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Stable matchings of teachers to schools ☆

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Several countries successfully use centralized matching schemes for school or higher education assignment, or for entry-level labour markets. In this paper we explore the computational aspects of a possible similar scheme for assigning teachers to schools. Our model is motivated by a particular characteristic of the education system in many countries where each teacher specializes in two subjects. We seek stable matchings, which ensure that no teacher and school have the incentive to deviate from their assignments. Indeed we propose two stability definitions depending on the precise format of schools' preferences. If the schools' ranking of applicants is independent of their subjects of specialism, we show that the problem of deciding whether a stable matching exists is NP-complete, even if there are only three subjects, unless there are master lists of applicants or of schools. By contrast, if the schools may order applicants differently in each of their specialization subjects, the problem of deciding whether a stable matching exists is NP-complete even in the presence of subject-specific master lists plus a master list of schools. Finally, we prove a strong inapproximability result for the problem of finding a matching with the minimum number of blocking pairs with respect to both stability definitions.

1. Introduction

In the organization of education, several countries or regions use various centralized schemes to allocate children to public schools (e.g., in Boston and New York [1,2]), students to universities (e.g., in Hungary [6]), and intending junior doctors to training positions in hospitals (e.g., in the USA [21]), etc. These schemes are usually not dictatorial in the sense that they take into account the wishes of both sides of the market: students may express their preferences over the universities they wish to attend, and the universities may order their applicants based on some kind of evaluation. After analyzing several successful and unsuccessful schemes Roth [17,18] convincingly argued that a crucial property for success is so-called *stability*, introduced in the seminal paper by Gale and Shapley [10]. Stability means that no unmatched student–school pair

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should simultaneously prefer each other to their current assignee(s) (if any). In many real markets, each instance not only admits a stable matching, but it is also possible to find such a matching efficiently.

However, sometimes there are circumstances leading to additional structural requirements. For example, married couples may wish to be allocated to the same hospital or at least to hospitals that are geographically close [15,7], or schools may wish to have the right to close a study programme if the number of applicants does not meet a certain lower quota [6]. In such cases, a suitable notion of stability has to be defined that really mirrors the intentions of the participants and motivates them to obey the recommended assignment. Alas, a stable matching is not necessarily bound to exist; and even worse, it is often a computationally difficult problem to decide whether in the given situation one does exist [16].

The topic of this paper is motivated by the problems arising in the labour market for teachers. Traditionally, a teacher for the upper elementary or lower secondary level of education in Slovakia and the Czech republic (and in fact in many other countries and regions, such as Germany [4] and Flanders [9]) specializes in two curricular domains (from now on called *subjects*), e.g., Mathematics and Physics, Chemistry and Biology, or Slovak language and English etc. When a school is looking for new teachers, it may have a limited number of lessons to cover (or teaching hours to fill) in each subject. Thus we suppose that each school has different capacity for each subject and that it will be willing to employ a set of teachers in such a way that these capacities will not be exceeded. Cechlárová et al. [8] studied a variant of this problem where the trainee teachers could only express which schools are acceptable for them, and which are not, without ordering them according to their preferences, and the schools had no input. In these settings, the aim was to assign as many trainee teachers as possible (ideally all of them) by respecting the schools' capacities.

The aim of this paper is to study algorithmic aspects of the problem of assigning teachers to schools within the framework of two-sided preferences. We suppose that teachers rank in order of preference their acceptable schools according to their own criteria, and vice versa, schools rank-order their applicants similarly [14]. In this context we suggest two stability definitions and study the computational complexity of problems concerned with finding stable matchings (or reporting that none exist). These definitions and the associated complexity results depend on the nature of the schools' preference lists.

The main results and the organization of the rest of the paper are as follows. In Section 2 we introduce relevant technical concepts and illustrate them by means of simple examples. In Section 3 we deal with the case when each school has a linear ordering on the set of teachers who apply for a position. We show that in this general case the problem of deciding whether a stable matching exists is NP-complete, even if there are only three subjects in total. This result is perhaps not unexpected, since the problem studied in this paper bears some resemblance to the Hospitals/Residents problem with Couples (HRC), and the problem of deciding whether a given instance of HRC admits a stable matching is NP-complete [16].

By contrast, we show in Section 4 that if either the preferences of schools are derived from a common master list of teachers, or vice versa if the preferences of teachers are derived from a common master list of schools, a unique stable matching exists and it can be found using straightforward extensions of the classical Serial Dictatorship mechanism [19]. Moreover, the problems with master lists are efficiently solvable without any restrictions on the number of subjects. In Section 5 we modify the stability definition to enable the schools to order the teachers differently according to their two specialization subjects. We show that in this case, the problem of deciding whether a stable matching exists is NP-complete even if there are only three subjects, there are master lists for each subject and there is also a master list of schools. Finally, problems involving finding matchings with the minimum number of blocking pairs are discussed in Section 6, where we show that, with respect to both stability definitions, the problem of finding a matching with the minimum number of blocking pairs is very difficult to approximate.

2. Preliminary definitions and observations

An instance of the Teachers Assignment Problem, TAP for short, involves a set A of applicants (teachers), a set S of schools and a set P of subjects. For ease of exposition, elements of the set P will sometimes be referred to by letters like M, F or I to remind the reader of real subjects taught at schools, such as Mathematics, Physics, or Informatics, etc.

Each applicant $a \in A$ is characterized by a pair of distinct subjects $\mathbf{p}(a) \subseteq P$, where $\mathbf{p}(a) = \{p_1(a), p_2(a)\}$, that define her *type*. Sometimes we shall also say that a particular applicant is of type FM, IM, or FI, etc. Corresponding to each applicant $a \in A$ there is a set $S(a) \subseteq S$ of schools that a finds *acceptable*. Moreover applicant a ranks $S(a)$ in strict order of preference.

Each school $s \in S$ has a certain capacity for each subject: the vector of capacities will be denoted by $\mathbf{c}(s) = (c_1(s), \ldots, c_k(s)) \in \mathbb{N}^k$, where $k = |P|$, and an entry of $\mathbf{c}(s)$ will be called the *partial capacity* of school s. Here, $c_i(s)$ is the maximum number of applicants, whose specialization involves subject p_i, that school s is able to take. Further, each school ranks its applicants in strict order of preference.

Let $S(A) = \{(a, s) : a \in A \wedge s \in S(a)\}$ denote the set of *acceptable* applicant–school pairs. An *assignment* $\mathcal{M}$ is a subset of $S(A)$ such that each applicant $a \in A$ is a member of at most one pair in $\mathcal{M}$. We shall write $\mathcal{M}(a) = s$ if $(a, s) \in \mathcal{M}$ and say that applicant a *is assigned to* school s, and write $\mathcal{M}(a) = \emptyset$ if there is no $s \in S$ with $(a, s) \in \mathcal{M}$. The set of applicants assigned to a school s will be denoted by $\mathcal{M}(s) = \{a \in A : (a, s) \in \mathcal{M}\}$. We shall also denote by $\mathcal{M}_p(s)$ the set of applicants assigned to s whose specialization includes subject p and by $\mathcal{M}_{p,r}(s)$ the set of applicants assigned to s whose specialization is exactly the pair $\{p, r\}$. More precisely,

$$\mathcal{M}_p(s) = \{a \in A : (a, s) \in \mathcal{M} \ \wedge \ p \in \mathbf{p}(a)\}$$

applicant	type	preferences	school	capacities F	I	M	preferences
a_1	FM	s_1, s_3	s_1	2	1	1	a_3, a_4, a_1, a_2
a_2	FM	s_1, s_3	s_2	1	1	1	a_4, a_3
a_3	IM	s_1, s_2	s_3	2	1	1	a_4, a_1, a_2
a_4	FI	s_3, s_2, s_1					

Fig. 1. Instance J_1 of TAP.

applicant	type	preferences	school	capacities F	I	M	preferences
a_1	FM	s_2, s_1	s_1	1	1	2	a_1, a_3, a_2
a_2	IM	s_1, s_2	s_2	1	1	1	a_2, a_1
a_3	FI	s_1					

Fig. 2. An instance J_2 of TAP with no stable matching.

and

$$\mathcal{M}_{p,r}(s) = \{a \in A : (a, s) \in \mathcal{M} \ \wedge \ \{p, r\} = \mathbf{p}(a)\}.$$

An assignment $\mathcal{M}$ is a *matching* if $|\mathcal{M}_p(s)| \le c_p(s)$ for each school s and each subject p. We say that an applicant a is *assigned* in $\mathcal{M}$ if $\mathcal{M}(a) \neq \emptyset$, otherwise she is *unassigned*. A school s is *full* in a matching $\mathcal{M}$ if it can admit no other applicant (irrespective of her specialization) and s is *undersubscribed* in subject p if $|\mathcal{M}_p(s)| < c_p(s)$.

Definition 1. Let $\mathcal{M}$ be a matching. We say that a pair (a, s) with $\mathbf{p}(a) = \{p_1, p_2\}$ and $s \in S(a)$ is *blocking* if either a is not assigned in $\mathcal{M}$ or a prefers s to $\mathcal{M}(a)$, and one of the following conditions holds:

(i) s is undersubscribed in both p_1 and p_2;
(ii) s is undersubscribed in p_i and it prefers a to one applicant in $\mathcal{M}_{p_{3-i}}(s)$ for some $i \in \{1, 2\}$;
(iii) s prefers a to one applicant in $\mathcal{M}_{p_1,p_2}(s)$;
(iv) s prefers a to two different applicants a_1, a_2 such that $a_1 \in \mathcal{M}_{p_1}(s)$ and $a_2 \in \mathcal{M}_{p_2}(s)$.

A matching is *stable* if it admits no blocking pair.

Example 1. Let J_1 be the instance of TAP with the set of subjects $P = \{F, I, M\}$ given in Fig. 1.

Consider the matching $\mathcal{M} = \{(a_1, s_3), (a_2, s_1), (a_3, s_2), (a_4, s_1)\}$. It is easy to see that $\mathcal{M}$ is not stable. Each of the conditions (i)–(iv) of Definition 1 is violated by the following blocking pairs, respectively:

(i) (a_4, s_3) is a blocking pair since a_4 prefers school s_3 to $\mathcal{M}(a_4) = s_1$, and s_3 is undersubscribed in both I and F;
(ii) (a_4, s_2) is a blocking pair since a_4 prefers s_2 to $\mathcal{M}(a_4) = s_1$, school s_2 is undersubscribed in F and it prefers a_4 to $a_3 \in \mathcal{M}_I(s_2)$;
(iii) (a_1, s_1) is a blocking pair since a_1 prefers s_1 to $\mathcal{M}(a_1) = s_3$, and school s_1 prefers a_1 to $a_2 \in \mathcal{M}(s_1)$ who is of the same type as a_1;
(iv) (a_3, s_1) is a blocking pair since a_3 prefers s_1 to $\mathcal{M}(a_3) = s_2$, and school s_1 prefers a_3 to both its assignees a_2 and a_4. □

Note that Definition 1 (iv), as also illustrated in Example 1, gives rise to the possibility that a school could drop two applicants and accept just one in order to satisfy a blocking pair. This is also a possibility in HRC that a single resident r can displace a couple c assigned to a hospital h if h prefers r to just one member of c [7].

We also remark that TAP bears a superficial resemblance to the variant of the Hospitals/Residents problem that modelled the problem of assigning junior doctors to hospitals in Scotland in years 2000–2005, where intending junior doctors sought not one position at hospitals, but two, namely a medical post and a surgical post [12]. They also typically had preferences over the half-years in which they would carry out each type of post, so the stability definition was different to the one given in Definition 1.

We next present two examples to show that a TAP instance need not admit a stable matching, and in such instances that do, stable matchings may have different sizes.

Example 2. Consider the instance J_2 of TAP given in Fig. 2. We show that J_2 admits no stable matching.

If a_1 is not assigned then (a_1, s_1) is a blocking pair as a_1 is the most preferred applicant for school s_1. If $\mathcal{M}(a_1) = s_2$ then assigning a_2 to s_1 leads to the blocking pair (a_3, s_1) and assigning a_3 to s_1 produces blocking pair (a_2, s_2). By contrast, if $\mathcal{M}(a_1) = s_1$ then a_2 must also be assigned to s_1 (this school is her first choice and it has enough room to accept her), which makes the pair (a_1, s_2) blocking. Hence no stable matching exists. □

applicant	type	preferences	school	capacities F	I	M	preferences
a_1	FM	s_2, s_1	s_1	1	1	2	a_1, a_2, a_3
a_2	FI	s_1, s_2	s_2	1	1	1	a_2, a_1
a_3	IM	s_1					

Fig. 3. An instance J_3 of TAP with stable matchings of different sizes.

applicant	type	preferences	school	capacities F	I	M	preferences
a_i^1	FI	s_i^1, s_i^3	s_i^1	1	1	2	a_i^4, a_i^1, a_i^3
a_i^2	FI	s_i^2, s_i^4	s_i^2	1	1	2	a_i^3, a_i^2, a_i^4
a_i^3	FM	s_i^1, s_i^2	s_i^3	1	1	0	a_i^1, a_i^5
a_i^4	IM	s_i^2, s_i^1	s_i^4	1	1	0	a_i^2, a_i^6
a_i^5	FI	s_i^3, s_i^T	s_i^T	1	1	2	a_i^5, x_i^1, x_i^2
a_i^6	FI	s_i^4, s_i^F	s_i^F	1	1	2	a_i^6, y_i^1, y_i^2
x_i^1	FM	$s_i^T, c(x_i^1), w_{i,3}^1$	$w_{i,1}^k$	1	1	2	$q_{i,1}^k, q_{i,3}^k, q_{i,2}^k$
x_i^2	IM	$s_i^T, c(x_i^2), w_{i,3}^2$	$w_{i,2}^k$	1	1	2	$q_{i,2}^k, q_{i,1}^k$
y_i^1	FM	$s_i^F, c(y_i^1), w_{i,3}^3$	$w_{i,3}^k$	1	1	1	$A(w_{i,3}^k), q_{i,3}^k$
y_i^2	IM	$s_i^F, c(y_i^2), w_{i,3}^4$	z_j	2	2	2	v_j^1, v_j^2, v_j^3
$q_{i,1}^k$	FM	$w_{i,2}^k, w_{i,1}^k$					
$q_{i,2}^k$	IM	$w_{i,1}^k, w_{i,2}^k$					
$q_{i,3}^k$	FI	$w_{i,3}^k, w_{i,1}^k$					

Fig. 4. The TAP instance constructed in the proof of Theorem 1.

Example 3. Consider the instance J_3 of TAP given in Fig. 3. It is straightforward to verify that $\mathcal{M}_1 = \{(a_1, s_2), (a_2, s_1)\}$, of size 2, and $\mathcal{M}_2 = \{(a_1, s_1), (a_2, s_2), (a_3, s_1)\}$, of size 3, are both stable in J_3. Hence J_3 admits stable matchings of different sizes. □

3. NP-hardness of TAP

In this section we show that it is hard to decide whether an instance of TAP admits a stable matching, even in the presence of restrictions on the number of subjects, the partial capacities of the schools and the lengths of the applicants' and schools' preference lists.

Theorem 1. *Given an instance of* TAP*, the problem of deciding whether a stable matching exists, is NP-complete. This result holds even if there are only three subjects, each partial capacity of a school is at most 2, and the preference list of each applicant and school is of length at most 3.*

Proof. It is easy to see that TAP belongs to NP, since when given an assignment, it can be checked in polynomial time that it is a matching and that it is stable. To prove completeness, we reduce from a restricted version of SAT. Let (2,2)-E3-SAT denote the problem of deciding, given a Boolean formula B in CNF in which each clause contains exactly 3 literals and, for each variable v_i, each of literals v_i and $\bar{v}_i$ appears exactly twice in B, whether B is satisfiable. Berman et al. [5] showed that (2,2)-E3-SAT is NP-complete.

Hence let B be an instance of (2,2)-E3-SAT. Let $V = \{v_1, v_2, \ldots, v_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ is the set of variables and clauses in B, respectively. Let us construct an instance J of TAP in the following way.

There are 3 subjects, namely F, I and M. For each variable v_i there are 6 applicants $a_i^1, a_i^2, \ldots, a_i^6$, 4 applicants $x_i^1, x_i^2, y_i^1, y_i^2$, 12 applicants $q_{i,1}^k, q_{i,2}^k, q_{i,3}^k$ $(1 \le k \le 4)$, 6 schools $s_i^1, s_i^2, s_i^3, s_i^4, s_i^T, s_i^F$ and 12 schools $w_{i,1}^k, w_{i,2}^k, w_{i,3}^k$ $(1 \le k \le 4)$. In addition, for each clause c_j there is one school z_j. Applicants x_i^1 and x_i^2 correspond to the first and to the second occurrence of literal v_i, and applicants y_i^1 and y_i^2 correspond to the first and to the second occurrence of literal $\bar{v}_i$, respectively.

The characteristics of applicants and schools and their preferences are given in Fig. 4. Here, the subscripts and superscripts involving i, j and k range over the following intervals: $1 \le i \le n$, $1 \le j \le m$ and $1 \le k \le 4$. In the preference list of school z_j, the symbol v_j^s means the x- or y-applicant that corresponds to the literal that appears in position s of clause c_j. Conversely, in the preference list of x- or y-applicants the symbol $c(.)$ denotes the z-school corresponding to the clause containing the corresponding literal. Also, in the preference list of $w_{i,3}^k$, the symbol $A(w_{i,3}^k)$ denotes x_i^k if $1 \le k \le 2$ and denotes y_i^{k-2} if $3 \le k \le 4$.

For each i ($1 \le i \le n$) let us denote

$$T_i = \{(x_i^1, s_i^T), (x_i^2, s_i^T), (a_i^6, s_i^F)\}, \qquad F_i = \{(y_i^1, s_i^F), (y_i^2, s_i^F), (a_i^5, s_i^T)\}.$$

Now, let f be a satisfying truth assignment of B. Define a matching $\mathcal{M}$ in J as follows. For each variable $v_i \in V$, if v_i is true under f, put the pairs T_i into $\mathcal{M}$ and if v_i is false under f put the pairs F_i into $\mathcal{M}$. In the former case add the pairs

$$(y_i^1, c(y_i^1)), (y_i^2, c(y_i^2)), (a_i^1, s_i^1), (a_i^2, s_i^4), (a_i^3, s_i^2), (a_i^4, s_i^2), (a_i^5, s_i^3),$$

and in the latter case add the pairs

$$(x_i^1, c(x_i^1)), (x_i^2, c(x_i^2)), (a_i^1, s_i^3), (a_i^2, s_i^2), (a_i^3, s_i^1), (a_i^4, s_i^1), (a_i^6, s_i^4).$$

Notice that as each clause $c_j \in C$ contains at most two false literals, school z_j has enough capacity for accepting all the allocated applicants. Finally, add the following pairs for each i ($1 \le i \le n$) and k ($1 \le k \le 4$):

$$(q_{i,1}^k, w_{i,2}^k), (q_{i,2}^k, w_{i,1}^k), (q_{i,3}^k, w_{i,3}^k).$$

It is obvious that the defined assignment is a matching; it remains to prove that it is stable. We show this by considering each type of applicants corresponding to variable v_i in turn. Firstly we remark that applicants $q_{i,1}^k, q_{i,2}^k, q_{i,3}^k$ each have their first choice school ($1 \le k \le 4$) so cannot be involved in a blocking pair. Now suppose that v_i is true under f. Then:

- applicants x_i^1, x_i^2, a_i^1, a_i^4 and a_i^5 have their most-preferred schools, so are not blocking;
- applicants y_i^1 and y_i^2 prefer school s_i^F, but this school is assigned a_i^6, whom it prefers;
- applicant a_i^2 prefers school s_i^2, but this school is assigned a_i^3, whom it prefers;
- applicant a_i^3 prefers school s_i^1, but this school is assigned a_i^1, whom it prefers;
- applicant a_i^6 prefers school s_i^4, but this school is assigned a_i^2, whom it prefers.

The case of a false variable can be proved similarly.

For the converse implication let us first prove two lemmata.

Lemma 1. *Each stable matching $\mathcal{M}$ in J contains for each i either all the pairs in T_i or all the pairs in F_i.*

Proof. Let $\mathcal{M}$ be a stable matching. Fix $i \in \{1, 2, \dots, n\}$. Notice first that both schools s_i^T and s_i^F must be full, otherwise either s_i^T will form a blocking pair with at least one of x_i^1 and x_i^2, or s_i^F will form a blocking pair with at least one of y_i^1 and y_i^2. Further, let us distinguish the following cases.

- $\{(a_i^5, s_i^T), (a_i^6, s_i^F)\} \subseteq \mathcal{M}$. Then, as there are no blocking pairs, $\{(a_i^1, s_i^3), (a_i^2, s_i^4)\} \subseteq \mathcal{M}$, which further implies $\{(a_i^3, s_i^2), (a_i^4, s_i^1)\} \subseteq \mathcal{M}$. This, however means that (a_i^3, s_i^1) and (a_i^4, s_i^2) are blocking pairs for $\mathcal{M}$, a contradiction.
- $\{(x_i^1, s_i^T), (x_i^2, s_i^T), (y_i^1, s_i^F), (y_i^2, s_i^F)\} \subseteq \mathcal{M}$. Now, to avoid blocking pairs, $\{(a_i^5, s_i^3), (a_i^6, s_i^4)\} \subseteq \mathcal{M}$, which further implies $\{(a_i^1, s_i^1), (a_i^2, s_i^2)\} \subseteq \mathcal{M}$. Then there are blocking pairs (a_i^3, s_i^2) and (a_i^4, s_i^1), again a contradiction.

The result follows. □

Lemma 2. *In each stable matching $\mathcal{M}$ in J, every applicant in the set $\{x_i^1, x_i^2, y_i^1, y_i^2 : 1 \le i \le n\}$ is assigned to her first- or second-choice school.*

Proof. For some $i \in \{1, 2, \dots, n\}$, consider applicant x_i^1 (the argument for x_i^2, y_i^1, y_i^2 is similar). Suppose firstly that x_i^1 is unassigned in $\mathcal{M}$. Then $(x_i^1, w_{i,3}^1)$ blocks $\mathcal{M}$, a contradiction. Now suppose that $(x_i^1, w_{i,3}^1) \in \mathcal{M}$. If $(q_{i,3}^1, w_{i,1}^1) \in \mathcal{M}$ then $(q_{i,1}^1, w_{i,2}^1) \in \mathcal{M}$, for otherwise $(q_{i,1}^1, w_{i,1}^1)$ blocks $\mathcal{M}$. But then $(q_{i,2}^1, w_{i,2}^1)$ blocks $\mathcal{M}$, a contradiction. Thus $q_{i,3}^1$ is unassigned in $\mathcal{M}$. Then $(q_{i,2}^1, w_{i,1}^1) \in \mathcal{M}$, for otherwise $(q_{i,2}^1, w_{i,1}^1)$ blocks $\mathcal{M}$. Also $(q_{i,1}^1, w_{i,2}^1) \in \mathcal{M}$, for otherwise $(q_{i,1}^1, w_{i,2}^1)$ blocks $\mathcal{M}$. Hence $(q_{i,3}^1, w_{i,1}^1)$ blocks $\mathcal{M}$, a contradiction. □

So, suppose that $\mathcal{M}$ is a stable matching in J. We form a truth assignment f in B as follows. Let $i \in \{1, 2, \dots, n\}$ be given. By Lemma 1, either $T_i \subseteq \mathcal{M}$ or $F_i \subseteq \mathcal{M}$. In the former case set $f(v_i) = \text{true}$, otherwise set $f(v_i) = \text{false}$. Now let $v_i \in V$ and suppose that $f(v_i) = \text{true}$. Then by Lemma 2, each of $y_{i,1}$ and $y_{i,2}$ is assigned to her second choice school. Now suppose that $f(v_i) = \text{false}$. Then again by Lemma 2, each of $x_{i,1}$ and $x_{i,2}$ is assigned to her second choice school. Now let $c_j \in C$ and suppose that all literals in c_j are false. By the preceding remarks about $x_{i,1}, x_{i,2}, y_{i,1}$ and $y_{i,2}$ we deduce that z_j is over-subscribed, a contradiction. Thus f is a satisfying truth assignment. □

```
begin
  M := ∅;
  for  i = 1, 2, ..., n
    if  a_i's list contains a school with enough free capacity for a_i {
      s:= first such school on a_i's list ;
      M := M ∪ {(a_i, s)};
    }
end
```

Fig. 5. Algorithm Serial Dictatorship.

```
begin
  M := ∅;
  for  j = 1, 2, ..., m
    /* let s_j's list be a_{i_1}, ..., a_{i_ℓ} */
    for  r = 1, 2, ..., ℓ
      if a_{i_r} is unassigned and s_j has enough capacity for a_{i_r} then
        M := M ∪ {(a_{i_r}, s_j)};
end
```

Fig. 6. Algorithm Dual Serial Dictatorship.

4. Master lists

In some centralized matching schemes all the applicants are ordered in a common *master list*. Although the criteria used for creating such lists are often subject to some controversy (see [11] for the description of the matching scheme for allocating medical students to hospital posts in England in 2005–2006 and [20] for the situation in the central allocation scheme of teachers in Portugal that was used prior to 2005), computationally the situation with master lists may be easier. A detailed study of stable matching problems with master lists from a computational point of view can be found in [13].

In this section we shall consider the case of a master list of applicants and the (perhaps slightly less realistic) case of a master list of schools. The problems of deciding the existence of a stable matching in these cases will be denoted by TAP-AM and TAP-SM, respectively. The phrase 's has enough capacity for a' used in the algorithms in this section means the following: if a is of type $\{p, r\}$ then $|\mathcal{M}_p(s)| < c_p(s)$ as well as $|\mathcal{M}_r(s)| < c_r(s)$.

Theorem 2. *Let J be an instance of* TAP-AM *with the master list $a_1, a_2, \ldots, a_n$ of applicants. Then J admits a unique stable matching that may be found by an application of Algorithm Serial Dictatorship as shown in Fig. 5.*

Proof. It is easy to see that Serial Dictatorship outputs a matching. We have to prove that this matching is stable and that it is the unique stable matching.

$\mathcal{M}$ **is stable.** Suppose that (a_i, s_j) is a blocking pair and that i is the smallest index of an applicant involved in a blocking pair. Since a_i has chosen the best available school from her list, s_j did not have enough capacity to accept a_i when it was a_i's turn in the algorithm. However, all the applicants that were assigned to s_j at that moment precede a_i in the master list, hence (a_i, s_j) cannot be a blocking pair.

Uniqueness. Let $\mathcal{M}' \neq \mathcal{M}$ be another stable matching and let a_i be the first applicant in the master list with $\mathcal{M}'(a_i) \neq \mathcal{M}(a_i)$. As Serial Dictatorship gave a_i her best available school and all applicants who precede a_i in the master list have the same assignments in $\mathcal{M}$ as in $\mathcal{M}'$, it must be the case that a_i prefers $s_j = \mathcal{M}(a_i)$ to $s_k = \mathcal{M}'(a_i)$. But this implies that (a_i, s_j) is a blocking pair for $\mathcal{M}'$, as s_j will be able to reject one or two applicants worse than a_i in order to free up sufficient capacity for a_i (for, s_j had enough room for a_i in $\mathcal{M}$ when it was a_i's turn during Serial Dictatorship, and any applicant that precedes a_i in the master list has the same assignment in $\mathcal{M}$ as she does in $\mathcal{M}'$). □

The situation with a master list of schools, although less likely to occur in practice, is also efficiently solvable.

Theorem 3. *Let J be an instance of* TAP-SM *with the master list of schools $s_1, s_2, \ldots, s_m$. Then J admits a unique stable matching that may be found by an application of Algorithm Dual Serial Dictatorship as shown in Fig. 6.*

Proof. Let us denote by $J(s_1)$ the subinstance of J containing just school s_1 and applicants who apply to s_1. $J(s_1)$ is an instance of TAP-AM, so it has a unique stable matching. This is obtained by Serial Dictatorship of applicants that is equivalent to the part of Dual Serial Dictatorship within one iteration of the **for**-loop for schools. Let us denote this matching by $\mathcal{M}_1$. Let us further observe that no applicant assigned to s_1 could be a member of a blocking pair, as she received her most preferred school. If we now denote by $J(-s_1)$ the subinstance of J with pairs of $\mathcal{M}_1$ deleted, the result follows by induction. □

applicant	type	preferences	school	capacities F	I	M	subject	preferences		
a_1	FM	s_1	s_1	1	1	1	F	a_1	a_2	a_3
a_2	FI	s_1					I	a_2	a_3	a_1
a_3	IM	s_1					M	a_3	a_1	a_2

Fig. 7. A small instance of TAP-SS with no stable matching.

5. Subject-specific preference lists

In this section we consider the variant of TAP in which a school can order applicants differently by subject. That is, each school has a preference list over the applicants for each subject. Let us denote this variant by TAP-SS. The definition of a blocking pair should now be modified in order to take account of this scenario.

Definition 2. Let $\mathcal{M}$ be a matching. We say that a pair (a, s) with $\mathbf{p}(a) = \{p_1, p_2\}$ and $s \in S(A)$ is *blocking* if a is not assigned in $\mathcal{M}$ or a prefers s to $\mathcal{M}(a)$, and one of the following conditions holds:

(i) s is undersubscribed in both p_1 and p_2;
(ii) s is undersubscribed in p_i and it prefers a in subject p_{3-i} to one applicant in $\mathcal{M}_{p_{3-i}}(s)$ for some $i \in \{1, 2\}$;
(iii) s prefers a in both subjects p_1, p_2 to one applicant in $\mathcal{M}_{p_1,p_2}(s)$;
(iv) s prefers a in subject p_1 to applicant $a_1 \in \mathcal{M}_{p_1}(s)$ and in subject p_2 to another applicant $a_2 \in \mathcal{M}_{p_2}(s)$.

We firstly observe that an instance of TAP-SS need not admit a stable matching, since TAP-SS is a generalization of TAP. We now give a simple instance of TAP-SS to illustrate this; the instance will be useful in the proof of Corollary 1 in Section 6.

Example 4. Consider the TAP-SS instance given by Fig. 7. For this small instance it may be verified that any matching containing (a_i, s_1), where $i \in \{1, 2, 3\}$, is blocked by (a_{i-1}, s_1), where addition is taken modulo 3. □

Given that TAP-SS is a generalization of TAP, Theorem 1 implies that the problem of deciding whether a stable matching exists, given an instance of TAP-SS, is NP-complete. In this section we will show that this result for TAP-SS holds even if each school's subject-specific preference list is derived from a subject-specific master list of the applicants. Define a matching $\mathcal{M}$ in a TAP-SS instance to be *applicant-complete* if every applicant is assigned in $\mathcal{M}$. We begin by showing that the problem of deciding whether a TAP-SS instance admits an applicant-complete stable matching is NP-complete.

Lemma 3. *Given an instance of* TAP-SS, *the problem of deciding whether an applicant-complete stable matching exists is NP-complete. This result holds even if there are only three subjects, each partial capacity of a school is at most 1, and the preference lists of the schools are derived from subject-specific master lists of the applicants.*

Proof. Clearly, this problem is in NP; to show completeness we reduce from (2,2)-E3-SAT (see the proof of Theorem 1). Let B be an instance of this problem, where $V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ be the set of variables and clauses respectively in B. We construct an instance J of TAP-SS in the following way.

There are 3 subjects, namely F, I and M. For each variable v_i $(0 \le i \le n-1)$ there are 4 applicants x_{4i+r} $(0 \le r \le 3)$, each of type FI, and 4 schools y_{4i+r} $(0 \le r \le 3)$. For each clause c_j $(1 \le j \le m)$ there are 4 applicants q_j and w_j^t $(1 \le t \le 3)$, each of type FM, and 4 schools s_j^t $(1 \le t \le 4)$. Let $X = \{x_i : 0 \le i \le 4n-1\}$, $Y = \{y_i : 0 \le i \le 4n-1\}$, $W = \{w_j^t : 1 \le j \le m \wedge 1 \le t \le 3\}$, $Q = \{q_j : 1 \le j \le m\}$, $S' = \{s_j^t : 1 \le j \le m \wedge 1 \le t \le 3\}$, $S'' = \{s_j^4 : 1 \le j \le m\}$ and $S = S' \cup S''$.

For each i $(0 \le i \le n-1)$, applicants x_{4i} and x_{4i+1} correspond to the first and second occurrences of literal v_i in B, and applicants x_{4i+2} and x_{4i+3} correspond to the first and second occurrences of literal $\bar{v}_i$ in B, respectively. For each $r \in \{0, 1\}$, let $s(x_{4i+r})$ denote the school s_j^t such that the $(r+1)$th occurrence of literal v_i appears in position t of clause c_j $(1 \le j \le m$, $1 \le t \le 3)$. Similarly, for each $r \in \{2, 3\}$, let $s(x_{4i+r})$ denote the school s_j^t such that the $(r-1)$th occurrence of literal $\bar{v}_i$ appears in position t of clause c_j $(1 \le j \le m$, $1 \le t \le 3)$.

The applicants' preferences, together with a summary of their types and a summary of the schools' partial capacities, are given in Fig. 8. Here, the subscripts and superscripts involving i, j and t range over the following intervals: $0 \le i \le n-1$, $1 \le j \le m$ and $1 \le t \le 3$.

We now construct the subject-specific master lists of applicants. Let $\langle X \rangle$ denote the elements of X in increasing order of subscript, and let $\langle \bar{X} \rangle$ denote the reverse of this order. Similarly let $\langle W \rangle$ denote the elements of W listed in increasing order of subscript, and within this ordering, those elements with equal subscript are listed in increasing order of superscript. Also let $\langle \bar{W} \rangle$ denote the reverse of $\langle W \rangle$. Finally let $\langle Q \rangle$ denote the elements of Q listed in increasing order of subscript. The master lists of the applicants with respect to subjects are shown in Fig. 9.

applicant	type	preferences	school	capacities F	I	M
x_{4i}	FI	$y_{4i}, s(x_{4i}), y_{4i+1}$	y_{4i}	1	1	0
x_{4i+1}	FI	$y_{4i+1}, s(x_{4i+1}), y_{4i+2}$	y_{4i+1}	1	1	0
x_{4i+2}	FI	$y_{4i+3}, s(x_{4i+2}), y_{4i+2}$	y_{4i+2}	1	1	0
x_{4i+3}	FI	$y_{4i}, s(x_{4i+3}), y_{4i+3}$	y_{4i+3}	1	1	0
q_j	FM	s_j^1, s_j^2, s_j^3	s_j^t	1	1	1
w_j^t	FM	s_j^t, s_j^4	s_j^4	1	0	1

Fig. 8. The TAP-SS instance constructed in the proof of Lemma 3.

$$\begin{array}{ll} F: & \langle W\rangle \quad \langle X\rangle \quad \langle Q\rangle \\ I: & \langle \bar{X}\rangle \\ M: & \langle Q\rangle \quad \langle \bar{W}\rangle \end{array}$$

Fig. 9. The master lists for the TAP-SS instance constructed in the proof of Lemma 3.

For each i $(0 \le i \le n-1)$, let us denote $T_i = \{(x_{4i+r}, y_{4i+r}) : 0 \le r \le 3\}$ and $F_i = \{(x_{4i+r}, y_{4i+r+1}) : 0 \le r \le 2\} \cup \{(x_{4i+3}, y_{4i})\}$. We claim that B has a satisfying truth assignment if and only if J has an applicant-complete stable matching.

For, let f be a satisfying truth assignment of B. Define a matching $\mathcal{M}$ in J as follows. For each variable $v_i \in V$, if $f(v_i) = \mathsf{true}$, add the pairs in T_i to $\mathcal{M}$ and if $f(v_i) = \mathsf{false}$, add the pairs in F_i to $\mathcal{M}$. Each clause $c_j \in C$ contains some literal in c_j that is true under f, let t be the position of c_j containing this true literal $(1 \le t \le 3)$. Add the following pairs to $\mathcal{M}$:

$$\{(w_j^{t'}, s_j^{t'}) : 1 \le t' \le 3 \wedge t \ne t'\} \cup \{(q_j, s_j^t), (w_j^t, s_j^4\}.$$

It is obvious that the defined assignment is an applicant-complete matching; it remains to prove that it is stable. It is straightforward to verify that no applicant in Q can be involved in a blocking pair of $\mathcal{M}$, and no pair in $X \times Y$ can block $\mathcal{M}$. Now suppose that (w_j^t, s_j^t) blocks $\mathcal{M}$ for some j $(1 \le j \le m)$ and t $(1 \le t \le 3)$. Then $(w_j^t, s_j^4) \in \mathcal{M}$ and $(q_j, s_j^t) \in \mathcal{M}$, and school s_j^t prefers q_j over w_j^t for subject M, so (w_j^t, s_j^t) cannot block $\mathcal{M}$ after all. Finally suppose that $(x_{4i+r}, s(x_{4i+r}))$ blocks $\mathcal{M}$ for some i $(0 \le i \le n-1)$ and r $(0 \le r \le 1)$. Then $f(v_i) = \mathsf{false}$ by construction of $\mathcal{M}$. Let $s_j^t = s(x_{4i+r})$. Then $(q_j, s_j^t) \in \mathcal{M}$, since $(x_{4i+r}, s(x_{4i+r}))$ blocks $\mathcal{M}$. But then by construction of $\mathcal{M}$, the tth literal of c_j is true under f, a contradiction. The argument is similar if $r \in \{2, 3\}$. Hence $\mathcal{M}$ is stable in J.

Conversely suppose that $\mathcal{M}$ is an applicant-complete stable matching in J. For any j $(1 \le j \le m)$, it follows that $(q_j, s_j^t) \in \mathcal{M}$ for some t $(1 \le t \le 3)$ and thus $(w_j^t, s_j^4) \in \mathcal{M}$, since q_j and w_j^t must be assigned. Moreover $(w_j^{t'}, s_j^{t'}) \in \mathcal{M}$ for each t' $(1 \le t' \le 3,\ t \ne t')$. Thus $\mathcal{M}$ contains no pair of the form $(x_{4i+r}, s(x_{4i+r}))$ $(0 \le i \le n-1,\ 0 \le r \le 3)$. Moreover, since each member of X must be assigned in $\mathcal{M}$, we have thus established that for each i $(0 \le i \le n-1)$, either $T_i \subseteq \mathcal{M}$ or $F_i \subseteq \mathcal{M}$.

Now we construct a truth assignment f in B as follows. If $T_i \subseteq \mathcal{M}$ set $f(v_i) = \mathsf{true}$ and if $F_i \subseteq \mathcal{M}$ set $f(v_i) = \mathsf{false}$. We claim that f is a satisfying truth assignment. For, suppose that some clause c_j contains no true literal. As $\mathcal{M}$ is an applicant-complete matching, $(q_j, s_j^t) \in \mathcal{M}$ for some t $(1 \le t \le 3)$. Now let x_{4i+r} be the applicant such that $s(x_{4i+r}) = s_j^t$ $(0 \le i \le n-1,\ 0 \le r \le 3)$. If $r \in \{0, 1\}$ then $f(v_i) = \mathsf{false}$, so $F_i \subseteq \mathcal{M}$. Hence $(x_{4i+r}, s(x_{4i+r}))$ blocks $\mathcal{M}$, a contradiction. Similarly if $r \in \{2, 3\}$ then $f(v_i) = \mathsf{true}$, so $T_i \subseteq \mathcal{M}$. Hence $(x_{4i+r}, s(x_{4i+r}))$ blocks $\mathcal{M}$, again a contradiction. □

We next show that the requirement of Lemma 3 for the stable matching to be applicant-complete can be dropped.

Lemma 4. *Given an instance of* TAP-SS, *the problem of deciding whether a stable matching exists is NP-complete. This result holds even if there are only three subjects, each partial capacity of a school is at most 1, and the preference lists of the schools are derived from subject-specific master lists of the applicants.*

Proof. We show how to modify the reduction presented in the proof of Lemma 3 in order to ensure that any stable matching in J is applicant-complete. We create a new TAP-SS instance J' from J as follows. For each applicant a in J, create two new applicants a' and a''. If a is of type FI, then a' and a'' are of type FM and IM respectively. If a is of type FM, then a' and a'' are of type FI and IM respectively. Create a new school $g(a)$ which has capacity 1 for each of subjects F, I and M. Append $g(a)$ to applicant a's preference list in J to obtain her preference list in J'. Each of applicants a' and a'' finds only $g(a)$ acceptable.

Let X' and X'' denote the sets of newly-created applicants in J' with single and double primes respectively that correspond to applicants in X. Define R' and R'' similarly for the newly-created applicants in J' that correspond to applicants in $Q \cup W$. For $A \in \{R, X\}$, let $\langle A'\rangle$ and $\langle A''\rangle$ denote arbitrary but fixed orderings of the applicants in A' and A'' respectively.

$$\begin{array}{lccccc} F: & \langle W\rangle & \langle X\rangle & \langle Q\rangle & \langle X'\rangle & \langle R'\rangle \\ I: & \langle X''\rangle & \langle R'\rangle & \langle R''\rangle & \langle \bar{X}\rangle & \\ M: & \langle R''\rangle & \langle Q\rangle & \langle \bar{W}\rangle & \langle X'\rangle & \langle X''\rangle \end{array}$$

Fig. 10. The master lists of the subjects in the TAP-SS instance constructed in the proof of Lemma 4.

The subject-specific master lists in J' are as shown in Fig. 10.

We show how to modify the proof of Lemma 3 to show that B has a satisfying truth assignment if and only if J' has a stable matching.

Firstly, if f is a satisfying truth assignment of B, construct the matching $\mathcal{M}$ in J as in the proof of Lemma 3. We then extend $\mathcal{M}$ to a matching $\mathcal{M}'$ in J' as follows. For each applicant a in J, add the pair $(a', g(a))$ to $\mathcal{M}$. Since $\mathcal{M}$ is applicant-complete in J, it is straightforward to verify that $\mathcal{M}'$ is stable in J'.

Conversely suppose that $\mathcal{M}'$ is a stable matching in J'. We firstly claim that each applicant a in J is assigned in $\mathcal{M}'$ to a school better than $g(a)$. For, suppose $(a, g(a)) \in \mathcal{M}'$. Then $(a'', g(a))$ blocks $\mathcal{M}'$, a contradiction. Now suppose that a is unassigned in $\mathcal{M}'$. Clearly some applicant is assigned to $g(a)$ in $\mathcal{M}'$, for otherwise $(a, g(a))$ blocks $\mathcal{M}'$. If $(a', g(a)) \in \mathcal{M}'$ then $(a, g(a))$ blocks $\mathcal{M}'$, whilst if $(a'', g(a)) \in \mathcal{M}'$ then $(a', g(a))$ blocks $\mathcal{M}'$. The claim is thus proved. It is then also straightforward to verify that $(a', g(a)) \in \mathcal{M}'$ for each applicant a in J, for otherwise $(a', g(a))$ blocks $\mathcal{M}'$.

Let $\mathcal{M}$ be the matching obtained from $\mathcal{M}'$ by removing all pairs of the form $(a', g(a))$, where a is an applicant in J. It follows by the previous paragraph that $\mathcal{M}$ is an applicant-complete stable matching in J. The remainder of the proof is then identical to the converse direction of the proof of Lemma 3. □

We finally present the main result of this section, which strengthens Lemma 4 to show that the result holds even if, additionally, the preference lists of applicants are derived from a master list of schools.

Theorem 4. *Given an instance of* TAP-SS*, the problem of deciding whether a stable matching exists is NP-complete. This result holds even if there are only three subjects, each partial capacity of a school is at most 1, the preference lists of the schools are derived from subject-specific master lists of the applicants, and the preference lists of the applicants are derived from a single master list of schools.*

Proof. We consider the reduction given by Lemmata 3 and 4, and show that the applicants' preference lists may be derived from a single master list of schools.

For each i $(0 \le i \le n-1)$, let $\langle S_i\rangle$ denote the sequence

$$\langle y_{4i}, s(x_{4i}), s(x_{4i+3}), y_{4i+3}, s(x_{4i+2}), y_{4i+1}, s(x_{4i+1}), y_{4i+2}\rangle.$$

Let $S^4 = \{s_j^4 : 1 \le j \le m\}$ and let $\langle S^4\rangle$ denote an arbitrary order of the schools in S^4. Let G denote the set of schools of the form $g(a)$ as introduced in the proof of Lemma 4 for each applicant a in the original TAP-SS instance as constructed in the proof of Lemma 3. Let $\langle G\rangle$ denote an arbitrary order of the schools in G. Define the following master list of schools:

$$\langle S_0\rangle \quad \langle S_1\rangle \quad \ldots \quad \langle S_{n-1}\rangle \quad \langle S^4\rangle \quad \langle G\rangle$$

In the proof of Lemma 3, let the preference list of each q_j $(1 \le j \le m)$ be reordered such that the relative ordering of the three schools s_j^1, s_j^2 and s_j^3 is derived from the above master list. This does not change the remainder of the proof of Lemma 3, nor the proof of Lemma 4. Moreover every other applicant's preference list is derived from the above master list of schools. The theorem then follows. □

6. "Most stable" matchings

Given an instance of TAP, we have already seen that a stable matching need not exist. In such cases it is natural to seek a matching that is "as stable as possible" in a precise sense. Here we regard such "most stable" matchings as those that admit the minimum number of blocking pairs. Note that this approach was also considered in [3].

Given a TAP instance I, denote by $opt(I)$ the minimum number of blocking pairs admitted by any matching in I. Let MIN BP TAP denote the problem of finding a matching M with $opt(I)$ blocking pairs in I. Clearly I admits a stable matching if and only if $opt(I) = 0$, hence MIN BP TAP is NP-hard by Theorem 1. This naturally leads to the consideration of the approximability of this problem.

If I is *solvable*, i.e., I admits a stable matching, then the standard notion of an approximation algorithm is not particularly meaningful in this case, since $opt(I) = 0$. However the concept of approximation is much more relevant in the case that I is unsolvable, i.e., $opt(I) > 0$.

In this section we prove that for unsolvable instances, MIN BP TAP is not approximable within $n^{1-\varepsilon}$, for any $\varepsilon > 0$, unless P=NP, where n is the number of agents (i.e., applicants plus schools). We show that similar observations hold for unsolvable instances of MIN BP TAP-SS, the problem of finding a matching with the minimum number of blocking pairs in an instance of TAP-SS.

Theorem 5. MIN BP TAP *is not approximable within* $n^{1-\varepsilon}$*, where n is the number of agents in a given instance, for any* $\varepsilon > 0$ *unless P=NP. The result holds even if the constructed instance* I' *is unsolvable, and in* I' *there are only three subjects, each partial capacity of a school is at most 2, and the preference list of each applicant is of length at most 3.*

Proof. Let I be an instance of TAP satisfying the restrictions given in the statement of the theorem. We know from Theorem 1 that the problem of deciding whether I admits a stable matching is NP-complete. Let n_0 be the number of agents in I. Choose $c = \lceil 2/\varepsilon \rceil$ and $k = n_0^c$. Now, let $I_1, I_2, \ldots, I_k$ be k disjoint copies of the instance I. Let I_{k+1} be the instance of TAP shown in Fig. 2 that admits no stable matching. (Note that I_{k+1} also satisfies the restrictions indicated in the theorem statement.) Let I' be the instance of TAP formed by taking the union of the sub-instances $I_1, I_2, \ldots, I_{k+1}$. Let $n = kn_0 + 5$ denote the number of agents in I'.

Clearly if I admits a stable matching then each I_r must admit a stable matching ($1 \le r \le k$), whilst I_{k+1} admits no stable matching but does admit a matching M with one blocking pair (namely $M = \{(a_1, s_2), (a_2, s_1)\}$). Hence $opt(I') = 1$. However, if I admits no stable matching, then each I_r ($1 \le r \le k+1$) admits only matchings with one or more blocking pair, and hence $opt(I') \ge k+1$. We now show that $n^{1-\varepsilon} \le k$.

Firstly $n = kn_0 + 5 \le 2kn_0 = 2n_0^{c+1}$, since we lose no generality by assuming that $n_0 \ge 5$. Hence

$$\frac{n}{2} \le n_0^{c+1}$$

which implies

$$\left(\frac{n}{2}\right)^{1/(c+1)} \le n_0.$$

Since $k = n_0^c$ it follows that

$$\left(\frac{n}{2}\right)^{c/(c+1)} \le k$$

and hence

$$2^{-c/(c+1)} n^{c/(c+1)} \le k. \tag{1}$$

We know that $n = kn_0 + 5 \ge k = n_0^c$, and by our earlier assumption, $n_0 \ge 2$. Hence $n \ge 2^c$ and it follows that $n^{-1} \le 2^{-c}$ and thus

$$n^{-1/(c+1)} \le 2^{-c/(c+1)}. \tag{2}$$

Inequality (2) implies that

$$n^{-1/(c+1)} n^{c/c+1} \le 2^{-c/(c+1)} n^{c/(c+1)} \tag{3}$$

and hence Inequalities (1) and (3) imply that

$$n^{(c-1)/(c+1)} = n^{c/(c+1)} n^{-1/(c+1)} \le 2^{-c/(c+1)} n^{c/(c+1)} \le k. \tag{4}$$

We now show that $n^{1-\varepsilon} \le n^{(c-1)/(c+1)}$. Observe that $c \ge 2/\varepsilon$ and thus $c + 1 \ge 2/\varepsilon$. Hence

$$1 - \varepsilon \le 1 - \frac{2}{c+1} \le \frac{c-1}{c+1}$$

and hence it follows from Inequality (4) that $n^{1-\varepsilon} \le k$.

Now, assume that X is an approximation algorithm for MIN BP TAP with a performance guarantee of $n^{1-\varepsilon}$. If I admits a stable matching, X must return a matching in I' with at most $opt(I') \cdot n^{1-\varepsilon} = n^{1-\varepsilon} \le k$ blocking pairs, since $opt(I') = 1$. Otherwise, I' does not admit a stable matching and, as shown above, X must return a matching with at least $k+1$ blocking pairs. Thus algorithm X may be used to determine whether I admits a stable matching in polynomial time, a contradiction to Theorem 1 unless P=NP. Hence, no such polynomial approximation algorithm for MIN BP TAP can exist unless P = NP. □

Corollary 1. MIN BP TAP-SS *is not approximable within* $n^{1-\varepsilon}$*, where n is the number of agents in a given instance, for any* $\varepsilon > 0$ *unless P=NP. The result holds even if the constructed instance* I' *is unsolvable, and in* I' *there are only three subjects, each partial capacity of a school is at most 1, the preference lists of the schools are derived from subject-specific master lists of the applicants, and the preference lists of the applicants are derived from a single master list of schools.*

Proof. This result follows using a similar proof to that Theorem 5, using the NP-completeness result of Theorem 4; in the modified proof of Theorem 5 we use as I_{k+1} the unsolvable instance of TAP-SS shown in Fig. 7 (for this small instance it may be verified that any matching containing (a_i, s_1), where $i \in \{1, 2, 3\}$, is blocked by (a_{i-1}, s_1) only, where addition is taken modulo 3). It is easy to see that I_{k+1} satisfies the restrictions on the instance specified in the statement of the corollary. The modifications to the derivation of $n^{1-\varepsilon} \le k$ are straightforward, and thus omitted. □

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References

[1] A. Abdulkadiroğlu, P.A. Pathak, A.E. Roth, The Boston Public School Match, Am. Econ. Rev. 95 (2) (2005) 368–371.
[2] A. Abdulkadiroğlu, P.A. Pathak, A.E. Roth, The New York City High School Match, Am. Econ. Rev. 95 (2) (2005) 364–367.
[3] D.J. Abraham, P. Biró, D.F. Manlove, "Almost stable" matchings in the Roommates problem, in: Proceedings of WAOA '05: the 3rd Workshop on Approximation and Online Algorithms, in: Lecture Notes in Comput. Sci., vol. 3879, Springer, 2006, pp. 1–14.
[4] J. Baumert, M. Kunter, W. Blum, M. Brunner, T. Voss, A. Jordan, U. Klusmann, S. Krauss, M. Neubrand, Y.M. Tsai, Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress, Am. Educ. Res. J. 47 (1) (2010) 133–180.
[5] P. Berman, M. Karpinski, Alexander D. Scott, Approximation hardness of short symmetric instances of MAX-3SAT, Electronic Colloquium on Computational Complexity Report, no. 49, 2003.
[6] P. Biró, T. Fleiner, R.W. Irving, D.F. Manlove, The College Admissions problem with lower and common quotas, Theoret. Comput. Sci. 411 (2010) 3136–3153.
[7] P. Biró, R.W. Irving, I. Schlotter, Stable matching with couples: an empirical study, ACM J. Exp. Algorithmics 16 (2011), section 1, article 2, 27 pages.
[8] K. Cechlárová, T. Fleiner, D.F. Manlove, I. McBride, E. Potpinková, Modelling practical placement of trainee teachers to schools, Cent. Eur. J. Oper. Res. 23 (3) (2015) 547–562, http://dx.doi.org/10.1007/s10100-014-0356-5.
[9] W. Van Dooren, L. Verschaffel, Pre-service teachers' preferred strategies for solving arithmetic and algebra word problems, J. Math. Teach. Educ. 6 (1) (2003) 27–52.
[10] D. Gale, L.S. Shapley, College admissions and the stability of marriage, Amer. Math. Monthly 69 (1962) 9–15.
[11] N. Hawkes, Online selection of new doctors grossly unfair, The Times, London, 4 March 2006.
[12] R.W. Irving, Matching medical students to pairs of hospitals: a new variation on a well-known theme, in: Proceedings of ESA '98: the 6th European Symposium on Algorithms, in: Lecture Notes in Computer Science, vol. 1461, Springer, 1998, pp. 381–392.
[13] R.W. Irving, D.F. Manlove, S. Scott, The stable marriage problem with master preference lists, Discrete Appl. Math. 156 (15) (2008) 2959–2977.
[14] D.F. Manlove, Algorithmics of Matching Under Preferences, World Scientific, 2013.
[15] E.J. McDermid, D.F. Manlove, Keeping partners together: algorithmic results for the hospitals/residents problem with couples, J. Comb. Optim. 19 (3) (2010) 279–303.
[16] E. Ronn, NP-complete stable matching problems, J. Algorithms 11 (1990) 285–304.
[17] A.E. Roth, The evolution of the labor market for medical interns and residents: a case study in game theory, J. Polit. Econ. 92 (6) (1984) 991–1016.
[18] A.E. Roth, Deferred acceptance algorithms: history, theory, practice, and open questions, Internat. J. Game Theory 36 (3–4) (2008) 537–569.
[19] A.E. Roth, M.A.O. Sotomayor, Two-Sided Matching: a Study in Game-Theoretic Modeling and Analysis, Econom. Soc. Monogr., vol. 18, Cambridge University Press, 1990.
[20] A.P. Tomás Emparelhamentos, Casamentos Estáveis e Algoritmos de Colocação de Professores, Technical Report DCC-2005-02, Departamento de Ciência de Computadores, Faculdade de Ciências, Laboratório de Inteligência Artificial e Ciência de Computadores, Universidade do Porto, 2005.
[21] http://www.nrmp.org (National Resident Matching Program website). Accessed 22 January 2015.