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Ordered Information Systems and Graph Granulation

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Abstract. The concept of an Information System, as used in Rough Set theory, is extended to the case of a partially ordered universe equipped with a set of order preserving attributes. These information systems give rise to partitions of the universe where the set of equivalence classes is partially ordered. Such ordered partitions correspond to relations on the universe which are reflexive and transitive. This correspondence allows the definition of approximation operators for an ordered information system by using the concepts of opening and closing from mathematical morphology. A special case of partial orders are graphs and hypergraphs and these provide motivation for the need to consider approximations on partial orders.

Keywords: Ordered Information System · Graph Granulation · Graph Partitioning

1 Introduction

From one perspective the theory of rough sets allows us to move between two levels of detail. Elements at the more detailed level are grouped together and these granules become elements at a more abstract (less detailed) level of detail. The process of granulation, that is the process of forming the granules, can be parameterized by a relation on a set. In the classic case described by Pawlak [8] the relation is an equivalence relation, but arbitrary relations give rise to several different granulations as described by Yao [18]. Stell [14] showed how these could be generalized to the case of a relation on a hypergraph, as opposed to the relations on a set considered in [18], by using operations of erosion and dilation from mathematical morphology. However, the treatment in [14] did not consider how to connect relations on a hypergraph with partitions of the underlying set of edges and nodes. The present paper is also related to the work of Lin [7] on granular computing and neighbourhood systems, since neighbourhood systems correspond to arbitrary relations. However, neighbourhood systems are structures on sets whereas here the more general case of partially ordered sets is studied.

The paper start by reviewing how hypergraphs can be seen as partial orders. This leads to a motivating example of a graph granulation which prompts an examination of ordered information systems. This shows how the partitions induced by such structures have corresponding reflexive and transitive relations. Using these relations we use the well-known operations of opening and closing, but novelly in the general case of a partial order rather than a set, to obtain appropriate approximation operators.

2 Background on Graphs and Hypergraphs

In this section we start by outlining the approach to graphs and hypergraphs that we use.

2.1 Graphs and Hypergraphs

We work with graphs which are undirected and which may have multiple edges between nodes as well as multiple loops on nodes. In a graph each edge is incident with one or two nodes, but consideration of binary relations on graphs leads naturally to hypergraphs as we will see later. In a hypergraph [1] there are edges and nodes, but each edge may be incident with any number of nodes. In our work we require that edges are incident with a non-zero number of nodes. One formalization of these structures is to have two disjoint sets for the nodes and for the edges. We use an alternative approach with a single set consisting of all the node and edges together with an incidence relation which expresses which edges are incident with which nodes. This has been used in [13, 14] and is based on using a similar approach to graphs in [3]. Relations appear in two ways in the paper: every hypergraph is treated as a set of edges and nodes equipped with an incidence relation on this set, and an indiscernibility relation on the hypergraphs is a further binary relation on the set of edges and nodes subject to appropriate constraints.

Definition 1. *A* ***hypergraph*** *consists of a set U and a reflexive incidence relation $H \subseteq U \times U$ such that for all $u, v, w \in U$, if $(u, v) \in H$ and $(v, w) \in H$ then $u = v$ or $v = w$.*

Given a hypergraph (U, H), an element $u \in U$ is an ***edge*** *if there is some $v \in U$ where $(u, v) \in H$ and $u \neq v$. An element which is not an edge is a* ***node****.*

It is straightforward to check that in a hypergraph (U, H) the incidence relation H will be transitive as well as reflexive so that it is a preorder and, in fact, a partial order too. Hypergraphs defined in this way may have edges that are incident with arbitrary non-empty sets of nodes and not just with one or two nodes as in the case of a graph. Graphs arise as a special case of hypergraphs as in the next definition.

Definition 2. *A* ***graph*** *is a hypergraph (U, H) which satisfies the constraint that for every $u \in U$ the set $\{v \in U \mid (u, v) \in H \text{ and } u \neq v\}$ has at most two elements.*

We use the terminology 'subgraph' for the structural parts of arbitrary hypergraphs.

Definition 3. *A* ***subgraph*** *of a hypergraph (U, H) is defined as a subset $K \subseteq U$ for which $k \in K$ and $(k, u) \in H$ imply $u \in K$.*

Figure 1 shows three different ways in which graphs may be visualised. The example shows a graph with two edges and three nodes. The node-edge visualisation is the most familiar depiction, with each edge drawn as line between two nodes. A useful alternative to this, and the only viable possibility once we are dealing with hypergraphs which are not graphs is the *Boundary Visualisation*. In this each edge is shown as a boundary enclosing all the nodes with which it is incident.

3 Granules in Graphs and Hypergraphs

Figure 2 provides a motivating example of a graph and its abstraction to a less detailed view. The left hand diagram provides the more detailed view. It shows an imaginary transport network of rail lines in a city. There are nine labelled stations: *West*, *Mid*, *C1*, *C2*, *C3*, *North*, *South*, *SouthEast*, and *SouthWest*. Five lines are labelled: a and b as well as p, q, and r.

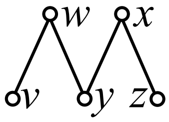
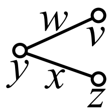
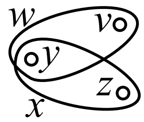
Poset Visualisation	Node-Edge Visualisation	Boundary Visualisation	Hypergraph
			$U = \{v, w, x, y, z\}$ $H = \{(x, x), (y, y), (z, z), (w, w), (v, v), (w, v), (w, y), (x, y), (x, z)\}$

Fig. 1. Three ways of visualising a graph

Now consider the representation on the right-hand side. This provides a less detailed view. Some of the stations are unchanged from the more detailed view. The three stations *C1*, *C2* and *C3* together with the three lines joining them have become a single node labelled *Centre*. This node does not represent a single station at the more detailed level; it represents a subgraph consisting of three nodes and three edges. The line labelled *westline* consists of two lines a and b together with the intermediate station *Mid*. Here the subgraph consisting of nodes *West*, *North*, *Mid* and edges a and b has become a subgraph consisting of one edge and two nodes.

The more detailed representation is a graph, with undirected edges which connect pairs of nodes. However, the less detailed representation is a *hypergraph* having two edges, one of which is incident with three nodes, *North*, *Centre* and *South*. This edge, labelled *circleline*, stands for a granule at the more detailed level consisting of the *SouthWest* and *SouthEast* nodes together with the edges connecting them to *South*, to *North* and to *C1*, *C2*, *C3*.

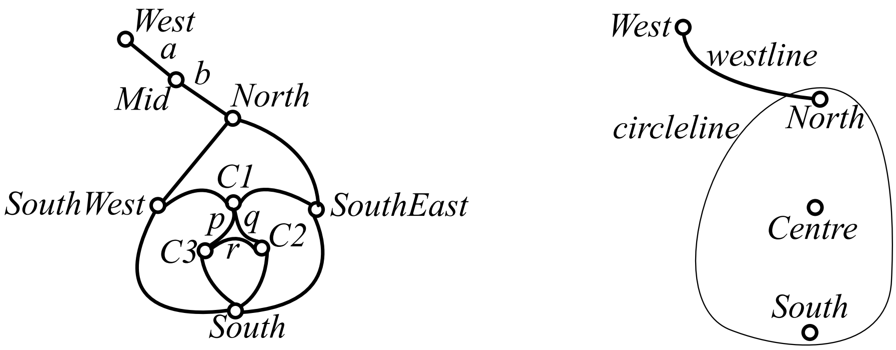


Fig. 2. Motivating example of coalescing edges and nodes

The clustering of separate entities at the more detailed level into a single entity at the less detailed level can be understood as the action of an attribute which assigns values in the lower level to entities at the higher level. To explain how this works in the case of a partially ordered set of entities, it is necessary next to introduce the concept of an ordered information system.

4 Ordered Information Systems

In this section we consider a generalization of the notion of Information System in which the universe is not merely a set but carries a partial order. In this setting the attributes defined on the universe are monotone, or order-preserving, functions to partially ordered sets of values. We see how this gives rise to a partition of the universe, as in the well-known set-based case, but with the additional structure of a partial order on the equivalence classes. In order to understand the appropriate way to define upper and lower approximations we then need to connect these partially ordered partitions with relations on the partially ordered universe, which we do in Theorem 2 below. Although our motivation used graphs and hypergraphs, we shall see that it is the partial order that provides the essential structure.

4.1 Information Systems on a Partially Ordered Universe

We recall the set case following the terminology of [9]. An **Information System**, $\mathcal{A} = (U, A)$, consists of a set U and a set A of functions called **attributes** defined on U where $\alpha : U \to V_\alpha$ for each $\alpha \in A$. Each subset $B \subseteq A$ gives rise to an **indiscernibility relation** IND_B where $(u_1, u_2) \in IND_B$ iff $\alpha(u_1) = \alpha(u_2)$ for all $\alpha \in B$. The relation IND_B is an equivalence relation on U, and we denote the equivalence class of u by $[u]_B$ or just $[u]$ where no ambiguity arises.

Each subset $X \subseteq U$ has an **upper approximation**, $\overline{B}(X)$, and a **lower approximation**, $\underline{B}(X)$, with respect to a given $B \subseteq A$ where,

$$\overline{B}(X) = \{u \in U \mid \exists w((w, u) \in IND_B \text{ and } w \in X)\}$$
$$\underline{B}(X) = \{u \in U \mid \forall w((u, w) \in IND_B \text{ implies } w \in X)\}$$

Now we generalize this to the case of a partially ordered set (U, H).

Definition 4. *An* ***Ordered Information System****,* $\mathcal{A} = (U, H, A)$*, consists of set* U *and partial order* H *on* U*, and a set,* A*, of order preserving functions called* ***ordered attributes*** *defined on* (U, H) *where* $\alpha : (U, H) \to (V_\alpha, K_\alpha)$

An ordered information system gives rise to an **indiscernibility relation** IND_B as before where $(u_1, u_2) \in IND_B$ iff $\alpha(u_1) = \alpha(u_2)$ for all $\alpha \in B$. The equivalence classes are partially ordered if we define $[u_1] \leqslant [u_2]$ iff $\forall \alpha \in B\ (\alpha u_1\ K_\alpha\ \alpha u_2)$, This defined ordering necessarily satisfies

$$u_1\ H\ u_2 \text{ implies } [u_1] \leqslant [u_2]. \tag{1}$$

As an example we can consider the change of level of detail in Figure 2 as an Ordered Information System. For simplicity, take just the part of the hypergraph at the detailed level consisiting of nodes $\{West, Mid, North\}$ and of edges $\{a, b\}$. The set U is then $\{West, Mid, North, a, b\}$, and the partial order H relates every edge in U to its two incident nodes. Thus $(a, West) \in H$ and $(a, Mid) \in H$ and so on. In addition H is defined to be reflexive. In this simple example there is just one attribute α where $V_\alpha = \{West, westline, North\}$ with the partial order K_α containing $(westline, North)$ and $(westline, West)$ as well as the identity pairs $(westline, westline)$ etc. The attribute α assigns the value *westline* to both a and b as well as to *Mid* because this is the less-detailed feature to which these entities belong, as well as satisfying $\alpha(West) = West$ and $\alpha(North) = North$.

In a partially ordered universe approximations need to respect structure. To do this we need to make use of definitions of approximations based on relations. So next we introduce relations on partial orders.

4.2 Relations on Partial Orders

A relation on a graph, a special case of Definition 5, is a relation on the set of all edges and nodes of the graph which in addition respects the incidence structure. The consequences of this definition have been explored in more detail in [15, 16], but all works in the more general setting of a partially ordered set (U, H).

Here we only need a few properties of these relations, which we set out in Theorem 1; proofs can be found in the above references. In the definition we write composition of relations as ; and we take $R \,;S$ to mean composition in the following order

$$R\,;S = \{(u, w) \in U \times U \mid \exists v \in U\ ((u, v) \in R \text{ and } (v, w) \in S)\}.$$

Definition 5. *A relation $R \subseteq U \times U$ is* ***stable*** *for a partial order H on U if $H\,;R\,;H \subseteq R$.*

The significance of Definition 5 is that stable relations on a hypergraph (U, H) correspond to union-preserving functions on the lattice of subgraphs [15]. Arbitrary relations correspond to union-preserving functions on the lattice of subsets, but here subgraphs are approximated.

Theorem 1 (Stell[15]). *Let R and S be stable relations on (U, H). We use $R;S$, to denote the composition, of R and S as relations on U.*

1. *$R\,;S$ is stable.*
2. *The equality relation $I \subseteq U \times U$ need not be stable, but H is a stable relation and satisfies $H\,;R = R = R\,;H$.*
3. *Neither the converse $\breve{R}$ nor the complement $\overline{R}$ need be stable, but stable relations are closed under the converse complement operation $\frown R = \overline{\breve{R}} = \breve{\overline{R}}$ and this satisfies $\frown\frown R = R$.*

Definition 6. *A relation R on (U, H) is* ***reflexive*** *if $H \subseteq R$, and is* ***transitive*** *if $R\,;R \subseteq R$.*

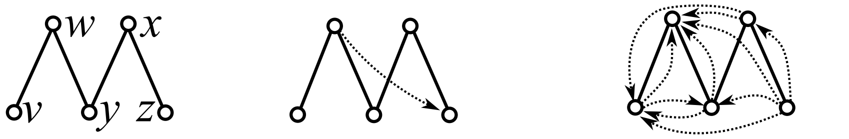
We note in passing that the notion of symmetry for such relations is not straightforward on account of the lack of an involutory converse operation [15]. In fact there are several different ways of defining symmetry but these are outside the scope of the present paper.

It can be seen that any relation on U (without any assumption of stability) which satisfies the reflexivity and transitivity conditions must actually be stable. This is because we would have $H\,;R\,;H \subseteq R\,;R\,;R \subseteq R$.

Figure 3 provides two examples of stable relations on the partial order, which is also more specifically a graph, from Figure 1. These two relations, which are described in detail as subsets of $U \times U$ in the figure, are reflexive which allows the convenient visualisation shown. In a reflexive relation we only need show those arrows (ordered pairs) which are present in addition to H. These added arrows are shown as dashed lines with arrow heads in the figure.

4.3 Correspondence between Partitions and Relations on Partial Orders

The well-known correspondence between partitions of a set and equivalence relations on a set allows us to switch between two ways of thinking about indiscernibility. The consideration of relations on sets that need not be equivalence relations has often been used in rough set theory [11,



(v,v) (w,w) (x,x) (y,y)
(z,z) (w,v) (w,y) (x,y)
(x,z) (w,z)

(v,v) (w,w) (x,x) (y,y)
(z,z) (w,v) (w,y) (x,y)
(x,z) (x,v) (x,w)
(v,y) (v,w) (y,w) (y,v)
(z,y) (z,v) (z,w) (z,x)

Fig. 3. Graph from Fig 1 with two examples of reflexive transitive stable relations

18, 12], but the use of relations on hypergraphs, or more generally on partial orders has been relatively unexplored. The approach in [14] deals with several approximation operators in terms of relations on hypergraphs, but did not consider what connection there might be between partitions (in some generalized sense) of a hypergraph and relations on the hypergraph with properties analogous to reflexivity, transitivity and symmetry. As already noted, symmetry will not be considered in this paper, so we will just deal with reflexive and transitive relations in the sense of Definition 6.

Definition 7. *For any relation R on U and any $u \in U$, define the u-**dilate** of R to be $uR = \{v \in U \mid u\ R\ v\}$. More generally, for any $X \subseteq U$, we will use XR to denote $\{v \in U \mid \exists x\ (x\ R\ v \text{ and } x \in X)\}$.*

The terminology ‘dilate’ is used as in mathematical morphology [10, 2] uR is the dilation by R of the set $\{u\}$ usually denoted $\{u\} \oplus R$. In the case that R is an equivalence relation on U, the u-dilates are just the equivalence classes. For more general R, two dilates can overlap without being equal and this can be seen in the examples in Figure 4.

Lemma 1. *Let R be a reflexive and transitive relation on U. Then for any $u, v \in U$ the following three statements are equivalent:*

$$v \in uR, \qquad u\ R\ v, \qquad vR \subseteq uR.$$

In generalizing the notion of a partition from a set to a hypergraph, and more generally to a partial order, it is clear that if the blocks of the partition are disjoint then they will not, except in trivial cases, respect the additional structure of the set. In the case of a graph, if we require that blocks are disjoint and that in addition they are always subgraphs then a connected graph will only have a single partition consisting of just one block. In the case of a partial order requiring blocks to both be disjoint and to be downward closed sets leads to a generalized form of this limitation.

Thus, we expect a general partition of a partial order either to have blocks that overlap or to have blocks that need not be downward closed sets. But a good notion of partition should also be connected with relations on the partial order and should be capable of supporting approximation operators with good properties. The following result demonstrates how these requirements are connected.

Theorem 2. *Let U be a set and H a partial order on U. The following three sets of structures are in bijective correspondence with each other.*

1. *Relations R on U such that $H \subseteq R$ and $R\,;R \subseteq R$.*
2. *Partitions of U equipped with a partial order, P, on the set of equivalence classes such that*
$$\forall u, v \in U\ (u\ H\ v \Rightarrow [u]\ P\ [v]), \tag{2}$$
where $[u]$ denotes the equivalence class of u.
3. *Sets $\mathcal{C}$ of subsets of U such that if we define for $u \in U$*
$$\ulcorner u \urcorner^{\mathcal{C}} = \bigcap\{B \in \mathcal{C} \mid u \in B\}$$
then for every $u \in U$ we have $\ulcorner u \urcorner^{\mathcal{C}} \in \mathcal{C}$ and if $u\ H\ v$ then $\ulcorner v \urcorner^{\mathcal{C}} \subseteq \ulcorner u \urcorner^{\mathcal{C}}$.

Proof. We show first that *1* corresponds with *2*.

Given a relation R as in *1*, define a relation $\equiv_R$ by $x \equiv_R y$ if $x\ R\ y$ and $y\ R\ x$. It is straightforward to check that this is an equivalence relation. From R we also define a relation P_R on the equivalence classes of $\equiv_R$ by $[x]\ P_R\ [y]$ if $x\ R\ y$. To check this is well-defined we need to check that if $x \equiv_R x'$ and $y \equiv_R y'$ then $x\ R\ y$ iff $x'\ R\ y'$, but this is routine. It is also clear that P_R satisfies the property stated of P in (2).

In the other direction, given an equivalence relation $\equiv$ and a partial order P satisfying (2), define a relation, $S_P^{\equiv}$, on U by $x\ S_P^{\equiv}\ y$ iff $[x]\ P\ [y]$. We can check that $S_P^{\equiv}$ contains H and is transitive.

To justify that we have a bijection, suppose first that we have a relation R as in *1*. We need to show that $S_{P_R}^{\equiv_R} = R$. Secondly, given an equivalence relation E and a partial order Q satisfying (2), we need to show that $\equiv_{S_Q^E} = E$ and that $P_{S_Q^E} = Q$. These are both routine calculations from the definitions.

We now show that *1* corresponds with *3*.

Given a relation R as in *1*, define a set of subsets of U by $\mathcal{B}_R = \{uR \subseteq U \mid u \in U\}$. The key observation here is that $\ulcorner u \urcorner^{\mathcal{B}_R} = uR$. To justify this, note that for any $w \in U$ the condition $u\ R\ w$ is equivalent, since R is reflexive and transitive, to
$$\forall v\ (v\ R\ u \Rightarrow v\ R\ w). \tag{3}$$
Now (3) is equivalent to $w \in \bigcap\{vR \subseteq U \mid u \in vR\}$ and hence to $w \in \ulcorner u \urcorner^{\mathcal{B}_R}$.

To map a set of subsets $\mathcal{C}$ as in *3* to a relation as in *1* we define the relation $T_{\mathcal{C}}$ by
$$u\ T_{\mathcal{C}}\ v \quad \text{iff} \quad \ulcorner v \urcorner^{\mathcal{C}} \subseteq \ulcorner u \urcorner^{\mathcal{C}}.$$

Finally, to show these two constructions provide a bijection, we have to check that given any relation R as in *1*, and any set of subsets $\mathcal{C}$ as in *3*, that $T_{\mathcal{B}_R} = R$ and that $\mathcal{B}_{T_{\mathcal{C}}} = \mathcal{C}$. These are routine calculations from the definitions. □

The above result shows how reflexive and transitive relations on a partial order are equivalent to two ways of weakening the usual notion of a partition on a set. Clearly, in the special case that H is the identity relation on U and in addition that R is symmetric, the structures in parts 2 and 3 both reduce to an ordinary partition on U.

Besides equivalence relations and partitions of a set U, functions defined on U provide another way of performing granulation. This too generalizes to the partial order case. We omit the proof as it uses similar techniques to that of Theorem 2.

Theorem 3. *Let (U, H) and (V, K) be posets and φ a function from U to V such that for all $u_1, u_2 \in U$ we have $u_1\ H\ u_2$ implies $\varphi u_1\ K\ \varphi u_2$. Then the relation $\mathcal{R}_\varphi$ on U defined by $u_1\ \mathcal{R}_\varphi\ u_2$ iff $\varphi u_1\ K\ \varphi u_2$ is transitive and contains H.* □

Figure 5 illustrates Theorem 5 by showing how the relations used in Figure 4 have corresponding order-preserving functions defined on the underlying partial orders.

4.4 Two Kinds of Granulation

In a set the notion of granulation, that is the formation of granules, involves grouping or clustering together subsets of the elements. This happens too with graphs. The lower example in Figure 4 shows a case where one edge and one node form one cluster and the remaining two nodes and edge form another cluster. However, granulating a partial order, such as a graph, is not just a matter of clustering elements together. Such clustering by itself only yields a discrete set and cannot produce non-trivial partial orders. The second component to granulation on a partial order is the provision of a partial order on the clusters of elements. Theorem 2 shows that reflexive and transitive relations on partial orders correspond to granulations consisting of the formation of clusters together with ordering the clusters.

The upper example in Figure 4 shows that even if the formation of clusters gives only singleton clusters we can still have an ordering on these clusters which strictly extends the original ordering. The cluster ordering is shown by bold arrows in the figure.

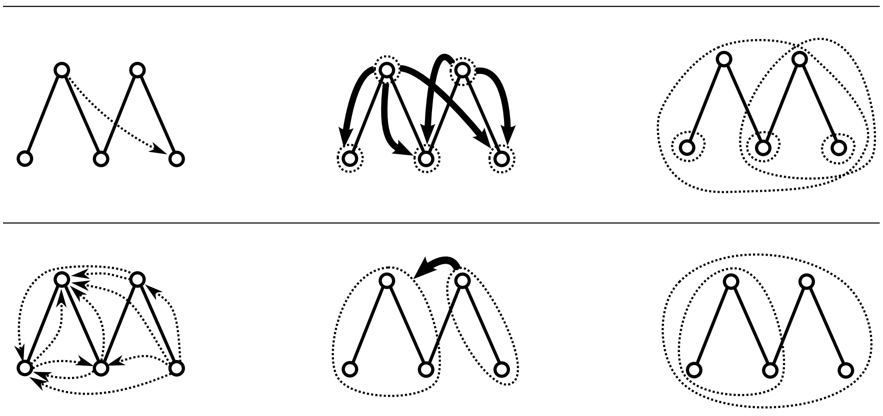


Fig. 4. Relations from Fig 3 with corresponding ordered partitions and overlapping dilates

5 Approximation Operators

Returning now to the definition of an ordered information system (U, H, A) in section 4.1 we can see that as each subset $B \subseteq A$ provides a partially ordered set of equivalence classes which

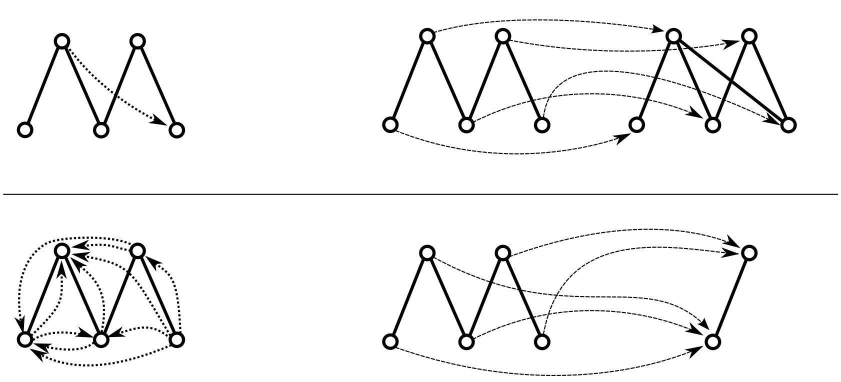


Fig. 5. Relations from Fig 3 with corresponding quotient structures

respects H as in equation (1). Thus by Theorem 2 we can construct a relation R_B which is stable with respect to H and is reflexive and transitive. The significance of this is that we can use R_B to define upper and lower approximation operators using opening and closing. Several different pairs of operators can be considered. The pair $\underline{B}$ and $\overline{B}$ suggested below does not generalize any of the three dual pairs discussed in [18] but generalizes $\underline{apr'_n}$ and $\overline{apr'_{\breve{n}}}$ in the notation of [18] where n is the neighbourhood operator arising from the relation R_B and $\breve{n}$ is that arising from the converse of R_B. The definition uses the **erosion** operator, defined for $X \subseteq U$ by $R_B \ominus X = \{u \in U : \forall v\ ((u, v) \in R_B \text{ implies } v \in X)\}$.

Definition 8. *Let (U, H, A) be an ordered information system and $B \subseteq A$. For any X where $XH \subseteq X$ we define the upper and lower approximation operators $\overline{B}$, $\underline{B}$ by $\overline{B}(X) = R_B \ominus (XR_B)$, and $\underline{B}(X) = (R_B \ominus X)R_B$.*

This departs from the more usual dualities of upper and lower in rough set theory by choosing an adjoint pair of operators. This reflects the preferred duality in mathematical morphology, and also the trend in some aspects of modal logic to consider adjoint pairs of modalities. Appropriate properties for approximation operators still hold with this choice including the following.

1. $\overline{B}\overline{B}(X) = \overline{B}(X)$, and $\underline{B}\underline{B}(X) = \underline{B}(X)$
2. $\underline{B}(X) \subseteq X \subseteq \overline{B}$
3. $\underline{B}(X \cap Y) \subseteq \underline{B}(X) \cap \underline{B}(Y)$ and $\underline{B}(X) \cup \underline{B}(Y) \subseteq \underline{B}(X \cup Y)$
4. $\overline{B}(X \cap Y) \subseteq \overline{B}(X) \cap \overline{B}(Y)$ and $\overline{B}(X) \cup \overline{B}(Y) \subseteq \overline{B}(X \cup Y)$
5. $X \subseteq Y$ implies $\overline{B}(X) \subseteq \overline{B}(Y)$ and $\underline{B}(X) \subseteq \underline{B}(Y)$

Some properties here are weaker than in the situation where the universe is unordered and where indiscernability is symmetric as well as reflexive and transitive. For example, we do not have $\overline{B}(X) \cup \overline{B}(Y) = \overline{B}(X \cup Y)$, nor do we have $\underline{B}(\overline{B}(X)) = \overline{B}(X)$. These weakenings are already well known in the the case of binary relations on a set (rather than a poset as here) in [18].

6 Conclusions and Further Work

This paper has used a correspondence between partially ordered partitions and certain reflexive transitive relations to find a simple way of defining approximation operators on ordered information systems. Further work is continuing on connections between these approximation operators and the bi-intuitionistic modal logic using relations on hypergraphs that was introduced in [17]. This is likely to provide a way of generalizing information logics [4] to the partially ordered setting. The notion of granulation used in this paper is not the only way parts of graphs can constitute granules. Further work will examine how ideas such as the tree-decompositions described in [5, p337] are connected with the reflexive and transitive relations studied here. In [6] Fan considers information systems where instead of functional attributes defined on the universe there are relations. Combining this approach with the present paper suggests it would be interesting to explore relational information systems in the more general case of a partially ordered universe.

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