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OPTIMIZATION OF BRANCHING PIPELINES ON BASIS OF DESIGN PRINCIPLES OF NATURE

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Abstract. *Structure of the optimal rectangular microcirculatory cell of a leaf is investigated on the model of liquid motion through the bifurcating tubes with permeable walls and its filtration into the cell filled with anisotropic porous biological medium. The relation between the diameters of the tubes in the bifurcation as well as coordinates of the bifurcation point at given width and length of the cell which provides minimum total energy consumptions are obtained.*

1 INTRODUCTION

Branching pipelines are widely used in Nature for long-distance liquid delivering. Special conducting structures in biological systems are designed for transport of liquid and dissolved substances on distances comparable to the characteristic size of the biological system. In animals they are represented by arterial, venous and lymphatic vessels. In plants special conducting systems consist of hollow rigid vessels in a series connection. In spite of the complicated topology of the networks the experimental investigations revealed some common design principles of the conducting systems construction in animals and plants [1-5]. The principles include certain relations between the lengths L_j , diameters d_j of the consecutive vessels and branching angles α_j in bifurcations. Here j is branching order that is introduced as follows (fig.1)

1. The largest vessel has $j=1$;
2. When two vessels with the orders l and k join into a parent vessel then its branching order is $j = \begin{cases} l-1 & \text{at } l=k \\ \min\{l, k\} & \text{at } l \neq k \end{cases}$

The statistical dependence $d_{k0}^\gamma = d_{k1}^\gamma + d_{k2}^\gamma$ (Murray's law) have been obtained for the arterial [1,4-6,8], venous [1,4-5], respiratory [1,8-9] mammal systems, astrorhizal systems in sponge [1], tree trunks and shoots [10-11], plant leaves of different types [2-3,12-13]. Here d_{k0}, d_{k1}, d_{k2} (fig.2) are the diameters of the parent and daughter's vessels at the bifurcation k . For the most part of the investigated networks $\gamma \approx 3$ ($\gamma = 2.55 - 3.02$ for arterial systems, $\gamma = 2.76 - 3.02$ for venous systems, $\gamma = 2.61 - 2.91$ for respiratory systems) and the higher the animal's position at the evolutionary scale, the closer γ to $g = 3$ [14]. For large vessels where flow is not laminar (aorta, respiratory trunk) it was obtained $\gamma \sim 2.33$ [7]. For small vessels where the fluid rheology should be taken into account $\gamma \sim 2.92$.

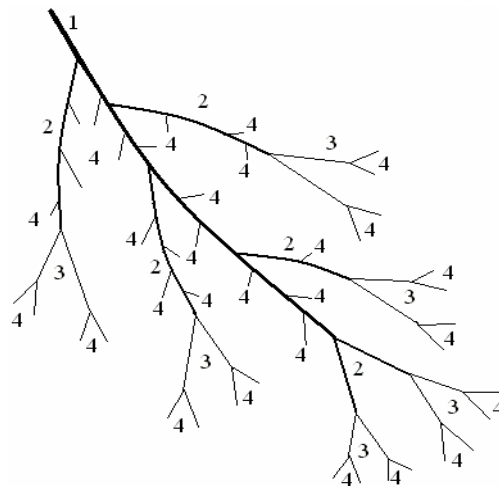


Figure 1: Branching orders $k = 1 - 4$ of the conducting system.

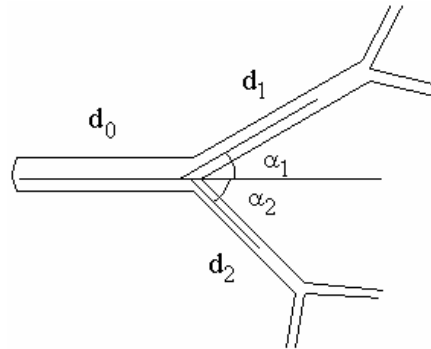


Figure 2: Geometry of the vascular bifurcation.

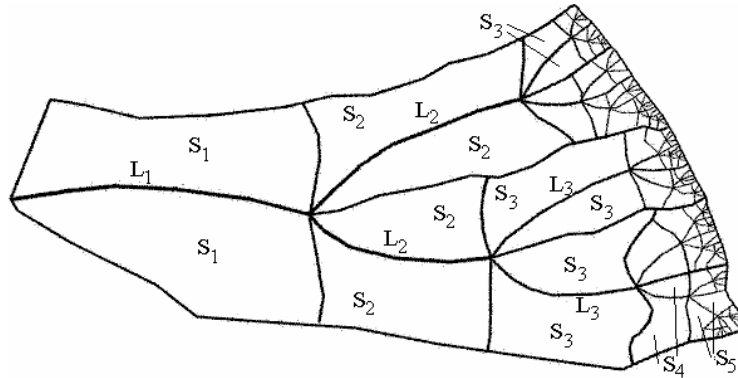


Figure 3: *Cotinus obovatus* leaf venation with vein segments L_j and the corresponding areas S_j of the leaf blade.

The branching angles α_{k1}, α_{k2} (fig.2) at the bifurcation k depend on the diameters d_{k0}, d_{k1}, d_{k2} as

$$\alpha_{j1} = \arccos \left(\frac{(1 + \xi_j^3)^{4/3} + 1 - \xi_j^4}{2(1 + \xi_j^3)^{2/3}} \right), \quad \alpha_{j2} = \arccos \left(\frac{(1 + \xi_j^3)^{4/3} + \xi_j^4 - 1}{2\xi_j^2(1 + \xi_j^3)^{2/3}} \right) \quad (1)$$

where $\xi_j = d_{j2}/d_{j1}$ asymmetry of the bifurcation, $d_{j2} = \min \{d_{j1}, d_{j2}\}$.

These relations between $d_{k0}, d_{k1}, d_{k2}, \alpha_{k1}, \alpha_{k2}$ correspond to the model of the optimal pipeline that minimizes the total energy costs W for the liquid delivering and the transport system construction and maintenance [1,2]. At the same time the relation $L_j = aS_j^{1/2}$ between the area S_j of a leaf blade and the length L_j of the corresponding vessel that provide the liquid delivering to the alive cells of the area has been revealed (fig.3), where $a \in [1.76; 2.25]$

for different types of leaves [3,17]. This relation is well-known in geology as Hack's law for river basins [18] that reflects general properties of branching transport systems construction in Nature.

2 GEOMETRY OF CONDUCTING SYSTEMS IN PLANT LEAVES

Here some statistical data on geometry of the *Cotinus obovatus* leaves (Fig.3) are presented. Geometrical parameters $L_{j,0-1}$, $D_{j,0-1}$ of the conducting system have been measured on the high-resolution digital pictures of the fresh-cutting leaves using image analysis software (SciImage 3b). The total numbers of elements of each branching order $i=1-5$ have been calculated. The results showed that the diameters $d_{j,0-2}$ in the bifurcations obey Murray's law with $\gamma \approx 3$ (fig.4). The numerical solution of the equation

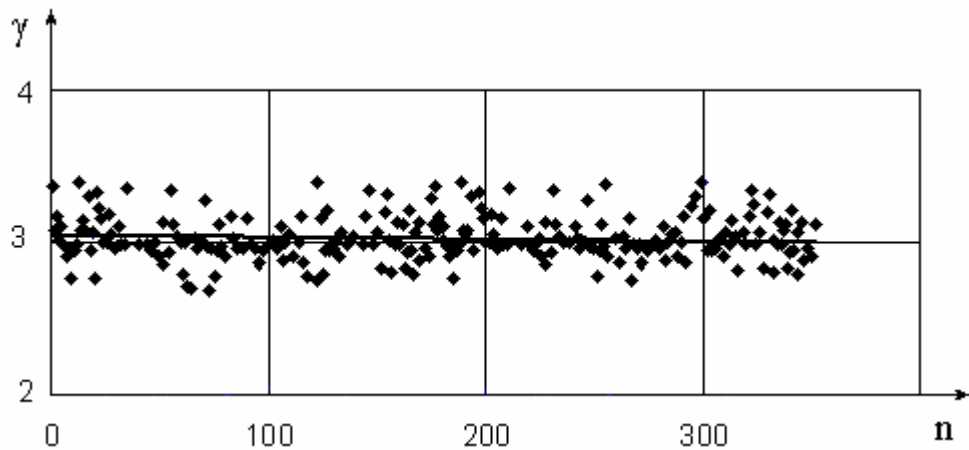


Figure 4: Calculated values γ for separate $n = 350$ vein bifurcations of *Cotinus obovatus* leaf.

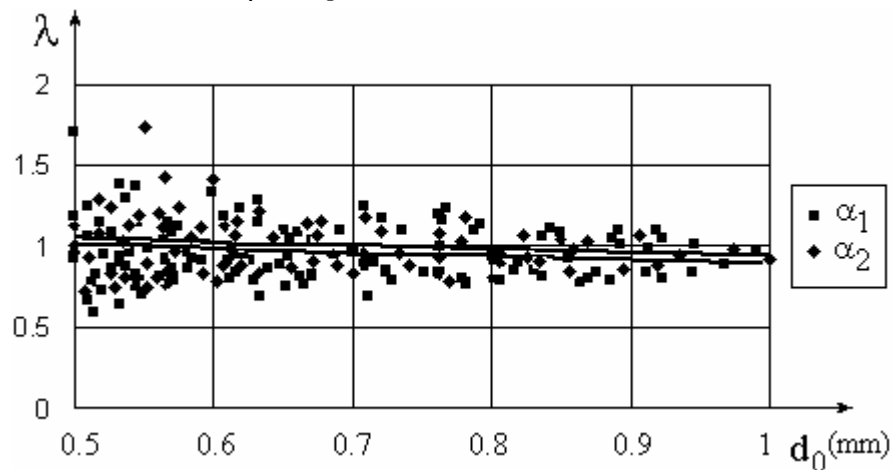


Figure 5: Dependences $\lambda(d_0)$ for α_1, α_2 *Cotinus obovatus* leaf.

$(1+(d_{j2}/d_{j1})^\gamma)(d_{j1}/d_{j0})^\gamma=1$ that has been derived from Murray's law, give $\gamma=2.69-3.72$ for 45 specimens of *Cotinus obovatus* leaf. For symmetrical dichotomous bifurcations $d_{j1}=d_{j2}$ the recurrent relations $d_i=2^{1/\gamma}d_{i+1}$ can be obtained from Murray's law. For $\gamma=3$ it gives $d_i=2^{1/3}D_{i+1}$ that had been used by Tomas Joung in the historically first model of a branching arterial tree. The same relation have been obtained for the conducting systems of several leaf types [2,3,12,13,17].

Statistical dependence $\lambda(d_0)$, where $\lambda=\alpha_{\text{meas}}/\alpha_{\text{opt}}$, α_{meas} are measured values, α_{opt} are optimal values for the corresponding point that have been calculated by (1) are presented in fig.5. Two black lines in fig.5 represent mean values λ for α_1 and α_2 values separately. Within the total range $d_0 \in [0.5;1]$ the mean values lie close to $\lambda=1$ when $\alpha_{\text{meas}} = \alpha_{\text{opt}}$.

The relation between the geometrical parameters L_j , D_j of each conducting element and its domain of influence (i.e. the area S_j of the leaf blade region provided with water by this element) have been investigated for *Cotinus obovatus* leaf as well [3]. The walls of the leaf vessels are permeable and the transporting fluid filters through the wall pores into the cells. The linear dependence $\sqrt{S_j}=k_jL_j$ has been found out for all the specimens. When $L_j=aR_j$ [3] hence it follows $S_i=k_j^2L_j^2=ak_j^2L_jD_j$. As $\Sigma_j=\pi d_jL_j$ is the lateral surface of the conducting element, then $S_j=ak_j^2\Sigma_j/(2\pi)$. It implies that $S_j \sim \Sigma_j$ for each conducting element and the inflow of the liquid in the conducting element and the outflow through its lateral surface of an element are balanced.

The comparative analysis of the data for *Cotinus obovatus* as well as for some other leaves [2,3,12,13,17] and mammal vascular systems [1,4-6] shows, that both transport structures are characterized by a similar kind of statistical dependences. It means that long-distance transport systems design principles in animals and high plants are the same.

3 LIQUID MOTION IN AN OPTIMAL BIFURCATION

Arterial system formation in a developing tissue is provided with mechanoreceptors in vascular vessel wall [1,7]. The mechanoreceptors exercise control over network self-organization by maintaining shear rate at the wall t_w in the certain limits [1]. Under steady flow condition (Poiseuille's flow in a cylindrical tube with rigid wall) the shear rate at the wall is $\tau_w=32\mu Q/(\pi d^3)$. At $t_w = \text{const}$ it gives $Q \sim d^3$, that corresponds to the optimal (in the meaning of Murray's law) tube. In such a manner the mechanoreceptors provided the local optimality conditions at each tube in the developing branching vascular tree. For a bifurcation j of the optimal vessels with $Q_{j,0-2} \sim d_{j,0-1}^3$ the continuity condition $Q_{j0}=Q_{j1}+Q_{j2}$,

where $Q_{j,0-2}$ - volumetric rate in the tube with diameter $d_{j,0-1}$ leads to the Murray's law for $d_{j,0-1}$ with $g = 3$. Under these circumstances the global optimality conditions for the whole transport system are the same [16] and the transport system possessing the minimal hydraulic resistance at a given total volume develops in the tissue.

The mechanisms of the vein systems with the same geometrical properties formation in plant tissues are unknown yet. The hypothesis of the optimal transport structure formation in plant tissues was proposed in our previous works [12-13,17]. It based on the model of the branching pipeline with permeable walls and $\Sigma_j = n Q_j$ for each tube. At such a condition the diameters d_i of the optimal pipeline with permeable wall obey Murray's law and similarity between the long-range transport systems in animal and plant tissues takes place. The control over the optimal transport system construction is exercised by alive cells in leaf blade by prevention the vessels' desolation by means of regulation of the balance between the plant sap inflow and consumption by the cells.

Steady motion of a viscous fluid through a single bifurcation (fig.2) of vessels with permeable walls is considered as a model of sap motion in the conducting system of the plant leaf. Each vessel is represented as a thin long round tube with radius R , $R/L \ll 1$. The mass and impulse continuity conditions in the cylindrical coordinate system at $Re \ll 1$ are

$$\frac{1}{r} \frac{\partial}{\partial r}(rV_r) + \frac{\partial V_x}{\partial x} = 0, \quad \frac{dp}{dx} = \mathbf{m} \left(\frac{\partial^2 V_x}{\partial r^2} + \frac{1}{r} \frac{\partial V_x}{\partial r} \right) \quad (2)$$

where p is pressure, $\vec{V} = (V_r, 0, V_x)$ is velocity, \mathbf{m} is viscosity of the fluid. The boundary conditions are defined as:

$$\begin{aligned} V_x|_{r=R} = 0, & \quad \frac{\partial V_x}{\partial r} \Big|_{r=0} = 0, \\ V_r|_{r=0} = 0, & \quad V_r|_{r=R} = w \end{aligned} \quad (3)$$

$$p|_{x=0} = p_1, \quad p|_{x=L} = p_2 \quad (4)$$

where w - outflow speed. By integrating (2) with respect to r taking into account (3), we shall receive:

$$\begin{aligned} V_x &= -\frac{1}{4\mathbf{m}} \frac{dp}{dx} (R^2 - r^2) \\ V_r &= \frac{1}{16\mathbf{m}} \frac{d^2 p}{dx^2} (2R^2 r - r^3) \end{aligned} \quad (5)$$

and the equation for the pressure field:

$$\frac{d^2 p}{dx^2} = \frac{16\mu w}{R^3} \quad (6)$$

Pressure is obtained from (6), (4) as

$$p^{(1)}(x) = p_1 - Z^p \left(\left(\frac{p_1 - p_2}{Z^p} + \frac{\Phi}{2} \right) y - \frac{\Phi}{2} y^2 \right) \quad (7)$$

at $w = w_0 = const$ and

$$p^{(2)}(x) = p_1 - Z^p \left(\frac{p_1 - p_2}{Z^p} y + \Phi \left(\frac{2}{3} y - y^2 + \frac{y^3}{3} \right) \right) \quad (8)$$

at $w = 2w_0(1 - y)$, where $y = x/L$ or, in other form

$$p^{(1)}(x) = p_1 - Z^p (Qy - \Phi y^2 / 2)$$

$$p^{(2)}(x) = p_1 - Z^p (Qy - \Phi(y^2 - y^3 / 3))$$

where $Q = 2p \int_0^R r V_x(r, 0) dr$, $Z^p = 8\mu L / (\pi R^4)$, $\Phi = \Sigma w_0$. In both cases of w distribution along the wall the full outflow $\Phi = 2\pi R L w_0$ remains the same. Substitution (7),(8) in (5) gives the velocity fields. Then we obtain that the hydraulic resistance $Z = (p_1 - p(L)) / Q$ of the tube is

$$Z^{(1)} = Z^p (1 - \Phi / (2Q)), \quad Z^{(2)} = Z^p (1 - 2\Phi / (3Q))$$

For a symmetrical bifurcation ($L_2 = L_1$, $d_2 = d_1$) of the tubes with diameters $d_{0,1}$ and lengths $L_{0,1}$ Poiseuille's law for the tubes in a bifurcation is

$$p_1 - p_2 = Q_1 Z_1, \quad p_2 - p_3 = Q_2 Z_2$$

where $Q_1, Q_2 = (Q_1 - \Phi_1) / 2$, $Z_{1,2}$ – the volumetric rates at the inlet and the hydraulic resistances of the first- and second-order tubes are $p_1 = p(0)$, $p_2 = p(L_1)$, $p_3 = p(L_1 + L_2)$. The x-axis is continuous along two tubes and $x \in [0, L_1]$ for the first-order tube and $x \in [L_1, L_1 + L_2]$ for the second-order ones. Then we obtain for the total hydraulic resistance of the bifurcation $Z = (p_1 - p_3) / Q_1$:

$$Z^{(1)} = Z_1^p + Z_2^p \frac{\Phi_1}{2Q_1}, \quad Z^{(2)} = Z_1^p + Z_2^p \frac{2\Phi_1}{3Q_1}$$

Here $Z = Z(L_{1,2}, R_{1,2}, w_0, Q_1)$ at the constant w_0, Q_1 depends on the bifurcation geometry only. The extreme problem for a bifurcation can be considered in the form of

$$Z(L_{1,2}, R_{1,2}) \rightarrow \min, \quad G(L_{1,2}, R_{1,2}) = const \quad (9)$$

where G is a geometrical restriction. Problem (9) for $G = \{V, S, D\}$, where V - full volume, S - full lateral surface, D - full dissipation in the system, and a few other criteria was solved for a single tube and a bifurcation of tubes with non-permeable walls as applied to the arterial vessels. As a result the criterion $G = V$ fits the experimental data best of all. Problem (9) with a number of criteria G was solved for a single tube with permeable walls as applied to plant leaves and the same result was obtained [10,11]. Taking into account this conclusion, we consider here the problem (9) with $G = V$ as applied to the bifurcation of tubes with permeable walls. The Lagrange function is $\Xi = Z + \mathbf{I}G$. The conditions $\Xi'_{L_{1,2}, R_{1,2}} = 0$ bring finally to the nonlinear system of equations $\{f_i(r, l) = 0\}$, where $r = d_0/d_1$, $l = L_0/L_1$. Here in contrast to the tubes with non-permeable walls the relative diameters and lengths are not independent. The results of the numerical solution of this system are presented in fig.6. Different curves correspond to the different pairs (Z_j, V) of optimal criteria in (9), where $j = 0$ correspond to the motion in the tube with non-permeable walls, $j = 1, 2$ – the tube with permeable walls at $w = w_0 = const$ and $w = 2w_0(1 - y)$ respectively. Two dashed lines in fig.6 correspond to the upper and lower boundaries of the measured data ($d_0/d_1 \in [1.52; 1.91]$, $L_0/L_1 \in [0.22; 2.81]$ [10,11]) for a number of leaves with different venation types including *Cotinus obovatus* leaf.

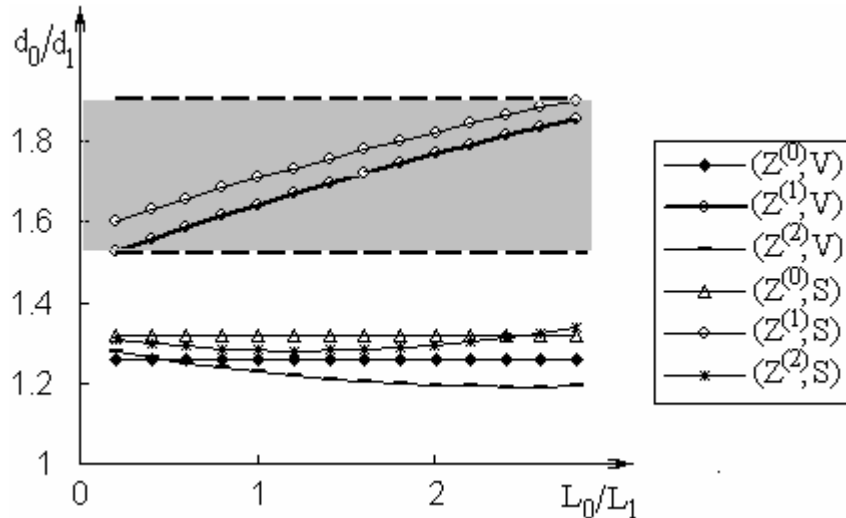


Figure 6: The dependences d_0/d_1 (L_0/L_1) for the optimal symmetrical bifurcation.

The models of the optimal bifurcation of the tubes with permeable walls both at constant and linearly decreasing functions w agree closely with the measured data. Two cases of $w(y)$ slightly differ. For comparison the results of solution of the problem (9) at $G = S$ are presented in fig.6 as well. All these curves lie closely to parameters of the optimal bifurcation of tubes with non-permeable walls and do not agree with the real data. Consequently the model of optimal bifurcation of the tubes with permeable walls, which

deliver the liquid with a minimal hydraulic resistance at a given volume is best suited to the measured data. This model perfectly corresponds to physics of the sap motion through the vein systems of the leaves.

4 POSSIBLE MECHANISMS OF THE OPTIMAL BIFURCATION FORMATION

In contrast to the tubes with non-permeable walls (blood vessels) the total resistances $Z^{(1,2)}$ depend on the relation between the inflow Q_1 at the point of entry and the outflow Φ_1 through the permeable wall. In the event of $\Phi_1/Q_1 \ll 1$ or $\Phi_1/Q_1 = const$ in each tube of the branching system the total resistances of the bifurcation differ from $Z^{(0)} = Z_1^p + Z_2^p / 2$ by the constant only. Thus the optimal bifurcation of the tubes with the permeable walls obeys the same Murray's law with $g = 3$ as described in the section 2 and the similarity between the geometry of mammal and plant transport systems take place. The first condition ($\Phi/Q \ll 1$) is possible for the 1-2 order vessels but is quite impossible for the last-order vessels where $\Phi \approx Q$. The second limitation can underlie the formation of the self-similar transport system in plant leaves.

Unlike the mammal vasculature the plant vessels are empty rigid tubes without alive cell contents so they can not estimate either wall shear stress or any other mechanical parameters. Only the alive plant cells in the influence domains would do that. The possible mechanism of the optimal transport system formation can be connected with the balance between the sap amount that is needed for all the cells in the domain S_j and the corresponding Φ_j that can be provided by the lateral surface pD_jL_j of the tube. Assuming $w_j = S_j / (2\pi d_j L_j)$ the model of the branching system of the tubes with optimal bifurcations where Φ_j/Q_j is the same for each vessel a constant was constructed (fig.3).

On the basis of the model of the viscous liquid motion in a tube with permeable walls the generalization of the Murray's law can be obtained [19].

5 LIQUID MOTION IN AN ELEMENTARY MICROCIRCULATORY CELL OF THE LEAF

A part of the leaf blade which is bounded either veins or leaf edge can be considered as an elementary physiological unit of the leaf construction. Measurements of the branching angles in the leaf venation revealed a scatter away from the theoretical optimal values (figs 4,5). The scatter can indicate the real cost of the optimality for the plant. In the chapter the cost penalty of nonoptimal branching angles is estimated for the rectangular model of the elementary microcirculatory cell of the plant leaf (fig.7). Here AB is a parent vessel, BC,BL are daughters' vessels, B is a branching point with coordinates $(x, y) = (p, q)$, $|OA| = a$ is given from the experimental data. In the cell HC, CL and OL are impermeable volumes that is the boarders HC, CL and OL separate different influence domains so that the liquid moves through the tubes AB, BC and BL and their permeable walls into the continuous media of the

main tissue of the plant leaf due to the active absorption of the alive cells of the rectangular OHCL.

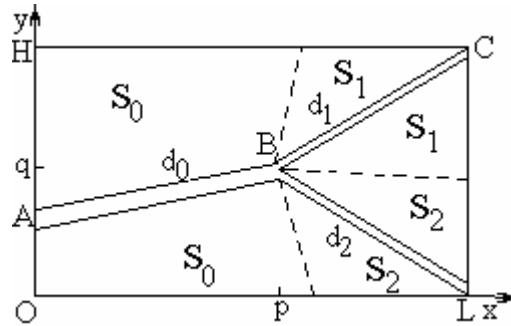


Figure 7: Schema of the elementary microcirculatory cell of the plant leaf.

The concept of the elementary microcirculatory cell is similar to the ‘‘junction box’’ that has been proposed for the arterial branching [5]. In a leaf the bifurcation ABCL is considered and the line OH is constructed parallel to the CL so that three values L,H,a are characterized the cell and a can vary so that for different bifurcations $a \in [0, H]$. For a given set L,H,a the dimensionless values $L^\circ = L/H, a^\circ = a/H$ can be introduced and $a^\circ \in [0,1]$. It is assumed that (p,q) are the coordinate of the optimal bifurcation so that

$$Z(L_0, R_0) + Z(L_1, R_1) + Z(L_2, R_2) \rightarrow \min \quad (10)$$

The problem (10) has been solved numerically for a given a° and the coordinates (p,q) have been determined on the basis of the theory introduced in chapter 3. Three tubes AB,BC,BL have been regarded as optimal in the meaning of the criterion (9). Then the total energy expenses

$$W = \sum_{j=0}^2 (Q_j (Z^{(j)})^2 + \chi \pi R_j^2 L_j)$$

where χ is a metabolic constant have been estimated for different small variations of the coordinates (p,q) of the bifurcation point. Some results of the calculations are presented in fig.8. Relatively large variations led to comparatively small increasing in the cost penalty.

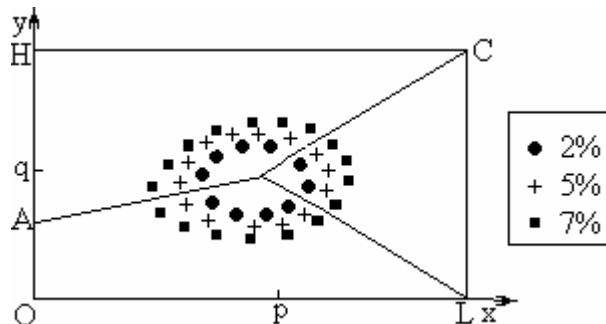


Figure 8: Cost of the nonoptimal branching angles of a bifurcation.

Large resulting variations in the branching angles $\alpha_{1,2} = \pm(20 - 25)\%$ will increase the total energy costs in 7% only. The result correspond to the calculations for the “junction box” and the bifurcation with impermeable walls [5]. So in terms of the energy expenses the measured branching angles in the plant leaf venations are highly consistent with the theoretical optimal principles.

6 CONCLUSIONS

- Relations between the diameters d_j , branching angles α_j and the influence domains S_j are the same for blood vessels in mammals, the vein systems of leaves, branching of the trees and shoots and correspond to the model of the optimal pipeline that provide liquid delivering at the minimum total energy costs for the liquid motion and the pipeline construction and maintenance.
- Control over the optimal pipeline formation in the developing plant tissue can be connected with the maintenance of the balance between the delivery and consumption, i.e. between form and function. The retardation of the cell divisions and growth in the domains of influence at the insufficient water delivering by a given vessel provides for the optimal transport system formation in plant leaves.
- Large scatter of the measured parameters of the geometrical parameters of the plant leaf veins can be explained by the relatively small energy costs of large variations in the branching angles of the bifurcation.

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