FUNDAMENTAL PRINCIPLES OF CLASSICAL MECHANICS
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In David Lodge's 1980 satire on academic mores, *Small World*, Perse McGarigle, an MSc student from Limerick, Ireland, entitles his thesis: *The Influence of T.S.Eliot on Shakespeare*. We are only able to see the past through the lens of our own present. It might be appropriate to subtitle the book under review as: *The influence of General Relativity and Quantum Field Theory on Classical Mechanics.*

Many of the most pressing and influential problems in the history of mathematical physics are to do with the behaviour of systems of finite numbers of particles or conintua interacting under the influence of external and interparticulate forces. This is the world of classical mechanics. When the numbers of particles, or degrees of freedom, become too large for detailed treatment we enter the world of statistical mechanics. When the particles are small enough that the act of measurement itself interferes with the system we are in the quantum world. When the scale of the problem is very large, the language of General Relativity takes over.

Fundamental progess in physics lies in the scrutiny of unspoken assumptions. So we might set up a problem using a particular, appropriate coordinate frame. Only when we start to worry about the way the physics appears to depend on our choices are we led to the idea of an inertial frame and is the principle of covariance able to take centre stage. Then the role of geometry becomes clear. But what is geometry? Geometry was for a long time an unspoken assumption about the space in which we move. When we question that assumption we can imagine other worlds, strange geometries. A geometry becomes a space of points admitting a class of transformations. Theories of relativity startle us by showing that these bizarrely connected spaces are, in fact, where we live, at least for the time being for we are not really sure what "points" are and whether continuity is the real deal.

In addition, paradigms of classical mechanics tend to be analytically tractable. This is because of a selection process imposed by the mathematical tools available. It doesn't represent the reality of even our mundane experience. The importance of the intractable problems and new mathematical tools have led to the modern theory of nonlinear dynamics. We tailor our questions to our tools even as we sharpen our tools to form new questions.

The mathematical study of mechanics started with Newton. The development of differential calculus allowed the formulation of universal laws of physics relating forces to changes in behaviour. This provided explanations of hitherto phenomenological laws like those of Kepler concerning periods of planetary orbits. The fundamental aspects of a theory are usually the ones least questioned: here the *kinematic* frame and the *dynamical* formulation. Ideas implicit here such as covariance and parallel transport are now seen as essential and definitive of the theory.

In Euclidean geometry we might imagine disembodied triangles hanging about in space. In order to make life simple we classify these into sets of congruent objects of equal side lengths and internal angles. Comparison of triangles requires that we move them around to check that they fit or have some other standard object, a frame, which we can slide about between them. In doing this we make assumptions about the way space is connected. This is the kinematic description. The book opens with a discussion of ideas of connection and covariance. The introduction of coordinate-free notation gives covariance centre stage and where possible the book exploits this although for the most part coordinate dependent descriptions are maintained.

That there exists a class of coordinate system in each of which dynamics takes a simple, identical form is hardly obvious to creatures who inhabit a rotating planet and see the effects of "fictitious" forces like the centrifugal and Coriolis forces. Covariance under an appropriate group of transformations is now a starting point of any fundamental theory and, indeed, in the sense of Klein it amounts to the *definition* of a geometry (Erlangen programme).

One nice feature at this early stage is the use of exterior differential algebra and Cartan's *moving frames*. This language is very unifying. Div, Grad and Curl all become expressions of a single exterior differential operator and Stokes' and Green's theorems become statements relating integrals over a set and the boundary of the set. Moving frames are used to describe general curvilinear coordinate systems are related to frame bundles and gauge transformations. Here the influence of Gauge Theories from Quantum Field Theory is apparent. Lam is coauthor of a very readable set of lecture notes on differential geometry of S.S.Chern [3] and that book makes a good companion to this volume.

The kinematics of constrained systems is treated and used as an opening for the Frobenius theorem on integrability. This answers the question: when can a set of relations between velocities be seen as a consequence of relations between coordinates? These are called *holonomic* constraints. No treatment of non-holonomic constraints is given.

The issue of comparison alluded to above is addressed in two distinct ways. One may specify an affine connection on the space of coordinates and velocities conceived as a *frame bundle* which allows us, for instance, to construct covariant derivatives. The other is Lie transport which however cannot produce a covariant comparison. The requirement of covariance can be seen as leading to a notion of the way space is connected in the sense that a free particle will naturally adopt a "straight" path of travel within the prescibed geometry. Differentiation of vectorial objects requires local comparison of elements belonging to distinct spaces and the connection is one way of describing this comparison. A choice of connection is a choice of smooth geometry. Another way of making the comparison is via Lie transport. The difference between these two methods is the torsion of the space.

When dynamical, that is to say non-kinematic, considerations like Newton's Laws are introduced care is taken to explain how the classical fictitious forces, for example, the Coriolis and centrifugal forces, arise from the connection. Later in the text there is a detailed treatment of the Foucault pendulum, the slow rotation of whose plane of oscillation is at once the preservation of an interial inside a non-inertial frame as well as a striking illustration of the illusory Coriolis force.

Plently of classical mechanics problems are discussed throughout: particulate systems, rigid bodies and continua: for instance, the tidal deformation of the surface of the sea using multipole expansions. One unusual example in a book of this sort is the use of transforms for turning linear differential equations for Green's functions into algebraic equations used to analyse the power spectrim of the classical Hydrogen atom.

Notions of fixed points for nonlinear dynamical systems, stability and Liapunov functions are discussed, including satbility issues to do with rigid body motions. In the rugby ball problem, for instance, rotation around the major axis (principle direction of the diagonalised inertia tensor) is stable but not about the other axes. The discussion of rigid body motion is an example which is picked up in a number of different places in the text which adds a nice thematic development but means the reader might miss material. For instance, precession is dealt with in a later chapter but does not feature in the index. Neither does the Foucault pendulum.

The language of fibre bundles is endemic. Essentially a bundle is a generalisation of a product space where the components of the product may vary from place to place. A projection maps points in the bundle to a base space. The inverse image of a point in the base space is the fibre and locally the bundle looks like a product of an open set in the base and the fibre itself. Depending on the nature of the fibre one has, for example, tangent and cotangent bundles of a manifold (base space), frame bundles or principle bundles, where the fibre is a Lie group. They are the natural context for questions of general covariance or equivariance and they can carry connections. Gauge theories are described kinematically by principle bundles. An important illustative example is the "falling cat problem": to understand the way a cat is able to reconfigure its body in falling in order to land on its feet. This is a composite body problem which can be described as a fibre bundle. The base space is the manifold of shapes and the fibre the orientation of the cat relative to the distinguished direction selected by the gravitational force.

The global properties of bundles give them non-trivial topological poperties. Again the text's use of differential forms pays off in a presentation of the theory of de Rham complexes on manifolds where obstructions to solving differential equations near singularities are related to the toplogy via cohomology theory. This comes about in the way that an integrals over open sets on the manifold can be described as a pairing between differential forms and chains. The obstruction is both a topological "hole" and a nonintegrabiliy condition. The presentation at these points tends to be impressionistic rather than precise but, in a book of this ambition at the advanced undergradute to beginning graduate level, it is appropriate.

The other important feature of the connection is that, although it is itself not a covariant thing, it gives rise to important covariant tensors like the torsion and curvature tensors which contain coordinate independent information about the space. A nice example of the influence of relativity on classical mechanics here lies in the fact that just as fictitious forces in the classical theory arise by writing down a connection associated with a rotating frame so, in a formal, mathematical sense, the gravitational force itself arises as a feature of a connection on space-time. The latter connection is tied to the presence of matter and energy which create the geometry of the universe and produce curvature invariants which cannot be annulled by a careful choice of coordinates. Of course, classical physics takes place in a Euclidean space, which is flat, and the connection giving rise to fictitious forces can be trivialised by a choice of inertial coordinates.

The Gauss-Bonnet theorem links the curvature contained inside a closed path to holonomy. For instance, a tangent vector transported around a loop on a sphere according to the prescription of the natural connection returns to its starting point rotated by an amount depending on the total curvature integrated over the area enclosed by the loop. In the case of the connection on a principle bundle this is important in quantum physics where the Lie group is the gauge group. Geometric phases arise this way.

The idea that the dynamical trajectory of a mechanical system extremises a functional defined on the set of all possible geometrical paths is an important one which gives rise to the Euler-Lagrange equations. These express themselves variously as the relations between forces and accelerations for finite numbers of particles (ordinary differential equations), continua (partial differential equations) or as equations for geodesic paths in curved spaces. It's not the case though that there is always a natural correspondence between sytems of particles and forces and a Riemannian geometry. The Lagrangian function, defined on the tangent space to a manifold and thus a function of coordinates and velocities, is a formulation provides a more-or-less algorithmic way of setting up the equations of motion. It is modified to accommodate constraints using Lagrange multipliers. It also allows a discussion symmetries via Noether's theorem.

The reformulation of Lagrangian to Hamiltonian mechanics is achieved by the Legendre transformation which maps points in the tangent bundle of a manifold to points in the cotangent bundle, replacing velocities with momenta. A relevant aspect of classical mechanics not discussed in the book is the theory of light. Light was a phenomenon of deep interest to Isaac Newton. But it was Hamilton who produced the first mathematical theory incorporating an appropriate geometry. The vectorial direction of a light ray is transformed when reflected or refracted by a symplectic map, that is, one preserving a fundamental generalised "area" form just as in classical mechanics there are geometrical invariants (Poincaré-Cartan) preserved by the time evolution of the mechanics. So from the beginning light and particle mechanics were intimately connected through the idea of an action function which is extremised by the trajectory of the system in spite of the fact that light has no mass. In Quantum Mechanics this becomes part of the incorporation of the dual wave-particle nature of light into a single mathematical theory. In Quantum Field Theory the idea is pushed further with path integrals. The classical trajectory is the extremising trajectory but arbitrary deviations from it play a role in generating quantum corrections and renomalising the valuse of fundamnetal constants like mass and charge. The role of light is also crucial in the theories of Relativity where it follows geodesic paths of least distance between points. The geometry of distance is defined by the presence of matter and energy.

Symplectic geometry is very different to Euclidean geometry. It is defined by an anti-symmetric, nondegenerate form. Hence the relation between a subspace and its "perpendicular" subspace is not a complementary one. A Lagrangian subspace, for example, is equal to its own perpendicular subspace. Non-degeneracy allows an isomorphism between a subspace and its dual or, in the context of particle mechanics, between tangent spaces (velocities) and cotangent spaces (momenta). In mechanics a symplectic form can be thought of as an analogue of "area".

As with light propagation, the Hamiltonian flow can be regarded as a symplectic map, also called a canonical transformation, for which generating functions can be written down. One such generating function is Hamilton's principal function which generates the flow as a map from initial to final conditions and satisfies a partial differential equation, the Hamilton-Jacobi equation, whose characteristic curves are the mechanical trajectories.

Mechanical and electromechanical paradigms of Hamiltonian systems tend to be "solvable" in some sense. The most studied kind of tractability is algebraic complete integrability. In this case there are a sufficient number of integrals (constants) of motion, functions of the coordinates and momenta, with values fixed by the initial conditions of the trajectory, that the motion is confined to the intersection of hypersurfaces. For bounded motions, this intersection is diffeomorphic to a higher dimensional torus. Natural coordinates are the "action variables" which identify the specific torus and "angle variables" on the torus itself. In these variables the integrable Hamiltonian system behaves like a set of simple, linear oscillators. The most natural functions on such tori are the theta-functions of Riemann. They are universal and allow the construction of multi-periodic functions in many variables through which the solutions can be described.

The existence itself of an integral of motion or conserved quantity is a consequence of some underlying geometrical symmetry. Thus, linear momentum is conserved when there is translational symmetry, angular momentum when there is rotational symmetry. The author describes such applications but without going into the geometry and application of Lie Groups more generally. The symmetric top is the standard example. But, infact, the relations between the three principal moments of inertia of a rigid body have to be veyy special for the simplifications of integrabilty to apply. Another example is the Kovalevskaya top.

Outside these special relations, symmetry breaks down and the motion becomes analytically untractable. If we imagine the phase space of an integrable system as being packed with concentric tori to which motion is confined, then the perturbed, non-symmetric system sees a breakdown of this structure in a very specfic manner, at resonances relating frequencies of the periodic motions, described by the Kolmogorov-Adler-Moser theorem. In order to simplify the situation sufficiently, one reduces to a Poincaré return map by taking a slice through phase space and defining an iterative map thereon by integarting the fow close to a periodic orbit. Such iterative maps arise in many other similar situations and are the main object of study in the modern theory of non-linear dynamics.

Without the analytic control of the classical theory methods of study are drawn from topology and versal deformation theory. The book describes witrh some care the phenomenon of the homoclinic tangle. In this case a homoclinic orbit, asymptotically connecting the fixed point to itself, breaks up under a small pertubation into stable and unstable manifolds which wrap around each other creating a situation where a trajectory is thrown back and forth in a chaotic manner. The fixed point has disintegrated into an attractor which is a product of Cantor sets.

As a final application the author discusses the restricted three body problem. Here a particle of negligible mass moves in the field of a rotating binary mass system but on a line perpendicular to their plane of motion passing through the centre of mass. Motions can be described by a symbolic dynamics of semi-infinite sequences and chaotic motions are possible.

Attractive features of the book include the presence of numerous problems throughout the text and the flexibility of the mathematical approach. Fairly sophisticated material is developed as far as is necesssary but no further and without losing sight of the physical requirements. The chapters may appear to jump back and forth between mathematics and physics but this engages the reader's attention on both fronts and breaks up what might otherwise be a rather uniform terrain. There are no serious proofs but there is some need for mathematical maturity so this is probably a text suited to a striong final year undergraduate or beginning postgraduate as introductory reading. But is it not a place to learn the mathematics in proper detail.

My only criticism, apart from wondering where "due East" is at the noth pole (p.253), is that the index is not as helpful as it might be. Without a fairly thorough reading I would not have known from the index that, say, precesion is dealt with or that the Foucault pendulum (mentioned only in a chapter heading) makes more than one appearance.

The earliest book on mechanics still used at the undergradute level is probably Whitaker's [6]. That's a very complete summary of the situation at the start of the twentieth century. Something like Goldstein [4], from which I learnt mechanics as an undergraduate, is a treatment with an eye on the intervening development of Quantum Mechanics. Marsden's book [5] is much more geometrical and uses the language of bundles and topology. The two books by Arnol'd [1, 2] have been very influential and bring mechanics up to the end of the twentieth century.

Lam's book is more generalist, perhaps, in its treatment. It doesn't surpass these other texts and certainly provides less detail but is broader in scope both in terms of subject matter and in its use of concepts under several disguises in different situations.

References

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