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The Restricted Stochastic User Equilibrium with Threshold model: Large-scale application and parameter testing

Rasmussen, T.K. [a], Nielsen, O.A. [a], Watling, D.P. [b], Prato, C.G. [c]

[a] *Department of Transport, Technical University of Denmark, Kgs. Lyngby, Denmark*
[b] *Institute of Transport Studies, University of Leeds, Leeds, United Kingdom*
[c] *School of Civil Engineering, The University of Queensland, Brisbane, Australia*

Abstract

This paper presents the application and calibration of the recently proposed Restricted Stochastic User Equilibrium with Threshold model (RSUET) to a large-scale case-study. The RSUET model avoids the limitations of the well-known Stochastic User Equilibrium model (SUE) and the Deterministic User Equilibrium model (DUE), by combining the strengths of the Boundedly Rational User Equilibrium model and the Restricted Stochastic User Equilibrium model (RSUE). Thereby, the RSUET model reaches an equilibrated solution in which the flow is distributed according to Random Utility Theory among a consistently equilibrated set of paths which all are within a threshold relative to the cost on the cheapest path and which do not leave any attractive paths unused.

Several variants of a generic RSUET solution algorithm are tested and calibrated on a large-scale case network with 18,708 arcs and about 20 million OD-pairs, and comparisons are performed with respect to a previously proposed RSUE model as well as an existing link-based mixed Multinomial Probit (MNP) SUE model. The results show that the RSUET has very attractive computation times for large-scale applications and demonstrate that the threshold addition to the RSUE model improves the behavioural realism, especially for high congestion cases. Also, fast and well-behaved convergence to equilibrated solutions among non-universal choice sets is observed across different congestion levels, choice model scale parameters, and algorithm step sizes. Clearly, the results highlight that the RSUET outperforms the MNP SUE in terms of convergence, calculation time and behavioural realism. The choice set composition is validated by using 16,618 observed route choices collected by GPS devices in the same network and observing their reproduction within the equilibrated choice sets generated by the RSUET model. Relevantly, the RSUET model is very successful in reproducing observed link counts.

1 Introduction and Motivation

As the need for large-scale transport models has increased in recent years in coincidence with the development of metropolitan, regional and national models worldwide, the need for computationally efficient and behaviourally realistic traffic assignment models stands tall.

Most traffic assignment models adopt variants of either the Deterministic User Equilibrium model (DUE, Wardrop, 1952) or the Stochastic User Equilibrium model (SUE, Daganzo and Sheffi, 1977) framework. The DUE has been widely applied in large-scale applications, mainly because of its computational attractiveness in that it distinguishes implicitly between potentially used routes and definitely unused routes, thereby circumventing the computationally intractable enumeration of the universal choice set. However, the drawback of the DUE is that it is based on an assumption of perfect traveller rationality and choice of only minimum cost paths, an implausible assumption especially in cases where there exist paths with costs only slightly greater than the minimum. The SUE removes this non-realistic assumption via the adaptation of Random Utility Maximisation (RUM) models. Under the commonly adopted assumptions on the perception errors, RUM models suffer from the theoretical need to allocate flow to all available alternatives (paths), no matter how non-sensible they may be (Watling et al., 2015). This not only constitutes a behavioural limitation, but also poses some distinct challenges for the theoretical consistent integration of state-of-the-art RUM models into practical large-scale traffic assignment problems. Moreover, the universal choice set may consist of millions of routes for each OD-pair and hence it becomes intractable to enumerate and assign traffic to the universal choice set. Consequently, SUE is usually calculated on a subset of the universal choice set in real-life applications, and this induces a theoretical inconsistency with the underlying model framework. The generation of the subset is not trivial, as the SUE does not provide any conditions/requirements to help distinguishing between relevant and irrelevant routes. Rather, the issue of sampling the choice sets in such a way that they are composed of all relevant alternatives, while leaving out non-relevant alternatives, is left to the modeller (Bovy, 2009; Prato, 2009).

The limitations of the DUE and SUE models have led to the development of alternative models. The Boundedly Rational User Equilibrium (BRUE) model relaxes the assumption of perfect rationality in the DUE, allowing for the use of some non-optimal paths within an excess cost within some often relative 'indifference band' to the cost of the minimum cost path. A recent review (Di and Liu, 2016) covers the BRUE from its original formulation (Mahmassani and Chang, 1987) to its most recent mathematical formulation (Di et al., 2013). The BRUE model assumption implies the possibility to obtain solutions using non-optimal paths, and typically the result is a space of possible flow solutions for possible path sets rather than a unique flow solution for a unique path set.

More recently the Restricted Stochastic User Equilibrium model was proposed (RSUE, Watling et al., 2015), which removes the need to use all paths in SUE. The RSUE allows to determine consistently unused alternatives from the equilibrium conditions while allocating flow among used paths according to Random Utility Theory. Thus, a SUE-style flow solution is found, however among an internally equilibrated and consistent non-universal choice set.

Motivated by the BRUE and RSUE, Watling et al. (2016) proposed the Restricted Stochastic User Equilibrium with Threshold (RSUET) model that combines the advantages of these two approaches, in terms of the behavioural realism they add. From the BRUE model, it uses the notion that there might be a tolerance on how large detours/deviations from the optimal equilibrium cost that route travellers would consider, and that routes outside the tolerance would not be used. From the RSUE model, it uses the possibility to integrate random utility theory (for splitting traffic between used alternatives) with the possibility to exclude some unreasonably costly routes from the equilibrated choice set in a consistent way. In this way, the RSUET model can be viewed as either a stochastic version of the BRUE model or a bounded rationality inspired version of the RSUE model.

The RSUET model is motivated not only from a behavioural perspective, but also from the need of traffic assignment models to be applicable to large-scale studies. The implicit treatment of the choice set plays a paramount role. Whereas the issue of distinguishing between used and unused paths are left to the modeller in the SUE, he/she can use the conditions underlying the RSUET model to ensure that the choice sets are consistently generated as the algorithm iterates and that they are equilibrated upon termination of the algorithm. This determination of the choice sets induces also some additional advantages compared to SUE approaches. Firstly, path-based approaches can be consistently applied. This allows the use of state-of-the-art choice models such as the path-size logit (Ben-Akiva and Bierlaire, 1999) and a more flexible specification of the cost function such that path-based reliability measures can be included. Secondly, simulation can be avoided in the generation of paths and allocation of flow. This not only improves computation time but also removes the stochasticity in the outputs, which may have a major implication in project appraisals for large-scale models (Manzo et al., 2015; Rich and Nielsen, 2015). Thirdly, convergence in both the allocation of flow among used paths and the generation of the final choice sets can be consistently evaluated. Watling et al. (2016) devised a companion generic solution method to the RSUET exploiting these advantages, where the algorithm is an extension of the algorithm presented for the RSUE (Rasmussen et al., 2015a).

Watling et al. (2016) demonstrated the applicability of the RSUET algorithm on the Sioux Falls network, but did not pursue to demonstrate efficient and consistent large-scale applicability. The present paper contributes by demonstrating this by presenting the tests of the novel model framework and solution algorithm for a large-scale application, and using a dataset consisting of observed routes collected by GPS devices to validate and calibrate the model, thereby demonstrating one possible utilisation of the increasingly available large-scale data sources on individual behaviour.

The remainder of the paper is structured as follows. Section 2 introduces the notation and restates the RSUET model and the solution algorithm proposed in Watling et al. (2016).

Section 3 introduces the different specifications of the algorithm and the evaluation criteria used in the present study. Section 4 presents the large-scale case-study and the results of the numerical tests, including the calibration of the parameters and the evaluation of the model performance at various network congestion levels. Last, section 5 draws the main conclusions and outlines future research directions.

2 Notation and Model Definition

Consider a network as a directed graph consisting of origin-destination (OD) pairs m (m=1, 2, ..., M) and links a (a=1, 2, ..., A). Define the demand for OD-pair m as d_m composing a non-negative M-dimensional vector $\mathbf{d}$, where d_m refers to element number m in $\mathbf{d}$. Define the index set R_m of all simple paths (without cycles) for each OD-pair m and the number N_m of paths in R_m. The union of the sets R_m is defined as R and the route index sets are constructed so that R = {1,2,...,N}, where $N = \sum_{m=1}^{M} N_m$. Denote the flow on path $r \in R_m$ between OD-pair m as x_{mr} and let $\mathbf{x}$ be the N-dimensional flow-vector on the universal choice set across all M OD-pairs, so that the notation x_{mr} refers to element number $r + \sum_{k=1}^{m-1} N_m$ in the N-dimensional vector $\mathbf{x}$. The convex set of demand-feasible non-negative path flow solutions G is given by:

$$G = \left\{\mathbf{x} \in \mathbf{R}_{+}^{N} : \sum_{r=1}^{N_m} x_{mr} = d_m, m = 1,2, \dots, M\right\} \tag{1}$$

where $\mathbf{R}_{+}^{N}$ denotes the N-dimensional, non-negative Euclidean space.

Denote the flow on link a (a=1,2,...,A) as f_a and let $\mathbf{f} = (f_1, f_2, \dots, f_a, \dots, f_A)$ be the A-dimensional link flow-vector where f_a refers to element number a in $\mathbf{f}$. Next, define δ_{amr} equal to 1 if link a is part of path r for OD-pair m and zero otherwise. Then the convex set of demand-feasible link flows is:

$$F = \left\{\mathbf{f} \in \mathbf{R}_{+}^{A} : f_a = \sum_{m=1}^{M} \sum_{r=1}^{N_m} \delta_{amr} \cdot x_{mr}, a = 1,2, \dots, A, \mathbf{x} \in G\right\} \tag{2}$$

In vector/matrix notation, let $\mathbf{f}$ and $\mathbf{x}$ be column vectors, and define $\boldsymbol{\Delta}$ as the $A \times N$-dimensional link-path incidence matrix. Then the relationship between link and path flows may be written as $\mathbf{f} = \boldsymbol{\Delta}\mathbf{x}$. It is then assumed that the travel cost on path r for OD-pair m is additive in the link travel costs of the links used:

$$c_{mr}(\mathbf{x}) = \sum_{a=1}^{A} \delta_{amr} \cdot t_a(\Delta\mathbf{x}) \qquad (r \in R_m;\ m = 1,2,\dots,M;\ \mathbf{x} \in \mathrm{G}) \tag{3}$$

Define $\mathbf{t}(\mathbf{f})$ ($\mathbf{t} : \mathbf{R}_{+}^{A} \rightarrow \mathbf{R}_{+}^{A}$) as the vector of generalised link travel cost functions, and $\mathbf{c}(\mathbf{x})$ ($\mathbf{c} : \mathbf{R}_{+}^{N} \rightarrow \mathbf{R}_{+}^{N}$) as the vector of generalised route travel cost functions. The relationships between path and link flows, and between link and path costs, then become:

$$\mathbf{f} = \Delta\mathbf{x} \quad \text{and} \quad \mathbf{c}(\mathbf{x}) = \Delta^{T}\mathbf{t}(\Delta\mathbf{x}) \tag{4}$$

For SUE-style models, U_{mr} denotes the random utility for path r between OD-pair m:

$$U_{mr} = -\theta \cdot c_{mr}(\mathbf{x}) + \xi_{mr} \qquad (r \in R_m\,; m = 1,2,\dots,M) \tag{5}$$

where $\boldsymbol{\xi} = \{\xi_{mr} : r \in R_m, m = 1,2,\dots,M\}$ are continuous random variables following some given joint probability distribution, and $\theta > 0$ is a given parameter. The following functions are then defined as the probability relations:

$$P_{mr}\big(\mathbf{c}(\mathbf{x})\big) = \Pr(-\theta \cdot c_{mr}(\mathbf{x}) + \xi_{mr} \ \geq\ -\theta \cdot c_{ms}(\mathbf{x}) + \xi_{ms}, \forall s \in R_m)$$
$$(r \in R_m\,; m = 1,2,\dots,M) \tag{6}$$

These relations express the probability that path r between OD-pair m will have a perceived utility greater than or equal to the utilities of all alternative paths in the universal set of routes for that OD-pair, when the random utilities are $-\theta \cdot \mathbf{c}(\mathbf{x}) + \boldsymbol{\xi}$ and the generalised path travel costs are $\mathbf{c}(\mathbf{x})$. for any non-empty subset $\tilde{R}_m$of R_m (m = 1,2,…,M), define also:

$$P_{mr}\big(\mathbf{c}(\mathbf{x})|\tilde{R}_m\big) = \Pr\big(-\theta \cdot c_{mr}(\mathbf{x}) + \xi_{mr} \ \geq\ -\theta \cdot c_{ms}(\mathbf{x}) + \xi_{ms}, \forall s \in \tilde{R}_m\big)$$
$$\big(r \in \tilde{R}_m \subseteq R_m\,; m = 1,2,\dots,M\big) \tag{7}$$

That is to say, whenever such a subset is not specified, P_{mr} refers to the universal set. The definition of the RSUET model is then as follows (Watling et al., 2016).

Definition: Restricted Stochastic User Equilibrium with Threshold (RSUET(Φ,Ω))
The route flow $\mathbf{x} \in G$ *is a RSUET(*Φ*,*Ω*) if and only if for all* $r \in R_m$ *and* m = 1,2,…,M*:*

$$x_{mr} > 0 \ \Rightarrow r \in \tilde{R}_m \wedge x_{mr} = d_m \cdot P_{mr}\big(\mathbf{c}(\mathbf{x})\big|\tilde{R}_m\big) \wedge c_{mr}(\mathbf{x}) \leq \Omega\big(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}; \boldsymbol{\varsigma}_m\big) \tag{8}$$
$$x_{mr} = 0 \ \Rightarrow r \notin \tilde{R}_m \wedge c_{mr}(\mathbf{x}) \geq \Phi\big(\{c_{ms}(\mathbf{x}) : s \in \tilde{R}_m\}; \boldsymbol{\xi}_m\big) \tag{9}$$

where Φ and Ω are exogenously defined functions which uniquely defines an 'external reference cost' and a threshold value ('internal reference cost') per OD movement, respectively.

Watling et al. (2016) proposed a corresponding generic solution algorithm, and demonstrated its applicability to the Sioux Falls network. The algorithm consists of an initialisation step identifying an initial feasible flow solution, followed by 5 sequential steps which are iterated until convergence: (i) column generation phase; (ii) restricted master problem phase; (iii) network loading phase; (iv) threshold condition phase; (v) convergence evaluation phase. In the *column generation phase*, the choice sets are grown in a systematic way that ensures the fulfilment of the second RSUET condition (9) at convergence, and thus that no attractive paths are left unused. The search for paths may be done in several ways, but Watling et al. (2016) suggest to use a single shortest path search for the RSUET(min, Ω) and a k-shortest path search for the RSUET(max, Ω). The *restricted master problem phase* allocates flow among the set of *used* paths according to the underlying choice model, to ensure that the part of the first

RSUET condition (8) concerning the flow allocation are fulfilled at convergence. The flow allocation can be done by using traditional path-based SUE allocation methods, or, using the cost transformation functions introduced in Rasmussen et al. (2015a), by DUE methods. The *network loading phase* loads route flows to the network to obtain the resulting link flows, link costs and route costs. The *threshold condition phase* identifies and removes any paths which violate the threshold condition and redistribute the flow among the remaining paths. Thereby the second part of condition (8) is fulfilled at convergence, ensuring that no unattractive paths are used. Lastly, the *convergence evaluation phase* uses a two-part gap measure and checks whether any violating paths have been removed to consistently evaluate whether the algorithm has converged to a solution fulfilling the underlying RSUET conditions.

3 Application of the RSUET Model to large-scale problems

The application of the RSUET model requires decisions about various specification and algorithmic details as described in the following.

3.1 Algorithm specification

The tests presented in the present paper focused on the RSUET(min, $\mathbf{\Omega}$) formulation rather than the RSUET(max, $\mathbf{\Omega}$). This was because (i) the corresponding RSUE(min) formulation was found promising in Rasmussen et al. (2015a), (ii) it ensures at least the minimum cost path to be used, but does not induce all paths with cost below the most costly used route to necessarily be used (e.g., multiple variants of routes repeatedly getting on and off motorways at ramps), and (iii) the RSUET(max, $\mathbf{\Omega}$) is a lot more computationally demanding than the RSUET(min, $\mathbf{\Omega}$) (Watling et al., 2016). In relation to the computational requirements, then if z refers to the number of zones (origins), then the max-formulation requires z^2 k-shortest path searches to cover all OD-pairs, as opposed to z searches for the single shortest path method of the min-formulation. Additionally, the k-shortest path search algorithm has a calculation complexity of $O(k_m \cdot V \cdot (V \cdot \log(V) + A))$ for each search, as opposed to $O(V \cdot \log(V) + A)$ for the single shortest path search method (see Rasmussen et al., 2015a)[1].

Given the min-formulation of the Φ-function, the *column generation phase* was based on single shortest path searches (see Watling et al., 2016). The implementation allowed the evaluation of two approaches in the *restricted master problem phase*. The first approach (referred to as the Path Swap variant) utilised the cost transformation functions introduced in Rasmussen et

[1] We implemented and did some initial tests of the k-shortest path algorithm. While managing to improve the computation time considerably compared to a first 'non-optimised' implementation, it still took approximately 2 seconds to compute the k=100 shortest paths between Rome and Copenhagen in the TransTools road network for passenger cars (Rich et al., 2009). This has roughly the same size as the Danish National Transport Model network, and the calculation time is about 100,000 times longer than the time required to compute the single shortest path between the same OD-pairs. Consequently, we did not manage to reach sufficiently low computation time levels to facilitate implementation in the iterative RSUET(max, Ω) algorithm, and believe that there is a need for further research to make the RSUET(max, Ω) operational for large-scale cases.

al. (2015a) to identify an auxiliary solution using the pairwise path-swapping strategy described in Carey and Ge (2012). See Rasmussen et al. (2015a) for more information on the integration of the path-swapping strategy and the cost transformation functions. The second approach (referred to as the Inner Logit variant) identified the auxiliary solution by directly using the closed-form MNL or PSL choice probability expressions.

The implementation facilitated the use of the Method of Successive Weighted Averages for the step-size strategy (MSWA, Liu et al., 2009). While being pre-defined, the MSWA allows giving more weight to auxiliary flow patterns found in later iterations, defining the step-size γ_n at iteration n as:

$$\gamma_n = \frac{n^d}{1^d+2^d+\ldots+n^d} \tag{10}$$

where $d \geq 0$ is a real number.

The tests of the RSUET(min, $\mathbf{\Omega}$) all utilise the threshold condition $\Omega\left(\{c_{ms}(\mathbf{x}): s \in \tilde{R}_m\}; \boldsymbol{\varsigma}_m\right) = \Omega\left(\{c_{ms}(\mathbf{x}): s \in \tilde{R}_m\}; \tau_m\right) = \tau_m \cdot \min\{c_{ms}: s \in \tilde{R}_m\}$ for each m=1, 2,..., M. The implementation steps were outlined in Watling et al. (2016): (i) paths could only be removed from iteration 15 onwards; (ii) at most one violating path could be removed from the choice set for each OD-pair in each iteration; (iii) flows on violating paths were redistributed among the remaining paths according to the flow distribution on these. This setup is referred to as the RSUET(min, $\boldsymbol{\tau}$·min) in the remainder of the paper.

3.2 Evaluation criteria

The MNL and PSL RSUET(min, $\boldsymbol{\tau}$·min) solution algorithms have been evaluated in various ways. Firstly, convergence was evaluated by using the two-part convergence measure proposed in Rasmussen et al. (2015a), consisting of a first part measuring the convergence to satisfy the underlying choice model among the used routes and a second part measuring the convergence to fulfil the criteria on unused routes:

$$Rel.\,gap_n^{Used} = \frac{\sum_{m=1}^{M}\sum_{r\in\tilde{R}_m} x_{mr,n}\cdot\left(x_{mr,n}\cdot\exp(\theta\cdot c_{mr}(\mathbf{x}_n)) - \tilde{c}_{m,\min}(\mathbf{x}_n)\right)}{\sum_{m=1}^{M}\sum_{r\in\tilde{R}_m} x_{mr,n}\cdot x_{mr,n}\cdot\exp(\theta\cdot c_{mr}(\mathbf{x}_n))} \tag{11}$$

$$Rel.\,gap_n^{Unused} = \frac{\sum_{m=1}^{M} d_m\cdot\left(\min_{\forall r\in R_m, x_{mr}>0}(c_{mr}(\mathbf{x}_n)) - \min_{\forall r\in R_m}(c_{mr}(\mathbf{x}_n))\right)}{\sum_{m=1}^{M} d_m\cdot\min_{\forall r\in R_m, x_{mr}>0}(c_{mr}(\mathbf{x}_n))} \tag{12}$$

where $\tilde{c}_{m,\min}(\mathbf{x}_n) = \min_{\forall r\in\tilde{R}_m}\{x_{mr,n}\cdot\exp(\theta\cdot c_{mr}(\mathbf{x}_n))\}$.

It is important to note that the two gap-measures proposed above have been developed solely for closed-form logit-type choice models in the RSUE and RSUET. They can thus not be used to evaluate convergence of solution algorithms of a link-based multinomial probit (MNP) SUE or a mixed MNP SUE. There is not an equivalent *consistent* measure available for such algorithms, and most applications evaluate the convergence by using 'stability' measures that do not evaluate the convergence to equilibrium directly, but rather the stability in solutions from iteration to iteration. One such measure is the link flow stability, weighted by flow and length:

$$Stability_n = \frac{\sum_{a=1}^{A} f_{a,n}(\mathbf{x}) \cdot l_a \cdot \frac{|f_{a,n}(\mathbf{x}) - f_{a,n-1}(\mathbf{x})|}{f_{a,n}(\mathbf{x})}}{\sum_{a=1}^{A} f_{a,n}(\mathbf{x}) \cdot l_a} = \frac{\sum_{a=1}^{A} l_a \cdot |f_{a,n}(\mathbf{x}) - f_{a,n-1}(\mathbf{x})|}{\sum_{a=1}^{A} f_{a,n}(\mathbf{x}) \cdot l_a} \tag{13}$$

Moreover, it is also important to evaluate whether the different model setups generate route choice sets of reasonable sizes containing relevant routes and leaving out non-sensible routes. This evaluation can be performed by computing the overlap between any observed route $r \in R_{obs}$ and any corresponding generated route $s \in \tilde{R}$ as follows (Ramming, 2002):

$$O_{rs} = \frac{L_{rs}}{L_r} = \frac{\sum_{a=1}^{A} \delta_{ar} \cdot \delta_{as} \cdot l_a}{\sum_{a=1}^{A} \delta_{ar} \cdot l_a} \tag{14}$$

where L_{rs} is the sum of length of overlapping elements between the observed path r and the generated path s. The overlap measure (14) can be computed for each generated path s for observation r, and let $O_{\max\{r\}}$ denote the highest overlap among the paths generated for observation r. Then the coverage using an overlap-threshold λ (e.g. 80%) can be computed as (Ramming, 2002):

$$Cov(\lambda) = \frac{\sum_{r \in R_{obs}} I(O_{\max\{r\}} \geq \lambda)}{|R_{obs}|} \tag{15}$$

where $I(\cdot)$ is an indicator equal to 1 when the criteria is fulfilled and zero otherwise.

Combining the development in the choice set size and coverage, an efficient algorithm should generate a few routes inducing a high coverage level within the first few iterations. The size of the choice sets should then stabilise, indicating that all relevant routes have been generated. Bekhor and Prato (2009) sought to combine these two components by proposing an efficiency index measure accounting for both behavioural consistency (coverage) and computational efficiency (choice set size). The measure thus supplements the two-part analysis above, and the efficiency index (EI) of an algorithm can be computed as:

$$EI = \frac{1}{|R_{obs}|} \cdot \sum_{r \in R_{obs}} \left\{ \left[I(O_{\max\{r\}} \geq \lambda) + \left(1 - \frac{\tilde{N}_r - R_{rel,r}}{\tilde{N}_r}\right) \right] / 2 \right\} = \frac{1}{2} \cdot Cov(\lambda) + \frac{1}{2 \cdot |R_{obs}|} \cdot \sum_{r \in R_{obs}} \frac{R_{rel,r}}{\tilde{N}_r} \tag{16}$$

where $R_{rel,r}$ is the number of relevant routes of observation r and $\tilde{N}_r$ is the number of used routes for the OD-pair corresponding to observation r. The number of relevant routes is difficult to specify in real-life large-scale networks, but this study used $R_{rel,r}$=2 for all observations as this was also used in Bekhor and Prato (2009). Additionally, counts of observed flows on links in the case-study area could be used to analyse the realism of the link flow distribution generated by the different algorithm variants.

Finally, the computational burden of the algorithms should also be evaluated. Other studies have found that the computational efforts required per iteration may vary greatly across different algorithm specifications and choice models (e.g., Rasmussen et al., 2015a). It is therefore important to not only evaluate the convergence as a function of the number of iterations, but also consider the computational burden *per iteration* when evaluating the performance of an algorithm. Therefore, the evolvement of the computation time per iteration across the algorithm variants and reported convergence etc. was also evaluated as a function of computation time rather than iteration number.

3.3 Specification of choice function and parameters

The model was implemented as a multi-class model that allows distinguishing between different trip purposes and vehicle classes (categories). The utility (cost) function considered several variables, and the cost of alternative *r* on OD movement *m* was specified as:

$$c_{mr}(\mathbf{x}) = \beta_{FreeTT,m} \cdot t_{FreeTT,mr}(\mathbf{x}) + \beta_{CongTT,m} \cdot t_{CongTT,mr}(\mathbf{x}) + \beta_{l,m} \cdot l_{mr} + \varepsilon_{mr} \quad (17)$$

where $\beta_{FreeTT,m}$, $\beta_{CongTT,m}$ and $\beta_{l,m}$ are the respective parameters associated with the free-flow travel time, congestion travel time and driving distance for the category associated with OD movement *m*. The distributed error term ε_{mr} expresses unobserved components and perception errors. The time-variables are measured in minutes, whereas all variables associated with length are measured in kilometres.

For the link-based MNP SUE and mixed MNP SUE, the error-term and (relevant only for the mixed MNP SUE) parameters associated to travel time were simulated from the gamma and the log-normal distribution, respectively. The parameters were simulated at the OD-level to account for taste heterogeneity across individuals, whereas the error-term was simulated at the link level per OD-pair. The mean of the error-term was zero, and the variance was specified as proportional to the mean cost (using scale parameter $\beta_{m\varepsilon}$) to ensure consistent aggregation from link- to path-level (see Nielsen and Frederiksen, 2006).

The choice function consisted of an additional term for the PSL RSUET(min, $\boldsymbol{\tau}$·min)-application, seeking to account for the effect of path overlapping. The term $\beta_{\mathrm{PS,m}} \cdot \ln\left(\mathrm{PS}_{\mathrm{mr}}\right)$ was added to the cost function (17), where $\beta_{\mathrm{PS,m}}$ was a non-positive OD-specific parameter and PS_{mr} was defined as proposed by Ben-Akiva and Bierlaire (1999):

$$PS_{mr} = \sum_{a \in \Gamma_{mr}} \frac{l_a}{L_{mr}} \cdot \frac{1}{\sum_{k \in \tilde{R}_m} \delta_{amk}} \quad (18)$$

where l_a and L_{mr} are measures of impedance on link a and on route r on OD movement m, and can either be measured as distance or cost (l_a=$t_a(\mathbf{f})$ and L_{mr}=$c_{mr}(\mathbf{x})$). Distance was used as a measure of impedance in the present application.

The values of the parameters used in the cost function (17) were transferred directly from the multi-class link-based mixed MNP SUE model applied in the Danish National Model. No re-calibration was done to fit these to each of the RSUET solution algorithms tested, as the issue of parameter estimation and how this might be done in a consistent way for the RSUET framework is beyond the scope of this paper[2].

Neither the parameters nor the error-terms are simulated in the RSUET(min, $\boldsymbol{\tau}$·min) application. This not only 'removes' the need for simulation, but also requires less parameters, as variances do not have to be specified. However, there was a need to specify the scale parameter θ_m, the threshold values τ_m, and the step-size parameter d. The path-size parameter β_{PS} also needed to be specified when applying the PSL choice model.

4 Case-study and Numerical Results

The present study uses a case-study covering the Danish Zealand Area to evaluate the RSUET model framework and demonstrate the applicability of several variants of the solution algorithm. A main objective has been to evaluate how large an impact the addition of the threshold condition has on the computation time as well as the equilibrium solution for different configurations of the model and for different network conditions (congestion levels). Among others, the evaluation has used real life observed data to assess the realism of the solutions.

4.1 Network, demand and observed data

The case network covers the eastern part of Denmark (primarily Zealand) with approximately 2.5 million inhabitants. The network consists of 12,451 links corresponding to 18,706 one-directional links in the network graph being a geographically limited subset of the network used in the Danish National Transport Model. The demand also stems from the Danish National Transport Model, and the demand matrices includes a total of 3.2 million daily trips categorised into 19 different user classes and three vehicle types (car, van and lorry) with approximately 20 million OD-cells in total (Nielsen & Pedersen, 2016). The network and demand is the same as used in Rasmussen et al. (2015), which verified that levels and the locations of congestion are realistic.

A total of 16,618 GPS traces from car trips within the study area were utilised to perform a disaggregate evaluation of the algorithms' ability to reproduce observed route choices; the origin and destination of each trip were added to the network and corresponding trips were

[2] For a discussion on calibration and estimation issues for the RSUE, see Watling et al. (2015).

appended to the demand matrix (with zero demand to not cause additional congestion in the network). The GPS data stem from two data-sources: 554 observations were collected in a person-based data collection in which travellers carried the GPS unit with them during all their travels (across modes of transport, see Rasmussen et al., 2015b). The remaining 16,064 observations were collected in a vehicle-based data collection among a sample of employees of the Municipality of Copenhagen. While the second source is richer in the number trips, the first also contains information on the personal characteristics of the car drivers.

4.2 Software implementation and configuration

The solution algorithm outlined in section 2 was implemented into the C#-based Traffic Analyst software package (Rapidis, 2015) that also implements solutions to the link-based MNP SUE and the mixed MNP SUE models (Nielsen et al., 2002).

Several variants of the implemented algorithm were tested on different configurations of the network demand. It was found that performing 100 iterations was sufficient to induce $Rel.gap_n^{Used}$ as well as $Rel.gap_n^{Unused}$ to reach a value below $1.3 \cdot 10^{-3}$ and $1.0 \cdot 10^{-12}$, respectively, for all applications. The analyses have been performed by using both the Path Swap as well as the Inner Logit variants for the determination of the auxiliary flow solution in the *restricted master problem phase* of the solution algorithm. Both approaches showed the same overall patterns, however the results of the Inner Logit variant are the only one reported because of the faster convergence.

4.3 Threshold and choice set composition

Routes were only allowed to be removed from a choice set if it contained at least $\widetilde{N}_{min}$=2 routes. Subsequently, it was verified that this did not give rise to unreasonably large fluctuations in flows when removing a route for any OD-pair. In order for the flows to stabilise in the initial iterations before removing any routes, routes were only allowed to be removed from iteration K_{min}=15 onwards.

4.3.1 Determination of threshold from revealed choices

The threshold specifies the maximum route cost relative to the cheapest used path. Its value was specified by analysing the choice of non-optimal paths in real-life observed route choices and on the basis of a comparison between costs on observed paths and costs on the corresponding minimum cost path. Figure 1 illustrates the cumulative share of observations as a function of the ratio between the cost on the observed path (path obtained from GPS data) and the cost on the minimum cost path between the corresponding locations. The observed paths were constituted by the 16,618 routes obtained from the GPS data. For each GPS trip, the cheapest path was found by performing a shortest-path search in the congested network between the origin and destination of the corresponding GPS trip. It can be seen, for example, that 71% of the observed paths were less than 5% longer than the corresponding optimal path.

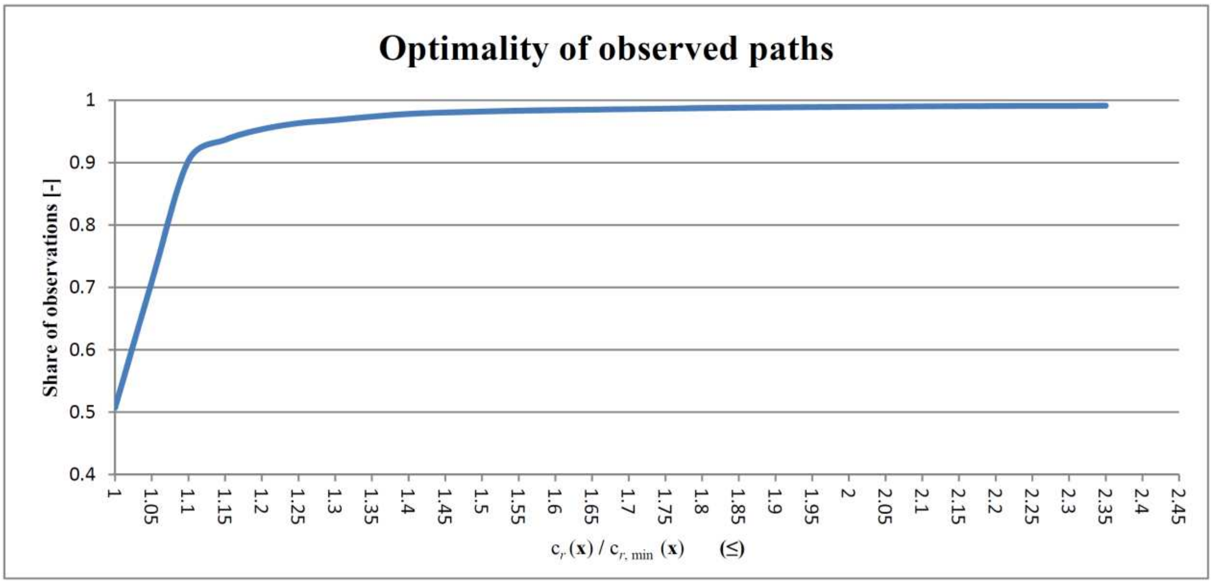

Figure 1 – Cumulative share of observations as a function of the ratio between the cost on the observed path r and the cost on the corresponding minimum cost path $c_{r,\min}$ **(x)**

The distribution of the 'non-optimality' of the observed routes is assumed to be representative of how (relatively) expensive paths have to be in order for the travellers not to consider and use them. The threshold was specified by using a 95% interval induces a choice of τ=1.2 (i.e. the relative cost on 95% of all observed paths is within this threshold), which has then been used in the remainder of the paper.

4.3.2 Example of route exclusion, threshold condition

1,989 unique routes were removed by the threshold condition when using τ=1.2, d=4 and the MNL choice model with θ=0.2. Note, however, that the same unique path may have been generated and subsequently excluded several times during the iterations of the solution algorithm. This section presents an example of an OD movement (commercial business trip undertaken in van), for which a previously generated route was removed by the threshold condition at equilibrium.

Figure 2 illustrates the four unique routes generated for the same OD-pair, where each of these has been the most attractive at some iteration of the solution algorithm. Table 1 reports the corresponding equilibrium cost components, generalised cost and route flow share on each of these. All four routes were however not included in the equilibrated choice set, as flow was only distributed among paths 1, 2 and 4. Path 3 is considerably more expensive than the others (32%), and the threshold condition therefore removed it from the final choice set.

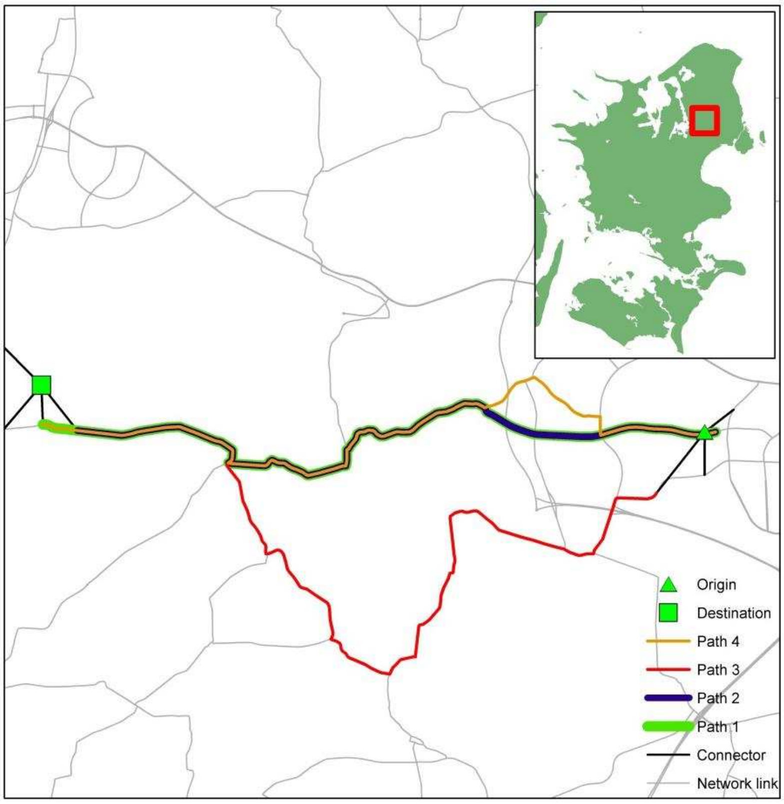

Figure 2 – Example of excluded route: 4 paths generated, but 3 utilised at convergence, MNL RSUET(*min*, *1.2·min*), Zealand application

Table 1 – Specification of cost components, generalised costs, relative costs as well as flows at equilibrium. MNL RSUET(*min*, *1.2·min*), Zealand application. l_{1r}, $t_{FreeTT,\,1r}$ and $t_{CongTT,\,1r}$ refer to the length, free-flow travel time and congested travel time of route *r*, respectively. $c_{1r}(\mathbf{x})$ and $c_{1,min}(\mathbf{x})$ refer to the cost on route r and the minimum cost across the used routes, respectively

Path	Category ID	l_{1r} [km]	$t_{FreeTT,\,1r}(\mathbf{x})$ [min]	$t_{CongTT,\,1r}(\mathbf{x})$ [min]	$c_{1r}(\mathbf{x})$	$c_{1r}(\mathbf{x})/c_{1,min}(\mathbf{x})$	Flow [%]
1	6	13.80	12.85	16.39	81.40	1.01	32.23
2	6	13.61	13.42	15.40	81.82	1.02	29.64
3	6	18.02	17.09	20.24	106.07	1.32	-
4	6	14.43	13.64	16.44	80.56	1.00	38.13

4.4 *Step-size strategy*

The step-size parameter *d* specifies the 'trust' in the auxiliary solution and may thus influence the convergence speed (Rasmussen et al., 2015a). Posing a higher trust in the auxiliary solution may also lead to higher fluctuations in the path-flows between iterations, which may possibly cause additional/other paths to be attractive. The choice of *d* may thus influence not only the convergence speed, but also the solution in terms of the composition of the choice sets. The converged solutions should however all be RSUET solutions.

If the model parameters τ and θ are kept constant (τ=1.2, θ=0.2), the convergence measures for the flow distribution (11) and the choice set composition (12) can be directly compared across d-values for the RSUET and RSUE. Figure 3 and Figure 4 illustrate the convergence pattern of the MNL RSUET(min, 1.2·min) for different step-size strategies.

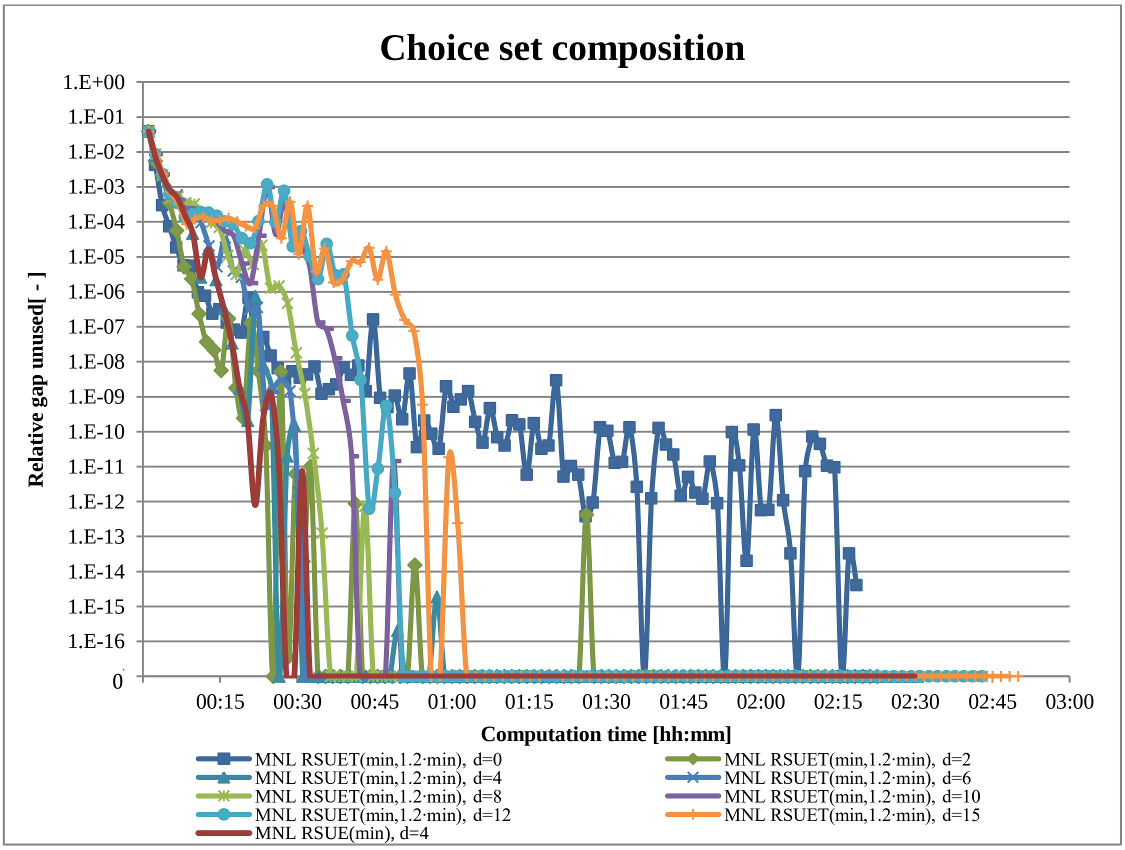

Figure 3 – Relative gap measure (12) for convergence of choice set composition as function of computation time, Zealand application. MNL RSUET(*min*, *1.2·min*) for various values of step-size parameter d as well as the MNL RSUE(min) with d=4. All with θ=0.2. Notice the log-scale on the vertical axis

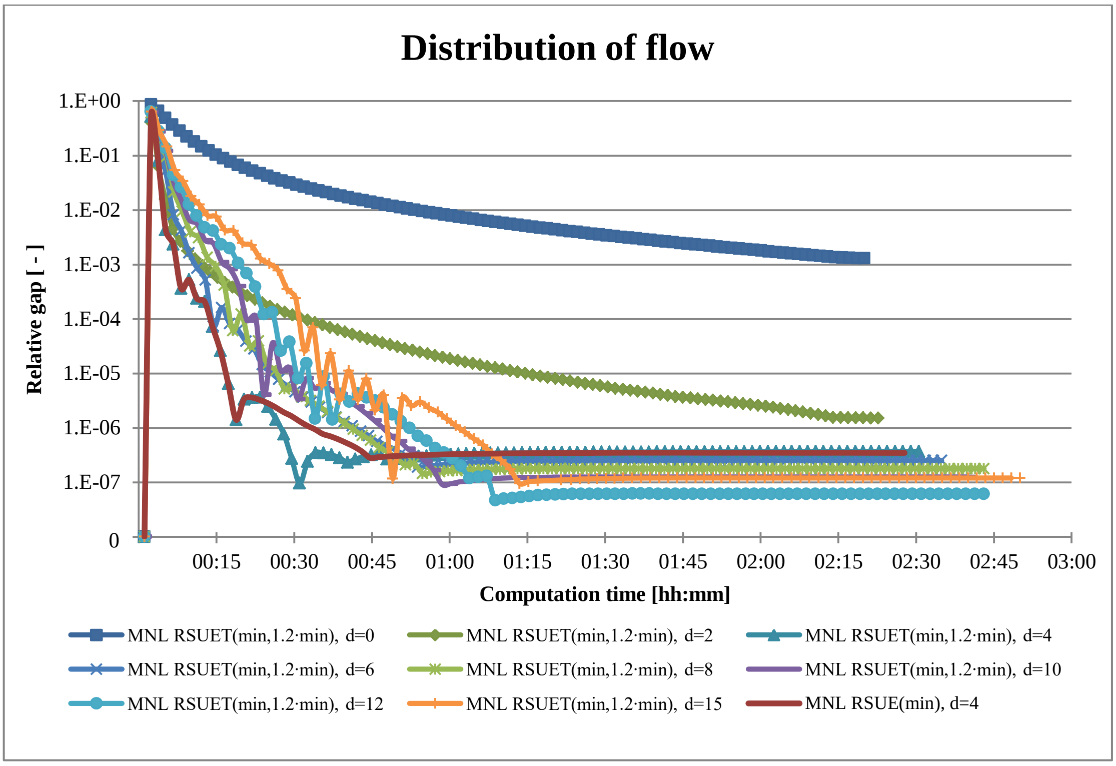

Figure 4 – Relative gap measure (11) for convergence of flow distribution among routes in the choice set as a function of computation time, Zealand application. MNL RSUET(*min*, *1.2·min*) for various values of step-size parameter d as well as the MNL RSUET(min) with *d*=4. All with θ=0.2. Notice the log-scale on the vertical axis

Initially, some fluctuations can be seen in both convergence measures. This is due to the introduction of new paths to the choice set – for $Rel.gap_n^{Unused}$ the increases arise because some currently unused paths become attractive to introduce into the choice set, and for $Rel.gap_n^{Used}$ the increases arise due to the need to redistribute flow towards equilibrium in a 'new' choice set consisting of also the path recently introduced. The choice set composition converged fast for all step-sizes, however with *d*=0 (MSA) being somewhat slower. Also the distribution of the flow among the paths in the choice set converged to a stable low level of approximately $1.0\text{-}3.5\cdot10^{-7}$, except for low values of *d* (*d*=0 and *d*=2) which were far from reaching this level at termination. Using *d*=4 caused the fastest convergence, as the final choice sets were generated within less than 30 minutes and the flow distribution converged within 35-40 minutes of calculation time. Consequently, the analyses presented in the remainder of the paper have been done using *d*=4.

The convergence pattern of the RSUET(min, 1.2·min) was identical to that of the corresponding RSUE(min) application during the first 15 iterations and seemed reasonable since K_{min}=15. From iteration 15 onwards the convergence pattern was also very similar, converging to almost identical values of the relative gap measures. This is because only a very small share of the routes was removed by the threshold condition (e.g. 1,989 routes across 1,621,201 OD-pairs in the case of *d*=4). Consequently, Figure 3 and Figure 4 do not report the results for other applications of the RSUE(min) than the one using *d*=4.

The relative gap associated with the distribution of flow among paths did not seem to converge to zero, but rather stabilised at approximately 1-$3.5\cdot10^{-7}$. The stabilization to a very small non-zero value is not an indication of the algorithm not converging, but rather an issue arising due to the limitations of the computer used[3].

4.5 Scale parameter

The scale parameter reflects the dispersion in the perception of costs among drivers: a low value reflects large variation in the perception error of drivers (with complete 'random' allocation in the extreme case of $\theta \rightarrow 0$), and a high value reflects small variation in the perception error of drivers (with the limit of DUE when $\theta \rightarrow \infty$). Several different values of the scale parameter were tested, each application using the same value across all OD movements, i.e. $\theta_m = \theta$ for $m=1,2,\ldots,M$. The relative gap measures were used to verify that all tests converged within reasonable computation time. The convergence measures can however not be compared across applications, as the scale parameter influences the relative gap measure. Therefore, a series of alternative analyses was performed to evaluate the performance of the solution algorithm for different values of the scale parameter. This also facilitated the comparison to the link-based MNP SUE and mixed MNP SUE solution methods.

1,169 observed daily link counts were available, and these were distributed throughout the case-study area. Figure 5 reports the Root Mean Square Error (RMSE) between the modelled and observed link counts (normalised by range of observed flows). In general, very high correspondence was observed (all normalised RMSE<0.052), demonstrating that the RSUE/RSUET applications are successful in distributing the flow in a way that matches the observed counts. Noting the tight range of the vertical axis it can be seen that only minor differences are seen between corresponding RSUE/RSUET applications, and the best normalised RMSE was obtained when using θ=0.2. It is prevailing that both MNP SUE applications performed worse than all RSUE/RSUET applications in reproducing link counts, even though prior studies have invested large efforts into calibrating the MNP SUE models to the case study.

[3] The relative gap is computed using exponential functions of the costs, which causes very small deviations to be amplified into large numbers. We performed a disaggregate analysis of the changes in flow and costs on routes between iterations when d=4, which showed that the average/maximum change in absolute cost and flow on the paths across all OD movements is a very low 2.9·10-12/2.3·10-10 for cost and 6.2·10-12/1.0·10-9 for flow. These numbers are at the limit of the accuracy of computation of real numbers in the C# software used, and the non-zero gap measure can be seen as a consequence hereof.

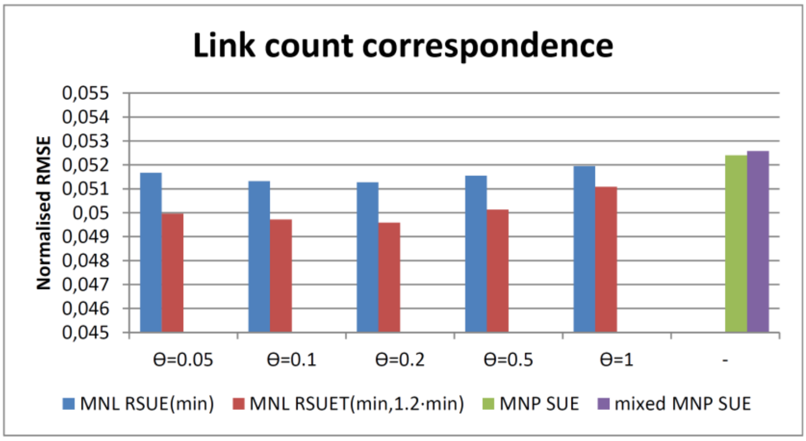

Figure 5 – Correspondence between modelled and observed daily link flows for various RSUE and RSUET configurations as well as the MNP SUE and mixed MNP SUE. Iteration 100, Zealand application

The analysis above showed good performance of the RSUE/RSUET on an aggregate level, by showing that these distribute flow in a way that reproduces link counts accurately. Moving to a disaggregate level, the models should also be able to reproduce rational real-life route choices. Their ability to do so was evaluated by comparing with 16,618 observed route choices collected via GPS, under the hypothesis that the observed routes should be represented in the corresponding choice sets generated. The coverage measure captures this, and this constitutes an important measure to use in the calibration of especially the scale parameter, as the scale parameter represents the heterogeneity in route choice. Figure 6 reports the coverage measure as a function of the overlap threshold λ, and shows a decreasing coverage with increasing λ, as expected. Also, it can be seen that the 'relative' performance of the different θ values was somewhat the same across λ values.

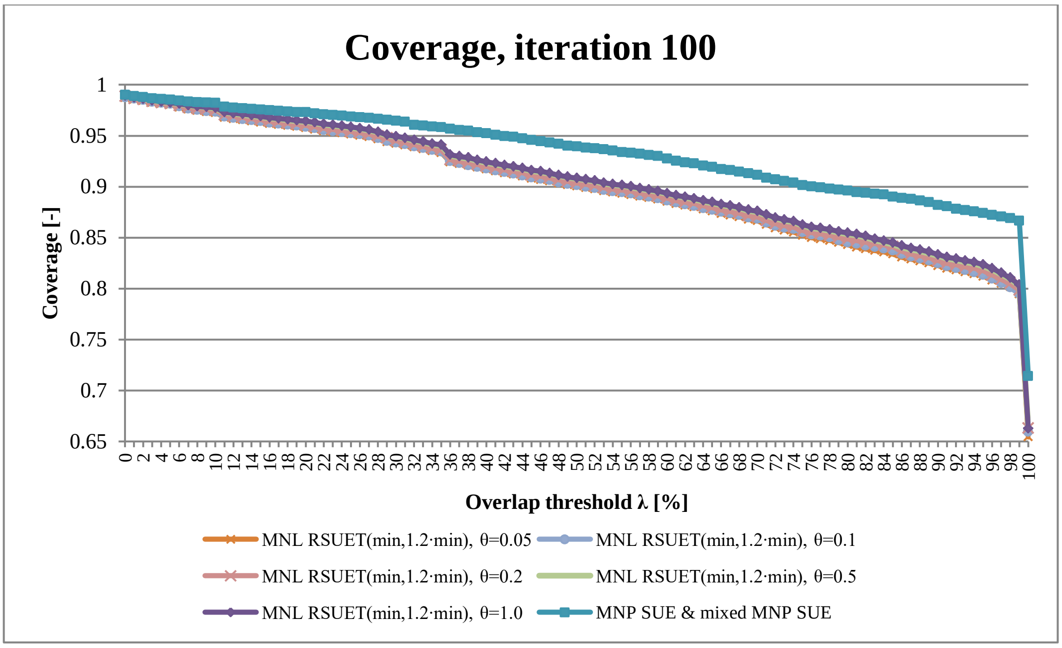

Figure 6 – Coverage as function of overlap threshold λ for various scale parameters in MNL RSUET(*min*, *1.2·min*), iteration 100. Zealand application

Table 2 reports various characteristics of the solution generated, including the coverage obtained at iterations 25 and 100 when using a 80% overlap threshold. In general, high coverage levels were produced for all θ. It can be seen that adding the threshold on the relative costs does not seem to reduce the coverage for any of the chosen θ. This indicates that the paths removed by the threshold condition are in general non-relevant. Furthermore, the coverage seems to increase with increasing scale parameter. This increase is probably related to the larger fluctuations in flow in the initial iterations caused by the larger scale parameter; more weight is put on differences in costs (closer to DUE), leading to more 'extreme' auxiliary flows and thereby also larger fluctuations. These fluctuations cause more routes to be generated (seen through larger average choice set sizes) but also more routes to violate the threshold at equilibrium (and thus be removed, see Table 2). The number of paths removed was however at a very low level, considering that the network contains 1.6 million OD-pairs.

Table 2 – Coverage, choice set size, efficiency index and number of routes removed (when relevant) for various scale parameters in MNL RSUET(*min*, *1.2·min*) and the MNL RSUE(*min*). The relevant measures are also reported for the MNP SUE and the mixed MNP SUE. Zealand application

		Coverage, λ=0.8		Choice set size			Efficiency index	Excluded paths
		Ite 25	Ite 100	Min.	Avg.	Max.		
θ=0.05	RSUE	0.8431	0.8431	1	2.364	10	0.9859	-
	RSUET	0.8431	0.8431	1	2.367	10	0.9859	1165
θ=0.1	RSUE	0.8452	0.8452	1	2.484	10	0.9733	-
	RSUET	0.8452	0.8452	1	2.484	10	0.9734	1180
θ=0.2	RSUE	0.8487	0.8487	1	2.696	13	0.9541	-
	RSUET	0.8487	0.8487	1	2.695	12	0.9543	1989
θ=0.5	RSUE	0.8535	0.8535	1	2.968	14	0.9335	-
	RSUET	0.8535	0.8535	1	2.967	13	0.9338	3784
θ=1.0	RSUE	0.8548	0.8548	1	3.059	13	0.9162	-
	RSUET	0.8548	0.8548	1	3.057	13	0.9165	4640
MNP SUE		0.8959	0.8959	1	14.894	100	0.6540	-
mixed MNP SUE		0.8959	0.8959	1	25.365	100	0.5460	-

The MNP SUE and mixed MNP SUE produced coverage levels which were considerably better than those of the RSUE and RSUET applications. This was however at the cost of generating large choice sets, which continued to grow without any clear tendency towards stabilisation. An average size of 37.0 routes was seen at iteration 200 for the mixed MNP SUE. The RSUE and RSUET on the other hand produced choice sets having a very computationally reasonable size, and which are equilibrated. The equilibrated choice sets were generated within a few iterations, which is also indicated by non-changing coverage from iteration 25 to iteration 100 (Table 2). The flow distribution also converged within a few iterations, highlighting that there is no need to perform many iterations to obtain an equilibrated RSUE/RSUET solution.

An efficient solution algorithm should produce a high coverage level while generating choice sets which are computationally attractive by containing only (a few) relevant routes. The efficiency index (16) captures this, and the RSUE/RSUET solution algorithms reached efficiency indexes ranging from 91.7% to 98.6%. The index for the RSUET is slightly better than the index generated by the corresponding RSUE formulations. This is due to the (slightly) smaller choice sets. The RSUE/RSUET solution algorithms generated significantly higher efficiency indexes than the MNP SUE and mixed MNP SUE. This highlights the weakness of the MNP SUE approaches, namely that they generated their high coverage levels at the cost of generating large choice sets.

The convergence pattern cannot be directly compared across θ-values, as mentioned earlier. In order to facilitate comparisons, the measures reported in Table 2 were supplemented by analyses of the link flow stability and the ability to reproduce observed link counts. This also facilitated the comparison to the MNP SUE and the mixed MNP SUE. It is however important to note that stability in link flows does not necessarily induce that an equilibrated solution has been found.

Figure 7 illustrates a very fast link flow stabilisation across iterations for all RSUE/RSUET applications. The effect of adding the threshold can clearly be seen, especially when $\theta \leq 0.2$, through a destabilisation of link flows at iteration 15 (~20min of computation time). Using $\theta=0.1$ or $\theta=0.2$ induces the best link flow stability. The stability of the MNP SUE and the mixed MNP SUE was considerably lower, indicating that convergence was not yet reached at iteration 100. This was also suggested by continuously increasing choice sets and is furthermore supported by a maximum relative deviation in link flow between iterations 99 and 100 of a very high 18.8%. This value was considerably lower for all the RSUET(min, 1.2·min) applications, e.g. $2.14 \cdot 10^{-5}$ % when $\theta=0.2$.

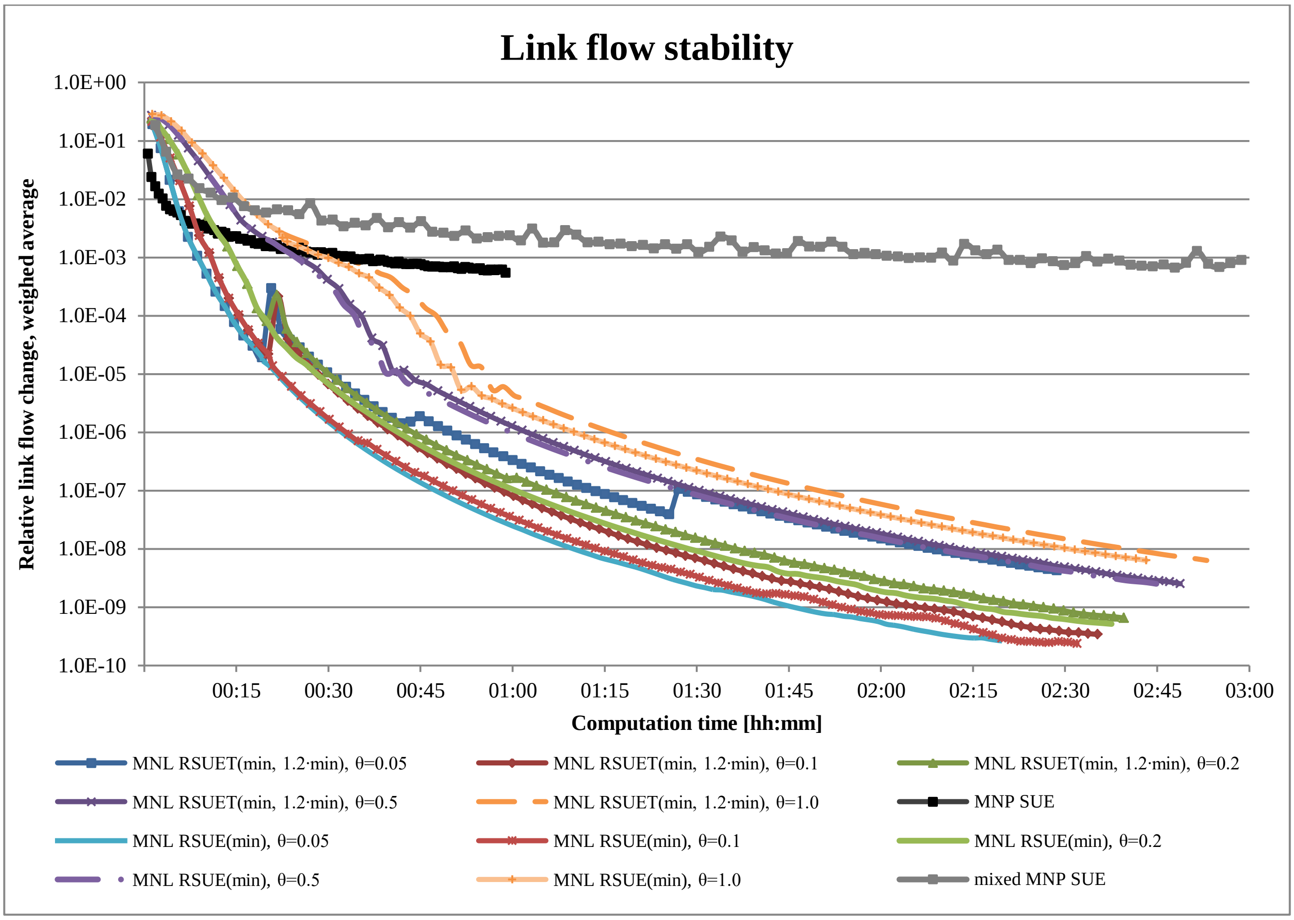

Figure 7 – Link flow stability across iterations, Zealand application. Notice the log-scale

Summarising, all tested values of θ produced good results for all evaluation criteria used. The best link count correspondence was however seen when using θ=0.2, and the analyses in the remainder of the paper have adopted this value.

4.6 Path overlap correction

The MNL choice model fails to account for path overlapping. Accordingly, this study applied also the PSL choice model to investigate the impact of accounting for this. This involved the specification of the parameter associated to the path-size correction factor. The identification of the optimal parameter value is not a one-dimensional problem, as e.g. the choice of a path-size parameter may influence the optimal value of θ and vice versa. The study did not seek to solve the resulting multidimensional optimisation problem. Rather, the PSL RSUET(min, 1.2·min) was applied for both d=0 and d=4, using θ=0.2 and $\beta_{PS,m} = \beta_{PS} = -3$. This parameter setting is assumed to be reasonable; section 4.5 found good performance when using θ=0.2 in the corresponding MNL RSUET, and Rasmussen et al. (2015a) tested different values of β_{PS} for the PSL RSUE(min) on the same network (also using θ=0.2, but with d=2) and found best performance when $\beta_{PS} = -3$.

Equilibrated solutions were found, with convergence patterns almost identical to the pattern of the corresponding MNL application (and therefore not reported here). The same choice sets were generated across the choice models for almost all OD-pairs. This is supported by a difference in average choice set size of 0.001 and 0.002 routes when comparing corresponding applications across choice models for d=0 and d=4, respectively. The high similarity of choice sets seems reasonable, as the same path generation technique was used in the solution algorithm for the two choice models (deterministic shortest path search). The choice set composition however varied in a few cases. This was a consequence of the different flow distribution across the two choice models (due to the correction for path overlapping), which (in some cases) caused other routes to be attractive.

The similarity of the choice sets also led to almost identical coverage levels. Accounting for path overlapping does not improve coverage, but allows the distribution of flow among routes *in* the choice sets to be more behaviourally realistic.

Figure 8 reports the computation time per iteration of the application of the MNL and PSL RSUET(*min*, *1.2·min*) solution algorithms. The computation time increased during the first iterations of the MNL RSUET(*min*, *1.2·min*) applications due to the path-based approach: the choice sets were generated within the first iterations, and storing an increasing number of paths in-memory and (re)distributing flow between these requires increasing memory and computational effort. The final choice sets were, generally, generated within the first 5-10 iterations when d=0 and d=4, and computation times per iteration stabilised from this point on.

The computation times in the initial iterations of the MNL and PSL applications were different: the computation time of the MNL was strictly increasing until a certain level, whereas the computation time of the PSL increased rapidly in the initial iterations and then reduced to the level of the corresponding MNL application. This is directly linked to the computation of the path-size correction factors. Since these were based on overlap in length, they only need to be recomputed when a route is added to or removed from the choice set. The choice sets were formed in the initial iterations, and the path-size correction term thus had to be computed for many paths in these (the choice set changed for many OD movements and the correction terms had to be recomputed for all routes in each of these choice sets). This is computationally expensive (especially as the number of routes in the choice sets grows) and explains the steep increase in computation time in the initial iterations. After a few iterations (iterations 4 and 6 for d=0 and d=4, respectively) new routes were generated for fewer OD movements, and fewer path-size correction terms thus had to be (re)computed. This reduced the computational effort. After the final choice sets were (more or less) generated at iteration 11, no further recalculation of path-size correction terms was needed. Therefore, the computation time reduced to that of the corresponding MNL formulation.

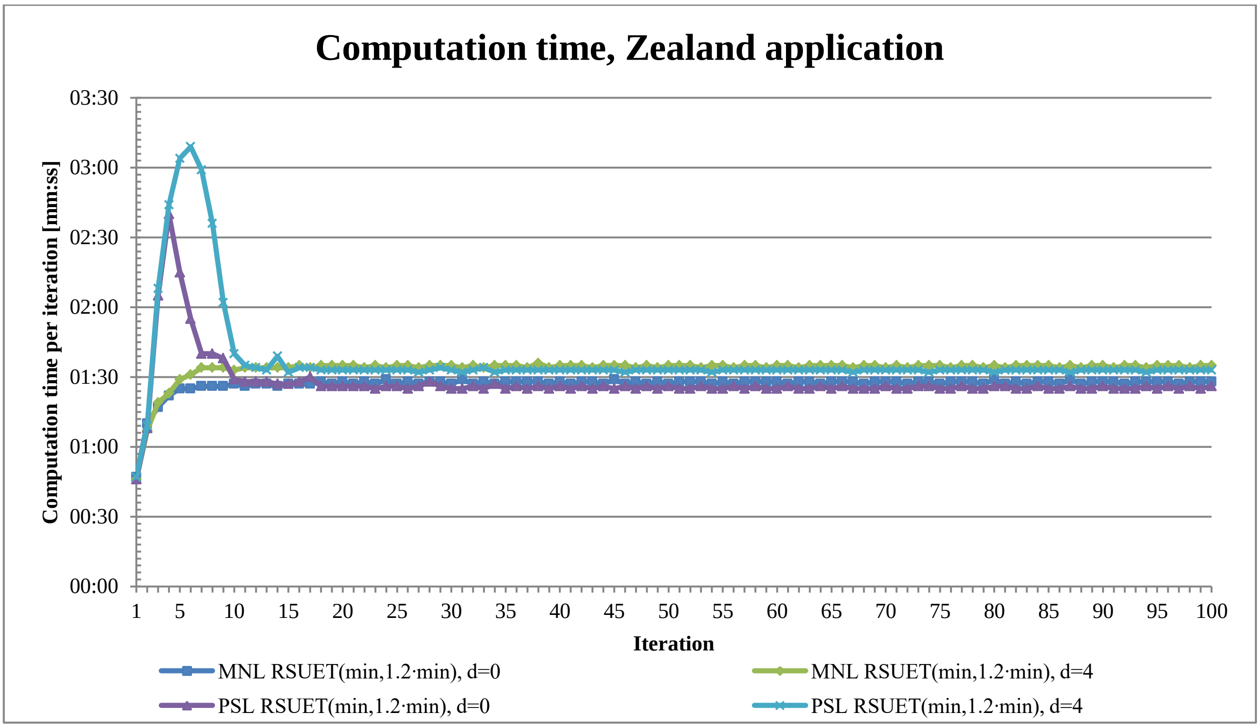

Figure 8 – Computation time per iterations for the MNL as well as PSL RSUET(min, 1.2·min) with d=0 and d=4. Zealand application

4.7 Stability to congestion level

The analyses above showed that the tested variants of the solution algorithm provide fast convergence to a stable solution which fulfils the RSUET(*min*, *1.2·min*) conditions. However, good performance in the Zealand application does not guarantee good performance when applied to other case-studies. One of the typical major challenges for solution algorithms is to also provide nice convergence patterns in high congestion real-life cases. The tested variant of the proposed solution algorithm with *d*=4 was applied to a variety of scaled versions of the original demand matrices (the scale-factors tested are 1.25, 1.5, 1.75 and 2.0). This was done to test the robustness towards the general congestion level in the network. Figure 9 illustrates the volume-capacity ratio in the network links for the different demand levels.

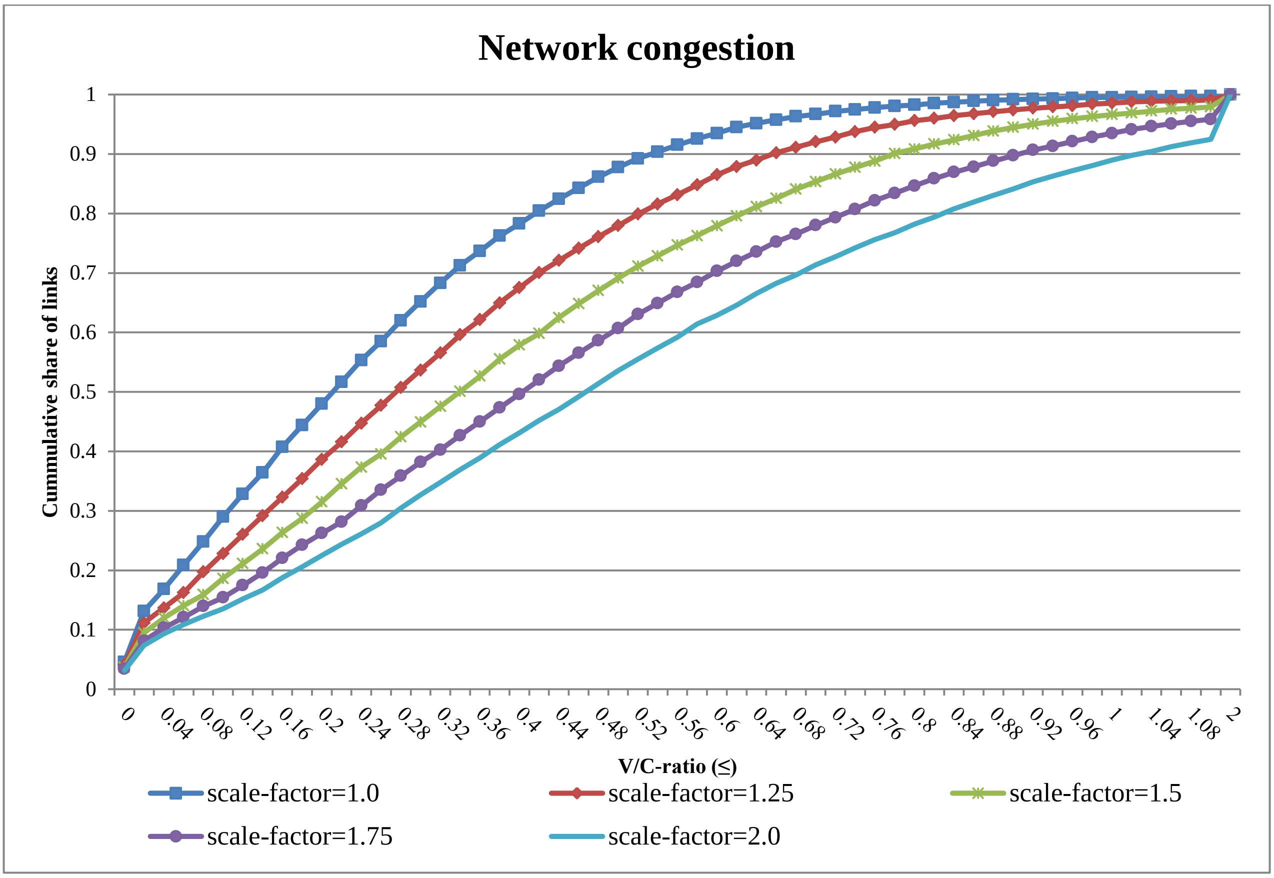

Figure 9 – Network congestion at various demand levels. Cumulative share of links as function of volume to capacity ratio, Zealand application

4.7.1 Convergence

Figure 10 and Figure 11 report the convergence measures for varying demand when performing 100 iterations. There was a clear tendency for slower convergence as the demand increased, in terms of both the number of iterations needed as well as the calculation time. However, a nice convergence pattern was seen for all the tested levels of demand. The travel times in the network fluctuated more in the initial iterations due to the larger demand which caused the choice set composition to require more iterations to converge and larger choice sets to be generated. The higher fluctuations and travel time differences in the network also caused the distribution of flow among paths to require more iterations to converge for increasing demand levels, but even the highest congestion case (demand scale-factor 2.0) converged nicely once the final choice sets were generated. Longer calculation time to converge for increasing demand level is however not only due to the need for more iterations. The calculation time *per iteration* also increased, due to the larger choice sets and hence more paths to store in memory and assign traffic between. Consequently, the average calculation time per iteration was approximately 90/105/130/145/180 seconds for scale parameters 1.0/1.25/1.5/1.75/2.0, respectively.

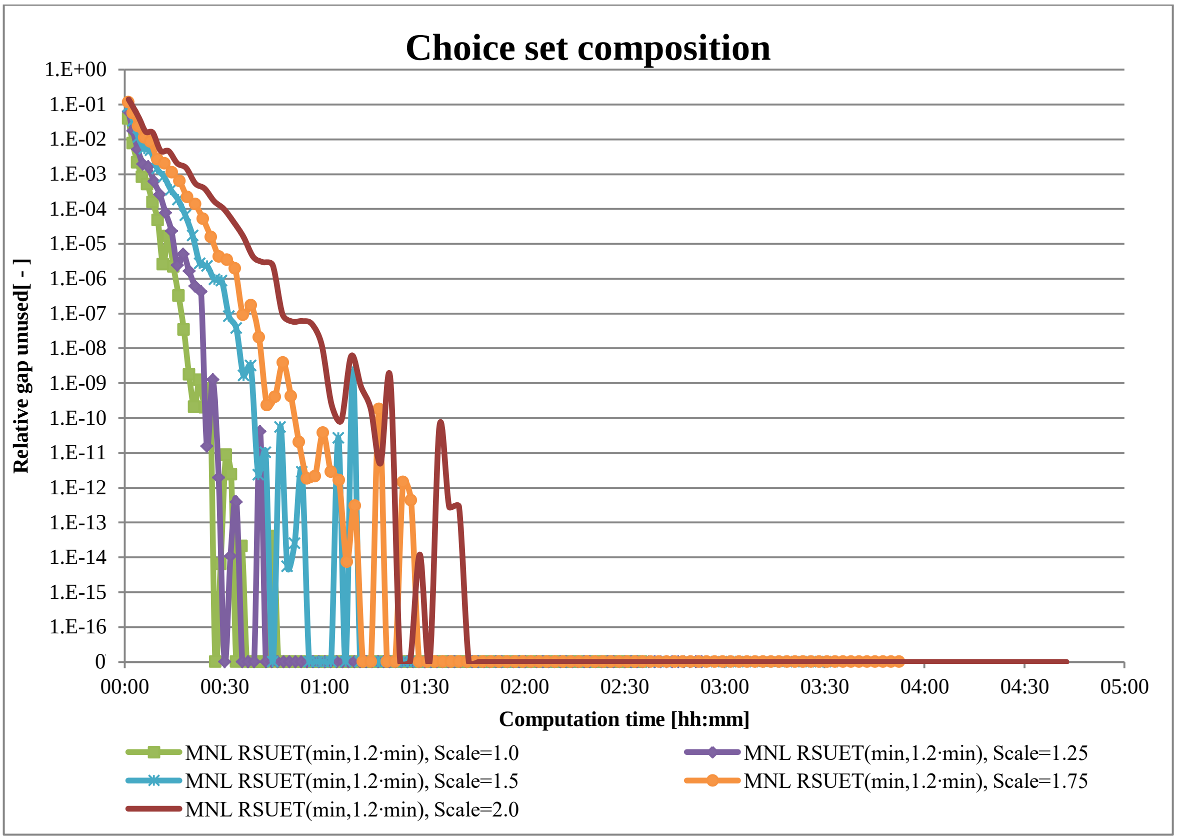

Figure 10 – Development of relative gap (12) measuring convergence of the choice sets for various values of the factor scaling the demand, MNL RSUET(min, 1.2·min) with d=4, Zealand application

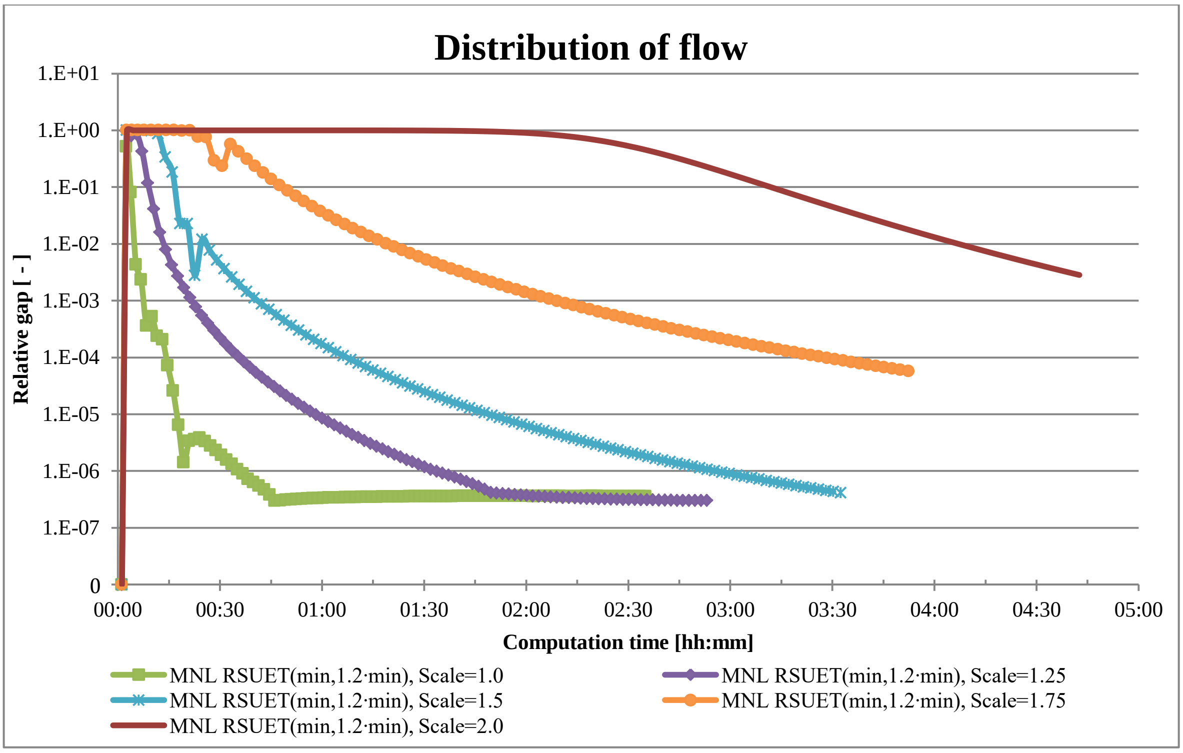

Figure 11 – Development of relative gap (11) measuring convergence of the distribution of flow between paths for various 'scaled' demands, MNL RSUET(min, 1.2·min) with d=4, Zealand application

4.7.2 Choice set size, route exclusion and cost distribution

Section 4.7.1 showed that more iterations were required for the choice set composition to converge when increasing the demand. This indicates that more routes – larger choice sets – were generated as the demand increased, as verified by Figure 12. The average choice set size grew larger and required more iterations to stabilise when increasing the demand, but after iteration 13-30 (depending on demand level) no major changes of the average and maximum choice set size occurred. Furthermore, it can be seen that the choice sets had a very reasonable and computationally attractive size across all demand levels. For some movements only one route was generated, even for a very high demand (the minimum choice set size was equal to 1 for all demand levels, and is thus not reported in Figure 12). This also seems justifiable, since for some movements, such as e.g. neighbouring zones in rural areas, only one alternative may be reasonable, even at a high congestion level. Even doubling the demand does not cause congestion on some (primarily rural) roads, as suggested by Figure 9.

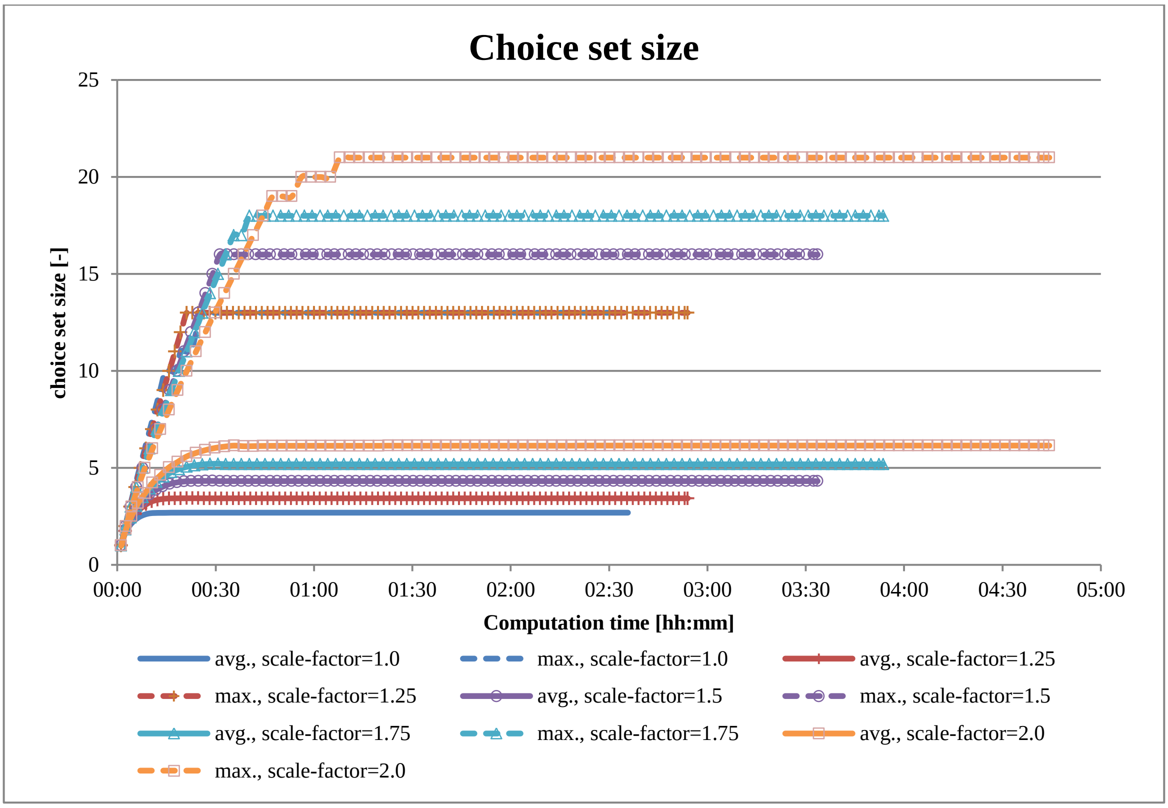

Figure 12 – Choice set characteristics for various values of the factor scaling the demand, MNL RSUET(min, 1.2·min) with d=4, Zealand application

Only a few routes violated the threshold condition by being more than 20% more costly than the cheapest path in the 'unscaled' Zealand application. *Step 5* of the solution algorithm did thus not remove many routes[4], at termination only 1,989 unique routes had been generated and removed again from the choice sets[5]. The corresponding numbers were 10,744, 34,519, 85,478 and 160,192 routes when the demand scale factor was 1.25, 1.5, 1.75 and 2.0, respectively. The threshold condition thus removed more paths as the network congestion increased, and one route was, on average, removed for each tenth OD movement when using a scale factor of 2.0. At this demand level the maximum number of unique paths removed for a single OD movement was 4 (this OD movement had 9 used paths in the resulting choice set at equilibrium). The increase in the number of paths removed for increasing demand seems reasonable, as link travel times fluctuate much more and thereby the route costs more 'easily' violate the threshold condition. The larger fluctuations in link travel times occur due to (i) the larger demand on OD-level, causing more flow to be reassigned in each iteration, and (ii) higher sensitivity to flow changes in the travel time functions when the general flow level is higher. One would therefore expect a larger variation on the relative costs among the routes left in the choice set at equilibrium. This is verified by Figure 13, where e.g. 7% of the routes were more than 4% more costly than the corresponding cheapest path in the 'unscaled' case, whereas it was 27% of the routes in the case where the scale-factor was equal to 2.0.

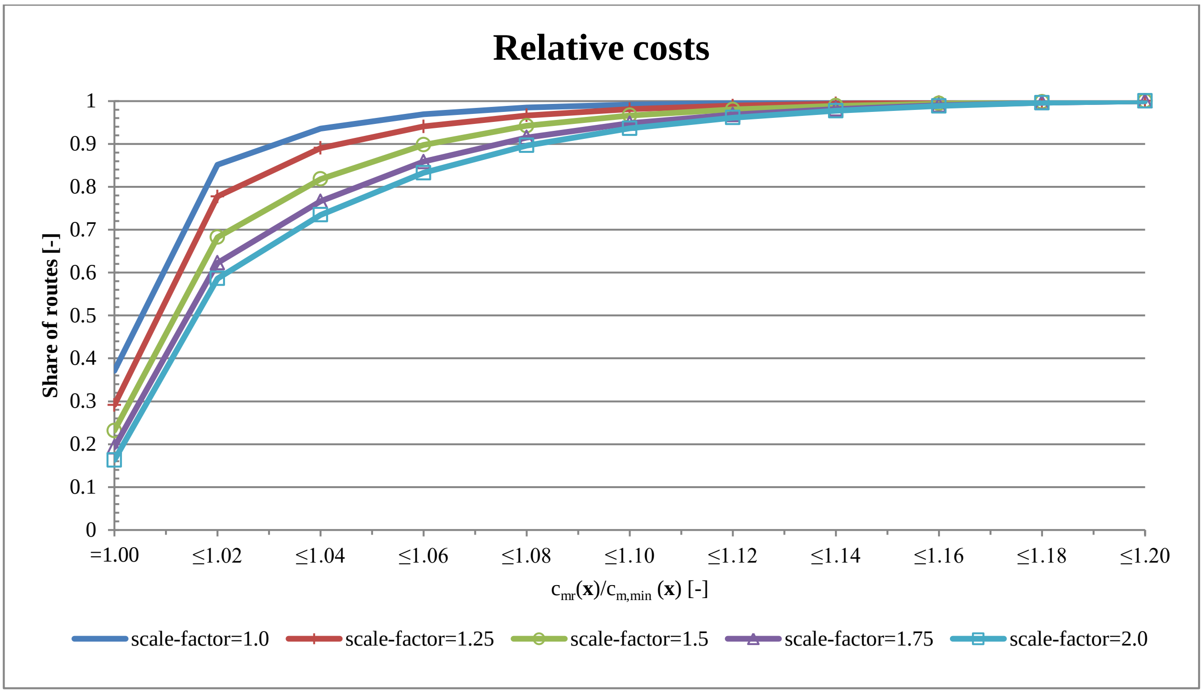

Figure 13 – Distribution of relative costs at convergence (iteration 100). Share of routes as a function of relative cost to the cheapest route in the corresponding choice set. MNL RSUET(min, 1.2·min) d=4 for varying values of the factor scaling the original demand, Zealand application

[4] Note on implementation: Paths to be removed are not discarded/flushed from memory but rather flagged as 'inactive'. This is done because these might again become attractive in a later iteration, and 'reactivating' an inactive path requires far less computational effort than assigning a new route to memory.

[5] The same unique route may however have been introduced and subsequently removed again several times as the algorithm progressed.

5 Conclusions

The study tackles the challenge of obtaining equilibrated RUM flow solutions among choice sets which do not leave attractive paths unused and contain only attractive paths in large-scale problems. Several variants of the RSUET solution algorithm proposed in Watling et al (2016) were applied on the large-scale Zealand network and compared to real life observed route choice data, link counts and an existing MNP-based SUE model.

The study found well-behaved and extremely fast convergence patterns to equilibrated solutions satisfying the underlying conditions across different scale parameters, step-sizes, and congestion levels. Comparisons to observed routes and observed link flows verified that the composition of the choice sets and the distribution of flow are very reasonable.

The effect of adding the threshold was investigated under different conditions. It was found that the threshold condition did not cause any of the 16,618 observed paths to be removed. This documented that paths violating the threshold were not used of any real car drivers in the specific case, which seems to be a strong argument for adding a threshold to SUE models. Moreover, the importance of the threshold increased as congestion increased. A comparison to two commonly adopted simulation-based SUE algorithms clearly highlighted the benefits of the RSUE/RSUET by showing that the SUE algorithms (i) generated choice sets which continued to grow in size without showing signs of stabilisation, and (ii) did not stabilise in link flows nearly as fast as the RSUE/RSUET, indicating much slower convergence.

Numerous different specifications of the threshold can be formulated, but the focus of the present study was on a formulation which specifies the threshold as relative to the cost of the least costly used route(s). The rationale is that there must be a limit to how large detours travellers find reasonable. The RSUET model thereby provides a very behaviourally realistic interpretation of the mechanism which distinguishes attractive and non-attractive paths. Many other models do not provide such a plausible interpretation, for example models based on random walk with loops (e.g., Fosgerau et al., 2013) or simulation-based models, where the draws may induce the use of highly unattractive paths (e.g. multinomial Probit as in Sheffi, 1985). The algorithm of Dial (1971) does have a behavioural interpretation of the mechanism, namely that only *efficient* paths are used. The approach is however quite different from that of the present paper in that paths are not explicitly enumerated and it is only applicable for MNL choice models. Also, *efficient* paths correspond to paths including only links that take the traveller further away from the origin and closer to the destination, which induces the risk of excluding some potentially attractive paths (e.g., Si et al., 2010).

If reformulated for dynamic rather than static assignment, the RSUET model framework and solution methods fit especially well in combination with disaggregate activity-based models. The activity-based models operate at an individual level and, when the utility functions become individual-based, this removes the need to account for taste heterogeneity in the assignment model and thereby enables the application of the proposed RSUET solution methods. Not only this allows a rich and consistent specification of the utility function that can improve significantly the behavioural realism, but also the extremely fast convergence of the RSUET solution algorithm allows for low computation times in the integrated model framework. An additional benefit is the absence of stochasticity in the output of the model as simulation is avoided. While the solution algorithm fits particularly well with individual-based approaches, they can also be used to approximate mixed logit models and, thereby, represent taste heterogeneity. This can be done by generating quantiles of the distribution of the preferences – e.g. of the value-of-time – and then consider each of these as separate user classes in the solution algorithm (parameters specified as mean value for the corresponding quantile).

The current paper has thus demonstrated the applicability and behavioural realism of several variants of the RSUET model and solution methods in a highly complex network. The algorithm managed to reproduce link counts and observed routes and converged extremely fast to an equilibrated solution fulfilling the underlying conditions, even in large-scale case-studies and for high-demand cases.

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