappa$ and μ sedimentation in a magnetic field can be made stable for fairly large values of the aggregation rate K. Thus, sedimentation will be stable for $d^* > 52.5$ ($K < 4.74 \cdot 10^6$) when $\varkappa = -0.95$, $\mu = \mu^*$, and for $d^* > 0.53$ ($K < 1.5 \cdot 10^8$) when $\varkappa = -0.999$, $\mu = \mu^*$ (Fig. 1). If we assume that the field of the forces Ψ_x and Ψ_z can be prescribed correct to 10^{-3}, then we must conclude that for $K > 1.5 \cdot 10^8$ sedimentation cannot be stabilized by means of an external magnetic field.

The sedimentation stability also depends on the initial erythrocyte concentration. The maximum increase in the interval of values of K for which gravitational sedimentation is stable is obtained for $C_0 = 0.32$. In this case sedimentation is stable for $d^* > 6.31 \cdot 10^3$ ($K < 1.29 \cdot 10^5$).

Thus, the instability of the gravitational erythrocyte sedimentation process with respect to small perturbations over a fairly wide range of values of the aggregation constant K may be responsible for the unexplained fluctuations of the ESR test. As shown above, by carrying out the ESR test in a magnetic field it is possible to obtain a stable process for fairly small values of the aggregation rate, which improves the diagnostic accuracy of the test. Thus, stabilization can be obtained for $K < 1.5 \cdot 10^8$ in the case $\varkappa = -0.999$, $\mu = \mu^*$. This corresponds to an external magnetic field with the mean values $\Psi_x \sim 10^{12}-10^{13}$ A^2/m^3 and $|\Psi| \lesssim 3 \cdot 10^{13}$ A^2/m^3, which is obtainable in practice.

If Ψ_x and Ψ_z are functions of the coordinates, the numerical calculation of the quantities ω and ω_b must be carried out for given distributions $\Psi_x(x, z)$ and $\Psi_z(x, z)$.

For calculation purposes we used values of K obtained in almost all gravitational sedimentation experiments on human blood [2]. A further improvement in the accuracy of the calculations requires the setting up of experiments on sedimentation in a magnetic field for the purpose of determining the values of t_a and K, since the aggregation rate constant may depend on the strength of the external magnetic field.

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