Endowment additivity and the weighted proportional rules for adjudicating conflicting claims

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Abstract

We propose and study a new axiom, restricted endowment additivity, for the problem of adjudicating conflicting claims. This axiom requires that awards be additively decomposable with respect to the endowment whenever no agent's claim is filled. For two-claimant problems, *restricted endowment additivity* essentially characterizes weighted extensions of the proportional rule. With additional agents, however, the axiom is satisfied by a great variety of rules. Further imposing versions of *continuity* and *consistency*, we characterize a new family of rules which generalize the proportional rule. Defined by a priority relation and a weighting function, each rule aims, as nearly as possible, to assign awards within each priority class in proportion to these weights. We also identify important subfamilies and obtain new characterizations of the constrained equal awards and proportional rules based on *restricted endowment additivity*.

Keywords: claims problem, restricted endowment additivity, weighted proportional rule, priority-augmented weighted proportional rule.

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1 Introduction

Claims problems arise in various guises throughout history with fascinating recommendations described even in the Babylonian Talmud (O'Neill 1982; Aumann and Maschler 1985). A claims problem consists of a group of agents, each with a claim on an endowment, and a rule specifies how the endowment will be divided. Capturing a fundamental economic problem of resource allocation, the simple model has spawned a vast and growing literature[1] and provides a testing ground for theories of equity.[2] In this setting, we propose and study a new additivity axiom. Intimately linked to the principle of proportionality, our results uncover a family of rules which allow diverse answers to the question *Proportional to what?*

Our primary axiom adapts the familiar notion of additivity, a property commonly sought for strategic and practical reasons[3] and now a bedrock principle for cost sharing.[4] The general principle requires that problems be additively decomposable along some dimensions. A first possibility is to require additivity on all dimensions, both the claims and the endowment in our setting. We will argue that, at least in some settings, claims represent intrinsic features which should not be decomposed. Even when desirable, additivity on both the claims and the endowment is incompatible with our feasibility constraints.[5]

We instead take as our starting point *endowment additivity*, which requires that problems be additively decomposable with respect only to the endowment. More concretely, when two problems are formed from a third problem by dividing its endowment into two parts while retaining the original vector of claims, *endowment additivity* requires that the award of each agent in the original problem equal the sum of her awards in the auxiliary problems. Consistent with feasibility but still very strong, this axiom alone characterizes the proportional rule (Moulin 1987).[6] Because *endowment additivity* applies simultaneously to large and small endowments, much of its strength stems from the feasibility requirements. Our axiom aims to capture the spirit of *endowment additivity* without sacrificing all flexibility by limiting the scope: *Restricted endowment additivity* requires additive decomposability for those problems in which no agent is fully compensated.[7] These are often the problems of interest, particularly with a taxation gloss: Our axiom applies except in those unfortunate cases in which some agents have already contributed their entire incomes.

The normative appeal for our axiom, as well as *endowment additivity*, depends on how we

[1]See Thomson (2003, 2014, 2015a) for surveys.

[2]See Moreno-Ternero and Roemer (2006), for example.

[3]See, for example, Moulin (1987) and Ju et al. (2007).

[4]Shapley (1953) provides the classic axiomatization invoking additivity and Moulin (2013) discusses recent applications.

[5]See Bergantiños and Méndez-Naya (2001) and the discussion in Thomson (2014).

[6]As described in Remark 4 of Moulin (1987), this characterization is a corollary of his Theorem 5 which applies this property to surplus sharing problems.

[7]Restricted versions of joint additivity in claims and the endowment lead to additional rules (Bergantiños and Vidal-Puga 2004, 2006). With other axioms, notions along these lines lead to the "minimal overlap" rule (Alcalde et al. 2008; Marchant 2008).

interpret claims. When we allow asymmetric treatment among agents, we acknowledge that their names convey relevant information. For example, we may differentiate claims of individuals from those of corporations or of the government. By formulating subproblems with a common claims vector as in *restricted endowment additivity*, we extend this idea to claims, taking the position that they represent intrinsic features and may not be divided arbitrarily. This is the correct approach when claims are partitioned into classes. For example, small claims may be eligible for small-claims court and so warrant uniform treatment, whereas large claims require different treatment. Similarly, a claim below a certain threshold may be eligible for class action or a claim above a different threshold may be eligible for private arbitration; dividing a large claim or consolidating several small claims would erase important information and be invalid. Taking instead a taxation perspective, eligibility for a particular government program or liability for a given tax may depend on one's tax bracket or whether one's income falls below a poverty line; spreading one's income across several jobs should not change the tax bracket to which one belongs. Just as age, disability, or socio-economic status may be relevant to classification, so too may be an agent's income.[8]

Continuing with the taxation interpretation, we describe practical advantages enjoyed by rules satisfying *restricted endowment additivity*. Suppose that the total burden derives from two taxing entities, perhaps a city and a state, or as requisitions for two separate projects.[9] *Endowment additivity* ensures that these problems may be handled independently; *restricted endowment additivity* requires coordination only to check whether the joint burden will require some agents to contribute their entire incomes. By decentralizing the process, the city and state can determine liabilities knowing only each agent's income. By contrast, consider the familiar "composition" axiom.[10] It also permits these problems to be considered separately, but requires that one problem be labeled "first" so that incomes may be updated in the second problem. Applying a rule satisfying this property, the city may assign liabilities differently among the agents depending on whether it acts first or second, and the two entities may not act simultaneously. Although the entities collect and agents pay the same total amounts, the necessity of coordination presents practical difficulties and the dependence on the temporal order appears arbitrary.

In conjunction with *restricted endowment additivity*, we study a weaker version of the familiar *consistency* principle. The general principle considers a situation in which one agent leaves with her award, creating a reduced problem in which the endowment is reduced by this amount. *Consistency* requires that each agent receive the same award in the reduced problem as in the original prob-

[8]As another example, some cultural norms call for the highest status or wealthiest person to bear the cost of a communal meal or celebration. Although paying for a large banquet may reverse the ex-post wealth ranking, responsibility for future expenses depends on the original ranking and the same individual will pay, provided her wealth is not entirely exhausted.

[9]From a claims perspective, these might be two funding agencies from which researchers have requested funds or departments with overlapping responsibilities for reimbursing costs included by individuals.

[10]A class of "baseline extension operators" generalize the logic of composition and define new rules from existing rules by first making a baseline award and then applying the original rule to an updated auxiliary problem (Hougaard et al. 2012, 2013). Similar observations apply to these rules as well.

lem. Our axiom, *full compensation consistency*, applies the principle to those cases in which the departing agent receives her full claim. As these cases constitute a small proportion of problems, *full compensation consistency* substantially weakens *consistency*. As we will see, *full compensation consistency* and *restricted endowment additivity* reinforce each other, as *full compensation consistency* gains traction precisely when *restricted endowment additivity* falls silent. Finally, we impose *endowment continuity* throughout.[11]

The implications of *restricted endowment additivity* are easiest to describe geometrically as restrictions on a "path of awards", the locus of awards vectors that a rule assigns for a given claims vector as the endowment varies. For two-claimant problems, *restricted endowment additivity* is very strong, requiring that the path follow a ray from the origin until one agent's claim is filled. With *endowment continuity*, the remainder of the path consists of either a vertical or horizontal segment (Proposition 2). On the other hand, with three or more claimants, *restricted endowment additivity* has limited force. It continues to require that each path of awards initially follow a ray from the origin, but is mute once one agent receives her full claim. In particular, *restricted endowment additivity* has no force in problems where any agent has a zero claim. Here *full compensation consistency* plays an essential role, duly extending the scope of *restricted endowment additivity* to all problems.

Building on the intuition from two-claimant problems, we briefly describe the families of rules arising from our analysis. First are the weighted proportional (W-proportional) rules, each defined by a function which assigns weights to agent-claim pairs. A W-proportional rule assigns awards proportionally with respect to that weighting function to the greatest extent possible. Intuitively, the path of awards moves toward a target determined by the weights, deviating only to ensure feasibility after some agents' claims are filled. While W-proportional rules allow for an essentially arbitrary weighting function, they exclude limiting cases in which some agents have full priority over others. The priority-augmented weighted proportional (PW-proportional) rules generalize the W-proportional rules by "filling in the gaps". Two parameters, a weighting function and a priority relation over agent-claim pairs, describe a PW-proportional rule.[12] The priority relation partitions the claims in each problem into priority classes, and awards within each class are made according to the weighting function as with W-proportional rules. The family includes, naturally, the proportional rule, as well as the constrained equal awards rule, the priority rules, and many new rules.

The combination of *restricted endowment additivity* and *full compensation consistency*, together with *endowment continuity*, implies that a rule follows a PW-proportional rule for each fixed popu-

[11] *Continuity* is typically required jointly for the endowment and claims, something we do not impose. This accords our interpretation of claims as intrinsic features not assumed to be decomposable.

[12] As with weights, it is important that the priorities be over agent-claim *pairs*. This allows, for example, the same agent to be in different tax brackets when her income changes. Similarly, two creditors with the same numerical claims, perhaps representing an individual and a corporation, may be in different priority classes.

lation (Theorem 1). Strengthening *full compensation consistency* to *consistency* imposes uniformity across populations, thereby characterizing the PW-proportional rules (Theorem 2). Notably, although the axiom is not imposed in our characterization, all PW-proportional rules are *endowment monotonic*, meaning that an agent's award never decreases when the endowment increases (Proposition 1).[13] Within the family, additional axioms place restrictions on the priority orders and weight functions (Corollary 2). Most significantly, we distinguish the W-proportional rules by *minimal sharing*, the very mild requirement that no agent with a positive claim receive her full claim while another agent with a positive claim receives nothing (Theorem 3). Our final result considers the distribution of losses as well as awards: *Restricted endowment additivity* and *self-duality*,[14] the requirement that losses be allocated according to the same principle as awards, characterize the proportional rule (Theorem 4).

It will be easier to compare our rules to families identified previously after formalizing our results, so we postpone further discussion of related literature and turn immediately to the model. In Section 2, we introduce our axioms and the PW-proportional rules. Section 3 presents our characterizations and Section 4 situates our work within the literature. All proofs appear in the appendix.

2 Model

A claims problem consists of a finite set of agents with conflicting claims over an amount to divide. Formally,[15] there is a countable set of potential agents $\boldsymbol{\mathcal{N}} \subseteq \mathbb{N}$ with $|\mathcal{N}| \geq 2$. For $N \subseteq \mathcal{N}$, $N' \subseteq N$, and $x \in \mathbb{R}^N_+$, let $x_{N'} \equiv (x_i)_{i \in N'}$ denote the components of x corresponding to N'. We write x_i for $x_{\{i\}}$ and also x_{-i} for $x_{N \setminus \{i\}}$. To denote vector inequalities, for each $N \subseteq \mathcal{N}$ and each pair $x, y \in \mathbb{R}^N_+$, we write $x \ll y$ if for each $i \in N$, $x_i < y_i$; $x \leq y$ if for each $i \in N$, $x_i \leq y_i$; and $x < y$ if $x \leq y$ and there is $i \in N$ such that $x_i < y_i$.

Given $N \subseteq \mathcal{N}$, a **claims problem for $\boldsymbol{N}$** is a pair $\boldsymbol{(c, E)} \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $E \leq \sum_N c_i$. For each $i \in N$, c_i is agent i's **claim** and E is the **endowment**. The set of all claims problems for N is $\boldsymbol{\mathcal{C}^N} \equiv \left\{(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+ : E \leq \sum_N c_i\right\}$. A **rule** is a mapping $\boldsymbol{\varphi}$ defined on $\bigcup_{N \subseteq \mathcal{N}} \mathcal{C}^N$ such that for each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $0 \leq \varphi(c, E) \leq c$ and $\sum_N \varphi_i(c, E) = E$.

Some additional notation will be useful. Let $\boldsymbol{\mathcal{U}}$ be the collection of functions $u \colon \mathcal{N} \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $i \in \mathcal{N}$, $u(i, \cdot)$ is strictly positive on $\mathbb{R}_{++}$. Within this collection, we distinguish those which are symmetric in the first argument: Let $\boldsymbol{\mathcal{U}^*} \subseteq \mathcal{U}$ consist of those functions $u \in \mathcal{U}$ such that for each pair $i, j \in \mathcal{N}$, $u(i, \cdot) = u(j, \cdot)$. Finally, we denote a complete and transitive relation

[13]For another case in which *endowment monotonicity* is implied by the combination of *consistency* and other axioms, none of which directly imply the property, see Young (1987b).

[14]Aumann and Maschler (1985) introduce this property and Young (1987b) also uses it to characterize the proportional rule. Via duality, our results identify and characterize the "dual" families of our rules (Remark 4).

[15]Let $\mathbb{N}$ denote the natural numbers and let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{R}_{++}$ denote respectively the real, non-negative real, and positive real numbers.

on $\mathcal{N} \times \mathbb{R}_+$ by $\prec$ and write $\mathbf{\Pi}$ for the collection of all such relations.

2.1 Properties

To introduce our properties, let φ be a rule. We begin with the central axiom of our study: If no agent's claim is filled by applying the rule, then the awards vector for that problem may be equivalently obtained by dividing the endowment and adding the awards vectors when the rule is applied separately to each resulting problem.[16]

Restricted endowment additivity: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair $E^1, E^2 \in \mathbb{R}_+$ with $E = E^1 + E^2$, if $\varphi(c, E) \ll c$, then $\varphi(c, E) = \varphi(c, E^1) + \varphi(c, E^2)$.

Importantly, the claims vector is the same in all three problems. This contrast with the hypotheticals defining "full" additivity, which no rule satisfies (Bergantiños and Méndez-Naya 2001), as well as the composition axioms (Young 1987a; Moulin 2000).

Our second concern is the behavior of the rule when some agents depart with their awards. We begin with the standard requirement[17] in this situation: If one agent departs with her award, then the awards of the remaining agents should be unchanged when the rule is reapplied to distribute the reduced endowment among those agents.

Consistency: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $\varphi_{-i}(c, E) = \varphi\big(c_{-i}, E - \varphi_i(c, E)\big)$.

By iteration, *consistency* further implies the same conclusion when a group of agents depart with their awards.

Our next axiom limits the requirements of *consistency* to those cases in which the departing agent receives her full claim.[18]

Full compensation consistency: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $\varphi_i(c, E) = c_i$, then $\varphi_{-i}(c, E) = \varphi(c_{-i}, E - c_i)$.

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[16] For comparison and future reference, we state (unrestricted) *endowment additivity* and the composition axioms together in parallel fashion: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each pair $E^1, E^2 \in \mathbb{R}_+$ with $E = E^1 + E^2$,

Endowment additivity: $\varphi(c, E^1 + E^2) = \varphi(c, E^1) + \varphi(c, E^2)$.

Composition up: $\varphi(c, E^1 + E^2) = \varphi(c, E^1) + \varphi\big(c - \varphi(c, E^1), E^2\big) = \varphi\big(c - \varphi(c, E^2), E^1\big) + \varphi(c, E^2)$.

Composition down: $\varphi(c, E^1) = \varphi\big(\varphi(c, E^1 + E^2), E^1\big)$ and $\varphi(c, E^2) = \varphi\big(\varphi(c, E^1 + E^2), E^2\big)$.

The axioms are similar in that they each decompose a given problem into subproblems, but differ in how they treat claims. In contrast with with *restricted endowment additivity*, these axioms apply to all claims problems.

[17] See Thomson (2012) for a thorough normative analysis of the consistency principle and Thomson (2015b) for a survey on its applications.

[18] Thomson (2014) mentions this property in passing, and Ju and Moreno-Ternero (2014) introduce an adapted version for an augmented model with exchange.

As with *consistency*, *full compensation consistency* further implies that awards remain unchanged when several agents whose claims are filled leave simultaneously. As its limited applicability suggests, *full compensation consistency* is a much milder requirement than *consistency*; an important example satisfying *full compensation consistency* but not *consistency* is the "random arrival" rule (O'Neill 1982; Chun and Thomson 2005). Similarly, a rule constructed by applying different "equal sacrifice" rules (Young 1987a) across populations satisfies *full compensation consistency* but not *consistency*.

A more familiar weakening of *consistency*, **null claims consistency**, further limits the requirement to cases in which the departing agent has a zero claim.[19]

Our third concern is how a rule responds to small changes in the endowment. Our requirement here is standard: Small changes in the endowment should lead to at most small changes in awards.

Endowment continuity: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each sequence $E^\nu \in \mathbb{R}_+$ with $(c, E^\nu) \in \mathcal{C}^N$, if $E^\nu \to E$, then $\varphi(c, E^\nu) \to \varphi(c, E)$.

As we will see, *restricted endowment additivity* implies a limited form of *endowment continuity* which applies when the endowment is below a threshold (Lemma 1).[20] Another notion, **claims continuity**, applies to small changes in the claims vector. Also common is the stronger property **joint continuity** which considers simultaneous changes in the claims and endowment.[21]

Our next axiom also pertains to changes in the endowment and requires that all agents' awards move in the same direction: If the endowment decreases, no agent's award should increase.

Endowment monotonicity: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' \in \mathbb{R}_+$, if $E' \leq E$, then $\varphi(c, E') \leq \varphi(c, E)$.

Endowment monotonicity implies *endowment continuity*.

Our next axioms concern the equity of the awards vectors. Our first requirement limits the extent to which one agent may be favored over another: If two agents have positive claims, then neither agent should be fully compensated while the other receives a null award.[22]

Minimal sharing: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each pair $i, j \in N$, if $\varphi_i(c, E) = c_i > 0$, then either $\varphi_j(c, E) > 0$ or $c_j = 0$.

[19] The formal requirement is: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, if $c_i = 0$, then $\varphi_{-i}(c, E) = \varphi(c_{-i}, E)$.

[20] A previous version of this paper considered a weaker requirement, parallel to *full compensation consistency*, which avoids this overlap. Details are available from the author.

[21] See, for example, Young (1987b).

[22] A similar, slightly stronger condition requires that in each problem with a positive endowment, those individuals with positive claims receive positive awards. This axiom is studied as "positive share" by Moulin (2002) and "positive awards" by Flores-Szwagrzak (2015).

Minimal sharing allows for considerable asymmetry, even among agents with equal claims. Our second requirement says that agents with equal claims should receive equal awards.

Equal treatment of equals: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, each pair $i, j \in N$, if $c_i = c_j$, then $\varphi_i(c, E) = \varphi_j(c, E)$.

While the implication of *equal treatment of equals* is more substantive, *minimal sharing* applies to a wider range of problems and so the axioms are logically distinct. A related equity requirement requires symmetric treatment of the agents: A rule is **anonymous** if permuting the components of the claims vector leads to a corresponding permutation of components of the awards vector).

We conclude with a standard axiom which plays a minor rule in our analysis.[23] It says that the awards assigned by the rule should not change when each agent's claim is truncated at the endowment.

Claims truncation invariance: For each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $\varphi\big((\min\{c_i, E\})_{i \in N}, E\big) = \varphi(c, E)$.

2.2 Proportional family

Our rules arising in our study generalize the notion of proportional division. A rule in the family is calibrated by a function $u \in \mathcal{U}$ which assigns to each agent-claim pair a fixed weight. The rule then divides the endowment proportionally to these weights to the greatest extent possible given feasibility constraints.

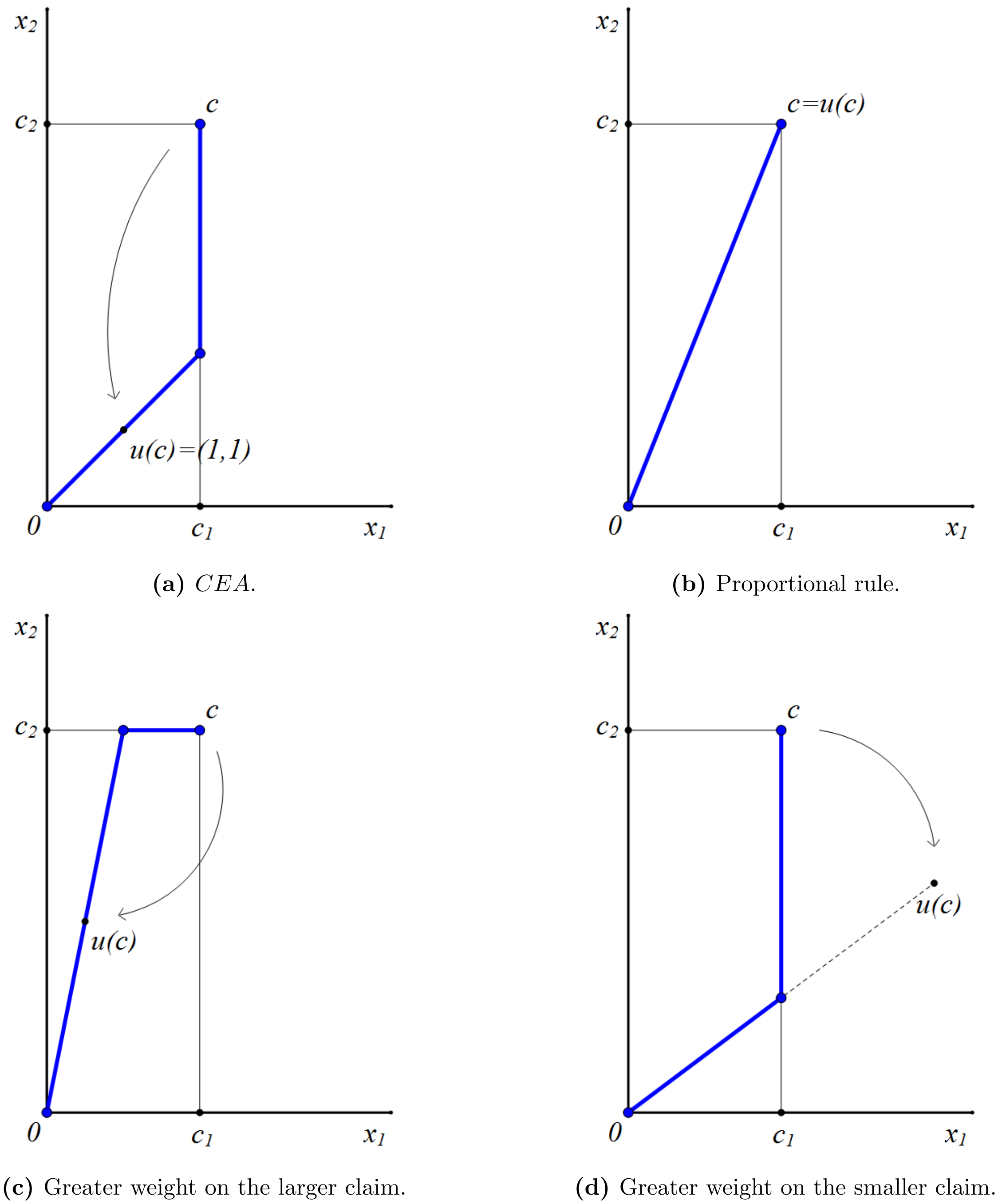
Weighted proportional rule with weights $\boldsymbol{u} \in \boldsymbol{\mathcal{U}}$, $\boldsymbol{P^u}$: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, $P_i^u(c, E) \equiv \min\{\lambda u(i, c_i), c_i\}$ where $\lambda \in \mathbb{R}_+$ is chosen so that $\sum_N P_i^u(c, E) = E$.

For short, we call these **W-proportional rules**. Figure 1 depicts the paths of awards[24] for several examples of W-proportional rules applied to two-claimant problems. In general, the paths of awards are piecewise linear with at most $|N|$ segments. Whenever all agents receive awards less than their claims, a W-proportional rule assigns awards proportionally according to u: for each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, if $P^u(c, E) \ll c$, then $P_i^u(c, E) = \frac{u(i, c_i)}{\sum_N u(j, c_j)} \cdot E$. Letting $x \equiv P^u(c, E)$, for each pair $i, j \in N$, $\frac{x_i}{u(i, c_i)} = \frac{x_j}{u(j, c_j)}$. By comparison, applying the weighting function in the spirit of Young (1987a) and Chambers and Moreno-Ternero (2015), we would have instead $\frac{u(i, x_i)}{u(i, c_i)} = \frac{u(j, x_j)}{u(j, c_j)}$.

[23] Curiel et al. (1987) introduce this property to distinguish rules which correspond to cooperative game solutions and Dagan (1996) studies the axiom under the name "independence of irrelevant claims". It is implied by "contraction independence" studied by Kıbrıs (2012), Kıbrıs (2013), and Stovall (2014b).

[24] For each $c \in \mathbb{R}_+^N$, the **path of awards of $\boldsymbol{\varphi}$ for $\boldsymbol{c}$** consists of the locus of awards vectors as the endowment varies: $\{\varphi(c, E) : 0 \leq E \leq \sum_N c_i\}$.

Remark 1. Each W-proportional rule admits multiple representations in terms of weighting functions. As only the relative weights assigned by u are relevant to determining the awards, various normalizations are possible.



(a) CEA. **(b)** Proportional rule. **(c)** Greater weight on the larger claim. **(d)** Greater weight on the smaller claim.

Figure 1: Illustrating W-proportional rules. In the figure, $N \equiv \{1, 2\}$ and $u(c) \equiv (u(1, c_1), u(2, c_2))$. (a) If for each $(i, c_0) \in \mathcal{N} \times \mathbb{R}_+$, $u(i, c_0) = 1$, then $P^u = CEA$. (b) If for each $(i, c_0) \in \mathcal{N} \times \mathbb{R}_+$, $u(i, c_0) = c_i$, then $P^u = P$. (c,d) Compared to the proportional rule, a W-proportional rule may be more favorable to agents with large claims or to agents with small claims.

Remark 2. If for each $(i, c_0) \in \mathcal{N} \times \mathbb{R}_+$, $u(i, c_0) = c_0$, then P^u is the **proportional rule**. If for each $(i, c_0) \in \mathcal{N} \times \mathbb{R}_+$, $u(i, c_0) = 1$, then P^u is the **constrained equal awards rule (CEA)**.[25] We denote these rules by $\boldsymbol{P}$ and $\boldsymbol{CEA}$ respectively.

The W-proportional family is diverse and, because the weighting functions are essentially unrestricted, contains rules that treat agents highly asymmetrically. However, the family excludes rules which give full priority to some agents or to some numerical claims. Intuitively, our next family "fills in the gaps" by including these limiting cases. Formally, we introduce priority classes through an order $\prec \in \Pi$. Building from the W-proportional rules, rules in the next family are parameterized jointly by a priority relation and a weighting function.

Priority-augmented weighted proportional rule with priority order $\boldsymbol{\prec \in \Pi}$ and weights $\boldsymbol{u \in \mathcal{U}}$, $\boldsymbol{P^{\prec,u}}$

For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $i \in N$, let $N_i^{\prec} \equiv \{j \in N : (j, c_j) \prec (i, c_i)\}$, $N_i^{\sim} \equiv \{j \in N : (j, c_j) \sim (i, c_i)\}$, and define

$$P_i^{\prec,u}(c, E) \equiv \begin{cases} 0 & \text{if } E \leq \sum_{N_i^{\prec}} c_j \\ c_i & \text{if } \sum_{N_i^{\prec} \cup N_i^{\sim}} c_j \leq E \\ P_i^u\big(c_{N_i^{\sim}}, E - \sum_{N_i^{\prec}} c_j\big) & \text{otherwise} \end{cases} .$$

We refer to these rules briefly as **PW-proportional rules**. When $\prec$ is the complete indifference relation, so all agent-claim pairs are in the same priority class, these are simply W-proportional rules. At the other extreme, if for each pair $i, j \in \mathcal{N}$ either (i) for each pair $x, y \in \mathbb{R}_+$, $(i, x) \prec (j, y)$ or (ii) for each pair $x, y \in \mathbb{R}_+$, $(j, x) \prec (i, y)$, then we obtain a standard priority rule.

The priorities and weights defining a PW-proportional rule may depend on the identities agents as well as their claims, but these priorities and weights are uniform across populations. Relaxing this uniformity, we obtain rules that coincide with a PW-proportional rule for each fixed population, though possibly a different PW-proportional rule for each population. A rule φ is a **collection of fixed-population PW-proportional rules** if for each $N \subseteq \mathcal{N}$, there are $\prec^N \in \Pi$ and $u^N \in \mathcal{U}$ such that for each $(c, E) \in \mathcal{C}^N$, $\varphi(c, E) = P^{\prec^N, u^N}(c, E)$. That is, $\varphi = \big(P^{\prec^N, u^N}\big)_{N \subseteq \mathcal{N}}$.

Our leading axioms will impose some structure on the priorities and weights employed across populations, although less than the uniformity defining PW-proportional rules. We distinguish collections of priorities and weights such that the departure of an agent whose claim is filled does not change the relative priorities and weights assigned to the agents who remain. Formally, a collection $(\prec^N, u^N)_{N \subseteq \mathcal{N}}$ is **path consistent** if: For each $N \subseteq \mathcal{N}$, each triple $i, j, k \in N$, and each triple $c_i, c_j, c_k \in \mathbb{R}_+$, if either (i) $(i, c_i) \prec^N (j, c_j)$ or (ii) $u^N(i, c_i) \cdot c_j > u^N(j, c_j) \cdot c_i$, then both (a) $j \prec^{N \setminus \{i\}} k \Leftrightarrow j \prec^N k$ and (b) $u^N(j, c_j) \cdot u^N(k, c_k) = u^{N \setminus \{i\}}(j, c_j) \cdot u^{N \setminus \{i\}}(k, c_k)$. Conditions (i)

[25] Another leading rule, the **constrained equal losses rule (CEL)**, is not a W-proportional rule. It is the "dual" of CEA and is defined for each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$ by $CEL(c, E) = c - CEA(c, \sum_N c_i - E)$.

and (ii) identify an agent i whose claim is filled before that of another agent j. Then (a) and (b) require that removing agent i does not change the priority or weight assigned to agent j relative to any other agent k. Finally, a **path-consistent collection of PW-proportional rules** is a collection of fixed-population PW-proportional rules with path-consistent priorities and weights.

Remark 3. Conditions (i) and (ii) for path consistency are never met when the proportional rule is applied. Therefore, redefining a path-consistent rule to coincide with the proportional rule for a given population and all supersets of that population preserves path consistency.

3 Main results

Our main results characterize those rules satisfying *restricted endowment additivity*, *full compensation consistency*, and *endowment continuity* (Theorem 1) as well as the subfamilies of PW-proportional rules (Theorem 2) and W-proportional rules (Theorem 3). First, we elaborate on properties shared by all W-proportional rules.

Proposition 1. *Each W-proportional rule satisfies* minimal sharing*,* endowment monotonicity*,* consistency*, and* restricted endowment additivity*.*

We omit the straightforward, if tedious, verifications.[26] *Endowment monotonicity* implies *endowment continuity*, so the W-proportional rules satisfy this property as well. On the other hand, W-proportional rules need not be continuous in the vector of claims.

As the PW-proportional rules are defined from the W-proportional rules, they inherit some of their properties.

Corollary 1. *Each PW-proportional rule satisfies* endowment monotonicity*,* consistency*, and* restricted endowment additivity*.*

In contrast with the properties in Corollary 1, PW-proportional rules do not generally satisfy *minimal sharing* as this property precludes non-trivial priorities. Next we introduce a technical lemma which provides a useful reformulation of *restricted endowment additivity*: Beginning from a problem in which no claim is filled, scaling down the endowment leads to a proportional scaling down of awards.

Lemma 1. *A rule φ satisfies* restricted endowment additivity *if and only if for each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and $\alpha \in [0, 1]$, if $\varphi(c, E) \ll c$, then $\varphi(c, \alpha E) = \alpha\varphi(c, E)$.*

As a consequence of Lemma 1, *restricted endowment additivity* implies a limited form of *endowment continuity*, namely for those endowments smaller than the smallest claim.

[26] A complete proof appears in the working paper version of this paper available from the author.

3.1 Two-claimant problems

We next consider two-claimant problems. On this domain, *restricted endowment additivity* is very strong. Together with *endowment continuity*, it requires that each of the rule's paths of of awards follow that of a PW-proportional rule.

Proposition 2. *On the domain of two-claimant problems, a rule satisfies* restricted endowment additivity *and* endowment continuity *if and only if it is a collection of fixed-population PW-proportional rules.*

The combination of *endowment continuity* and *restricted endowment additivity* implies *endowment monotonicity*, although neither axiom implies it independently. By *restricted endowment additivity* alone, if one agent's claim is filled in a problem, then the claim of at least one agent must be filled at each larger endowment. For endowments larger than the largest claim, however, which agent's claim is filled may vary arbitrarily as the endowment increases. *Endowment continuity* prevents this. Figure 2 illustrates the path of awards for some discontinuous rules satisfying *restricted endowment additivity.* Example 1 illustrates the role of *endowment continuity* in Proposition 2.

Example 1. A rule satisfying restricted endowment additivity but not endowment continuity. For each $N \subseteq \mathcal{N}$ with $|N| = 2$ and each $(c, E) \in \mathcal{C}^N$, label the agents so that $c_1 \leq c_2$ and let

$$\varphi(c,E) \equiv \begin{cases} CEA(c,E) & \text{if } E < c_2 \\ (E - c_2, c_2) & \text{if } c_2 \leq E \end{cases}.$$

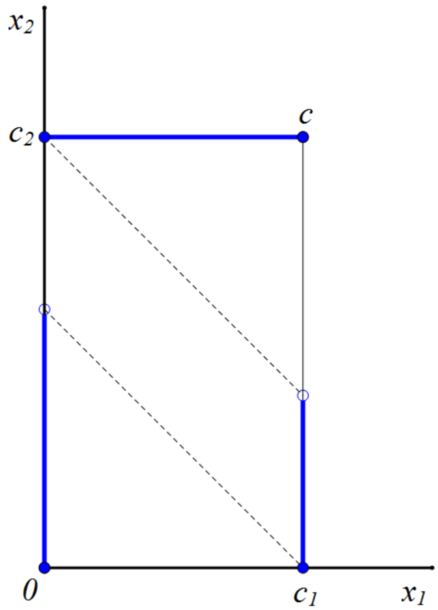
For each $N \subseteq \mathcal{N}$ with $|N| = 2$ and each $(c, E) \in \mathcal{C}^N$, if $\varphi(c, E) \ll c$, then $\varphi(c, E) = CEA(c, E)$, so φ satisfies *restricted endowment additivity.* However, φ violates *endowment continuity*: Let $N \equiv \{1, 2\}$ and $c_N \equiv (2, 4)$. Then for each $E \in \mathbb{R}_+$ with $E < 4$, $\varphi(c, E) = (\frac{E}{2}, \frac{E}{2})$ and $\lim_{E\to 4-} \varphi(c, E) = (2, 2)$. However, $\varphi(c, 4) = (0, 4)$.

3.2 Problems with an arbitrary number of claimants

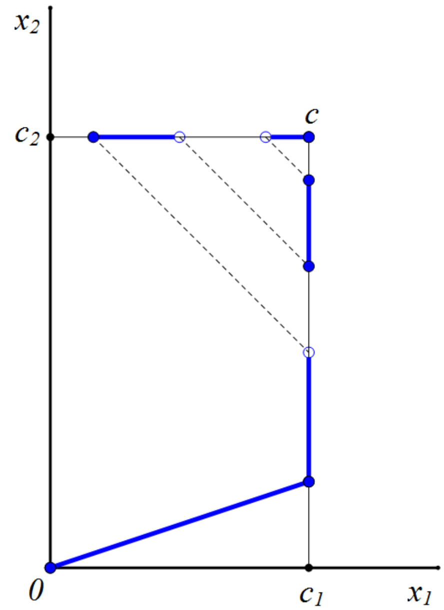
Our general results build on Proposition 2. By invoking *full compensation consistency*, we characterize those rules whose components form path-consistent collections of PW-proportional rules.

Theorem 1. *A rule satisfies* restricted endowment additivity*,* full compensation consistency*, and* endowment continuity *if and only if it is a path-consistent collection of fixed-population PW-proportional rules.*

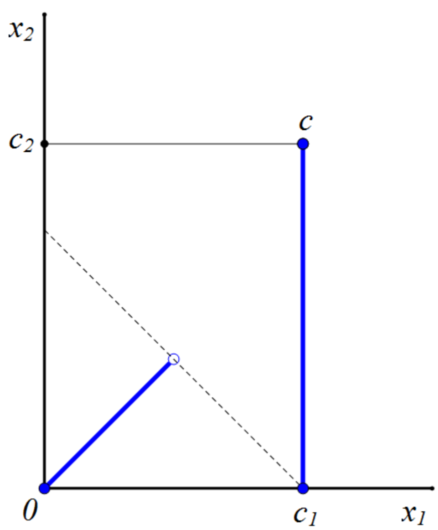
For two-claimant problems, coincidence follows from *restricted endowment additivity* and *endowment continuity* alone (Proposition 2), but this is not true for larger populations. Instead, *full compensation consistency* plays an essential role. Most simply, *full compensation consistency*



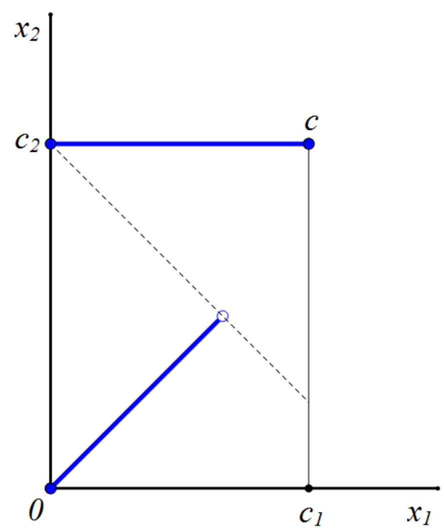
(a) Discontinuities when the endowment equals a claim.



(b) Multiple discontinuities.



(c) CEA modified to favor agent 1.



(d) CEA modified to favor agent 2.

Figure 2: Illustrating independence of axioms in Proposition 2: Discontinuous rules satisfying restricted endowment additivity. (a) The rule awards the entire endowment to agent 2 until the endowment is sufficient to fill the smallest claim, at which point the rule awards the full endowment to the agent with that claim. The rule switches again when it is possible to fill the claim of the agent with the larger endowment. (b) Provided that at least one claim remains filled, *restricted endowment additivity* is consistent with arbitrarily "switching" between filling one agent's claim and the other agent's claim. (c) The rule fills agent 1's claim as soon as it is feasible to do so and follows CEA otherwise. (d) The rule fills agent 2's claim as soon as it is feasible to do so and follows CEA otherwise.

ensures that the collection of fixed-population rules is path consistent, but its implications run deeper because the force of *restricted endowment additivity* and *endowment continuity* is greatly reduced in problems with three or more agents. Although *full compensation consistency* relates only in a small subset of problems, its force complements that of *restricted endowment additivity* so that their combination imposes considerably greater structure on rules than either axiom alone.

Our next examples verify independence of the axioms in Theorem 1. Of course, a great many rules satisfy *continuity* and *consistency*, but not *restricted endowment additivity.* These include the all equal sacrifice rules (Young 1987a) besides the proportional rule, as well as all members of the Talmud family (Moreno-Ternero and Villar 2006) besides the constrained equal awards rule. Example 2 illustrates the diversity of rules which become admissible without *full compensation consistency*, even in the presence of additional axioms.

Example 2. A family of rules satisfying restricted endowment additivity, endowment monotonicity, anonymity, and minimal sharing, but not full compensation consistency. Let $u \in \mathcal{U}$ and let $\bar{\varphi}$ be a rule satisfying *endowment monotonicity*, *anonymity*, and *minimal sharing.* For example, $\bar{\varphi}$ could be CEA, CEL, or in fact any parametric rule (Young 1987b). For each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, let $\bar{E}(c,E) \equiv \sup\{E' \in [0, \sum_N c_i] : P^u(c, E') \ll c\}$ and

$$\varphi(c,E) \equiv \begin{cases} P^u(c,E) & \text{if } E \leq \bar{E}(c,E) \\ P^u\big(c, \bar{E}(c,E)\big) + \bar{\varphi}\big(c - P^u(c, \bar{E}(c,E)), E - \bar{E}(c,E)\big) & \text{otherwise} \end{cases}.$$

Since both P^u and $\bar{\varphi}$ satisfy *endowment monotonicity*, *anonymity*, and *minimal sharing*, so does φ. For each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, if $\varphi(c, E) \ll c$, then $\varphi(c, E) = P^u(c, E)$. Therefore, since P^u satisfies *restricted endowment additivity*, φ does as well.

In general, φ violates *full compensation consistency.* For example, let $N \equiv \{1, 2, 3\}$, $c_N \equiv (2, 4, 8)$, $E \equiv 8$, $P^u = CEA$, and $\bar{\varphi} = CEL$. Then

$$\varphi(c,E) = CEA((2,4,6),6) + CEL((0,2,4),2) = (2,2,4).$$

However, $\varphi(c_{\{2,3\}}, E - \varphi_1(c,E)) = CEA((4,6),6) = (3,3) \neq (2,4) = \varphi_{\{2,3\}}(c,E)$.

Endowment continuity is also essential to the characterization. Demonstrating the importance of *endowment continuity* in Theorem 1, Example 3 presents a family of rules satisfying *restricted endowment additivity* and *full compensation consistency*, in fact even *consistency.* Although discontinuous, these rules have a natural interpretation. Each rule proceeds according to a priority order. If the highest priority agent-claim pair can be filled completely, then the rule does so and otherwise distributes no award at this point. With the remaining endowment, it proceeds to next highest priority agent-claim pair and so on. After considering each agent-claim pair, the rule distributes the remainder of the endowment according to a W-proportional rule, for example CEA or P.

Example 3. A family of rules satisfying restricted endowment additivity and consistency, but not continuity. Let $\prec \in \Pi$ and $u \in \mathcal{U}$. For each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, label the agents so that $(1,c_1) \prec (2,c_2) \prec \cdots \prec (n,c_n)$. Let $N_0(c,E) \equiv \emptyset$ and for each $k = 1,\ldots,n$, recursively define $N_k(c,E) \equiv N_{k-1} \cup \{k\}$ if $c_k \le E - \sum_{N_{k-1}(c,E)} c_j$. Now for each $i \in N$, let

$$\varphi_i^{\prec}(c,E) \equiv \begin{cases} c_i & \text{if } i \in N_n(c,E) \\ P_i^u\big(c_{N\setminus N_n(c,E)}, E - \sum_{N_n(c,E)} c_j\big) & \text{otherwise} \end{cases}.$$

That is, $\varphi^{\prec}$ gives priority to agent-claim pairs according to $\prec$, but only if the claim can be filled. To distribute the remainder of the endowment, $\varphi^{\prec}$ follows a W-proportional rule.

For each $N \subseteq \mathcal{N}$ and $(c,E) \in \mathcal{C}^N$, if $\varphi^{\prec}(c,E) \ll c$, then $N_n(c,E) = \emptyset$ and $\varphi^{\prec}(c,E) = P^u(c,E)$. Therefore, $\varphi^{\prec}$ satisfies *restricted endowment additivity*. To see that it is *consistent*, let $N \subseteq \mathcal{N}$, $(c,E) \in \mathcal{C}^N$, and $i \in N$. If $i \in N_n(c,E)$, then $\varphi_i(c,E) = c_i$ and $N_{n-1}(c_{-i}, E - c_i) = N_n(c,E)\setminus\{i\}$. Then also $E - \sum_{N_n(c,E)} c_j = (E - c_i) - \sum_{N_{n-1}(c_{-i},E-c_i)} c_j$, so $\varphi_{-i}(c,E) = \varphi(c_{-i}, E - c_i)$. Now suppose $i \notin N_n(c,E)$ and let $x_i \equiv P_i^u\big(c_{N\setminus N_n(c,E)}, E - \sum\big)N_n(c,E)c_j\big)$. Then $\varphi_i(c,E) = x_i$ and $N_{n-1}(c_{-i}, E - x_i) = N_n(c,E)$ so $\varphi_{N_n(c,E)}(c,E) = \varphi_{N_n(c,E)}(c_{-i}, E - c_i)$. Also, $E - \sum_{N_n(c,E)} c_j = E - \sum_{N_{n-1}(c_{-i},E-x_i)} c_j$ and P^u is *consistent*, so

$$\begin{aligned}
\varphi_{N\setminus(N_n(c,E)\cup\{i\})}(c,E) &= P^u_{N\setminus(N_n(c,E)\cup\{i\})}\left(c, E - \sum_{N_n(c,E)} c_j\right) \\
&= P^u_{N\setminus(N_{n-1}(c_{-i},E-P_i^u(c,E))\cup\{i\})}\left(c, E - \sum_{N_{n-1}(c_{-i},E-P_i^u(c,E))} c_j\right) \\
&= \varphi_{N\setminus(N_{n-1}(c,E)\cup\{i\})}\big(c_{-i}, E - x_i\big).
\end{aligned}$$

Thus, $\varphi_{-i}(c,E) = \varphi(c_{-i}, E - P_i^u(c,E))$ and $\varphi^{\prec}$ is *consistent*.

In general, $\varphi^{\prec}$ violates *endowment continuity*. For example, let $N \equiv \{1,2\}$, $c_N \equiv (2,2)$, $(1,2) \prec (2,2)$, and $P^u = CEA$. Then for $E \in \mathbb{R}_+$ with $E < 2$, $\varphi^{\prec}(c,E) = (\frac{E}{2}, \frac{E}{2})$ and $\lim_{E\to 2-} \varphi^{\prec}(c,E) = (1,1)$. However, $\varphi^{\prec}(c,2) = (2,0)$.

With additional properties, we characterize subclasses. First, if *full compensation consistency* is strengthened to *consistency*, we ensure uniformity across all populations, characterizing the PW-proportional rules.

Theorem 2. *A rule satisfies* restricted endowment additivity, consistency, *and* endowment continuity, *if and only if it is a PW-proportional rule.*

Perhaps most importantly, adding *minimal sharing* to the axioms of Theorem 2 leads to a characterization of the W-proportional rules.

Theorem 3. *A rule satisfies* restricted endowment additivity, consistency, endowment continuity, *and* minimal sharing *if and only if it is a W-proportional rule.*

Within the W-proportional family, additional properties[27] restrict the function u.

Corollary 2. *For each $u \in \mathcal{U}$, P^u satisfies*

1. equal treatment of equals *if and only if $u \in \mathcal{U}^*$.*
2. claims monotonicity *if and only if for each $i \in \mathcal{N}$, $u(i, \cdot)$ is non-decreasing.*
3. claims continuity *if and only if for each $i \in \mathcal{N}$, $u(i, \cdot)$ is continuous.*
4. homogeneity *if and only if for each $i \in \mathcal{N}$, $u(i, \cdot)$ is homogeneous.*
5. claims truncation invariance *if and only if for each $i \in N$, $u(i, \cdot)$ is constant.*

With *claims truncation invariance*, we obtain a new characterization of the constrained equal awards rule. Compared with Theorem 3, we also strengthen *minimal sharing* to *equal treatment of equals*, but no longer impose *endowment continuity.*

Proposition 3. *The constrained equal awards rule is the unique rule satisfying* restricted endowment additivity*,* consistency*,* equal treatment of equals*, and* claims truncation invariance*.*

The constrained equal awards rule is also characterized by replacing *restricted endowment additivity* in Proposition 3 with *composition up* (Dagan 1996; Flores-Szwagrzak 2015). In fact, *consistency* is redundant in this case (Dagan 1996). Thus, under *equal treatment of equals* and *claims truncation invariance*, *composition up* implies *restricted endowment additivity*, whereas *restricted endowment additivity* does not imply *composition up.*

3.3 Duality

So far, we have applied the principle of proportionality to awards. A complementary line of inquiry applies proportionality to losses. The dual of a rule applies the procedure of the original rule, but to determine the allocation of losses instead of the allocation of awards. Formally,[28] the **dual of $\boldsymbol{\varphi}$, $\dot{\boldsymbol{\varphi}}$**, is defined for each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, by $\dot{\varphi}(c, E) = c - \varphi\big(c, \sum_N c_i - E\big)$. This notion immediately extends our results to the allocation of losses.

Remark 4. By replacing each axiom in Theorems 1, 2, and 3, and Proposition 3 with their duals, characterizes of the duals of the corresponding rules.

Duality also leads to a natural additional axiom: A rule should distribute gains and losses according to the same procedure. That is, the rule should coincide with its dual.[29]

[27] A rule φ is: **claims monotonic** if an increase in an agent's claim never leads to a decrease in the agent's award; **claims continuous** if it is continuous in the vector of claims; and **homogeneous** if after scaling the claims and endowment by the same constant, the awards vector is scaled by the same constant.

[28] We depart from the standard notation for the dual of a rule, φ^d, to avoid confusion that may arise with either P^{ud} or $(P^u)^d$.

[29] Aumann and Maschler (1985) introduced this property for claims problems.

Self-duality: For each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, $\varphi(c, E) = \dot{\varphi}(c, E)$.

Together with *self-duality* and *null claims consistency*, *restricted endowment additivity* characterizes the proportional rule.[30]

Theorem 4. *The proportional rule is the unique rule satisfying either (i)* restricted endowment additivity*,* null claims consistency*, and* self-duality*; or (ii) the dual of* restricted endowment additivity*, the dual of* null claims consistency*, and* self-duality*.*

Interestingly, although *null claims consistency* is very mild and implied by *full compensation consistency*, Theorem 4 fails without it: Otherwise, any *self-dual* rule may be applied in problems in which an agent has a zero claim. A second characterization instead uses *minimal sharing*.

Proposition 4. *The proportional rule is the unique rule satisfying* restricted endowment additivity*,* null claims consistency*,* minimal sharing*, and the* dual of minimal sharing*.*

Excluding agents with zero claims, the combination of *minimal sharing* and its dual requires that the path of awards be interior: No agent's claim is filled until the endowment reaches the sum of the claims. *Restricted endowment additivity* then applies over the full range of endowments, and this implies that the path of awards is a single segment from the origin to the point representing the claims.

4 Discussion

We now know a great deal about rules satisfying diverse combinations of desirable properties, and the rules arising from our analysis can be better understood by comparison to families of rules identified by other criteria. Considering the natural description of the PW-proportional rules, their overlap with previously studied families is surprisingly small.

The intuitive description of our rules is most directly comparable to the equal sacrifice rules (Young 1987a). Recalling a previous observation, for each $u \in \mathcal{U}$, $N \subseteq \mathcal{N}$, $i, j \in N$, and $(c, E) \in \mathcal{C}^N$ such that $x \equiv P^u(c, E) \ll c$, $\frac{x_i}{u(i,c_i)} = \frac{x_j}{u(j,c_j)}$. In contrast, the equal sacrifice method would require $\frac{u(i,x_i)}{u(i,c_i)} = \frac{u(j,x_j)}{u(j,c_j)}$. These conditions are jointly satisfied only when $u(i, x_i) = x_i$ so that u is linear in its second component. Since equal sacrifice rules require the corresponding ratios hold in all problems, this singles out the proportional rule. Generalizing so that the equal sacrifice ratio only applies within "brackets" leads to a larger family including constrained versions of these rules such as the constrained equal awards and constrained equal losses rules (Chambers and Moreno-Ternero 2015).

[30] A similar result characterizes the proportional rule by *self-duality* together with either composition axiom (Young 1987a). The proportional rule is also singled out by the full strength of *endowment additivity* (Moulin 1987; Chun 1988), by applying order preservation to groups (Chambers and Thomson 2002), as well as by strategic considerations (Ju et al. 2007).

This family is still essentially disjoint from the W-proportional and PW-proportional families, the proportional and constrained equal awards rules being the only common members.

We observe greater overlap with the more general family of parametric rules characterized by *equal treatment of equals*, *consistency*, and *continuity* (Young 1987b). Each parametric rule is defined by a continuous function $f\colon \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+$ non-decreasing in its second component.[31] For each $N \subseteq \mathcal{N}$ and each $(c, E) \in \mathcal{C}^N$, the corresponding rule φ^f makes awards such that for each $i \in N$, $\varphi_i^f(c, E) = f(c_i, \lambda)$ where $\lambda \in \mathbb{R}$ is chosen so that $\sum_N \varphi_i^f(c, E) = E$. A parametric rule is a W-proportional rule if there is a continuous function $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$ such that for each $(c_0, \lambda) \in \mathbb{R}_+ \times \mathbb{R}$, $f(c_0, \lambda) = \max\{\min\{\alpha(c_0)\lambda, c_0\}, 0\}$ so that f is linear in its second component over the relevant range (i.e. $f(c_0, \lambda) = \alpha(c_0)\lambda$). Allowing discontinuous α continues to define a W-proportional rule, although no longer a parametric one; if α is allowed to depend on λ, then the rule is a parametric rule but not a W-proportional rule.

The parametric rules can be generalized to allow asymmetric treatment among agents by choosing individual-specific functions (Stovall 2014a). This larger family includes many additional W-proportional and also PW-proportional rules. Since the PW-proportional rules are *consistent*, those satisfying additional continuity properties are members of this family. The additional properties exclude some PW-proportional rules, however, which may be discontinuous with respect to the claims vector. Within the family of generalized parametric rules, *restricted endowment additivity* requires that the defining functions be essentially linear.

Next consider the family of rules distinguished by the two composition axioms together with *consistency* (Moulin 2000; Chambers 2006).[32] Among the W-proportional rules, the combination of the two composition axioms essentially implies *claims truncation invariance*[33]. From Corollary 2, P^u satisfies *claims truncation invariance* if and only if u is independent of claims. In fact, each of these rules satisfies both composition axioms as well as *homogeneity* and so is a member of the family identified by Moulin (2000). Moreover, these are precisely the "weighted constrained equal awards rules" ($WCEA$) studied by Flores-Szwagrzak (2015). The members common to these families, then, consist of the $WCEA$ rules together with the proportional rule. Flores-Szwagrzak (2015) characterizes the $WCEA$ rules by *composition up*, *consistency*, *claims truncation invariance*, and *positive awards*.[34] Since the $WCEA$ rules are W-proportional rules, comparison of the characterizing axioms shows that *composition up* implies *restricted endowment additivity* under the other axioms. The converse implication is not true, however, as the W-proportional family includes more than the $WCEA$ rules.

[31] Also required is that for each $c_0 \in \mathbb{R}_+$, $\lim_{\lambda \to -\infty} f(\cdot, \lambda) = 0$ and $\lim_{\lambda \to \infty} f(c_0, \lambda) = c_0$.

[32] The difference between the families is whether *homogeneity* is also imposed. Since a weak form of *homogeneity* follows from *restricted endowment additivity*, the difference is insubstantial for comparison with our rules.

[33] The exception is the proportional rule itself.

[34] This axiom is similar in spirit to and implies *minimal sharing*. It requires that in each problem with a positive endowment, all agents with positive claims receive positive awards. The main theorem in Flores-Szwagrzak (2015) characterizes "priority-augmented" extensions of $WCEA$ rules by the first three axioms.

	PW-prop.	W-prop.	CEA	P
rest. endow. add.	$\boxed{+}$	$\boxed{+}$	$\boxed{+}$	$\boxed{+}$
endow. add.	(P)	(P)	$-$	$+$
homogeneity	$*$	$*$	$+$	$+$
full comp. continuity	$\boxed{+}$	$\boxed{+}$	$+$	$+$
endow. continuity	$\boxed{+}$	$+$	$+$	$+$
endow. monotonicity	$+$	$+$	$+$	$+$
null claims consistency	$+$	$+$	$+$	$\boxed{+}$
full comp. consistency	$+$	$+$	$+$	$+$
consistency	$\boxed{+}$	$\boxed{+}$	$\boxed{+}$	$+$
minimal sharing	$*$	$\boxed{+}$	$+$	$+$
equal treatment	$*$	$*$	$\boxed{+}$	$+$
claims trunc. inv.	$*$	$*$	$\boxed{+}$	$-$
self-duality	$*$	$*$	$+$	$\boxed{+}$
	Thm 2	Thm 3	Prop 3	Thm 4

Table 1: Properties satisfied by PW-proportional rules. Symbols $+$, $*$, and $-$ indicate that a property is satisfied by all, some, or none of the rules in a family. Characterization results are noted in the final line with boxes indicating the characterizing axioms.

Distinguishing another subfamily of the W-proportional rules, we find the "compromise" rules recently introduced by Thomson (2015c). These rules award a weighted average of the recommendations of the proportional and constrained equal awards rules. When extended from two to many claimants by *consistency*, these rules are precisely the W-proportional rules which preserve order in the sense that agents with higher claims receive higher awards. Interestingly, another method of compromise that instead takes a weighted average of the paths of awards of the proportional and constrained equal awards rules is disjoint from the PW-proportional rules aside from the limiting rules themselves (Thomson 2015d).[35] Other families extend the "Talmud" rule.[36] The Talmud rule is not a PW-proportional rule; in fact, the overlap with each of these families consists of only the constrained equal awards rule.

Beyond claims problems, our approach also applies to sharing a surplus where even the full strength of *endowment additivity* no longer characterizes a single rule (Moulin 1987). Although we do not pursue it here, this line of inquiry will be a productive avenue for future research. To conclude, we summarize our result in Table 4.

[35] A related approach leads to an "egalitarian rule", which is also not a PW-proportional rule (Giménez-Gómez and Peris 2014).

[36] This rule is introduced by Aumann and Maschler (1985) and generalized variously by Hokari and Thomson (2003), Moreno-Ternero and Villar (2006), and Thomson (2008)

A Omitted proofs

A.1 Proof of Lemma 1

Proof. For reference in the proof, we call the condition described in the lemma the "scaling property". First suppose that φ satisfies the scaling property. Let $N \subseteq \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ be such that $\varphi(c, E) \ll c$ and let $E^1, E^2 \in \mathbb{R}_+$ be such that $E^1 + E^2 = E$. Let $\alpha \equiv \frac{E^1}{E}$ so $1 - \alpha = \frac{E^2}{E}$. Then $\alpha \in [0, 1]$, so by the scaling property, $\varphi(c, E^1) = \varphi(c, \alpha E) = \alpha\varphi(c, E)$ and $\varphi(c, E^2) = \varphi(c, (1 - \alpha)E) = (1 - \alpha)\varphi(c, E)$. Therefore,

$$\varphi(c, E^1) + \varphi(c, E^2) = \alpha\varphi(c, E) + (1 - \alpha)\varphi(c, E) = \varphi(c, E).$$

Therefore, φ satisfies *restricted endowment additivity.*

Conversely, suppose that φ satisfies *restricted endowment additivity.* Let $N \subseteq \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$ be such that $\varphi(c, E) \ll c$. First we show that $\varphi(c, \cdot)$ is continuous on $\{E' : \varphi(c, E') \ll c\}$. Let $\varepsilon \in \mathbb{R}_{++}$ and let $E^1, E^2 \in \{E' : \varphi(c, E') \ll c\}$ be such that $E^1 \leq E^2$ and $|E^1 - E^2| < \frac{\varepsilon}{|N|}$. By *restricted endowment additivity*, $\varphi(c, E^2) = \varphi(c, E^1) + \varphi(c, E^2 - E^1)$. Also, for each $i \in N$, $0 \leq \varphi_i(c, E^2 - E^1) \leq E^2 - E^1$, so

$$\left|\varphi_i(c, E^2) - \varphi_i(c, E^1)\right| = \varphi_i(c, E^2 - E^1) \leq E^2 - E^1 < \frac{\varepsilon}{|N|}.$$

Then $\|\varphi(c, E^2) - \varphi(c, E^1)\| < \sum_N \frac{\varepsilon}{|N|} = \varepsilon$. Therefore, $\varphi(c, \cdot)$ is continuous on $\{E' : \varphi(c, E') \ll c\}$.

Now we verify the scaling property. By *restricted endowment additivity*, $\varphi(c, E) = \varphi(c, \frac{E}{2}) + \varphi(c, \frac{E}{2}) = 2\varphi(c, \frac{E}{2})$. By repeated application of *restricted endowment additivity*, for each $k \in \mathbb{N}$, $\varphi(c, E) = k\varphi(c, \frac{E}{k})$. Similarly, $\varphi(c, \frac{2E}{k}) = \varphi(c, \frac{E}{k}) + \varphi(c, \frac{E}{k}) = 2\varphi(c, \frac{E}{k})$. Again by repeated application of *restricted endowment additivity*, for each $l \in \mathbb{N}$ with $l \leq k$, $\varphi(c, E) = k\varphi(c, \frac{E}{k}) = \frac{k}{l}\varphi(c, \frac{lE}{k})$. That is, for each $q \in \mathbb{Q}_+ \cap [0, 1]$, $q\varphi(c, E) = \varphi(c, qE)$. Then by continuity on $\{E' : \varphi(c, E') \ll c\}$, for each $\alpha \in [0, 1]$, $\alpha\varphi(c, E) = \varphi(c, \alpha E)$. □

A.2 Proof of Theorem 2

Proof. Each PW-proportional rule satisfies the axioms of the proposition, so we prove the converse. Let φ be a rule satisfying the axioms of the proposition. Let $N \subseteq \mathcal{N}$ with $|N| = 2$ and $c \in \mathbb{R}^N_+$. If either agent's claim is zero, then the awards are completely specified by feasibility and all rules coincide, so suppose $0 \ll c$. To calibrate the candidate PW-proportional rule, let $c_0 \equiv \frac{1}{2}\min_N c_i > 0$ and $w(N, c) \equiv \frac{\varphi(c, c_0)}{c_0}$. That is, $w(N, c)$ represents the fractions of the endowment awarded to each agent at (c, c_0). By construction, $\sum_N w_i(N, c) = 1$ and $\varphi(c, c_0) \ll c$. Labeling the agents i and j, we choose $P^{\prec, u}$ so that: (i) if $0 \ll w(N, c)$, then $(i, c_i) \sim (j, c_j)$, $u(i, c_i) = w_i(N, c)$, and $u(i, c_i) = w_i(N, c)$; and (ii) if $w_i(N, c) = 0$ so $w_j(N, c) = 1$, then $(i, c_i) \prec (j, c_j)$. By construction,

$\varphi(c, c_0) = P^{\prec,u}(c, c_0)$.

We show that φ coincides with $P^{\prec,u}$. Let $\bar{E} \equiv \sup\{E' \in \mathbb{R}_+ : \varphi(c, E') \ll c\}$. By feasibility and definition of c_0, $0 < c_0 < 2c_0 \leq \bar{E}$. Let $E \in \mathbb{R}_+$ with $E \leq \sum_N c_i$ so $(c, E) \in \mathcal{C}^N$. If $E \leq c_0$, then by Lemma 1,

$$\varphi(c, E) = \varphi(c, \tfrac{E}{c_0} c_0) = \tfrac{E}{c_0} \varphi(c, c_0) = w(N, c)E = \tfrac{E}{c_0} P^{\prec,u}(c, c_0) = P^{\prec,u}(c, E).$$

If $c_0 < E < \bar{E}$, then $\varphi(c, E) \ll c$ and $P^{\prec,u}(c, E) \ll c$ so again by Lemma 1,

$$\tfrac{c_0}{E} \varphi(c, E) = \varphi(c, \tfrac{c_0}{E} E) = \varphi(c, c_0) = w(N, c)c_0 = P^{\prec,u}(c, c_0) = P^{\prec,u}(c, \tfrac{c_0}{E} E) = \tfrac{c_0}{E} P^{\prec,u}(c, E)$$

and $\varphi(c, E) = P^{\prec,u}(c, E)$. By *endowment continuity*, $\varphi(c, \cdot)$ is continuous at $\bar{E}$, so $\varphi(c, \bar{E}) = w(N, c)\bar{E} = P^{\prec,u}(c, \bar{E})$ as well. Therefore, $\varphi(c, \cdot)$ coincides with $P^{\prec,u}(c, \cdot)$ on $[0, \bar{E}]$.

Finally, suppose $\bar{E} < E$. Then $\bar{E} < \sum_N c_i$. By *endowment continuity*, there is $i \in N$ such that $\varphi_i(c, \bar{E}) = c_i = P^{\prec,u}(c, \bar{E})$. Let $j \in N \backslash \{i\}$. Then $P^{\prec,u}(c, E) = (c_i, E - c_i)$. Suppose by way of contradiction that $\varphi(c, \bar{E}) \neq (c_i, E - c_i)$. By definition of $\bar{E}$, $\varphi(c, E) \not\ll c$, so $\varphi(c, E) = (E - c_j, c_j)$. Let $\hat{E} \equiv \inf\{E' \in [\bar{E}, E] : \varphi(c, E') = (E' - c_j, c_j)\}$. Now $\bar{E} < \hat{E}$, so $\varphi(c, \hat{E}) \in \{(c_i, \hat{E} - c_i), (\hat{E} - c_j, c_j)\}$. But $(c_i, \hat{E} - c_i) \neq (\hat{E} - c_j, c_j)$, so this violates *endowment continuity*. Instead, $\varphi(c, E) = (c_i, E - c_i) = P^{\prec,u}(c, E)$. □

A.3 Proof of Theorem 1

Proof. Each collection of fixed-population PW-proportional rules satisfies *restricted endowment additivity* and *endowment continuity*. If the weights are path-consistent, then the weights are proportional in all pairs of problems meeting the hypothesis of *full compensation consistency*, so each path-consistent collection of PW-proportional rules satisfies *full compensation consistency* as well.

For the converse, let φ be a rule satisfying the axioms of the theorem. By Proposition 2, φ coincides with a fixed-population PW-proportional rule for each two-claimant problem. First we show that once an agent's claim is filled, it remains filled as the endowment increases. With this fact in hand, we show that the rule coincides with a PW-proportional rule for each fixed population and verify path consistency.

Step 1: For each $N \subseteq \mathcal{N}$, each $(c, E) \in \mathcal{C}^N$, and each $E' \in \mathbb{R}_+$ with $E' \leq E$, $\{i \in N : \varphi(c, E') = c_i\} \subseteq \{i \in N : \varphi(c, E) = c_i\}$. We proceed by induction on the number of agents. By Proposition 2, the assertion is true for all populations of size at most two. Let $N \subseteq \mathcal{N}$ and suppose that the assertion is true for all populations smaller than N: for each $\hat{N} \subseteq \mathcal{N}$ with $|\hat{N}| < |N|$, each $(\hat{c}, \hat{E}) \in \mathcal{C}^{\hat{N}}$, and each $\hat{E}' \in \mathbb{R}_+$ with $\hat{E}' \leq \hat{E}$, $\{i \in N : \varphi(\hat{c}, \hat{E}')\} \subseteq \{i \in N : \varphi(\hat{c}, \hat{E})\}$.

Let $(c, E) \in \mathcal{C}^N$ and $E' \in \mathbb{R}_+$ with $E' < E$. Let $N_0 \equiv \{i \in N : \varphi(c, E) = c_i\}$ and $N_0' \equiv \{i \in$

$N : \varphi(c, E') = c_i\}$ and suppose by way of contradiction that $N_0' \not\subseteq N_0$. Then $|N| > 2$, $N_0' \neq \emptyset$, and since $E' < E \leq \sum_N c_i$, $N_0' \neq N$. Let $\bar{E} \equiv \inf\left\{\hat{E} \in [E', E] : \{i \in N : \varphi_i(c, \hat{E}) = c_i\} \neq N_0'\right\}$ and let $\bar{N}_0 \equiv \{i \in N : \varphi(c, \bar{E}) = c_i\}$. By *endowment continuity*, $\varphi(c, \cdot)$ is continuous at $\bar{E}$, so $E' \leq \bar{E} < E$ and $\bar{N}_0 = N_0'$. Also, by *endowment continuity* and the definition of $\bar{E}$, there is $E'' \in [\bar{E}, E]$ such that $\{i \in N : \varphi(c, E'') = c_i\} \neq \bar{N}_0$ and for each $j \in N \backslash \bar{N}_0$, $\varphi_j(c, \bar{E}) - \varphi_j(c, E'') < c_j - \varphi_j(c, \bar{E})$. Let $N_0'' \equiv \{i \in N : \varphi(c, E'') = c_i\}$, $N'' \equiv N \backslash N_0''$, and $E_0'' \equiv \sum_{N_0''} c_j$. Then $N_0'' \subseteq \bar{N}_0$. By *full compensation consistency*,

$$\varphi_{N''}(c, \bar{E}) = \varphi\left(c_{N''}, \bar{E} - \sum_{N \backslash N''} \varphi_j(c, \bar{E})\right) = \varphi\left(c_{N''}, \bar{E} - \sum_{N_0''} c_j\right) = \varphi(c_{N''}, \bar{E} - E_0'') \text{ and}$$

$$\varphi_{N''}(c, E'') = \varphi\left(c_{N''}, E'' - \sum_{N \backslash N''} \varphi_j(c, E'')\right) = \varphi\left(c_{N''}, E'' - \sum_{N_0''} c_j\right) = \varphi(c_{N''}, E'' - E_0'').$$

Now $\bar{E} < E''$, so by hypothesis,

$$\{j \in N'' : \varphi_j(c_{N''}, \bar{E} - E_0'') = c_j\} \subseteq \{j \in N'' : \varphi_j(c_{N''}, E'' - E_0'') = c_j\}.$$

But $N_0'' \subseteq \bar{N}_0$ and $N_0'' \neq \bar{N}_0$, so there is $i \in \bar{N}_0 \backslash N_0''$. Then $i \in N''$ and

$$\varphi_i(c_{N''}, E'' - E_0'') = \varphi_i(c, E'') < c_i = \varphi_i(c, \bar{E}) = \varphi_i(c_{N''}, \bar{E} - E_0'').$$

That is, $i \in \{j \in N'' : \varphi_j(c_{N''}, \bar{E} - E_0'') = c_j\}$ and $i \notin \{j \in N'' : \varphi_j(c_{N''}, E'' - E_0'') = c_j\}$, which contradicts $\{j \in N'' : \varphi_j(c_{N''}, \bar{E} - E_0'') = c_j\} \subseteq \{j \in N'' : \varphi_j(c_{N''}, E'' - E_0'') = c_j\}$. Instead, $N_0' \subseteq N_0$.

Step 2: For each $N \subseteq \mathcal{N}$, φ is a fixed-population PW-proportional rule. First, agents with zero claims are fully compensated in each problem. By *full compensation consistency*, the awards of agents with positive claims are unchanged when these agents are excluded, so it suffices to consider problems in which all claims are positive. To calibrate the candidate PW-proportional rule, for each $N \subseteq \mathcal{N}$ and each $c \in \mathbb{R}^N_{++}$, let $c_0 \equiv \frac{1}{2}\min_N c_i$ and $w(N, c) \equiv \frac{\varphi(c, c_0)}{c_0}$. Then for each $i \in N$, $w_i(N, c)$ is the fraction of the endowment that φ awards to agent i at (c, c_0). By construction, $\sum_N w_i(N, c) = 1$ and $\varphi(c, c_0) \ll c$.

We argue by induction on the number of agents. By Proposition 2, the assertion is true for populations of size at most two. Now let $N \subseteq \mathcal{N}$ with $|N| > 2$ and suppose that φ coincides with a fixed-population PW-proportional rule for each population smaller than N: for each $\hat{N} \subseteq \mathcal{N}$ with $|\hat{N}| < |N|$, there are $\prec \in \Pi$ and $u \in \mathcal{U}$ such that for each $(\hat{c}, \hat{E}) \in \mathcal{C}^{\hat{N}}$, $\varphi(\hat{c}, \hat{E}) = P^{\prec, u}(\hat{c}, \hat{E})$.

Let $c \in \mathbb{R}^N_{++}$. Define $\bar{E} \equiv \inf\{E \in \mathbb{R}_+ : \exists i \in N, \varphi_i(c, E) = c_i\}$ and $\bar{N} \equiv \{i \in N : \varphi_i(c, \bar{E}) = c_i\}$. If $\bar{N} = N$, then $\bar{E} = \sum_N c_i$ and for each $i \in N$, $w_i(N, c) = \frac{c_i}{\sum_N c_i}$ and φ coincides with the proportional rule: for each $E \in \mathbb{R}_+$ with $E \leq \sum_N c_i$, $\varphi(c, E) = P(c, E)$. Suppose instead that $\bar{N} \neq N$ and let $E_0 \equiv \sum_{\bar{N}} c_i$.

By Lemma 1, for each $E \in \mathbb{R}_+$ with $E < c_0$, $\varphi(c, E) = \varphi(c, \frac{E}{c_0} c_0) = \frac{E}{c_0}\varphi(c, c_0) = w(N, c)E$. Similarly, for each $E \in \mathbb{R}_+$ with $c_0 < E < \bar{E}$, $\varphi(c, E) \ll c$, so again by Lemma 1, $\frac{c_0}{E}\varphi(c, E) = \varphi(c, \frac{c_0}{E}E) = \varphi(c, c_0) = w(N, c)c_0$. Also, by *endowment continuity*, $\varphi(c, \bar{E}) = w(N, c)\bar{E}$. Altogether, for each $E \in \mathbb{R}_+$ with $E \leq \bar{E}$, $\varphi(c, E) = w(N, c)E$.

By hypothesis, there are $\prec \in \Pi$ and $u \in \mathcal{U}$ such that φ coincides with $P^{\prec,u}$ for each $(\hat{c}, \hat{E}) \in \mathcal{C}^{N \backslash \bar{N}}$. Re-scaling if necessary, we may suppose that $\sum_{N\backslash\bar{N}} u(i, c_i) = \sum_{N\backslash\bar{N}} w_i(N, c)$. We now extend $P^{\prec,u}|_{N\backslash\bar{N}}$ from $N\backslash\bar{N}$ to N. For each $i \in \bar{N}$, let $u(i, c_i) \equiv w_i(N, c)$ and let $\prec$ be such that for each $j \in N$, if $w_j(N, c) > 0$, $(i, c_i) \sim (j, c_j)$ and if $w_j(N, c) = 0$, then $(i, c_i) \prec (j, c_j)$. Now let $E \in \mathbb{R}_+$ with $E \leq \sum_N c_i$. By construction, if $E \leq \bar{E}$, then $\varphi(c, E) = P^{\prec,u}(c, E)$. Suppose instead that $\bar{E} < E$. By Step 1, $\varphi_{\bar{N}}(c, E) = c_{\bar{N}} = P_{\bar{N}}(c, E)$. It remains to consider $N\backslash\bar{N}$.

By *full compensation consistency*, $\varphi_{N\backslash\bar{N}}(c, E) = \varphi(c_{N\backslash\bar{N}}, E - E_0) = P^{\prec,u}(c_{N\backslash\bar{N}}, E - E_0)$. Now $\varphi(c_{N\backslash\bar{N}}, \bar{E} - E_0\big) \ll c_{N\backslash\bar{N}}$ and $\varphi(c_{N\backslash\bar{N}}, \bar{E} - E_0) = \varphi_{N\backslash\bar{N}}(c, \bar{E})$, so

$$w(N\backslash\bar{N}, c_{N\backslash\bar{N}})(\bar{E} - E_0) = \varphi(c_{N\backslash\bar{N}}, \bar{E} - E_0) = \varphi_{N\backslash\bar{N}}(c, \bar{E}) = w_{N\backslash\bar{N}}(N, c)\bar{E}.$$

Since $E_0 = \sum_{\bar{N}} \varphi_i(c, \bar{E}) = \sum_{\bar{N}} w_i(N, c)\bar{E}$ and $\sum_N w_i(N, c) = \sum_{N\backslash\{\bar{N}\}} w_i(N\backslash\bar{N}, c_{N\backslash\bar{N}})$, this implies that for each pair $i, j \in N\backslash\bar{N}$,

$$\frac{w_i(N\backslash\bar{N}, c_{N\backslash\bar{N}})}{w_i(N, c)} = \frac{\bar{E} - E_0}{\bar{E}} = \frac{w_j(N\backslash\bar{N}, c_{N\backslash\bar{N}})}{w_j(N, c)}.$$

Furthermore, by the definition of $P^{\prec,u}$,

$$\frac{u(i, c_i)}{u(j, c_j)} = \frac{\varphi_i(c_{N\backslash\bar{N}}, \bar{E} - E_0)}{\varphi_j(c_{N\backslash\bar{N}}, \bar{E} - E_0)} = \frac{w_i(N\backslash\bar{N}, c_{N\backslash\bar{N}})}{w_j(N\backslash\bar{N}, c_{N\backslash\bar{N}})}.$$

By our normalization, $\sum_{N\backslash\bar{N}} u(i, c_i) = \sum_{N\backslash\bar{N}} w_i(N, c)$, so in fact for each $i \in N\backslash\bar{N}$, $u(i, c_i) = w_i(N, c)$. Therefore, for each $E \in \mathbb{R}_+$ with $E \leq \sum_N c_i$, $\varphi(c, E) = P^{\prec,u}(c, E)$.

Step 3: φ is a path-consistent collection of PW-proportional rules. By Step 2, φ is a collection of fixed-population PW-proportional rules. To see that the collection is path-consistent, let $N \subseteq \mathcal{N}$, $(c, E) \in \mathcal{C}^N$, and $i \in N$. Suppose that $\varphi_i(c, E) = c_i$ and let $P^{\prec,u}$ and $P^{\hat{\prec},\hat{u}}$ be the components of φ associated with N and $N\backslash\{i\}$ respectively. Then by *full compensation consistency*,

$$P^{\prec,u}_{-i}(c, E) = \varphi_{-i}(c, E) = \varphi\big(c_{-i}, E - \varphi_i(c, E)\big) = P^{\hat{\prec},\hat{u}}\big(c_{-i}, E - \varphi_i(c, E)\big).$$

Let $j, k \in N\backslash\{i\}$. Then $P^{\prec,u}_j(c, E) = 0$ if and only if $P^{\hat{\prec},\hat{u}}_j\big(c_{-i}, E - \varphi_i(c, E)\big) = 0$ and $P^{\prec,u}_k(c, E) = 0$ if and only if $P^{\hat{\prec},\hat{u}}_k\big(c_{-i}, E - \varphi_i(c, E)\big) = 0$, so $(j, c_j) \prec (k, c_k)$ if and only if $(j, c_j)\hat{\prec}(k, c_k)$. Also, if

$P_j^{\prec,u}(c,E) > 0$, then

$$\frac{u(k,c_k)}{u(j,c_j)} = \frac{P_k^{\prec,u}(c,E)}{P_j^{\prec,u}(c,E)} = \frac{P_k^{\hat{\prec},\hat{u}}\big(c_{-i},E-\varphi_i(c,E)\big)}{P_j^{\hat{\prec},\hat{u}}\big(c_{-i},E-\varphi_i(c,E)\big)} = \frac{\hat{u}(k,c_k)}{\hat{u}(j,c_j)}.$$

Therefore, u and $\hat{u}$ are related by a re-scaling on $(N\backslash\{i\}, c_{-i})$. Since this is true for each population and each problem, the collection of fixed-population PW-proportional rules is path-consistent. □

A.4 Proof of Theorem 2

Proof. We have seen that each PW-proportional rule satisfies the axioms, so let φ be a rule satisfying the axioms of the theorem. By Theorem 1, φ is a path-consistent collection of fixed-population PW-proportional rules. Let $N \subseteq \mathcal{N}$, $N' \subseteq N$. Let $P^{\prec,u}$ and $P^{\prec',u'}$ be the fixed-population PW-proportional rules which coincide with φ on N and N' respectively.

Step 1: $\prec|_{N'} = \prec'|_{N'}$. Let $c \in \mathbb{R}_+^N$ and $j,k \in N'$. First suppose that $(j,c_j) \prec^N (k,c_k)$. Then there is $E \in \mathbb{R}_+$ with $E \leq \sum_N c_i$ such that $\varphi_j(c,E) > 0 = \varphi_k(c,E)$. By repeated application of *consistency*, $\varphi_j\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) = \varphi_j(c,E) > 0$ and $\varphi_k\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) = \varphi_k(c,E) = 0$. Therefore, $(j,c_j) \prec^{N'} (k,c_k)$. If instead $(j,c_j) \sim^N (k,c_k)$. Then there is $E \in \mathbb{R}_+$ with $E \leq \sum_N c_i$ such that $0 < \varphi_j(c,E) < c_j$ and $0 < \varphi_k(c,E) < c_k$. By repeated application of *consistency*, $\varphi_j\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) = \varphi_j(c,E)$ and $\varphi_k\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) = \varphi_k(c,E)$, so $0 < \varphi_j\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) < c_j$ and $0 < \varphi_k\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) < c_k$. Therefore, $(j,c_j) \sim^{N'} (k,c_k)$. Altogether, $\prec'|_{N'} = \prec|_{N'}$.

Step 2: $u|_{N'}$ and $u'|_{N'}$ are proportional. Let $c \in \mathbb{R}_+^N$ and $j,k \in N'$ with $(j,c_j) \sim^N (k,c_k)$. By Step 1, $(j,c_j) \sim^{N'} (k,c_k)$ as well. There is $E \in \mathbb{R}_+$ with $E \leq \sum_N c_i$ such that $0 < \varphi_j(c,E) < c_j$ and $0 < \varphi_k(c,E) < c_k$. Then $\frac{\varphi_j(c,E)}{\varphi_k(c,E)} = \frac{u(j,c_j)}{u(k,c_k)}$. By repeated application of *consistency*, $\varphi_j\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) = \varphi_j(c,E)$ and $\varphi_k\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) = \varphi_k(c,E)$, so $0 < \varphi_j\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) < c_j$ and $0 < \varphi_k\big(c_{N'}, E - \sum_{N\backslash N'} c_i\big) < c_k$. Then $\frac{\varphi_j\left(c_{N'}, E - \sum_{N\backslash N'} c_i\right)}{\varphi_j\left(c_{N'}, E - \sum_{N\backslash N'} c_i\right)} = \frac{u'(j,c_j)}{u'(k,c_k)}$. Combining results, $\frac{u(j,c_j)}{u(k,c_k)} = \frac{u'(j,c_j)}{u'(k,c_k)}$ and so $u|_{N'}$ and $u'|_{N'}$ are proportional.

By Steps 1 and 2, $\prec^{N'}$ and u' may be replaced by $\prec^N$ and u without changing any awards. Moreover, this is true for each pair $N, N' \subseteq \mathcal{N}$ with $N' \subseteq N$. In general, let $N, N' \subseteq \mathcal{N}$ and consider $N'' \equiv N \cup N'$. Then by Steps 1 and 2, replacing $\prec^{N'}$ and $\prec^{N''}$ with $\prec^N$ and replacing u' and u'' with u does not change any awards. Continuing in this fashion, we conclude that there are $\prec \in \Pi$ and $u \in \mathcal{U}$ such that for each $N \subseteq \mathcal{N}$, $\prec^N|_N = \prec|_N$ and $u^N|_N$ is proportional to $u|_N$. Thus, $\varphi = P^{\prec,u}$. □

A.5 Proof of Theorem 3

Proof. We have seen that each W-proportional rule satisfies the axioms of the theorem, so let φ be a rule satisfying these axioms. By Theorem 2, φ is a PW-proportional rule: there are $\prec \in \Pi$ and $u \in \mathcal{U}$ such that $\varphi = P^{\prec,u}$. To show that φ is a W-proportional rule, let $N \subseteq \mathcal{N}$ and $(c, E) \in \mathcal{C}^N$. Let $i, j \in N$ and suppose that $c_i > 0$, $c_j > 0$, and $E > 0$. Since $E > 0$, there is $k \in N$ such that $\varphi_k(c, E) > 0$. Then by *minimal sharing*, $\varphi_i(c, E) > 0$ and $\varphi_j(c, E) > 0$. Therefore, $(i, c_i) \sim (j, c_j)$. Since this is true for each pair of agents with positive claims and in each problem, $\prec$ is the complete indifference relation and $\varphi = P^{\prec,u} = P^u$. □

A.6 Proof of Proposition 3

Proof. The constrained equal awards rule satisfies the axioms, so let φ be a rule satisfying the axioms of the proposition.

Step 1: Coincidence for small endowments. Let $N \subseteq \mathcal{N}$ and $c \in \mathbb{R}^N_+$. By *consistency*, the presence of agents with zero claims has no bearing on the awards of the remaining agents, so we may suppose $c \in \mathbb{R}^N_{++}$. Let $c_0 \equiv \frac{1}{2} \min_N c_i$. By *claims truncation invariance* and *equal treatment of equals*,

$$\varphi(c, c_0) = \varphi\big((c_0, \ldots, c_0), c_0\big) = \big(\tfrac{c_0}{|N|}, \ldots, \tfrac{c_0}{|N|}\big) = CEA(c, c_0).$$

Let $E \in \mathbb{R}_+$ with $E \leq \sum_N c_i$. First suppose that $\varphi(c, E) \ll c$. If $E \leq c_0$, then by Lemma 1, $\frac{E}{c_0}\,\varphi(c, c_0) = \varphi\big(c, \frac{E}{c_0}\, c_0\big) = \varphi(c, E)$. If $c_0 < E$, then by Lemma 1, $\frac{c_0}{E}\,\varphi(c, E) = \varphi\big(c, \frac{c_0}{E}\, E\big) = \varphi(c, c_0)$. Therefore, for each $E \in \mathbb{R}_+$ such that $\varphi(c, E) \ll c$, $\varphi(c, E) = \frac{E}{c_0}\,\varphi(c, c_0) = \frac{E}{c_0}\, CEA(c, c_0) = CEA(c, E)$. In particular, this is true for each $E \in \mathbb{R}_+$ such that $E < |N| \min_N c_i$.

Step 2: Coincidence for two-claimant problems. Let $N \subseteq \mathcal{N}$ with $|N| = 2$ and $c \in \mathbb{R}^N_+$ and label the agents so that $c_1 \leq c_2$. By Step 1, for each $E \in \mathbb{R}_+$ such that $\varphi(c, E) \ll c$, $\varphi(c, E) = CEA(c, E)$. Suppose by way of contradiction that there is $E \in \mathbb{R}_+$ with $E \leq c_1 + c_2$ such that $\varphi(c, E) \neq CEA(c, E)$. Then $\varphi(c, E) \not\ll c$ and $E \geq 2c_1$. If $c_1 = c_2$, then $E = 2c_1$ and by feasibility, $\varphi(c, E) = (c_1, c_1) = CEA(c, E)$. Instead, $c_1 < c_2$. Then $\varphi(c, E) \neq CEA(c, E) = (c_1, E - c_1)$, so $\varphi(c, E) = (E - c_2, c_2)$. Since $E - c_2 < c_1$, there is $k \in \mathbb{N}$ such that $\frac{E}{k} < c_1 + c_2 - E$. Let $N' \subseteq \mathcal{N}$ with $|N'| = k$ and for each $i \in N'$, let $c'_i = c_1$. By *consistency* and *equal treatment of equals*,

$$\varphi\big((c_1, c_2, c'_{-12}), c_2 + (k+1)(E - c_2)\big) = (E - c_2, c_2, E - c_2, \cdots, E - c_2).$$

By choice of k, $E < k(c_1 + c_2 - E)$, so $c_2 + (k+1)(E - c_2) < kc_1$. But then by Step 1,

$$\varphi\big((c_1, c_2, c'_{-12}), c_2 + (k+1)(E - c_2)\big) = CEA\big((c_1, c_2, c'_{-12}), c_2 + (k+1)(E - c_2)\big) = \big(\tfrac{E}{k+2}, \ldots, \tfrac{E}{k+2}\big),$$

a contradiction. Instead, $\varphi(c, E) = (c_1, E - c_1) = CEA(c, E)$. Therefore, φ coincides with CEA on the domain of two-claimant problems. Since CEA is the unique *consistent* rule with this property, $\varphi = CEA$. □

A.7 Proof of Theorem 4

Proof. We prove statement (i); since the properties in the statements are dual, (ii) follows immediately by duality. The proportional rule satisfies the properties of statement (i), so let φ be a rule satisfying these properties. By *self-duality*, $\dot{\varphi}$ satisfies *restricted endowment additivity* as well. Let $N \subseteq \mathcal{N}$ and $c \in \mathbb{R}^N_+$. By *null claims consistency*, the presence of agents with zero claims has no bearing on the awards of the remaining agents, so we may suppose $c \in \mathbb{R}^N_{++}$. By *self-duality*, $\varphi\big(c, \frac{1}{2}\sum_N c_i\big) = \frac{c}{2}$. Then by Lemma 1, for each $\alpha \in [0,1]$,

$$\varphi\left(c, \frac{\alpha}{2}\sum_N c_i\right) = \alpha\varphi\left(c, \frac{1}{2}\sum_N c_i\right) = \frac{\alpha c}{2} \text{ and}$$
$$\dot{\varphi}\left(c, \frac{\alpha}{2}\sum_N c_i\right) = \alpha\dot{\varphi}\left(c, \frac{1}{2}\sum_N c_i\right) = \frac{\alpha c}{2}.$$

Then furthermore, for each $\alpha \in [0, \frac{1}{2}]$,

$$\varphi\left(c, \alpha\sum_N c_i\right) = \alpha c,$$
$$\dot{\varphi}\left(c, \alpha\sum_N c_i\right) = \alpha c, \text{ and}$$
$$\varphi\left(c, (1-\alpha)\sum_N c_i\right) = c - \dot{\varphi}\left(c, \alpha\sum_N c_i\right) = c - \alpha c = (1-\alpha)c.$$

Combining results, for each $\alpha \in [0,1]$, $\varphi\big(c, \alpha\sum_N c_i\big) = \alpha\varphi\big(c, \sum_N c_i\big) = \alpha c$. That is, φ is the proportional rule. □

References

Alcalde, J., del Carmen Marco-Gil, M., and Silva, J. A. (2008). The minimal overlap rule revisited. *Social Choice and Welfare*, 31:109–128.

Aumann, R. J. and Maschler, M. (1985). Game theoretic analysis of a bankruptcy problem from the Talmud. *Journal of Economic Theory*, 36:195–213.

Bergantiños, G. and Méndez-Naya, L. (2001). Additivity in bankruptcy problems and in allocation problems. *Spanish Economic Review*, 3(3):223–229.

Bergantiños, G. and Vidal-Puga, J. J. (2004). Additive rules in bankruptcy problems and other related problems. *Mathematical Social Sciences*, 47:87–101.

Bergantiños, G. and Vidal-Puga, J. J. (2006). Additive rules in discrete allocation problems. *European Journal of Operational Research*, 172:971–978.

Chambers, C. (2006). Asymmetric rules for claims problems without homogeneity. *Games and Economic Behavior*, 54(5):241–260.

Chambers, C. and Moreno-Ternero, J. (2015). Taxation and poverty. *Social Choice and Welfare*, pages 1–23.

Chambers, C. and Thomson, W. (2002). Group order preservation and the proportional rule for the adjudication of conflicting claims. *Mathematical Social Sciences*, 44:235–252.

Chun, Y. (1988). The proportional solution for rights problems. *Mathematical Social Sciences*, 15:231–246.

Chun, Y. and Thomson, W. (2005). Convergence under replication of rules to adjudicate conflicting claims. *Games and Economic Behavior*, 50(2):129–142.

Curiel, I., Maschler, M., and Tijs, S. (1987). Bankruptcy games. *Zeitschrift für Operations Research*, 31(5):A143–A159.

Dagan, N. (1996). New characterizations of old bankruptcy rules. *Social Choice and Welfare*, 13:51–59.

Flores-Szwagrzak, K. (2015). Priority classes and weighted constrained equal awards rules for the claims problem. *Journal of Economic Theory*, 160:36–55.

Giménez-Gómez, J.-M. and Peris, J. E. (2014). A proportional approach to claims problems with a guaranteed minimum. *European Journal of Operational Research*, 232:109–116.

Hokari, T. and Thomson, W. (2003). Claims problems and weighted generalizations of the Talmud rule. *Economic Theory*, 21:241–261.

Hougaard, J. L., Moreno-Ternero, J., and Østerdal, L. P. (2012). A unifying framework for the problem of adjudicating conflicting claims. *Journal of Mathematical Econoimcs*, 48:107–114.

Hougaard, J. L., Moreno-Ternero, J., and Østerdal, L. P. (2013). Rationing in the presence of baselines. *Social Choice and Welfare*, 40:1047–1066.

Ju, B.-G., Miyagawa, E., and Sakai, T. (2007). Non-manipulable division rules in claim problems and generalizations. *Journal of Economic Theory*, 232:109–116.

Ju, B.-G. and Moreno-Ternero, J. (2014). Fair allocation of disputed properties. CORE discussion paper.

Kıbrıs, Ö. (2012). A revealed preference analysis of solutions to simple allocation problems. *Theory and Decision*, 72:509–523.

Kıbrıs, Ö. (2013). On recursive solutions to simple allocation problems. *Theory and Decision*, 75:449–463.

Marchant, T. (2008). Scale invariance and similar invariance conditions for bankruptcy problems. *Social Choice and Welfare*, 31(5):693–707.

Moreno-Ternero, J. and Roemer, J. (2006). Impartiality, priority, and solidarity in the theory of justice. *Econometrica*, 74(5):1419–1427.

Moreno-Ternero, J. and Villar, A. (2006). The TAL-family of rules for bankruptcy problems. *Social Choice and Welfare*, 27:231–249.

Moulin, H. (1987). Equal or proportional division of a surplus and other methods. *International Journal of Game Theory*, 16(5):161–186.

Moulin, H. (2000). Priority rules and other asymmetric rationing methods. *Econometrica*, 68:643–684.

Moulin, H. (2002). The proportional random allocation of indivisible units. *Social Choice and Welfare*, 19:381–413.

Moulin, H. (2013). Cost sharing in networks: some open questions. *International Game Theory Reivew*, 15:134–144.

O'Neill, B. (1982). A problem of rights arbitration from the Talmud. *Mathematical Social Sciences*, 2:345–371.

Shapley, L. (1953). A value for n-person games. *Annals of Mathematical Studies*, 28:307–318.

Stovall, J. E. (2014a). Asymmetric parametric division rules. *Games and Economic Behavior*, 84:87–110.

Stovall, J. E. (2014b). Collective rationality and monotone path division rules. *Journal of Economic Theory*, 154:1–24.

Thomson, W. (2003). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: a survey. *Mathematical Social Sciences*, 45:249–297.

Thomson, W. (2008). Two families of rules for the adjudication of conflicting claims. *Social Choice and Welfare*, 31:667–692.

Thomson, W. (2012). Lorenz rankings of rules for the adjudication of conflicting claims. *Economic Theory*, 50:547–569.

Thomson, W. (2014). How do divide when there isn't enough: From the Talmud to game theory. book manuscript, University of Rochester.

Thomson, W. (2015a). Axiomatic and game-theoretic analysis of bankruptcy and taxation problems: an update. *Mathematical Social Sciences*, 74:41–59.

Thomson, W. (2015b). Consistent allocation rules. book manuscript, University of Rochester.

Thomson, W. (2015c). A consistent compromise between the constrained equal awards and proportional rules. *Economic Theory*, 60(3):495–520.

Thomson, W. (2015d). For claims problems, another compromise between the proportional and constrained equal awards rules. *Journal of Dynamics and games*, 2(3&4):363–382.

Young, P. (1987a). Distributive justice in taxation. *Journal of Economic Theory*, 43:321–335.

Young, P. (1987b). On dividing an amount according to individual claims or liabilities. *Mathematics of Operations Research*, 12:398–414.