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A Way of Relating Instantaneous and Finite Screws
based on the Screw Triangle Product

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Abstract: It has been a desire to unify the models for structural and parametric analyses and design in the field of robotic mechanisms. This requires a mathematical tool that enables analytical description, formulation and operation possible for both finite and instantaneous motions. This paper presents a method to investigate the algebraic structures of finite screws represented in a quasi-vector form and instantaneous screws represented in a vector form. By revisiting algebraic operations of screw compositions, this paper examines associativity and derivative properties of the screw triangle product of finite screws and produces a vigorous proof that a derivative of a screw triangle product can be expressed as a linear combination of instantaneous screws. It is proved that the entire set of finite screws forms an algebraic structure as Lie group under the screw triangle product and its time derivative at the initial pose forms the corresponding Lie algebra under the screw cross product, allowing the algebraic structures of finite screws in quasi-vector form and instantaneous screws in vector form to be revealed.

Keywords: Robotic mechanisms, Screw theory, Finite screw, Screw triangle, Lie group and Lie algebra

1. Introduction

The instantaneous screw [1] has been considered as a powerful tool and extensively applied in analysis and design of serial and parallel mechanisms. By taking twist of an end link as a linear combination of instantaneous screws produced by joints, Hunt [2] and Duffy [3] developed a method to formulate Jacobian matrix. Angeles [4] presented an alternative way to do so by taking all actuation wrenches as instantaneous screws. These works were extended by Joshi and Tsai [5], Huang and Liu [6] in dealing with lower mobility serial and parallel mechanisms, resulting in the overall Jacobian and generalized Jacobian matrix that allow velocity, force, stiffness and rigid body dynamic analyses to be unified using instantaneous screw theory. Another use of screw theory in analysis and synthesis of robotic mechanisms is mobility analysis and type synthesis. Regarding small and finite displacements or velocities and constraint forces of a mechanism in a systematic frame of a screw system and its reciprocal system, Dai and Rees Jones [7] revealed the interrelationship and intersection of all screw systems and their reciprocal systems by introducing set theory in screw theory. In the meantime, they [8] developed a linear algebraic procedure to obtain a reciprocal screw system and its basis in a null space construction using cofactors from a screw algebra context. These theories were directly used in mobility analysis of different types of parallel mechanisms, including those having overconstraints [9]. Utilizing annihilator property between a screw system and its reciprocal system, Huang and Li [10], Fang [11], Kong [12] and colleagues proposed a number of simple and effective approaches for type synthesis of various parallel mechanisms. However, since instantaneous screws are not valid to represent finite motions, final verification is required to ensure consistency between finite and instantaneous motions [13-15].

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In order to describe finite motions precisely, the finite screw was proposed. Investigation into finite screws can be traced back to the pioneering work of Dimentberg [16-17] and Roth [18] in dealing with composition of finite motions of a rigid body, leading to development of the screw triangle product that provides a sound geometric interpretation to the Rodrigues formula with dual angles [17]. Investigations into finite screws were developed by Roth [18], Parkin [19], Huang [20-21] and Dai [22-23]. They proposed a simple and widely accepted form of finite screws that can be analogously expressed as a quasi-vector with six algebraic entries. A finite screw in this form looks extraordinarily similar to an instantaneous one even if it is nonlinear in nature. This form allows the linear format of a screw triangle to be made in such a way that the composition of two finite screws can be expressed as the sum of five meaningful terms [21, 24]. Based upon the algebraic characteristic of finite screws, Dai, Holland and Kerr [22] and Dai [25-26] revealed the relationship between finite screws and instantaneous screws through solving the eigenscrew and derivative of the finite displacement screw matrix [25]. Utilizing finite screw theory, finite motion analyses of different geometrical elements, such as points, lines and planes, as well as simple open loop and closed loop mechanisms are carried out by Huang [20, 27], Hunt [28].

It has been a desire to develop a theoretical package that enables analysis and synthesis of robotic mechanisms to be integrated into a unified and consistent process [29]. This issue needs to relate nonlinear finite to linearized instantaneous motions of rigid body systems. Hence, a preliminary and essential step to achieve the aforementioned goal is to develop a general and effective method that enables the description, formulation and operation of finite and instantaneous motions to be implemented under a consistent and unified mathematical tool, a fundamental and challenging issue in the field of mechanisms and robotics. The methods available at hand can be roughly divided into three categories, i.e. matrix group based method, dual quaternion based method and screw theory based method.

The matrix group based method can be traced back to the Erlangen program proposed by Klein [30]. By utilizing matrix groups to describe finite motions of rigid body systems, Brockett [31] applied the exponential map between Lie group SE(3) and Lie algebra se(3) to relating models for finite motions to that for instantaneous motions [32]. However, two barriers are encountered in the use of matrix groups for finite motion composition. The first barrier arises from implementation of matrix groups for affine transformations where finite motions of a rigid body cannot be directly represented by Chasles' axis [17] as well as by the angular and/or linear displacement about the axis, leading to a complicated description of rigid body motion. The second barrier comes from that the finite motion composition cannot be algebraically derived by Baker-Campbell-Hausdorff formula [33]. Consequently, motion patterns of a number of parallel mechanisms cannot precisely be described using the existing matrix group based method since they can no longer be represented by products of several Lie subgroups [34].

The dual quaternion based method can be traced back to description of rotations of a rigid body by means of Euler's four-square identity, Euler-Rodrigues parameters [23, 35] and Hamilton quaternions [36]. Perez and McCarthy [37] seem to be the first to use the dual quaternions to do analyses for finite and instantaneous motions of serial kinematic chains. In their work, unit dual quaternions and unit pure dual quaternions were used for describing finite and instantaneous motions, for the algebraic structure of the former is a double cover of Lie group SE(3) whose Lie algebra in turn constitutes the latter as by Selig [38]. With the aid of group theory, Selig [39-40] and Dai [26, 35] investigated the algebraic properties of the exponential and Cayley maps between unit dual quaternions and unit pure dual quaternions. By introducing the notation of high-dimensional Clifford algebra, Selig [40] and Featherstone [41] extended the dual quaternions representation to deal with rigid body dynamics. However, a unit dual quaternion is not the simplest form of a rigid body motion. The redundancy in dual quaternion representation may cause complexity in analytical expressions of the finite motion operations. In addition, the Rodrigues formula with dual angles is not the simplest form of the Baker-Campbell-Hausdorff formula when it is applied to composition of finite motions of a rigid body [42].

The screw theory based method depends on the development of instantaneous screws and finite screws. Considering

the algebraic characteristic of finite screws, Dai [22] demonstrated the relationship between finite displacement screw operation and the different matrix representations of SE(3) elements as well as quaternions [25, 35]. By solving the eigenscrew and derivative of finite displacement screw matrix, Dai [25] formulated the eigen and differential mappings between finite screws and instantaneous screws. Meanwhile, the consistency between these mappings and the exponential mapping of matrix Lie group/Lie algebra or the Euler-Rodrigues formula was revealed [25-26, 35]. The correspondence between vector subspaces and screw systems was discussed by Huang, Sugimoto and Parkin [43] to differentiate finite screw systems from screw systems arising from instantaneous kinematics and statics. Although the composition of finite motions can be expressed as a screw triangle product of finite screws in quasi-vector form [21, 24], and that of instantaneous motions as the linear combination of instantaneous screws in vector form, the algebraic structures of two kinds of screws remain an open issue to be investigated.

Addressing on the need to integrate models for analysis and synthesis of robotic mechanisms in a unified mathematic framework, this paper intends to reveal algebraic structures of finite screws in a quasi-vector form and instantaneous screws in a vector form. The paper is organized as follows. Having a brief review of the state-of-the-art of finite and instantaneous screw theory in Section 1, Section 2 presents the derivation of finite screws from dual quaternions and addresses the description of finite screws in quasi-vector form and instantaneous screws in vector form. Section 3 explores the associativity and derivative properties of the screw triangle product, resulting in that the set of finite screws is an associative and differentiable algebraic structure. This leads to a vigorous proof in Section 4 that the entire set of finite screws forms a Lie group under the screw triangle product and its time derivative at the identity forms the corresponding Lie algebra under the screw cross product before the conclusions are drawn in Section 5.

2. The Description of Finite and Instantaneous Screws

2.1 The derivation of finite screws from dual quaternions

The finite motion of a body from its initial pose to an arbitrary pose can be parameterized as a unit dual quaternion [26, 36]

$$\boldsymbol{D} = \cos\frac{\hat{\theta}}{2} + \sin\frac{\hat{\theta}}{2}\boldsymbol{S} \tag{1}$$

where $\boldsymbol{S} = \boldsymbol{s}_f + \varepsilon\left(\boldsymbol{r}_f \times \boldsymbol{s}_f\right)$ is the dual vector describing the finite motion axis (the Chasles' axis), $\boldsymbol{s}_f$ is the unit vector of this axis, $\boldsymbol{r}_f$ is the position vector pointing from the fixed reference point to an arbitrary point on the axis, ε is defined as the dual unit with the property $\varepsilon^2 = 0$; $\hat{\theta} = \theta + \varepsilon t$ is the dual angle, θ and t are the angular displacement about and linear displacement along the Chasles' axis with respect to the initial pose.

The linearization of sine and cosine of $\hat{\theta}$ allows $\boldsymbol{D}$ to be expressed as the sum of a dual scalar and a dual vector

$$\boldsymbol{D} = \left(\cos\frac{\theta}{2} - \varepsilon\frac{t}{2}\sin\frac{\theta}{2}\right) + \left(\sin\frac{\theta}{2}\boldsymbol{s}_f + \varepsilon\left(\sin\frac{\theta}{2}\left(\boldsymbol{r}_f \times \boldsymbol{s}_f\right) + \frac{t}{2}\cos\frac{\theta}{2}\boldsymbol{s}_f\right)\right) \tag{2}$$

It is easy to see that the dual vector contains all the elements, i.e. the Chasles' axis, the angular and linear displacement necessary to describe a finite motion. Thus, dividing the dual vector by $0.5\cos\left(\theta/2\right)$ leads to a finite displacement screw (or finite screw for simplicity) in six-dimensional quasi-vector form, which can be considered as the non-redundant minimal description of finite motion.

$$S_f = 2\tan\frac{\theta}{2}\begin{pmatrix} s_f \\ r_f \times s_f \end{pmatrix} + t\begin{pmatrix} 0 \\ s_f \end{pmatrix} \tag{3}$$

2.2 The relationship between a finite screw and an instantaneous screw

In this section, we develop the relationship between a single finite screw in quasi-vector form and a single instantaneous screw in vector form. Assuming that the arbitrary pose of a body varies with time, the time derivative of S_f gives

$$\dot{S}_f = \frac{\dot{\theta}}{\cos^2\left(\frac{\theta}{2}\right)}\begin{pmatrix} s_f \\ r_f \times s_f \end{pmatrix} + 2\tan\frac{\theta}{2}\begin{pmatrix} \dot{s}_f \\ \dot{r}_f \times s_f + r_f \times \dot{s}_f \end{pmatrix} + \dot{t}\begin{pmatrix} 0 \\ s_f \end{pmatrix} + t\begin{pmatrix} 0 \\ \dot{s}_f \end{pmatrix} \tag{4}$$

where $\dot{\theta}$ and $\dot{t}$ are the angular velocity about and linear velocity along the screw axis at the instant considered. Note that at the initial pose where $\theta = 0$ and $t = 0$, the finite displacement axis (the Chasles's axis) is coincident with the instantaneous velocity axis (the instantaneous screw axis or the Mozzi's axis) at the instant. Thus, Eq. (4) can be rewritten as

$$\dot{S}_f\big|_{\substack{\theta=0 \\ t=0}} = \dot{\theta}\begin{pmatrix} s_t \\ r_t \times s_t \end{pmatrix} + \dot{t}\begin{pmatrix} 0 \\ s_t \end{pmatrix} \tag{5}$$

This means that the time derivative of S_f at the initial pose is exactly the instantaneous screw (or twist) S_t at the same pose

$$S_t = \dot{S}_f\big|_{\substack{\theta=0 \\ t=0}} = \begin{pmatrix} \omega \\ v \end{pmatrix} \tag{6}$$

where s_t denotes the unit vector of the instantaneous screw axis, r_t denotes the position vector pointing from the fixed reference point to an arbitrary point on the axis. ω and v should be understood as the angular velocity of the body and the linear velocity of the point instantaneously coincident with the fixed reference point at the initial pose, and s_t, r_t, $\dot{\theta}$ and $\dot{t}$ appearing in Eq. (5) should be understood as the corresponding quantities at the same pose.

3. Associativity and Derivative Laws of Screw Triangle Products

Lie group theory [44-45] shows that a representation of a group should be a subgroup of a general linear group acting on a vector space. In this sense, the entire set of finite screws in 6×1 quasi-vector form is neither SE(3) nor its representation because a finite screw is not a linear transformation acting on any point or line coordinate systems. Therefore, unlike matrix or dual quaternion representations of SE(3) or its double cover, the set of finite screws cannot simply inherit all the properties of SE(3). Thus, the algebraic structure of this set under screw triangle product needs to be revealed. For a given set and a product, four check points must be examined to show that this is a group, i.e. closure, associativity, existence of an identity, and existence of inverses. In addition, differentiability should be ensured to prove it is a Lie group. In this section, we firstly investigate closure and associativity of finite screws under the screw triangle product. Then, we will investigate the relationship between finite motions generated by a number of finite screws and

instantaneous motions produced by the same number of instantaneous screws. The outcomes are useful for revealing algebraic structures of finite screws in quasi-vector form and instantaneous screws in vector form.

Assume that the finite motion of a rigid body is generated by a number of successive one-parameter finite screws (1-DOF finite motion). Each has the form given in Eq. (3)

$$S_{f,j} = 2\tan\frac{\theta_j}{2}\begin{pmatrix} s_{f,j} \\ r_{f,j} \times s_{f,j} \end{pmatrix} + t_j\begin{pmatrix} 0 \\ s_{f,j} \end{pmatrix}, \quad j = 1, 2, 3, \cdots \tag{7}$$

where $\theta_j \equiv 0$ for pure translation, otherwise the ratio $t_j/\theta_j = h_j$ is set to be a constant for screw motion. Note that $s_{f,j}$ and $r_{f,j}$ are constant vectors since the finite motion is accounted from the initial pose where $s_{f,j} = s_{t,j}$ and $r_{f,j} = r_{t,j}$.

The composition of two successive finite screws can be derived from the algebraic product of two dual quaternions D_1 and D_2 which have the form given in Eq. (1) [26, 36]

$$D_{12} = D_2 D_1 = \cos\frac{\hat{\theta}_{12}}{2} + \sin\frac{\hat{\theta}_{12}}{2} S_{12} \tag{8}$$

where

$$\cos\frac{\hat{\theta}_{12}}{2} = \cos\frac{\hat{\theta}_1}{2}\cos\frac{\hat{\theta}_2}{2} - \sin\frac{\hat{\theta}_1}{2}\sin\frac{\hat{\theta}_2}{2} S_1 \square S_2 \tag{9}$$

$$\sin\frac{\hat{\theta}_{12}}{2} S_{12} = \sin\frac{\hat{\theta}_1}{2}\cos\frac{\hat{\theta}_2}{2} S_1 + \cos\frac{\hat{\theta}_1}{2}\sin\frac{\hat{\theta}_2}{2} S_2 + \sin\frac{\hat{\theta}_1}{2}\sin\frac{\hat{\theta}_2}{2} S_2 \times S_1 \tag{10}$$

Linearizing sine and cosine of the dual angles, and equating the real and dual parts on both sides of Eq. (10), gives

$$\sin\frac{\theta_{12}}{2} s_{12} = \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} s_{f,1} + \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2} s_{f,2} + \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(s_{f,2} \times s_{f,1}\right) \tag{11}$$

$$\begin{aligned}
&\sin\frac{\theta_{12}}{2}\left(r_{f,12} \times s_{f,12}\right) + \frac{t_{12}}{2}\cos\frac{\theta_{12}}{2} s_{f,12} \\
&= \left(\frac{t_1}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \frac{t_2}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right) s_{f,1} + \sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2}\left(r_{f,1} \times s_{f,1}\right) + \left(\frac{t_2}{2}\cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \frac{t_1}{2}\sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right) s_{f,2} \\
&+ \cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(r_{f,2} \times s_{f,2}\right) + \left(\frac{t_2}{2}\sin\frac{\theta_1}{2}\cos\frac{\theta_2}{2} + \frac{t_1}{2}\cos\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\right)\left(s_{f,2} \times s_{f,1}\right) \\
&+ \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\left(s_{f,2} \times\left(r_{f,1} \times s_{f,1}\right) + \left(r_{f,2} \times s_{f,2}\right) \times s_{f,1}\right)
\end{aligned} \tag{12}$$

Dividing both sides of Eqs. (11)-(12) by $0.5\cos\left(\theta_{12}/2\right)$, respectively, and rewriting them into a finite screw form, finally results in

$$S_{f,12} = S_{f,1} \square S_{f,2} = \frac{1}{1 - \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2} s_{f,1}^T s_{f,2}}\left(S_{f,1} + S_{f,2} - \frac{1}{2} S_{f,1} \times S_{f,2} - \tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\left(t_2\begin{pmatrix} 0 \\ s_{f,1} \end{pmatrix} + t_1\begin{pmatrix} 0 \\ s_{f,2} \end{pmatrix}\right)\right) \tag{13}$$

where symbol "$\square$" is referred to as screw triangle product [21], and $S_{f,1} \times S_{f,2}$ is defined as the screw cross product of $S_{f,1}$ and $S_{f,2}$

$$S_{f,1} \times S_{f,2} = \begin{pmatrix} 2\tan\dfrac{\theta_1}{2}s_{f,1} \times 2\tan\dfrac{\theta_2}{2}s_{f,2} \\ 2\tan\dfrac{\theta_1}{2}s_{f,1} \times \left(2\tan\dfrac{\theta_2}{2}r_{f,2}\times s_{f,2}+t_2 s_{f,2}\right)+\left(2\tan\dfrac{\theta_1}{2}r_{f,1}\times s_{f,1}+t_1 s_{f,1}\right)\times 2\tan\dfrac{\theta_2}{2}s_{f,2} \end{pmatrix} \tag{14}$$

Obviously, $S_{f,12}$ satisfies the closure of finite motion composition since it can be rewritten into the standard form given by Eq. (3) *via* adequate algebraic operation.

3.1 The associativity law of successive screw triangle products

The associativity law of composition of finite screws can directly be proved using Eq. (13). Consider the composition of three arbitrary finite screws

$$S_{f,123} = S_{f,1}\,\square\,S_{f,2}\,\square\,S_{f,3} \tag{15}$$

By assuming that $S_{f,12} = S_{f,1}\,\square\,S_{f,2}$ is the screw triangle product of the first two, the composition of $S_{f,12}$ and $S_{f,3}$ yields

$$S_{f,(12)3} = S_{f,12}\,\square\,S_{f,3} \tag{16}$$

Hence, visualizing $S_{f,12}$ as the first finite screw given in Eq. (13), leads to

$$S_{f,(12)3} = \frac{1}{1-\tan\dfrac{\theta_{12}}{2}\tan\dfrac{\theta_3}{2}s_{f,12}^{\mathrm{T}}s_{f,3}}\left(S_{f,12}+S_{f,3}-\frac{1}{2}S_{f,12}\times S_{f,3}-\tan\dfrac{\theta_{12}}{2}\tan\dfrac{\theta_3}{2}\left(t_3\begin{pmatrix}\mathbf{0}\\s_{f,12}\end{pmatrix}+t_{12}\begin{pmatrix}\mathbf{0}\\s_{f,3}\end{pmatrix}\right)\right) \tag{17}$$

or

$$S_{f,(12)3} = \frac{1}{A}\begin{pmatrix} S_{f,1}+S_{f,2}+S_{f,3}-\dfrac{1}{2}S_{f,1}\times S_{f,2}-\dfrac{1}{2}S_{f,1}\times S_{f,3}-\dfrac{1}{2}S_{f,2}\times S_{f,3} \\ +\dfrac{1}{4}\left(S_{f,1}\times S_{f,2}\right)\times S_{f,3}-\left(1-B\right)S_{f,3}+\dfrac{1}{2}S_{fp,12}\times S_{f,3}-S_{fp,12}-BS_{fp,(12)3} \end{pmatrix} \tag{18}$$

where

$$A = 1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}s_{f,1}^{\mathrm{T}}s_{f,2}-\tan\frac{\theta_1}{2}\tan\frac{\theta_3}{2}s_{f,1}^{\mathrm{T}}s_{f,3}-\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}s_{f,2}^{\mathrm{T}}s_{f,3}-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}\tan\frac{\theta_3}{2}s_{f,3}^{\mathrm{T}}\left(s_{f,2}\times s_{f,1}\right),$$

$$B = 1-\tan\frac{\theta_1}{2}\tan\frac{\theta_2}{2}s_{f,1}^{\mathrm{T}}s_{f,2}, \quad S_{fp,12} = \begin{pmatrix} \mathbf{0} \\ \tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_2}{2}t_2 s_{f,1}+\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_2}{2}t_1 s_{f,2} \end{pmatrix},$$

$$S_{fp,(12)3} = \begin{pmatrix} \mathbf{0} \\ \tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_3}{2}t_3 s_{f,1}+\tan\dfrac{\theta_2}{2}\tan\dfrac{\theta_3}{2}t_3 s_{f,2}+\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_2}{2}\tan\dfrac{\theta_3}{2}t_3\left(s_{f,2}\times s_{f,1}\right) \\ +2\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_2}{2}\tan\dfrac{\theta_3}{2}\left(r_{f,1}-r_{f,2}\right)^{\mathrm{T}}\left(s_{f,1}\times s_{f,2}\right)s_{f,3}+\left(\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_3}{2}+\tan\dfrac{\theta_2}{2}\tan\dfrac{\theta_3}{2}s_{f,1}^{\mathrm{T}}s_{f,2}\right)t_1 s_{f,3} \\ +\left(\tan\dfrac{\theta_2}{2}\tan\dfrac{\theta_3}{2}+\tan\dfrac{\theta_1}{2}\tan\dfrac{\theta_3}{2}s_{f,1}^{\mathrm{T}}s_{f,2}\right)t_2 s_{f,3} \end{pmatrix}$$

Similarly, assuming that $S_{f,23} = S_{f,2} \Box S_{f,3}$ is the screw triangle product of the last two finite screws in Eq. (15), and replacing the subscripts '(12)' by '1' and '3' by '(23)' in Eq. (17), yields

$$S_{f,1(23)} = S_{f,1} \Box S_{f,23} = \frac{1}{A} \left(\begin{array}{l} S_{f,1} + S_{f,2} + S_{f,3} - \frac{1}{2} S_{f,1} \times S_{f,2} - \frac{1}{2} S_{f,1} \times S_{f,3} - \frac{1}{2} S_{f,2} \times S_{f,3} \\ + \frac{1}{4} S_{f,1} \times \left(S_{f,2} \times S_{f,3} \right) - \left(1 - C \right) S_{f,1} + \frac{1}{2} S_{f,1} \times S_{fp,23} - S_{fp,23} - C S_{fp,1(23)} \end{array} \right) \tag{19}$$

It is easy to prove that the following identity holds

$$\frac{1}{4} \left(S_{f,1} \times S_{f,2} \right) \times S_{f,3} - \left(1 - B \right) S_{f,3} + \frac{1}{2} S_{fp,12} \times S_{f,3} - S_{fp,12} - B S_{fp,(12)3}$$
$$\equiv \frac{1}{4} S_{f,1} \times \left(S_{f,2} \times S_{f,3} \right) - \left(1 - C \right) S_{f,1} + \frac{1}{2} S_{f,1} \times S_{fp,23} - S_{fp,23} - C S_{fp,1(23)} \tag{20}$$

Thus

$$S_{f,123} = S_{f,(12)3} = S_{f,1(23)} \tag{21}$$

This means that the screw triangle product satisfies the associativity, i.e. in the composition of a set of successive finite screws, they can be divided into a number of groups first so long as their sequences within and amongst groups are unchanged. In this way, we can prove that the entire set of finite screws under the screw triangle product is of a closed and associative algebraic structure.

3.2 The derivative law of screw triangle product

In order to investigate the differential property of screw triangle product, let the screw triangle product of two finite screws be written as

$$S_{f,12} = S_{f,1} \Box S_{f,2} = \frac{E}{F} \tag{22}$$

where

$$E = \left(S_{f,1} + S_{f,2} - \frac{1}{2} S_{f,1} \times S_{f,2} - G \left(t_2 \binom{0}{s_{f,1}} + t_1 \binom{0}{s_{f,2}} \right) \right), \quad F = 1 - G s_{f,1}^{\mathrm{T}} s_{f,2}, \quad G = \tan \frac{\theta_1}{2} \tan \frac{\theta_2}{2}$$

Taking time derivative of Eq. (22) and noting that $s_{f,j}$ and $r_{f,j}$ ($j = 1, 2$) are constant vectors, leads to

$$\dot{S}_{f,12} = \frac{\dot{E}}{F} - \frac{E \dot{F}}{F^2} \tag{23}$$

where

$$\dot{E} = \left(\dot{S}_{f,1} + \dot{S}_{f,2} - \frac{1}{2} \left(\dot{S}_{f,1} \times S_{f,2} + S_{f,1} \times \dot{S}_{f,2} \right) - \dot{G} \left(t_2 \binom{0}{s_{f,1}} + t_1 \binom{0}{s_{f,2}} \right) + G \left(\dot{t}_2 \binom{0}{s_{f,1}} + \dot{t}_1 \binom{0}{s_{f,2}} \right) \right), \quad \dot{F} = -\dot{G} s_{f,1}^{\mathrm{T}} s_{f,2}$$

At the initial pose where $\theta_j = 0$ and $t_j = 0$, we have

$$E = \mathbf{0}, \quad G = 0, \quad F = 1, \quad \dot{G} = 0, \quad \dot{F} = 0, \quad \dot{S}_{f,1} \times S_{f,2} + S_{f,1} \times \dot{S}_{f,2} = \mathbf{0}$$

Thus

$$\dot{S}_{f,12}\Big|_{\substack{\theta_j=0 \\ t_j=0}}^{j=1,2} = \dot{S}_{f,1}\Big|_{\substack{\theta_j=0 \\ t_j=0}}^{j=1,2} + \dot{S}_{f,2}\Big|_{\substack{\theta_j=0 \\ t_j=0}}^{j=1,2} = S_{t,1} + S_{t,2} \tag{24}$$

Note that $\dot{S}_{f,12}$ can also be rewritten into the standard form given in Eq. (6), it thereby satisfies the closure of instantaneous motion composition. Hence, the property given in Eq. (24) can be extended to a general form for the system composed of n 1-DOF motion generators

$$\dot{S}_{f,12...n}\Big|_{\substack{\theta_j=0 \\ t_j=0}}^{j=1,2,\cdots,n} = \sum_{j=1}^{n} \dot{S}_{f,j}\Big|_{\substack{\theta_j=0 \\ t_j=0}}^{j=1,2,\cdots,n} = \sum_{j=1}^{n} S_{t,j} \tag{25}$$

where

$$S_{t,j} = \dot{\theta}_j \begin{pmatrix} s_{t,j} \\ r_{t,j} \times s_{t,j} \end{pmatrix} + \dot{t}_j \begin{pmatrix} \mathbf{0} \\ s_{t,j} \end{pmatrix} = \begin{pmatrix} \boldsymbol{\omega}_j \\ v_j \end{pmatrix}, \quad j = 1, 2, \cdots, n$$

It is clear that any curve in the entire set of finite screws is continuous and differentiable, and its time derivative forms an instantaneous screw system. The foregoing associativity and derivative properties allow the algebraic structures of finite and instantaneous screws as addressed in what follows.

4. Algebraic Structures of Finite and Instantaneous Screws

On the basis of Sections 2 and 3, we intend to reveal the specific algebra structures of finite and instantaneous screws. Let $\{S_f\}$ be the entire set of finite screws in the form given by Eq. (3)

$$\{S_f\} = \left\{ 2\tan\frac{\theta}{2} \begin{pmatrix} s_f \\ r_f \times s_f \end{pmatrix} + t \begin{pmatrix} \mathbf{0} \\ s_f \end{pmatrix} \Big| s_f, r_f \in \Box^3, \ |s_f| = 1, \ \theta, t \in \Box \right\} \tag{26}$$

In the above sections, the closure of this set under screw triangle product and the associativity and derivative properties of this product are proved. And let $S_{f0} \in \{S_f\}$ at $\theta = 0$ and $t = 0$ be the identity element of $\{S_f\}$, i.e.

$$S_{f,0} = \begin{pmatrix} \mathbf{0} \\ \mathbf{0} \end{pmatrix} \tag{27}$$

Note that $-S_f \in \{S_f\}$ for $\forall S_f \in \{S_f\}$ such that

$$S_f \Box S_{f,0} = S_{f,0} \Box S_f = S_f,$$

$$S_f \Box (-S_f) = (-S_f) \Box S_f = S_{f,0} \tag{28}$$

This means that there exists a unique inverse for each finite screw.

Building upon the above proofs and analyses, we can conclude that the algebraic structure of $\{S_f\}$ is a Lie group

under the screw triangle product according to the Lie group theory [44-45]. Here, we refer $\{S_f\}$ to as Lie group of finite screws. Obviously, a set of one-parameter finite screws given by Eq. (7) forms a subgroup of this Lie group.

In the neighborhood of the identity, let $\{S_t\} = \left\{ \dot{S}_f \big|_{\substack{\theta=0 \\ t=0}} \right\}$ be the entire set of instantaneous screws given in Eq. (6)

$$\{S_t\} = \left\{ \dot{S}_f \big|_{\substack{\theta=0 \\ t=0}} \right\} = \left\{ \dot{\theta} \begin{pmatrix} s_t \\ r_t \times s_t \end{pmatrix} + \dot{t} \begin{pmatrix} 0 \\ s_t \end{pmatrix} \middle| s_t, r_t \in \square^3, |s_t|=1, \dot{\theta}, \dot{t} \in \square \right\} = \left\{ \begin{pmatrix} \omega \\ v \end{pmatrix} \middle| \omega, v \in \square^3 \right\} \tag{29}$$

This set spans a vector space having the following properties under screw cross product: bilinear (Eqs. (30) and (31)), noncommutative (Eq. (32)) and Jocobi identity (Eq. (33))

$$\left(S_{t,1} + S_{t,2} \right) \times S_{t,3} = S_{t,1} \times S_{t,3} + S_{t,2} \times S_{t,3},$$

$$S_{t,1} \times \left(S_{t,2} + S_{t,3} \right) = S_{t,1} \times S_{t,2} + S_{t,1} \times S_{t,3} \tag{30}$$

$$\left(kS_{t,1} \right) \times S_{t,2} = S_{t,1} \times \left(kS_{t,2} \right) = k \left(S_{t,1} \times S_{t,2} \right), \quad \forall k \in \square \tag{31}$$

$$S_{t,1} \times S_{t,2} = -S_{t,2} \times S_{t,1} \tag{32}$$

$$S_{t,1} \times \left(S_{t,2} \times S_{t,3} \right) + S_{t,2} \times \left(S_{t,3} \times S_{t,1} \right) + S_{t,3} \times \left(S_{t,1} \times S_{t,2} \right) = 0 \tag{33}$$

Thus, $\{S_t\}$ is a Lie algebra under the screw cross product in the light of [44-45]. So, we refer $\{S_t\}$ to as Lie algebra of instantaneous screws.

Finally, consider a continuous curve $\{S_f\}^C \subseteq \{S_f\}$ composed by n Lie subgroups of one-parameter finite screws, i.e.

$$\{S_f\}^C = \{S_{f,12\cdots n}\} = \{S_{f,1} \square S_{f,2} \square \cdots \square S_{f,n}\} \tag{34}$$

Note that the time derivative of $S_{f,12\cdots n} \in \{S_f\}^C$ at the neighborhood of the identity element has the form (see Eq. (25))

$$\dot{S}_{f,12\ldots n} \big|_{\substack{\theta_j=0 \\ t_j=0}, j=1,2,\cdots,n} = \sum_{j=1}^{n} S_{t,j} \tag{35}$$

Thus, the tangent space of $\{S_f\}^C$ in the neighborhood of the identity element forms an instantaneous screw sub-system of $\{S_t\}$. So, we conclude that $\{S_t\}$ is the corresponding Lie algebra of Lie group $\{S_f\}$.

5. Conclusions

This paper presents a way of relating instantaneous and finite screws based on screw triangle product. The following conclusions are drawn.

1) The screw triangle product of a number of one-parameter finite screws in 6×1 quasi-vector form satisfies the associativity, and its time derivative at initial pose can be expressed as a linear combination of instantaneous screws in 6×1 vector form.

2) By examining the closure, associativity, existence of an identity, and existence of inverses of the entire set of finite

screws under screw triangle product, we have proven that the algebraic structure of the set forms a Lie group, and its time derivative at the identity forms the corresponding Lie algebra of instantaneous screws under the screw cross product.

3) The revealed relationship between finite and instantaneous screws provides a possibility to model finite and instantaneous motions in a consistent manner, allowing type synthesis and performance analysis of robotic mechanisms to be unified into the concise framework of screw theory.

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