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Bosonic Loop Soups and Their Occupation Fields

by

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Declarations

The work in this thesis was conducted by the author during the period October 2012 - September 2015 at the University of Warwick, in collaboration with Dr Stefan Adams. Elements of the introduction were previously used in my Masters theses [Dan11, Dan12]. Where we make use of work not our own, or rework established arguments, we write (for instance): "we follow [Szn12], p.13".

To the best of my knowledge, the material contained in this thesis is original and my own work except where otherwise stated. This thesis has not been submitted for a degree at any other university.

Abstract

We consider a model for random loops on graphs which is inspired by the Feynman–Kac formula for the grand canonical partition function of an ideal gas. We associate to this model a corresponding occupation field, which is a positive random field detailing the total time spent by loops at each vertex. We argue that well known critical phenomena for the ideal gas can be reinterpreted in terms of random variables of this occupation field. We also argue that higher order correlations, such as the existence of off-diagonal long-range order, can only be seen in the occupation field by studying a modified space–time model of loops. We provide an isomorphism theorem for this model to a complex Gaussian field, and derive a version of Symanzik's formula which describes the ideal gas interacting with a random background environment. Finally we consider the effect of interactions on the gas, and present a large deviations analysis of the cycle distribution of the loop model under two mean field Hamiltonians.

List of Notation

Mathematics

$[N]$ The set of integers $1, 2, \ldots, N$.

$\mathbb{R}_+$ Positive real numbers.

$\mathbb{R}_-$ Negative real numbers.

$\mathbb{H}$ Negative half plane.

$\mathrm{Spec}(A)$ Spectrum of A.

$\sqsubset$ Multi-subset.

$\#$ Cardinality of a discrete set.

Re, Im Real and imaginary part.

S_N Symmetric group on $[N]$.

Probability

$\mathbf{1}_{\{A\}}$ Indicator variable of an event A.

$\overset{(d)}{=}$ Equality in distribution.

$\overset{(d)}{\longrightarrow}$ Convergence in distribution.

δ_a Dirac δ-measure at a.

$\mathcal{M}_1$ Space of probability measures on a measure space.

Statistical Mechanics

β Inverse temperature.

h Chemical potential.

$Z_\Lambda(\beta, N)$ Canonical partition function .

$\Xi_\Lambda(\beta, h)$ Grand canonical partition function.

$\rho_c(\beta)$ Critical density.

$\tilde{\sigma}$ 1-particle reduced density matrix.

Graph Theory

(Λ, w, κ) Weighted graph.

λ Rate vector.

P (Sub)-stochastic transition matrix.

$\mathbf{P}, \mathbf{E}$ Law and expectation of a random walk.

l_x^T, l_x Local time at x.

m_Λ Spectral measure.

Loop Models

$(\Gamma, \mathcal{G})$ Measure space of loops.

μ_h Markov loop measure.

$\mu_{\beta,h}^B$ Bosonic loop measure.

$\mathbb{P}, \mathbb{E}$ Law and expectation of the Markov loop soup.

$\mathbb{P}^B, \mathbb{E}^B$ Law and expectation of the Bosonic loop soup.

$\mathcal{L}$ Occupation field of a Poisson loop soup.

$\overline{\mathcal{L}}$ Mean occupation.

$\underline{n} = (n^{(j)})_{j\geq 1}$ Cycle distribution.

$\Omega = \mathbb{R}^{\mathbb{Z}^d}$ Space of fields on $\mathbb{Z}^d$.

$\mathbf{T}$, $\mathbf{T}_N$ Continous and discrete torus.

X^N Space–time random walk.

$\mathcal{L}^N$ Space–time occupation field.

$\mathcal{L}^{\downarrow N}$ Projected occupation field.

Introduction

Statistical mechanics can be summarised as the study of macroscopic models via a microscopic description; that is, we define a model locally (on the micro-scale) and study global (or macroscopic) changes as we vary some model parameters. To motivate this description we turn to perhaps the best known model of statistical mechanics, the Ising model. This is a model for magnetism, where the polarity of any particular site in a material is assumed to be influenced by nearby sites. In the presence of a strong external magnetic field the polarity of the sites align and the material is considered to be magnetized, moreover there is long-range correlation between sites. As the strength of the external field reduces to zero, intuition suggests that this long-range correlation should also vanish to 0. This, however, is not always the case: depending on the model temperature the material can retain a magnetic force even in the absence of an external field, seen by the continued presence of long-range correlations. This is an example of a phase transition, where for one range of the parameters we see typical behaviour whilst for another range we see other behaviour entirely (in this case above the critical temperature there is no residual magnetism, whilst below it there is). Defining and proving the existence of phase transitions is of central importance to the study of statistical mechanics.

In the instance of the Ising model the search for a phase transition was first undertaken by Ising [Isi25] who showed that in the 1-dimensional model no such phase transition occurs. It would take the best part of two decades before Onsager [Ons44] provided the first proof that the Ising model in 2-dimensions does undergo a phase transition, and provided an exact solution for the critical temperature for the nearest neighbour model on the square lattice. Onsager's work is seen by many to be the starting point of the rigorous mathematical study of phase transitions, and to this day the Ising model remains one of the most studied models of statistical mechanics.

At the same time as Ising was working on his eponymous model[1], a different type of phase transition was being proposed by physicists Satyendra Nath Bose and Albert Einstein. They considered a gas of particles distributed in a box, where each particle is considered to be at a certain energy level. Einstein [Ein24] defined a model in which there was no particle interaction, and observed that at moderate temperatures

[1] That the model takes Ising's name is somewhat contentious since the model was first proposed by his supervisor Lenz, [Len20].

only a microscopic proportion of the atoms were designated to any given energy state; however, given a low enough temperature a macroscopic proportion occupied the lowest of the energy states, the ground state. Einstein remarked: "A separation is effected; one part condenses, the rest remains a 'saturated ideal gas'." Unlike the Ising model, where the physical phase transition was known to be exhibited in real magnets well before the mathematical phase transition was proven to occur, prior to Einstein's observation nobody had considered that such a condensation phenomenon could occur, and for some time little attention was paid to the problem. It was only after London [Lon38] observed that a similar transition occurs in liquid helium that Einstein's suggestion was given its due attention. Even then the occurence of this transition, which had now become known as the Bose–Einstein condensation phenomenon, remained a purely theoretical construct. It was not until 1995 that this would change. With the advent of new cooling technology two independent teams demonstrated the existence of the Bose–Einstein condensate: the group of Eric Cornell and Carl Wieman condensed a vapor of rubidium 87 atoms [CW02], and shortly after Wolfgang Ketterle's team condensed a gas of sodium-23 atoms. The breakthroughs of these two groups lead to all three physicists being awarded the 2001 Nobel Prize for Physics dedicated to "the achievement of Bose–Einstein condensation in dilute gases of alkali atoms, and for early fundamental studies of the properties of the condensates", [Nob01].

I Probabilistic Approaches to the Bose Gas

In the following section we give an overview of the mathematical study of the Bose gas, in particular highlighting several probabilistic approaches. We start our journey by introducing the classical model for the Bose gas, before deriving the Feynman–Kac formula, which is at the heart of the probabilistic analysis.

The Bose Gas As a Quantum Mechanical Model

The following section is adapted from our previous work, [Dan11, Dan12]. Before entering into the quantum world we recall some classical mechanics. The classical canonical ensemble is described as a collection of $N \gg 1$ particles in a box $\Lambda \subset \mathbb{R}^d$, $|\Lambda| < \infty$, $d \geq 1$. The state of the particles is an element of the phase space $\Gamma_\Lambda = (\Lambda \times \mathbb{R}^d)^N$ of pairs (x, p) where $x \in \Lambda^N$ correspond to particle positions, and $p \in \mathbb{R}^{dN}$ describe the momenta. The energy of a configuration $(x, p) \in \Gamma_\Lambda$ is described by a Hamiltonian

$$H_N(x,p) := \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{1\leq i<j\leq N} V\big(|x_i - x_j|\big), \qquad (x,p) \in \Gamma_\Lambda, \tag{0.1}$$

where m is the mass of a particle, and the potential $V : \mathbb{R}_+ \to \mathbb{R}$ describes how pairs of particles interact. The Boltzmann weight $\exp(-\beta H_N(x,p))$ describes the mass associated to the configuration (x,p), and we define the canonical partition function to be the average of these weights

$$Z_\Lambda(\beta, N) := \frac{1}{N!} \int_{\Gamma_\Lambda} \exp\big(-\beta H_N(x,p)\big) \mathrm{d}x \mathrm{d}p. \tag{0.2}$$

We note that the unexpected factorial term arises as a resolution of the Gibbs paradox, and is justified by the heuristic assumption that we cannot distinguish between particles [Ada06]; this is better justified in quantum mechanics, where bosons are known to be indistinguishable.

In quantum mechanics, the Heisenberg uncertainty principle dictates that it is not possible to simultaneously measure the position and momentum of a particle; resultingly it no longer makes sense to consider particle configurations as elements of the phase space Γ_Λ. Instead, the state of a system is described by a wave function $\psi(x)$, a complex valued function such that $|\psi(x)|^2$ describes a distribution of particle locations in Λ. The wave function plays the role of a distribution for the particles: in the case of a single particle without spin, this simply means the probability that the particle is in a measurable set $A \subset \Lambda$ is given to be $\int_A |\psi(x)|^2 \mathrm{d}x$. The wave function itself is defined to be a solution to the time-independent Schrödinger equation

$$E\psi(x) = H_N \psi(x),$$

where E is the *separation constant* describing the energy of ψ. The Hamiltonian H_N is given by a *Schrödinger operator* acting on $L^2(\Lambda^N)$

$$H_N \psi(x) := -\frac{\hbar^2}{2m} \sum_{i=1}^{N} \Delta_i^{(\mathrm{bc})} \psi(x) + \sum_{1 \le i < j \le N} V(|x_i - x_j|) \psi(x), \qquad \psi \in L^2\left(\Lambda^N\right), \tag{0.3}$$

where $\hbar \approx 1.05 \times 10^{-34}$ is Planck's constant, and $\Delta_i^{(\mathrm{bc})}$ is the Laplacian associated to particle i under some boundary conditions 'bc', which henceforth we assume to be periodic.

The definition of H_N is analogous to (0.1), where we replace the momenta with momentum operators $p_i \mapsto -i\hbar\nabla p_i$. For interacting bosons the Hamiltonian, H_N, acts on the space of symmetric wave functions (a consequence of bosons having integer spins): that is we only consider those wave functions that are equivalent under permutation of the particle indices. The eigenvalues of the operator H_N describe the possible energy states that the system can occupy, so the natural analogue of (0.2) is the quantum canonical partition function

$$Z_\Lambda(\beta, N) := \operatorname{Tr}_{L^2_+(\Lambda^N)}\Big(\exp\big(-\beta H_N\big)\Big), \tag{0.4}$$

where $L^2_+(\Lambda^N) \subset L^2(\Lambda^N)$ is the sub-Hilbert space of symmetric L^2-functions, and Tr denotes the trace of an operator. In working with the canonical ensemble we assert that the total number of particles is fixed, and given by $N \gg 1$; alternatively we may be interested in allowing the particle number to fluctuate around a mean particle number $\langle N \rangle$. To achieve this we introduce a chemical potential[2] $h \in \mathbb{R}$ and give a wave function $\psi \in L^2_+(\Lambda^n)$ the weighting $\exp(-\beta(H_n\psi - hn)) =: \sigma_\Lambda^{(n)}$. These weights are used to define the density matrix operator $\sigma_\Lambda = \oplus_{n=1}^\infty \sigma_\Lambda^{(n)}$, a Boltzmann distribution on the Fock space of symmetric wave functions, obtained by taking the direct sum of n-particle spaces, $\oplus_{n=1}^\infty L^2_+(\Lambda^n)$. The partition function is then defined by summing over all $n \geq 1$ as well as all symmetric wave functions

$$\begin{aligned}\Xi_\Lambda(\beta, h) &:= \sum_{n=0}^{\infty} \mathrm{Tr}_{L^2_+(\Lambda^n)}\, \sigma_\Lambda^{(n)} \\ &= \sum_{n=0}^{\infty} \mathrm{Tr}_{L^2_+(\Lambda^n)} \Big(\exp\big(-\beta(H_n - hn)\big)\Big) \\ &= \sum_{n\geq 0} Z_\Lambda(\beta, n) e^{\beta h n}. \end{aligned} \tag{0.5}$$

The resulting model is known as the grand-canonical ensemble. A classical analysis of the Bose gas now proceeds, see eg. [ZUK77], by analysing thermodynamic functions defined in terms of the partition functions $Z_\Lambda(\beta, N)$ and $\Xi_\Lambda(\beta, h)$. For instance, from equation (0.5) we infer the mean particle number is given by

$$\langle N \rangle = \frac{1}{\beta}\partial_h \log \Xi_\Lambda(\beta, h). \tag{0.6}$$

In the case of the ideal gas, Einstein [Ein24] derived an expression for the grand canonical partition function factorised as

$$\Xi_\Lambda(\beta, h) = \prod_{i\geq 0} \Big(1 - \exp\big(-\beta(\varepsilon_i - h)\big)\Big)^{-1}, \tag{0.7}$$

which is defined for $h < \varepsilon_0$, where $0 < \varepsilon_0 < \varepsilon_1 < \cdots$ describe the energy levels that the particles can occupy, and are the eigenvalues of the Laplace operator (under prescribed boundary conditions). In dimension $d = 3$, combining (0.7) with (0.6) and taking the thermodynamic limit $|\Lambda| \to \infty$ the particle density is seen to satisfy

$$\rho(\beta, h) := \lim_{|\Lambda|\to\infty} \frac{\langle N \rangle}{|\Lambda|} = \int_{\mathbb{R}^3} \frac{1}{e^{\beta(\frac{1}{2m}|x|^2 - h)} - 1} \mathrm{d}x,$$

[2] The standard notation for chemical potential is μ, however we reserve this letter for the loop measures which will be central to this thesis. The choice of h is not itself random, when considering models for magnetism the letter h is regularly used to denote an external field, which plays the same role as the chemical potential here

and for all $h < 0$ we have

$$\rho(\beta, h) < \left(\frac{2\pi\beta\hbar^2}{m}\right)^{-\frac{3}{2}} \zeta\left(\frac{3}{2}\right) =: \rho_c(\beta), \tag{0.8}$$

where $\zeta(s) := \sum_{j\geq 1} j^{-s}$ denotes the Riemann-zeta function. This suggests that the gas cannot achieve densities above a fixed critical density $\rho_c(\beta)$. Seemingly a paradox, we can explain away this bound by considering separately the density of particles in the lowest energy state. Taking the thermodynamic limit in such a way that $\langle N\rangle/|\Lambda| = \rho > 0$, so that $h = h_\Lambda$ is now a sequence of chemical potentials chosen to preserve this equality, then for $\rho > \rho_c(\beta)$ we can write $\rho = \rho_c(\beta) + \rho_0$, where

$$\rho_0 = \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} \frac{1}{\exp(\beta(\varepsilon_0 - h)) - 1},$$

which is the density of the ground state. We say that Bose–Einstein condensation (BEC) occurs when $\rho_0 > 0$, and refer to this as the condensate density.
The derivation above follows the classic approach of Einstein, which is valid for describing the phase transition of an ideal gas. For interacting gasses, however, the energy levels no longer factorise as single-particle energies, and Einstein's definition no longer has a meaning. A definition of BEC for interacting gases was first provided by Penrose and Onsager [PO56], who studied the 1-particle reduced density matrix. Just as the partition function was defined as the trace of the density matrix σ_Λ, the 1-particle reduced density matrix is given by the partial trace after integrating out all but one of the particles

$$\tilde{\sigma}_\Lambda := \sum_{n=0}^{\infty} n \operatorname{Tr}_{L^2_+(\Lambda^{n-1})} \sigma_\Lambda^{(n)}. \tag{0.9}$$

Following [LSSY05], for suitably nice potentials V at 'zero' temperature, i.e. $\beta = \infty$, for $x, x' \in \mathbb{R}^3$

$$\tilde{\sigma}_\Lambda(x, x') = N \int_{\Lambda^{N-1}} \psi_0(x, y_1, \ldots, y_N)\psi_0(x', y_1, \ldots, y_N) \mathrm{d}y_2 \cdots \mathrm{d}y_N, \tag{0.10}$$

where ψ_0 is the ground state wave function, which minimizes $\int \phi H_N(\phi)$. Taking the thermodynamic limit, such that $N/|\Lambda| \to \rho$, Penrose and Onsager said that the reduced density matrix has *off-diagonal long-range order* (ODLRO) if the largest eigenvalue of $\tilde{\sigma}_\Lambda$ is of the order N as $|\Lambda| \to \infty$. It can be shown, [PS08] pp.396-7, that this is in fact equivalent to the requirement that $\lim_{|x-x'|\to\infty} \lim_{|\Lambda|\to\infty} \tilde{\sigma}_\Lambda(x, x') \neq 0$.

Moreover, in the case of the ideal gas, this limit agrees with the condensate fraction

$$\lim_{|x-x'|\to\infty} \lim_{|\Lambda|\to\infty} \tilde{\sigma}_\Lambda(x,x') = \rho_0.$$

The above motivates the common definition that BEC is said to occur in an interacting gas, if and only if the 1-particle reduced density matrix exhibits ODLRO. Proving the existence of ODLRO remains a challenge, and has only been achieved in a handful of cases. Notably, [LS02] provides the only rigorous proof of BEC in the continuum for a class of trap potentials, whilst [DLS78, KLS88] prove BEC for a lattice gas at half filling, that is the density of the gas is equal to half the number of lattice sites.
To this point we have described 'classical' quantum mechanics: whilst the language of probability is used, at this level we need little probabilistic machinery. This changes in the next section where we provide a probabilistic formula for the partition function of the Bose gas.

Feynman–Kac Formulae

Again we follow the description in [Dan12]. Feynman-Kac formulae were introduced by Feynman [Fey48, Fey53] as a tool to make rigorous his abstract path integral. In the latter of these papers, Feynman derived a formula for the partition function of the Bose gas as an integral over a collection of particle trajectories, where these trajectories are distributed according to interacting Brownian bridges.
Working in the canonical ensemble, Feynman–Kac formulae allow us to derive stochastic representations for kernels of exponential operators. Given the operator $\exp(-tH_N)$, we wish to find a function $K_t(x,y)$ such that

$$\exp\Big(-tH_N\Big)f(x) = \int_{\Lambda^N} K_t(x,y)f(y)dy, \qquad f \in L^2\big(\Lambda^N\big). \tag{0.11}$$

To simplify notation we write $H := H_0 + V$ in place of H_N. In the simple case of the ideal gas, $H = H_0 = -\frac{1}{2}\Delta$ and in the infinite volume limit, it is well known that the kernel $K_t(x,y) = p_t(x,y)$ satisfying (0.11) is the heat-kernel

$$p_t(x,y) := (2\pi t)^{-\frac{d}{2}} \exp\left(-\frac{|x-y|^2}{2t}\right).$$

On realising that this is the transition kernel of a d-dimensional Brownian motion, the relationship between Hamiltonian operators and stochastic processes is less mystical. Feynman–Kac type formulae extend for interacting gases, $V \not\equiv 0$. A prototypical result is

Proposition 0.1. *For the Hamiltonian $H = -\frac{1}{2}\Delta + V$, with $V\colon \mathbb{R}^d \to \mathbb{R}$ bounded* *and smooth*

$$\exp(-tH)f(x) = \mathbb{E}_x\Big[\exp\Big(-\int_0^t V(B_s)ds\Big)f(B_t)\Big], \qquad g \in L^2(\mathbb{R}^d). \tag{0.12}$$

where $\mathbf{E}_x$ *is the expectation with respect to the Wiener measure* $\mathbf{P}_x$ *of a Brownian motion started at* $x \in \mathbb{R}^d$, $B_0 = x$.

A proof is given in [Dan12], whilst an in depth treatment of Feynman–Kac formulae under weaker assumptions is given in [LHB11]. The Feynman–Kac formula for the partition function of a Bose gas is then obtained by applying (0.12) inside the trace (0.4). In the following we let S_N denote the symmetric group on $[N] := \{1, \dots, N\}$, ie. the set of permutations $\pi\colon [N] \to [N]$, and write $\mathbf{P}^t_{x,y}[\,\cdot\,] = \mathbf{P}_x[\,\cdot\,, B_t = y]$ for the non-normalised Brownian bridge measure from x to y over time horizon $t > 0$. Whilst we assumed below (0.3) that the box has periodic boundary conditions, with suitable changes to the definition of $\mathbf{P}_x$ the following holds for free and Dirichlet boundary conditions as well.

Theorem 0.2 (Feynman–Kac Representation of the Bose Gas)**.** *Let* $H_N = H_0 + V$, *be the Hamiltonian of a Bose gas where* V *decays sufficiently fast. The partition function has the representation*

$$Z_\Lambda(\beta, N) =$$
$$\frac{1}{N!}\sum_{\pi\in S_N}\int_\Lambda dx_1 \cdots \int_\Lambda dx_N \times \bigotimes_{i=1}^N \mathbf{P}^\beta_{x_i,\pi(x_i)}\Big[\exp\Big(-\sum_{1\le i<j\le N}\int_0^\beta V(|B^{(i)}_s - B^{(j)}_s|)ds\Big)\Big] \tag{0.13}$$

See [Fey53] for the classical reference, or [Gin71] for a rigorous account. Feynman recognised (0.13) as the partition function of a probabilistic model of random permutations $\pi \in S_N$, whose law we denote $P_{\Lambda,N}$, and conjectured that the occurrence of BEC is signaled by the existence of macroscopic cycles in the random permutation model (i.e. those which grow with the volume $|\Lambda|$).

One approach to studying critical phenomena is through the analysis of thermodynamic functions such as the canonical specific free energy or the grand canonical pressure, defined respectively as

$$\begin{aligned} f_\rho(\beta) &:= \lim_{|\Lambda|\to\infty} -\frac{1}{\beta|\Lambda|}\log Z_\Lambda(\beta, N), \\ p(\beta,\mu) &:= \lim_{|\Lambda|\to\infty} \frac{1}{\beta|\Lambda|}\log \Xi_\Lambda(\beta,\mu), \end{aligned} \tag{0.14}$$

where in the case of the specific free energy it is assumed that the limit is taken such that it preserves the density $N/|\Lambda| \to \rho$. In either case, understanding the singularities of the thermodynamic function can be used to indicate the existence of a phase transition. No explicit formula can be derived for the free energy at a fixed particle density and temperature, but from using (0.13) Adams, Collevecchio and König [ACK11] derive a variational formula for $f_\rho(\beta)$ under general requirements on the potential V. The variational problem is posed over a space of probability measures which describe marked Poisson point processes on $\mathbb{R}^d$, where the marks are looped trajectories corresponding to the loops over which we integrate in the Feynman–Kac representation. In this description, BEC is recognised via a loss of probability mass in the minimiser of the variational problem, with the interpretation that such a probability distribution puts some mass on infinite cycles.

Feynman's notion of infinite cycles, or cycle percolation, was made rigorous in a series of papers by Sütő [Süt93, Süt02], who took as an order parameter the length ξ_1 of the cycle containing the element $1 \in [N]$. Writing $\mathbf{P}_{\Lambda,N}$ for the probability measure on S_N induced by (0.13), he showed that in the thermodynamic limit $N/|\Lambda| \to \rho$

$$\sum_{j\geq 1} \lim_{|\Lambda|\to\infty} \mathbf{P}_{\Lambda,N}[\xi_1 = j] \leq 1,$$

with strict inequality when $\rho > \rho_c(\beta)$, given in (0.8). The interpretation here is that there is a loss of probability mass, i.e. with non-zero probability the cycle size ξ_1 is infinite. In the second of the two papers this argument is strengthened to say that infinite cycles occur if and only if there is BEC; moreover it is claimed that the proof also holds for the mean-field gas, described by the Hamiltonian $H_N = H_0 + \frac{a}{|\Lambda|}N^2$ for some constant $a > 0$.

These papers show that in the ideal gas, presence of macroscopic cycles is equivalent to BEC and hence equivalent to ODLRO. However, to make the cycle order parameter valid for interacting gases a direct relationship to ODLRO must be derived. Letting $\rho(n)$ denote the density of particles belonging to cycles of length n,

$$\rho(n) := \lim_{|\Lambda|\to\infty} \frac{1}{|\Lambda|} n\, \mathbf{E}_{\Lambda,N}\big[\#\{\text{cycles } c \in \pi \text{ st. } |c| = n\}\big],$$

Ueltschi [Uel06a, Uel06b] considers the problem of finding a sequence of correlation functions $c_n(x)$ and $c_\infty(x)$ such that

$$\lim_{|\Lambda|\to\infty} \tilde{\sigma}_\Lambda(x,y) = \sum_{n\geq 1} c_n(x-y)\rho(n) + c(x-y)\rho(\infty),$$

where $\tilde{\sigma}_\Lambda$ is the reduced density matrix introduced in (0.10) and one would hope (at least it is commonly assumed) that $\rho(\infty) = \rho_0$ is the condensate fraction. This equality can be shown to hold in finite volume (i.e. before taking the Λ limit) for interacting gases, and the coefficients $c_{n,\Lambda}(x-y)$ are given as expectations of single particle trajectories from x to y, $x, y \in \Lambda$. At high temperatures, and for suitably fast decaying potentials, Ueltschi demonstrates that this can be carried through to the thermodynamic limit. On the other hand, he provides heuristic arguments suggesting that in a crystalline phase then one can simultaneously have $\rho_0 = 0$ and $\rho(\infty) > 0$.

Rather than working directly with the permutations which arise from the Feynman–Kac formula, we can work instead with partitions. To a permutation $\pi \in S_N$ we associate an integer partition $\lambda = \lambda(\pi)$, where $\lambda = (\lambda_i)_{i\geq 1}$ and λ_i is the number of cycles of length i in π. We associate to each partition its empirical shape measure, the rescaled Young tableaux $Q_\lambda(k) = N^{-1}\sum_{j\geq k}\lambda_j$, which describes a probability measure on $\mathbb{N}$. In [Ada08, Dan11] a large deviations analysis is undertaken for the shape measures in the thermodynamic limit $|\Lambda| \to \infty$, and a variational problem is derived. As in [ACK11], BEC is seen through the solution to this variational problem being a sub-probability measure. Whilst the analysis of shape measures for $P_{\Lambda,N}$ was novel to [Ada08], Vershik [Ver96] had previously performed a similar analysis for the partitions that arise from the momentum space description of the ideal Bose gas, that is the sequences $(n_i)_{i\geq 0}$, where n_i denotes the number of particles in energy state ε_i. Vershik demonstrates that in the thermodynamic limit the Young tableaux of the typical partition converges to a smooth curve, and identified an exact expression for the limit shape of the 'mean' scaled tableaux. A similar analysis is possible in the grand-canonical ensemble, with [Lew86, vdBLP88] considering the mean-field and hard core models from the momentum space description, and [BCMP05] analysing the mean-field model from the loop (Feynman–Kac) perspective.

If we consider instead the box Λ to be a subset of $\mathbb{Z}^d$, we can derive the lattice analogue of the partition function (0.13), where the probability measures $\mathbf{P}_x$ no longer denote Wiener measure, but rather the distribution of a continuous time simple random walk. Tóth [Tót93] considers the lattice gas with a discrete approximation to the Lennard–Jones potential, and shows that the grand canonical partition function $\Xi_\Lambda(\beta, h)$ is in fact equivalent to the partition function of the spin-1/2 Heisenberg ferromagnet. The specific choice of potential allows for a series of manipulations which rewrite the partition function as an expectation with respect to a new random permutation model: the random stirring, or interchange, process. This is a model of a time evolving random permutation, $(\pi_t)_{t\geq 0}$, where $\pi_t \colon \Lambda \to \Lambda$. Each edge in the graph Λ is equipped with a unit rate Poisson process, and if at time t the edge $(x, y) \in \Lambda$ 'rings', then we update $\pi_{t+} = (x, y) \circ \pi_t$. The partition function can then be equated to

$$\Xi_\Lambda(\beta, h) = \mathbb{E}\left[\prod_{n\geq 1}(1 + e^{\beta h n})^{l_\beta(n)}\right],$$

where $l_\beta(n)$ denotes the number of n-cycles in the permutation π_β. Once again the

Bose condensate can be related to macroscopic cycles, but moreover the equivalence to the spin-1/2 model means that infinite cycles also correspond to spontaneous symmetry breaking, and the Mermin–Wagner theorem [MW66]. A detailed survey of the random stirring process is the content of [Dan12].

In the next section we describe the Markov loop soup: a Poisson point process on a space of lattice loops, which will be the starting point for our own analysis of the Bose gas. We have already mentioned that [ACK11] considered the Bose gas as a marked Poisson point process on $\mathbb{R}^d$, where the marks are Brownian loops. A point process approach was also taken by Rafler [Raf09], for the ideal gas in $\mathbb{R}^d$. Rafler studies the Martin–Dynkin boundary of the point process: heuristically, the collection of all other point processes which locally resemble the one of interest. In the grand canonical ensemble, it is shown that this set contains only a single process, and says there is no phase transition. In the canonical and microcanonical ensembles, Rafler proves that the Martin–Dynkin boundary is a convex (non-singleton) set of mixed Poisson processes, and says a phase transition occurs. Rafler also considers geometric aspects of the 'typical' loop: such as the location of the barrycentre, and percolation questions. In some respects this has the closest similarity to our own work, where we will study the geometry of the Poisson point process via its associated occupation field, and relate the thermodynamic functions of the grand canonical Bose gas with correlations in this field.

II A Survey of Markov Loop Soups

Just as the probabilistic models described above have a physical derivation, the Markov loop soup also owes its conception to the physics community, where it arises via a functional integral description of a lattice model.

In [Sym66, Sym69], Symanzik provided a heuristic description of φ^4-quantum field theory in terms of a gas of interacting Brownian loops. On considering lattice field theories in place of Symanzik's continuum model, Brydges, Fröhlich and Spencer [BFS82] were able to make rigorous the connection between the two models. A concise version of this equivalence can be described for the Gaussian case.

Let P be the transition matrix of a symmetric random walk, $X = (X_t)_{t\geq 0}$, on a lattice box $\Lambda \subset \mathbb{Z}^d$, $d \geq 1$, and consider the Gaussian field described by

$$P^G(\mathrm{d}\varphi) = Z_\Lambda^{-1} e^{-\langle \varphi,(I-P)\varphi\rangle} \mathrm{d}\varphi,$$

with $\mathrm{d}\varphi = \prod_{x\in\Lambda} \mathrm{d}\varphi_x$ and Z the normalisation constant which makes P^G a probability measure. The covariance of the field φ can be related to the local time of the random walk

$$\text{Cov}(\varphi_x, \varphi_y) \;=\; (I-P)^{-1}_{xy} \;=\; \mathbf{E}_x\left[\int_0^\infty \mathbf{1}_{\{X_t=y\}}\mathrm{d}t\right] \;=:\; G_{xy}, \tag{0.15}$$

which is immediate from $G_{xy} = \sum_{n\geq 1} P^n_{xy}$, Corollary 1.5. Symanzik's formula provides a deeper understanding of the link between Gaussian fields and random walks, notably relating the partition function Z_Λ of the Gaussian field to a sum over families of random loops.

Theorem 0.3 (Symanzik's Formula)**.** *The partition function Z_Λ of the law P^G can be expressed as*

$$Z_\Lambda = \sum_{n\geq 0} \frac{1}{n!}\left(\sum_{x\in\Lambda}\int_0^\infty \frac{1}{t}\mathbf{P}_x[X_t = x]\mathrm{d}t\right)^n \tag{0.16}$$

See [BFS82]. This formula holds in the greater generality of the partition function of a φ^4-theory, and we return to this in Section 3.3.1 where we discuss an interpretation for the Bose gas. Inspired by the work of Symanzik and Brydges et al., Dynkin [Dyn83] provided an extension of (0.15) for correlations for the square of a Gaussian field. Defining the *local time* at 'infinite time' of a random walk $X = (X_t)_{t\geq 0}$ to be the random variable $l_x = \int_0^\infty \mathbf{1}_{\{X_s = x\}}\mathrm{d}t$, then under the measure $\mathbf{P}_{xy}[\,\cdot\,] = \int_0^\infty \mathbf{P}_x[\,\cdot\,, X_t = y]$, Dynkin's theorem says.

Theorem 0.4 (Dynkin's Isomorphism)**.** *For any bounded measurable $F\colon \mathbb{R}^\Lambda \to \mathbb{R}$*

$$\mathbf{E}_{xy}\otimes E^G\left[F\left(l_x + \frac{1}{2}\varphi_x^2\right)\right] = E^G\left[\varphi_x\varphi_y F\left(\frac{1}{2}\varphi_x^2\right)\right].$$

See [Szn12], pp.35-6. An extension to complex Gaussian measures was given by Brydges, [Bry92]. Symanzik's work for Euclidean quantum fields, and Feynman's description of the Bose gas are both examples of the functional integral approach to statistical mechanics. Other important examples are Aizenman's random walk description of the Ising model [Aiz82], and the more recent work of Brydges and Slade (along with an ensemble of collaborators) regarding the functional integral description of the self avoiding random walk, for a survey see [BIS09].
Independently of the relevance to statistical mechanics, ensembles of loops have been the focus of recent work in probability. Letting Γ_t denote the collection of continuous time loops on a graph Λ: i.e. càdlàg paths $\gamma\colon [0,t] \to \Lambda$ with $\gamma(0) = \gamma(t)$, Le Jan defined a measure μ_Λ on the space $\Gamma = \cup_{t>0}\Gamma_t$ by

$$\mu_\Lambda(G) := \sum_{x\in\Lambda}\int_0^\infty \frac{1}{t}\mathbf{P}_x[G,\, X_t = x]\mathrm{d}t, \tag{0.17}$$

with $\mathbf{P}_x$ the law of a symmetric random walk. In [LeJ10, LeJ11], Le Jan provides a comprehensive analysis of the *Markov loop soup*: the family $(\mathbb{P}_\alpha)_{\alpha>0}$ of Poisson point processes on the set Γ described by the intensity measures $\alpha\mu_\Lambda$. A configuration $S \subset \Gamma$ (where we abuse notation and write '$\subset$' although S may in fact be a multiset under the law $\mathbb{P}_\alpha$) defines a random field $\mathcal{L} = (\mathcal{L}_x)_{x\in\Lambda}$ called the occupation field, where

$$\mathcal{L}_x = \mathcal{L}_x(S) := \sum_{\gamma\in S} \int_0^{|\gamma|} \mathbf{1}_{\{\gamma(s)=x\}} \mathrm{d}s, \qquad x \in S,$$

with $|\gamma|$ the length of the loop (i.e. the unique $t > 0$ for which $\gamma \in \Gamma_t$). Amongst many results regarding the properties of $\mathcal{L}$, Le Jan provides an interpretation to Symanzik's formula and Dynkin's isomorphism via the loop soup.

Theorem 0.5 (Le Jan's Isomorphism)**.** *Let $G = (G_{xy})_{x,y\in\Lambda}$ be the Green's function associated with the random walk $\mathbf{P}$, and P^G the law of the associated centred Gaussian field. Then*

$$\Big((\tfrac{1}{2}\varphi_x^2)_{x\in\Lambda},\, P^G \Big) \overset{\text{(d)}}{=} \Big((\mathcal{L}_x)_{x\in\Lambda},\, \mathbb{P}_{\alpha=\frac{1}{2}} \Big).$$

That is, the occupation field of the Markov loop soup at intensity $\alpha = \frac{1}{2}$ is equal in distribution and the square of a Gaussian field.

See [Szn12], pp.90–1. Whilst the essence of this theorem was already present in the work of Symanzik and Brydges et al., their work does not recognise the right hand side of (0.16) as the normalising constant of a Poisson point process.

It would be amiss at this point not to mention that prior Le Jan's work, Lawler and Werner had already sparked interest in loop soups. In [LW04] they consider the loop measure[3] (0.17) but with $\mathbf{P}_x$ now the distribution of a Brownian motion in $\mathbb{C}$, which they relate to the Schramm–Leowner evolution (SLE) processes. Later papers of Sheffield and Werner [SW12] and Qian and Werner [QW15] relate the Brownian loop soup to the conformal loop ensembles (CLE), a non-Poissonian collection of mutually and self avoiding random loops in $\mathbb{C}$: these arise as the outer boundaries of clusters in the Brownian loop soup.

Another variant on the loop soup is to consider the discrete time random walk loop soup. An n-loop is a finite sequence $z = (z(i))_{i=0}^n$ with $z(i) \in \Lambda$, and $z(0) = z(n)$. A discrete loop measure, μ^D, is defined on the collection of all discrete loops in Λ via

$$\mu^D(z) := \frac{1}{n} P_{z(0)z(1)} \cdots P_{z(n-1)z(n)} P_{z(n)z(0)},$$

[3]Strictly speaking, neither Le Jan or Lawler and Werner study the measure μ, rather they consider an 'unrooted' version, in which the loop forgets its starting position. This distinction does not affect many of the properties of the soup, in particular the occupation field, and so we gloss over this detail.

with P a symmetric transition matrix on Λ; basic properties of this measure are outlined in [LL10]. Lawler and Trujillo Ferreras [LTF07] proved that the Brownian loop soup can be derived as the scaling limit of a discrete time random walk loop soup on $\mathbb{Z}^2$, whilst Lawler and Perlman [LP14] define the occupation field of discrete loops, from which they derive an alternative proof of Theorem 0.5. This proof is adapted in [Cam15] to provide an isomorphism theorem between a Gaussian free field (not the square of the field!) and the loop occupation field where the sign of the occupation field $\mathcal{L}_x$ is changed according to an Ising type interaction.
Another random field associated with the discrete (and for that matter the continuous) loop soup is the covering field $\mathcal{C} = (\mathcal{C}_e)_{e\in\Lambda} \in \{0,1\}^\Lambda$, which is now indexed by the edges of the graph. For a configuration of discrete loops S we set

$$\mathcal{C}_e = \begin{cases} 1 & \text{if } \exists\, z \in S \text{ st. } e \in z, \\ 0 & \text{else.} \end{cases}$$

That is, the edge e is open, $\mathcal{C}_e = 1$, if and only if there is a loop which crosses it. When $\Lambda = \mathbb{Z}^d$, $d \geq 1$, as for Bernoulli bond percolation, we can define the probability $\theta(\alpha)$ that under the measure $\mathbb{P}_\alpha$ the cluster of $\mathcal{C}_e$ which contains the origin has infinitely many edges. Le Jan and Lemaire [LeJL13] prove that θ is increasing in $\alpha > 0$, and via a simple coupling with bond percolation provide a lower bound on the critical intensity $\alpha_c \coloneqq \inf\{\alpha > 0 : \theta(\alpha) > 0\}$. Chang and Sapozhnikov [CS14] proved that for $d = 1, 2$, $\alpha_c = 0$, whilst for $d \geq 3$ the phase transition is non-trivial: $\alpha_c > 0$, and using a coupling to the Gaussian free field Lupu [Lup14] proved that in fact $\alpha_c > \frac{1}{2}$.

III Summary of Contents and Structure

In the preceding sections we saw how models for random loops have arisen in two distinct contexts: the probabilistic analysis of the Bose gas, and the study of Gaussian fields and isomorphism theorems. Introducing a *Bosonic* loop measure, our analysis aims to concurrently develop the literature of the Bose gas, and loop soups. We approach these topics from two directions:

- How do functionals of the Bosonic occupation field relate to the thermodynamic properties of the ideal gas? Moreover, can we characterise BEC in terms of behaviour of the occupation field?
- To what extent can we carry through the analysis of the occupation field under μ_Λ to the Bosonic loop measure? In particular, does the Bosonic loop measure also induce an isomorphism theorem to a Gaussian free field?

The loop measure we consider is given by

$$\mu^{B}_{\beta,h,\Lambda}(G) := \sum_{j\geq 1}\sum_{x\in\Lambda} \frac{e^{\beta hj}}{j}\mathbf{P}^{\beta j}_{xx}[G].$$

This differs from the measure μ in two aspects. The most immediate difference is that rather than allowing loops of all durations $t > 0$, we restrict to those loops whose duration is exactly an integer multiple of $\beta > 0$. The second distinction is the addition of the term $e^{\beta hj}$, and we will look to study the behaviour of the loop model as we vary $h \leq 0$. This is in contrast to Le Jan, who considers varying the intensity with the factor α which is independent of the loop lengths.
Letting $\Gamma = \cup_{j\geq 1}\Gamma_{\beta j}$ denote the space of all β loops, we will see that

$$\exp\left(\mu^{B}_{\beta,h,\Lambda}(\Gamma)\right) = \Xi_{\Lambda}(\beta, h),$$

where Ξ is now the grand canonical partition function of an ideal lattice Bose gas. Interpreting the partition function on the right hand side as the normalisation constant of a probability measure, we recognise this as none other than the normalisation of a Poisson variable with intensity $\mu^{B}_{\beta,h,\Lambda}(\Gamma)$. Moreover, letting S denote the Poisson point process induced by $\mu^{B}_{\beta,h,\Lambda}$, we have

$$\mathbb{P}\left[\sum_{\gamma\in S}|\gamma| = \beta N\right] = \frac{e^{\beta hN}Z_{\Lambda}(\beta, N)}{\Xi_{\Lambda}(\beta, h)},$$

which should be compared to (0.5). The above formulae, which we prove in Section 1.2 and Section 2.2 respectively, give the interpretation of the loop soup as a *poissonization* of the canonical ensembles of the Bose gas, which is exactly to say that it describes the grand canonical ensemble.

Thesis Outline

In Chapter 1 we formally introduce the Bosonic loop measure, and its related loop soup and occupation field. In this section we also clarify our definition of graphs and their spectra, and detail the sense in which we will consider thermodynamic limits in this thesis.
We start our analysis of the Bosonic occupation field in Chapter 2 where we consider the mean occupation on the graph, and prove that in the limit this converges to a degenerate distribution. In turn this is related to the density of the ideal Bose gas, and we provide a definition for BEC of an ideal Bose gas on a graph.
In Chapter 3 we address the problem of finding an isomorphism theorem for the Bosonic occupation field; following the approach of Le Jan, we derive the Laplace transform for the occupation field, but see that in the case of the Bosonic measure

this does not agree with that of a Gaussian process. Introducing a different *space-time* loop measure we show that this measure does have a description in terms of complex Gaussian fields. Further we see that the space-time occupation field provides an interpretation to the 1-particle reduced density matrix of the ideal gas. Having established several results for the ideal gas, in Chapter 4 we consider possible Hamiltonians defined on the occupation field, in particular focusing on two mean-field models. We present a large deviations analysis for the mean-field models, focused on deriving expressions for the critical density.
Finally, Chapter 5 gives an overview of further topics for consideration, and provide some closing remarks.
Whilst for the most part the text is self contained and can be read in a linear manner, we defer the proofs of some technical statements to a series of appendices.

Chapter 1

Definitions and Preliminary Results

We commence by introducing some basic terminology and definitions. In the first section we describe graphs and their associated Markov processes, we introduce the local time and Green's function of a process, and define the sense in which we will take thermodynamic limits of graphs. In the second section, we define formally both the Markov and Bosonic loop measures, their associated Poisson processes, and occupation fields.

1.1 Random Walks on Graphs, and Their Limits

Throughout this thesis when we refer to a graph we will mean not only the graph structure (i.e. edge and vertex sets), but also to a random walk defined on the structure. As a consequence, we define a graph to be a triple (Λ, w, κ), with Λ a finite set, $w \colon \Lambda \times \Lambda \to \mathbb{R}_+$ a *weight function*, and $\kappa \colon \Lambda \to \mathbb{R}_+$ the *killing vector*, where we use $\mathbb{R}_+$ to denote the positive reals. The set Λ corresponds to the vertices of the graph, whilst for pairs $x, y \in \Lambda$ with $w_{xy} := w(x, y) > 0$, we say there is an edge from x to y. The weights themselves, together with the vector κ, determine a family of Markov processes, as described in the coming section. To simplify notation, henceforth we will simply write Λ to denote the triple $\Lambda = (\Lambda, w, \kappa)$.

1.1.1 Weighted Graphs and Their Markov Generators

A graph induces a family of Markov processes whose jump distributions are determined by the functions w, κ. To facilitate the definition we enlarge the vertex set to $\Lambda^* = \Lambda \cup \{\dagger\}$, where the additional vertex $\dagger$ is called the *cemetery state*. The discrete time Markov chain $(Z^*_n)_{n\geq 0}$ on Λ^* is defined via the stochastic matrix

$P^* = (P^*_{xy})_{x,y\in\Lambda^*}$ defined for $x, y \neq \dagger$ to be

$$P^*_{xy} = \frac{w_{xy}}{\lambda_x}, \qquad P^*_{x\dagger} = \frac{\kappa_x}{\lambda_x}, \qquad P^*_{\dagger y} = 0, \qquad P^*_{\dagger\dagger} = 1,$$

where $\lambda_x := \kappa_x + \sum_{y\in\Lambda} w_{xy}$ for $x \neq \dagger$, and we set $\lambda_\dagger = 0$. We refer to $\lambda\colon \Lambda \to \mathbb{R}_+$ as the *rate vector*. In turn, P^* is used to define a continuous time Markov process on Λ^* with unit exponential jump rates. Denoted $(X^*_t)_{t\geq 0}$, the process is determined by its generator $(P^* - I)$, with I the $|\Lambda^*| \times |\Lambda^*|$ identity matrix. We also define $(\overline{X}^*_t)_{t\geq 0}$ with variable exponential jump rates, which leaves site $x \in \Lambda^*$ at rate λ_x: it is determined by the generator $Q^* = \lambda(P^* - I)$. In the case that $\kappa \not\equiv 0$, all three processes above will almost surely visit the site $\dagger$, after which we say that they are *killed*.

We work throughout with the induced sub-stochastic processes on Λ, and refer to these as *random walks*. We define the law and expectation of the processes $Z, X, \overline{X}$ conditioned to start from $x \in \Lambda$ by $\mathbf{P}_x$, $\mathbf{E}_x$, and writing P, Q for the cofactors of P^* and Q^* obtained by deleting the row and column corresponding to $\dagger$, we have

$$\mathbf{P}_x\big[Z_n = y\big] = \big(P^n\big)_{xy}, \qquad \mathbf{P}_x\big[X_t = y\big] = \big(e^{t(P-I)}\big)_{xy}, \qquad \mathbf{P}_x\big[\overline{X}_t = y\big] = \big(e^{tQ}\big)_{xy}.$$

We stress that these are not true probability measures, in that if $\kappa \not\equiv 0$, then summing over $y \in \Lambda$ the expressions may total less than 1. We briefly describe the standard coupling of the walks $X, \overline{X}$ to Z. Given a path $Z = (Z_n)_{n\geq 1}$ distributed according to $\mathbf{P}_x$, define $J_0 = 0$, and $J_n \sim \mathrm{Exp}(1)$ i.i.d. exponential variables, $n \geq 1$. Then the path

$$Y_t = Z_n, \qquad \text{for } J_n \leq t \leq J_{n+1}$$

is equal in distribution to X under $\mathbf{P}_x$. If instead $I_0 = 0$ and $I_n \sim \mathrm{Exp}(\lambda_{Z_n})$ are independent and exponentially distributed according to the rate of the current state, the resulting path Y_t has the distribution of $\overline{X}$ under $\mathbf{P}_x$. Throughout we will assume that graphs are loop free, $w_{xx} = 0$ for all $x \in \Lambda$, and *irreducible*: for all $x, y \in \Lambda$ there is an $n \geq 0$ such that $P^n_{xy} > 0$.

1.1.2 Random Walk Local Time and the Green's Function

For $x \in \Lambda$ we define the *local time* at x of the walk $\overline{X}$ to be the random variable

$$l^T_x\big(\overline{X}\big) := \int_0^T \mathbf{1}_{\{\overline{X}_s = x\}} \mathrm{d}s.$$

We define the local time of X similarly, and replacing the integral with a sum, the local time of Z. Similarly we define the local time 'at infinity' by

$$l_x\left(\overline{X}\right) = \int_0^\infty \mathbf{1}_{\{\overline{X}_s = x\}} \mathrm{d}s,$$

where as before we can interchange $\overline{X}$ for either of X or Z; this variable exists as an extended real number, and satisfies.

Proposition 1.1. *Suppose $\kappa \not\equiv 0$, then for $x, y \in \Lambda$, l_y is $\mathbf{P}_x$-a.s. finite, and $\mathbf{E}_x[l_y] < \infty$. Conversely, if $\kappa \equiv 0$, then $\mathbf{P}_x$-a.s. $l_y = \infty$.*

Proof. The proof is classical, and for brevity we prove it only in the case w_{xy}, $\kappa_x > 0$ for all $x, y \in \Lambda$; see [LPW09] Lemma 1.13 pp.11-2 for a more detailed proof (though not in the context of local times). In light of the standard coupling it suffices to prove the proposition only in the case of the discrete walk Z, so that $l_y = \#\{n \geq 0 : \ Z_n = y\}$. Moreover, we couple Z with the walk Z^* on the space $\Lambda \cup \{\dagger\}$, and defining $T = \inf\{n \geq 0 : \ Z^*_n = \dagger\}$, then

$$\mathbf{E}_x\left[\sum_{y \in \Lambda} l_y\right] = \mathbf{E}_x[T - 1].$$

Let $\varepsilon := \inf_y P^*_{y\dagger} > 0$, which is strictly positive due to our assumption $\kappa_x > 0$ for all $x \in \Lambda$. In particular, for all $k \geq 0$, $\mathbf{P}_x[Z_{k+1} \neq \dagger \,|\, Z_k \neq \dagger] \leq (1 - \varepsilon)$. Then

$$\begin{aligned}
\mathbf{E}_x[T] &= \sum_{n \geq 0} \mathbf{P}_x[T > n] \\
&= \sum_{n \geq 0} \mathbb{P}_x[Z^*_1, \ldots, Z^*_n \neq \dagger] \\
&\leq \sum_{n \geq 0} (1 - \varepsilon)^n \\
&= \varepsilon^{-1}
\end{aligned}$$

Hence $\mathbf{E}_x[l_y] \leq \sum_y \mathbf{E}[l_y] \leq \varepsilon^{-1} < \infty$.

In the case $\kappa \equiv 0$, let $T_y := \inf\{n \geq 0 \colon Z_n = y\}$, then the same proof as above asserts that $\mathbf{E}_x[T_y] < \infty$, and in particular $\mathbf{P}_x[T_y < \infty] = 1$. Consequently, not only does Z_n almost surely visit y, it visits infinitely often, and hence $\mathbb{E}_x[l_y] = \infty$. $\square$

Henceforth we say that the graph Λ is recurrent if $\kappa \equiv 0$, else we say it is transient. The *Green's function* associated to a walk $\overline{X}$ is the matrix of expected local times at infinity

$$G_{xy}\left(\overline{X}\right) := \mathbf{E}_x\left[l_y\left(\overline{X}\right)\right], \qquad x\, y \in \Lambda, \tag{1.1}$$

which exists as an extended real number. Defining $G_{xy}(X)$, $G_{xy}(Z)$ analogously, if the graph is recurrent then

$$G_{xy}\big(\overline{X}\big) = G_{xy}\big(X\big) = G_{xy}\big(Z\big) = \infty.$$

The following proposition allows us to extend this equality to the transient case. In the following we use $\overset{(d)}{=}$ to denote equality in distribution.

Proposition 1.2. *For $x, y \in \Lambda$, under the law $\mathbf{P}_x$*

$$l_y\Big(\overline{X}\Big) \overset{(d)}{=} \lambda_y^{-1} l_y(X).$$

Proof. It suffices to show that the two variables have the same cumulative distribution function: $\mathbf{P}_x\Big[l_y^T\Big(\overline{X}\Big) \leq t\Big] = \mathbf{P}_x\big[\lambda^{-1} l_y^T(X) \leq t\big]$, $t \in \mathbb{R}$. Let N (respectively $\overline{N}$) denote the total number of visits that X (resp. $\overline{X}$) makes to y in time T. From the standard coupling $N \overset{(d)}{=} \overline{N}$, so

$$\begin{aligned}
\mathbf{P}_x\Big[l_y^T(\overline{X}) \leq t\Big] &= \sum_{n=0}^{\infty} \mathbf{P}_x\Big[l_y^T(\overline{X}) \leq t \,\Big|\, \overline{N} = n\Big] \mathbf{P}_x\Big[\overline{N} = n\Big] \\
&= \sum_{n=0}^{\infty} \mathbf{P}_x\Big[l_y^T(\overline{X}) \leq t \,\Big|\, \overline{N} = n\Big] \mathbf{P}_x\big[N = n\big],
\end{aligned}$$

so it suffices to show: $\mathbf{P}_x\big[l_y^T\big(\overline{X}\big) \leq t \,\big|\, \overline{N} = n\big] = \mathbf{P}_x\big[\lambda_y^{-1} l_y^T(X) \leq t \,\big|\, N = n\big]$, $n \geq 0$. But this follows since on the event $\overline{N} = N = n$, the coupling gives

$$l_y^T\Big(\overline{X}\Big) = \sum_{k=1}^{n} I_k = \lambda_y^{-1} \sum_{k=1}^{n} J_k = \lambda_y^{-1} l_y^T(X),$$

where $I_k \sim \mathrm{Exp}(\lambda_y)$ and $J_k \sim \mathrm{Exp}(1)$ are all i.i.d. and we used the scaling relation for exponential variables: $\mathrm{Exp}(\lambda_y) = \lambda_y^{-1}\mathrm{Exp}(1)$. ☐

Since equality in distribution implies that the expectations agree, it follows that

$$G_{xy}\big(\overline{X}\big) = \lambda_y^{-1} G_{xy}\big(X\big) < \infty. \tag{1.2}$$

Moreover, since $\mathbf{E}[\mathrm{Exp}(1)] = 1$, then

$$G_{xy}(X) = G_{xy}(Z).$$

1.1.3 Graph Spectra and Spectral Convergence

In this section we provide several basic facts about the spectra of the matrices P, Q, as well as describing the notion of graph convergence which will be used in

Chapter 2. For a square matrix, A, we denote $\mathrm{Spec}(A)$ for its spectrum, i.e. the set of eigenvalues of A.

Theorem 1.3. *The spectrum of P satisfies*

$$\mathrm{Spec}(P) \subset \{z \in \mathbb{C} : |z| \leq 1\}.$$

Furthermore, $\mathrm{Spec}(P) \subset \{z \in \mathbb{C} : |z| < 1\}$ *if and only if* $\kappa \not\equiv 0$.

Proof. Since Λ is irreducible, P satisfies the conditions of the Perron–Frobenius theorem, [HJ13] p.534. Defining the *spectral radius* to be $\rho = \max\{|\eta| : \eta \in \mathrm{Spec}(P)\}$, there is a positive vector $v_x \geq 0$ for which ρ is a corresponding eigenvalue: $Pv = \rho v$, and

$$0 \leq \min_x \sum_y P_{xy} \leq \rho \leq \max_x \sum_y P_{xy} \leq 1.$$

If $\kappa \equiv 0$, then $\sum_y P_{xy} = 1$ for all $x \in \Lambda$, and so 1 is an eigenvalue, and $\rho = 1$. Now suppose $\kappa \not\equiv 0$, and that there is a positive vector v such that $Pv = v$; in particular since Λ is loop free we have $v_x = \sum_{y \neq x} P_{xy} v_y$, for any $x \in \Lambda$. Choosing z such that v_z is maximal

$$v_z = \sum_{y \neq z} P_{zy} v_y \leq v_z \sum_{y \neq z} P_{zy} \leq v_z,$$

which is a contradiction unless both inequalities are in fact equalities. But since v is positive, $\sum_{y \neq z} P_{zy} v_y = v_z \sum_{y \neq z} P_{zy}$ holds only if v is a constant, $v \equiv c > 0$. But then for any $x \in \Lambda$

$$c = v_x = c \sum_y P_{xy},$$

so that P is stochastic, which contradicts $\kappa \not\equiv 0$. □

We say that a random walk is *reversible* if it satisfies

$$\lambda_x P_{xy} = \lambda_y P_{yx}, \qquad x,\, y \in \Lambda.$$

Corollary 1.4. *If P is reversible, then the eigenvalues of P are real, and* $\mathrm{Spec}(P) \subset [-1, 1]$. *If in addition $\kappa \not\equiv 0$, then* $\mathrm{Spec}(P) \subset (-1, 1)$.

Proof. In light of the previous theorem it suffices to prove that P has real eigenvalues. Defining the inner-product

$$\langle u, v \rangle_\lambda := \sum_x \lambda_x u_x v_x, \qquad u, v \in \mathbb{R}^{|\Lambda|},$$

we note that P is self-adjoint for this inner product

$$\langle u, Pv\rangle_\lambda = \sum_x \lambda_x u_x \left(\sum_y P_{xy} v_y\right) = \sum_y \lambda_y \left(\sum_x P_{yx} u_x\right) v_y = \langle Pu, v\rangle_\lambda.$$

It follows immediately that P has real eigenvalues. □

As a consequence we also obtain the following representation of the Green's function of the random walk.

Corollary 1.5. *If $\kappa \not\equiv 0$, then the Green's function is given by*

$$G(X) = G(Z) = (I - P)^{-1}.$$

Proof. Working with the discrete walk we note

$$G_{xy} = \mathbf{E}\left[\sum_{n=0}^{\infty} \mathbf{1}_{\{Z_n = y\}}\right] = \sum_{n=0}^{\infty} \mathbf{P}[Z_n = y] = \sum_{n=0}^{\infty} P^n_{xy}.$$

From Theorem 1.3 the spectral radius of P is strictly less than 1, and so the matrix power series above converges to $(I - P)^{-1}$ by Proposition B.20. □

Turning to the generator of a continuous time process, we observe that it is no longer possible to find a uniform bound on the spectrum. We denote $\mathbb{H} := \{z \in \mathbb{C} : \operatorname{Re}(z) \le 0\}$.

Theorem 1.6. *Let $\lambda^* = \max_{x\in\Lambda} \lambda_x$. The spectrum of Q satisfies*

$$\operatorname{Spec}(Q) \subset \{z \in \mathbb{C} : |z + \lambda^*| \le |\lambda^*|\} \subset \mathbb{H}.$$

Further, if $\kappa \not\equiv 0$ then Q is non-singular, and $\operatorname{Re}(\eta) < 0$, for all $\eta \in \operatorname{Spec}(Q)$.

Proof. The first part is an application of the Geršgorin circle theorem, which gives that

$$\operatorname{Spec}(Q) \subset \bigcup_{x\in\Lambda} \left\{z \in \mathbb{C} : |z + \lambda_x| \le \left|\lambda_x \textstyle\sum_{y\in\Lambda} P_{xy}\right|\right\},$$

see [HJ13] pp.388-9. Since P is (sub)-stochastic, $\sum_y P_{xy} \le 1$, and

$$\subset \bigcup_{x\in\Lambda} \{z \in \mathbb{C} : |z + \lambda_x| \le |\lambda_x|\},$$

which in turn is a subset of the largest disc centred at λ^*.
For the case $\kappa \not\equiv 0$, we appeal to [HJ13] Corollary 6.2.9 p.399, which in our context says that a matrix is non-singular so long as it is irreducible, diagonally dominant $|Q_{xx}| \ge \sum_{y\in\Lambda\setminus\{x\}} |Q_{xy}|$, and strictly dominant in at least one index: there exists

$x \in \Lambda$ such that $|Q_{xx}| > \sum_{y \in \Lambda \setminus \{x\}} |Q_{xy}|$. We have already assumed that P (and hence Q) is irreducible, and choosing $x \in \Lambda$ such that $\kappa_x \neq 0$

$$Q_{xx} = \lambda_x > \lambda_x \sum_{y \in \Lambda \setminus \{x\}} P_{xy} = \sum_{y \in \Lambda \setminus \{x\}} Q_{xy}. \qquad \square$$

As we will see in Chapter 2, several statistics of loop occupation fields are determined entirely by the spectrum of the graph. Consequently in taking graph limits it will often suffice only to study the limit of the spectra; in the following we provide the notion of convergence which will be used in later chapters.

Definition 1.7. *The (empirical) spectral measure of a finite graph Λ is defined as the measure m_Λ on $\mathbb{H}$ equipped with the Borel σ-algebra*

$$m_\Lambda(\mathrm{d}x) := \frac{1}{|\Lambda|} \sum_\eta \delta_\eta(\mathrm{d}x),$$

where the sum runs over the eigenvalues $\eta \in \mathrm{Spec}(Q)$, and δ_a denotes the Dirac (or degenerate) distribution with atom at $a \in \mathbb{R}$.

Note that integration against the spectral measure is no more than a summation

$$\int_{\mathbb{H}} f(x) m_\Lambda(\mathrm{d}x) = \frac{1}{|\Lambda|} \sum_\eta \int_{\mathbb{H}} f(x) \delta_\eta(x) \mathrm{d}x = \frac{1}{|\Lambda|} \sum_\eta f(\eta). \tag{1.3}$$

A sequence of probability measures $(m_n)_{n \geq 1}$ on $\mathbb{C}$ is said to converge *in distribution* (or *weakly*) to a measure m_∞ if given any bounded continuous function $f \colon \mathbb{C} \to \mathbb{R}$

$$\lim_{n \to \infty} \int f(x) m_n(\mathrm{d}x) = \int f(x) m_\infty(\mathrm{d}x).$$

In this case, we write $m_n \xrightarrow{(\mathrm{d})} m_\infty$. Noting that $f(x) \equiv 1$ is a bounded continuous function on $\mathbb{C}$, then

$$m_\infty(\mathbb{C}) = \int_{\mathbb{C}} 1 \, m_\infty(\mathrm{d}u) = \lim_{n \to \infty} \int_{\mathbb{C}} 1 \, m_n(\mathrm{d}u) = 1$$

so that any weak limit of a sequence of probability measures is itself a probability measure. Our notion of graph convergence will be through convergence in distribution of the associated spectral measures.

Definition 1.8. *Let $\Lambda_n = (\Lambda_n, w_n, \kappa_n)_{n=1}^\infty$ be a sequence of graphs, and write $m_n = m_{\Lambda_n}$ for the spectral measures. We say that the sequence $(\Lambda_n)_{n \geq 1}$ is a (spectrally) convergent graph sequence if there exists a measure m_∞ to which the spectral measures converge in distribution, $m_n \xrightarrow{(\mathrm{d})} m_\infty$.*

In practice, when proving convergence of graph sequences, rather than considering the m_Λ directly it is easier to work with a representative *distribution* of the measures.

Given a bounded Lebesgue measurable domain $D \subset \mathbb{R}^d$ with Lebesgue measure $|D| = 1$, we say that a measurable function $\phi_\Lambda \colon D \to \mathbb{H}$ is a distribution for Λ if m_Λ is obtained as the pushforward measure of Lebesuge measure under ϕ_Λ

$$m_\Lambda(B) = \big|\phi_\Lambda^{-1}(B)\big|, \qquad B \subseteq \mathbb{H} \text{ measurable.}$$

We recall that the change of variables formula for pushforward measures allows us to write integrals against m_Λ as

$$\int_{\mathbb{H}} f(y) m_\Lambda(\mathrm{d}y) = \int_D \big(f \circ \phi_\Lambda\big)(x)\mathrm{d}x, \tag{1.4}$$

where $f \colon \mathbb{H} \to \mathbb{R}$ is measurable, and the integral on the right hand side is with respect to Lebesgue measure. The following proposition enables us to confirm convergence of the spectral measures by studying associated distribution functions.

Proposition 1.9. *Let (Λ_n) be a graph sequence, and $\phi_n \colon D \to \mathbb{H}$ a sequence of distribution functions on the same domain $D \subset \mathbb{R}^d$. If there exists $\phi_\infty \colon D \to \mathbb{H}$ such that $\phi_n \to \phi_\infty$ pointwise almost everywhere, then the graph sequence Λ_n converges, and the limit measure m_∞ is given by the pushforward of ϕ_∞*

$$m_\infty(B) := |\phi_\infty^{-1}(B)|, \qquad B \subseteq \mathbb{H} \text{ measurable.}$$

Proof. Almost everywhere pointwise convergence of the sequence $(\phi_n)_{n\geq 1}$ implies that for any continuous bounded $f \colon \mathbb{H} \to \mathbb{R}$ the composition $f \circ \phi_n \to f \circ \phi_\infty$ almost everywhere. Boundedness of f ensures that $f \circ \phi_n$ is uniformly bounded, that is there is an M such that $(f \circ \phi_n)(x) \leq M$ for all $x \in D$ and $n \geq 1$. Moreover, boundedness of the domain D ensures

$$\int_D (f \circ \phi_n)(x)\mathrm{d}x \leq M \int_D 1\mathrm{d}x = M|D|,$$

from which the claim follows via the dominated convergence theorem. □

For a graph Λ with spectrum $\mathrm{Spec}(Q) = (\eta_j)_{j=1}^{|\Lambda|}$, the simplest choice of distribution function is $\phi_\Lambda \colon (0,1] \to \mathbb{H}$ defined to be

$$\phi_\Lambda(u) = \eta_{\lceil |\Lambda| u \rceil}.$$

We refer to this as the *canonical* distribution function, and for most cases it suffices to study only this function. The greater generality in which we defined distribution functions will however be useful when proving convergence for lattice boxes $\Lambda \subset \mathbb{Z}^d$, $d \geq 1$. We now provide two examples of graph sequences, which will be referenced in later chapters. Derivations of the spectra, and proofs of convergence are deferred to Appendix A.

Complete Graph

For $N \geq 2$, we define the complete graph on N vertices by $K_N = ([N], w, \kappa)$, where for all $x, y \in [N]$

$$w_{xy} = \kappa_x = \frac{1}{N}.$$

Subsequently the continuous time walk $\overline{X}$ is the simple random walk with unit jump rate and which is killed according to a geometric random variable with mean N. The eigenvalues of Q are

$$\eta_1 = \cdots = \eta_{N-1} = -(1 + \frac{1}{N}),\ \eta_N = -\frac{1}{N}.$$

The sequence $(K_N)_{N\geq 2}$ is spectrally convergent, and has limit measure $m_\infty = \delta_{-1}$.

Lattice Boxes

For $N \geq 1$ define the lattice box $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$, for $d \geq 1$. We will consider two different random walks on Λ_N. The first is defined by the weight function

$$w_{xy}^{(dir)} = \begin{cases} \frac{1}{2d} & \text{if } |x - y| = 1, \\ 0 & \text{else.} \end{cases}$$

and killing vector $\kappa_x^{(dir)} = 1 - \sum_{y\in\Lambda_N} w_{xy}$. This defines $\overline{X}$, the continuous time simple random walk killed on exiting the box Λ_N. We say that $\Lambda_N^{(dir)} = (\Lambda_N, w^{(dir)}, \kappa^{(dir)})$ is the lattice box with *Dirichlet* boundary conditions. Alternatively we can define the weighting

$$w_{xy}^{(per)} = \begin{cases} \frac{1}{2d} & \text{if } \exists\, 1 \leq i \leq d \text{ st. } x_i = -y_i = \pm N, \text{ and } x_j = y_j,\ j \neq i, \\ 0 & \text{else.} \end{cases}$$

with $\kappa^{(per)} \equiv 0$, for which $\overline{X}$ is the continuous time simple random walk on the d-torus, and we call $\Lambda_N^{(per)} = (\Lambda_N, w^{(per)}, \kappa^{(per)})$ the lattice box with *periodic* boundary conditions. In the case of $\Lambda_N^{(per)}$, the spectrum is

$$\mathrm{Spec}(Q^{(per)}) = \left\{ \frac{1}{d} \sum_{i=1}^{d} \cos\left(2\pi \frac{j_i}{2N+1} \right) - 1 :\ j = (j_i)_{i=1}^{d} \in \{1, \ldots, 2N+1\}^d \right\}.$$

The sequence $\Lambda_N^{(per)}$ converges in distribution, and the limit measure m_∞ is defined via its distribution function $\phi_\infty\colon [0,1)^d \to [-2,0]$

$$\phi_\infty^{(per)}(u) = \frac{1}{d}\sum_{i=1}^{d} \cos(2\pi u_i) - 1.$$

Rather than defining the spectrum of $\Lambda_N^{(dir)}$ directly, a comparison argument yields convergence of the sequence $\Lambda_N^{(dir)}$, and $\phi_\infty^{(dir)} = \phi_\infty^{(per)}$.

1.2 Loop Measures, Soups, and Their Occupation Fields

Given $t > 0$, a càdlàg function $p\colon [0,t] \to \Lambda$ with finitely many points of discontinuity $0 = t_0 < t_1 < \cdots < t_n < t$ is said to be a *path* on Λ if the sequence of vertices $x_j = p(t_j)$ are such that $w(x_j, x_{j+1}) > 0$ for $j = 0, \dots n-1$. Let D_t denote the set of all paths of duration t, and $D = \cup_{t>0} D_t$ the set of all paths. Given $p \in D$, we denote $|p|$ for its duration: the unique t such that $p \in D_t$. A path whose start and end points agree is called a *loop*, we denote

$$\Gamma_t \coloneqq \{\gamma \in D_t : \ \gamma(0) = \gamma(t)\}$$

for the set of length $t > 0$ loops, and $\Gamma = \cup_{t>0}\Gamma_t$ for the set of all loops. A path with no discontinuities is by default a loop, and we call this a *point loop*.
Following [Szn12] pp.35-6, we construct a σ-algebra on D (respectively, Γ) as follows. We define a bijection $\Phi\colon D \to D_1 \times (0,\infty)$, where $p \mapsto (\tilde{p}, t)$ with $t = |p|$, and $\tilde{p}(s) = p(st)$. We endow $D_1 \times (0,\infty)$ with the product σ-algebra $\mathcal{A} \times \mathcal{B}$, where $\mathcal{B}$ is the Borel σ-algebra on $(0,\infty)$, and $\mathcal{A}$ is the σ-algebra generated by the family of sets $A_{s,x} = \{p \in D_1 \colon p(s) = x\}$, with $s \in [0,1]$, $x \in \Lambda$. We observe that $\mathcal{A}$ is a natural choice as it is none other than the Borel σ-algebra induced by the Skorokhod topology on D_1, see [Bil99] pp.134–5. Finally, using the projection Φ we define

$$\mathcal{D} = \left\{\Phi^{-1}(B) : \ B \in \mathcal{A} \times \mathcal{B}\right\},$$

which is a σ-algebra on D. In the same way we define $\mathcal{G}$, a σ-algebra on Γ.

1.2.1 The Measures μ and μ^B

As before, let $\mathbf{P}$ denote the law of a random walk on Λ. For $t > 0$, we define a measure on paths from x to y by

$$\mathbf{P}_{xy}^t[G] \coloneqq \mathbf{P}_x\Big[G \cap \Big\{\overline{X}_t = y\Big\}\Big], \qquad G \in \mathcal{D}.$$

We call this the (non-normalised) random walk bridge measure, where the term non-normalised stems from the fact that this is not a probability measure: $\mathbf{P}^t_{xy}[D] = \mathbf{P}_x\left[\overline{X}_t = y\right] \le 1$. We define two families of measures on Γ, the *Markov* loop measures

$$\mu_{h,\Lambda}(G) := \sum_{x\in\Lambda}\int_0^\infty \frac{e^{ht}}{t}\mathbf{P}^t_{xx}[G]\mathrm{d}t, \qquad G\in\mathcal{G}. \tag{1.5}$$

and the *Bosonic* loop measures

$$\mu^B_{\beta,h,\Lambda}(G) := \sum_{x\in\Lambda}\sum_{j\ge 1}\frac{e^{\beta hj}}{j}\mathbf{P}^{\beta j}_{xx}[G], \qquad G\in\mathcal{G}. \tag{1.6}$$

The parameter $h < 0$ is called the *chemical potential*, whilst $\beta > 0$ is the *inverse temperature*. We note that $\mu_{h,\Lambda}$ depends only on h, since adding the relevant terms in $\beta > 0$ act only as a change of variables in the definition. Henceforth we freely denote μ, μ^B for the Markov and Bosonic loop measures respectively, including the various subscripts only when we wish to highlight the dependence on the graph structure, or values of β and h. We remark that in the case $h = 0$, the measure μ_0 is exactly the loop measure studied by Le Jan [LeJ10, LeJ11], and many of the properties which he studies can be generalised for μ_h, $h < 0$: one of these will be the isomorphism theorem, which we discuss in Chapter 3.
We briefly comment that our convention differs from that used elsewhere in the literature. As remarked in the introduction, in particular the footnote on page xviii, for most authors a loop is defined as a conjugacy class of $\Gamma/\sim$ where $\sim$ is the equivalence relation which equates all loops which can be obtained from one another by a time shift (i.e. by forgetting their root); the loop measure μ_h is constant on conjugacy classes, and so determines a measure on the space $\Gamma/\sim$. Since the random variables we consider will not be effected by whether or not our loops are rooted, we will not make use of this equivalence relation, and keep the convention that a loop is endowed with its root, $\gamma(0)$.

Lemma 1.10. *Assume $\kappa \not\equiv 0$. Fix $\beta > 0, h \le 0$, then*

(i) μ is a σ-finite measure.

(ii) μ^B is finite with

$$\mu^B(\Gamma) = -\operatorname{Tr}\Big(\log\Big(I - e^{\beta(Q+hI)}\Big)\Big) < \infty.$$

If $\kappa \equiv 0$, then the above hold if and only if $h < 0$.

Proof. For a proof that μ_0 is σ-finite, see [Szn12], p.63; from the definition, for $h < 0$, μ_h is dominated by μ_0, i.e. $\mu_h(G) \le \mu_0(G)$, so it follows that μ_h is σ-finite.

For (ii), we have

$$\begin{aligned}\mu^B(\Gamma) &= \sum_{x\in\Lambda}\sum_{j\geq 1}\frac{e^{\beta h j}}{j}\mathbf{P}_x\left[\overline{X}_{\beta j} = x\right] \\ &= \sum_{x\in\Lambda}\sum_{j\geq 1}\frac{e^{\beta h j}}{j}\left(e^{\beta j Q}\right)_{xx} \\ &= \sum_{x\in\Lambda}\sum_{j\geq 1}\frac{1}{j}\left(e^{\beta(Q+hI)}\right)^j_{xx}.\end{aligned}$$

Appealing to Proposition B.22, the series over $j \geq 1$ converges if the spectral radius of $\exp(\beta(Q+hI))$ is less than 1. Letting $\mathrm{Spec}(Q) = \{\eta_j\}_{j=1}^{|\Lambda|}$, we require

$$\max_j |e^{\beta(\eta_j+h)}| = \max_j e^{\beta(\mathrm{Re}(\eta_j)+h)} < 1,$$

where Re denotes the real part of a complex number, and we relied on the fact that for $z \in \mathbb{C}$, $|\exp(z)| = \exp(\mathrm{Re}(z))$. But from Theorem 1.6, $\mathrm{Re}(\eta_j) \leq 0$ holds for all $j = 1, \ldots, |\Lambda|$, and so the inequality above holds whenever $h < 0$. On the other hand, if $\kappa \not\equiv 0$, then the same theorem for the spectrum asserts that $\mathrm{Re}(\eta_j) < 0$, so that the inequality holds for $h = 0$. □

On recalling that $\mathrm{Tr}(\log(I - A)) = \log\det(I - A)$, justified below Proposition B.22, the formula above gives

$$\exp\left(\mu^B(\Gamma)\right) = \prod_{j=1}^{|\Lambda|}\left(1 - e^{\beta(\eta_j+h)}\right)^{-1},$$

where as above we denote η_j for the eigenvalues of Q. On comparing this with (0.7) we see that this is none other than Einstein's formula for the grand canonical partition function of the ideal gas. In the following we give a direct argument of this fact, which does not rely on the spectral representation.

Theorem 1.11. *For any $\beta > 0$ and $h < 0$*

$$\exp\left(\mu^B(\Gamma)\right) = \Xi_\Lambda(\beta, h).$$

Proof. To simplify notation in the following we write $z = \exp(\beta h)$, which is known as the *fugacity*, and write $\mathbf{P}_j = \sum_{x\in\Lambda}\mathbf{P}_x\left[\overline{X}_{\beta j} = x\right]$, given these notational simplifications, the total mass of the loop measure becomes $\mu^B(\Gamma) = \sum_{j\geq 1}\frac{z^j}{j}\mathbf{P}_j$. Expanding the exponential power series

$$\exp\left(\mu^B(\Gamma)\right) = \sum_{m\geq 0}\frac{1}{m!}\left(\sum_{j\geq 1}\frac{z^j}{j}\mathbf{P}^{\beta j}\right)^m,$$

each power in m can further be expanded using the multinomial theorem for power series

$$= \sum_{m\geq 0} \frac{1}{m!} \sum_{\sum k_j = m} \binom{m}{\underline{k}} \prod_{j\geq 1} \left(\frac{z^j \mathbf{P}_j}{j} \right)^{k_j},$$

where the sum is over sequences $\underline{k} = (k_j)_{j\geq 1}$ with $\sum_j k_j = m$, and $\binom{m}{\underline{k}} = m!/\prod_j k_j!$ is the infinite multinomial coefficient. Canceling the factorial terms, we see that the sum depends on m only through the sequences $\underline{k}$, so we write

$$= \sum_{\sum k_j < \infty} \prod_{j\geq 1} \frac{1}{k_j!} \left(\frac{z^j \mathbf{P}_j}{j} \right)^{k_j},$$

where the sum now runs over all terminating sequences $\underline{k} = (k_j)_{j\geq 1}$. We factor this as a summation over integer partitions: i.e. fixing $n \geq 0$, consider those sequences $\underline{k}$ such that $\sum_j jk_j = n$ with the interpretation that there are k_j blocks of length j. Now

$$\begin{aligned} &= \sum_{n\geq 0} \sum_{\sum jk_j = n} \prod_{j\geq 1} \frac{1}{k_j!} \left(\frac{z^j \mathbf{P}_j}{j} \right)^{k_j} \\ &= \sum_{n\geq 0} \frac{1}{n!} \sum_{\sum jk_j = n} \prod_{j\geq 1} \frac{n!}{k_j! j^{k_j}} z^{jk_j} \mathbf{P}_j^{k_j}. \end{aligned}$$

Recognising the combinatorial factor $n!/(k_j! j^{k_j})$ as the number of permutations $\pi \in S_n$ with exactly k_j cycles of length j, we can instead sum over permutations. Denoting c for a cycle in a permutation π, and n_c for the length of the cycle

$$\begin{aligned} &= \sum_{n\geq 0} \frac{1}{n!} \sum_{\pi \in S_n} \prod_{c\in\pi} z^{n_c} \mathbf{P}_{n_c} \\ &= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\pi \in S_n} \prod_{c\in\pi} \mathbf{P}_{n_c}. \end{aligned} \tag{1.7}$$

Working in the opposite direction, we note that for a permutation $\pi \in S_n$ with cycles $c = (c(1), \ldots, c(n_c))$

$$\sum_{x_1,\ldots,x_n \in \Lambda} \prod_{i=1}^{n} \mathbf{P}_{x_i} \left[\overline{X}_\beta = x_{\pi(i)} \right] = \sum_{x_1,\ldots,x_n \in \Lambda} \prod_{c\in\pi} \prod_{i=1}^{n_c} \mathbf{P}_{x_{c(i)}} \left[\overline{X}_\beta = x_{c(i+1)} \right], \tag{1.8}$$

with the convention $c(n_c+1) = c(1)$. The second sum depends only on $x_{c(1)}, \ldots, x_{c(n_c)}$ and so we can factorise the sum as

$$= \prod_{c\in\pi} \sum_{x_1,\ldots,x_{n_c}} \prod_{i=1}^{n_c} \mathbf{P}_{x_{c(i)}} \left[\overline{X}_\beta = x_{c(i+1)} \right],$$

and the Chapman–Kolmogorov equation gives

$$\begin{aligned}
&= \prod_{c\in\pi} \sum_x \mathbf{P}_x\left[\overline{X}_{\beta n_c} = x\right] \\
&= \prod_{c\in\pi} \mathbf{P}_{n_c}.
\end{aligned}$$

Substituting (1.8) into (1.7) we obtain

$$\begin{aligned}
\exp(\mu^B(\Gamma)) &= \sum_{n\geq 0} \frac{z^n}{n!} \sum_{\pi\in S_n} \sum_{x_1,\dots,x_n\in\Lambda} \prod_{i=1}^n \mathbf{P}_{x_i}\left[\overline{X}_\beta = x_{\pi(i)}\right] \\
&= \sum_{n\geq 0} z^n Z_\Lambda(\beta, n),
\end{aligned}$$

where $Z_\Lambda(\beta, n)$ is recognised as the graph analogue of the canonical partition function (0.13), and the result follows from (0.5). □

1.2.2 The Poisson Loop Soup

We start by providing a formal definition of the Poisson point process for loop measures as a random measure, for which we follow [Kal01] pp.225-6, after which we give a more user friendly description.

A *counting* measure on $(\Gamma, \mathcal{G})$ is a measure taking integer values, $\xi\colon \mathcal{G} \to \mathbb{N}$, and we denote $\mathcal{N} := \mathcal{N}(\Gamma, \mathcal{G})$ for the set of all σ-finite counting measures on $(\Gamma, \mathcal{G})$. For $G \in \mathcal{G}$, we define the evaluation map $\pi_G\colon \mathcal{N} \to \mathbb{N}$ by $\pi_G(\xi) = \xi(G)$, and let $\mathcal{F} := \sigma(\pi_G : G \in \mathcal{G})$ be the smallest σ-algebra for which the evaluation maps are measurable. Thus, we have defined a measure space $(\mathcal{N}, \mathcal{F})$ of counting measures. A *point process* is any probability measure $\mathbb{P}$ on $(\mathcal{N}, \mathcal{F})$. Given a measurable function $F\colon \mathcal{N} \to \mathbb{R}$, we denote the expectation with respect to $\mathbb{P}$ by

$$\mathbb{E}[F] := \int_{\mathcal{N}} F(\xi)\mathbb{P}[\mathrm{d}\xi].$$

We consider point processes as the law of a random measure ξ: that is, rather than considering events $\{\pi_G^{-1}(C)\}$, we write the equivalent $\{\xi(G) \in C\}$. We state without proof the following uniqueness criteria for point processes [Kal01], Lemma 12.1 pp.225-6.

Lemma 1.12. *A point process is uniquely determined by its finite dimensional distributions. That is the point processes ξ with law $\mathbb{P}$, and η with law $\widetilde{\mathbb{P}}$ are equal in distribution, $\xi \overset{(\mathrm{d})}{=} \eta$ if and only if for all $n \geq 1$, $G_1, \dots, G_n \in \mathcal{G}$*

$$\big(\xi(G_1), \dots, \xi(G_n)\big) \overset{(\mathrm{d})}{=} \big(\eta(G_1), \dots, \eta(G_n)\big).$$

Let ν denote a σ-finite measure on $(\Gamma, \mathcal{G})$. The *poisson point process* with intensity

ν is the law $\mathbb{P}$ on $(\mathcal{N}, \mathcal{F})$ with the defining properties

Poisson property. For $G \in \mathcal{G}$ and $n \geq 0$

$$\mathbb{P}[\xi(G) = n] = e^{-\nu(G)} \frac{\nu(G)^n}{n!}.$$

Independent increments. For pairwise disjoing sets $G_1, \ldots, G_k \in \mathcal{G}$, the variables $\xi(G_1), \ldots \xi(G_k)$ are pairwise independent under $\mathbb{P}$.

Any $\xi \in \mathcal{N}$ can be associated with a countable (or finite, if ξ is finite) collection $S = S_\xi \subset \Gamma \times \mathbb{N}$, where $(\gamma, n_\gamma) \in S$ if and only if $\xi(\gamma) = n_\gamma$. The collection of pairs S can in turn be identified with a multiset of loops, with the loop γ occurring with multiplicity n_γ, and abusing notation we denote S for this multiset. When S is a multiset with elements in Γ we write $S \sqsubset \Gamma$.
In this form we see that $\mathbb{P}$, the Poisson point process with intensity ν, is none other than a law over random multisets $S \sqsubset \Gamma$. The Poisson property, defined above, becomes

$$\mathbb{P}[\#(S \cap G) = n] = e^{-\nu(G)} \frac{\nu(G)^n}{n!}, \qquad G \in \mathcal{G},\, n \geq 0,$$

whilst the independence property reads: for $G, H \in \mathcal{G}$ if $G \cap H = \varnothing$, then $\#(S \cap G)$ is independent of $\#(S \cap H)$, under $\mathbb{P}$.
The Poisson point process with intensity μ will henceforth be denoted by $\mathbb{P}$, respectively that of the measure μ^B will be $\mathbb{P}^B$. The corresponding expectations are $\mathbb{E}$ and $\mathbb{E}^B$; as before, when we wish to stress the dependence on the parameters Λ, β, h, we add the relevant subscripts. We colloquially refer to the law $\mathbb{P}$ as the *Markov loop soup*, and $\mathbb{P}^B$ as the *Bosonic loop soup*.

1.2.3 Occupation Times and the Occupation Field

Finally we come to describe the occupation field of the loop soup, which is analogous to the local field $(l_x)_{x \in \Lambda}$ associated with a random walk. For $x \in \Lambda$ we define the functional $L_x \colon \Gamma \to \mathbb{R}_+$ by

$$L_x(\gamma) := \int_0^{|\gamma|} \delta_x\big(\gamma(s)\big) \mathrm{d}s,$$

with $\delta_x \colon \mathbb{R} \to \mathbb{R}$ the Kronecker delta function taking the value 1 at x, and 0 elsewhere. We refer to $L_x(\gamma)$ as the *occupation time* of γ at $x \in \Lambda$, and define the field $L \colon \Gamma \to \mathbb{R}_+^\Lambda$ by $L(\gamma) = \big(L_x(\gamma)\big)_{x \in \Lambda}$, which is the *occupation field* of the loop γ.

Proposition 1.13. *The occupation field $L \colon \Gamma \to \mathbb{R}_+$ is a $\mathcal{G}$-measurable map.*

Proof. It suffices to prove $\mathcal{G}$-measurability of L_x for all $x \in \Lambda$. Recalling the definition of $\mathcal{G}$ on page 10, define $\tilde{L}_x \colon \Gamma_1 \times (0,\infty) \to \mathbb{R}_+$ by

$$\tilde{L}_x(\tilde{\gamma}, t) := t \int_0^1 \delta_x\big(\tilde{\gamma}(s)\big)\mathrm{d}s,$$

so that $L_x(\gamma) = \tilde{L}_x(\Phi(\gamma))$, from which measurability of L_x is equivalent to that of $\tilde{L}_x$. Moreover, since the identity map $t \mapsto t$ is measurable in $\mathcal{B}(\mathbb{R})$, and a product of measurable functions is measurable, it suffices to show the restriction $L_x \colon \Gamma_1 \to \mathbb{R}_+$ is measurable. But in general if $v \colon \Lambda \to \mathbb{R}$, then $\gamma \mapsto \int_0^1 v\big(\gamma(s)\big)\mathrm{d}s$ is $\mathcal{A}$-measurable (where we have used the fact that since Λ is finite v is necessarily measurable and bounded, and so the integral exists), [Bil99] pp.246–9. □

For $S \sqsubset \Gamma$ we write $L(S) := \sum_{\gamma \in S} L(\gamma)$. In the case that S is a random set under the measure $\mathbb{P}$ or $\mathbb{P}^B$, by virtue of Proposition 1.13, its local field is a random field and we use caligraphic font $\mathcal{L} = L(S) \in \mathbb{R}^\Lambda$ to denote this variable.

Chapter 2

The Mean Occupation Under μ^B

We start our analysis of the occupation field of the Bosonic loop soup by considering the mean occupation of a site, which we define by

$$\overline{\mathcal{L}} := \frac{1}{|\Lambda|} \sum_{x \in \Lambda} \mathcal{L}_x, \tag{2.1}$$

which is a positive random variable under the law $\mathbb{P}^B$. Our emphasis is on studying the behaviour of $\overline{\mathcal{L}}$ in the thermodynamic limit: i.e. under the sequence $\mathbb{P}^B_{\Lambda_N}$, with Λ_N a convergent graph sequence. Before proceeding, we state the following assumptions, which we maintain throughout the chapter.

A1 A graph sequence $(\Lambda_N)_{N \geq 1}$ will always denote a spectrally convergent sequence. The spectral measure of Λ_N is denoted m_N, with limit $m_N \xrightarrow{\text{(d)}} m_\infty$. The associated Bosonic loop measure is $\mu^B_N = \mu^B_{\Lambda_N}$, and the law of the associated Bosonic loop soup is denoted $\mathbb{P}^B_N = \mathbb{P}^B_{\Lambda_N}$.

A2 The inverse temperature is strictly positive, $\beta > 0$, and the chemical potential is strictly negative $h < 0$. When no subscripts are given, e.g. $m, \mu^B, \mathbb{P}^B$ then the results are understood to be in the context of some unspecified graph.

As a consequence of **A2** and Lemma 1.10, the loop soups $\mathbb{P}^B_N$ are well defined.

2.1 The Mean Occupation in the Thermodynamic Limit

The aim of this section is to prove convergence in distribution of $\overline{\mathcal{L}}$ to a degenerate random variable, and to describe a central limit theorem for the fluctuations. Then in the following section, Section 2.2, we relate the limiting value of the mean occupation to the density of an ideal Bose gas defined on the graph.

2.1.1 Convergence in Distribution of $\overline{\mathcal{L}}$

Our analysis of the mean occupation on a graph Λ will be via the cumulant generating function, defined for $s \in \mathbb{R}$ by

$$F(s) := \log \mathbb{E}^B\left[e^{s\overline{\mathcal{L}}}\right], \qquad s \in \mathbb{R}.$$

We derive an expression for F by using the independence of the loop soup on each of the sets $\Gamma_{\beta j}$. We will need the following result.

Proposition 2.1. *For $j \geq 1$*

$$\mu^B(\Gamma_{\beta j}) = \frac{|\Lambda|}{j} \int_{\mathbb{H}} e^{\beta j(h+u)} m(\mathrm{d}u).$$

Proof. As in Lemma 1.10, we have

$$\begin{aligned}\mu^B(\Gamma_{\beta j}) &= \sum_{x \in \Lambda} \frac{1}{j}\left(e^{\beta j(hI+Q)}\right)_{xx} \\ &= \frac{1}{j} \operatorname{Tr}\left[e^{\beta j(hI+Q)}\right].\end{aligned}$$

Since the trace can alternatively be formulated as the sum over eigenvalues $\eta \in \operatorname{Spec}(Q)$

$$= \frac{1}{j} \sum_{\eta} e^{\beta j(h+\eta)}$$

the claim follows by definition of the spectral measure m, and (1.3). □

Lemma 2.2. *The expectation of the mean occupation is*

$$\mathbb{E}^B\left[\overline{\mathcal{L}}\right] = \beta \int_{\mathbb{H}} \frac{1}{e^{-\beta(h+u)} - 1} m(\mathrm{d}u).$$

Moreover, for $s < |\Lambda||h|$ the cumulant generating function of $\overline{\mathcal{L}}$ exists and is given by

$$F(s) = |\Lambda| \int_{\mathbb{H}} \log\left(\frac{1 - e^{-\beta(h+u)}}{e^{\beta s/|\Lambda|} - e^{-\beta(h+u)}}\right) m(\mathrm{d}u). \tag{2.2}$$

Proof. Rather than summing the mean occupation over vertices in the graph Λ, we recognise that the total occupation is none other than the total of all loop lengths. Consequently we have

$$\sum_{x \in \Lambda} \mathcal{L}_x = \sum_{\gamma \in S} |\gamma| = \sum_{j \geq 1} (\beta j) \# \{S \cap \Gamma_{\beta j}\}.$$

The last of these expressions has the benefit of summing over the disjoint sets $\Gamma_{\beta j}$, so that appealing to the independence property of the Poisson point process the expectation is

$$\begin{aligned}\mathbb{E}^{B}\big[\overline{\mathcal{L}}\big] &= \frac{\beta}{|\Lambda|}\sum_{j\geq 1} j\,\mathbb{E}^{B}\big[\#\{S\cap\Gamma_{\beta j}\}\big],\\ &= \frac{\beta}{|\Lambda|}\sum_{j\geq 1} j\mu^{B}(\Gamma_{\beta j}).\end{aligned}$$

Applying Proposition 2.1, and changing the order of summation and integration this is

$$= \beta\int_{\mathbb{H}}\sum_{j\geq 1} e^{\beta j(h+u)} m(\mathrm{d}u). \tag{2.3}$$

As before, since for $z\in\mathbb{C}$, $|\exp(z)| = \exp(\mathrm{Re}(z))$, we have that $|\exp(\beta(h+u))| < 1$ for all $u\in\mathbb{H}$ (since $h, \mathrm{Re}(u) < 0$), and the sum converges to the desired expression. Similarly, the cumulant generating function becomes

$$\begin{aligned}F(s) &= \log\mathbb{E}^{B}\left[\exp\left(\frac{s}{|\Lambda|}\sum_{j\geq 1}\beta j\#(S\cap\Gamma_{\beta j})\right)\right]\\ &= \sum_{j\geq 1}\log\mathbb{E}^{B}\left[\exp\left(\frac{s}{|\Lambda|}\beta j\#(S\cap\Gamma_{\beta j})\right)\right].\end{aligned}$$

Each term in $j\geq 1$ is now the cumulant generating function of a Poisson distributed random variable. Since the cumulant generating function of a Poisson $\alpha > 0$ variable is $s\mapsto\alpha(e^{s}-1)$ we have

$$\begin{aligned}&= \sum_{j\geq 1}\mu^{B}(\Gamma_{\beta j})\left[\exp\left(\frac{s}{|\Lambda|}\beta j\right) - 1\right]\\ &= |\Lambda|\int_{\mathbb{H}}\sum_{j\geq 1}\left(\frac{1}{j}e^{\beta j(h+u+\frac{s}{|\Lambda|})} - e^{\beta j(h+u)}\right)m(\mathrm{d}u).\end{aligned}$$

So long as the series converges we have (2.2). We have already seen in computing the expectation that $|\exp(\beta(h+u))| < 1$, so it remains to confirm that the first of the two series also converges in a domain around 0, i.e.

$$\left|e^{\beta(h+u+\frac{s}{\Lambda})}\right| = e^{\beta(h+\mathrm{Re}(u)+\frac{s}{\Lambda})} < 1,$$

for $u\in\mathbb{H}$. Since $\mathrm{Re}(u)\leq 0$, this holds at least for $s < |h||\Lambda|$. □

We note that from the formula for the expectation, and since $(e^{-\beta(h+u)}-1)^{-1}$ is a bounded continuous function, then we already have that for a sequence of graphs

$\lim_{N\to\infty} \mathbb{E}_N^B[\overline{\mathcal{L}}]$ converges. This is similar to saying that the mean occupation converges in expectation, however since each occupation field is defined on a different probability space, this is not a well defined notion. Our aim therefore will be to show that the laws of the mean occupation converge in distribution. Before proving this we state the following analytical lemma which we will call on.

Lemma 2.3. *For a positive constant $c > 0$, the function $x(e^{c/x}-1)$ is a non-negative decreasing function of $x > 0$, and has limit*

$$\lim_{x\to\infty} x(e^{c/x} - 1) = c.$$

Proof. That the function is decreasing is seen on differentiating

$$\frac{\mathrm{d}}{\mathrm{d}x} x(e^{c/x} - 1) = \frac{1}{x}\Big((e^{c/x} - 1)x - ce^{c/x})\Big),$$

on expanding the exponentials, the bracketed term is given by the power series

$$\sum_{n=1}^{\infty} \frac{c^{n+1}}{x^n}\left(\frac{1}{n!} - \frac{1}{(n-1)!}\right) < 0.$$

The limit is computed by appealing to l'Hopital's rule

$$\begin{aligned}\lim_{x\to\infty} x(e^{c/x} - 1) &= \lim_{y\to 0} \frac{e^{cy} - 1}{y} \\ &= c \lim_{y\to 0} e^{cy}.\end{aligned}$$

□

Theorem 2.4. *The law of $\overline{\mathcal{L}}$ under the sequence $(\mathbb{P}_N^B)_{N\geq 1}$ converges in distribution to a degenerate distribution, $\overline{\mathcal{L}} \xrightarrow{(\mathrm{d})} \delta_{a_\infty}$, with atom*

$$a_\infty := a_\infty(\beta, h) = \beta \int_{\mathbb{H}} \frac{e^{\beta h}}{e^{-\beta u} - e^{\beta h}} m_\infty(\mathrm{d}u). \tag{2.4}$$

Proof. Our proof relies on the fact that convergence in distribution of $\big(\overline{\mathcal{L}}, \mathbb{P}_N^B\big)$ is equivalent to pointwise convergence of the cumulant generating functions $F_N = F_{\Lambda_N}$ on a domain of the origin, [Bil95] p.390. We identify a suitable domain to be $(-\infty, |h|)$ since Lemma 2.2 asserted that $F_N(s) < \infty$ throughout this domain. For $N \geq 1$ we write $F_N(s) = \sum_{j\geq 1} f_N^{(j)}(s)$, where for each $j \geq 1$

$$f_N^{(j)}(s) := \mu_N^B(\Gamma_{\beta j})\Big(e^{\beta j s/|\Lambda|} - 1\Big).$$

Our aim is to show that for $s < |h|$, the terms $f_N^{(j)}$ satisfy the requirements of the dominated convergence theorem. From Proposition 2.1

$$\begin{aligned}\left|f_N^{(j)}(s)\right| &= \frac{|\Lambda_N|}{j}\left|e^{\beta js/|\Lambda_N|}-1\right|\left|\int_{\mathbb{H}} e^{\beta j(h+u)} m_N(\mathrm{d}u)\right| \\ &\leq \frac{|\Lambda_N|}{j}\left|e^{\beta js/|\Lambda_N|}-1\right|\int_{\mathbb{H}}\left|e^{\beta j(h+u)}\right| m_N(\mathrm{d}u) \\ &= \frac{|\Lambda_N|}{j}\left|e^{\beta js/|\Lambda_N|}-1\right|\int_{\mathbb{H}} e^{\beta j(h+\mathrm{Re}(u))} m_N(\mathrm{d}u) \\ &\leq \frac{|\Lambda|}{j}\left|e^{\beta js/|\Lambda_N|}-1\right| e^{\beta jh},\end{aligned}$$

where we used the fact that m_N is a probability measure, and that $\mathrm{Re}(u) \leq 0$ for $u \in \mathbb{H}$. Finally, in light of Lemma 2.3 we have the uniform bound

$$\left|f_N^{(j)}(s)\right| \leq \frac{1}{j} e^{\beta j(h+s)},$$

which we note is summable for $s < |h|$. Consequently the dominated convergence theorem allows

$$\begin{aligned}F_\infty(s) &:= \lim_{N\to\infty} F_N(s) \\ &= \lim_{N\to\infty} \sum_{j\geq 1} f_N^{(j)}(s) \\ &= \sum_{j\geq 1} \lim_{N\to\infty} f_N^{(j)}(s),\end{aligned}$$

and the final line is finite for $s < |h|$. In particular we have pointwise convergence of the cumulant generating functions, and the mean occupation converges in distribution. We now identify the limit as the degenerate distribution. Note

$$\begin{aligned}f_\infty^{(j)}(s) &:= \lim_{N\to\infty} f_N^{(j)}(s) \\ &= \lim_{N\to\infty} \frac{|\Lambda_N|}{j}\left(\int_{\mathbb{H}} e^{\beta j(h+u)} m_N(\mathrm{d}u)\right)\left(e^{\beta j \frac{s}{|\Lambda_N|}}-1\right).\end{aligned}$$

Since the integrand over $\exp(\beta j(h+u))$ is a bounded continuous function on $\mathbb{H}$, it follows from $m_N \xrightarrow{(\mathrm{d})} m_\infty$ that we can take the limit inside the integral

$$= \left(\int_{\mathbb{H}} e^{\beta j(h+u)} m_\infty(\mathrm{d}u)\right) \lim_{N\to\infty} \frac{|\Lambda_N|}{j}\left(e^{\beta j \frac{s}{|\Lambda_N|}}-1\right).$$

The remaining limit is handled by Lemma 2.3

$$= \beta s \int_{\mathbb{H}} e^{\beta j(h+u)} m_\infty(\mathrm{d}u).$$

Finally summing over $j \geq 1$

$$F_\infty(s) \;=\; \sum_{j\geq 1} f_\infty^{(j)}(s) \;=\; \left(\beta \sum_{j\geq 1} \int_{\mathbb{H}} e^{\beta j(h+u)} m_\infty(\mathrm{d}u)\right) s,$$

which is the form taken by the cumulant generating function of a degenerate distribution. Moreover, on taking the summation inside the integral, we obtain the atom given by (2.4). □

An immediate consequence is the following analogue to a weak law of large numbers.

Corollary 2.5 (Weak Law of Large Numbers)**.** *Let a_∞ be the atom* (2.4)*. For all $\varepsilon > 0$*

$$\lim_{N\to\infty} \mathbb{P}_N^B\left[\left|\bar{\mathcal{L}} - a_\infty\right| > \varepsilon\right] = 0.$$

Proof. Using the Portmanteau theorem, Theorem 4.25 [Kal01] pp.75-6, we see

$$\limsup_{N\to\infty} \mathbb{P}_N^B\left[\bar{\mathcal{L}} \in (a_\infty - \varepsilon, a_\infty + \varepsilon)^c\right] \;\leq\; \delta_{a_\infty}\Big((a_\infty - \varepsilon, a_\infty + \varepsilon)^c\Big) \;=\; 0.$$

In particular the limit exists, and is 0. □

The functions $f^{(j)}$ defined in the proof of Theorem 2.4 are themselves the cumulant generating functions of the scaled occupation of βj-loops

$$f^{(j)}(s) = \mathbb{E}^B\left[e^{s\beta j \#(S\cap\Gamma_{\beta j})/|\Lambda|}\right].$$

Defining the variables $n^{(j)} := \#(S \cap \Gamma_{\beta j})$ for the number of loops of length βj, then we have the following.

Corollary 2.6. *For $j \geq 1$, the random variables $|\Lambda|^{-1} n_N^{(j)}$ converge in distribution to a degenerate distribution with atom*

$$n_\infty^{(j)} := \frac{e^{\beta j h}}{j} \int_{\mathbb{H}} e^{\beta j u} m_\infty(\mathrm{d}u).$$

Proof. The variables $|\Lambda_N|^{-1} n_N^{(j)}$ have cumulant generating functions $\tilde{f}_N \colon \mathbb{R} \to \mathbb{R}$ given by

$$\tilde{f}_N(s) \;:=\; \log \mathbb{E}^B\left[e^{s n_N^{(j)}/|\Lambda_N|}\right] \;=\; f_N(s/\beta j) \;=\; \mu_N^B(\Gamma_{\beta j})\Big(e^{s/|\Lambda_N|} - 1\Big).$$

Unlike the case for F_N, the functions $\tilde{f}_N$ exist for all $s \in \mathbb{R}$, and that they converge pointwise follows from the same argument presented in Theorem 2.4. □

Before we move on, we note that in the case of the complete graph K_N, defined on page 9, the formulae for a_∞, $n_\infty^{(j)}$ simplify to

$$\begin{aligned} n_\infty^{(j)} &= \frac{1}{j} e^{\beta j(h-1)}, \\ a_\infty &= \beta \Big(e^{\beta(1-h)} - 1 \Big)^{-1}. \end{aligned} \tag{2.5}$$

In particular, as one would expect from their definitions: $a_\infty = \beta \sum_{j \geq 1} j n_\infty^{(j)}$.

2.1.2 Fluctuations from the Average and Large Deviations

To complement the proof that the mean occupation converges in distribution we provide a central limit theorem for the fluctuations. The method of proof is similar to that of Theorem 2.4.

Theorem 2.7 (Central Limit Theorem)**.** *Under the sequence of measures* $(\mathbb{P}_N^B)_{N \geq 1}$

$$|\Lambda_N|^{1/2} \Big(\overline{\mathcal{L}} - \mathbb{E}_N^B \big[\overline{\mathcal{L}} \big] \Big) \xrightarrow{(\mathrm{d})} \mathcal{N}(0, \sigma^2),$$

where $\mathcal{N}(0, \sigma^2)$ *is the centred normal distribution with variance*

$$\sigma^2 := \sigma^2(\beta, h) := \beta^2 \int_{\mathbb{H}} \frac{e^{\beta(h+u)}}{(e^{\beta(h+u)} - 1)^2} m_\infty(\mathrm{d}u). \tag{2.6}$$

Proof. As before, writing $F_N(s) = \sum_{j \geq 1} f_N^{(j)}(s)$ for the cumulant generating function of $\overline{\mathcal{L}}$ under $\mathbb{P}_N^B$, we have

$$\begin{aligned} \log \mathbb{E}_N^B &= \Big[e^{s|\Lambda_N|^{1/2} (\overline{\mathcal{L}} - \mathbb{E}_N^B[\overline{\mathcal{L}}])} \Big] = F_N(s|\Lambda_N|^{1/2}) - s|\Lambda_N|^{1/2} \mathbb{E}_N^B[\overline{\mathcal{L}}] \qquad (2.7) \\ &= \sum_{j \geq 1} \Big(\int_{\mathbb{H}} e^{\beta j(h+u)} m_N(\mathrm{d}u) \Big) \Big(\frac{|\Lambda_N|}{j} \Big(e^{\beta j s |\Lambda_N|^{-1/2}} - 1 \Big) - \beta s |\Lambda_N|^{1/2} \Big). \end{aligned}$$

As in the proof of Theorem 2.4, we show that each of the terms in the summation is uniformly bounded in Λ_N, and apply the dominated convergence theorem. To this end, denoting $g_N^{(j)}(s)$ for the j-th summand in the expression above, and using the same bound as before for the integral term

$$\Big| g_N^{(j)}(s) \Big| \leq e^{\beta j h} \Big| \frac{|\Lambda_N|}{j} \Big(e^{\beta j s |\Lambda_N|^{-1/2}} - 1 \Big) - \beta s |\Lambda_N|^{1/2} \Big|.$$

An analogous result as in Lemma 2.3 for the function $x \mapsto x(e^{cx^{-1/2}} - 1) - cx^{1/2}$ ensures that this function is decreasing in x, and so we have the bound

$$\begin{aligned}
&\leq e^{\beta jh} \left| \frac{1}{j}(e^{\beta js} - 1) - \beta s \right| \\
&\leq \frac{1}{j} e^{\beta j(h+s)},
\end{aligned}$$

from which it follows that we can apply the dominated convergence theorem, and the cumulant generating functions (2.7) converge pointwise for $s < |h|$. Considering the N limit of $g_N^{(j)}(s)$ we have on expanding the exponential

$$g_N^{(j)}(s) = \frac{|\Lambda_N|}{j} \left(\frac{1}{2|\Lambda_N|}(\beta js)^2 + \frac{1}{3!|\Lambda_N|^{3/2}}(\beta js)^3 + \cdots \right) \int_{\mathbb{H}} e^{\beta j(h+u)} m_N(\mathrm{d}u).$$

On taking the limit in Λ_N, all but the first term of the power series vanish, leaving

$$\lim_{N\to\infty} g_N^{(j)}(s) = \left(\frac{1}{2}\beta^2 j \int_{\mathbb{H}} e^{\beta j(h+u)} m_\infty(\mathrm{d}u) \right) s^2.$$

And from the dominated convergence theorem

$$\lim_{N\to\infty} \log \mathbb{E}_N^B \left[e^{s|\Lambda_N|^{1/2}(\overline{\mathcal{L}} - \mathbb{E}_N^B[\overline{\mathcal{L}}])} \right] = \left(\beta^2 \int_{\mathbb{H}} \sum_{j\geq 1} j e^{\beta j(h+u)} m_\infty(\mathrm{d}u) \right) \frac{s^2}{2}.$$

Rearranging the bracketed term yields (2.6). Moreover this is the cumulant generating function of a centered normal distribution. □

A similar argument to Corollary 2.6 provides the analogous result for the scaled cycle distributions $|\Lambda_N|^{-1} n_N^{(n)}$. We omit the proof.

Corollary 2.8. *For $j \geq 1$, the random variables $|\Lambda|^{-1} n_N^{(j)}$ satisfy a central limit theorem under the measures $(\mathbb{P}_N^B)_{N\geq 1}$*

$$|\Lambda_N|^{-1/2} \left(n_N^{(j)} - \mathbb{E}_N^B \left[n_N^{(j)} \right] \right) \xrightarrow{(\mathrm{d})} \mathcal{N}\left(0, \frac{1}{j} \int_{\mathbb{H}} e^{\beta j(h+u)} m_\infty(\mathrm{d}u) \right).$$

Finally we make some remarks about the probability of rare events; since the limit distribution of $\overline{\mathcal{L}}$ is concentrated on the point a_∞, the event that $\overline{\mathcal{L}} > a > a_\infty$ becomes increasing unlikely under $\mathbb{P}_N^B$ as $N \to \infty$. To understand the degree to which this is a rare event, we consider the probabilities on a logarithmic scale. Using a Chernoff bound, the following calculation provides an upper bound on the probability the mean occupation exceeds a value $a > 0$

$$\mathbb{P}_N^B \left[\overline{\mathcal{L}} > a \right] = \mathbb{P}_N^B \left[e^{s|\Lambda_N|\overline{\mathcal{L}}} > e^{s|\Lambda_N|a} \right],$$

which holds for all $s \in \mathbb{R}$, and by Markov's inequality

$$\leq \mathbb{E}_N^B\left[e^{s|\Lambda_N|\overline{\mathcal{L}}}\right]e^{-a|\Lambda_N|s}.$$

Then we have

$$\lim_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathbb{P}_N^B\big[\overline{\mathcal{L}}>a\big] \leq \left(\lim_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathbb{E}_N^B\left[e^{s\overline{\mathcal{L}}}\right]\right) - as.$$

Supposing that the function[1] $\lim_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathbb{E}_N^B\left[e^{s|\Lambda_N|\overline{\mathcal{L}}}\right] =: \mathbf{\Lambda}(s)$ exists as an extended real number, and optimising over the values $s \in \mathbb{R}$

$$\lim_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathbb{P}_N^B\big[\overline{\mathcal{L}}>a\big] \leq \inf_{s\in\mathbb{R}}\big(\mathbf{\Lambda}(s)-as\big) =: -I(a).$$

The calculation above is the standard computation for an upper bound of a *large deviation principle* (LDP). The function I is known as the *rate function* of the LDP. Heuristically an LDP for $(\overline{\mathcal{L}}, \mathbb{P}_N^B)_{N\geq 1}$ says

$$\mathbb{P}_N^B\big[\overline{\mathcal{L}}\in A\big] \sim e^{-|\Lambda_N|I(A)}, \tag{2.8}$$

where $I(A) = \inf_{a\in A} I(a)$. In Chapter 4 we will consider the LDP for the cycle distribution $\underline{n} = (n^{(j)})_{j\geq 1}$; since this result will subsume the LDP for the mean occupation (in that the LDP for $\overline{\mathcal{L}}$ can be derived from that of $\underline{n}$) we will defer the rigorous definition of an LDP for later. The simplest LDP is Cramér's theorem for the mean of a sum of i.i.d. random variables, $S_N = \frac{1}{N}(X_1+\cdots+X_N)$. Whilst this is not so far from the situation we are in, since $\overline{\mathcal{L}}$ can be expressed as a sum of the independent variables $(n^{(j)})_{j\geq 1}$, there are two distinctions. The first of these is simply that whilst the variables $n^{(j)}$ are independent, they are not identically distributed. The more significant difference is the sense in which the limit is taken. Cramérs theorem deals with a scaled limit of a finite summation, whereas we consider an infinite summation, but with each summand converging in the limit. To derive an LDP for the mean occupation we must instead use the Gärtner–Ellis theorem, [dH00] Theorem I.4 pp.54-7. The following lemma proves the requisite conditions for this theorem.

Lemma 2.9. *The limit*

$$\mathbf{\Lambda}(s) := \lim_{N\to\infty}\frac{1}{|\Lambda_N|}\log\mathbb{E}_N^B\left[e^{s|\Lambda_N|\overline{\mathcal{L}}}\right],$$

exists in $\mathbb{R}\cup\{+\infty\}$. Denoting $\mathcal{D}_{\mathbf{\Lambda}} = \{s\in\mathbb{R} : \mathbf{\Lambda}(s)<\infty\}$, then $0\in\mathcal{D}_{\mathbf{\Lambda}}$, and $\mathbf{\Lambda}$ is

[1] We use the bold font $\mathbf{\Lambda}$ since this is the accepted notation used in the literature of large deviations, of course this is at odds with our choice of denoting graphs by Λ. We hope that the bold type face will be sufficient to avoid any confusion!

differentiable on the interior $\operatorname{int}\mathcal{D}_{\mathbf{\Lambda}}$.

Proof. Using the notation established in the previous sections, we have

$$\begin{aligned}\mathbf{\Lambda}(s) &= \lim_{N\to\infty}\frac{1}{|\Lambda_N|}F_N(|\Lambda_N|s)\\ &= \sum_{j\geq 1}\frac{1}{j}\left(\int_{\mathbb{H}} e^{\beta j(h+u)}m_\infty(\mathrm{d}u)\right)\left(e^{\beta js}-1\right),\end{aligned}$$

where we justify taking the limit via the dominated convergence theorem, as in Theorem 2.4. Referencing this theorem again, we also have that $\mathbf{\Lambda}(s) < \infty$ for $s < |h|$, so that $0 \in \mathcal{D}_{\mathbf{\Lambda}}$, and in particular on the domain of convergence we have

$$\mathbf{\Lambda}(s) = \int_{\mathbb{H}} \log\left(\frac{1-e^{\beta(h+u)}}{1-e^{\beta(h+u+s)}}\right)m_\infty(\mathrm{d}u), \qquad s < |h|.$$

That $\mathbf{\Lambda}$ is differentiable throughout $\operatorname{int}\mathcal{D}_{\mathbf{\Lambda}}$ follows as an application of differentiating under the (complex) integral sign; this is justified in [Mat01]. □

The existence of $\mathbf{\Lambda}$, together with $0 \in \operatorname{int}\mathcal{D}_{\mathbf{\Lambda}}$ is in fact sufficient to derive a weak form of the Gärtner–Ellis theorem, [dH00] Theorem V.6 pp.54–7, but we will not consider this here. Unfortunately to move away from this weaker form, in addition to the claims of Lemma 2.9 we require that $\mathbf{\Lambda}$ is *steep*: that is either $\mathcal{D}_{\mathbf{\Lambda}} = \mathbb{R}$, or if $s^* > 0$ is on the boundary of convergence of $\mathbf{\Lambda}$, then $\lim_{s\nearrow s^*}\mathbf{\Lambda}'(s) = +\infty$. For general graph sequences we do not have a way to prove this condition, and must include it as an assumption in the following theorem.

Theorem 2.10. *Suppose* $\mathbf{\Lambda}$ *exists and is steep. The sequence* $\left(\overline{\mathcal{L}}, \mathbb{P}_N^B\right)_{N\geq 1}$ *satisfies an LDP with rate* $|\Lambda_N|$ *and good rate function*

$$I(x) = \sup_{s\in\mathbb{R}}\left(sx - \mathbf{\Lambda}(s)\right). \tag{2.9}$$

Taking as our graph sequence $(K_N)_{N\geq 1}$ the complete graph, we can solve this variational problem explicitly, in this case

$$\mathbf{\Lambda}'(s) = \beta\sum_{j\geq 1}\left(\int_{\mathbb{H}} e^{\beta j(h+u+s)}\delta_{-1}(\mathrm{d}u)\right) = \frac{\beta}{e^{-\beta(h-1+s)}-1},$$

and $\mathbf{\Lambda}$ is steep at the boundary $s = 1-h$. Then for $x \in \mathbb{R}$ we wish to find s_x which solves $\mathbf{\Lambda}'(s_x) = x$, which is satisfied for $x > 0$ by

$$s_x = \beta(1-h) + \log\left(\frac{x+\beta}{x}\right).$$

For $x \leq 0$ there is no solution, and $\sup_{s\in\mathbb{R}}\left(xs - \mathbf{\Lambda}(s)\right) = \infty$. On checking that s_x

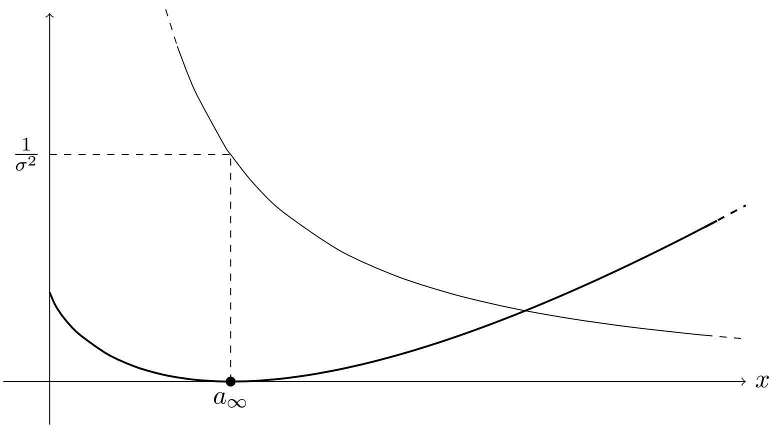

Figure 2.1: The rate function $I(x)$ (in bold) and its second derivative for the graph sequence Λ_N with $\beta = 1$, $h = -1$. I has its unique zero at the atom a_∞, and at this point the second derivative is given by the reciprocal of the variance of the central limit theorem, $I''(a_\infty) = \frac{1}{\sigma^2}$.

is in fact a maxima for $xs - \mathbf{\Lambda}(s)$, and after a deal of rearranging, we obtain

$$I(x) = \begin{cases} x(1-h) + \frac{x}{\beta}\log\left(\frac{x}{\beta+x}\right) + \log\left(\frac{\beta}{\beta+x}\right) - \log(1 - e^{-\beta(1-h)}) & x > 0, \\ +\infty & x \leq 0. \end{cases}$$

Following [dH00] p.8, we remark that $I \geq 0$, and has a unique zero. In particular this occurs at $x^* = a_\infty$ given by (2.5), the atom of the limit distribution, which is as one should expect since the expression (2.8) implies that I should be small on events with high probability. Further, on taking the second derivative one has $I''(a_\infty) = \frac{1}{\sigma^2}$, the variance given in (2.6). See Figure 2.1.
For now we will leave our large deviations analysis at this point, returning to it rigorously in Chapter 4. In the following section, however, we will provide a context in which we can interpret both the steepness condition of $\mathbf{\Lambda}$, and also the variational problem for I.

2.2 Mean Occupation as the Density of the Ideal Bose Gas

Our intention now is is to develop the link between the Bosonic loop soup and the grand canonical ensemble of the ideal Bose lattice gas. Our focus will be on the density of the Bose gas, but we begin by providing expressions for a variety of thermodynamic functions.

2.2.1 Loop Soup Descriptions of Thermodynamic Functions

In Theorem 1.11 we proved directly that $\exp\Big(\mu^B_{\beta,h,\Lambda}(\Gamma)\Big) = \Xi_\Lambda(\beta,h)$, where the right hand side is the partition function for an ideal gas in the grand canonical ensemble, (0.7). This is the first hint that we can relate the Bosonic loop soup to the ideal gas. In this section we develop the links between thermodynamic functions and the Bosonic loop soup. As a starting point we recall the derivation of the mean particle number in the grand-canonical ensemble. From (0.5) we have

$$\begin{aligned}\langle N\rangle &= \sum_{n\geq 0} n\frac{e^{\beta hn}Z_\Lambda(\beta,n)}{\Xi_\Lambda(\beta,h)}\\ &= \frac{1}{\beta}\frac{1}{\Xi_\Lambda(\beta,h)}\partial_h\sum_{n\geq 0}e^{\beta hn}Z_\Lambda(\beta,n)\\ &= \frac{1}{\beta}\frac{\partial_h\Xi_\Lambda(\beta,h)}{\Xi_\Lambda(\beta,h)}\\ &= \frac{1}{\beta}\partial_h\log\Xi_\Lambda(\beta,h).\end{aligned}$$

The function $p_\Lambda(\beta,h) = \beta^{-1}\log\Xi_\Lambda(\beta,h)$ is the *pressure* of the Bose gas. In light of Theorem 1.11, we can consider $\mu^B(\Gamma)$ to be the 'loop soup pressure'. The following theorem justifies this, and demonstrates that the loop soup is none other than a geometric representation of the grand-canonical ensemble.

Theorem 2.11. *For* $n \geq 0$

$$\mathbb{P}^B\left[\sum_{\gamma\in S}|\gamma| = \beta n\right] = \frac{e^{\beta hn}Z_\Lambda(\beta,n)}{\Xi_\Lambda(\beta,h)},$$

with $Z_\Lambda(\beta,n)$ *the canonical partition function for the ideal lattice gas,* (0.13).

Proof. The manipulation is similar to that employed in Theorem 1.11. Fixing $n \geq 0$, we sum over all integer sequences k_j such that $\sum_{j\geq 1} jk_j = n$, with the interpretation that k_j is the number of loops chosen from $\Gamma_{\beta j}$

$$\begin{aligned}\mathbb{P}^B\Big[\textstyle\sum_{\gamma\in S}|\gamma| = \beta n\Big] &= \sum_{\sum jk_j=n}\mathbb{P}^B[\#(S\cap\Gamma_{\beta j}) = k_j,\ j\geq 1]\\ &= \sum_{\sum jk_j=n}\prod_{j\geq 1}\mathbb{P}^B[\#(S\cap\Gamma_{\beta j}) = k_j]\\ &= \sum_{\sum jk_j=n}\prod_{j\geq 1}e^{-\mu^B(\Gamma_{\beta j})}\frac{\mu^B(\Gamma_{\beta j})^{k_j}}{k_j!}\\ &= \frac{e^{-\mu^B(\Gamma)}e^{\beta hn}}{n!}\sum_{\sum jk_j=n}\prod_{j\geq 1}\frac{n!}{k_j!j^{k_j}}\left(\frac{j}{e^{\beta hj}}\mu^B(\Gamma_{\beta j})\right)^{k_j}.\end{aligned}$$

As before, the combinatorial factor corresponds to the number of permutations $\pi \in S_n$ with cycle structure $\underline{k} = (k_j)_{j\geq 1}$. Using the notation established in Theorem 1.11 for permutations $\pi \in S_N$, we have

$$= \frac{e^{\beta h n}}{n!\,\Xi} \sum_{\pi \in S_n} \prod_{c \in \pi} \Big(\frac{j}{e^{\beta h j}} \mu^{B}(\Gamma_{\beta j}) \Big)^{n_c}.$$

Finally on recognising that

$$\begin{aligned} e^{-\beta h j} j \mu^{B}(\Gamma_{\beta j}) &= \sum_{x \in \Lambda} \mathbf{P}_x[X_{\beta j} = x] \\ &= \sum_{x_1, \dots, x_j \in \Lambda} \prod_{i=1}^{j} \mathbf{P}_{x_i}[X_\beta = x_{i+1}], \end{aligned}$$

with $x_{j+1} = x_1$, then we can write

$$\prod_{c \in \pi} \big(j \mu^{B}(\Gamma_{\beta j}) \big)^{n_c} = \sum_{x_1, \dots, x_n \in \Lambda} \prod_{i=1}^{n} \mathbf{P}_{x_i}[X_j = x_{\pi(i)}].$$

The result now follows on comparison to the canonical partition function, (0.13). □

As a consequence we have the following analogous result to $\langle N \rangle = \beta^{-1} \partial_h \log \Xi_\Lambda$ in the context of the loop soup.

Corollary 2.12. $\mathbb{E}^{B}\big[\sum_{x \in \Lambda} \mathcal{L}_x\big] = \partial_h \mu^{B}(\Gamma)$.

Proof. The proof follows by exactly the same steps as taken for $\langle N \rangle$. Alternatively we can derive the equation from the cumulant generating function of the previous section. Let F be as before then

$$\begin{aligned} \log \mathbb{E}^{B}\Big[e^{s \sum_x \mathcal{L}_x} \Big] &= F(|\Lambda| s) \\ &= \sum_{j \geq 1} \mu^{B}(\Gamma_{\beta j}) \big(e^{\beta j s} - 1 \big), \end{aligned}$$

and

$$\begin{aligned} \mathbb{E}^{B}[\textstyle\sum_{x \in \Lambda} \mathcal{L}_x] &= \frac{\mathrm{d}}{\mathrm{d}s} F(|\Lambda| s)_{|s=0} \\ &= \beta \sum_{j \geq 1} j \mu(\Gamma_{\beta j}) \\ &= \beta \sum_{j \geq 1} \sum_{x \in \Lambda} e^{\beta h j} \mathbf{P}_x[X_{\beta j} = x] \\ &= \sum_{j \geq 1} \partial_h \mu(\Gamma_{\beta j}). \end{aligned}$$

□

In light of these results, we can consider the particle density of the ideal gas to be the same as the mean occupation, up to a factor of β^{-1}. That is we make the ansatz

$$\rho_\Lambda := \frac{\langle N \rangle}{|\Lambda|} = \frac{1}{\beta}\mathbb{E}_N^B\big[\bar{\mathcal{L}}\big],$$

and we can apply the results of the preceding section to analyse the behaviour of the density in the thermodynamic limit.

2.2.2 The Intrinsic Equation of the Ideal Gas

The (thermodynamic) density of the Bose gas is defined as

$$\begin{aligned}\rho(\beta, h) &= \lim_{N\to\infty} \rho_{\Lambda_N}(\beta, h) \\ &= \lim_{N\to\infty} \frac{1}{\beta}\mathbb{E}_N^B\big[\bar{\mathcal{L}}\big],\end{aligned}$$

and applying Theorem 2.4, the density exists and is given by

$$\rho(\beta, h) = \int_{\mathbb{H}} \frac{e^{\beta h}}{e^{-\beta u} - e^{\beta h}} m_\infty(\mathrm{d}u). \tag{2.10}$$

In this section we study the behaviour of ρ as a function of h, and discuss how BEC is exhibited in the grand canonical ensemble. Considering for a minute the canonical ensemble described by $Z_\Lambda(\beta, N)$ we recall that the thermodynamic limit in this context is taken by simultaneously sending $N, |\Lambda| \to \infty$, in such a way that the density converges: $N/|\Lambda| \to \rho \in \mathbb{R}_+$, where we use ρ to distinguish between the function ρ defined above. A natural question is given a density $\rho > 0$, at what value of $h < 0$ does $\rho(\beta, h) = \rho$. The heuristic is that at this value h, it is 'equivalent' to study the grand canonical ensemble, instead of the canonical ensemble: '$\Xi_\infty(\beta, h) = Z_\infty(\beta, \rho)$'. Of course we have not defined the functions above, and this is simply a heuristic description. A rigorous account is afforded by the theory of *equivalence of ensembles*, [Hua87] Chapter 7.6. Formally we phrase this question as the following intrinsic equation.

$$\textbf{Fix } \beta > 0\textbf{. For } \rho > 0 \textbf{ find } h^* = h^*(\rho) \in (-\infty, 0) \textbf{ for which } \rho(\beta, h^*) = \rho. \tag{2.11}$$

For the time being, we will keep $\beta > 0$ fixed, and denote $\rho(h)$ for the density leaving the dependence on β implicit. The following proposition allows us to concentrate our study of the variational problem on the extreme values as h approaches $-\infty$ and 0.

Proposition 2.13. *For $h \in (-\infty, 0)$, the map $h \mapsto \rho(h)$ is a differentiable, convex, strictly increasing function.*

Proof. That the map is differentiable follows on checking the conditions which allow for differentiation under the (measure theoretic) integral sign. Let $p(h,u) = (e^{-\beta(u+h)} - 1)^{-1}$ denote the integrand in our expression for the density, then it is immediate that $u \mapsto p(h,u)$ is integrable on $u \in \mathbb{H}$, and that $h \mapsto p(h,u)$ is differentiable for $h < 0$. Furthermore, we have

$$\partial_h p(h,u) = \beta \frac{e^{-\beta(u+h)}}{(e^{-\beta(u+h)} - 1)^2},$$

which is positive and integrable, from which differentiability is confirmed [Kle08] Theorem 6.28 pp.140-2. Moreover strict positivity of $\partial_h p(h,u)$ confirms that $\rho(h)$ is strictly increasing in $h < 0$. Similarly one can check that the second derivative also exists, and is positive, and so the density is convex. □

Applying this proposition, it follows that as we take the limit $h \nearrow 0$, the density has a limit

$$\rho_c := \rho_c(\beta) = \lim_{h \nearrow 0} \rho(\beta, h),$$

which exists as an extended real number $\rho_c \in [0, \infty) \cup \{+\infty\}$, we refer to this as the *critical density*. Since the integrand $p(h,u)$, introduced in the previous proof, is strictly increasing in $h < 0$ the monotone convergence theorem gives

$$\begin{aligned}\rho_c &= \lim_{h \nearrow 0} \int_{\mathbb{H}} \frac{e^{\beta h}}{e^{-\beta u} - e^{\beta h}} m_\infty(\mathrm{d}u) \\ &= \int_{\mathbb{H}} \lim_{h \nearrow 0} \frac{e^{\beta h}}{e^{-\beta u} - e^{\beta h}} m_\infty(\mathrm{d}u) \\ &= \int_{\mathbb{H}} \frac{1}{e^{-\beta u} - 1} m_\infty(\mathrm{d}u). \end{aligned} \tag{2.12}$$

Theorem 2.14. *For all $\rho \in (0, \rho_c)$ there is a unique solution to the intrinsic equation* (2.11)*.*

Proof. It follows from Proposition 2.13 that there is a bijection between $h \in (-\infty, 0)$ and the values ρ for which there is a solution to the intrinsic equation, ensuring uniqueness of the solution. Moreover, since the density is strictly increasing, the density cannot attain either of the limit values ρ_c, or $\lim_{h \searrow -\infty} \rho(\beta, h)$. That the lower limit is in fact 0, follows on applying the monotone convergence theorem (this time for non-negative decreasing sequences of functions) to the integrand $p(h, u)$

$$\begin{aligned}\lim_{h\searrow-\infty}\rho(\beta,h) &= \lim_{h\searrow-\infty}\int_{\mathbb{H}} p(h,u)m_\infty(\mathrm{d}u)\\ &= \int_{\mathbb{H}}\lim_{h\searrow-\infty} p(h,u)m_\infty(\mathrm{d}u)\\ &= 0,\end{aligned}$$

which is the expected lower limit. □

We briefly remark that we exclude in our discussion the possibility that $h = 0$, since the Bose gas is ill-defined at this value. However, as we saw in Lemma 1.10, so long as $\kappa \not\equiv 0$, then we can define the Bosonic loop soup at $h = 0$; in this case the result above changes in that we can now solve the intrinsic equation for values $\rho \in (-\infty, \rho_c]$.

Of primary interest to us is the value $\rho_c(\beta)$, and specifically whether or not this value is finite. In case $\rho_c < \infty$ then we see that there is a break down in the equivalence of ensembles. That is, supposing that $\rho > \rho_c$, then we can clearly still define the thermodynamic limit of the canonical ensemble with $N/|\Lambda| \to \rho$, however there is no longer a grand canonical ensemble with which to relate it. This is an indicator of the BEC phase transition, where we interpret the density of particles which are not ascribed to loops as the condensate. We now ask how the geometric properties of the graph sequence Λ_N effect whether or not BEC occurs. Recalling (2.5), when $\Lambda_N = K_N$ then the critical density is finite and given by

$$\rho_c = \frac{1}{e^\beta - 1},$$

which we note agrees with Tóth's [Tót90] derivation of the critical density in the canonical ensemble on the complete graph. Einstein [Ein24], demonstrated that in the case of the ideal gas in $\mathbb{R}^3$ the critical density $\rho_c(\beta)$ is finite, and given by

$$\rho_c(\beta) = \zeta\left(\frac{3}{2}\right)\left(\frac{2\pi\hbar^2\beta}{m}\right)^{-3/2} < \infty, \tag{2.13}$$

with $\zeta(s) = \sum_{n\geq 1} n^{-s}$, the Riemann ζ-function. In the following section we consider the physically realistic case for lattice boxes converging to $\mathbb{Z}^d$, $d \geq 1$. In light of this we will then provide a characterisation for when BEC occurs for arbitrary graph sequences.

2.2.3 The Critical Density for $\mathbb{Z}^d$

Our aim in this section is to study the critical density of the ideal gas on the integer lattice $\mathbb{Z}^d$, and we prove that

$$\rho_c(\beta) := \lim_{h \nearrow 0} \rho(\beta, h) \begin{cases} = \infty & \text{if } d = 1, 2, \\ < \infty & \text{if } d \geq 3, \end{cases}$$

so that condensation of the ideal gas occurs only in dimensions 3 and above. From (2.12) it is sufficient to ask whether or not

$$\int_{\mathbb{H}} \frac{1}{e^{-\beta u} - 1} m_\infty(\mathrm{d}u) < \infty.$$

We consider first the case $d = 1$, and the vertex set of a graph will be $[-N, N] \subset \mathbb{Z}$. On page 9 we defined two different graph sequences $\Lambda_N^{(dir)}$, $\Lambda_N^{(per)}$ and commented that in the limit these are spectrally equivalent, in as much that $m_\infty^{(dir)} = m_\infty^{(per)}$; in the following when we refer to properties of $\mathbb{Z}^d$, we mean with respect to either of these limits. Due to the spectral equivalence our results will hold for either definition, and as a matter of taste we will work with the periodic graph, $\Lambda^{(per)}$.
Since the generator matrix Q associated with $\Lambda_N^{(per)}$ is circulant, its eigenvalues take on a particularly simple form (our reason for choosing the periodic box, over the case with Dirichlet boundaries)

$$\mathrm{Spec}(Q) = \left\{ \cos\left(2\pi \frac{j}{2N+1}\right) - 1 : \, j = 1, \ldots, 2N+1 \right\}.$$

and the canonical spectral distributions converge to $\phi_\infty(u) = \cos(\pi u) - 1$, $u \in (0, 1]$, see Appendix A.

Proposition 2.15. *For $h < 0$, the density $\rho(h)$ of $\mathbb{Z}$ admits the expressions*

$$\begin{aligned} \rho(h) &= \int_{[0,1]} \frac{e^{\beta h}}{e^{\beta(1-\cos(\pi u))} - e^{\beta h}} \mathrm{d}u \\ &= \int_{[-2,0]} \frac{e^{\beta h}}{e^{\beta u} - e^{\beta h}} \frac{1}{\sqrt{u(2-u)}} \mathrm{d}u. \end{aligned}$$

In addition it has the power series representation

$$= \sum_{j \geq 1} I_0(\beta j) e^{-\beta j(1-h)},$$

where I_0 is the modified Bessel function of the first kind.

Proof. The first expression follows on using the change of variables formula (1.4) with the integral representation for the density, (2.10). For the second expression we consider the cumulative density function

$$m_\infty(\{u \le x\}) = \int_0^1 \mathbf{1}_{\{\cos(\pi u)-1\le x\}}\mathrm{d}u,$$

solving $\cos(\pi u) = x - 1$, we find

$$= 1 - \frac{1}{\pi}\cos^{-1}(x-1),$$

which we recognise as the cumulative distribution function of an arc-sine random variable on $[-2, 0]$. Hence m_∞ has probability density function $(-u(2+u))^{-1/2}$, on $[0, 2]$. Writing this instead on $[-2, 0]$, we have the given formula. Finally, to derive the power series expansion we recall that $\rho = \sum_{j\ge1} jn_\infty^{(j)}$, where $n_\infty^{(j)}$ are the limit densities of βj loops, Corollary 2.6. Then

$$\begin{aligned} n_\infty^{(j)}(h) &= \frac{1}{j}e^{\beta jh}\int_{[0,1]} e^{-\beta j(1-\cos(\pi u))}\mathrm{d}u \\ &= \frac{1}{j}e^{\beta j(h-1)}\int_{[0,1]} e^{\beta j\cos(\pi u)}\mathrm{d}u. \end{aligned}$$

The integral in the second line can be recognised as $I_0(\beta j)$, the modified Bessel function of the first kind, [AS64] formula 9.6.16. □

The derivation of the spectrum for $\Lambda_N^{(per)}$ when $d \ge 2$ is done in detail in Appendix A, and we only briefly mention the approach here. We rely on the fact that the interpretation of $\mathbb{Z}^2 = \mathbb{Z}\times\mathbb{Z}$ as a cartesian product has an analogue description for graphs. Moreover, the transition matrix of a cartesian product of two graphs is the Kronecker sum (defined in the appendices) of the two matrices, from which we can derive the spectrum of the product graph. In turn this carries through to d-products of graphs, and in the context of $\Lambda^{(per)}$ we find that the spectrum is given by

$$\mathrm{Spec}(Q) = \left\{\frac{1}{d}\sum_{i=1}^{d}\cos\left(2\pi\frac{j_i}{2N+1}\right) - 1 :\ j = (j_i)_{i=1}^d \in \{1,\dots,2N+1\}^d\right\},$$

from which we derive spectral convergence, and in particular we have the limiting distribution function $\phi_\infty\colon [0,1)^d \to [-2,0]$ defined by

$$\phi_\infty(u) = \frac{1}{d}\sum_{i=1}^{d}\cos(2\pi u_i) - 1.$$

where $u = (u_1,\dots,u_d)$.

Proposition 2.16. *For $h < 0$, the density $\rho(h)$ of $\mathbb{Z}^d$, $d \ge 1$ has the power series* *expansion*

$$\rho(\beta, h) = \sum_{j\geq 1} e^{\beta j(h-1)} I_0\Big(\frac{\beta}{d} j\Big)^d.$$

Proof. The case for $d = 1$ has already been shown in Proposition 2.15. Following the same method

$$\begin{aligned}\rho(h) &= \sum_{j\geq 1} j n_\infty^{(j)}(h) \\ &= \sum_{j\geq 1} e^{\beta j(h-1)} \int_{[0,1)^d} \exp\Big(-\frac{\beta}{d} j \sum_{i=1}^d \cos(\pi u_i)\Big) \mathrm{d}u \\ &= \sum_{j\geq 1} e^{\beta j(h-1)} \Big(\int_{[0,1)} e^{-\frac{\beta}{d} j\cos(\pi u)} \mathrm{d}u\Big)^d \\ &= \sum_{j\geq 1} e^{\beta j(h-1)} I_0\Big(\frac{\beta}{d} j\Big)^d. \end{aligned}$$

□

We are now in a position to prove our result for the critical density in $\mathbb{Z}^d$.

Theorem 2.17. *For $\beta > 0$,*

$$\rho_c(\beta) := \lim_{h\nearrow 0} \rho(\beta, h) \begin{cases} = \infty & \text{if } d = 1, 2, \\ < \infty & \text{if } d \geq 3. \end{cases}$$

Proof. As previously remarked, it suffices to consider the integrability of

$$\rho_c(\beta) = \int_{\mathbb{H}} \frac{1}{e^{-\beta u} - 1} m_\infty(\mathrm{d}u) = \sum_{j\geq 1} e^{-\beta j} I_0\big(\tfrac{\beta}{d} j\big)^d.$$

Consider the function

$$f(w) := e^{-\beta w} I_0\Big(\frac{\beta}{d} w\Big)^d, \qquad w > 0$$

differentiating in $w > 0$ we have

$$f'(w) = \beta e^{-\beta w} I_0\Big(\frac{\beta}{d} w\Big)^{d-1} \Big(I_1\Big(\frac{\beta}{d} w\Big) - I_0\Big(\frac{\beta}{d} w\Big)\Big).$$

It follows from positivity of I_0, that $f(w) > 0$, and moreover since $I_0(w) > I_1(w)$ the bracketed term above is negative, $f'(w) < 0$, and f is a positive decreasing function in $w > 0$. Then the integral test asserts

$$\int_1^\infty f(w)\mathrm{d}w \leq \rho_c \leq \int_0^\infty f(w)\mathrm{d}w. \tag{2.14}$$

We compute the integrals above by the manipulating the following identity [AS64] 9.6.10,

$$I_0(w) = \sum_{n=0}^{\infty} \binom{2n}{n} \frac{\left(\frac{1}{2}w\right)^{2n}}{(2n)!},$$

Taking the d-th power we obtain

$$I_0\left(\frac{w}{d}\right)^d = \sum_{n=0}^{\infty} \sum_{\sum k_j = n} \binom{2k_1}{k_1} \cdots \binom{2k_d}{k_d} \frac{\left(\frac{1}{2d}w\right)^{2k_1} \cdots \left(\frac{1}{2d}w\right)^{2k_d}}{(2k_1)! \cdots (2k_d)!},$$

where the sum runs over $\underline{k} = (k_1, \dots, k_d)$ such that $k_1 + \cdots + k_d = n$

$$\begin{aligned} &= \sum_{n=0}^{\infty} \left(\frac{1}{2d}w\right)^{2n} \sum_{\sum k_j = n} \binom{2k_1}{k_1} \cdots \binom{2k_d}{k_d} \frac{1}{(2k_1)! \cdots (2k_d)!} \\ &= \sum_{n=0}^{\infty} \left(\frac{1}{2d}w\right)^{2n} \frac{1}{(2n)!} \sum_{\sum k_j = n} \binom{2k_1}{k_1} \cdots \binom{2k_d}{k_d} \binom{2n}{2k_1, \dots, 2k_d}. \end{aligned}$$

The final term is recognised as the combinatorial factor which counts the total number of d-dimensional lattice paths of length $2n$ which start and end at the same point. Together with the weighting $(2d)^{-2n}$, this is exactly the probability that the simple random walk on $\mathbb{Z}^d$ is at the origin after $2n$ steps. Moreover, since the walk can only be at the origin after an even number of steps we have

$$I_0\left(\frac{w}{d}\right)^d = \sum_{n=0}^{\infty} \mathbf{P}_0[Z_n = 0] \frac{w^n}{n!}.$$

Returning to the computation of the integral of $f(w)$

$$\begin{aligned} \int_0^{\infty} e^{-\beta w} I_0\left(\frac{\beta}{d}w\right)^d \mathrm{d}w &= \frac{1}{\beta} \int_0^{\infty} e^{-w} I_0\left(\frac{w}{d}\right)^d \mathrm{d}w \\ &= \frac{1}{\beta} \sum_{k=0}^{\infty} \mathbf{P}_0[Z_k = 0] \frac{1}{k!} \int_0^{\infty} e^{-w} w^k \mathrm{d}w \end{aligned}$$

and recognising this final integral as the Γ-function we have

$$\begin{aligned} &= \frac{1}{\beta} \sum_{k=0}^{\infty} \mathbf{P}_0[Z_k = 0] \\ &= \frac{1}{\beta} G(Z), \end{aligned}$$

with $G(Z)$ the Green's function of $\mathbb{Z}^d$, (1.1). Noting that $\lim_{w\downarrow 0} f(w) = 1$, then

$$\int_0^1 f(w) = \frac{1}{\beta}\int_0^1 e^{-w} I_0\Big(\frac{w}{d}\Big)^d =: \frac{1}{\beta} C_d < \infty.$$

Then (2.14) gives

$$\frac{1}{\beta}(G(Z) - C_d) \le \rho_c(\beta) \le \frac{1}{\beta} G(Z).$$

Finally we appeal to Polya's theorem, which asserts that $G(Z)$ is finite if and only if $d \le 3$ [LL10] Theorem 4.1.1 p.75, from which our claim follows. □

Further analysis of the critical density reveals the following asymptotic formula, which is comparable to the equation (2.13) derived by Einstein.

Theorem 2.18. *For $d \ge 3$,*

$$\rho_c(\beta) \sim \beta^{-\frac{d}{2}} \Big(\frac{d}{2\pi}\Big)^{\frac{d}{2}} \zeta\Big(\frac{d}{2}\Big), \qquad \beta \to \infty,$$

where $\zeta(s) := \sum_{k\ge 1} k^{-s}$ is the Riemann ζ-function.

Proof. As in the previous theorem we write

$$\begin{aligned}\rho_c(\beta) &= \sum_{j\ge 1} e^{-\beta j} I_0\big(\tfrac{\beta}{d} j\big)^d \\ &= \sum_{j\ge 1} \Big(e^{-\frac{\beta}{d} j} I_0\big(\tfrac{\beta}{d} j\big)\Big)^d.\end{aligned}$$

Using the asymptotics $I_0(z) \sim (2\pi z)^{-\frac{1}{2}} e^z$ [AS64] 9.7.1, we have for $\beta \to \infty$

$$\rho_c(\beta) \sim \Big(\frac{d}{2\pi\beta}\Big)^{\frac{d}{2}} \sum_{j\ge 1} j^{-\frac{d}{2}},$$

which is exactly as required. □

Ultimately we have demonstrated that for $\mathbb{Z}^d$, the critical density is closely related with the Green's function G of a random walk, and consequently with the transience or recurrence of the graph. We would like therefore to develop a similar relationship for more general graphs, however since our definition of graph convergence did not require the limit graph to exist, we cannot simply say that BEC is the same as transience of the limit graph. For example, we have already seen that BEC occurs when we consider the graph sequence K_N, however there is no limit object towards which this sequence converges, and so we cannot speak of transience. We can however describe a heuristic link between BEC and the underlying random walk. Defining the mean local time of the walk $\overline{X}$ on a graph Λ as

$$\overline{G}_\Lambda := \overline{G}_\Lambda(\overline{X}) = \frac{1}{|\Lambda|}\sum_{x\in\Lambda} G_\Lambda(\overline{X})_{xx},$$

then using Corollary 1.5, and that the generator of the walk $\overline{X}$ is $Q = \lambda(P - I)$

$$\begin{aligned}\overline{G}_\Lambda &= \frac{1}{|\Lambda|}\sum_{x\in\Lambda}(-Q)^{-1}_{xx}\\ &= -\frac{1}{|\Lambda|}\sum_{\eta\in\mathrm{Spec}(Q^{-1})}\eta\\ &= -\frac{1}{|\Lambda|}\sum_{\eta\in\mathrm{Spec}(Q)}\frac{1}{\eta},\end{aligned}$$

since the eigenvalues of Q^{-1} are the reciprocals of the eigenvalues of Q. Consequently we have the integral expression

$$\overline{G}_\Lambda = -\int_{\mathbb{H}}\frac{1}{u}m_\Lambda(\mathrm{d}u).$$

Up to this point the calculations are rigorous. Suppose that we can justify taking the limit inside the integral, for the sake of argument we define

$$\overline{G}_\infty = -\int_{\mathbb{H}}\frac{1}{u}m_\infty(\mathrm{d}u),$$

then $\overline{G}_\infty$ is in some sense an indicator of transience or recurrence of the graph limit. But recalling the integral form of the critical density

$$\rho_c = \int_{\mathbb{H}}\frac{1}{e^{-\beta u}-1}m_\infty(\mathrm{d}u),$$

and taking the power series expansion of the exponential

$$\begin{aligned}&= \frac{1}{\beta}\int_{\mathbb{H}}\frac{1}{-u+u^2-u^3+\cdots}m_\infty(\mathrm{d}u)\\ &= -\int_{\mathbb{H}}O\left(\frac{1}{u}\right)m_\infty(\mathrm{d}u),\end{aligned}$$

so that integrability of ρ_c is equivalent to finiteness of $\overline{G}_\infty$.
In the case that the limit graph is *recurrent*, that is u^{-1} is not integrable under m_∞, the argument above becomes rigorous since then Fatou's lemma gives

$$\liminf_{N\to\infty}\overline{G}_{\Lambda_N} = \liminf_{N\to\infty}\int_{\mathbb{H}}\frac{1}{\phi_N(u)}\mathrm{d}u \geq \int_{\mathbb{H}}\frac{1}{\phi_\infty(u)}\mathrm{d}u = \infty.$$

This is not the only case where the recurrence assumption facilitates our analysis.

Recall that in the previous section, Theorem 2.10, we proved an LDP for the density under the caveat that the function $\mathbf{\Lambda}$ must be steep at the boundary. Then from Lemma 2.9

$$\mathbf{\Lambda}'(s) \;=\; \int_{\mathbb{H}} \frac{1}{e^{-\beta(h+s+u)}-1} m_\infty(\mathrm{d}u) \;=\; \rho(h+s). \tag{2.15}$$

Assuming the graph is recurrent, the boundary is at $s=|h|$, and $\lim_{s\nearrow|h|}\rho(h+s)=\infty$, which is to say $\mathbf{\Lambda}$ is steep.

Corollary 2.19. *LDP of Theorem 2.10 holds whenever* $\rho_c=\infty$.

Recalling that the rate function $I(x)$ was given by the variational problem (2.9), then for a given $x\in\mathbb{R}$ the supremum is achieved at the value $s_x\in\mathbb{R}$ which solves

$$x \;=\; \mathbf{\Lambda}'(s_x) \;=\; \rho(h+s_x),$$

so that the LDP variational problem is none other than a shift of the intrinsic equation for the ideal gas (2.11).
As a consequence of Corollary 2.19 we have that the LDP holds for lattice graphs in $d=1,2$. Let us consider the case of $\mathbb{Z}^d$ further; from Lemma 2.9 , and the power series expression from Proposition 2.16 we have

$$\mathbf{\Lambda}(s)=\sum_{j\geq 1}\frac{1}{j}\left(e^{\beta j(h+s-1)}-e^{\beta j(h-1)}\right)I_0\left(\frac{\beta}{d}j\right)^d,$$

which converges whenever $\sum_{j\geq1} j^{-1}e^{\beta j(h+s-1)}I_0\left(\frac{\beta}{d}j\right)^d$ does. Setting $z=e^{\beta(h+s)}$, we have a power series in z^j, with coefficients $j^{-1}e^{-\beta j}I_0\left(\frac{\beta}{d}j\right)^d$, and

$$\begin{aligned}\sqrt[j]{\frac{1}{j}e^{-\beta j}I_0\left(\frac{\beta}{d}j\right)^d} &\sim \sqrt[j]{\frac{1}{j}e^{-\beta j}\left(e^{\frac{\beta j}{d}}\left(2\pi\frac{\beta}{d}j\right)^{-\frac{1}{2j}}\right)^d}\\ &= \left(\left(\frac{2\pi\beta}{d}\right)^{\frac{d}{2}} j^{-\frac{d+1}{2}}\right)^{\frac{1}{j}},\end{aligned}$$

where we applied the same asymptotics for I_0 as before. Since $j^{-\alpha/j}\to 1$ as $j\to\infty$ for all $\alpha>0$, the Cauchy root test ensures that the radius of convergence (as a function of z) is 1, or equivalently that the series converges for $s<|h|$. Moreover, Pringsheim's theorem for power series with positive coefficients [FS09] Theorem IV.6 pp.240-1, asserts that the series must diverge at $s=|h|$, so that $|h|$ is the boundary of convergence for $\mathbf{\Lambda}$. However for $d\geq 3$, using (2.15)

$$\lim_{s\nearrow|h|}\mathbf{\Lambda}'(s) \;=\; \lim_{s\nearrow|h|}\rho(h+s) \;=\; \rho_c \;<\; \infty.$$

In particular we have shown that $\mathbf{\Lambda}$ is not steep at its boundary, and so our LDP as stated cannot be applied in this context.
In the above we have provided two examples of how proofs are facilitated by working in the recurrent regime. Having said that, we still believe that they hold in both phases so long as $h < 0$. We state this in the following conjecture.

Conjecture 2.20. *For any spectrally convergent graph sequence:*

(i) The sequence $\big(\overline{\mathcal{L}}, \mathbb{P}_N^B\big)_{N\geq 1}$ *satisfies an LDP with rate* $|\Lambda_N|$ *and good rate function*

$$I(x) = \sup_{s\in\mathbb{R}} \big(sx - \mathbf{\Lambda}(s)\big).$$

(ii) The limit $\overline{G}_\infty = \lim_{N\to\infty} \overline{G}_{\Lambda_N}$ *exists as an extended real number, and is finite if and only if* $\rho_c < \infty$.

We return to study the first part (the LDP) in Chapter 4, where we will state a stronger form of this statement working on the space of cycle distributions. In this case, we will be able to prove the theorem in both the recurrent and transient setting.

Chapter 3

Bosonic Occupation Fields and their Isomorphism Theorems

In this chapter we look more closely at the geometry of the occupation field, and in particular we seek to describe it from a Gaussian perspective. The starting point for this analysis will be a study of the Laplace transform of the occupation field for which we derive an exact expression, which unfortunately we will not be able to relate to any well known distribution. In the second section we circumvent this by introducing a *space-time* random walk, for which the Markov loop soup can be related to the Bosonic soup. With this new occupation field we will not only be able to provide a Gaussian characterisation of the Bose gas, but also find a geometric interpretation of the 1-particle reduced density matrix as a correlation in the occupation field. Finally in the third section we relate our findings to Symanzik's formula to a model for an ideal gas in a space-time random environment.
Since we work with the distribution of the entire occupation field, we will not consider graph limits as in the previous section (since as mentioned before, the limit object need not exist), and for the most part we work on a single fixed graph Λ. Throughout the chapter we work under the following assumption

A2′ The inverse temperature is strictly positive $\beta > 0$, and either: $\kappa \equiv 0$ and $h < 0$, or $\kappa \not\equiv 0$ and $h \leq 0$.

The assumption **A2′** is a weakening of **A2** introduced in Chapter 2, where we insisted that $h < 0$ for any graph. This limitation was implemented due to the fact that the Bose gas is only defined for $h < 0$; in this chapter our focus is more on the occupation field, and so long as the measures μ, μ^B are σ-finite (respectively, finite) (which is the content of the assumption) the field is well defined.

3.1 The Laplace Transform of the Occupation Field

Let $f\colon \Gamma \to \mathbb{R}$ be a functional on the space of loops, and define F on configurations of loops $S \sqsubset \Gamma$ by

$$F(S) := \sum_{\gamma \in S} f(\gamma),$$

and say that F is the *additive* functional of loop configurations associated with f. The Campbell formula gives an expression for the *Laplace functional* associated with F. In the following we denote ν for an arbitrary σ-finite measure on $(\Gamma, \mathcal{G})$.

Lemma 3.1. *Let $f\colon \Gamma \to \mathbb{R}_+$ be positive and measurable, and denote F for the associated additive functional. Then*

$$\mathbb{E}_\nu\Big[e^{-F(S)}\Big] = \exp\Big(-\int_\Gamma 1 - e^{-f(\gamma)}\nu(\mathrm{d}\gamma)\Big).$$

If $\int(f \wedge 1)\mathrm{d}\nu < \infty$, then the above holds on replacing $f \mapsto if$.

See [Kal01], pp.227–8 for a proof for general Poisson point processes. Our analysis of the occupation field $\mathcal{L}$ will proceed by applying this formula in the case of the Laplace transform of the occupation field. Given $u, v\colon \Lambda \to \mathbb{R}$ we denote $\langle u, v\rangle = \sum_{x\in\Lambda} u(x)v(x)$ for their inner product; with regards to notation we will swap between writing v_x and $v(x)$, using whichever is clearer in the context. Given a positive function $v\colon \Lambda \to \mathbb{R}_+$, or equivalently a positive vector $v \in \mathbb{R}_+^\Lambda$, the Laplace transform of $\mathcal{L}$ against v is given by the expectation

$$\mathbb{E}_\nu\Big[e^{-\langle v, \mathcal{L}\rangle}\Big].$$

The Campbell formula allows us to derive the following expression.

Proposition 3.2. *For $v \in \mathbb{R}_+^\Lambda$*

$$\mathbb{E}_\nu\Big[e^{-\langle v, \mathcal{L}\rangle}\Big] = \exp\Big(-\int_\Gamma 1 - \exp\Big(-\textstyle\int_0^{|\gamma|} v(\gamma_s)\mathrm{d}s\Big)\nu(\mathrm{d}\gamma)\Big). \tag{3.1}$$

Proof. This follows from the definitions in Section 1.2.3, since for a configuration $S \sqsubset \Gamma$

$$\langle v, L(S)\rangle \;=\; \sum_{x\in\Lambda} v_x L_x(S) \;=\; \sum_{\gamma\in S}\sum_{x\in\Lambda} v_x L_x(\gamma) \;=\; \sum_{\gamma \in S}\langle v, L(\gamma)\rangle,$$

so that the occupation field is an additive functional. But further for $\gamma \in \Gamma$

$$\langle v, L(\gamma)\rangle \;=\; \sum_{x\in\Lambda} v_x \int_0^{|\gamma|} \delta_x\big(\gamma(s)\big)\mathrm{d}s \;=\; \int_0^{|\gamma|} v(\gamma(s))\mathrm{d}s. \qquad \square$$

In the case that the loop measure ν is either μ, or μ^B, the integral on the right hand side of (3.1) can be computed with the help of the Feynman–Kac formula for random walks. We appeal to the following version proven in [Szn12], pp.23–4, and p.29.

Theorem 3.3 (Feynman–Kac)**.** *Let $v, f\colon \Lambda \to \mathbb{R}$. For $x \in \Lambda$, and $T \geq 0$*

$$\mathbf{E}_x\left[f(\overline{X}_T)\exp\left(\int_0^T v(\overline{X}_s)\mathrm{d}s\right)\right] = \left(e^{T(Q+V)}f\right)(x), \tag{3.2}$$

where $V = \operatorname{diag}(v)$.

3.1.1 The Calculation for $\mathbb{P}^B$

This section is devoted to obtaining an expression for the Laplace transform of $\mathcal{L}$ under the Bosonic loop soup $\mathbb{P}^B$. We note that unlike the result of Le Jan [LeJ10, LeJ11] for the occupation field under $\mathbb{P}_0$, the Markov loop soup at $h = 0$, we do not put any restrictions on the graph structure. Le Jan's proof required that the graph be reversible, which will turn out not to be necessary here. We return to this remark in Section 3.2 where we present a form of Le Jan's theorem which holds for graphs with *normal* transition matrices, which will be important in our study of the space-time loop soup.
Before identifying the Laplace transform of the occupation field, we state the following proposition which will be required in the proof; we recall that for $v\colon \Lambda \to \mathbb{R}$ we denote $V = \operatorname{diag}(v)$ for the $|\Lambda| \times |\Lambda|$ matrix with v on the diagonal.

Proposition 3.4. *For $c \in \mathbb{H}$, $v\colon \Lambda \to \mathbb{R}_+$, the eigenvalues of $Q + hI + cV$ have strictly negative real part*

$$\operatorname{Spec}\left(Q + hI + cV\right) \subset \operatorname{int}\mathbb{H}.$$

Proof. The proof builds on that of Theorem 1.6. As in that result, it suffices to check that the union of the Geršgorin disks is contained in $\operatorname{int}\mathbb{H}$. We have

$$\operatorname{Spec}(Q + hI + cV) \subset \bigcup_{x\in\Lambda}\left\{z \in \mathbb{C} :\ |z + \lambda_x - h - cv_x| \leq \left|\lambda_x \textstyle\sum_{y\in\Lambda} P_{xy}\right|\right\}.$$

This is a subset of $\mathbb{H}$ if and only if for all $x \in \Lambda$

$$\operatorname{Re}\left(-(\lambda_x - h - cv_x) + \lambda_x \sum_{y\in\Lambda} P_{xy}\right) \leq 0,$$

or equivalently

$$h + \mathrm{Re}(c)v_x \le \lambda_x \left(1 - \sum_{y\in\Lambda} P_{xy} \right).$$

Noting that the left hand side is negative, whilst the right hand side is positive, it follows that $\mathrm{Spec}(Q + hI - cV) \subset \mathbb{H}$. According to [HJ13] Corollary 6.2.9 p.399, so long as the inequality above is strict for at least one $x \in \Lambda$, then $Q + hI + cV$ is non-singular, and hence $\mathrm{Spec}(Q + hI + cV) \subset \mathrm{int}\,\mathbb{H}$. But this is the case if either $h < 0$, or if $\kappa \not\equiv 0$, which is exactly assumption $\mathbf{A2'}$. □

Theorem 3.5. *For* $v \colon \Lambda \to \mathbb{R}_+$

$$\mathbb{E}^B\left[e^{-\langle v,\mathcal{L}\rangle}\right] = \frac{\det(I - e^{\beta(Q+hI)})}{\det(I - e^{\beta(Q+hI-V)})} = \frac{\det(e^{-\beta hI} - e^{\beta Q})}{\det(e^{-\beta hI} - e^{\beta(Q-V)})}. \tag{3.3}$$

where $V = \mathrm{diag}(v)$*. Moreover, the formulae continue to hold on replacing* $v \mapsto iv$*.*

Proof. We consider the integral expression in (3.1), and note that since $\mu^B(\Gamma) < \infty$ this expression continues to hold for iv, courtesy of Lemma 3.1. From the definition of the loop measure (1.6), we have

$$\begin{aligned}\int_\Gamma 1 - e^{-\langle v,L(\gamma)\rangle}\mu^B(\mathrm{d}\gamma) &= \sum_{x\in\Lambda}\sum_{j\ge1}\frac{e^{\beta hj}}{j}\mathbf{E}_x\left[\mathbf{1}_{\{\overline{X}_{\beta j}=x\}}\left(1 - e^{-\int_0^{\beta j} v(\overline{X}_s)\mathrm{d}s}\right)\right] \\ &= \sum_{x\in\Lambda}\sum_{j\ge1}\frac{e^{\beta hj}}{j}\left(\mathbf{E}_x\left[\mathbf{1}_{\{\overline{X}_{\beta j}=x\}}\right] - \mathbf{E}_x\left[\mathbf{1}_{\{\overline{X}_{\beta j}=x\}}e^{-\int_0^{\beta j} v(\overline{X}_s)\mathrm{d}s}\right]\right).\end{aligned}$$

Applying the Feynman–Kac formula to both expectations

$$= \sum_{x\in\Lambda}\sum_{j\ge1}\frac{e^{\beta hj}}{j}\left(\left(e^{\beta jQ}\right)_{xx} - \left(e^{\beta j(Q-V)}\right)_{xx}\right),$$

and since $e^{\beta hI} = e^{\beta h}I$ commutes with any matrix, we can take the product of matrix exponentials

$$\begin{aligned} &= \sum_{j\ge1}\frac{1}{j}\left(\mathrm{Tr}\left[e^{\beta j(Q+hI)}\right] - \mathrm{Tr}\left[e^{\beta j(Q+hI-V)}\right]\right) \qquad (3.4) \\ &= -\log\det\left(I - e^{\beta(Q+hI)}\right) + \log\det\left(I - e^{\beta(Q+hI-V)}\right).\end{aligned}$$

The final line follows on using that for $A \in \mathbb{C}^{n\times n}$

$$\sum_{j\ge1}\frac{1}{j}\mathrm{Tr}[A^j] = \sum_{\eta\in\mathrm{Spec}(A)}\sum_{j\ge1}\frac{1}{j}\eta^j = -\sum_{\eta\in\mathrm{Spec}(A)}\log(1-\eta) = -\log\det(I - A),$$

so long as $|\eta| < 1$ for all $\eta \in \mathrm{Spec}(A)$. In our context, as a consequence of Proposition 3.4 and that $\beta > 0$

$$\rho\Big(e^{\beta(Q+hI-V)}\Big) \;=\; \max_{\eta}|e^{\eta}| \;=\; \max_{\eta} e^{\mathrm{Re}(\eta)} \;<1. \qquad \square$$

Having derived an expression for the Laplace transform it makes sense to ask what we can do with it! Unfortunately the formula does not prove easy to wield, for instance even providing the distribution of the occupation field at a single point seems beyond possibility. As an example, consider the easier task of deriving the expected value of the occupation field at the point $x \in \Lambda$. Let $\tilde{v} \in \mathbb{R}^{\Lambda}$ be the vector which is zero in all entries except $\tilde{v}_x = v$. Then

$$\mathbb{E}^B[\mathcal{L}_x] \;=\; -\frac{\mathrm{d}}{\mathrm{d}v}\mathbb{E}^B\Big[e^{-\langle\tilde{v},\mathcal{L}\rangle}\Big]_{|v=0} \;=\; -\left(\frac{\mathrm{d}}{\mathrm{d}v}\prod_{i=1}^{\Lambda}\frac{1-e^{\beta(\eta_i+h)}}{1-e^{\beta(\eta_i^v+h)}}\right)\Bigg|_{v=0},$$

where we denoted $\mathrm{Spec}(Q) = \{\eta_i\}_{i=1}^{|\Lambda|}$, $\mathrm{Spec}(Q-\tilde{V}) = \{\eta_i^v\}_{i=1}^{\Lambda}$. The problem we face is that even when $\tilde{V}$ is as simple as it is in this case, it has only one non-zero diagonal entry, we cannot give an explicit formula for the eigenvalues of $Q-V$ in terms of those of Q. Since even this example proves troublesome, we have little hope for deriving higher order correlations. There is, perhaps unsurprisingly, one expression we can obtain from (3.3) however, which is the distribution of the mean occupation. Setting $\tilde{v}_x = v$ for all $x \in \Lambda$, the eigenvalues of $\eta_i^v \in \mathrm{Spec}(Q)$ are now given by $\eta_i^v = \eta_i - v$, so that

$$\mathbb{E}^B\Big[e^{-\langle\tilde{v},\mathcal{L}\rangle}\Big] \;=\; \frac{\det(I-e^{\beta(Q+hI)})}{\det(I-e^{\beta(Q+hI-V)})} \;=\; \prod_{i=1}^{|\Lambda|}\frac{1-e^{\beta\eta_i+h}}{1-e^{\beta\eta_i-v+h}},$$

which on taking logarithms, and rephrasing as an integral over the spectral measure, is seen to agree with (2.2) the cumulant generating function for the mean occupation $\overline{\mathcal{L}}$.

3.1.2 The Calculation for $\mathbb{P}$

In this section we provide the complementary result for the occupation field of the Markov loop soup, $(\mathcal{L},\mathbb{P})$, which will turn out to be easier to manipulate. The formula for the Laplace transform of the occupation field $\mathcal{L}$ under $\mathbb{P}$ was derived by Le Jan [LeJ10, LeJ11], though we reference [Szn12] Proposition 3.7 pp.76–80, and Theorem 4.3 pp.87–89, for a detailed derivation. Our purpose for outlining this theorem will become clear in Section 3.2, where we will re-envisage the Bosonic loop measure as a limit of a particular sequence of Markov loop measures. One small obstacle which we must overcome, however, is that as it stands, the proof of Le Jan holds under the assumption that the random walk is reversible, which we recall requires $\lambda_x P_{xy} = \lambda_y P_{yx}$, for all $x, y \in \Lambda$. As we will see, the random walks which will be of interest in our context will not have this property, and so we introduce an alternative condition.We will require that the transition matrix is *normal*, that is

$$P^*P = PP^*,$$

where P^* is the conjugate transpose of P, $P^*_{xy} = \overline{P}_{yx}$; of course since P is a real matrix, this simply says $P^TP = PP^T$. Following Le Jan, we make a change in our conventions: for this section we consider the Markov loop measure to be induced by the walk X, with unit jump rates, rather than the walk $\overline{X}$ which has been considered up until now. To balance this change, we will then scale the occupation field by considering $(\lambda_x^{-1}\mathcal{L}_x)_{x\in\Lambda}$. Recalling the scaling relation Proposition 1.2, this is equivalent to considering the unscaled field $\mathcal{L}$ under the Markov measure associated with $\overline{X}$.

Since our proof only deviates from that of Le Jan in a few places, we provide only a sketch proof and refer the reader to [Szn12] for the additional details. As with the case of the Bosonic loop measure, it suffices to study the integral expression in (3.1), where we replace ν with $\mu = \mu_h$, and we see for $v \in \mathbb{R}_+$

$$\begin{aligned}\int_\Gamma 1 - e^{-\langle v,\lambda^{-1}\mathcal{L}\rangle}\mu(\mathrm{d}\gamma) &= \int_\Gamma 1 - e^{-\langle v/\lambda,\mathcal{L}\rangle}\mu(\mathrm{d}\gamma)\\ &= \sum_{x\in\Lambda}\int_0^\infty \frac{e^{ht}}{t}\mathbf{E}_x\Big[\mathbf{1}_{\{X_t=x\}}\Big(1 - e^{-\int_0^t v/\lambda(X_s)\mathrm{d}s}\Big)\Big]\mathrm{d}t.\end{aligned}$$

As before we apply the Feynman–Kac formula, Theorem 3.3, to each of the two terms in the expectation

$$= \sum_{x\in\Lambda}\int_0^\infty \frac{e^{ht}}{t}\Big(\Big(e^{t(P-I)}\Big)_{xx} - \Big(e^{t(P-I-V/\lambda)}\Big)_{xx}\Big)\mathrm{d}t,$$

where we note that the expression is in terms of $P-I$ in place of Q since we assumed the measure is driven by the unit rate walk X; the notation V/λ is understood to mean the diagonal matrix with entries v_x/λ_x. Continuing as for the Bosonic case

$$= \int_0^\infty \frac{e^{(h-1)t}}{t}\Big(\mathrm{Tr}\big[e^{tP}\big] - \mathrm{Tr}\Big[e^{t(P-V/\lambda)}\Big]\Big)\mathrm{d}t.$$

This is the point at which the proof diverges from that of the Bosonic measure. The analogous line in the Bosonic case is given in (3.4), and we justified that we could perform the summation over $j \geq 1$. In the Markov case we must do some more work before getting an expression as a sum. In particular, expanding the matrix exponentials, and changing the order of summation we obtain

$$= \sum_{k\geq 1}\Big(\mathrm{Tr}[P^k] - \mathrm{Tr}[(P-V/\lambda)^k]\Big)\int_0^\infty \frac{e^{(h-1)t}}{t}\frac{t^k}{k!}\mathrm{d}t. \quad (3.5)$$

Pausing to consider the integral term, we note

$$\begin{aligned}\int_0^\infty \frac{e^{(h-1)t}}{t}\frac{t^k}{k!}\mathrm{d}t &= \frac{1}{(1-h)^k}\frac{1}{k!}\int_0^\infty e^{-t}t^{k-1}\mathrm{d}t \\ &= \frac{1}{k(1-h)^k},\end{aligned}$$

where we recognise the latter integral as that of $\Gamma(k)$, the Γ-function [AS64], 6.1.1. p.255. Consequently we have arived at

$$\int_\Gamma 1 - e^{-\langle v,L\rangle}\mathrm{d}\mu_h = \sum_{k\geq 1}\frac{1}{k}\Big(\mathrm{Tr}\Big[\big(\tfrac{1}{1-h}P\big)^k\Big] - \mathrm{Tr}\Big[\big(\tfrac{1}{1-h}(P-V/\lambda)\big)^k\Big]\Big),$$

which is now familiar on recalling (3.4), with the exception that the matrices are no longer exponentiated. It is at this point at which the proof for the Markov loop measure becomes somewhat more delicate. We recall that courtesy of Proposition 3.4, the expression (3.4) converged for any choice of generator Q and $v \in \mathbb{R}_+^\Lambda$. The difference now is that rather than wanting the spectral radius of an exponential matrix $\exp(P - V/\lambda)$ to be bounded by 1, we require that $\rho\Big(\frac{1}{1-h}\big(P - V/\lambda\big)\Big) < 1$. Supposing that this is the case (which in general is not true, but we return to that in the following paragraph), then the same argument as for the Bosonic case yields

$$\begin{aligned}\int_\Gamma 1 - e^{-\langle v,L\rangle}\mathrm{d}\mu_h &= -\log\det\Big(I - \tfrac{1}{1-h}P\Big) + \log\det\Big(I - \tfrac{1}{1-h}(P-V/\lambda)\Big) \\ &= -\log\left(\frac{\det\Big((1-h)I - P\Big)}{\det\Big((I-h)I - P + V/\lambda\Big)}\right). \quad (3.6)\end{aligned}$$

The sticking point is the bound on the spectrum, which fails to hold for general $v \in \mathbb{R}_+^\Lambda$ even in the simplest of cases. For instance, consider $P = \left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$, then $\lambda = 1$, and supposing $h = -1$, $v = (u, u)$ the eigenvalues of $\Big(\frac{1}{2}\big(P - V\big)\Big)$ are $\frac{1}{2}(-u \pm 1)$, and the spectral radius is greater than 1 for $u > 1$. Sznitman demonstrates [Szn12] pp.79-80 that this can be overcome so long as the bound holds for small enough

$v \in \mathbb{R}_+^\Lambda$. That is, if for some $\varepsilon > 0$

$$\rho\Big(\tfrac{1}{1-h}\big(P - V/\lambda\big)\Big) < 1 \quad \text{whenever } \max_{x\in\Lambda} |v_x| < \varepsilon, \tag{3.7}$$

then an argument via analytic extensions confirms that the identity (3.6) holds for all $v \in \mathbb{R}_+^\Lambda$ (whilst the constant $h < 0$ does not appear in the work of Le Jan, or the description of Sznitman, it does not feature in the proof of the analytic extension). In the case that P is reversible this is confirmed in [Szn12] pp.77–8 by observing that P is symmetric with respect to the inner product $\langle u, v\rangle_\lambda := \sum_{x\in\Lambda} \lambda_x u_x v_x$, from which the result follows from spectral bounds of symmetric operators. In the following we confirm that the condition (3.7) also holds if the transition matrix is normal.

Proposition 3.6. *Let P denote a normal transition matrix. Then there exists an $\varepsilon > 0$ for which condition* (3.7) *holds.*

Proof. The proof relies on a spectral bound for normal matrices which we derive in Appendix B, Proposition B.12; in our present context it says

$$\rho\Big(\tfrac{1}{1-h}\big(P - V/\lambda\big)\Big) \;=\; \tfrac{1}{1-h}\rho\big(P - V/\lambda\big) \;\leq\; \tfrac{1}{1-h}\big(\rho(P) + \rho(V/\lambda)\big),$$

where we have used the fact that since V/λ is diagonal, it is necessarily normal since $(V/\lambda)^T = V/\lambda$. Moreover, since V/λ is positive and diagonal, its spectral radius is given by its largest element

$$\rho(V/\lambda) \;=\; (v/\lambda)^* \;:=\; \max_{x\in\Lambda}\big(v_x/\lambda_x\big) \;\leq\; \Big(\max_{x\in\Lambda} v_x\Big)\Big/\Big(\min_{y\in\Lambda}\lambda_y\Big) \;=\; v^*\big(\lambda^{-1}\big)^*.$$

Substituting this into the inequality above, we have

$$\rho\Big(\tfrac{1}{1-h}\big(P - V/\lambda\big)\Big) \leq \frac{1}{1-h}\rho(P) + \frac{1}{1-h}v^*\big(\lambda^{-1}\big)^*.$$

and the right hand side is strictly less than 1 whenever

$$v^* < \frac{(1-h) - \rho(P)}{(\lambda^{-1})^*}.$$

If the right hand side of this expression is strictly positive, then it is a suitable candidate for ε. Noting that the denominator is strictly positive, this condition holds whenever the numerator $(1-h) - \rho(P) > 0$. But under assumption **A2′** this is always true since either $\rho(P) < 1$ and $h \leq 0$, or $\rho(P) = 1$ and $h < 0$, where we reference Theorem 1.3 for the bound on $\rho(P)$, and in either case the claim follows. □

Combining (3.6) along with Sznitman's argument for the analytic extension, we obtain the following representation of the Laplace transform.

Theorem 3.7. *Suppose that P is either reversible or normal. The Laplace transform of $\lambda^{-1}\mathcal{L}$ under $\mathbb{P}$ is given for $v \in \mathbb{R}_+^\Lambda$ to be*

$$\mathbb{E}\Big[e^{-\langle v, \lambda^{-1}\mathcal{L}\rangle}\Big] = \frac{\det\big((1-h)I - P\big)}{\det\big((1-h)I - P + \frac{V}{\lambda}\big)}$$

We recall that when defining the Markov loop measure, (1.5), we omitted any dependence on $\beta > 0$ since a change of variables demonstrates shows that it leaves the measure unchanged. The following corollary relates the measure $\mathbb{P}_h$ to the unweighted measure $\mathbb{P}_0$ which is studied by Le Jan.
For $c \leq 0$, we construct from $\Lambda = (\Lambda, w, \kappa)$ a new graph $\Lambda^c = \big(\Lambda, w^c, \kappa^c\big)$, where the weights remain the same $w^c_{xy} = w_{xy}$, but the killing is given by

$$\kappa^c_x = \kappa_x - c\lambda_x,$$

negativity of c ensures $\kappa^c_x \geq \kappa_x$. Let $\mu^c = \mu^c_0$, so that this is the true Markov loop measure considered by Le Jan (i.e. zero chemical potential), but with a reweighted random walk. We denote $\mathbb{P}^c$ for the law of the associated local field.

Corollary 3.8. *For $h \leq 0$*

$$\big(\lambda^{-1}\mathcal{L}, \mathbb{P}_h\big) \overset{(d)}{=} \big((\lambda^h)^{-1}\mathcal{L}, \mathbb{P}^h\big).$$

Moreover

$$\mathbb{E}_h\Big[e^{-\langle v, \lambda^{-1}\mathcal{L}\rangle}\Big] = \det(I + G^h V)^{-1},$$

where $G^h = G^h(\overline{X})$ is the Green's function of the walk $\overline{X}$ with variable jump rates on Λ^h.

Proof. Noting that

$$\lambda^h_x = \sum_{y\in\Lambda} w^h_{xy} + \kappa^h_x = \lambda_x - h\lambda_x = (1-h)\lambda_x,$$

then

$$P^h_{xy} := \frac{w^h_{xy}}{\lambda^h_x} = \frac{w_{xy}}{(1-h)\lambda_x} = \frac{1}{1-h}P_{xy}.$$

Now considering the Laplace transform of the occupation field $\lambda^{-1}\mathcal{L}$ under the measure $\mathbb{P}_h$, we have from the preceding theorem

$$\begin{aligned}\mathbb{E}_h\Big[e^{-\langle v,\lambda^{-1}\mathcal{L}\rangle}\Big] &= \frac{\det\big((1-h)I-P\big)}{\det\big((1-h)I-P+V/\lambda\big)}\\ &= \frac{\det\big(I-(1-h)^{-1}P\big)}{\det\Big(I-(1-h)^{-1}P+\frac{V}{(1-h)\lambda}\Big)}\\ &= \frac{\det(I-P^h)}{\det(I-P^h+V/\lambda^h)}.\end{aligned}$$

The final line is exactly $\mathbb{E}_0^h\Big[e^{-\langle v,(\lambda^h)^{-1}\mathcal{L}\rangle}\Big]$. Continuing to manipulate this expression

$$\begin{aligned}\frac{\det(I-P^h)}{\det(I-P^h+V/\lambda^h)} &= \det\Big((I-P^h)^{-1}(I-P^h+V/\lambda^h)\Big)^{-1}\\ &= \det\Big(I+(I-P^h)^{-1}V/\lambda^h\Big)^{-1},\end{aligned}$$

which is as desired, since $G^h\big(\overline{X}\big) = (\lambda^h)^{-1}(I-P^h)^{-1}$, from Corollary 1.5, and equation (1.2). □

We note that as a consequence of this corollary the two conditions of assumption **A2′** are actually equivalent: since if $h<0$, then necessarily $\kappa^h \not\equiv 0$.

3.2 Space-Time Realisations of the Ideal Bose Gas

In this section we present an alternative definition for a Bosonic loop measure, in which we massage the measure μ so as that it is 'close' to the measure μ^B. We commence by recalling the definitions of the respective loop measures

$$\mu(\,\cdot\,) := \sum_{x\in\Lambda}\int_0^\infty \frac{e^{ht}}{t}\mathbf{P}_{xx}^t[\,\cdot\,]\mathrm{d}t,$$

and on performing change of variables $t\mapsto\beta t$

$$= \sum_{x\in\Lambda}\int_0^\infty \frac{e^{\beta ht}}{t}\mathbf{P}_{xx}^{\beta t}[\,\cdot\,]\mathrm{d}t,$$

which we compare to the Bosonic loop measure

$$\mu^B(\,\cdot\,) := \sum_{x\in\Lambda}\sum_{j\geq 1}\frac{e^{\beta hj}}{j}\mathbf{P}_{xx}^{\beta j}[\,\cdot\,].$$

It is clear that the only distinction between the two measures μ and μ^B is that the first of the two allows loops of all lengths $\beta t>0$, whilst the second is concentrated on loops whose length is βj with $j\in\mathbb{N}_{\geq 1}$. One way to incorporate this restriction into the definition of μ, is to introduce a second *drift* process into the definition of $\mathbf{P}_x$. We derive this heuristically as follows; defining

$$\Theta_t := \beta^{-1} t \bmod 1,$$

which we view as a deterministic stochastic process on the torus $\mathbf{T} := \mathbb{R}/\mathbb{Z}$, so that the pair $\overline{X}_t^{(\beta)} := \big(\overline{X}_t, \Theta_t\big)$ defines a stochastic process on the space $\Lambda \times \mathbf{T}$, whose distribution we write as $\mathbf{Q}$. For $x \in \Lambda$, $\tau \in \mathbf{T}$ we denote the site $x_\tau := (x, \tau) \in \Lambda \times \mathbf{T}$, and then

$$\mathbf{Q}_{x_\tau}\Big[\overline{X}_t^{(\beta)} = y_{\tau'}\Big] = \delta_{\tau'}\big(\beta^{-1}(\tau + t) \bmod 1\big)\mathbf{P}_x\Big[\overline{X}_t = y\Big],$$

and integrating the expression on the right hand side

$$\int_{t>0} \delta_\tau\big(\beta^{-1}(t+\tau) \bmod 1\big)\mathbf{P}_x\Big[\overline{X}_t = x\Big]\mathrm{d}t = \sum_{j \geq 1} \mathbf{P}_x\Big[\overline{X}_{\beta j} = x\Big].$$

Consequently we can reformulate the Bosonic loop measure as

$$\begin{aligned}\mu^B(\,\cdot\,) &= \sum_{x\in\Lambda}\sum_{j\geq 1}\frac{e^{\beta h j}}{j}\mathbf{P}_{xx}^{\beta j}[\,\cdot\,] \\ &= \sum_{x\in\Lambda}\int_0^\infty \frac{e^{\beta h t}}{t}\mathbf{Q}_{x_0 x_0}^t[\,\cdot\,]\mathrm{d}t,\end{aligned}$$

and using invariance under torus translations: $\mathbf{Q}_{x_0} \overset{(\mathrm{d})}{=} \mathbf{Q}_{x_\tau}$, we have $\int_0^1 \mathbf{Q}_{x_\tau}\mathrm{d}\tau = \mathbf{Q}_{x_0}$ and

$$= \sum_{x\in\Lambda}\int_0^1\int_0^\infty \frac{e^{\beta h t}}{t}\mathbf{Q}_{x_\tau x_\tau}^t[\,\cdot\,]\mathrm{d}t\,\mathrm{d}\tau.$$

In particular we see that the Bosonic loop measure can be equated to a Markov loop measure defined on the space $\Lambda \times \mathbf{T}$, which we call the *space-time* realisation of the ideal Bose gas. The derivation above was somewhat non-rigorous, but can be made rigorous on definition of the suitable σ-algebras; we omit this here since we will not work directly with this process. Studying a loop measure on $\Lambda \times \mathbf{T}$ would of course require us to leave the discrete world, and certain technical constraints will become cumbersome: for instance the occupation field $\mathcal{L} = (\mathcal{L}_{x_\tau})_{x_\tau \in \Lambda\times\mathbf{T}}$ is no longer indexed by a discrete set, and each path $\tau \mapsto \mathcal{L}_{x_\tau}$ will not be guaranteed to be continuous. Moreover it is not clear that the techniques established for deriving the Laplace transform, which depended on the fact that we were manipulating finite matrices, will carry over to a continuous setting. Instead we choose to work with a sequence of discrete loop measures whose limit agrees with the continuous one.
As in the definition of the process $\overline{X}^{(\beta)}$, we consider a pair $\big(\overline{X}_t, \overline{\Theta}_t^N\big)$, where $\overline{\Theta}^N =$

$\big(\overline{\Theta}_t^N\big)_{t\geq 0}$ is now a random process on $\mathbf{T}_N = \mathbb{Z}/N\mathbb{Z}$, the discrete torus, or cycle, for $N \geq 2$. In fact if we define $\mathbf{T}_N = ([N], w_N, \kappa_N)$, with $\kappa_N \equiv 0$ and

$$w_N(\tau, \tau') = \begin{cases} \beta^{-1}N & \text{if } \tau' = \tau + 1 \bmod N, \\ 0 & \text{else.} \end{cases}$$

then the induced process $\overline{\Theta}^N$ is the walk which only takes clockwise steps on the torus (i.e. from j to $j+1 \bmod N$), and does so at rate $\beta^{-1}N$, so as that the expected time to cover the whole torus is β. Equivalently we can identify this with a rate $\beta^{-1}N$ Poisson jump process taken modulo N. As we discussed above, we will not actually take the limit of the process itself, however the following theorem justifies that the sequence $\big(\overline{X}_t, \Theta_t^N\big)$, $N \geq 1$, will be approximating the correct loop model. For $\lambda > 0$, let $P^\lambda = (P_t^\lambda)_{t\geq 0}$ denote the Poisson point process defined by its law $\mathbf{P}$ on the Skorokhod space $D[0,\infty)$ of càdlàg paths $x\colon [0,\infty) \to \mathbb{R}_+$ with $x(0) = 0$, see [Bil99] pp.135–6 for a detailed construction. We also define $\delta^\lambda = (\lambda t)_{t\geq 0}$ which is deterministic and can also be described as a process with paths in $D[0,\infty)$.

Theorem 3.9. *For $\lambda > 0$, the sequence of scaled Poisson processes $\left(\frac{1}{N}P^{\lambda N}\right)$ converges in distribution to the rate λ deterministic drift: $\frac{1}{N}P^{\lambda N} \xrightarrow{(\mathrm{d})} \delta^\lambda$.*

Before proving this result we state a technical lemma regarding the incomplete Gamma function; recall [AS64] 6.5.1 pp.260, that this is given by

$$\Gamma(s, x) := \int_x^\infty t^{s-1} e^{-t} \mathrm{d}t =: \Gamma(s) - \gamma(s, x),$$

and $\Gamma(s) = \Gamma(s, 0)$ is the Γ function.

Lemma 3.10. *For $c_1 > c_2 > 0$,*

$$\lim_{x\to\infty} \frac{\gamma(\lfloor c_1 x\rfloor, \lfloor c_2 x\rfloor)}{\Gamma(\lfloor c_1 x\rfloor)} = \lim_{x\to\infty} \frac{\gamma(\lfloor c_1 x\rfloor, c_2 x)}{\Gamma(\lfloor c_1 x\rfloor)} = 0.$$

Proof. Since both terms above are positive, and $\gamma(\lfloor c_1 x\rfloor, \lfloor c_2 x\rfloor) \leq \gamma(\lfloor c_1 x\rfloor, c_2 x)$, it suffices only to prove the later limit. We consider the integrand $t \mapsto e^{-t} t^{\lfloor c_1 x\rfloor - 1}$, differentiating this for $t > 0$ there is a unique critical point $t^* = \lfloor c_1 x\rfloor - 1$, at which the integrand is maximised. Subsequently, for sufficiently large $x > 0$, $t^* = \lfloor c_1 x\rfloor - 1 > c_2 x$ and hence

$$\begin{aligned} \gamma(\lfloor c_1 x\rfloor, c_2 x) &= \int_0^{c_2 x} e^{-t} t^{\lfloor c_1 x\rfloor - 1} \mathrm{d}t \\ &\leq \int_0^{c_2 x} e^{-c_2 x} (c_2 x)^{\lfloor c_1 x\rfloor - 1} \mathrm{d}t \\ &= e^{-c_2 x} (c_2 x)^{\lfloor c_1 x\rfloor}, \end{aligned}$$

where we used the fact that $c_2x < t^*$ and the integrand is increasing on the domain of integration.Appealing to Stirling's approximation

$$\Gamma(c_1x) = \sqrt{\frac{2\pi}{c_1x}}\left(\frac{\lfloor c_1x\rfloor}{e^1}\right)^{\lfloor c_1x\rfloor}\left(1+O\left(\frac{1}{\lfloor c_1x\rfloor}\right)\right),$$

we obtain the bound

$$\begin{aligned}\frac{\gamma(\lfloor c_1x\rfloor, c_2x)}{\Gamma(\lfloor c_1x\rfloor)} &\leq \sqrt{\frac{\lfloor c_1x\rfloor}{2\pi}}e^{\lfloor c_1x\rfloor - c_2x}\left(\frac{c_2x}{\lfloor c_1x\rfloor}\right)^{\lfloor c_1x\rfloor}\left(1+O\left(\frac{1}{\lfloor c_1x\rfloor}\right)\right)^{-1}\\ &\leq A\sqrt{\lfloor c_1x\rfloor}e^{\lfloor c_1x\rfloor - c_2x}\left(\frac{c_2x}{\lfloor c_1x\rfloor}\right)^{\lfloor c_1x\rfloor},\end{aligned}$$

for some constant $A > 0$ and sufficiently large x. Taking logarithms of the upper bound

$$\begin{aligned}&\log\left(A\sqrt{\lfloor c_1x\rfloor}e^{\lfloor c_1x\rfloor - c_2x}\left(\frac{c_2x}{\lfloor c_1x\rfloor}\right)^{\lfloor c_1x\rfloor}\right)\\ &\qquad = \tilde{A} + \frac{1}{2}\log\lfloor c_1x\rfloor + \lfloor c_1x\rfloor - c_2x + \lfloor c_1x\rfloor\log\frac{c_2x}{\lfloor c_1x\rfloor}\end{aligned}$$

using that $\lfloor c_1x\rfloor \leq c_1x$ we have

$$\leq \tilde{A} + \frac{1}{2}\log c_1x + (c_1 - c_2)x + c_1x\log c_2x - \lfloor c_1x\rfloor\log\lfloor c_1x\rfloor$$

similarly since $c_1x - 1 \leq \lfloor c_1x\rfloor$, and consequently $-(c_1x-1) \geq -\lfloor c_1x\rfloor$ we can remove the final instance of the floor terms

$$\begin{aligned}&\leq \tilde{A} + \frac{1}{2}\log c_1x + (c_1-c_2)x + c_1x\log c_2x\\ &\qquad\qquad - (c_1x-1)\log(c_1x-1)\\ &= \tilde{A} + \frac{1}{2}\log c_1x + x\Big((c_1-c_2) + c_1\log c_2x - c_1\log(c_1x-1)\Big)\\ &\qquad\qquad + \log(c_1x-1)\\ &\leq \tilde{A} + \frac{3}{2}\log c_1x + x\left((c_1-c_2) + c_1\log\frac{c_2}{(c_1 - x^{-1})}\right)\end{aligned}$$

which diverges to $-\infty$ so long as the coefficient of x is negative. But this is the case, since on rearranging

$$\lim_{x\to\infty}\left((c_1-c_2) + c_1\log\frac{c_2}{(c_1-x^{-1})}\right) = (c_1-c_2) + c_1\log\frac{c_2}{c_1}$$

and

$$(c_1-c_2) + c_1\log\frac{c_2}{c_1} < 0 \qquad \text{if and only if} \qquad \log\frac{c_2}{c_1} < \frac{c_2}{c_1} - 1,$$

which is true since $\log y < y - 1$ for all $y > 0$. □

We now return to prove the theorem.

Proof of Theorem 3.9. The proof follows the same steps as that of Donsker's theorem concerning the convergence of simple random walk to Brownian motion, [Bil99] Theorem 14.1 pp.146–7; as in that proof, it suffices to prove convergence of the f.d.d.s and tightness of the sequence of measures. For simplicity of notation we consider the case $\lambda = 1$.
Starting with convergence of the f.d.d.s, we fix $m \geq 1$, $x \in \mathbb{R}^m$ and $0 \leq t_1 \leq t_2 \leq \cdots \leq t_m < \infty$. By the Cramér–Wold theorem [Bil95] Theorem 29.4 p.383, it suffices to prove

$$\frac{1}{N}\sum_{j=1}^{m} x_j P^N_{t_j} \xrightarrow{\text{(d)}} \sum_{j=1}^{m} x_j t_j.$$

We prove the stronger statement of convergence in second moments. Note

$$\begin{aligned}\mathbf{E}_0\left[\left|\sum_{j=1}^{m} x_j\Big(\frac{1}{N}P^N_{t_j} - t_j\Big)\right|^2\right] =& \sum_{j=1}^{m} x_j^2 \mathbf{E}_0\left[\left|\frac{1}{N}P^N_{t_j} - t_j\right|^2\right] \\ &+ 2\sum_{1\leq i<j\leq m} x_i x_j \mathbf{E}_0\left[\Big(\frac{1}{N}P^N_{t_i} - t_i\Big)\Big(\frac{1}{N}P^N_{t_j} - t_j\Big)\right].\end{aligned}$$

Since $P^N_t \sim \mathrm{Poi}(Nt)$, we have $\mathbf{E}_0\Big[P^N_t\Big] = Nt$, we recognise each term as a covariance

$$\begin{aligned}&= \sum_{j=1}^{m} x_j^2 \operatorname{Var}\Big(\frac{1}{N}P^N_{t_j}\Big) + 2\sum_{1\leq i<j\leq m} x_i x_j \operatorname{Cov}\Big(\frac{1}{N}P^N_{t_i}, \frac{1}{N}P^N_{t_j}\Big) \\ &= \frac{1}{N^2}\left(\sum_{j=1}^{m} x_j^2 \operatorname{Var}\Big(P^N_{t_j}\Big) + 2\sum_{1\leq i<j\leq m} x_i x_j \operatorname{Cov}\Big(P^N_{t_i}, P^N_{t_j}\Big)\right).\end{aligned}$$

Using the fact that $\operatorname{Var}\Big(P^N_{t_j}\Big) = Nt_j$, and $\operatorname{Cov}\Big(P^N_{t_i}, P^N_{t_j}\Big) = N\min(t_i, t_j) = Nt_i$

$$= \frac{1}{N}\left(\sum_{j=1}^{m} x_j^2 t_j + 2\sum_{1\leq i<j\leq m} x_i x_j t_i\right).$$

Consequently the term inside the bracket is independent of N, and the expression above vanishes as $N \to \infty$, which is to say that we have convergence in second moments. Convergence in distribution now follows.
To establish tightness we need to show that the process $\frac{1}{N}P^N$ does not grow too fast, [Bil99] Lemma 3 pp.173–4, and Theorem 13.2 pp.139-40 property (i), and neither does it fluctuate quickly, property (ii). Since the Poisson process is increasing, the

definitions of these properties are simplified, and we require for all $T \geq 0$

$$\lim_{C\to\infty} \limsup_{N\to\infty} \mathbf{P}_0 \left[\frac{1}{N} P_T^N > C \right] = 0, \tag{3.8}$$

and letting $\Pi_{\delta,T}$ denote the set of partitions $0 = t_0 < t_1 < \cdots < t_m = T$ such that for $j = 1, \ldots, m$, $|t_j - t_{j-1}| > \delta$, then we require for all $\varepsilon > 0$

$$\lim_{\delta\to 0} \limsup_{N\to\infty} \mathbf{P}_0 \left[\inf_{\underline{t}\in\Pi_{\delta,T}} \max_{1\leq j\leq m} \frac{1}{N} \left(P_{t_j}^N - P_{t_{j-1}}^N \right) > \varepsilon \right] = 0. \tag{3.9}$$

The first of these, (3.8), is given exactly on applying the formula for the cumulative distribution function of a Poi(NT) variable

$$\mathbf{P}_0[P_T^N > CN] = 1 - \frac{\Gamma(\lfloor CN \rfloor, TN)}{\Gamma(\lfloor CN \rfloor)} = \frac{\gamma(\lfloor CN \rfloor, TN)}{\Gamma(\lfloor CN \rfloor)}.$$

Then since for $C > 2$, $\lfloor CN \rfloor > N + 1$ holds for all $N \geq 1$, on appealing to Lemma 3.10, the above expression converges to 0, as required.
Similarly we derive a bound on the second probability (3.9) in terms of the Γ function as follows. Let $m \geq 1$ be the largest value such that $2^{-m} > \delta$, so that the dyadic partition $\underline{t}(m) = [0, T] \cap 2^{-m}\mathbb{Z} \in \Pi_{\delta,T}$. Then

$$\begin{aligned} \mathbf{P}_0 \left[\inf_{\underline{t}\in\Pi_{\delta,T}} \max_{1\leq j\leq m} \frac{1}{N} \left(P_{t_j}^N - P_{t_{j-1}}^N \right) > \varepsilon \right] &\leq \mathbf{P}_0 \left[\max_{t_j\in\underline{t}(m)} \frac{1}{N} \left(P_{t_j}^N - P_{t_{j-1}}^N \right) > \varepsilon \right] \\ &= 1 - \mathbf{P}_0 \left[\max_{t_j\in\underline{t}(m)} \left(P_{t_j}^N - P_{t_{j-1}}^N \right) \leq N\varepsilon \right]. \end{aligned}$$

Using the independence of increments of a Poisson process, the variables $\left(P_{t_j}^N - P_{t_{j-1}}^N \right)$ are all independent, and since in general $\mathbb{P}[\max(X, Y) \leq \varepsilon] = \mathbb{P}[X \leq \varepsilon]\mathbb{P}[Y \leq \varepsilon]$ for independent variables, we have

$$= 1 - \prod_{t_j\in\underline{t}(m)} \mathbf{P}_0 \left[P_{t_j}^N - P_{t_{j-1}}^N \leq N\varepsilon \right].$$

The partition $\underline{t}(m)$ has at most $2^m(T + 1)$ intervals of width 2^{-m}, and all of these are identically distributed, hence

$$\begin{aligned} &\leq 1 - \mathbf{P}_0[P_{2^{-m}}^N \leq N\varepsilon]^{2^m(T+1)} \\ &= 1 - \left(\frac{\Gamma(\lfloor \varepsilon N \rfloor, \lfloor 2^{-m} N \rfloor)}{\Gamma(\lfloor \varepsilon N \rfloor)} \right)^{2^m(T+1)}. \end{aligned}$$

Recognising the term inside the product as

$$1 - \frac{\gamma(\lfloor \varepsilon N \rfloor, \lfloor 2^{-m} N \rfloor)}{\Gamma(\lfloor \varepsilon N \rfloor)},$$

and choosing m sufficiently small that $2^{-m} < \varepsilon$, then the bracketed term tends to 1 as $N \to \infty$. Note that since we take the limit in N first, the power of $2^m(T+1)$ does not affect the convergence. $\square$

3.2.1 The Space–Time Random Walk Measure

In light of the proof that the suitably scaled Poisson process converges to a deterministic drift, we proceed to describe a space-time random walk, for which the associated Markov loop model will approximate the Bosonic loop measure. We note that although their emphasis is quite different to our own, a similar discretization of the temporal process was used by Balaban, Feldman, Knörrer and Trubowitz [BFKT08a, BFKT08b] who study the Bose gas from a functional integral approach. Rather than defining the space-time random walk as a pair $(\overline{X}, \Theta^N)$, it will be easier for us to define it as a single process on the graph $\Lambda \times \mathbf{T}_N$. That is, if $\Lambda = (\Lambda, w, \kappa)$ is the graph which induces the walk $\overline{X}$, we define the new graph $\Lambda_N = (\Lambda \times \mathbf{T}_N, w^N, \kappa^N)$, where we adopt the same convention as before writing $x_\tau = (x, \tau) \in \Lambda \times \mathbf{T}_N$, and the weight function is given by

$$w^N(x_\tau, y_{\tau'}) = \begin{cases} w(x,y) & \text{if } \tau = \tau' \\ \beta^{-1}N & \text{if } x = y,\ \tau' = \tau + 1 \bmod N, \\ 0 & \text{else.} \end{cases} \tag{3.10}$$

and the killing vector is $\kappa^N_{x_\tau} = \kappa_x$. The induced process is denoted $\overline{X}^N$, and the normalised unit-jump rate walk is denoted X^N. Note that on checking the jump rates above, it is clear that the walk $\overline{X}^N$ defined above does agree with the process $(\overline{X}, \overline{\Theta}^N)$. When we wish to see the process as a product, we refer to $\overline{X}$ as the *spatial component*, and $\overline{\Theta}^N$ the *temporal component*. Before proceeding we update our collection of assumptions which will now be maintained for the remains of the chapter.

A1′ Graph sequences will always be of the form $\Lambda_N = \Lambda \times \mathbf{T}_N$, that is limits are only considered in the temporal dimension.

A2′ The inverse temperature is strictly positive $\beta > 0$, and either: $\kappa \equiv 0$ and $h < 0$, or $\kappa \not\equiv 0$ and $h \leq 0$.

A3 The weights w_{xy}, κ_x of the graph Λ are normalised so that $\lambda_x = \sum_{y \in \Lambda} w_{xy} + \kappa_x = 1$, or equivalently $P_{xy} = w_{xy}$. Moreover, P is normal.

A4 The Markov loop measure $\mu_N = \mu_{h,\Lambda_N}$ denotes the Markov loop measure defined on Λ_N, and is with respect to the normalised walk X^N, not the walk $\overline{X}^N$. We denote the law of the loup soup $\mathbb{P}_N$, and define the scaled occupation field $\mathcal{L}^N := (1 + N\beta^{-1})^{-1}\mathcal{L}$.

Whenever we omit the subscript N, e.g. Λ, μ, $\mathbb{P}$, we are referring to the Markov loop measure on Λ.

The first two assumptions speak for themselves. The later two recall the conditions under which we studied the Laplace transform for the loop soup under the Markov loop measure. The assumption **A4** is really only fixing notation, and reintroducing the scaling convention which was adopted in Section 3.1.1. Of greater interest is the restriction we introduce in assumption **A3**. Again, the fact that we choose P to be normal will be of importance in deriving the Laplace transform for the space–time random walk, the curiosity, however, is our insistence that the weights normalise to 1. Whilst not essential for deriving the Laplace transform of $\mathcal{L}^N$ itself, we will see that under this assumption we can prove convergence in distribution of a related 'projected' occupation field.

Our first order of business is to confirm that the transition matrix P^N induced by (3.10) is in fact a normal transition matrix. Just as the transition matrix of the d-dimensional lattice box can be derived as a *Kronecker sum* of matrices, we will prove that this is the case in our present context. Given two matrices $A \in \mathbb{C}^{m\times m}$ and $B \in \mathbb{C}^{n\times n}$ with $m, n \geq 1$, we define their *Kronecker product* to be the $mn \times mn$ matrix given in block form by

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1m}B \\ a_{21}B & a_{22}B & \cdots & a_{2m}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mm}B \end{pmatrix},$$

and entry wise this is $(A \otimes B)_{i_j k_l} = a_{i_k} b_{j_l}$, $1 \leq i, k \leq m$ and $1 \leq j, l \leq n$. Then the *Kronecker sum* is the $mn \times mn$ matrix

$$A \oplus B := \big(A \otimes I_n\big) + \big(I_m \otimes B\big),$$

and $(A \oplus B)_{i_j k_l} = a_{ik}\delta_j(l) + b_{jl}\delta_i(k)$. Both the Kronecker product and sum turn out to be pleasant to work with due to their distributivity properties, and their spectral representations. We defer proofs and statements of these to Appendix B, but will draw on certain results in the coming pages. Working from the definitions above we derive the following.

Lemma 3.11. *Let P denote the transition matrix of the walk X on Λ, and P^N that of the process $X^N = (X, \Theta^N)$ on Λ_N. Then*

$$P^N = \frac{1}{1 + N\beta^{-1}}\big(P \oplus N\beta^{-1}\Sigma\big),$$

where $\Sigma := \operatorname{circ}(0, 1, 0, \ldots, 0) \in \mathbb{C}^{N\times N}$ is the rightward shift of the identity matrix. Moreover P^N is a normal matrix.

Proof. Using the notation established above

$$\begin{aligned}\big(P \oplus N\beta^{-1}\Sigma\big)_{x_\tau y_{\tau'}} &= P_{xy}\delta_\tau(\tau') + N\beta^{-1}\Sigma_{\tau\tau'}\delta_x(y)\\ &= w_{xy}\delta_\tau(\tau') + N\beta^{-1}\delta_{\tau+1}(\tau')\delta_x(y),\end{aligned}$$

which we see agrees with (3.10); note that we adopted the convention here that $\tau+1$ is taken modulo N whenever necessary, which was to avoid cumbersome terms such as $\delta_{\tau+1 \bmod N}(\tau')$.
To see that P^N is normal, according to Proposition B.15 it suffices to show that both P and $N\beta^{-1}\Sigma$ are normal. Since, by assumption, P is normal, it remains to confirm that Σ is. However, on noting that $\Sigma^T = \operatorname{circ}(0,\dots,0,1)$, i.e. $\Sigma^T_{\tau\tau'} = \delta_{\tau-1}(\tau')$ we have

$$\begin{aligned}\big(\Sigma^T\Sigma\big)_{\tau\tau'} &= \sum_{\theta=1}^{N} \Sigma^T_{\tau\theta}\Sigma_{\theta\tau'}\\ &= \sum_{\theta=1}^{N} \delta_{\tau-1}(\theta)\delta_{\theta+1}(\tau'),\end{aligned}$$

noticing that this requires us to simultaneously have $\tau-1=\theta=\tau'-1$, i.e. $\tau=\tau'$

$$= \delta_\tau(\tau'),$$

i.e. $\Sigma^T\Sigma = I_N$, the identity matrix. Then since $\Sigma\Sigma^T = \big(\Sigma^T\Sigma\big)^T = I_N$, the claim follows. □

As a side note, perhaps a more intuitive justification for the normality of Σ comes from considering the related Markov chain which always jumps clockwise on the torus; similarly the Markov chain associated to Σ^T always jumps anti-clockwise. It follows then that a clockwise jump, followed by an anti-clockwise one leaves the process at its starting point (i.e. the identity matrix), as does an anti-clockwise jump followed by a clockwise one, from which we have that $\Sigma\Sigma^T = I_N$.
Courtesy of this proposition and Theorem 3.7 we are in a position to state the formula for the Laplace transform of the occupation field $\mathcal{L}^N$; having said that, at this stage there is nothing to be gained by repeating the formula. For the time being we develop the relationship between $\big(\mathcal{L}^N, \mathbb{P}_N\big)$ and $\big(\mathcal{L}, \mathbb{P}^B\big)$. We achieve this by projecting the occupation field $\mathcal{L}^N$ onto the spatial dimension; that is we define $\mathcal{L}^{\downarrow N} = \big(\mathcal{L}^{\downarrow N}_x\big)_{x\in\Lambda}$ via

$$\mathcal{L}^{\downarrow N}_x := \sum_{\tau=1}^{N} \mathcal{L}^N_{x_\tau}, \qquad x \in \Lambda,$$

which we refer to as the *projected* occupation field. For our voyage into the space–

time formulation to have any worth, we of course expect that this field can be related to the Bosonic occupation field, as $\mathbf{T}_N \to \mathbb{R}/\mathbb{Z}$. It will turn out that the fields are equivalent on accounting for a surplus field of point loops: those which do not complete a lap of the torus, but merely stay at their starting point. The main step towards deriving the relation is the following statement.

Theorem 3.12.

$$\lim_{N\to\infty} \mathbb{E}_N\left[e^{-\langle v,\mathcal{L}^{\downarrow N}\rangle}\right] = \frac{\det\left(I - e^{\beta(I-P)}\right)}{\det\left(I - e^{\beta(I-P+V)}\right)}. \tag{3.11}$$

Proof. In light of Corollary 3.8 it suffices to consider only the case $h = 0$, since any other $h < 0$ can be obtained by changing the definition of P. Let 0_N be the $N \times N$ matrix of all zeros; for a vector $v \in \mathbb{R}^\Lambda$, let $v^N := v \oplus 0_N$ be the length $N|\Lambda|$ vector with $v(x_t) = v(x)$, i.e. $v^N = (v_1, \ldots, v_1, \ldots, v_{|\Lambda|}, \cdots v_{|\Lambda|}) \in \mathbb{R}^{N|\Lambda|}$. From the definitions we have

$$\langle v, \mathcal{L}^{\downarrow N}\rangle \;=\; \sum_{x\in\Lambda} v_x \sum_{\tau=1}^{N} \mathcal{L}^N_{x_\tau} \;=\; \langle v^N, \mathcal{L}^N\rangle.$$

Then applying Theorem 3.7 to the space–time walk

$$\begin{aligned}\mathbb{E}_N\left[e^{-\langle v,\mathcal{L}^{\downarrow N}\rangle}\right] &= \mathbb{E}_N\left[e^{-\langle v^N,\mathcal{L}^N\rangle}\right] \\ &= \frac{\det\left(I - P^N\right)}{\det\left(I - P^N + \frac{1}{1+N\beta^{-1}}V^N\right)}.\end{aligned} \tag{3.12}$$

Our analysis of this term now proceeds by identifying the eigenvalues. As in Lemma 3.11 we can rewrite the matrix $-P^N + \left(1 + N\beta^{-1}\right)^{-1}V^N$ as a Kronecker sum

$$\begin{aligned}-P^N + \frac{1}{1+N\beta^{-1}}V^N &= -\left(P^N - \frac{1}{1+N\beta^{-1}}V^N\right) \\ &= -\frac{1}{1+N\beta^{-1}}\left((P-V)\oplus N\beta^{-1}\Sigma\right).\end{aligned}$$

We appeal to the fact that the eigenvalues of the Kronecker sum of two square matrices are given by the sum of all the pairs of eigenvalues, Lemma B.13, hence we have

$$\operatorname{Spec}\left(-P^N + \frac{1}{1+N\beta^{-1}}V^N\right) = \left\{-\frac{\eta^v_i + N\beta^{-1}\sigma_j}{1+N\beta^{-1}} \,:\, 1 \le i \le |\Lambda|,\, 1 \le j \le N\right\},$$

where $\eta_i^v \in \mathrm{Spec}(P - V)$, $\sigma_j \in \mathrm{Spec}(\Sigma)$. It follows that for any $v \in \mathbb{R}_+^\Lambda$

$$\det\left(I - P^N + \frac{1}{1+N\beta^{-1}}V^N\right) = \prod_{i=1}^{|\Lambda_N|}\prod_{j=1}^{N}\left(\left(1 - \frac{\eta_i^v}{1+N\beta^{-1}}\right) - \frac{N\beta^{-1}\sigma_j}{1+N\beta^{-1}}\right).$$

Since $\Sigma = \mathrm{circ}(0,1,0,\ldots,0)$ is a circulant matrix, its eigenvalues are given explicitly as the N-th roots of unity, Theorem B.18

$$\sigma_j = \omega_N^j := \exp\left(2\pi i \frac{j}{N}\right),$$

and we recall the following identity $\prod_{j=1}^N (a - c\omega_N^j) = a^N - c^N$, for $a \in \mathbb{R}$, $c \geq 0$, Proposition B.19. Applying this, the product over $j = 1,\ldots,N$ becomes

$$\begin{aligned}\prod_{j=1}^{N}\left(\left(1 - \frac{\eta_i^v}{1+N\beta^{-1}}\right) - \frac{N\beta^{-1}\sigma_j}{1+N\beta^{-1}}\right) &= \left(1 - \frac{\eta_i^v}{1+N\beta^{-1}}\right)^N - \left(\frac{N\beta^{-1}}{1+N\beta^{-1}}\right)^N \\ &= \left(\frac{N\beta^{-1}}{1+N\beta^{-1}}\right)^N\left(\left(1 + \frac{1-\eta_i^v}{N\beta^{-1}}\right)^N - 1\right).\end{aligned}$$

Substituting this into the expression (3.12), where the numerator is simply the case $v \equiv 0$, and we denote the corresponding eigenvalues $\eta_i = \eta_i^0$, we obtain

$$\mathbb{E}_N\left[e^{-\langle v, \mathcal{L}^{\downarrow N}\rangle}\right] = \prod_{i=1}^{|\Lambda|}\frac{\left(1 + \frac{1-\eta_i}{N\beta^{-1}}\right)^N - 1}{\left(1 + \frac{1-\eta_i^v}{N\beta^{-1}}\right)^N - 1}.$$

In taking limits we recognise the term in N in both the numerator and denominator to be convergent to an exponential

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}_N\left[e^{-\langle v, \mathcal{L}^{\downarrow N}\rangle}\right] &= \prod_{i=1}^{|\Lambda|}\lim_{N\to\infty}\frac{\left(1 + \frac{1-\eta_i}{N\beta^{-1}}\right)^N - 1}{\left(1 + \frac{1-\eta_i^v}{N\beta^{-1}}\right)^N - 1} \\ &= \prod_{i=1}^{|\Lambda|}\frac{e^{\beta(1-\eta_i)} - 1}{e^{\beta(1-\eta_i^v)} - 1} \\ &= \frac{\det\left(I - e^{\beta(I-P)}\right)}{\det\left(I - e^{\beta(I-P+V)}\right)},\end{aligned}$$

from which the claim follows on comparison with the Laplace transform of $\mathcal{L}$ under $\mathbb{P}^B$, Theorem 3.5. □

On comparing (3.11) with (3.3), we see that the two Laplace transforms agree up to a change in the sign preceeding the term in V. To reconcile this difference we must take into account the contribution of point loops to the field $\mathcal{L}^N$. For the purpose of simplicity we briefly return to considering general graphs Λ, before returning to the space–time context. As before we let Γ denote the set of all loops on Λ, the set of point loops is given to be $\Gamma^* \subset \Gamma$ with

$$\Gamma^* := \{\gamma \in \Gamma : \#(\gamma \cap \Lambda) = 1\},$$

i.e those loops that do not leave their starting point. The occupation field of point loops is denoted by $\mathcal{G} = (\mathcal{G}_x)_{x\in\Lambda}$, with $\mathcal{G}_x = \sum_{\gamma\in\Gamma^*} L(\gamma)$. Le Jan [LeJ10] Proposition 14, refers to the loops in Γ^* as *trivial loops*, and identified the distribution of $\mathcal{G}$ under the Markov loop measure μ as a Gamma random field. We prove this statement in the additional context of the parameters β and h, and also demonstrate that in the space–time limit the point loops contribute a deterministic factor to the occupation field $\mathcal{L}^{\downarrow N}$.

Lemma 3.13. *For $h < 0$, the occupation field of point loops satisfies*

$$(\mathcal{G}, \mathbb{P}) \overset{\text{(d)}}{=} \left(\frac{1}{1-h} G, P^{\otimes\Lambda}\right),$$

with P the law of $\Gamma(1,1)$ random variable.
Moreover, in the context of the space–time loop model, the field $\mathcal{G}^{\downarrow N} = \sum_{\tau=1}^{N} \mathcal{G}_{x_\tau}$ converges to a degenerate distribution

$$\left(\mathcal{G}^{\downarrow N}, \mathbb{P}_N\right) \xrightarrow{\text{(d)}} \left(\frac{\beta}{1-h}\mathbf{1}, \delta^{\otimes\Lambda}_{\frac{\beta}{1-h}}\right).$$

Proof. Since we are now somewhat accustomed to the computation of occupation fields, we describe the following in brief. Our usual application of the Campbell formula leads us to identify the following integral

$$\int_{\Gamma^*} 1 - e^{-\langle v,\lambda^{-1}L\rangle}\mu(d\gamma) = \sum_{x\in\Lambda}\int_0^\infty \frac{e^{ht}}{t}\mathbf{E}_x\Big[\mathbf{1}_{\{X_u=x,\forall 0\le u\le t\}}\Big(1 - e^{-\int_0^t v/\lambda(X_s)\mathrm{d}s}\Big)\Big]\mathrm{d}t,$$

where we brought the requirement that the loops are point loops into the indicator variable for the random walk. This then becomes

$$\begin{aligned}
&= \sum_{x\in\Lambda}\int_0^\infty \frac{e^{ht}}{t}(1 - e^{-tv_x/\lambda_x})\mathbf{P}_x[X_u = x,\, \forall 0 \le u \le t] \\
&= \sum_{x\in\Lambda}\int_0^\infty \frac{1}{t}\Big(e^{-(1-h)t} - e^{-(1+v_x/\lambda_x)t}\Big)\mathrm{d}t \\
&= \sum_{x\in\Lambda}\log\Big(\frac{1 + v_x/\lambda_x}{1-h}\Big)
\end{aligned}$$

It follows that the Laplace transform of the field $\mathcal{G}_x$ is given by

$$\mathbb{E}\Big[e^{-\langle v,\mathcal{G}\rangle}\Big] = \prod_{x\in\Lambda}\left(\frac{1+v_x/\lambda_x}{1-h}\right)^{-1},$$

from which the first claim follows.
Turning to the context of the space–time walk, we consider the case $h = 0$ for simplicity. We have

$$\begin{aligned}\lim_{N\to\infty}\mathbb{E}_N\left[e^{-\left\langle v,\mathcal{G}^{\downarrow N}\right\rangle}\right] &= \lim_{N\to\infty}\prod_{x\in\Lambda}\prod_{\tau=1}^{N}\left(1+\frac{v_x/\lambda_x}{1+N\beta^{-1}}\right)\\ &= \lim_{n\to\infty}\prod_{x\in\Lambda}\left(1+\frac{v_x/\lambda_x}{1+N\beta^{-1}}\right)^{-N}\\ &= \prod_{x\in\Lambda}e^{-\beta v_x/\lambda_x}.\end{aligned}$$ □

Combining the above expression for the occupation field of $\mathcal{G}^{\downarrow N}$ with (3.3), one immediately obtains.

Theorem 3.14 (The Bosonic Isomorphism Theorem)**.** *In the limit $N\to\infty$, the projected occupation field satisfies*

$$(\mathcal{L}^{\downarrow N},\mathbb{P}_N)\xrightarrow{(\mathrm{d})}\left(\mathcal{L}+\frac{\beta}{1-h}\mathbf{1},\mathbb{P}^B\right),$$

where $\mathbf{1}$ is the deterministic field with $\mathbf{1}_x = 1$ for all $x\in\Lambda$.

We mentioned previously that working with the field $\mathcal{L}$ under $\mathbb{P}^B$ presents challenges due to the fact that we cannot manipulate the Laplace transform easily; this provides some level of motivation to work instead with the field $\mathcal{L}^N$ and then project down to $\mathcal{L}^{\downarrow N}$, since the Laplace transform of $\mathcal{L}^N$ was given in terms of the Green's function of the walk P^h, and in particular did not involve exponential terms. In the following section we discuss the distribution of the space–time occupation field, and state the analogue of Le Jan's isomorphism theorem to the Gaussian free field.

3.2.2 Complex Gaussian Measures and The Space–Time Isomorphism

Having derived a random field, a natural question to ask is how the global structure of the field appears, and how correlations behave across it. As a simple calculation, we can consider the expected value at a point of the field, something which was outside of our scope when studying the mean occupation in Chapter 2. As before we assume that the term in $h\le 0$ has been absorbed into the definition of the random walk $P = P^h$, and for a point $x_\tau\in\Lambda_N$, choose $\tilde{v}\in\mathbb{R}^{|\Lambda|N}$ to be the vector of all zeros except $\tilde{v}_{x_\tau} = v$. Then

$$\begin{aligned}\mathbb{E}\big[\mathcal{L}^N_{x_\tau}\big] &= -\frac{\mathrm{d}}{\mathrm{d}v}\mathbb{E}_N\Big[e^{-\langle \tilde{v}, \mathcal{L}^N\rangle}\Big]_{|v=0} \\ &= -\frac{\mathrm{d}}{\mathrm{d}v}\det\big(I + G^N\widetilde{V}\big)^{-1}_{|v=0}.\end{aligned}$$

where $G^N = G\big(\overline{X}^N\big)$ is the Green's function of the variable jump rate walk. Since $\widetilde{V}$ only has one non-zero entry, $\big(G^N\widetilde{V}\big)_{x_\tau x_\tau} = G^N_{x_\tau x_\tau}v$, we have

$$\begin{aligned}&= -\frac{\mathrm{d}}{\mathrm{d}v}(1 + G^N_{x_\tau,x_\tau}v)^{-1}_{|v=0} \\ &= G^N_{x_\tau,x_\tau}.\end{aligned}$$

Comparing this with our attempt at a calculation using the Laplace transform of the Bosonic occupation field on p. 45, we note that the ease of computation here follows from the fact that the dependence on V becomes multiplicative. In this particular example we immediately derived the spectrum of $G^N\tilde{V}$, whereas previously we discussed that the spectrum of $Q - V$ remains mysterious. Continuing this example, we note that we need not stop at the expectation; writing

$$\mathbb{E}\Big[e^{-v\mathcal{L}^N_{x_\tau}}\Big] = \frac{1}{1 + G^N_{x_\tau x_\tau}v},$$

this is recognised as the Laplace transform of a Γ-distributed variable, $\mathcal{L}^N_{x_\tau} \sim \Gamma(1, G_{x_\tau,x_\tau})$. Of course turning this into a result about $\big(\mathcal{L}_x, \mathbb{P}^B\big)$ remains a challenge: although we have $\sum_{\tau=1}^N \mathcal{L}^N_{x_\tau} \xrightarrow{(\mathrm{d})} \mathcal{L}_x + \frac{\beta}{1-h}\mathbf{1}$, we cannot readily calculate the sum of a collection of dependent Gamma random variables.

That each site of $\mathcal{L}^N$ is Γ-distributed is a sub-result of a much more significant observation regarding the distribution of the entire field. In fact we are already in a position to provide a description of the space–time occupation field in terms of a recognisable distribution. In [VeJ97], Vere–Jones introduced a family of distributions known as *α-permanental processes*; letting $\mathcal{I}$ denote an arbitrary index set, and $U\colon \mathcal{I}\times\mathcal{I} \to \mathbb{R}_+$, the α-permanental process with kernel U and law $P_{\alpha,U}$ is, when it exists, the process $\theta = (\theta_t)_{t\in\mathcal{I}}$ such that for all finite collections $\underline{t} = (t_1, \dots t_m) \in \mathcal{I}^m$, and $v \in \mathbb{R}^m_+$

$$E_{\alpha,U}\Big[e^{-\frac{1}{2}\sum_{i=1}^m v_i\theta_{t_i}}\Big] = \det\big(I + U_{|\underline{t}}\,V\big)^{-\frac{1}{\alpha}}, \tag{3.13}$$

where we denote $U_{|\underline{t}}$ for the $m\times m$ matrix with entries $U_{|\underline{t}}(i,j) := U(t_i, t_j)$. In the special case that $\alpha = 1/2$ and U is positive and symmetric, the process θ agrees with the distribution of the square of half a Gaussian process. That is if $\phi = (\phi_t)_{t\in\mathcal{I}}$

is a centred Gaussian distribution on $\mathcal{I}$ with covariance U, and law P^U then

$$\left(\theta, P_{\alpha=\frac{1}{2},U}\right) \overset{(d)}{=} \left(\phi^2, P^U\right).$$

In general one can ask under what conditions the right hand side of (3.13) determines a distribution: i.e. when is this a valid Laplace transform for a random process? Vere–Jones provided necessary and sufficient conditions on the index $\alpha > 0$ and kernel U. Eisenbaum and Kaspi [EK09] establish that these conditions are satisfied whenever U is the kernel of a transient Markov process on $\mathcal{I}$, in which case the corresponding α-permanental process exists for all values $\alpha > 0$. Further they relate the α-permenantal process to Dynkin's isomorphism for the local field of a random walk: we return to this in the next section. As a side note we remark that in the same paper of Eisenbaum and Kaspi they go on to consider the Bosonic point process, which can be derived from the Feynman–Kac formula for the ideal gas (0.13). A loop of length βj is identified with j points $x_1, \ldots, x_j \in \mathbb{R}^d$ by forgetting the Brownian paths connecting them, and considering only the point locations. Note that this is somewhat perpendicular to our own study, where we study only statistics of the paths.

Returning to the occupation field of the space–time walk, we note on comparing Laplace transforms, that we have confirmed Eisenbaum and Kaspi's result on the existence of permanental processes in the special case of the space–time walk.

Theorem 3.15. *Let $G = G^N = \left(G^N_{x_\tau y_{\tau'}}\right)$ be the Green's function of the walk $\overline{X}^N$ on Λ_N, then for $\alpha > 0$ the permanental process $\theta = (\theta_{x_\tau})_{x_\tau \in \Lambda_N}$ with law $P_{\alpha,G}$ exists, and*

$$\left(\mathcal{L}^N, \mathbb{P}^\alpha_N\right) \overset{(d)}{=} \left(\tfrac{1}{2}\theta, P_{\alpha,G}\right),$$

where $\mathbb{P}^\alpha_N$ denotes the law of the occupation field with intensity measure $\alpha\mu_N$.

Note that the extension to all intensities $\alpha > 0$ follows immediately from Campbell's formula (3.1) once one has established the case for $\alpha = 1$. A consequence of identifying the occupation field as a permanental process is that we can in fact get all of the correlations as expressions in the Green's function. In the following we use $\operatorname{Per} A$ to denote the *permanent* of a square matrix $A \in \mathbb{C}^{m\times m}$, which is defined by

$$\operatorname{Per} A := \sum_{\pi\in S_m} \prod_{i=1}^m A_{i\,\pi(i)}.$$

Corollary 3.16. *For $x_1, \ldots, x_m \in \Lambda$, $\tau_1, \ldots, \tau_m \in \mathbf{T}_N$ let $\underline{x} = (x_{1\tau_1}, \ldots, x_{m\tau_m}) \in \Lambda_N^m$. Then*

$$\mathbb{E}_N\left[\mathcal{L}^N_{(x_1,\tau_1)} \cdots \mathcal{L}^N_{(x_m,\tau_m)}\right] = \operatorname{Per} G^N_{|\underline{x}}.$$

The moment formula for permanental processes is derived in [VeJ97] for general $\alpha > 0$.
As we have already mentioned, in the case that the generator of the random walk is in fact symmetric one can recognise the Laplace transform in terms of Gaussian fields. This is the content of Le Jan's isomorphism theorem, [LeJ10, LeJ11]. We recall that given a symmetric real matrix $U \in \mathbb{R}^{\Lambda\times\Lambda}$ with non-negative entries, the centred *(discrete) Gaussian free field* is the field $\phi = (\phi_x)_{x\in\Lambda}$ with law P^U, where

$$P^U(\mathrm{d}\phi) := \frac{1}{(2\pi)^{\frac{1}{2}|\Lambda|}\sqrt{\det U}} e^{-\frac{1}{2}\langle\phi, U^{-1}\phi\rangle}\mathrm{d}\phi,$$

and $\mathrm{d}\phi := \prod_{x\in\Lambda}\mathrm{d}\phi_x$. P^U is determined by the fact that it is the unique law on $\mathbb{R}^\Lambda$ for which $E^U[\phi_x\phi_y] = U(x,y)$, see [Szn12] pp.31-2.

Theorem 3.17 (Le Jan's Isomorphism)**.** *Let Λ be such that the associated walk is reversible, and let $\mathbb{P}^\alpha$ denote the law of the loop soup with intensity measure $\alpha\mu$. Then*

$$\left(\lambda^{-1}\mathcal{L}, \mathbb{P}^{\alpha=\frac{1}{2}}\right) \overset{(\mathrm{d})}{=} \left(\tfrac{1}{2}\phi, P^G\right),$$

where P^G is the law of the Gaussian field with covariance $G = G\big(\overline{X}\big)$.
Moreover, if $\alpha = k/2$ for some $k = 1, 2, \ldots$, then

$$\left(\lambda^{-1}\mathcal{L}, \mathbb{P}^{\alpha=\frac{k}{2}}\right) \overset{(\mathrm{d})}{=} \left(\tfrac{1}{2}\big(\phi^{(1)} + \cdots + \phi^{(k)}\big),\ P^G\otimes\cdots\otimes P^G\right),$$

where $\phi^{(j)}$ are independent P^G Gaussian fields.

A proof that $(I+GV)^{-\frac{1}{2}}$ is the Laplace transform of a square of a Gaussian process is given in [Szn12] Proposition 2.14 pp.47-9; the extension to $\alpha = \frac{k}{2}$ is immediate from the superposition of Poisson processes, and follows on inspection of Campbell's formula for the Laplace functionals, Lemma 3.1.
Implicit in the above was that the Green's function $G = G\big(\overline{X}\big)$ is symmetric, else the occupation field $\lambda^{-1}\mathcal{L}$ cannot possibly be equivalent to a Gaussian field. This is a consequence of reversibility of the walk, since recalling the conventions of Section 1.1

$$G_{xy} = \lambda_y^{-1}G(Z)_{xy} = \lambda_y^{-1}\sum_{n\geq 0}P^n_{xy} = \lambda_x^{-1}\sum_{n\geq 0}P^n_{yx} = G_{yx}.$$

In the absence of reversibility, however, one can no longer hope for a Gaussian description. In particular for the space time occupation field it would appear as though a permanental description, Theorem 3.15, is as best as we can do. This is in fact not quite true, so long as one is happy to leave the realm of real probability measures, and consider instead complex measures. In the following we provide a brief introduction to complex Gaussian measures, and closely follow the exposition

of Brydges, Imbrie and Slade [BIS09].
We consider distributions on complex fields $(\psi_x)_{x\in\Lambda} \in \mathbb{C}^\Lambda$, with $\psi_x = u_x + iv_x$, and we adopt the convention of always writing the field as the pair $(\psi_x, \overline{\psi}_x)$, where $\overline{\psi}_x = u_x - iv_x$ is the complex conjugate. We use the notation ψ to distinguish from the real fields ϕ discussed above. Integration on $\mathbb{C}^\Lambda$ is defined via the differential forms

$$\mathrm{d}\psi_x = \mathrm{d}u_x + i\mathrm{d}v_x, \qquad \mathrm{d}\overline{\psi}_x = \mathrm{d}u_x - i\mathrm{d}v_x,$$

and we take products of differential forms via the anti-commutative wedge product, which satisfies: $\mathrm{d}u_x \wedge \mathrm{d}v_y = -\mathrm{d}v_y \wedge \mathrm{d}u_x$. Henceforth we omit the wedge symbol, $\mathrm{d}u_x\mathrm{d}v_y := \mathrm{d}u_x \wedge \mathrm{d}v_y$. A consequence of anticommutativity is

$$\begin{aligned} \mathrm{d}u_x\mathrm{d}u_x &= -\mathrm{d}u_x\mathrm{d}u_x = 0 = \mathrm{d}v_x\mathrm{d}v_x, \\ \mathrm{d}\overline{\psi}_x\mathrm{d}\psi_x &= \big(\mathrm{d}u_x - i\mathrm{d}v_x\big)\big(\mathrm{d}u_x + i\mathrm{d}v_x\big) = 2i\mathrm{d}u_x\mathrm{d}v_x. \end{aligned}$$

The differential form on $\mathbb{C}^\Lambda$ is then given by

$$\mathrm{d}\overline{\psi}\mathrm{d}\psi := \prod_{x\in\Lambda} \mathrm{d}\overline{\psi}_x\mathrm{d}\psi_x,$$

and using the calculation above, this is

$$= (2i)^{|\Lambda|} \prod_{x\in\Lambda} \mathrm{d}u_x\mathrm{d}v_x,$$

so that integration on $\mathbb{C}^\Lambda$ can be done against real variables $u_x, v_x \in \mathbb{R}$. We saw in the definition of the (real) Gaussian field ϕ that it was determined by a symmetric matrix, where symmetry ensured the existence of its inverse which was required to define the normalisation constant. The complex equivalent is to require a matrix $A \in \mathbb{C}^{|\Lambda|\times|\Lambda|}$ to be *Hermitian*: $A = \big(\overline{A}\big)^T =: A^*$. We assume throughout that A is positive-definite, and consequently invertible; we denote $C := A^{-1}$. The following proposition will enable us to define a complex valued equivalent to the Gaussian free field.

Proposition 3.18. *Let $A \in \mathbb{C}^{|\Lambda|\times|\Lambda|}$ be Hermitian, with inverse $C = A^{-1}$. Then*

$$Z_C := \int e^{-\langle\psi, A\overline{\psi}\rangle}\mathrm{d}\overline{\psi}\mathrm{d}\psi = \frac{(2\pi i)^{|\Lambda|}}{\det A}.$$

Moreover, this continues to hold if A is not Hermitian, but has positive definite Hermitian part: $H_A := \frac{1}{2}(A + A^*)$.

Proof. We consider only the case of A Hermitian, and follow the calculations of [BIS09] Lemma 2.1 p.37-8. As a consequence of A being Hermitian, its eigenvalues are necessarily real, and moreover it can be diagonalised by a unitary matrix. That is, there is a $U \in \mathbb{C}^{|\Lambda|\times|\Lambda|}$ with $UU^* = U^*U = I$, and $A = UDU^*$, with $D = \mathrm{diag}(\eta_x)$ the diagonal matrix of eigenvalues $\eta_x \in \mathrm{Spec}(A)$, which are in an arbitrary order. Performing a change of variables $\varphi = U\psi$

$$\begin{aligned}\int e^{-\langle\psi,A\overline{\psi}\rangle}\mathrm{d}\overline{\psi}\mathrm{d}\psi &= \int e^{-\langle\varphi,D\overline{\varphi}\rangle}\mathrm{d}\overline{\varphi}\mathrm{d}\varphi \\ &= (2\pi i)^{|\Lambda|}\int e^{-\sum(u_x+iv_x)\eta_x(u_x-iv_x)}\mathrm{d}u_1\ldots\mathrm{d}u_{|\Lambda|}\mathrm{d}v_1\ldots\mathrm{d}v_{|\Lambda|} \\ &= (2\pi i)^{|\Lambda|}\prod_{x\in\Lambda}\int_{\mathbb{R}} e^{-\eta_x(u_x^2+v_x^2)}\mathrm{d}u_x\mathrm{d}v_x,\end{aligned}$$

which we recognise as a standard Gaussian integral

$$= (2\pi i)^{|\Lambda|}\prod_{x\in\Lambda}\frac{1}{\eta_x}. \qquad \square$$

As a consequence of this proposition we define a *complex Gaussian measure* with covariance $C = A^{-1}$ by

$$P^C\big(\mathrm{d}\psi, \mathrm{d}\overline{\psi}\big) := \frac{1}{Z_C}\int e^{-\langle\psi,A\overline{\psi}\rangle}\mathrm{d}\overline{\psi}\mathrm{d}\psi. \tag{3.14}$$

Integrals of this form have been widely studied by physicists under the guise of path integrals, and often go under the name of Grassman integration. Recent work by Brydges and Slade, along with a myriad of co-authors, have applied this formalism to a variety of statistical mechanical models: notably the φ^4-field theory, and the study of self avoiding walk models. Remarkably they derive a complex Gaussian integral representation for the weakly self-avoiding walk model, for which the two point correlation function is seen to agree with the two point function of a φ^4-field theory; see [BIS09] for a survey. To model self avoiding walks, additional machinery is required to define suitable differential forms, and we will not make use of these here. However, at the heart of these arguments, will be an integration by parts formula for complex Gaussian measures, which will be reminiscent of the moment formula for a permanental field, Corollary 3.16. We define differentiation of complex fields via

$$\frac{\partial}{\partial\psi_x} := \frac{1}{2}\bigg(\frac{\partial}{\partial u_x} - i\frac{\partial}{\partial v_x}\bigg).$$

Lemma 3.19. *Let A have positive definite Hermitian part H_A, $C = A^{-1}$ and let P^C denote the induced complex Gaussian measure. For $F\colon \mathbb{C}^{|\Lambda|} \to \mathbb{C}$ smooth, and*

$x \in \Lambda$

$$\int \overline{\psi}_x F \, P^C\big(\mathrm{d}\overline{\psi}\mathrm{d}\psi\big) = \sum_{y\in\Lambda} C_{xy} \int \frac{\partial F}{\partial \psi_y} \, P^C\big(\mathrm{d}\psi, \mathrm{d}\overline{\psi}\big). \tag{3.15}$$

Proof. Again we follow [BIS09] Lemma 2.2, p.38. As a preliminary calculation we note

$$\begin{aligned}
\frac{\partial}{\partial u_y} e^{-\langle \psi, A\overline{\psi}\rangle} &= \frac{\partial}{\partial u_y} e^{-\sum_{y,z\in\Lambda}(u_y+iv_y)A_{yz}(u_z-iv_z)} \\
&= \sum_{z\in\Lambda} A_{yz}(u_z - iv_z) e^{-\sum_{y,z\in\Lambda}(u_y+iv_y)A_{yz}(u_z-iv_z)} \\
&= e^{-\langle \psi, A\overline{\psi}\rangle} \sum_{z\in\Lambda} A_{yz}\overline{\psi}_z,
\end{aligned}$$

and similarly

$$\frac{\partial}{\partial v_y} e^{-\langle \psi, A\overline{\psi}\rangle} = i e^{-\langle \psi, A\overline{\psi}\rangle} \sum_{z\in\Lambda} A_{yz}\overline{\psi}_z.$$

Consequently

$$\frac{\partial}{\partial \psi_y} e^{-\langle \psi, A\overline{\psi}\rangle} = e^{-\langle \psi, A\overline{\psi}\rangle} \sum_{z\in\Lambda} A_{yz}\overline{\psi}_z.$$

Now considering the integral term on the right hand side of (3.15), and applying the integration by parts formula for real integrals (against u_x, v_x)

$$\begin{aligned}
\int \frac{\partial F}{\partial \psi_y} P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) &= \int \frac{\partial F}{\partial \psi_y} e^{-\langle \psi, A\overline{\psi}\rangle} \mathrm{d}\overline{\psi}\mathrm{d}\psi \\
&= \int F \frac{\partial}{\partial \psi_y} e^{-\langle \psi, A\overline{\psi}\rangle} \mathrm{d}\overline{\psi}\mathrm{d}\psi \\
&= \sum_z A_{yz} \int \overline{\psi}_z F \, P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}).
\end{aligned}$$

Summing both sides over C_{xy}

$$\begin{aligned}
\sum_{y\in\Lambda} C_{xy} \int \frac{\partial F}{\partial \psi_y} P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) &= \sum_{y\in\Lambda}\sum_{z\in\Lambda} C_{xy}A_{yz} \int \overline{\psi}_z F \, P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) \\
&= \sum_{z\in\Lambda} \left(\sum_{y\in\Lambda} C_{xy}A_{yz} \right) \int \overline{\psi}_z F \, P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}).
\end{aligned}$$

Since $C = A^{-1}$, the summation over y is equal to $\delta_x(y)$ and all of the terms on the right hand side vanish except for the term with $y = x$

$$= \int \overline{\psi}_x F \, P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}). \qquad \square$$

Following the parlance of statistical mechanics, we use $\langle \cdot \rangle_C$ to denote 'expectation' against the complex Gaussian measure P^C; when the expectation is against variables of the form $\psi_x \overline{\psi}_x \in \mathbb{R}$ we will also write E^C. As an application of the integration by parts formula, we have the following calculations. For $x, y \in \Lambda$

$$\langle \overline{\psi}_x \psi_y \rangle_C \; = \; \int \overline{\psi}_x \psi_y P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) \; = \; \sum_z G_{xz} \int \frac{\partial}{\partial \psi_z} \psi_y P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) \; = \; G_{xy} \tag{3.16}$$

whilst

$$\begin{aligned}
\langle \psi_x \overline{\psi}_x \psi_y \overline{\psi}_y \rangle_C &= E^C\left(\psi_x \overline{\psi}_x \psi_y \overline{\psi}_y\right) \\
&= \int \psi_x \overline{\psi}_x \psi_y \overline{\psi}_y P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) \\
&= \sum_z G_{xz} \int \frac{\partial}{\partial \psi_z} \psi_x \psi_y \overline{\psi}_y P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) \\
&= G_{xx} \int \psi_y \overline{\psi}_y P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) + G_{xy} \int \psi_x \overline{\psi}_y P^C(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) \\
&= G_{xx} G_{yy} + G_{xy} G_{yx}.
\end{aligned}$$

We recognise these as agreeing with the moments of an $\alpha = 1$ permanental process, and in fact we have the following more general result, [BIS09] Lemma 2.3 pp.39.

Lemma 3.20. *For $x_1, \ldots, x_m \in \Lambda$*

$$\langle \textstyle\prod_{i=1}^m \psi_{x_i} \overline{\psi}_{x_i} \rangle_C \; = \; \mathrm{Per}\, G_{|\{x_1, \ldots, x_m\}}.$$

The above is sufficient to confirm that the $\alpha = 1$ permanental field agrees with a complex Gaussian, so long as the generator of the permanental field has positive definite Hermitian part. As in [BIS09] we omit the proof of this moments lemma, since it is a rather drawn out inductive argument. Instead we prove the equality in distribution by showing equivalence of the Laplace transforms. We prove this in the context of interest to us, the space–time occupation field.

Theorem 3.21. *Let $G = G^N$ be the Green's function of the walk $\overline{X}^N$ on Λ^N. Then*

$$\left(\mathcal{L}^N, \mathbb{P}\right) = \left(\psi\overline{\psi}, P^G\right),$$

where $(\psi\overline{\psi})_{x_\tau} := \psi_{x_\tau} \overline{\psi}_{x_\tau} = u_{x_\tau}^2 + v_{x_\tau}^2$, for $x_\tau \in \Lambda_N$.

Proof. Denoting $\lambda_N = 1 + N\beta^{-1}$, we recall that $G = G^N = \lambda_N^{-1}\left(I - P^N\right)^{-1}$, we first confirm that $\lambda_N(I-P^N)$ and $\lambda_N(I-P^N+V)$ have positive definite Hermitian part, where $V=\mathrm{diag}(v)$ for some $v\in\mathbb{R}_+^{\Lambda_N}$. We have

$$H_{\lambda(I-P^N)+V} = \frac{1}{2}\Big((\lambda(I-P^N)+V)+(\lambda(I-P^N)+V)^T\Big) = I-H_{P^N}+V.$$

The matrix H_{P^N} is symmetric, and is the average of two transition matrices, and hence is itself a transition matrix. Moreover, by assumption **A2** the matrix is sub-stochastic. It follows that $\mathrm{Spec}(I-H_{P^N})\subset(0,2)$, so that $I-H_{P^N}$ is positive definite. Moreover, positivity of $v\in\mathbb{R}_+^{\Lambda}$ ensures that V is also positive definite, and hence so is the sum $I-H_{P^N}+V$, Proposition B.2. Positivity of λ_N then gives the desired result. Now

$$\left\langle e^{-\langle\psi,V\overline{\psi}\rangle}\right\rangle_G = Z_G^{-1}\int e^{-\langle\psi,\lambda_N(I-P+V/\lambda_N)\overline{\psi}\rangle}\mathrm{d}\overline{\psi}\mathrm{d}\psi,$$

and since $\lambda_N(I-P+V/\lambda_N)$ has positive definite Hermitian part, by Proposition 3.18 the integral above exists and is given by $Z_{\lambda_N^{-1}(I-P+V/\lambda_N)^{-1}}$, from which

$$=\frac{\det(I-P)}{\det(I-P+V/\lambda_N)}. \qquad \square$$

As mentioned above, the theorem is not limited to the case of the space–time walk, and holds for any permanental process whose generator has positive definite Hermitian part. In terms of the loop soup isomorphism however, to date we have only established the Laplace transform of the occupation field for generators $P-I$ which are normal or reversible. We do, however, believe that an extension to processes with H_{I-P} positive definite is possible. A further question is whether this can in turn be extended to the generator $Q=\lambda(P-I)$ with continuous jump rates.
In light of this isomorphism and the fact that $\mathcal{L}^{\downarrow N}\to\mathcal{L}+\frac{\beta}{1-h}\mathbf{1}$, the occupation field under $\mathbb{P}^B$, we have the interpretation of the Bosonic occupation field as the field obtained on integrating out the spatial component of a space–time complex Gaussian field plus the additon of a deterministic factor. In the next section we continue to work with the space–time description of the loop model, but provide further relationships with the ideal Bose gas.

3.3 Space Time Loops And The Bose Gas

In the final section of this chapter we provide two applications of the space–time formulation of the loop soup to the study of the ideal Bose gas. The first of these continues on the theme of the previous section, where we derive a version of Symanzik's formula for complex Gaussian fields, which we then provide a physical interpretation for. The second section considers space–time correlations in the loop soup, and relates these to the 1-particle reduced density matrix of an ideal gas.

3.3.1 Symanzik's Formula for Bosons in a Random Environment

It was mentioned in the introduction that the study of loop measures was started by Symanzik [Sym66, Sym69] who used them to derive a path integral representation for Euclidean quantum field theory. One particular result relates the correlations of a quantum field under a potential to an expectation with respect to an occupation field of random loops and paths. In this section we derive Symanzik's formula for complex measures in the context of the space–time loop soup, and will relate the formula to a model of ideal bosons interacting with a space–time random environment. Our steps follow those of Sznitman [Szn12] Section 4.3, pp.91–5, and before engaging specifically with the space–time world we recall some definitions.

Up until now we have considered only measures on closed loops. In this section we also consider measures on paths, and define for $x, y \in \Lambda$ the measure

$$\mathbf{P}_{xy}[B] = \frac{1}{\lambda_x} \int_0^\infty P_x\big[\mathbf{1}_{\{X_t=y\}}(B \cap D_t)\big]\mathrm{d}t, \qquad D \in \mathcal{D},$$

where $\mathcal{D}$ is the σ-algebra on the space of paths D, introduced on p. 10. This is a finite measure on 'open ended bridges': i.e. paths from x to y but with no prescribed duration. The additional factor of λ^{-1} is similar to the convention adopted in the previous section of working with the walk X, and then scaling the occupation field by $\mathcal{L} \mapsto \lambda^{-1}\mathcal{L}$; in this context it ensures that the total mass of the measure is

$$\mathbf{P}_{xy}[D] \;=\; \frac{1}{\lambda_x} \int_0^\infty P_x[X_t = y]\mathrm{d}t \;=\; G\big(\overline{X}\big)_{xy}.$$

We stress that this is not a probability measure, however we still write expressions such as $\mathbf{P}_{xy}[X_t = z]$ to stand for the mass of the set $\{\gamma \in D : \gamma_t = z\}$. We define the local time under the measure $\mathbf{P}_{xy}$ analogously to how occupation times for the loop soup were defined in Section 1.2.3, but denote local fields of random walk by lowercase $l = (l_x)_{x\in\Lambda}$; we also make the distinction of referring to local fields when speaking of random walks, and occupation fields when referring to the loop soup. As discussed above, since we have defined the measure in terms of the walk X, we will consider the scaled local field $\lambda^{-1}l$: in fact we maintain all the assumptions **A1′-4** introduced on p.56. The following lemma is the equivalent of the derivation for the Laplace transform of the Markov loop soup.

Lemma 3.22. *For Λ reversible or normal, and $v \in \mathbb{R}_+^\Lambda$*

$$\mathbf{E}_{xy}\Big[e^{-\langle v,\lambda^{-1}l\rangle}\Big] = (V - Q)^{-1}_{xy}.$$

Proof. We argue via the Feynman–Kac formula (3.3). We proceed as in the proof for the loop soup occupation field, noting

$$\begin{aligned}\mathbf{E}_{xy}\Big[e^{-\langle v,\lambda^{-1}l\rangle}\Big] &= \int_0^\infty \mathbf{E}_x\Big[\mathbf{1}_{\{X_t=y\}}e^{-\int_0^t V/\lambda(X_s)\mathrm{d}s}\Big]\\ &= \int_0^\infty \Big(e^{t(P-I-V/\lambda)}\Big)_{xy}.\end{aligned}$$

If we consider the truncated integral on $[0,T]$, for any $T>0$

$$\begin{aligned}\int_0^T e^{t(P-I-V/\lambda)} &= \sum_{n=0}^\infty \frac{(P-I-V/\lambda)^n}{n!}\int_0^T t^n\mathrm{d}t\\ &= \sum_{n=0}^\infty \frac{(P-I-V/\lambda)^n}{(n+1)!}T^{n+1}\\ &= (P-I-V/\lambda)^{-1}\left(\sum_{n=0}^\infty \frac{(P-I-V/\lambda)^n}{n!}-I\right)\\ &= (P-I-V/\lambda)^{-1}\Big(e^{T(P-I-V/\lambda)}-I\Big).\end{aligned}$$

Appealing to Proposition 3.6, for $v\in\mathbb{R}_+^\Lambda$ sufficiently small the eigenvalues of $P-I-V$ have negative real part and consequently $\lim_{T\to\infty}e^{T(P-I-V)}=0$, [HJ13] Theorem 5.6.12 pp.348–9. Hence

$$\mathbf{E}_{xy}\Big[e^{-\langle v,\lambda^{-1}l\rangle}\Big] = \lambda_x^{-1}(I-P+V/\lambda)^{-1} = (V-Q)^{-1}.$$

This confirms the result for sufficiently small v, the extension to all positive v is given in [Szn12] Proposition 3.10 p.81. □

Denoting $G_V=(V-Q)^{-1}$ (note that when $V=0$, $G_V=G$), then so long as $\lambda(I-P)$ has positive Hermitian part, the complex Gaussian measure P^{G_V} is well defined, and from the lemma above, and (3.16) we have the relation

$$\mathbf{E}_{xy}\Big[e^{-\langle v,\lambda^{-1}l\rangle}\Big] = G_V(x,y) = \big\langle\overline{\psi}_x\psi_y\big\rangle_{G_V}. \tag{3.17}$$

This can be seen as a basic form of Dynkin's isomorphism theorem for complex measures; more generally this says for bounded measurable functions $F\colon\mathbb{R}^\Lambda\to\mathbb{R}$

$$\mathbf{E}_{xy}\otimes E^{G_V}\Big[F\big(\lambda^{-1}l+\psi\overline{\psi}\big)\Big] = \Big\langle\overline{\psi}_x\psi_yF\big(\psi\overline{\psi}\big)\Big\rangle_{G_V},$$

see [Bry92] Theorem 3.2 pp.21–22. The content of (3.17) is that the moments of a Gaussian field are characterised by the local field of a random walk. Symanzik's formula generalises this result so that the left hand side is now an expectation with respect to a perturbation of a Gaussian.

Let $C=A^{-1}$ be the covariance matrix of a complex Gaussian field, and let $f\colon\mathbb{R}_+^\Lambda\to\mathbb{R}_+$ be measurable and integrable with respect to P^C, $E_C\big[f\big(\psi\overline{\psi}\big)\big]<\infty$. We define the perturbed Gaussian measure $P^{C,h}$ to be the normalised measure

$$P^{C,f}(\mathrm{d}\psi, \mathrm{d}\overline{\psi}) = \frac{1}{Z_{C,h}} e^{-\langle \psi, A\overline{\psi}\rangle} f\Big(\psi\overline{\psi}\Big) \mathrm{d}\overline{\psi}\mathrm{d}\psi.$$

In the case that $f(\underline{t}) = \exp\big(-g\sum_{x\in\Lambda} t_x^2 - \nu\sum_{x\in\Lambda} x\big)$, this is exactly the law of the φ^4-field theory studied by Symanzik, [Sym66, Sym69]. As with this example, we consider the case that f is multiplicative: $f(\underline{t}) = \prod_{x\in\Lambda} f_x(t_x)$, for some measurable $f_x \colon \mathbb{R}_+ \to \mathbb{R}_+$; moreover we assume that each of the f_x can be derived as the Laplace transform of a random variable on $\mathbb{R}_+$; that is there is a law ν_x on $\mathbb{R}_+$ for which

$$f_x(u) = \int_0^\infty e^{-vu}\nu_x(\mathrm{d}v).$$

In this case we write $P^{C,\nu}$ in place of $P^{C,f}$, and we can view the perturbation f as being the effect of a randomisation in the Gaussian, since for example

$$\begin{aligned}
Z_{G,\nu} &= \int \left(\prod_{x\in\Lambda}\int_0^\infty e^{-\psi_x\overline{\psi}_x v_x}\nu_x(\mathrm{d}v_x)\right) P^C\big(\mathrm{d}\psi\mathrm{d}\overline{\psi}\big) \\
&= \int \nu_1(\mathrm{d}x_1)\cdots\int \nu_{|\Lambda|}(\mathrm{d}x_{|\Lambda|})\bigg(\int e^{-\langle\psi, A-V\overline{\psi}\rangle} P^C\big(\mathrm{d}\psi\mathrm{d}\overline{\psi}\big)\bigg) \\
&= E^{\otimes\nu}\big[Z_{(A-V)^{-1}}\big] \\
&= (2\pi i)^{|\Lambda|}\mathbb{E}^{\otimes\nu}\left[\frac{1}{\det(A-V)}\right], \qquad (3.18)
\end{aligned}$$

where we use $E^{\otimes\nu} = \bigotimes_{x\in\Lambda} E^{\nu_x}$ to denote the product measure, and $V = \mathrm{diag}(v_x)$ is now a random diagonal matrix. We now state Symanzik's formula for the space–time walk; as with the derivation of the space–time isomorphism, this result can be generalised so long as the Gaussian measure exists and the Laplace transform of the Markov loop occupation field is defined. As we have done throughout the section, we write $\mathcal{L}^N = (1+N\beta^{-1})^{-1}\mathcal{L}$, and similarly we denote $l^N := (1+N\beta^{-1})^{-1}l$.

Theorem 3.23 (Symanzik's Formula)**.** *Let $G = G^N$ be the Green's function of the walk $\overline{X}^N$ on Λ_N, and for $x_\tau \in \Lambda_N$ let ν_{x_τ} be the law of a positive random variable v_{x_τ}. Then for x_τ, $y_{\tau'} \in \Lambda_N$*

$$\langle\overline{\psi}_{x_\tau}\psi_{y_{\tau'}}\rangle_{G,\nu} = \frac{E^{\otimes\nu}\otimes\mathbf{E}_{x_\tau y_{\tau'}}\otimes\mathbb{E}_N\Big[e^{-\langle v,(\mathcal{L}^N+l^N)\rangle}\Big]}{E^{\otimes\nu}\otimes\mathbb{E}_N\Big[e^{-\langle v,\lambda^{-1}\mathcal{L}^N\rangle}\Big]}.$$

Proof. Our proof follows that given in [Szn12] pp.78–9, with the distinction that we work with complex Gaussian measures and that the loop soup is considered at intensity 1. From the definition of the measure $P^{G,\nu}$,

$$\begin{aligned}\langle\overline{\psi}_{x_\tau}\psi_{y_{\tau'}}\rangle_{G,\nu} &= \frac{1}{Z_{G,\nu}}\int \overline{\psi}_{x_\tau}\psi_{y_{\tau'}} e^{-\langle\psi,-Q\overline{\psi}\rangle}\left(\prod_{z_\theta\in\Lambda_N}\int_0^\infty e^{-\psi_{z_\theta}\overline{\psi}_{z_\theta}v_{z_\theta}}\nu_{z_\theta}(\mathrm{d}v_{z_\theta})\right)\mathrm{d}\overline{\psi}\mathrm{d}\psi\\ &= \frac{1}{Z_{G,\nu}}E^{\otimes\nu}\left[\frac{Z_{G,V}}{Z_{G,V}}\int \overline{\psi}_{x_\tau}\psi_{y_{\tau'}} e^{-\langle\psi,(V-Q)\overline{\psi}\rangle}\mathrm{d}\overline{\psi}\mathrm{d}\psi\right],\end{aligned}$$

we recognise the integral term as agreeing with the right hand side of (3.17), so that

$$= \frac{1}{Z_{G,\nu}}E^{\otimes\nu}\otimes\mathbf{E}_{xy}\left[Z_{G_V}e^{-\langle v,l^N\rangle}\right].$$

Using the identity for the partition function $Z_{G_V} = Z_{G_V} = (2\pi i)^{|\Lambda|}\det G_V$, this becomes

$$= \frac{(2\pi i)^{|\Lambda|}}{Z_{G,\nu}}E^{\otimes\nu}\otimes\mathbf{E}_{xy}\left[\det G_V e^{-\langle v,l^N\rangle}\right],$$

and then multiplying by $\det G/\det G$

$$= \frac{(2\pi i)^{|\Lambda|}\det G}{Z_{G,\nu}}E^{\otimes\nu}\otimes\mathbf{E}_{xy}\left[\frac{\det G_V}{\det G}e^{-\langle v,l^N\rangle}\right].$$

Considering the quotients of determinants

$$\begin{aligned}\frac{\det G_V}{\det G} &= \frac{\det\left(\lambda_N I\left(I-P^N\right)\right)}{\det\left(\lambda_N I\left(I-P^N+V/\lambda\right)\right)}\\ &= \frac{\det\left(\lambda_N I\right)\det\left(I-P^N\right)}{\det\left(\lambda_N I\right)\det\left(I-P^N+V/\lambda\right)}\\ &= \frac{\det\left(I-P^N\right)}{\det\left(I-P^N+V/\lambda\right)},\end{aligned}$$

where we recognise the right most expression as the Laplace transform of the field $\mathcal{L}^N$, Theorem 3.7. Hence

$$\langle\overline{\psi}_{x_\tau}\psi_{y_{\tau'}}\rangle_{G,\nu} = \frac{(2\pi i)^{|\Lambda|}\det G}{Z_{G,\nu}}E^{\otimes\nu}\otimes\mathbf{E}_{xy}\otimes\mathbb{E}_N\left[e^{-\langle v,\mathcal{L}^N+l^N\rangle}\right],$$

which is the desired numerator. The calculation for the denominator proceeds along similar lines, from (3.18)

$$\frac{Z_{G,\nu}}{(2\pi i)^{|\Lambda|}\det G} = E^{\otimes\nu}\left[\frac{\det G_V}{\det G}\right] = E^{\otimes\nu}\otimes\mathbb{E}_N\left[e^{-\langle v,\mathcal{L}^N\rangle}\right]. \qquad \square$$

We now provide a heuristic derivation of how Symanzik's theorem can be interpreted in the context of the ideal gas. Recalling the definition of the 1-particle reduced density matrix $\tilde{\sigma}_\Lambda$, given by (0.9), the Onsager–Penrose criterion for BEC was described as the existence of a non-vanishing limit of $\tilde{\sigma}_\Lambda(x, y)$ as $|\Lambda| \to \infty$, and then $|x - y| \to \infty$. Ginibre [Gin71] provides a description of the 1-particle density matrix which is amenable to our study of occupation fields. Before stating this we provide some details regarding an interacting gas. Assuming that the graph Λ has been endowed with a metric $d_\Lambda \colon \Lambda \times \Lambda \to \mathbb{R}_+$, we define a pair potential $\Phi \colon \mathbb{R}_+ \to \mathbb{R}$, which is assumed to be integrable. Given a collection of paths $p_1, \ldots, p_m \in D_{\beta j}$, we define the weight

$$U(p_1, \ldots, p_m) := \int_0^\beta \sum_{i<j} \Phi\Big(d_\Lambda\big(p_i(t), p_j(t)\big)\Big) \mathrm{d}t,$$

and the partition function for the Bose gas with pair potential Φ is then given by

$$\Xi_{\Lambda,\Phi}(\beta, h) = \sum_{n=0}^{\infty} \frac{e^{\beta h n}}{n!} \sum_{x_1,\ldots,x_n \in \Lambda} \bigotimes_{i=1}^{n} \mathbf{E}^{\beta}_{x_i x_i}\left[e^{-U\left(\overline{X}^{(1)},\ldots,\overline{X}^{(n)}\right)}\right],$$

where we recall that $\mathbf{P}^{\beta}_{xx}[\,\cdot\,] = \mathbf{P}_x\big[\mathbf{1}_{\{\overline{X}_\beta = x\}}\,\cdot\,\big]$, and $\overline{X}^{(i)}$ is the walk associated with the measure $\mathbf{P}^{\beta}_{x_i,x_i}$. Ginibre's expression for the 1-particle reduced density matrix is as follows, we explain the notation beneath.

Theorem 3.24. *The grand canonical partition function of the interacting Bose gas with suitably defined pair potential* $\Phi \colon \mathbb{R}_+ \to \mathbb{R}$ *is given by*

$$\Xi_{\Lambda,\Phi}(\beta, h) = \sum_{n \geq 0} \frac{1}{n!} \big(\mu^B\big)^{\otimes n}\left[e^{-U(\gamma_1,\ldots,\gamma_n)}\right],$$

whilst the 1*-particle reduced density matrix of an ideal gas on a graph* Λ *is given by*

$$\tilde{\sigma}_\Lambda(x, y) = \Xi_{\Lambda,\Phi}(\beta, h)^{-1} \sum_{j \geq 1} \sum_{n \geq 0} \frac{e^{\beta h j}}{n!} \mathbf{E}^{\beta j}_{xy} \otimes \big(\mu^B\big)^{\otimes n}\left[e^{-U(\overline{X},\gamma_1,\ldots,\gamma_n)}\right].$$

See [Gin71] pp.355-9, or [BR02] Theorem 6.3.14, pp.385-6. The term over n is an expectation with respect to n independent loops $\gamma_1, \ldots, \gamma_n$ chosen according to the Bosonic loop measure μ^B, whilst the expectation is with respect to a single path from x to y, whose length is determined according to the weighted sum $\sum_{j \geq 1} e^{\beta h j} \mathbf{E}^{\beta j}_{xy}$. Finally the exponential term in U is understood to integrate over all 'legs' of the walks. That is if p is a path of length βj, then it is understood that p contributes j terms to be integrated over in U: one corresponding to each interval $[i\beta, (i+1)\beta)$, $i = 0, \ldots, j-1$. The summation over $n \geq 0$ is none other than the Poisson point process under μ^B, so that we can rewrite these expressions as

$$\begin{aligned} \Xi_{\Lambda,\Phi}(\beta,h) &= \mathbb{E}^B\left[e^{-U(S)}\right] \\ \tilde{\sigma}_\Lambda(x,y) &= \frac{1}{\Xi_{\Lambda,\Phi}}\sum_{j\geq 1} e^{\beta hj}\mathbf{E}^{j\beta}_{xy}\otimes\mathbb{E}^B\left[e^{-U(\overline{X},S)}\right], \end{aligned} \tag{3.19}$$

where we dropped the dependence of $\Xi_{\Lambda,\Phi}$ on $\beta > 0$ and $h < 0$ for clarity, and S is the Bosonic loop soup. In the case that $\Phi \equiv 0$, we have the ideal Bose gas, and since $\exp\left(\mu^B(\Gamma)\right) = \Xi_\Lambda(\beta,h)$

$$\tilde{\sigma}_\Lambda(x,y) \;=\; \frac{1}{\Xi_{\Lambda,\Phi}}\left(\sum_{j\geq 1} e^{\beta hj}\mathbf{E}^{j\beta}_{xy}\right)\left(\sum_{n\geq 0}\frac{1}{n!}\mu^B(\Gamma)^n\right) \;=\; \sum_{j\geq 1} e^{\beta hj}\mathbf{E}^{j\beta}_{xy}. \tag{3.20}$$

Whilst at the present time we have no way of handling pair interactions, the equation (3.19) has its similarities to the expression of Symanzik's formula, we develop this relationship in the following, though we do so somewhat heuristically. Rather than considering the potential U to be defined via a pair interaction, instead we suppose that $\Phi\colon \Lambda \to \mathbb{R}_+$, and then for paths $p_1,\ldots,p_m \in D_\beta$ define

$$U(p_1,\ldots,p_m) = \sum_{i=1}^m \int_0^\beta \Phi(p_i(t))\mathrm{d}t.$$

In the corresponding Bosonic model, paths are no longer weighted via their interaction with one another, but rather with their interaction with a background environment. For a path $p \in D_{\beta j}$ made up of the legs $p_i\colon [0,\beta) \to \Lambda$, $p_i(t) = p(\beta i + t)$, $i = 0,\ldots,j-1$, then we have

$$U(p) = U(p_0,\ldots,p_{j-1}) = \sum_{i=0}^{j-1}\int_0^\beta \Phi(p_i(t))\mathrm{d}t \;=\; \int_0^{\beta j}\Phi(p(t))\mathrm{d}t \;=\; \langle \Phi, l(p)\rangle,$$

where $l(p)$ is the local field of the path p. It follows that for such a choice of U, the formula on the right hand side of (3.19) becomes

$$\frac{1}{\Xi_{\Lambda,\Phi}}\sum_{j\geq 1} e^{\beta hj}\mathbf{E}^{j\beta}_{xy}\otimes\mathbb{E}^B\left[e^{-U(\overline{X},S)}\right] = \frac{1}{\Xi_{\Lambda,\Phi}}\sum_{j\geq 1} e^{\beta hj}\mathbf{E}^{\beta j}_{xy}\otimes\mathbb{E}^B\left[e^{-\langle\Phi,\mathcal{L}+l\rangle}\right],$$

with $\mathcal{L}$ the occupation field under $\mathbb{P}^B$, and l the local field of the path under $\mathbf{P}^{j\beta}_{xy}$. We formally identify this as the 1-particle reduced density matrix of a model of ideal bosons interacting with a background environment Φ, and denote this by $\tilde{\sigma}$ as in the rigorously derived case. A particular feature of the function $\tilde{\sigma}(x,y)$ is that the sum over $j \geq 1$ considers only walks which terminate at y at times which are an integer multiples of β; although a path from x to y is not a loop (unless $x = y$), the paths of interest still have the 'space–time property' of needing to do full cycles of the torus. As such it is natural to derive $\tilde{\sigma}_\Lambda(x,y)$ as an observable of the space–time loop model. In particular we look to relate the 1-particle reduced density matrix to Symanzik's formula. We make the following 'exchanges', which are now natural after some experience with the space–time loop model

$$\begin{aligned} \Phi\colon \Lambda \to \mathbb{R}_+ &\quad\longleftrightarrow\quad \Phi_N : \Lambda_N \to \mathbb{R}_+ \\ \mathbb{E}^B &\quad\longleftrightarrow\quad \mathbb{E}_N \\ \mathcal{L}, l &\quad\longleftrightarrow\quad \mathcal{L}^N, l^N \\ \sum_{j\geq 1} e^{\beta h j}\mathbf{E}^{\beta j}_{xy} &\quad\longleftrightarrow\quad \mathbf{P}_{xy}. \end{aligned}$$

We note that in moving to the space–time model we have performed our usual change from using the variable jump rate walk $\overline{X}$ to working with X the unit jump rate process. Moreover, we have also assumed the convention of taking $P = P^h$, the walk scaled by $h < 0$ introduced in Corollary 3.8, so as that we can drop the dependency on h in all of the expressions. We have therefore the following heuristic derivation of the 1-particle reduced density matrix of a Bose gas interacting with a space–time background field

$$\tilde{\sigma}_\Lambda(x,y) = \lim_{N\to\infty} \frac{1}{\Xi_{\Lambda_N,\Phi_N}} \mathbf{E}_{x_0 y_0} \otimes \mathbb{E}_N\left[e^{-\left\langle \Phi_N, \mathcal{L}^N + l^N \right\rangle}\right]$$

and performing similar substitutions for the partition function term

$$= \lim_{N\to\infty} \frac{\mathbf{E}_{x_0 y_0} \otimes \mathbb{E}_N\left[e^{-\left\langle \Phi_N, \mathcal{L}^N + l^N \right\rangle}\right]}{\mathbb{E}_N\left[e^{-\langle \Phi_N, \mathcal{L}^N\rangle}\right]}.$$

The fact that we take $\mathbf{E}_{x_0 y_0}$ is exactly the condition which was previously required of the paths under $\sum_{j\geq 1} \mathbf{E}^{\beta j}_{xy}$: that they have duration equal to an integer multiple of β. Moreover we stress that the right hand side of the above is an observable which is defined purely on the graph Λ, whereas the right hand side is a space–time expression. On appealing to Symanzik's formula we can therefore relate the 1-particle reduced density matrix to the correlations of a space–time complex Gaussian field. We state the following heuristic theorem, or *heurum*, summarising our findings.

Heurum 3.25. *Let $\tilde{\sigma}_\Lambda\colon \Lambda \times \Lambda \to \mathbb{R}_+$ be the 1-particle reduced density matrix of an ideal Bose gas interacting with a background potential $\Phi\colon \Lambda \to \mathbb{R}_+$, and let $\Phi_N\colon \Lambda_N \to \mathbb{R}_+$ be such that $\Phi_N(x_\tau) = \Phi(x)$, $x_\tau \in \Lambda_N$. Then*

$$\tilde{\sigma}_\Lambda(x,y) = \lim_{N\to\infty} \left\langle \overline{\psi}_{x_0} \psi_{y_0} \right\rangle_{G_N,\Phi_N}.$$

Moreover this expression continues to hold on replacing Φ_N above with a space–time *random environment $\Phi_N \sim E^{\otimes \nu}$, so long as the limit exists.*

3.3.2 The 1-particle Reduced Density Matrix and Leg Walker

We saw in (3.20) that the 1-particle reduced density matrix of an ideal gas is given by

$$\tilde{\sigma}_\Lambda(x, y) = \sum_{j \geq 1} e^{\beta h j} \mathbf{P}_x \left[\overline{X}_{\beta j} = y \right].$$

Penrose and Onsager [PO56] provided the general criterion that the Bose gas in $\mathbb{R}^d$ undergoes BEC if it shows *off-diagonal long-range order* (ODLRO), which in the grand canonical ensemble, is to say

$$\lim_{|x-y| \to \infty} \lim_{|\Lambda| \to \infty} \tilde{\sigma}(x, y) = C > 0.$$

This condition can be reformulated for sequences of graphs, so long as the limit graph is well defined, where the term $|x - y|$ is replaced with a graph metric. We choose to work with an alternative criterion of Yang [Yan62], who considered instead the condition

$$\tilde{\sigma}^2(x) := \lim_{|\Lambda| \to \infty} \int_\Lambda \tilde{\sigma}(x, y) \tilde{\sigma}(y, x) \mathrm{d}y = C.$$

This definition has its advantage in our context as we need not consider graph metrics. We make the following definitions for graphs

$$\tilde{\sigma}^2_\Lambda(x) := \sum_{y \in \Lambda} \tilde{\sigma}_\Lambda(x, y) \tilde{\sigma}_\Lambda(y, x),$$

and if the graph is vertex transitive, such as the periodic lattice box $\Lambda_N^{(per)}$, this is the same as

$$\begin{aligned} &= \frac{1}{|\Lambda|} \sum_{x,y} \tilde{\sigma}_\Lambda(x, y) \tilde{\sigma}_\Lambda(y, x) \\ &=: \tilde{\sigma}^2_\Lambda. \end{aligned}$$

In the vertex transitive case, Yang's criterion says $\lim_{|\Lambda| \to \infty} \tilde{\sigma}^2_\Lambda = C$. In the following we return to the notation of Chapter 2, and denote Λ_N for a convergent graph sequence (rather than the space–time graph considered elsewhere in this chapter).

Theorem 3.26. *Let Λ_N be a convergent graph sequence, with reversible transition* *matrices. For $h < 0$*

$$\tilde{\sigma}_\Lambda^2 = \int \frac{1}{\left(e^{-\beta(h+u)} - 1\right)^2} m_\infty(\mathrm{d}u).$$

Before proving the theorem, we note that the diagonal terms of the 1-particle density matrix are given by

$$\begin{aligned}\tilde{\sigma}_\Lambda(x,x) &= \sum_{j\geq 1} e^{\beta hj} \mathbf{P}_x\left[\overline{X}_{\beta j} = x\right] \\ &= \sum_{j\geq 1} j\mu^B(\Gamma_{x,\beta j}),\end{aligned}$$

with $\Gamma_{x,\beta j} := \{\gamma \in \Gamma_{\beta j} : \gamma_0 = x\}$. And consequently

$$\mathrm{Tr}[\tilde{\sigma}_\Lambda] \; = \; \sum_{x\in\Lambda}\sum_{j\geq 1} j\mu^B(\Gamma_{x,\beta j}) \; = \; \rho_\Lambda,$$

with $\rho_\Lambda = \rho_\Lambda(\beta, h)$ the density of the ideal gas. That is the density is the trace of the 1-particle reduced density matrix, whereas $\tilde{\sigma}_\Lambda^2 = \mathrm{Tr}[\tilde{\sigma}_\Lambda\tilde{\sigma}_\Lambda]$. On recalling that the integral expression for the thermodynamic limit of the density was given as

$$\rho = \int \frac{1}{e^{-\beta(h+u)} - 1} m_\infty(\mathrm{d}u),$$

it is perhaps no surprise that $\tilde{\sigma}^2$ has the same integrand but squared. We make this rigorous below.

Proof of Theorem 3.26. We write $z = e^{\beta h}$ for the fugacity of the Bose gas, so as that $\tilde{\sigma}_\Lambda(x)$ becomes

$$\tilde{\sigma}_\Lambda^2(x) = \sum_{y\in\Lambda}\sum_{j,k\geq 1} z^{j+k} \mathbf{P}_x\left[\overline{X}_{\beta j} = y\right] \mathbf{P}_y\left[\overline{X}_{\beta k} = x\right],$$

and applying the Markov property

$$= \sum_{j,k\geq 1} z^{j+k} \mathbf{P}_x\left[\overline{X}_{\beta j+k} = x\right].$$

For each n there are exactly $n-1$ pairs $1 \leq j,k \leq n-1$ such that $j+k = n$, so that

$$= \sum_{n=1}^{\infty} (n-1) z^n \mathbf{P}_x\left[\overline{X}_{\beta n} = x\right].$$

Denoting $G_\beta(x;z) := \sum_{n=0}^{\infty} z^n \mathbf{P}_x\big[\overline{X}_{\beta n} = x\big]$, we have from the above

$$\tilde{\sigma}_\Lambda^2 = \frac{1}{|\Lambda|}\sum_{x\in\Lambda}\bigg(z\frac{\mathrm{d}}{\mathrm{d}z}G_\beta(x;z) - G_\beta(x;z) + 1\bigg), \tag{3.21}$$

and writing $G_\beta(x;z)$ in terms of the generator Q

$$\begin{aligned}\sum_{x\in\Lambda} G_\beta(x;z) &= \mathrm{Tr}\big[\textstyle\sum_{n=0}^{\infty} z^n e^{\beta n Q}\big] \\ &= \mathrm{Tr}\left[(I - ze^{\beta Q})^{-1}\right].\end{aligned}$$

Then considering the derivative term

$$\frac{\mathrm{d}}{\mathrm{d}z}\sum_{x\in\Lambda} G_\beta(x;z) = \mathrm{Tr}\bigg[\frac{\mathrm{d}}{\mathrm{d}z}(I - ze^{\beta Q})^{-1}\bigg],$$

which on using the identity $\frac{\mathrm{d}}{\mathrm{d}z}A^{-1}(z) = -A^{-1}(z)\big(\frac{\mathrm{d}}{\mathrm{d}z}A(z)\big)A^{-1}(z)$,

$$= \mathrm{Tr}\Big[(I - ze^{\beta Q})^{-1}e^{\beta Q}(I - ze^{\beta Q})^{-1}\Big].$$

Combining the last two equations with (3.21)

$$\tilde{\sigma}_\Lambda^2 = \frac{1}{|\Lambda|}\,\mathrm{Tr}\Big[z(I - ze^{\beta Q})^{-1}e^{\beta Q}(I - ze^{\beta Q})^{-1} - (I - ze^{\beta Q})^{-1} + I\Big].$$

To make the following manipulations easier to follows we write $R = (I - ze^{\beta Q})^{-1}$ and $S = e^{\beta Q}$, we note that $R^{-1} - I = -zS$. The above becomes

$$\begin{aligned}\tilde{\sigma}_\Lambda^2 &= \frac{1}{|\Lambda|}\,\mathrm{Tr}[zRSR - R + I] \\ &= \frac{1}{|\Lambda|}\,\mathrm{Tr}\big[R(zSR - I + R^{-1}\big] \\ &= \frac{1}{|\Lambda|}\,\mathrm{Tr}[R(zSR - zS)] \\ &= \frac{1}{|\Lambda|}\,\mathrm{Tr}[zRS(R - I)].\end{aligned}$$

On confirming the identity: $R - I = zRS$ we have

$$\begin{aligned}&= \frac{1}{|\Lambda|}\,\mathrm{Tr}\big[z^2 RSRS\big] \\ &= \frac{z^2}{|\Lambda|}\,\mathrm{Tr}\big[(RS)^2\big].\end{aligned}$$

Writing

$$RS \;=\; (I - ze^{\beta Q})e^{\beta Q} \;=\; (e^{-\beta Q} - z)^{-1},$$

we get

$$\begin{aligned}\tilde{\sigma}_\Lambda^2 &= \int \frac{z^2}{(e^{-\beta u} - z)^2} m_\Lambda(\mathrm{d}u) \\ &= \int \frac{1}{\left(e^{-\beta(h+u)} - 1\right)^2} m_\Lambda(\mathrm{d}u),\end{aligned}$$

and the result follows on taking the limit in Λ. □

The function $G_\beta(x; z)$ defined above is itself the Green's function of a walk, which we call the *leg walker*. This is the discrete time walk $Z^\beta = \left(Z_n^\beta\right)_{n\geq 0}$, whose transition matrix is given by $P^\beta = e^{\beta Q}$. The walk is equivalent to the process $(\overline{X}_{\beta j})_{j\geq 1}$, where we only 'observe' the continuous time walk at times which are integer multiples of β: the name leg comes from the fact that we do not see the processes behavior between successive 'legs' of its path. The 1-particle reduced density matrix is then exactly the Green's function

$$\tilde{\sigma}_\Lambda(x, y) \;=\; \sum_{j\geq 1} z^j \mathbf{P}_x[Z_j = y] \;=:\; G_\beta(x, y; z).$$

In the case that Λ is a subset of $\mathbb{Z}^d$, we can derive the behavior of $\tilde{\sigma}_\Lambda(x, y)$ at $h = 0$ by appealing to established facts for the Green's function of random walks on lattices. As a first point we note that it is sufficient to consider the Green's function of the limit graph $\mathbb{Z}^d$, since the limit of the Green's functions of the boxes $[-N, N]^d$ agree with this, see [LL10] p.101. Henceforth let $G_\beta(x) := G_\beta(0, x; 1)$ be the Green's function for the infinite lattice, providing the expected local time at x given a walk started from $0 \in \mathbb{Z}^d$; we are now interested in taking the limit as $|x| \to \infty$. For $d = 1, 2$, we know that $G_\beta(x) = \infty$ for all $x \in \mathbb{Z}^d$, we concentrate therefore on the case that $d \geq 3$.
Whilst the walk Z_β is no longer nearest neighbour, it retains several desirable properties of the nearest neighbour walk. In the following we define the first-step covariance matrix $C = C^\beta \in \mathbb{R}^{d\times d}$ to be

$$C_{jk} = \mathbf{E}_0\Big[Z_1^\beta(j) Z_1^\beta(k)\Big],$$

where $Z_1^\beta = \left(Z_1^\beta(1), \ldots, Z_1^\beta(d)\right) \in \mathbb{Z}^d$.

Lemma 3.27. *The random walk $Z^\beta = \left(Z_n^\beta\right)_{n\geq 1}$ satisfies the following properties.*

(i) The walk is centred, $\mathbf{E}_0\Big[Z_1^\beta\Big] = \underline{0}$, *with* $\underline{0} = (0, \ldots, 0) \in \mathbb{R}^d$,

(ii) has finite first moment $\mathbf{E}_0\Big[\big|Z_1^\beta\big|^2\Big] = \beta < \infty$,

(iii) and covariance matrix $C = \frac{\beta}{d} I$.

Proof. All three statements follow from properties of the continuous time walk $\overline{X}_\beta$. For (i), the symmetry of X_β implies

$$\begin{aligned}
\mathbf{E}_0\Big[Z_1^\beta\Big] &= \sum_{x\in\mathbb{Z}^d} xP(0,x) \\
&= \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\big(xP(0,x)+(-x)P(0,-x)\big) \\
&= \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\big(xP(0,x)+(-x)P(0,x)\big) \\
&= 0.
\end{aligned}$$

Finiteness of the mean-squared displacement follows since

$$\mathbf{E}_0\Big[|Z_1^\beta|^2\Big] = e^{-\beta}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\mathbf{E}_0\Big[|Z_n^\beta|^2\Big],$$

where Z_n is the simple random walk on $\mathbb{Z}^d$. And following the standard calculation, letting $Z_n = \sum_{i=1}^n Y_i$ with $Y_i \sim Z_1$ i.i.d.

$$\mathbf{E}_0\big[|Z_n|^2\big] \;=\; \sum_{1\le k,l\le n}\mathbf{E}\big[Y_kY_l\big] \;=\; \sum_{k=1}^{n}\mathbf{E}\big[Y_k^2\big] \;=\; n,$$

from which

$$\mathbf{E}_0\big[|Z_1|^2\big] \;=\; e^{-\beta}\sum_{n=0}^{\infty}\frac{\beta^{n+1}}{n!} \;=\; \beta.$$

Finally considering the covariance matrix, we generalise the argument from above

$$C_{ij} \;=\; \mathbf{E}_0\big[X_1(i)X_1(j)\big] \;=\; e^{-\beta}\sum_{n=0}^{\infty}\frac{\beta^n}{n!}\mathbf{E}_0\big[Z_n(i)Z_n(j)\big],$$

and

$$\begin{aligned}
\mathbf{E}_0\big[Z_n(i)Z_n(j)\big] &= \sum_{1\le k,l\le n}\mathbf{E}_0\big[Y_k(i)Y_l(j)\big] \\
&= n\mathbf{E}_0\big[Y_1(i)Y_1(j)\big] \\
&= \frac{n}{d}\delta_i(j),
\end{aligned}$$

where the final line follows since $Y_1(i)Y_1(j) = 0$ if $i \neq j$ since in a single step, Z moves in only one of the coordinate directions, and $\mathbf{E}_0\big[Y_1(i)Y_1(i)\big] = 1/d$, since the coordinate direction in which the walk does move is chosen uniformly amongst the d available options. The result now follows on substituting back into the exponential series. □

As a consequence of the fact that the walk Z^β is suitably regular, we can apply the following result from [Spi64], P1 pp.308–10, which we state in the context of the 1-particle reduced density matrix.

Theorem 3.28. *Let $\Lambda_N = [-N,N]^d \subset \mathbb{Z}^d$, with $d \geq 3$. The 1-particle reduced density matrices $\tilde{\sigma}_N = \tilde{\sigma}_{\Lambda_N}$ at chemical potential $h \nearrow 0$ are such that*

$$\lim_{N\to\infty} \tilde{\sigma}_N(0,x) = \frac{d}{2\pi\beta}|x|^{-(d-2)} + O\Big(|x|^{-d}\Big).$$

This result suggests that we cannot see the BEC phenomenon in the grand–canonical ensemble, at any temperature $1/\beta > 0$, and is a manifestation of the break down in the equivalence of ensembles at the point of phase transition. To resolve this, we would need to work with loop soups where the total loop length is fixed: but this would mean parting ways with the Poissonian nature of the model, which has been fundamental to our study of the ideal gas.

Chapter 4

Bosonic Loop Soups Under Cycle Distribution Hamiltonians

In the preceding chapters our attention has been focused on studying the ideal Bosonic loop soup, and on understanding the distribution of its occupation field. In Section 3.3.1, we described how Symanzik's theorem for the space-time loop model can be seen as a formulation of the Bosonic loop model interacting with a space-time random environment. In this chapter we look to study a different class of interactions, in which a Hamiltonian reweights configurations according to interactions between loops.

A Hamiltonian is a mapping H of a configuration of loops $S \sqsubset \Gamma$, to the real numbers, $H(S) \in \mathbb{R}$, we will be more specific about the form which H takes in the subsequent sections. Given a Hamiltonian we define the loop soup $\mathbb{Q}^H := \mathbb{Q}^H_{\beta,h,\Lambda}$ by its Radon–Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}^H}{\mathrm{d}\mathbb{P}^B}(\cdot) := \frac{1}{Z^H} e^{-\beta H(\cdot)}, \tag{4.1}$$

where $Z^H = Z^H_{\beta,h} = \mathbb{E}^B\big[e^{-\beta H}\big]$ is the partition function. We write $\mathbb{E}^H$ to denote expectation against the loop soup with law $\mathbb{Q}^H$. If the Hamiltonian is additive, that is $H(S) = \sum_{\gamma \in S} H(\gamma)$, then Campbell's formula, Lemma 3.1, provides an expression for the partition function as

$$Z^H = \exp\biggl(-\int_\Gamma 1 - e^{-\beta H(\gamma)} \mu^B(\mathrm{d}\gamma)\biggr).$$

This can in turn be extended to provide an expression for the Laplace transform of the occupation field

$$\begin{aligned}\mathbb{E}^{H}\left[e^{-\langle v,\mathcal{L}\rangle}\right] &= \frac{1}{Z^{H}}\int e^{-\langle v,\mathcal{L}\rangle+\beta H}\mathrm{d}\mathbb{P}^{B}\\ &= \frac{1}{Z^{H}}\mathbb{E}^{B}\left[e^{-\langle v,\mathcal{L}\rangle+\beta H}\right],\end{aligned}$$

and since both H and $\mathcal{L}$ are additive we can apply Campbell's formula again

$$= \frac{1}{Z^{H}}\exp\left(-\int_{\Gamma}1-e^{-(\langle v,L(\gamma)\rangle+\beta H(\gamma))}\mu^{B}(\mathrm{d}\gamma)\right),$$

and combining this with the expression for the partition function

$$= \exp\left(\int e^{-\beta H(\gamma)}\left(e^{-\langle v,L\rangle}-1\right)\mu^{B}(\mathrm{d}\gamma)\right). \tag{4.2}$$

Proposition 4.1. *The expected mean occupation under an additive Hamiltonian $H\colon\Gamma\to\mathbb{R}$ is given by*

$$\mathbb{E}^{H}\left[\overline{\mathcal{L}}\right] = \frac{1}{|\Lambda|}\int|\gamma|e^{-\beta H(\gamma)}\mu^{B}(\mathrm{d}\gamma). \tag{4.3}$$

Proof. The proof follows as an application of (4.2); we note that the desired expectation is given by

$$\mathbb{E}^{H}\left[\overline{\mathcal{L}}\right] = -\frac{1}{|\Lambda|}\frac{\mathrm{d}}{\mathrm{d}v}\mathbb{E}^{H}\left[e^{-v\sum_{x\in\Lambda}\mathcal{L}_x}\right]\Big|_{v=0}.$$

Considering the derivative

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}v}\mathbb{E}^{H}\left[e^{-v\sum_{x\in\Lambda}\mathcal{L}_x}\right] &= \frac{\mathrm{d}}{\mathrm{d}v}\exp\left(\int e^{-\beta H(\gamma)}\left(e^{-v|\gamma|}-1\right)\mu^{B}(\mathrm{d}\gamma)\right)\\ &= \left(\frac{\mathrm{d}}{\mathrm{d}v}\int e^{-\beta H(\gamma)}\left(e^{-v|\gamma|}-1\right)\mu^{B}(\mathrm{d}\gamma)\right)\mathbb{E}^{H}\left[e^{-v\sum_{x\in\Lambda}\mathcal{L}_x}\right]\\ &= \left(\int|\gamma|e^{-\beta H+v|\gamma|}\mu^{B}(\mathrm{d}\gamma)\right)\mathbb{E}^{H}\left[e^{-v\sum_{x\in\Lambda}\mathcal{L}_x}\right].\end{aligned}$$ □

Unfortunately additive Hamiltonians are limited in their scope: ultimately additivity implies that the model has only *self interactions*. These can of course be of interest: for instance the self-intersection Hamiltonian

$$H(\gamma) := \int_0^{|\gamma|}\int_0^{|\gamma|}\delta_{\gamma(s)}\big(\gamma(t)\big)\mathrm{d}s\mathrm{d}t,$$

however in such a case we cannot expect to be able to solve the integral (4.3). We consider instead Hamiltonians which are not additive, but which depend on the geometry of the loop soup to a lesser extent. In particular we will study two Hamiltonians which depend on the loop soup only through its cycle distribution.

We recall that this was given in Chapter 2 to be the sequence $\underline{n} = \underline{n}(S) := (n_j)_{j\geq 1}$ with

$$n_j := \#\{S \cap \Gamma_{\beta j}\}, \tag{4.4}$$

with $\Gamma_{\beta j}$ the set of loops of duration βj, $j \geq 1$. In the canonical ensemble of N particles, the cycle distribution is a random partition of the integer N; this model has been previously studied in [Ver96, Ada08, Dan11]. In the grand-canonical ensemble, the sequence $\underline{n}$ is a random bounded integer sequence. As recorded in the introduction, similar work has been carried out in [BCMP05] for the cycle distribution, and in [Lew86, vdBLP88] for the momenta distribution for the continuum mean field model. Our contribution is a rigorous and self contained large deviations analysis for two lattice mean field models, and focuses on identifying expressions for the density of the loop soup: this complements the analysis undertaken in Chapter 2 where the density of an ideal gas was studied.
In Corollary 2.6 we saw that for spectrally convergent graph sequences, the re-scaled cycle counts $|\Lambda|^{-1}n_j$ converge in distribution to a degenerate variable. Our aim now is to strengthen this result in two regards: namely we consider the distribution of the entire sequence, rather than the individual entries, and we also provide a rigorous account of the large deviations principle which was referenced at the conclusion of Chapter 2. In our present context, the heuristic understanding of an LDP is that we can find a function $I\colon \mathbb{R}^{\mathbb{N}} \to [0,\infty) \cup \{+\infty\}$ such that for a set $E \subset \mathbb{R}^{\mathbb{N}}$, the probability that the cycle distribution of the loop soup lies in E scales like

$$\mathbb{P}_\Lambda[E] \sim e^{-|\Lambda| I(E)}.$$

As we will discuss later in the chapter, the minimizer of the rate function can in turn be related to the density of the Bose gas.

4.1 The Cycle Distribution of an Ideal Gas

Before studying the mean field models we warm up by describing the LDP for the ideal gas. Not only will this be an easier setting within which to introduce the methods we employ, but the result will be relied on when generalising to the case with interactions.
Central to our large deviations analysis is the choice of measure space on which we define the law of $\underline{n}$, and this proves to be a rather delicate matter. Billingsley [Bil99], pp.9–10 proposes a metric, d, on the space $\mathbb{R}^{\mathbb{N}}$ of all real sequences, under which $(\mathbb{R}^{\mathbb{N}}, d)$ is Polish. Whilst this is often a sufficient condition for proving LDPs, the technical lemma (Baldi's Lemma to be described below) on which our large deviations analysis depends requires that the space is also a Banach space: which

$(\mathbb{R}^{\mathbb{N}}, d)$ is not. Consequently we turn to the classical sequence spaces, $\ell_p(\mathbb{R})$ which are Banach spaces for $1 \leq p \leq \infty$. Our choice is again restricted since Baldi's Lemma also requires that the sequence of laws satisfies the *exponential tightness* property. We defer the definition of this until later, but remark that this effectively restricts us to considering our sequences as elements of $\ell_1(\mathbb{R})$, the space of all absolutely convergent sequences. For the most part we hide the technical problems imposed by this choice of space in Appendix C.

Let $\underline{m} = (m_j)_{j\geq 1}$ be a summable sequence of positive weights: $m_j > 0$ and $\sum_{j\geq 1} m_j = M < \infty$. Denoting P_{m_j} for the law of the Poisson variable with intensity m_j, in Appendix C we construct the product measure

$$P_{\underline{m}} := \bigotimes_{j=1}^{\infty} P_{m_j}, \tag{4.5}$$

on the measure space $(\ell_1(\mathbb{R}), \mathcal{B})$, where $\mathcal{B}$ denotes the Borel σ-algebra of $\ell_1(\mathbb{R})$. Choosing the sequence $m_j = \mu^B_\Lambda\big(\Gamma_{\beta j}\big)$, the resulting measure agrees with the law of the cycle distribution of the Bosonic loop soup, $\big(\underline{n}, \mathbb{P}^B_\Lambda\big)$. Abusing notation, throughout this chapter we will let $\mathbb{P}^B_\Lambda$ denote the law of the scaled cycle distribution, $|\Lambda|^{-1}\underline{n}$. We work under the following assumptions.

A1 A graph sequence $(\Lambda_N)_{N\geq 1}$ will always denote a spectrally convergent sequence. The spectral measure of Λ_N is denoted m_N, with limit $m_N \xrightarrow{(\mathrm{d})} m_\infty$. The associated Bosonic loop measure is $\mu^B_N = \mu^B_{\Lambda_N}$, and the law of the associated Bosonic loop soup is denoted $\mathbb{P}^B_N = \mathbb{P}^B_{\Lambda_N}$.

A2 The inverse temperature is strictly positive, $\beta > 0$, and the chemical potential is strictly negative $h < 0$. When no subscripts are given, e.g. $m, \mu^B, \mathbb{P}^B$ then the results are understood to be in the context of some unspecified graph.

A5 Since we will not make use of the Markov loop measure μ in this section, we abuse notation and for $j \geq 1$ denote

$$\mu^N_j := \mu^B_N\big(\Gamma_{\beta j}\big), \qquad \mu^\infty_j := \lim_{N\to\infty} \frac{1}{|\Lambda_N|}\mu^N_j, \qquad \mu^\infty := \sum_{j\geq 1} \mu^\infty_j.$$

In light of the previous assumptions all these values exist and are finite. Moreover, when we wish to stress the dependence on the chemical potential $h < 0$, we write $\mu^N_j(h)$ etc.

Given a graph sequence $(\Lambda_N)_{N\geq 1}$, we say that the sequence of measures $\mathbb{P}^B_N = \mathbb{P}^B_{\Lambda_N}$ satisfies an LDP with rate function $I\colon \ell_1(\mathbb{R}) \to \mathbb{R}_+ \cup \{+\infty\}$ if $I \not\equiv +\infty$, is lower semi-continuous: that is given a sequence $\underline{t}^{(n)} \to \underline{t} \in \mathbb{R}^{\mathbb{N}}$ then $\liminf I(\underline{t}^{(n)}) \geq I(\underline{t})$, and satisfies

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{P}_N^B[C] \le -I(C), \qquad\qquad C \subset \ell_1(\mathbb{R}) \text{ closed,}$$
$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{P}_N^B[O] \ge -I(O), \qquad\qquad O \subset \ell_1(\mathbb{R}) \text{ open.}$$

If in addition the level sets $\mathcal{C}_a := \{\underline{t} : I(\underline{t}) = a\}$ are compact then I is said to be a *good* rate function. A detailed introduction to large deviations is given in [dH00]. For real valued random variables, a candidate rate function for an LDP is often given by the Legendre transform of the cumulant generating function. In our context, the cumulant generating function is given to be

$$F(\underline{t}) := \lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_N^B\Big[e^{|\Lambda_N|\langle \underline{t}, \underline{n}\rangle}\Big], \qquad \underline{t} \in \ell_\infty(\mathbb{R}), \tag{4.6}$$

where $\langle \underline{t}, \underline{n}\rangle = \sum_{j\ge 1} t_j n_j$, and $\ell_\infty(\mathbb{R})$ is chosen as it is the dual space to $\ell_1(\mathbb{R})$. After confirming certain technical requirements, detailed in Lemma 4.2, we will identify the rate function as the Legendre transform of F

$$I(\underline{x}) = \sup_{t\in\ell_\infty(\mathbb{R})} \{\langle \underline{t}, \underline{x}\rangle - F(\underline{t})\}.$$

Before proceeding we recall a few definitions. The sequence of measures $\mathbb{P}_N^B$ is said to be *exponentially tight* if for any $\alpha > 0$, we can find a compact set $K = K_\alpha$, such that

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{P}_N^B(K) < -\alpha.$$

A function $f : \ell_\infty(\mathbb{R}) \to \mathbb{R}$ is said to be *Gâteaux differentiable* if for all $\underline{t}$, $\underline{s} \in \ell_\infty(\mathbb{R})$ the map $\varepsilon \mapsto f(\underline{t} + \varepsilon \underline{s})$ is differentiable at $\varepsilon = 0$. In that case we define the Gâteaux derivative of f in the direction $\underline{s}$ to be

$$\mathrm{d}f(\underline{t}\,;\,\underline{s}) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(\underline{t} + \varepsilon\underline{s})\Big|_{\varepsilon=0}.$$

Finally, a function $f : \ell_\infty(\mathbb{R}) \to \mathbb{R}$ is said to be *lower semi-continuous* if for all sequences $\underline{t}^{(n)} \to t \in \ell_\infty(\mathbb{R})$ then

$$\liminf_{n\to\infty} f\big(\underline{t}^{(n)}\big) \ge f(\underline{t}).$$

Our LDP relies on confirming the criteria of the following result, which holds in the more general context of measures on Banach spaces, [DZ98] pp.160–1.

Lemma 4.2 (Baldi's Theorem)**.** *Suppose $\mathbb{P}_N^B$ is an exponentially tight sequence of measures on $\ell_1(\mathbb{R})$. Let $F(\underline{t})$ be as in (4.6), and suppose that it exists and is finite* *for bounded $\underline{t} \in \ell_\infty(\mathbb{R})$, the dual space of $\ell_1(\mathbb{R})$. If F is Gâteaux differentiable, and lower semi-continuous on $\ell_\infty(\mathbb{R})$, then $\mathbb{P}_N^B$ satisfies an LDP with rate function*

$$I(\underline{x}) = \sup_{t \in \ell_\infty(\mathbb{R})} \big\{ \langle \underline{t}, \underline{x} \rangle - F(\underline{t}) \big\}. \tag{4.7}$$

In the following results we confirm the requisite conditions of Lemma 4.2, starting by proving exponential tightness of the measures. Of course this requires us to find suitable sets K_α which satisfy the definition, which calls on a combination of both probabilistic and topological intuition. In the following proof we provide a suitable family of sets, but defer the proof that they are compact in $\ell_1(\mathbb{R})$ to Lemma C.3, since the proof somewhat distracts from the flow of the probabilistic argument.

Proposition 4.3. $\mathbb{P}_N^B$ *is an exponentially tight sequence of measures.*

Proof. Suppose we can find a sequence $\underline{x} = \underline{x}(\alpha) \in \ell_1(\mathbb{R})$ such that for all $j \geq 1$

$$\limsup_{N \to \infty} \frac{1}{|\Lambda_N|} \log \mathbb{P}_N^B \big[|\Lambda_N|^{-1} n_j > x_j \big] < -2^{-j} \alpha, \tag{4.8}$$

and define the set $K_{\underline{x}} := \{y \in \ell_1(\mathbb{R}) : \ |y_j| \leq |x_j| \, \forall j \geq 1\}$. Then, by independence of the n_j

$$\limsup_{N \to \infty} \frac{1}{|\Lambda_N|} \log \mathbb{P}_N^B (K_{\underline{x}}^c) \ = \ \limsup_{|\Lambda_N| \to \infty} \frac{1}{|\Lambda_N|} \sum_{j \geq 1} \log \mathbb{P}_N^B \big[|\Lambda_N|^{-1} n_j > x_j \big] \ < \ -\alpha.$$

Appealing to Lemma C.3 in the appendix, we assert that such a set is compact, and hence the above assures that $\mathbb{P}_N^B$ is an exponentially tight sequence. Subsequently, our focus is on constructing such a sequence $\underline{x}$. Fixing $j \geq 1$, we derive a suitable candidate for $x_j = x_j(\alpha)$ via a Chernoff bound. For all constants $c \geq 0$, and $t \geq 0$

$$\begin{aligned} \mathbb{P}_N^B \big[|\Lambda_N|^{-1} n_j > c \big] &= \mathbb{P}_N^B \Big[e^{t n_j / |\Lambda|} > e^{tc} \Big] \\ &\leq e^{-tc} \mathbb{E}_N^B \Big[e^{t n_j / |\Lambda|} \Big], \end{aligned}$$

where we have applied Markov's inequality. The remaining expectation is none other than the moment generating function of a Poisson variable with mean μ_j^N

$$= \exp\Big(- tc \Big) \exp\Big(\mu_j^N (e^{t/|\Lambda|} - 1) \Big).$$

The inequality above holds for all $t \geq 0$, and differentiating in t we obtain the minimum at $t^* = |\Lambda_N| \log(c|\Lambda_N| / \mu_j^N)$. Hence

$$\mathbb{P}_N^B \big[\, |\Lambda_N|^{-1} n_j > c \big] \leq \left(\frac{c|\Lambda_N|}{\mu_j^N} \right)^{-|\Lambda_N| c} \exp\Big(|\Lambda_N| c - \mu_j^N \Big).$$

And taking the limit in $N \to \infty$

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{P}_N^B[|\Lambda_N|^{-1} n_k \in K_j^c] \\ \leq c - c\log c - \limsup_{N\to\infty} \left\{ \frac{1}{|\Lambda_N|} \mu_j^N - c\log(\mu_j^N/|\Lambda_N|) \right\}.$$

Appealing to **A1**, we have $|\Lambda_N|^{-1}\mu_j^N \to \mu_j^\infty$. And so

$$= c - \mu_j^\infty - c\log(c/\mu_j^\infty).$$

We note that for $c > 0$, the map

$$c \mapsto c - \mu_j^\infty - c\log(c/\mu_j^\infty) - 2^{-j}\alpha,$$

is differentiable, decreasing and has a unique zero c_j^*. Consequently (4.8) holds for any $x_j > c_j^*$. It remains to show that x_j can be chosen so that $\sum_{j\geq 1} |x_j| < \infty$. If we choose $x_j = c_j^* + 2^{-j}$, then $\sum_{j\geq 1} x_j = 1 + \sum_{j\geq 1} c_j^*$, subsequently is suffices to check summability of c_j^*. Since the c_j^* solve

$$c_j^*\big(1 - \log(c_j^*/\mu_j^\infty)\big) = \mu_j^\infty - 2^{-j}\alpha,$$

then summing over $j \geq 1$, the right hand side is convergent and

$$\sum_{j\geq 1} c_j^*\big(1 - \log(c_j^*/\mu_j^\infty)\big) = \mu^\infty - \alpha < \infty.$$

Suppose, for a contradiction, that $\sum_{j\geq 1} c_j^* = \infty$, then for the left hand side of the above equation to converge, we require $1 - \log(c_j^*/\mu_j^\infty) \to 0$, and consequently we have $c_j^*/\mu_j^\infty \to e^1$, as $j \to \infty$. In particular there is a $J \geq 1$ such that for $j \geq J$, $c_j^*/\mu_j^\infty < 3$, and hence

$$\sum_{j\geq J} c_j^* \;\leq\; 3\sum_{j\geq J} \mu_j^\infty \;<\; 3\mu^\infty,$$

which is a contradiction. In particular $\sum_{j\geq 1} c_j^* < \infty$, as required. □

We now turn to the cumulant generating function.

Proposition 4.4. *The limit cumulant generating function* (4.6) *exists and is given by*

$$F(\underline{t}) = \sum_{j\geq 1} \mu_j^\infty e^{t_j} - \mu^\infty < \infty, \qquad \underline{t} \in \ell_\infty(\mathbb{R}).$$

Moreover, F is Gâteaux differentiable, lower semi-continuous, and strictly convex.

Proof. Using independence of the n_j we write

$$\frac{1}{|\Lambda_N|}\log\mathbb{E}_N^B\left[e^{|\Lambda_N|\langle \underline{t},\underline{n}\rangle}\right] = \frac{1}{|\Lambda_N|}\sum_{j\geq 1}\log\mathbb{E}_N^B\left[e^{|\Lambda_N|t_j n_j}\right]$$
$$= \left(\sum_{j\geq 1}\frac{1}{|\Lambda_N|}\mu_j^N e^{t_j}\right) - \left(\sum_{j\geq 1}\frac{1}{|\Lambda_N|}\mu_j^N\right).$$

Both sets of summands are positive, and have limits as $N \to \infty$. Subsequently Fatou's lemma ensures that we can interchange the summation and limit (note that in this instance Fatou's lemma gives equality since the limsup and liminf agree)

$$F(\underline{t}) = \lim_{|\Lambda|\to\infty}\frac{1}{|\Lambda|}\sum_{j\geq 1}\log\mathbb{E}_N^B\left[e^{|\Lambda|t_j n_j}\right]$$
$$= \left(\sum_{j\geq 1}\mu_j^\infty e^{t_j}\right) - \left(\sum_{j\geq 1}\mu_j^\infty\right)$$
$$= \sum_{j\geq 1}\mu_j^\infty(e^{t_j}-1).$$

Since $\underline{t} \in \ell_\infty(\mathbb{R})$ is bounded we can choose $T = \sup_{j\geq 1} t_j$, and then

$$\leq \mu^\infty(e^T - 1).$$

confirming that $F(\underline{t})$ is finite.
To confirm Gateaux differentiability, let $\underline{t}, \underline{s} \in \ell_\infty(\mathbb{R})$, and consider

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}F(\underline{t}+\varepsilon\underline{s}) = \sum_{j\geq 1}\mu_j^\infty s_j e^{t_j+\varepsilon s_j} < \infty.$$

In particular the derivative is defined at $\varepsilon = 0$, and hence F is Gateaux differentiable. Lower semi-continuity is an immediate consequence of Fatou's lemma since for any sequence $\underline{t}^{(n)} \to \underline{t}$, then

$$\liminf_{n\to\infty}\sum_{j\geq 1}\mu_j^\infty(e^{t_j^{(n)}}-1) \geq \sum_{j\geq 1}\mu_j^\infty(e^{t_j}-1) = F(\underline{t}).$$

Finally we see that F is strictly convex since for any distinct $\underline{t}, \underline{s} \in \ell_\infty(\mathbb{R})$, and $\lambda \in [0, 1]$

$$F\big(\lambda\underline{s} + (1-\lambda)\underline{t}\big) = \sum_{j\geq 1}\mu_j^\infty e^{\lambda s_j + (1-\lambda)t_j} - \mu^\infty.$$

Considering each term of the sum individually, since the exponential function is convex

$$
\begin{aligned}
&< \lambda \sum_{j\geq 1} \mu_j^\infty e^{s_j} + (1-\lambda) \sum_{j\geq 1} \mu_j^\infty e^{t_j} - \mu^\infty \\
&= \lambda F(\underline{s}) + (1-\lambda) F(\underline{t}). \qquad \square
\end{aligned}
$$

This pair of propositions confirm that the sequence $\mathbb{P}_N^B$ satisfies the conditions of Baldi's theorem. We complete the derivation of the LDP by solving the variational problem (4.7), for which we analyse the zeros of the Gateaux derivative. In particular we appeal to the following result of convex analysis.

Lemma 4.5. *Let $f\colon \ell_1(\mathbb{R}_+) \to \mathbb{R}$ be strictly convex and Gâteaux differentiable. The point $\underline{x} \in \operatorname{int} \ell_1(\mathbb{R}_+)$ is the unique minimum of f if and only if $\mathrm{d}f(\underline{x}; \underline{y}) = 0$ for all $\underline{y} \in \ell_1(\mathbb{R})$.*

That the minimum is unique (if it is achieved at all) is a consequence of strict convexity. A proof of the lemma is given in Appendix C see Proposition C.4 and Lemma C.5. In the above we can of course interchange convex functions for concave, with minima being changed for maxima in the statement. It is in this form that we use the lemma to derive the following expression for the large deviation rate function.

Theorem 4.6. *The sequence $\mathbb{P}_N^B$ satisfies an LDP with rate $|\Lambda_N|$ and good rate function $I\colon \ell_1(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ given by*

$$
I(\underline{x}) = \begin{cases} \sum_{j\geq 1} x_j \log\left(\frac{x_j}{\mu_j^\infty}\right) - \sum_{j\geq 1} x_j + \mu^\infty & , \underline{x} \in \ell_1(\mathbb{R}_+), \\ +\infty & , \textit{else.} \end{cases} \tag{4.9}
$$

Proof. As already remarked, Propositions 4.3 and 4.4 confirm the conditions of Baldi's Theorem, Lemma 4.2; it remains to derive (4.9) from the variational problem (4.7).

Let $G_{\underline{x}}(\underline{t}) := \langle \underline{x}, \underline{t} \rangle - F(\underline{t})$ denote the functional which we wish to maximize. We consider first the case $\underline{x} \in \ell_1(\mathbb{R}) \backslash \ell_1(\mathbb{R}_+)$; let j be such that $x_j < 0$, and consider $\underline{t}' = \underline{t}'(\underline{x}) = (0, \ldots, t, 0, \ldots) \in \ell_\infty(\mathbb{R})$, the sequence with all entries equal to 0 except in the j-th position. Then

$$
\begin{aligned}
\sup_{\underline{t} \in \ell_\infty(\mathbb{R})} G_{\underline{x}}(\underline{t}) &\geq \sup_{t' \in \mathbb{R}} G_{\underline{x}}(\underline{t}') \\
&= \sup_{t' \in \mathbb{R}} x_j t' - \mu_j^\infty e^{t'} + \mu_\infty
\end{aligned}
$$

Taking $t' \to -\infty$, since $x_j < 0$ the above diverges to $+\infty$.

For the remaining case, note that strict convexity of F implies that $G_{\underline{x}}$ is strictly concave. We find the supremum of $G_{\underline{x}}$ by computing the Gâteaux derivative $\mathrm{d}G_{\underline{x}}$

$$\mathrm{d}G_{\underline{x}}(\underline{t};\, \underline{s}) = \sum_{j\geq 1} s_j\big(x_j - \mu_j^{\infty} e^{t_j}\big),$$

and setting $t_j^* = \log(x_j/\mu_j^{\infty})$, then $\mathrm{d}G_{\underline{x}}(\underline{t}^*;\, \underline{s}) = 0$, for all $\underline{s} \in \ell_{\infty}(\mathbb{R})$. Hence according to Lemma 4.5, $G_{\underline{x}}$ is maximised at $\underline{t}^*$, and so for $x \in \ell_1(\mathbb{R}_+)$

$$\begin{aligned} I(\underline{x}) &= G_{\underline{x}}(t^*) \\ &= \sum_{j\geq 1} x_j\big(\log\big(x_j/\mu_j^{\infty}\big) - 1\big) + \mu^{\infty}. \end{aligned}$$ □

The following corollary proves our intuition that the minimiser of the rate function provides the 'mean' sequence of the cycle structure $|\Lambda|^{-1}n_j$, in the limit $|\Lambda| \to \infty$, where we recall that in Corollary 2.6 we already saw that $\mathbb{E}_{\Lambda}\big[|\Lambda|^{-1}n_j\big] \to \mu_j^{\infty}$. The proof provides a template which we will follow when finding the minimisers of the rate function of an LDP with interactions.

Corollary 4.7. *The rate function* $I(\underline{x})$ *given by* (4.9) *is strictly convex on* $\ell_1(\mathbb{R}_+)$*, positive, and has a unique zero (and hence minimum) at* $\underline{x}^* = (x_j^*)_{j\geq 1}$ *given by*

$$x_j^* = \mu_j^{\infty}.$$

Proof. We confirm that I is strictly convex on $\ell_1(\mathbb{R}_+)$ by checking that each term of the summation over $j \geq 1$ is convex. i.e. $x_j \mapsto x_j \log(x_j/\mu_j^{\infty}) - x_j$ is convex for $x_j \geq 0$, $j \geq 1$. Then since I is a linear combination of strictly convex functions, it follows that I too is strictly convex. That each of the individual terms is convex is immediate on checking that the second derivative is positive.
The Gâteaux derivative of I is given by

$$\begin{aligned} \mathrm{d}I(\underline{x};\underline{y}) &:= \frac{\mathrm{d}}{\mathrm{d}\varepsilon} I(\underline{x} + \varepsilon\underline{y})\Big|_{\varepsilon=0} \\ &= \Big(\textstyle\sum_{j\geq 1} y_j \log \frac{x_j+\varepsilon y_j}{\mu_j^{\infty}}\Big)\Big|_{\varepsilon=0} \\ &= \sum_{j\geq 1} y_j\big(\log x_j - \log \mu_j^{\infty}\big), \end{aligned}$$

setting $x_j^* = \mu_j^{\infty}$, then clearly $\mathrm{d}I(\underline{x}^*;\underline{y}) = 0$ for all $\underline{y} \in \ell_1(\mathbb{R})$. □

Recognising the values of x_j^* to be exactly the mean number of βj loops, calculated in Corollary 2.6, we obtain a new expression for the density of the ideal gas as

$$\rho(\beta, h) = \sum_{j\geq 1} j x_j^*. \tag{4.10}$$

In the following section we introduce two models for the cycle distribution under a

Hamiltonian, and apply (4.10) to derive an expression for the density.

4.2 Mean Field Hamiltonians

We move away from the study of the ideal gas, and introduce two Hamiltonians which are of interest: the particle mean field Hamiltonian denoted H_Λ^{PMF}, and the cycle mean field Hamiltonian, H_Λ^{CMF} , which are defined as

$$H_{\Lambda_N}^{\text{PMF}}(\underline{x}) := \frac{1}{2|\Lambda_N|}\left(\sum_{j\geq 1} jx_j\right)^2,$$
$$H_{\Lambda_N}^{\text{CMF}}(\underline{x}) := \frac{1}{2|\Lambda_N|}\left(\sum_{j\geq 1} x_j\right)^2.$$

The factor of 1/2 present in the above is chosen to simplify later expressions, and corresponds to fixing the interaction strength of the model to 1, [BCMP05]. The Hamiltonian H^{PMF} has been previously studied in [BCMP05] where the authors employ a large deviations analysis to study the occurrence of long loops in the Bose gas, supporting the arguments of [Süt93, Süt02] that Bose–Einstein condensation is equivalent to the occurrence of infinite cycles. In turn their work is developed from earlier studies [Lew86, vdBLP88] which work with the classical momentum-space description of the Bose gas. In this case they work with integer sequences $(\tilde{n}_j)_{j\geq 1}$, with $\tilde{n}_j$ corresponding to the number of particles in the j-th energy level, and prove the existence of BEC by studying the occupancy of the ground state. Our aim in the following is to make the large deviations analysis for the cycle structure more transparent, and to focus on deriving explicit formulae for the density of the Bose gas under the mean field models.
Let $\mathbb{Q}_N^{\text{PMF}}$, $\mathbb{Q}_N^{\text{CMF}}$ be the change of measures induced by H^{PMF} and H^{CMF} respectively, as in (4.1). In the case of the particle mean field model, configurations are down-weighted according to the total particle number, or density, of the Bose gas. On the other hand, the cycle mean field penalises configurations which have many cycles, but does not differentiate between the length of these cycles: this model is specific to the functional integral description which we have followed throughout this work. We consider first the cycle mean field model.

4.2.1 The Cycle Mean Field Model

We will see that the LDP for the sequence $\mathbb{Q}_N^{\text{CMF}}$ can be obtained from that of the ideal gas as an application of Varadhan's lemma, which we state below.

Lemma 4.8 (Varadhan)**.** *Suppose $\mathbb{P}_N^{B}$ satisfies an LDP on $\ell_1(\mathbb{R})$ with rate function*

I, and let $H\colon \ell_1(\mathbb{R}) \to \mathbb{R}$ be continuous and bounded below. Then

$$\lim_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_N^B\left[e^{-|\Lambda_N|\beta H}\right] = -\inf_{\underline{y}\in\ell_1(\mathbb{R})} \big\{\beta H(\underline{y}) + I(\underline{y})\big\}. \tag{4.11}$$

Moreover, the sequence $\left(\mathbb{Q}_\Lambda^{|\Lambda_N|H}\right)_{N\geq 1}$ satisfies an LDP on $\ell_1(\mathbb{R})$ with rate function

$$I^H(\underline{x}) = \beta H(\underline{x}) + I(\underline{x}) - \inf_{\underline{y}\in\ell_1(\mathbb{R})} \big\{\beta H(\underline{y}) + I(\underline{y})\big\}. \tag{4.12}$$

See [dH00], pp.32–4; the second statement follows from the first, and is known as the *tilted* LDP. Before we can apply Varadhan's lemma, we must massage the Hamiltonian into a form which resembles that in Lemma 4.8. To this end, we note that we can rewrite

$$H_{\Lambda_N}^{\text{CMF}}(\underline{n}) \;=\; \frac{|\Lambda_N|}{2}\left(\sum_{j\geq 1} |\Lambda_N|^{-1} n_j\right)^2 \;=:\; |\Lambda_N| H^{\text{CMF}}\left(|\Lambda_N|^{-1}\underline{n}\right). \tag{4.13}$$

That is we have replaced the (graph dependent) Hamiltonian $H_{\Lambda_N}^{\text{CMF}}$ with the scale-free Hamiltonian H^{CMF} given by

$$H^{\text{CMF}}(\underline{x}) := \frac{1}{2}\Big(\textstyle\sum_{j\geq 1} x_j\Big)^2.$$

Proposition 4.9. *The Hamiltonian $H^{\text{CMF}} : \ell_1(\mathbb{R}) \to \mathbb{R}$ is bounded below, and continuous.*

Proof. Clearly H^{CMF} is bounded below by 0. To see that it is sequentially continuous, and hence continuous, let $\underline{x}^{(n)} \to \underline{x}$ be a convergent sequence in $\ell_1(\mathbb{R})$, $\lim_{n\to\infty} \sum_{j\geq 1} |x_j^{(n)} - x_j| = 0$. The function $S(\underline{x}) := \sum_{j\geq 1} x_j$ is continuous since

$$\begin{aligned}\left|S\big(\underline{x}^{(n)}\big) - S\big(\underline{x}\big)\right| &= \lim_{n\to\infty}\left|\textstyle\sum_{j\geq 1} x_j^{(n)} - \sum_{j\geq 1} x_j\right| \\ &\leq \lim_{n\to\infty} \sum_{j\geq 1} |x_j^{(n)} - x_j| \\ &= 0.\end{aligned}$$

Hence H^{CMF} is continuous as it is a composition of continuous functions: $H^{\text{CMF}} = T\circ S$, where $T\colon \mathbb{R}\to\mathbb{R}$ is simply $T(x) = x^2$. □

In light of this, we are in a position to apply Varadhan's lemma to derive the LDP for the sequence $\mathbb{Q}_N^{\text{CMF}}$.

Theorem 4.10. *The sequence $\mathbb{Q}^{\text{CMF}}$ satisfies an LDP on $\ell_1(\mathbb{R})$ with rate $|\Lambda_N|$, and* *rate function*

$$I^{\text{CMF}}(\underline{x}) = \begin{cases} \beta\Big(\sum_{j\geq 1} x_j\Big)^2 + \Big(\sum_{j\geq 1} x_j \log \frac{x_j}{\mu_j^\infty} - x_j\Big) \\ \qquad\qquad\qquad + \frac{1}{\beta}W(\beta\mu^\infty) + \frac{1}{2\beta}W(\beta\mu^\infty)^2 & , \underline{x} \in \ell_1(\mathbb{R}_+) \\ +\infty & , \textit{else}, \end{cases}$$

where W denotes the Lambert-W function, defined by the relation $z = W(z)e^{W(z)}$ for $z \geq 0$. In particular I^{CMF} is strictly convex on $\ell_1(\mathbb{R}_+)$, positive, and has a unique zero at $\underline{x}^* = (\underline{x}_j^*)_{j\geq 1}$ given by

$$\underline{x}_j^* := \mu_j^\infty \frac{W(\beta\mu^\infty)}{\beta\mu^\infty}. \tag{4.14}$$

Proof. Appealing to (4.13) and Proposition 4.9, as well as the fact that we have already established an LDP for the measures $\mathbb{P}_N^B$ in Theorem 4.6, the sequence of measures $\mathbb{Q}_\Lambda^{\text{CMF}}$ satisfy the requirements of Varadhan's lemma.
It remains to solve the variational problem (4.12). Let $F(\underline{y}) := \beta H^{\text{CMF}}(\underline{y}) + I(\underline{y})$ be the function which we want to minimize, and I the rate function for $\mathbb{P}_N^B$, given in (4.9). Written explicitly

$$F(\underline{y}) = \begin{cases} \frac{\beta}{2}\Big(\sum_{j\geq 1} y_j\Big)^2 + \sum_{j\geq 1}\Big(y_j \log\Big(\frac{y_j}{\mu_j^\infty}\Big) - y_j\Big) + \mu^\infty & , \underline{y} \in \ell_1(\mathbb{R}_+) \\ +\infty & , \text{else}. \end{cases} \tag{4.15}$$

Note that for $\underline{x}, \underline{y} \in \ell_1(\mathbb{R})$ and $\lambda \in [0,1]$ then

$$H^{\text{CMF}}(\lambda\underline{x} + (1-\lambda)\underline{y}) = \frac{1}{2}\left(\lambda \sum_{j\geq 1} x_j + (1-\lambda)\sum_{j\geq 1} y_j\right)^2,$$

and since $x \mapsto x^2$ is strictly convex throughout $\mathbb{R}$

$$\leq \frac{\lambda}{2}\left(\sum_{j\geq 1} x_j\right)^2 + \frac{1-\lambda}{2}\left(\sum_{j\geq 1} y_j\right)^2,$$

so that H^{CMF} is convex on $\ell_1(\mathbb{R})$. Having previously shown that $-I$ is strictly convex on $\ell_1(\mathbb{R}_+)$, Corollary 4.7, then F (and subsequently I^{CMF}) is strictly convex on $\ell_1(\mathbb{R})$ as it is a sum of a convex and strictly convex function. We proceed to find the minimizer of F by calculating the Gâteaux derivative on $\ell_1(\mathbb{R}_+)$

$$\begin{aligned} \mathrm{d}F(\underline{x};\, \underline{y}) &:= \left\{\frac{\beta}{2}\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(\sum_{j\geq 1} x_j + \varepsilon y_j\Big)^2 + \sum_{j\geq 1} y_j \log \frac{x_j + \varepsilon y_j}{\mu_j^\infty}\right\}\Big|_{\varepsilon=0} \\ &= \beta\Big(\sum_{j\geq 1} y_j\Big)\Big(\sum_{j\geq 1} x_j\Big) + \sum_{j\geq 1} y_j \log \frac{x_j}{\mu_j^\infty}. \end{aligned}$$

To find a candidate $\underline{x}^*$ for which $\mathrm{d}F(\underline{x}^*;\, y) = 0$ for all $y \in \ell_1(\mathbb{R}_+)$ we require

$$\log \frac{x_j^*}{\mu_j^\infty} = -c,$$

for some constant $c \in \mathbb{R}$, i.e. $x_j^* = \mu_j^\infty e^{-c}$. Factoring out the summation over the y_j

$$\mathrm{d}F(\underline{x}^*;\, \underline{y}) = \Big(\textstyle\sum_{j\geq 1} y_j\Big)\Big(\beta \sum_{j\geq 1} x_j^* - c\Big).$$

The above is equivalent to 0 for all $\underline{y}$ if and only if $\beta \sum_{j\geq 1} x_j^* = c$, that is

$$\beta \sum_{j\geq 1} \mu_j^\infty e^{-c} = \beta\mu^\infty e^{-c} = c.$$

Recalling that for $z > 0$ the Lambert-W function at z is defined by the relation $z = W(z)e^{W(z)}$ [Wri59], then we have $c = W(\beta\mu^\infty)$. Consequently using the identity $e^{-W(z)} = z^{-1}W(z)$

$$\begin{aligned} x_j^* &= \mu_j^\infty e^{-W(\beta\mu^\infty)} \\ &= \mu_j^\infty \frac{W(\beta\mu^\infty)}{\beta\mu^\infty}, \end{aligned}$$

as claimed. To derive the expression for the rate function we evaluate the following expressions

$$\begin{aligned} \sum_{j\geq 1} x_j^* &= \frac{1}{\beta} W(\beta\mu^\infty). \\ \sum_{j\geq 1} x_j^* \log \frac{x_j^*}{\mu_j^\infty} &= \frac{W(\beta\mu^\infty)}{\beta\mu^\infty} \log \frac{W(\beta\mu^\infty)}{\beta\mu^\infty} \sum_{j\geq 1} \mu_j \\ &= \frac{W(\beta\mu^\infty)}{\beta} \log \frac{W(\beta\mu^\infty)}{\beta\mu^\infty} \\ &= -\frac{W(\beta\mu^\infty)^2}{\beta}. \end{aligned}$$

Substituting into (4.15)

$$F(\underline{x}^*) = \mu^\infty - \frac{1}{2\beta} W(\beta\mu^\infty)^2 - \frac{1}{\beta} W(\beta\mu^\infty),$$

from which the identity for I^{CMF} now follows immediately. □

Recalling the discussion below Corollary 4.7, we derive an expression for the density of Bosonic loop soup under the cycle mean field Hamiltonian. As with the discussion of the density of the ideal gas in Section 2.2, we interchangeably write $\rho^{\mathrm{CMF}} = \rho^{\mathrm{CMF}}(\beta, h) = \rho^{\mathrm{CMF}}(\beta) = \rho^{\mathrm{CMF}}(h)$, including and excluding the arguments as is relevant to the context. In the following we write $\mu_h^\infty = \mu^\infty(h)$ for ease of reading.

Corollary 4.11. *The density of the Bose gas under the Hamiltonian H^{CMF} is given by*

$$\rho^{\text{CMF}}(h) = e^{-W(\beta\mu^\infty(h))}\rho(h),$$

where ρ denotes the density of the ideal gas. Consequently,

1. *$\rho^{\text{CMF}}(h)$ is monotone increasing in $h < 0$.*
2. *$\rho_c^{\text{CMF}}(\beta) := \lim_{h\nearrow 0}\rho^{\text{CMF}}(\beta, h)$ exists and is finite whenever $\rho_c(\beta) < \infty$. In particular*

$$\rho_c^{\text{CMF}}(\beta) = w(\beta)\rho_c(\beta).$$

with $w(\beta) = \lim_{h\nearrow 0}\exp(-W(\beta\mu^\infty))$.

Proof. Employing (4.10), and the formula for $\underline{x}^*$, (4.14)

$$\begin{aligned}\rho^{\text{CMF}} &= \sum_{j\geq 1} jx_j^* \\ &= \frac{W(\beta\mu^\infty)}{\beta\mu^\infty}\sum_{j\geq 1} j\mu_j^\infty \\ &= e^{-W(\beta\mu^\infty)}\rho,\end{aligned}$$

with $\rho = \rho(\beta, h)$ the density of the ideal gas.
We will employ the following formula for the derivative of $W : \mathbb{R}_+ \to \mathbb{R}$

$$\frac{\mathrm{d}}{\mathrm{d}z}W(z) = \frac{W(z)}{z(1 + W(z))}$$

which holds for all $z \neq -e^{-1}$, and for differentiable $f, g : \mathbb{R} \to \mathbb{R}$, combining this with the product and chain rules so long as $f(z) \neq -e^{-1}$

$$\left(e^{-W(f)}g\right)' = e^{-W(f)}\left(g' - \frac{gf'W(f)}{f(1 + W(f))}\right), \tag{4.16}$$

where we drop the z-dependence for clarity. Now, fixing $\beta > 0$, and writing $\mu^\infty = \mu_h^\infty$, $\rho^{\text{CMF}} = \rho^{\text{CMF}}(h)$ then (4.16) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}h}\rho^{\text{CMF}}(h) &= e^{-W(\beta\mu_h^\infty)}\left(\rho_h' - \frac{\rho_h\beta(\mu_h^\infty)'W(\beta\mu_h^\infty)}{\beta\mu_h^\infty\left(1 + W\left(\beta\mu_h^\infty\right)\right)}\right) \\ &= e^{-W(\beta\mu_h^\infty)}\left(\rho_h' - \frac{\rho_h(\mu_h^\infty)'W(\beta\mu_h^\infty)}{\mu_h^\infty\left(1 + W\left(\beta\mu_h^\infty\right)\right)}\right).\end{aligned}$$

Since the prefactor of $\exp(-W(\beta\mu_h)) > 0$, we require that the bracketed term is also positive, which is equivalent to

$$\rho_h'\mu_h^\infty(1+W(\beta\mu_h^\infty)) > \rho_h(\mu_h^\infty)'W(\beta\mu_h^\infty).$$

Further, if we expand the brackets on the left-hand side, then since $\rho_h'\mu_h^\infty > 0$ (because ρ_h is increasing in h, and μ_h^∞ is positive) it suffices to show that $\rho_h'\mu_h^\infty W(\beta\mu_h^\infty) > \rho_h(\mu_h^\infty)'W(\beta\mu_h^\infty)$ or

$$\rho_h'\mu_h^\infty > \rho_h(\mu_h^\infty)'. \tag{4.17}$$

Note that

$$\begin{aligned}(\mu_h^\infty)' &= \frac{\mathrm{d}}{\mathrm{d}h}\left(\lim_{|\Lambda_N|\to\infty}\sum_{x\in\Lambda}\sum_{j\geq 1}\frac{e^{\beta hj}}{j}\mathbf{P}_x^{\beta j}\right)\\ &= \sum_{j\geq 1}\left(\frac{\mathrm{d}}{\mathrm{d}h}\frac{e^{\beta hj}}{j}\right)\lim_{|\Lambda_N|\to\infty}\sum_{x\in\Lambda}\mathbf{P}_x^{\beta j}\\ &= \beta\sum_{j\geq 1}e^{\beta hj}\lim_{|\Lambda_N|\to\infty}\sum_{j\geq 1}\mathbf{P}_x^{\beta j}\end{aligned}$$

Letting $m_j^\infty = \lim_{|\Lambda_N|\to\infty}\sum_{x\in\Lambda}\mathbf{P}_x^{\beta j}$, which was seen to converge in Chapter 2,

$$\begin{aligned}&= \beta\sum_{j\geq 1}e^{\beta hj}m_j^\infty \qquad (4.18)\\ &= \beta\rho(h).\end{aligned}$$

A similar computation for the density yields

$$\rho_h' = \beta\sum_{j\geq 1}je^{\beta hj}m_j^\infty. \tag{4.19}$$

Using the power series representations of $\mu_h^\infty, (\mu_h^\infty)'$ and ρ_h' we have

$$\begin{aligned}\rho_h'\mu_h^\infty &= \beta\left(\sum_{j\geq 1}je^{\beta hj}m_j^\infty\right)\left(\sum_{j\geq 1}\frac{e^{\beta hj}}{j}m_j^\infty\right)\\ &= \beta\sum_{j\geq 1}\left(e^{\beta hj}m_j^\infty\right)^2 + \beta\sum_{i<j}\left(\frac{i}{j}+\frac{j}{i}\right)e^{\beta h(i+j)}m_i^\infty m_j^\infty\end{aligned}$$

where the sum runs over all pairs of integers $1 \leq i < j \leq \infty$, and given that $i/j + j/i > 2$ for all such pairs

$$> \beta \sum_{j \geq 1} \left(e^{\beta h j} m_j^\infty \right)^2 + \beta \sum_{i<j} 2 e^{\beta h(i+j)} m_i^\infty m_j^\infty$$

$$= \beta \left(\sum_{j \geq 1} e^{\beta h j} m_j^\infty \right)^2$$

$$= \rho_h (\mu_h^\infty)',$$

which is inequality (4.17). □

4.2.2 The Particle Mean Field Model

We now turn to analyse the large deviations for the particle mean field model; as with the cycle mean field model we can reformulate the Hamiltonian $H_{\Lambda_N}^{\text{PMF}}$ so as to be scale independent

$$H_{\Lambda_N}^{\text{PMF}}(\underline{n}) = \frac{|\Lambda_N|}{2} \left(\sum_{j \geq 1} |\Lambda_N|^{-1} j n_j \right)^2 =: |\Lambda_N| H^{\text{PMF}}(|\Lambda_N|^{-1} \underline{n}), \tag{4.20}$$

with

$$H^{\text{PMF}}(\underline{x}) := \frac{1}{2} \left(\sum_{j \geq 1} j x_j \right)^2.$$

Whilst on first inspection it would appear as though the analysis follows from a similar application of Varadhan's lemma, this is not quite the case. To see this, we note that H^{PMF} is no longer continuous as a function from $\ell_1(\mathbb{R})$ to $\mathbb{R}$. For instance, considering the sequence $\underline{x}^{(n)}$ with $x_j^{(n)} = 1/(nj^2)$, then we have $\underline{x}^{(n)} \to \underline{0}$, the sequence of all zeros, since

$$\lim_{n \to \infty} \left| \sum_{j \geq 1} \underline{x}_j^{(n)} - \underline{0} \right| = \lim_{n \to \infty} \tfrac{\pi^2}{6n} = 0.$$

On the other hand

$$\lim_{n \to \infty} \left| H^{\text{PMF}}(\underline{x}^{(n)}) - H^{\text{PMF}}(\underline{0}) \right| = \lim_{n \to \infty} \left| \tfrac{1}{2} \left(\sum_{j \geq 1} \tfrac{1}{nj} \right)^2 - 0 \right| = \infty.$$

We can however establish lower semi-continuity.

Proposition 4.12. *The Hamiltonian* $H^{\text{PMF}} \colon \ell_1(\mathbb{R}) \to \mathbb{R} \cup \{+\infty\}$ *is bounded below and lower semi-continuous.*

Proof. It is clear that H^{PMF} is bounded from below by 0. To see lower semi-continuity, note that if $\underline{x}^{(n)} \to x$ in $\ell_1(\mathbb{R})$, then we have for each $j \geq 1$: $x_j^{(n)} \to x_j$, and consequently $jx_j^{(n)} \to jx_j$. Then applying Fatou's lemma

$$\liminf_{n\to\infty} \sum_{j\geq 1} jx_j^{(n)} \geq \sum_{j\geq 1} jx_j,$$

and since the map $T(x) = x^2$ is continuous

$$\liminf_{n\to\infty} H^{\text{PMF}}\big(\underline{x}^{(n)}\big) = \frac{1}{2}T\Big(\liminf_{n\to\infty} \textstyle\sum_{j\geq 1} jx_j^{(n)}\Big) \geq \frac{1}{2}T\Big(\textstyle\sum_{j\geq 1} jx_j\Big) = H^{\text{PMF}}(\underline{x}),$$

which is to say that H^{PMF} is lower semi-continuous. □

Lower semi-continuity is sufficient to prove 'half' of the equality in Varadhan's lemma, (4.11). We state the following proposition as it applies in our context; a general proof is given in [DZ98] Lemma 4.3.6 pp.138-9.

Proposition 4.13. *Let I denote the rate function for the sequence $\big(\mathbb{P}_N^B\big)_{N\geq 1}$. The Hamiltonian H^{PMF} satisfies the upper bound*

$$\limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_N^B\Big[e^{-|\Lambda_N|\beta H^{\text{PMF}}}\Big] \leq -\inf_{\underline{y}\in\ell_1(\mathbb{R})} \big\{\beta H^{\text{PMF}}(\underline{y}) + I(\underline{y})\big\}.$$

Appealing to the tilted LDP, the second part of Lemma 4.8, the LDP for $\mathbb{Q}_N^{\text{PMF}}$ will follow if we can establish the corresponding lower bound

$$\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_N^B\Big[e^{-|\Lambda_N|\beta H^{\text{PMF}}}\Big] \geq -\inf_{\underline{y}\in\ell_1(\mathbb{R})} \big\{\beta H^{\text{PMF}}(\underline{y}) + I(\underline{y})\big\}. \tag{4.21}$$

Since the derivation of this will be done in several steps, we outline the programme below. Fixing $J \geq 1$, we derive an LDP for the truncated cycle distribution $(n_j)_{j=1}^J$, first in the case of the ideal gas, Proposition 4.15, and then for the particle mean field model, Lemma 4.16. The latter will follow by a standard application of Varadhan's lemma. Following an argument of [ACK11], we then demonstrate that the left hand side of (4.21) is in fact bounded from below by the equivalent statement for the truncated sequences. In the truncated case we already know that Varadhan's lemma is satisfied, and optimising over $J \geq 1$ we obtain the desired expression on the right hand side of (4.21), Theorem 4.18.
For $J \geq 1$, define the projection map $\pi_J : \ell_1(\mathbb{R}) \to \mathbb{R}^J$ which takes the first J coordinates of $\underline{x} \in \ell_1(\mathbb{R})$

$$\pi_J(\underline{x}) := (x_1, \ldots, x_J) \in \mathbb{R}^J.$$

We note that π_J is continuous, since if $\underline{x}^{(n)} \to \underline{x}$ in $\ell_1(\mathbb{R})$, then we have pointwise convergence $x_j^{(n)} \to x_j$, for all $j \geq 1$, and consequently $\big(x_1^{(n)}, \ldots, x_J^{(n)}\big) \to (x_1, \ldots, x_J)$ in $\mathbb{R}^J$. Let $\mathbb{P}_{N,J}^B$ be the law of the properly scaled truncated cycle distributions $|\Lambda_N|^{-1}(n_1, \ldots, n_J)$, which is obtained from $\mathbb{P}_N^B$ via the pushforward

measure $\mathbb{P}^B_{N,J}[\,\cdot\,] := \mathbb{P}^B_N\big[\pi_J^{-1}(\,\cdot\,)\big]$.
To establish an LDP for $\mathbb{P}^B_{N,J}$ we will need the *contraction principle* for LDPs. In our context this reads.

Lemma 4.14 (Contraction Principle)**.** *Let $F\colon \ell_1(\mathbb{R}) \to \mathbb{R}^n$ be a continuous map, then the sequence of pushforward probability measures $\big(\mathbb{P}^B_N\big[F^{-1}(\,\cdot\,)\big]\big)_{N\geq 1}$ satisfies an LDP with rate $|\Lambda_N|$ and rate function*

$$I_F(y) := \inf_{\underline{x}\in\ell_1(\mathbb{R})\colon F(\underline{x})=y} I(\underline{x}),$$

where I is the rate function for the sequence $\big(\mathbb{P}^B_N\big)_{N\geq 1}$, given by (4.9)*.*

See [dH00] Theorem III.20 p.35.

Proposition 4.15. *For $J \geq 1$, the sequence $\mathbb{P}^B_{N,J}$ satisfies an LDP on $\mathbb{R}^J$ with rate $|\Lambda_N|$, and rate function*

$$I_J(\underline{x}) = \begin{cases} \sum_{j=1}^J \Big(x_j \log\Big(\frac{x_j}{\mu_j^\infty}\Big) - x_j + \mu_j^\infty\Big) & , x \in \mathbb{R}^J_+ \\ +\infty & , \textit{else.} \end{cases}$$

Proof. Since $\pi_J\colon \ell_1(\mathbb{R}) \to \mathbb{R}^J$ is continuous, we are in a position to apply the contraction principle, Lemma 4.14, which asserts that $\mathbb{P}^B_{N,J}$ satisfies an LDP, and identifies the rate function I_J as

$$I_J(\underline{x}) := \inf_{y\in\ell_1(\mathbb{R})\colon \pi_J(y)=x} I(\underline{y}),$$

with I the rate function of the sequence $\mathbb{P}_N$, given in (4.9).
If $\underline{x} \in \mathbb{R}^J\backslash\mathbb{R}^J_+$, then any element $\underline{y} \in \ell_1(\mathbb{R})$ with $\pi_J(\underline{y}) = \underline{y}$ is such that $\underline{y} \in \ell_1(\mathbb{R})\backslash\ell_1(\mathbb{R}_+)$, and hence $I(\underline{y}) = +\infty$. This establishes the identity for $\underline{x} \in \mathbb{R}^J\backslash\mathbb{R}^J_+$.
Now suppose that $\underline{x} \in \mathbb{R}^J_+$; a similar argument to the above allows us to only consider those $\underline{y} \in \ell_1(\mathbb{R})$ with positive entries, since any $\underline{y}$ with a negative entry has $I(\underline{y}) = +\infty$, and hence will not be a candidate for the infimum. So let $\underline{y} \in \ell_1(\mathbb{R}_+)$ with $\pi_J(\underline{y}) = \underline{x}$. We write

$$\begin{aligned} I(\underline{y}) &= \sum_{j=1}^J y_j\left(\log\frac{y_j}{\mu_j^\infty} - 1\right) + \sum_{i>J} y_i\left(\log\frac{y_i}{\mu_i^\infty} - 1+\right) + \mu^\infty \\ &= \sum_{j=1}^J x_j\left(\log\frac{x_j}{\mu_j^\infty} - 1\right) + \sum_{i>J} y_i\left(\log\frac{y_i}{\mu_i^\infty} - 1\right) + \mu^\infty, \end{aligned}$$

where the first and last terms are constant, so it suffices to minimise only the second sum, which we do term wise. In particular we look to minimise an expression of the form $y \mapsto y\big(\log\frac{y}{m} - 1 + m\big)$, which on differentiating is seen to have its unique minimum at $y^* = m$. Consequently the infimum is given by the sequence $\underline{y}^*$ with $y^*_j = x_j$, for $j \le J$ and $y^*_j = \mu^\infty_j$ for $j > J$. Hence

$$I_J(\underline{x}) \;=\; I(\underline{y}^*) \;=\; \sum_{j=1}^{J} x_j\left(\log\frac{x_j}{\mu^\infty_j} - 1\right) - \sum_{j>J}\mu^\infty_j + \mu^\infty,$$

which is exactly as desired. □

Now let $H^{\text{PMF}}_J \colon \mathbb{R}^J \to \mathbb{R}\cup\{+\infty\}$ denote the truncated Hamiltonian

$$H^{\text{PMF}}_J(\underline{x}) := \frac{1}{2}\Big(\textstyle\sum_{j=1}^{J} j x_j\Big)^2.$$

Since $\mathbb{R}^J$ is finite dimensional we no longer have any problems with continuity of H^{PMF}, which is now immediate as it is obtained from elementary combinations of continuous functions. Following the notation above, we let $\mathbb{Q}^{\text{PMF}}_{N,J}$ denote the law of the truncated cycle sequence with Radon–Nikodym derivative

$$\frac{\mathrm{d}\mathbb{Q}^{\text{PMF}}_{N,J}}{\mathrm{d}\mathbb{P}^{B}_{N,J}}(\,\cdot\,) := \frac{1}{Z^{\text{PMF}}_{N,J}} e^{-\beta H^{\text{PMF}}_J(\,\cdot\,)}.$$

Lemma 4.16. *For $J \ge 1$, the sequence $\mathbb{Q}^{\text{PMF}}_{N,J}$ satisfies an LDP on $\mathbb{R}^J$ with rate $|\Lambda_N|$, and rate function*

$$I^{\text{PMF}}_J(\underline{x}) = \begin{cases} \frac{\beta}{2}\Big(\sum_{j=1}^{J} j x_j\Big)^2 + \sum_{j=1}^{J} x_j\Big(\log\Big(\frac{x_j}{\mu^\infty_j}\Big) - 1\Big) + A_J & ,\underline{x}\in\mathbb{R}^J_+ \\ +\infty & ,\textit{else}, \end{cases}$$

where A_J is the constant

$$A_J = \frac{c_J^2}{2\beta} + \sum_{j=1}^{J}\mu^\infty_j\Big(h - \frac{c_J}{\beta}\Big), \tag{4.22}$$

and c_J is the unique solution to

$$\sum_{j=1}^{J} j\mu^\infty_j\Big(h - \frac{c_J}{\beta}\Big) = \frac{c_J}{\beta}.$$

Proof. As a consequence of the continuity of the truncated Hamiltonian H_J, it is immediate from Varadhan's lemma, Lemma 4.8 that $\mathbb{Q}^{\text{PMF}}_{N,J}$ satisfies an LDP. It remains to solve the variational problem for I_J, (4.12). As in the case for the ideal gas, when $\underline{y} \notin \mathbb{R}^J_+$, then the rate function is easily seen to be $I^{\text{PMF}}_J(\underline{x}) = +\infty$.
Let $F(\underline{y}) := \beta H^{\text{PMF}}_J(\underline{y}) + I_J(\underline{y})$; since $\mathbb{R}^J$ is now a finite dimensional space, the theory of Gâteaux derivatives is now replaced with the equivalent theory for directional

derivatives. We maintain the notation established elsewhere, and compute

$$\mathrm{d}F(\underline{x};\underline{y}) = \beta\Big(\sum_{j=1}^J jy_j\Big)\Big(\sum_{j=1}^J jx_j\Big) + \sum_{j=1}^J y_j \log\tfrac{x_j}{\mu_j^\infty}.$$

Considering $\underline{x}^* = (x_j^*)_{j\geq 1}$ of the form $x_j^* = \mu_j^\infty e^{-cj}$ we can write

$$\begin{aligned}\mathrm{d}F(\underline{x}^*;\,\underline{y}) &= \beta\Big(\sum_{j=1}^J jy_j\Big)\Big(\sum_{j=1}^J jx_j^*\Big) - c\Big(\sum_{j=1}^J jy_j\Big)\\ &= \Big(\sum_{j=1}^J jy_j\Big)\Big\{\beta\Big(\sum_{j=1}^J jx_j^*\Big) - c\Big\},\end{aligned}$$

which is equal to 0 for all $y \in \ell_1(\mathbb{R}_+)$ if and only if $\beta\sum_{j=1}^J jx_j^* = c_J > 0$, noting that $c_J > 0$ must be the case since $x_j^* \in \mathbb{R}_+^J$. Substituting the expression for x_j^* the required condition becomes

$$\sum_{j=1}^J j\mu_j^\infty e^{-jc_J} = \frac{c_J}{\beta}. \tag{4.23}$$

Recalling that the definition of μ_j^N, for finite $N \geq 1$, was given to be $\mu_j^N = \mu_j^N(h) = \frac{1}{j}e^{\beta h}\sum_{x\in\Lambda_N}\mathbf{P}_x[X_{\beta j} = x]$, we can absorb the term e^{-jc_J} into the measure with a change in the chemical potential:

$$\mu_j^N(h)e^{-jc_J} = \mu_j^N(h - c_J/\beta).$$

Note that since $c_J > 0$, we have that $h - c_J/\beta < 0$ so that the left hand side remains well defined. Carrying this through to the limit we have

$$\sum_{j=1}^J j\mu_j^\infty(h)e^{-jc_J} = \sum_{j=1}^J j\mu_j^\infty\Big(h - c_J/\beta\Big) =: \rho_J\Big(h - \frac{c_J}{\beta}\Big),$$

where the right hand side is the density of loops of length $\leq J$ in the ideal gas. Returning to (4.23), we are thus looking to confirm that there is a $c_J > 0$ which solves

$$\rho_J\left(h - \frac{c_J}{\beta}\right) = \frac{c_J}{\beta}.$$

It is immediate from the definition of ρ_J above that the map $c \mapsto \rho_J(h - c/\beta)$ is continuous, strictly decreasing, and has limit 0 as $c \to \infty$, from which the existence of a unique fixed point c_J is guaranteed. Finally we evaluate $F(\underline{x}^*)$, from (4.23)

$$\beta H_J(\underline{x}^*) = \frac{\beta}{2}\left(\sum_{j=1}^J j\mu_j^\infty e^{-jc_J}\right)^2 = \frac{\beta}{2}\left(\frac{c_J}{\beta}\right)^2 = \frac{c_J^2}{2\beta},$$

whilst

$$
\begin{aligned}
I_J(\underline{x}^*) &= \sum_{j=1}^{J} \mu_j^\infty e^{-jc_J} \log(e^{-jc_J}) - \mu_j e^{-jc_J} + \sum_{j=1}^{J} \mu_j^\infty \\
&= -c_J \sum_{j=1}^{J} j\mu_j e^{-jc_J} - \sum_{j=1}^{J} \mu_j^\infty e^{-jc_J} + \sum_{j=1}^{J} \mu_j^\infty \\
&= -\frac{c_J^2}{\beta} - \sum_{j=1}^{J} \big(\mu_j^\infty(h) - \mu_j^\infty(h - c_J/\beta)\big),
\end{aligned}
$$

so that

$$
\begin{aligned}
F(\underline{x}^*) &= \sum_{j=1}^{J} \big(\mu_j^\infty(h) - \mu_j^\infty(h - c_J/\beta)\big) - \frac{c_J^2}{2\beta} \\
&= -A_J + \sum_{j=1}^{J} \mu_j^\infty(h),
\end{aligned}
$$

from which we obtain the formula for I_J^{PMF}. □

Having established the LDP for the truncated cycle distributions, we are now in a position to derive a lower bound for the LDP on the full cycle distribution.

Proposition 4.17. *For $J \geq 1$, let A_J be as given in* (4.22)*. Then*

$$
\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_N^B\Big[e^{-|\Lambda_N|\beta H^{\mathrm{PMF}}}\Big] \geq \limsup_{J\to\infty} A_J.
$$

Proof. Let $\mathbb{R}^J \times \underline{0} \subset \ell_1(\mathbb{R})$ be the subset of sequences which are 0 after at most their first J entries, that is $\underline{x} \in \mathbb{R}^J \times \underline{0}$ if and only if $x_j = 0$, $j > J$. This set is immediately seen to be closed in $\ell_1(\mathbb{R})$, and hence is measurable. Consequently we can write

$$
\mathbb{E}_N^B\Big[e^{-|\Lambda_N|\beta H^{\mathrm{PMF}}}\Big] \geq \mathbb{E}_N^B\Big[e^{-|\Lambda_N|\beta H^{\mathrm{PMF}}} \mathbf{1}_{\{\mathbb{R}^J\times\underline{0}\}}\Big] = \mathbb{E}_N^B\Big[e^{-|\Lambda_N|\beta H_J^{\mathrm{PMF}}} \mathbf{1}_{\{\mathbb{R}^J\times\underline{0}\}}\Big],
$$

where we used the shorthand $\big\{R^J \times \underline{0}\big\} = \big\{\underline{n} \in R^J \times \underline{0}\big\}$ to denote the event that their are no loops in the soup S with length greater than βJ, and then noted that on this event we can replace H with H_J.
The Hamiltonian H_J only affects the first J entries in any $\underline{x} \in \ell_1(\mathbb{R})$, so that in this expectation the additional entries x_j, $j > J$, remain independent. Hence

$$
\begin{aligned}
&\mathbb{E}_N^B\Big[e^{-|\Lambda_N|\beta H_J^{\mathrm{PMF}}} \mathbf{1}_{\{\mathbb{R}^J\times\underline{0}\}}\Big] \\
&\quad= \left(\int_{\mathbb{R}^J} e^{-|\Lambda_N|\beta H_J^{\mathrm{PMF}}(\underline{y})} \bigotimes_{j=1}^{J} P_{\mu_j^N}(\mathrm{d}y_j)\right)\left(\int \prod_{j>J} \mathbf{1}_{\{y_j=0\}} \bigotimes_{j>J} P_{\mu_j^N}(\mathrm{d}y_j)\right),
\end{aligned}
$$

where $P_{\mu_j^N}$ are independent Poisson distributions with mean μ_J^N. Consequently the latter of the two terms simplifies to give

$$= \left(\int_{\mathbb{R}^J} e^{-|\Lambda_N|\beta H_J^{\text{PMF}}(\underline{y})} \bigotimes_{j=1}^{J} P_{\mu_j^N}(\mathrm{d}y_j) \right) \exp\left(-\sum_{j>J} \mu_j^N \right).$$

Moreover, since the integrand in the first term depends only on the first J entries, we can reintroduce the integration against the full measure

$$\begin{aligned}
&= \mathbb{E}_N^B\left[e^{-|\Lambda_N|\beta H_J^{\text{PMF}}(\underline{y})} \right] \exp\left(-\sum_{j>J} \mu_j^N \right) \\
&= \mathbb{E}_{N,J}^B\left[e^{-|\Lambda_N|\beta H_J^{\text{PMF}}(\underline{y})} \right] \exp\left(-\sum_{j>J} \mu_j^N \right).
\end{aligned}$$

Therefore we have

$$\mathbb{E}_N^B\left[e^{-|\Lambda_N|\beta H^{\text{PMF}}} \right] \geq \mathbb{E}_{N,J}^B\left[e^{-|\Lambda_N|\beta H_J^{\text{PMF}}(\underline{y})} \right] \exp\left(-\sum_{j>J} \mu_j^N \right),$$

and taking the appropriate limit infimum

$$\begin{aligned}
\liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_N^B\left[e^{-|\Lambda_N|\beta H^{\text{PMF}}} \right] \geq \liminf_{N\to\infty} \frac{1}{|\Lambda_N|} \log \mathbb{E}_{N,J}^B\left[e^{-|\Lambda_N|\beta H_J^{\text{PMF}}(\underline{y})} \right] \\
- \limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \sum_{j>J} \mu_j^N.
\end{aligned}$$

Since we have already established the LDP for the sequence of measures $\mathbb{Q}_{N,J}^{\text{PMF}}$ by Varadhan's lemma, the first term is given exactly by A_J

$$\begin{aligned}
&= A_J - \limsup_{N\to\infty} \frac{1}{|\Lambda_N|} \sum_{j>J} \mu_j^N \\
&= A_J - \sum_{j>J} \mu_j^\infty.
\end{aligned}$$

Since $\sum_{j>J} \mu_j^\infty = \mu^\infty - \sum_{j=1}^J \mu_j^\infty \to 0$, the result follows on taking limits in J on the right hand side. □

Finally we complete the proof of the lower bound for Varadhan's lemma by identifying the limit of the sequence A_J.

Theorem 4.18. *For $J \geq 1$, let A_J be as in* (4.22). *Then*

$$\lim_{J\to\infty} A_J = \frac{c^2}{2\beta} + \mu^\infty\Big(h - \frac{c}{\beta}\Big),$$

where c is the unique solution to

$$\rho\Big(h - \frac{c}{\beta}\Big) = \frac{c}{\beta}.$$

Proof. Recall that we defined

$$\rho_J(h) := \sum_{j=1}^{J} j\mu_j^{\infty}(h),$$

and that c_J was the unique solution to

$$\rho_J\Big(h - \frac{c_J}{\beta}\Big) = \frac{c_J}{\beta}.$$

We first confirm that c_J converges to some c. From the definition we see that $\rho_{J+1}(h) > \rho_J(h)$ for all $h < 0$ and $J \geq 1$, and it follows that the fixed points c_J form an increasing sequence since

$$\rho_{J+1}\Big(h - \frac{c_J}{\beta}\Big) > \rho_J\Big(h - \frac{c_J}{\beta}\Big) = \frac{c_J}{\beta}.$$

Moreover, since $\rho_J(h) \to \rho(h)$, the density of the ideal gas, it follows that $c_J < \rho(h)$, so that c_J is in fact a bounded increasing sequence, and hence has a limit $c = \lim_{J\to\infty} c_J$.To identify the limit we note that since $\rho_J(h) < \rho(h)$

$$\frac{c_J}{\beta} = \rho_J\Big(h - \frac{c_J}{\beta}\Big) < \rho\Big(h - \frac{c_J}{\beta}\Big),$$

and so taking the limit in $J \to \infty$

$$\frac{c}{\beta} = \lim_{J\to\infty} \frac{c_J}{\beta} \leq \lim_{J\to\infty} \rho\Big(h - \frac{c_J}{\beta}\Big) = \rho\Big(h - \frac{c}{\beta}\Big),$$

by continuity of ρ, Proposition 2.13. Similarly since $c_J < c$, and ρ_J is increasing

$$\rho_J\Big(h - \frac{c}{\beta}\Big) < \rho_J\Big(h - \frac{c_J}{\beta}\Big) = \frac{c_J}{\beta},$$

and in the limit

$$\rho\Big(h - \frac{c}{\beta}\Big) = \lim_{J\to\infty} \rho_J\Big(h - \frac{c}{\beta}\Big) \leq \lim_{J\to\infty} \frac{c_J}{\beta} = \frac{c}{\beta},$$

which gives the equality $\frac{c}{\beta} = \rho\Big(h - \frac{c}{\beta}\Big)$.
We apply a similar argument to confirm convergence of the second term of A_J. First

of all we have

$$\sum_{j=1}^{J} \mu_j^{\infty}\Big(h - \frac{c_J}{\beta}\Big) < \sum_{j=1}^{\infty} \mu_j^{\infty}\Big(h - \frac{c_J}{\beta}\Big),$$

so that

$$\lim_{J\to\infty} \sum_{j=1}^{J} \mu_j^{\infty}\Big(h - \frac{c_J}{\beta}\Big) \;\le\; \lim_{J\to\infty} \sum_{j=1}^{\infty} \mu_j^{\infty}\Big(h - \frac{c_J}{\beta}\Big) \;=\; \mu^{\infty}\Big(h - \frac{c}{\beta}\Big),$$

where we justify taking the limit inside the summation by the dominated convergence theorem (since $\mu^{\infty}(h) < \infty$). For the corresponding lower bound we note

$$\sum_{j=1}^{J} \mu_j^{\infty}\Big(h - \frac{c_J}{\beta}\Big) > \sum_{j=1}^{J} \mu_j^{\infty}\Big(h - \frac{c}{\beta}\Big),$$

and then

$$\lim_{J\to\infty} \sum_{j=1}^{J} \mu_j^{\infty}\Big(h - \frac{c_J}{\beta}\Big) \;\ge\; \lim_{J\to\infty} \sum_{j=1}^{J} \mu_j^{\infty}\Big(h - \frac{c}{\beta}\Big) \;=\; \mu^{\infty}\Big(h - \frac{c}{\beta}\Big). \qquad \square$$

The LDP for H^{PMF} now follows immediately.

Corollary 4.19. *The sequence $\mathbb{Q}_N^{\mathrm{PMF}}$ satisfies an LDP on $\ell_1(\mathbb{R})$ with rate $|\Lambda_N|$, and rate function*

$$I^{\mathrm{PMF}}(\underline{x}) = \begin{cases} \frac{\beta}{2}\Big(\sum_{j\ge1} j x_j\Big)^2 + \sum_{j\ge1} x_j\Big(\log\Big(\frac{x_j}{\mu_j^{\infty}}\Big) - 1\Big) + A & ,\underline{x} \in \ell_1(\mathbb{R}_+) \\ +\infty & ,\textit{else,} \end{cases}$$

where

$$A = \frac{c^2}{2\beta} + \mu^{\infty}\Big(h - \frac{c}{\beta}\Big),$$

and c is the unique solution to

$$\rho\Big(h - \frac{c}{\beta}\Big) = \frac{c}{\beta}.$$

In particular I^{PMF} is strictly convex on $\ell_1(\mathbb{R}_+)$, positive, and has a unique zero at $\underline{x}^* = (\underline{x}_j^*)_{j\ge1}$ given by

$$\underline{x}_j^* := \mu_j\Big(h - \frac{c}{\beta}\Big), \tag{4.24}$$

Proof. All of the statements are immediate from the preceding analysis. That an LDP is satisfied follows since we have confirmed the limit of Varadhan's Lemma, (4.11). As in the case of the cycle mean field, strict convexity of I^{PMF} is established since I is strictly convex, and βH is convex: so that their sum, and hence I^{PMF}, is strictly convex. Finally the uniqueness of the zero follows from the uniqueness of the minimum, which was established in the previous results. Alternatively this can be derived directly by computing the Gâteaux derivative of I^{PMF}. □

We conclude by analysing the density of the particle mean field model.

Corollary 4.20. *For $\beta > 0$, $h < 0$, the density of the Bose gas under the Hamiltonian H^{PMF} is given by $\rho^{\text{PMF}}(\beta, h) = p$, where p is the unique solution to*

$$p = \rho(\beta, h - p), \tag{4.25}$$

where ρ denotes the density of the ideal gas. Consequently,

(i) ρ^{PMF} is monotone increasing in $h < 0$, and monotone decreasing in $\beta > 0$.

(ii) For all $\beta > 0$, $h < 0$: $\rho^{\text{PMF}}(\beta, h) < \rho(\beta, h)$.

(iii) $\rho_0^{\text{PMF}}(\beta) := \lim_{h \nearrow 0} \rho^{\text{PMF}}(\beta, h)$ exists and is finite. In particular it solves

$$\rho_0^{\text{PMF}}(\beta) = \rho(\beta, -\rho_0^{\text{PMF}}(\beta)).$$

Proof. The formula for the density follows immediately from (4.10) with

$$\rho^{\text{PMF}}(\beta, h) = \sum_{j \geq 1} j x_j^*$$

which was defined in Theorem 4.19 to be

$$\begin{aligned} &= \frac{c}{\beta} \\ &= \rho\left(\beta, h - \frac{c}{\beta}\right). \end{aligned}$$

Fixing $\beta > 0$, let $c = c_h$ and consider ρ^{PMF} as a function of h, $h \mapsto \rho^{\text{PMF}}(h)$, we show that this function is increasing in $h < 0$. We have

$$\begin{aligned} \frac{d}{dh} \rho^{\text{PMF}}(h) &= \frac{1}{\beta} \frac{d}{dh} c_h \\ &= \frac{d}{dh} \rho\left(\beta, h - \frac{c_h}{\beta}\right) \end{aligned}$$

using the multivariate chain rule, along with the fact that the first argument of ρ is independent of h

$$= \left(1 - \frac{1}{\beta} \frac{d}{dh} c_h\right) \partial^{(0,1)} \rho\left(\beta, h - \frac{c_h}{\beta}\right)$$

where $\partial^{(1,0)}\rho$, $\partial^{(0,1)}\rho$ respectively denote the partial derivative of ρ in its first and second arguments. Equating the right hand side of the first and third lines, implicit differentiation yields

$$\frac{\mathrm{d}}{\mathrm{d}h}\rho^{\mathrm{PMF}}(h) \;=\; \frac{1}{\beta}\frac{\mathrm{d}}{\mathrm{d}h}c_h \;=\; \frac{\partial^{(0,1)}\rho(\beta, h - c_h/\beta)}{1 + \partial^{(0,1)}\rho(\beta, h - c_h/\beta)}.$$

Since $c_h > 0$, as explained in Lemma 4.16 and $h - c_h < 0$, it follows that the right hand side is positive, since the map $x \mapsto \partial^{(0,1)}\rho(\beta, x)$ is positive for $x < 0$, Proposition 2.13. It follows that ρ^{PMF} is increasing in h.
Now fixing $h < 0$, setting $c = c_\beta$ and considering $\beta \mapsto \rho^{\mathrm{PMF}}(\beta)$ similar manipulation gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\beta}\rho^{\mathrm{PMF}}(\beta) &= \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{c_\beta}{\beta}\\ &= \frac{\mathrm{d}}{\mathrm{d}\beta}\rho\left(\beta, h - \frac{c_\beta}{\beta}\right)\\ &= \partial^{(1,0)}\rho\left(\beta, h - \frac{c_\beta}{\beta}\right) - \frac{\mathrm{d}}{\mathrm{d}\beta}\left(\frac{c_\beta}{\beta}\right)\partial^{(0,1)}\rho\left(\beta, h - \frac{c_\beta}{\beta}\right).\end{aligned}$$

As in the previous calculation, after rearranging

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\rho^{\mathrm{PMF}}(\beta) \;=\; \frac{\mathrm{d}}{\mathrm{d}\beta}\frac{c_\beta}{\beta} \;=\; \frac{\partial^{(1,0)}\rho\left(\beta, h - \frac{c_\beta}{\beta}\right)}{1 + \partial^{(0,1)}\rho\left(\beta, h - \frac{c_\beta}{\beta}\right)}.$$

The denominator of the right hand expression is positive whilst the numerator is negative, again appealing to Proposition 2.13, from which it follows that ρ^{PMF} is decreasing in $\beta > 0$. The bound for part (ii) is immediate from the fact that ρ is increasing in h

$$\rho^{\mathrm{PMF}}(\beta, h) \;=\; \rho(\beta, h - \rho^{\mathrm{PMF}}) \;<\; \rho(\beta, h).$$

For part (iii), since ρ^{PMF} is increasing in h then it must converge as $h \nearrow 0$, possibly to $+\infty$, which is to say that $\rho_0^{\mathrm{PMF}}(\beta) := \lim_{h\nearrow 0}\rho^{\mathrm{PMF}}(\beta, h)$ exists as an extended real number. Then, by continuity of $\rho(\beta, h)$ in h,

$$\begin{aligned}\rho_0^{\mathrm{PMF}}(\beta) &= \lim_{h\nearrow 0}\rho^{\mathrm{PMF}}(\beta, h)\\ &= \lim_{h\nearrow 0}\rho(\beta, h - \rho^{\mathrm{PMF}}(\beta, h))\\ &= \rho\left(\beta, \lim_{h\nearrow 0} h - \rho^{\mathrm{PMF}}(\beta, h)\right)\\ &= \rho(\beta, -\rho_0^{\mathrm{PMF}}(\beta)).\end{aligned}$$

But suppose that $\lim_{h\nearrow 0}\rho^{\text{PMF}}(\beta,h)=\infty$, then the right hand side of the last line above would be

$$
\begin{aligned}
&= \rho\Big(\beta, -\lim_{h\nearrow 0}\rho^{\text{PMF}}(\beta,h)\Big)\\
&= 0,
\end{aligned}
$$

since $\lim_{h\to-\infty}\rho(\beta,h)=0$, which is a contradiction. □

Note that in the above we refrained from denoting ρ_c^{PMF} for $\lim_{h\nearrow 0}\rho^{\text{PMF}}(h) =: \rho_0^{\text{PMF}}$. As we see from Corollary 4.20 part (iii), the value $\rho_0^{\text{PMF}} < \infty$ regardless of the convergent sequence of graphs taken, and so unlike ρ_c, the critical density of the ideal gas, ρ_0^{PMF} does not exhibit a phase-transition. In particular, the corollary suggests that although the reference measure $\mathbb{P}_{\beta,h}$ is only well defined for $h<0$, the density ρ^{PMF} has an extension to positive h. As such the intrinsic equation (2.11) is no longer an indicator of the presence of the Bose–Einstein condensation phase transition. We relax the assumption in this intrinsic equation that $h<0$, and now ask.

$$
\textbf{Fix } \beta>0\textbf{. For } \rho>0 \textbf{ find } h^*=h^*(\rho)\in\mathbb{R} \textbf{ for which } \rho^{\text{PMF}}(\beta,h^*)=\rho\textbf{.} \quad (4.26)
$$

As with the ideal gas, we define the critical density for the particle mean field model to be the supremum of those densities which can be achieved:

$$
\rho_c^{\text{PMF}}(\beta) := \sup\{\varrho :\ \exists\, h\in\mathbb{R}\,\text{st.}\ \varrho=\rho^{\text{PMF}}(\beta,h)\}.
$$

The following corollary is the equivalent of Theorem 2.14 in the case of the particle mean field Bose gas.

Corollary 4.21. *The intrinsic equation* (4.26) *is such that*

(i) For all $h<\rho_c(\beta)$, there exists a solution $p=p(\beta,h)$.

(ii) For $h>\rho_c(\beta)$ there is no solution.

(iii) As $h\nearrow\rho_c(\beta)$, the solution satisfies $\lim_{h\to\rho_c(\beta)}p(\beta,h)=\rho_c(\beta)$.

In particular, there is a unique solution to (4.26) *for $\varrho<\rho_c(\beta)$, and*

$$
\rho_c^{\text{PMF}}(\beta)=\rho_c(\beta).
$$

Proof. In the preceding corollary we have already shown that a unique solution exists whenever $h\le 0$. As a function of p, $\rho(h-p)$ exists for $p>h$, and is strictly

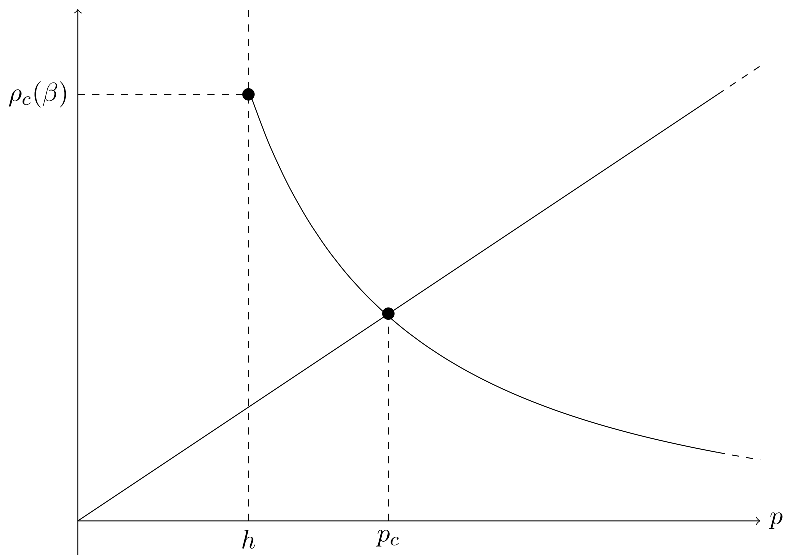

Figure 4.1: Illustrating the proof of Corollary 4.21. If $0 < h < \rho_c(\beta)$, then the curve $\rho(h-p)$ will always intersect the line $y = p$ at some point $p > h$.

decreasing with limit 0 as $p \to \infty$. Existence of a solution to (4.25) is equivalent to an intersection of the line $y = p$ with the curve $\rho(h-p)$, at some point $p > h$. Since the first of these is increasing and the second decreasing, this is in turn equivalent to $\lim_{p \searrow h} y(p) \leq \lim_{p \searrow h} \rho(h-p)$, but

$$\lim_{p \searrow h} y(p) = h \leq \lim_{p \searrow h} \rho(h-p) = \rho_c(\beta),$$

which confirms parts (i) and (ii). Finally (iii) follows since we have

$$h < p = \rho(h-p) < \lim_{\varepsilon \nearrow 0} \rho(\varepsilon) = \rho_c(\beta),$$

and in taking the limit $h \nearrow \rho_c(\beta)$ the above are all equalities. $\square$

Not only does the above result assert that there is a critical density for the particle mean field, but moreover that this is achieved at a critical chemical potential:

$$h_c^{\mathrm{PMF}}(\beta) = \sup\{h : \exists\, \text{soln. } p(h) \text{ to } (4.25)\}.$$

Furthermore we have $h_c^{\mathrm{PMF}} = \rho_c^{\mathrm{PMF}} = \rho_c$.

Chapter 5

Topics for Further Study

Before concluding we share some thoughts about two further topics which we have partially explored, and which remain open for further study. The first of these looks to establish that the occupation field for the Bosonic loop measure can in fact be extended to a law defined on the entirety of $\mathbb{Z}^d$. The ground work for this result has been established, but we have found it challenging to prove convergence of the Laplace transforms in the limit. The second section considers the *hole distribution* of the occupation field: this is the probability that a given site is not visited by any loops, $\mathbb{P}^B[\mathcal{L}_x = 0]$; this is an interesting problem since such sites cannot occur in a Gaussian field, and highlights the non-Gaussian nature of the measure $\mathbb{P}^B$. Moreover a question of percolation can be posed in terms of the *vacant set* of the occupation field.

In addition to the two topics studied here, together with Stefan Adams we are in the process of preparing a pre-print [AD15] concerning a mixture of the two mean-field models studied in Chapter 4. This model is inspired by the Huang-Yang-Luttinger model which is defined in the momentum space description of the bose gas; this model has a pay-off between wanting to achieve an optimal density, whilst penalising configurations with many loops.

5.1 The Bosonic Occupation Field of $\mathbb{Z}^d$

In Theorem 3.5 we established a formula for the Laplace transform of the occupation field $\mathcal{L}$ under the Bosonic loop measure $\mathbb{P}^B$. If we wish to establish a result for the existence of a limiting occupation field, it does not make sense to consider graph limits in the sense of spectral convergence: since as we saw, the limit graph may not exist. For this reason we restrict our attention to the case of the lattice box $\Lambda_N^{(per)}$, where each box is embedded in $\mathbb{Z}^d$, and so the limit graph is well defined. In this section we ask whether the corresponding random field $(\mathcal{L}_x)_{x\in\Lambda_N}$ also has a limit as $N \to \infty$, as a random field on $\mathbb{Z}^d$. Throughout this section we work with the following iteration of our standard assumption.

A2″ The inverse temperature is strictly positive $\beta > 0$, and either: the dimension of the lattice is $d = 1, 2$ and and $h < 0$, or $d \geq 3$ and $h \leq 0$.

Before considering the laws $\mathbb{P}_N^B$ we make some general remarks concerning convergence to a field defined on $\mathbb{Z}^d$. Henceforth we consider $d \geq 1$ to be fixed and denote

$$\Omega := \{\phi\colon \mathbb{Z}^d \to \mathbb{R}\},$$

for the collection of all functions defined on the graph; this is none other than $\Omega = \mathbb{R}^{\mathbb{Z}^d}$, however this notation becomes rather unsightly on repetition. Billingsley [Bil99] pp.9-10, establishes that there is a metric on Ω, for which Ω becomes a complete separable metric space, i.e. a *Polish space*. Moreover, the topology induced is that of pointwise convergence: $\phi_n \to \phi$ in Ω if and only if $\phi_n(x) \to \phi(x)$ for all $x \in \mathbb{Z}^d$. We denote $\mathcal{M}_1(\Omega) = \mathcal{M}_1(\Omega, \mathcal{B})$ for the space of all probability measures on Ω with respect to the Borel σ-algebra $\mathcal{B} = \mathcal{B}(\Omega)$. This space is itself a Polish space, and the associated metric is such that convergence is equivalent to convergence in distribution.

The outcome of the above is that convergence of distributions in $\mathcal{M}_1$ can be characterised by two properties: tightness, and convergence of finite dimensional distributions (f.d.d.s), which we define in the lemma to follow. Let $\Lambda \subset \mathbb{Z}^d$, and define the projection $\pi_\Lambda\colon \Omega \to \mathbb{R}^\Lambda$ by the map $(\phi_x)_{x\in\mathbb{Z}^d} \mapsto (\phi_x)_{x\in\Lambda}$. Given a measure $P \in \mathcal{M}_1(\Omega)$, we denote $P\pi_\Lambda^{-1}[\,\cdot\,] = P(\pi_\Lambda^{-1}[\,\cdot\,])$ for the pushforward measure on $\mathcal{M}_1(\mathbb{R}^\Lambda, \mathcal{B})$, and refer to it as the f.d.d. supported on Λ.

Lemma 5.1. *A sequence of probability measures $(P_n)_{n\geq 1} \in \mathcal{M}_1(\Omega)$ converge in distribution if and only if they satisfy*

(i) ***Tightness.*** *For each $x \in \mathbb{Z}^d$*

$$\lim_{C\to\infty} \lim_{n\to\infty} P_n\big[|\phi(x)| \geq C\big] = 0.$$

(ii) ***Convergence of f.d.d.s.*** *Fix $M \geq 1$ and let $\Lambda = [-M, M]^d$. There is a measure $Q_\Lambda \in \mathcal{M}_1(\mathbb{R}^\Lambda, \mathcal{B})$ such that*

$$P_n\pi_\Lambda^{-1} \xrightarrow{(\mathrm{d})} Q_\Lambda$$

as $n \to \infty$.

Moreover, the limit measure $P := \lim_{n\to\infty} P_n$ has the property $P\pi_\Lambda^{-1} = Q_\Lambda$.

We make some remarks concerning this lemma. The first requirement says that each of the 1-dimensional f.d.d.s is *tight*. This effectively means that probability mass does not get put on elements of Ω which are unbounded. That it is sufficient to confirm tightness for the 1-dimensional f.d.d.s is a consequence of the fact that Ω

is equipped with the topology of pointwise convergence, [Gia01]. Requirement (ii) on the other hand says that all of the f.d.d.s converge weakly to some limit. The existence of the limit measure, and that it satisfies $P\pi_\Lambda^{-1} = Q_\Lambda$ is then a consequence of the Kolmogorov extension theorem, [Bil95] pp.482–92.
To actually apply Lemma 5.1 we still require a way to prove convergence in distribution of the f.d.d.s. This is routine if one knows the distribution of the limiting variable, since then it suffices to prove convergence of the respective Laplace transforms to the limit transform. In our instance we do not, however, have control of the Laplace transforms of $P_n\pi_\Lambda^{-1}$; a priori it is not clear that the limit of a sequence of Laplace transforms necessarily defines a Laplace transform itself. In fact this statement is not true for any sequence of probability measures, but is in the presence of tightness.

Lemma 5.2. *Let $(P_n)_{n\geq 1}$ be a sequence of probability measures on $\mathbb{R}^k$, and denote $(L_n)_{n\geq 1}$ for the associated sequence of Laplace transforms, where $L_n\colon \mathbb{R}_+^k \to \mathbb{R}$. Suppose that the sequence P_n is tight, and that $\lim_{n\to\infty} L_n(v) := L(v)$ exists, then there exists a measure P on $\mathbb{R}^k$ whose Laplace transform is L, and moreover $P_n \xrightarrow{\text{(d)}} P$.*

Proof. Since the sequence P_n is tight, there is a subsequence $(P_{n_k})_{k\geq 1}$ which converges to some probability measure Q, see [Bil99] pp.57–63. Let L^Q denote the Laplace transform of Q, and note that we have $L_{n_k} \to L^Q$. But then by the assumption that $L_n \to L$, we have $L = L^Q$, and in particular L is a Laplace transform. Consequently, $P_n \to Q$. □

We now return to the setting of the Bosonic occupation field. Henceforth we write $P_N := P_{\Lambda_N} \in \mathcal{M}_1(\Omega)$ for the law on Ω which satisfies

$$P_N\pi_{\Lambda_N}^{-1}[\,\cdot\,] = \mathbb{P}_N^B[\mathcal{L} \in \cdot\,], \tag{5.1}$$

which defines the law of $\mathcal{L}$ under $\mathbb{P}_N^B$ on the larger space Ω; our aim then is to prove that $P_N \xrightarrow{\text{(d)}} P_\infty$, for some distribution P_∞, which we would then identify as being the Bosonic occupation field of $\mathbb{Z}^d$. We first demonstrate the the sequence P_N is tight, before proceeding to explain why we have been unable to derive convergence of the f.d.d.s.

Lemma 5.3. *For all $x \in \mathbb{Z}^d$,*

$$\lim_{C\to\infty}\lim_{N\to\infty} P_N[|\phi(x)| \geq C] = 0,$$

which is to say P_N is a tight sequence of probability measures on Ω.

Proof. We choose $N \geq 1$ sufficiently large that $x \in \Lambda_N$, and in this case we have

$(\phi(x), P_N) \overset{(d)}{=} (\mathcal{L}, \mathbb{P}_N^B)$. Applying Markov's inequality, we obtain the upper bound

$$\begin{aligned} P_N\big[|\phi(x)| \geq C\big] &= \mathbb{P}_N^B[\mathcal{L}_x \geq C] \\ &\leq \frac{\mathbb{E}_N^B[\mathcal{L}_x]}{C}. \end{aligned}$$

Since the graph $\Lambda_N^{(per)}$ is invariant under translations, it follows that $\mathcal{L}_x \overset{(d)}{=} \mathcal{L}_y$ for all $x, y \in \Lambda_N$. As a consequence we have

$$\mathbb{E}_N^B[\mathcal{L}_x] \;=\; \frac{1}{|\Lambda|}|\Lambda|\mathbb{E}_N^B[\mathcal{L}_x] \;=\; \frac{1}{|\Lambda|}\mathbb{E}_N^B\Big[\textstyle\sum_{y\in\Lambda}\mathcal{L}_y\Big] \;=\; \mathbb{E}_N^B\big[\bar{\mathcal{L}}\big].$$

Then returning to Markov's inequality

$$\begin{aligned} \lim_{N\to\infty} P_N\big[|\phi(x)| \geq C\big] &\leq \lim_{N\to\infty} \frac{\mathbb{E}_N^B\big[\bar{\mathcal{L}}\big]}{C} \\ &= \frac{a_\infty}{C}. \end{aligned}$$

where $a_\infty = a_\infty(\beta, h)$ is the atom in Theorem 2.4, and under assumption **A2′′** this is finite. The lemma follows on taking the limit $C \to \infty$. □

We move on to consider the f.d.d.s. We make some prior remarks concerning the notation used in the proof. Let $\Lambda = [-M, M] \subset \mathbb{Z}^d$ be a fixed lattice box, $M \geq 1$. Our aim is to prove that the laws $P_N\pi_\Lambda^{-1} \in \mathcal{M}_1(\Omega)$ converge, by showing convergence of their Laplace transforms L_N, defined by

$$L_N(v) := E_N\pi_\Lambda^{-1}\Big[e^{-\langle v, \pi_\Lambda(\phi)\rangle}\Big], \qquad v \in \mathbb{R}_+^\Lambda,$$

with $E_N\pi_\Lambda^{-1}$ denoting expectation with respect to $P_N\pi_\Lambda^{-1}$. The following proposition rephrases convergence of the L_N as convergence of the Laplace transforms for the field $\mathcal{L}$ under $\mathbb{P}_N^B$.

Proposition 5.4. *Suppose $N \geq M$, $\Lambda = [-M, M]^d$. Let $v \in \mathbb{R}^\Lambda$, and let $v_N \in \mathbb{R}^{\Lambda_N}$ be the vector such that $\pi_\Lambda v_N = v$, and $v_N \equiv 0$ outside of Λ. Then*

$$L_N(v) = \mathbb{E}_N^B\Big[e^{-\langle v_N, \mathcal{L}\rangle}\Big].$$

Proof. From (5.1) we have

$$\begin{aligned} L_N(v) &:= \mathbb{E}_N^B\pi_\Lambda^{-1}\Big[e^{-\langle v, \pi_\Lambda(\phi)\rangle}\Big] \\ &= \mathbb{E}_N^B\Big[e^{-\langle v, \pi_\Lambda(\mathcal{L})\rangle}\Big], \end{aligned}$$

and then the claim follows since $\langle v, \pi_\Lambda(\mathcal{L})\rangle = \sum_{x\in\Lambda} v_x\mathcal{L}_x = \langle v_N, \mathcal{L}\rangle$. □

The end result of the above is that we have massaged the Laplace transform defined in terms of measures on $\mathcal{M}_1(\Omega)$ into a form to which we can apply Theorem 3.5, the equation for the Laplace transform of $\mathcal{L}$ under $\mathbb{P}_N^B$. It remains then to confirm pointwise convergence of the Laplace transforms. Since we have not been able to complete a proof of this, we outline some thoughts on the matter. Recalling the notation of Proposition 5.4, let $V_N = \operatorname{diag}(v_N) \in \mathbb{R}^{|\Lambda_N|\times|\Lambda_N|}$, and define the following spectra

$$\operatorname{Spec}(Q_N) = \left(\eta_N^{(i)}\right)_{i=1}^{|\Lambda_N|}, \quad \operatorname{Spec}(Q_N - V_N) = \left(\xi_N^{(i)}\right)_{i=1}^{|\Lambda_N|}, \quad \operatorname{Spec}(V_N) = \left(\nu_N^{(i)}\right)_{i=1}^{|\Lambda_N|},$$

with Q_N the generator of the continuous time walk on Λ_N. According to Theorem 3.5

$$L_N(v) = \frac{\det\left(e^{-\beta h} I - e^{\beta(Q_N)}\right)}{\det\left(e^{-\beta h} I - e^{\beta(Q_N - V_N)}\right)} = \prod_{i=1}^{|\Lambda_N|} \frac{e^{-\beta h} - e^{\eta^{(i)}}}{e^{-\beta h} - e^{\xi^{(i)}}}. \tag{5.2}$$

To simplify expressions in the following, we write

$$\widehat{\eta}_N^{(i)} := \exp\Big(-\beta h\Big) - \exp\Big(\beta \eta_N^{(i)}\Big), \qquad \widehat{\xi}_N^{(i)} := \exp\Big(-\beta h\Big) - \exp\Big(\beta \xi_N^{(i)}\Big),$$

so that the product above becomes $\prod_{i=1}^{|\Lambda_N|} \widehat{\eta}_N^{(i)} / \widehat{\xi}_N^{(i)}$, and according to Lemma 5.2 convergence of the f.d.d.s is equivalent to the limit of this product existing.

We have at our disposal two eigenvalue inequalities which we make use of to analyse this product, these are the Weyl inequalities, and the Cauchy interlacing inequalities. Unfortunately, to the best of our knowledge, these are not strong enough to prove convergence of the product above; we can however use them to show boundedness.

Lemma 5.5. *Fix $M \geq 1$, $\Lambda = [-M, M]^d$ and $v \in \mathbb{R}_+^\Lambda$. Let $L_N \colon \mathbb{R}_+^\Lambda \to \mathbb{R}$ be as above. Then there exists a $U \colon \mathbb{R}_+^\Lambda \to \mathbb{R}$, such that*

$$0 \leq L_N(v) \leq U(v), \qquad \forall v \in \mathbb{R}_+^\Lambda.$$

Proof. Consider the spectra to be ordered, so that $\eta_N^{(1)} \geq \eta_N^{(2)} \geq \cdots \geq \eta_N^{(|\Lambda_N|)}$, and similarly for $\xi_N^{(i)}$. Since V_N is diagonal its eigenvalues are exactly the entries of v_N, so that $\nu_N^{(i)} = 0$ for $i > |\Lambda|$. Since the matrices Q_N and V_N are both symmetric, and hence Hermitian, we are in a position to apply Weyl's inequality, Theorem B.16 to $Q_N - V_N$. In particular we have

$$\eta_N^{(i+|\Lambda|)} \leq \xi_N^{(i)} \leq \eta_N^{(i)}, \qquad \text{for } i = 1, \ldots, |\Lambda_N| - |\Lambda|, \tag{5.3}$$
$$\eta_N^{(i)} - \nu_N^{(1)} \leq \xi_N^{(i)} \leq \eta_N^{(i)}, \qquad \text{for } i = (|\Lambda_N| - |\Lambda|) + 1, \ldots, |\Lambda_N|. \tag{5.4}$$

Since the function $x \mapsto e^{-\beta h} - e^{\beta x}$ is decreasing in x, the inequality above hold for

$\widehat{\eta}_N^{(i)}$, $\widehat{\xi}_N^{(i)}$ on reversing the inequality signs, and consequently from (5.3) we obtain

$$1 \leq \frac{\widehat{\eta}_N^{(i+|\Lambda|)}}{\widehat{\xi}_N^{(i)}} \leq \frac{\widehat{\eta}_N^{(i+|\Lambda|)}}{\widehat{\eta}_N^{(i)}}, \qquad \text{for } i = 1, \dots, |\Lambda_N| - |\Lambda|. \tag{5.5}$$

Returning to the determinant expression, we write the product as

$$L_N(v) \;=\; \prod_{i=1}^{|\Lambda_N|} \frac{\widehat{\eta}_N^{(i)}}{\widehat{\xi}_N^{(i)}} \;=\; \left(\prod_{i=1}^{|\Lambda|} \frac{\widehat{\eta}_N^{(i)}}{\widehat{\xi}_N^{(-i)}} \right) \left(\prod_{i=1}^{|\Lambda_N|-|\Lambda|} \frac{\widehat{\eta}_N^{(i+|\Lambda|)}}{\widehat{\xi}_N^{(i)}} \right),$$

where we use the shorthand notation $(-i) := (i + |\Lambda_N| - |\Lambda|)$, and the eigenvalues $\eta_N^{(-i)}$ etc. are the corresponding i-th smallest eigenvalues. Let $R_N(v)$ denote the first of the two products, and $S_N(v)$ the second. Our aim is to show

$$\lim_{N\to\infty} R_N(v) = R_\infty(v),$$
$$\lim_{N\to\infty} S_N(v) \leq S_\infty(v),$$

so that $L_\infty(v) \leq R_\infty(v) S_\infty(v) =: U(v)$.

Proof for $\mathbf{R_N}$. Since Λ is a fixed box, the product above is over finitely many terms, and we can take the limit in each term of the product

$$\lim_{N\to\infty} R_N(v) = \prod_{i=1}^{|\Lambda|} \frac{\widehat{\eta}_\infty^{(i)}}{\widehat{\xi}_\infty^{(-i)}}$$
$$= \prod_{i=1}^{|\Lambda|} \frac{\exp\Big(-\beta h\Big) - \exp\Big(\beta \eta_\infty^{(i)}\Big)}{\exp\Big(-\beta h\Big) - \exp\Big(\beta \xi_\infty^{(-i)}\Big)},$$

with $\eta_\infty^{(i)} := \lim_{N\to\infty} \eta_N^{(i)}$, and similarly for $\xi_\infty^{(i)}$, $\widehat{\eta}_\infty^{(i)}$, $\widehat{\xi}_\infty^{(i)}$. In the following we first confirm that $\widehat{\eta}_\infty^{(i)} = 1$, before showing that the limit $\widehat{\xi}_\infty^{(-i)}$ exists[1]. Recalling that the spectrum of Q_N is given in Appendix A to be

$$\mathrm{Spec}(Q_N) = \left\{ \eta_N(\underline{k}) = \frac{1}{d} \sum_{j=1}^{d} \cos\left(2\pi \frac{k_j}{2N+1}\right) - 1 : \; \underline{k} \in \{1, \dots, 2N+1\}^d \right\},$$

choosing $\underline{k}(i) = (i, 1, \dots, 1)$, then for large enough N, $i < 2N+1$, so that $\eta_N(\underline{k}(i)) \in \mathrm{Spec}(Q_N)$. Moreover $\eta_N(k(1)) \geq \eta_N(k(2)) \geq \cdots \geq \eta_N(k(i))$. It follows that $\eta_N(k(i))$ is at most the i-th largest eigenvalue of Q_N, and so $\eta_N^{(i)} \geq \eta_N(k(i))$, and

[1] A word of caution regarding the interpretation of these claims. The statement that $\widehat{\eta}_\infty^{(i)} = 1$ for $i \geq 1$ is understood to say that having fixed an i, in the limit the i-th largest eigenvalue is 1. It does not say that all of the eigenvalues are 1 in the limit.

hence

$$\begin{aligned}\lim_{N\to\infty}\eta_N^{(i)} &\geq \lim_{N\to\infty}\eta_N(k(i))\\ &= \lim_{N\to\infty}\frac{1}{d}\left(\cos\left(2\pi\frac{i}{2N+1}\right)+\sum_{j=1}^{d-1}\cos\left(2\pi\frac{1}{2N+1}\right)\right)-1\\ &= 0.\end{aligned}$$

Since $\mathrm{Spec}(Q_N) \subset [-2, 0]$, it follows that $\lim_{N\to\infty}\eta_\infty^{(i)} = 0$, and hence $\widehat{\eta}_\infty^{(i)} = 1$. Turning to $\xi_\infty^{(-i)}$, we note that the matrix $Q_N - V_N$ can be obtained from $Q_{N+1} - V_{N+1}$ by deleting the rows and columns which correspond to vertices $x \in \Lambda_{N+1}\backslash\Lambda_N$. As a result we are in a position to apply the Cauchy interlacing theorem, Theorem B.17, which gives

$$\xi_N^{(-i)} = \xi_N^{i+|\Lambda_N|-|\Lambda|} \geq \xi_{N+1}^{i+|\Lambda_{N+1}|-|\Lambda|} = \xi_{N+1}^{(-i)},$$

and in particular $\xi_N^{(-i)}$ is a decreasing sequence in $N \geq 1$. Recalling the inequality (5.4), and combining this with the fact that $\eta_N^{(-i)} \in [-2, 0]$, and $\nu_N^{(1)} = \max_{x\in\Lambda} v_x = v^*$ which is independent of N, then we have the lower bound

$$\xi_N^{(-i)} \geq -2 - v^*,$$

so that $\xi_N^{(-i)}$ is a bounded decreasing sequence, and hence converges to some $\xi_\infty^{(-i)}$. It follows that

$$\lim_{N\to\infty} R_N(v) = \prod_{i=1}^{|\Lambda|}\frac{1}{\xi_\infty^{(-i)}} =: R_\infty(v).$$

Proof for $\mathbf{S_N}$. From (5.4) we have

$$1 \;\leq\; S_N \;\leq\; \prod_{i=1}^{|\Lambda_N|-|\Lambda|}\frac{\widehat{\eta}_N^{(i+|\Lambda|)}}{\widehat{\eta}_N^{(i)}} =: U_N,$$

so it suffices to prove that the upper bound U_N converges. Taking logarithms

$$\log U_N = \left(\sum_{i=1}^{|\Lambda_N|-|\Lambda|}\log\widehat{\eta}_N^{(i+|\Lambda|)}\right) - \left(\sum_{i=1}^{|\Lambda_N|-|\Lambda|}\log\widehat{\eta}_N^{(i)}\right),$$

and since the terms $i = |\Lambda|+1, \ldots, |\Lambda_N| - |\Lambda|$ in both sums agree

$$= \sum_{i=1}^{|\Lambda|}\log\widehat{\eta}_N^{(-i)} - \log\widehat{\eta}_N^{(i)}.$$

Using that the logarithm is an increasing function, we can bound the sum by

$$\leq |\Lambda|\Big(\log \widehat{\eta}_N^{(-|\Lambda|)} - \log \widehat{\eta}_N^{(|\Lambda|)}\Big),$$

and in the limit $N \to \infty$

$$\lim_{N\to\infty} S_N \leq |\Lambda| \lim_{N\to\infty} \log \frac{\widehat{\eta}_N^{(-|\Lambda|)}}{\widehat{\eta}_N^{(|\Lambda|)}} = |\Lambda| \log\bigg(\frac{e^{-\beta h} - e^{-2\beta}}{e^{-\beta h} - 1}\bigg) \qquad \square$$

The factorisation $L_N = R_N S_N$ was performed to separate off the part of the spectrum of $Q_N - V_N$ which is not close to that of Q_N, i.e. the terms which contribute to R_N, and then to show that the remaining part of the spectra converge: i.e. $S_N \to 1$. That this part converges was anticipated since the Weyl inequality (5.3) implies that for any $i \geq 1$, $\xi_N^{(i)} \sim \eta_N^{(i)}$. Unfortunately it is not in fact the case that S_N converges to 1, as seen by approximating this limit using a computer. This lemma could still be of use if one is able to show that the sequence S_N is monotonic, from which convergence to some function S_∞ will then follow by boundedness; to date our attempts have proved fruitless.

An alternative characterisation of convergence of the Laplace transforms is obtained by taking logarithms of the product, and confirming that the corresponding series converge. Considering this

$$\log L_N(v) = \sum_{i=1}^{|\Lambda_N|} \log \widehat{\eta}_N^{(i)} - \log \widehat{\xi}_N^{(i)},$$

which we can rewrite as integrals against spectral measures m_N, and m_N^V, where the latter is the spectral measure of $Q_N - V_N$. Then

$$= |\Lambda_N|\bigg(\int \log\Big(e^{-\beta h} - e^{\beta u}\Big) m_N(\mathrm{d}u) - \int \log\Big(e^{-\beta h} - e^{\beta u}\Big) m_N^V(\mathrm{d}u)\bigg).$$

A similar proof as in the above lemma can be used to show that $m_N^V \xrightarrow{(\mathrm{d})} m_\infty$, since the 'bulk' of the eigenvalues converge via Weyl's inequality, and the remaining $|\Lambda|$ terms which are not suitably bounded are negligible in the N limit. However, due to the additional pre-factor of $|\Lambda_N|$ this is not a strong enough statement to confirm convergence of the above. For this to hold we need to understand the relative rates of convergence of m_N, m_N^V to m_∞.

Conjecture 5.6. *Fix $M \geq 1$, $\Lambda = [-M, M]^d$ and $v \in \mathbb{R}_+^\Lambda$. Let $L_N \colon \mathbb{R}_+^\Lambda \to \mathbb{R}$ be as defined in Proposition 5.4. Then $\lim_{N\to\infty} L_N(v) = L_\infty(v)$ exists.*
Consequently, there exists a law $P_\infty \in \mathcal{M}_1$ such that $P_N \xrightarrow{(\mathrm{d})} P_\infty$.

5.2 The Hole Distribution and Vacant Set Percolation

We move away from the topic of limits of graphs, and throughout this section consider Λ to be a fixed graph. We recall that below the proof of Theorem 3.5 we remarked about the difficulty of describing such simple correlations as the expected occupation at a site. Fixing some $x \in \Lambda$, we are interested in the law of $\mathcal{L}_x$, which is a positive random variable. In the case of the Markov loop measure we saw in Section 3.2.2 that

$$\mathbb{E}\big[e^{-v\mathcal{L}_x}\big] = \frac{1}{1 + G_{xx}v},$$

so that at each site the occupation field has a Γ-distribution. A partial explanation for the difficulty in deriving a similar result for the Bosonic occupation field is afforded by the fact that the corresponding distribution is no longer continuous. In particular the law $(\mathcal{L}_x, \mathbb{P}^B)$ has an atom at 0. This is a result of the fact that the loop measure μ^B is finite, rather than σ-finite, and hence

$$\mathbb{P}^B[\mathcal{L} \equiv 0] \;=\; \mathbb{P}^B[S = \varnothing] \;=\; e^{-\mu^B(\Gamma)} \;=\; \frac{1}{\Xi_\Lambda(\beta, h)} > 0. \tag{5.6}$$

It is no longer surprising that we could not find a Gaussian description of the Bosonic occupation field, since in particular the two distributions are not absolutely continuous: $\mathbb{P}^B \ll \mathbb{P}$.

We refer to those sites with 0 occupation as 'holes' for the occupation field, and can define the *vacant set* $\mathcal{V} \subset S$ to be the set of all holes

$$\mathcal{V} := \big\{x \in \Lambda : \; \mathcal{L}_x = 0\big\}.$$

In light of the previous section, and assuming that Conjecture 5.6 is true, then a natural question arises concerning whether or not the random subset $\mathcal{V} \subset \mathbb{Z}^d$ percolates or not. We do not expect this to be a simple problem, and as indicated in the introduction, active research regarding a similar problem for discrete Markov loop soups is still ongoing, see for example [LeJL13, CS14, Lup14].

Identifying the probability that a given site is a hole does not require us to derive a formula for the full distribution of the occupation at that site. Rather we can compute it using the formula

$$\mathbb{P}^B[\mathcal{L}_x = 0] \;=\; \mathbb{P}^B[S \cap \Gamma_x = \varnothing] \;=\; e^{-\mu^B(\Gamma_x)},$$

where $\Gamma_x := \{\gamma \in \Gamma : \; \gamma(t) = x \text{ for some } t \in [0, |\gamma|)\}$ is the set of loops which visit site x. Alternatively it can be computed from the Laplace transform, since

$$\begin{aligned}\lim_{v\to\infty}\mathbb{E}^B\left[e^{-v\mathcal{L}_x}\right] &= \mathbb{P}^B[\mathcal{L}_x=0]+\lim_{v\to\infty}\mathbb{E}^B\left[e^{-v\mathcal{L}_x}\,\middle|\,\mathcal{L}_x>0\right]\\ &= \mathbb{P}^B[\mathcal{L}_x=0].\end{aligned}$$

In general we do not have a way to compute this value from either of these expressions, though in the case that the graph is vertex transitive then we can derive a crude upper bound. Rather than defining vertex transitivity, we consider the example of $\Lambda_N^{(per)}$ for the lattice box with periodic boundaries. Clearly the set Γ_x contains as a subset $\Gamma_{\circ x}$, the set of loops which are rooted at x, $\Gamma_{\circ x}:=\{\gamma\in\Gamma_x:\ \gamma(0)=x\}$; note that this should not be confused with our previous notation $\Gamma_{\beta j}$ for loops of length βj, which will not be used here. The mass of this set under the Bosonic loop measure can be computed explicitly as

$$\begin{aligned}\mu^B(\Gamma_{\circ x}) &= \sum_{j\geq 1}\frac{e^{\beta hj}}{j}\mathbf{P}_x\left[\overline{X}_{\beta j}=x\right]\\ &= \frac{1}{|\Lambda_N^{(per)}|}\sum_{x\in\Lambda_N^{(per)}}\sum_{j\geq1}\frac{e^{\beta hj}}{j}\mathbf{P}_x\left[\overline{X}_{\beta j}=x\right]\\ &= \frac{1}{|\Lambda_N^{(per)}|}\mu^B(\Gamma),\end{aligned}$$

where we used vertex transitivity of the lattice box with periodic boundaries to introduce the sum over $x\in\Lambda_N^{(per)}$. Then in terms of the spectral measure this is

$$= -\int\log\left(1-e^{\beta(h+u)}\right)m_N(\mathrm{d}u).$$

Using the fact that $\Gamma_{\circ x}\subset\Gamma_x$, we obtain the bound

$$\mathbb{P}^B\left[\mathcal{L}_x=0\right]\leq\exp\left(\int\log\left(1-e^{\beta(h+u)}\right)m_N(\mathrm{d}u)\right).$$

Returning to general graphs Λ, a complimentary lower bound can be obtained as follows. Given a site $x\in\Lambda$, and a set $A\subset\Lambda$ with $x\in\Lambda$, let $\Gamma_A:=\cup_{y\in A}\Gamma_y$ denote the set of loops which visit A. By definition we have $\Gamma_x\subset\Gamma_A$ and so

$$\mathbb{P}^B\left[\mathcal{L}_x=0\right]\ =\ \mathbb{P}^B\left[S\cap\Gamma_x=\varnothing\right]\geq\mathbb{P}^B\left[S\cap\Gamma_A=\varnothing\right]\ =\ \exp\left(-\mu^B(\Gamma_A)\right).\qquad(5.7)$$

We now look to choose A in such a way as that we can compute the left most expression; to this end, we note that by the definition of a measure

$$\mu^B\left(\Gamma_A\right)=\mu^B(\Gamma)-\mu^B\left(\Gamma_{A^c}\right)$$

We already have an expression for $\mu^B(\Gamma)$ in terms of the spectral distribution, so the first term is easily computed, whilst the second set Γ_A^c is exactly the set of loops which are contained in $\Lambda\backslash A$. In fact, consider the graph Λ^* which has vertex set $\Lambda\backslash A$, weight function $w^*_{xy} = w_{xy}$, for $x, y \in \Lambda\backslash A$, where w_{xy} is the weight function of Λ, and killing

$$\kappa_x^* := \kappa_x + \sum_{y \in A} w_{xy}.$$

Comparing the walk X^* on Λ^* with X on Λ, we note that X^* can be coupled to X in such a way as that they agree up to the point at which X first visits a site in A, at which point X^* is killed. It follows that any loop $\gamma \in \Gamma_A^c$, i.e. a loop which does not visit A, is given the same mass under the measure μ_Λ^B and $\mu_{\Lambda^*}^B$, and moreover

$$\mu_\Lambda^B\big(\Gamma_A^c\big) = \mu_{\Lambda^*}^B\big(\Gamma_A^c\big) \; = \; \mu_{\Lambda^*}^B\big(\Gamma\big).$$

This expression can now be given exactly in terms of the spectral measure m^* associated with the graph Λ^*.

Whilst the above has outlined a programme for computing a lower bound, the challenge remains to actually identify the spectral measure μ^*. In the case of the lattice, whilst it was not overly taxing to identify the spectral measure m_N for a lattice box $\Lambda_N^{(per)}$, we do not have any general method to compute the spectrum of the graph on removal of some subset A, and hence identifying m_N^* will prove challenging. Suppose that we can identify the eigenvalues of Λ^*, then using our usual expansion of the loop measure in terms of the spectrum we obtain

$$\mathbb{P}^B[\mathcal{L}_x = 0] \; \geq \; e^{\mu_{\Lambda^*}^B(\Gamma) - \mu_\Lambda^B(\Gamma)} \; = \; \frac{\prod_{\eta^*}\big(e^{-\beta h} - e^{\beta \eta^*}\big)}{\prod_\eta \big(e^{-\beta h} - e^{\beta \eta}\big)}, \tag{5.8}$$

where the products run over the spectrum of Λ, respectively Λ^*. In this form it is clear that this is not far from the issues we faced in the previous section, where we could not handle the similar expression (5.2) for a small perturbation in the definition of the spectral measure.

Having stressed negative outcomes so far, we conclude by providing two examples where this technique does in fact give exact expressions for the probability that a site is a hole: we consider first the complete graph K_N and then the 1-dimensional lattice box with periodic boundaries. Both examples proceed from the fact that we can compute exactly the spectral distribution of Λ^* when $A = \{x\}$, in which case the inequality (5.7) is an equality. For the first of these cases, we recall that we identify the vertex set of the complete graph K_N with the set $[N] := \{1, \ldots, N\}$; the weights are defined in Appendix A.

Proposition 5.7. *Let $\mathbb{P}_N^B$ be the law of the Bosonic loop soup on K_N. The proba-*

bility that the site $1 \in K_N$ is a hole is given by

$$\mathbb{P}^B_N\big[\mathcal{L}_1 = 0\big] = \frac{\Big(1 - e^{\beta\left(h-\frac{1}{N}\right)}\Big)\Big(1 - e^{\beta\left(h-\frac{N+1}{N}\right)}\Big)}{\Big(1 - e^{\beta\left(h-\frac{2}{N}\right)}\Big)},$$

and in the limit $N \to \infty$

$$\lim_{N\to\infty} \mathbb{P}^B_N\big[\mathcal{L}_1 = 0\big] = 1 - e^{\beta(h-1)}.$$

Proof. For any $N \geq 2$, we note that the graph Λ^* on $K_N \backslash \{1\}$ is the complete graph K_{N-1} with the weights

$$w^*_{xy} = \frac{1}{N}, \qquad \kappa^*_{xy} = \frac{2}{N}.$$

Denoting $\mathrm{Spec}(Q_N) = \big\{\eta_i\big\}_{i=1}^{N}$, and $\mathrm{Spec}(Q^*_N) = \big\{\eta^*_i\big\}_{i=1}^{N-1}$, these are given to be

$$\begin{aligned} \eta_1 = \cdots = \eta_{N-1} &= -\frac{N+1}{N}, & \eta_N &= -\frac{1}{N}, \\ \eta^*_1 = \cdots = \eta^*_{N-2} &= -\frac{N+1}{N}, & \eta^*_{N-1} &= -\frac{2}{N}, \end{aligned}$$

see Appendix A. Then from (5.8)

$$\mathbb{P}^B\big[\mathcal{L}_1 = 0\big] = \frac{\Big(1 - e^{\beta\left(h-\frac{N+1}{N}\right)}\Big)^{N-1}\Big(1 - e^{\beta\left(h-\frac{1}{N}\right)}\Big)}{\Big(1 - e^{\beta\left(h-\frac{N+1}{N}\right)}\Big)^{N-2}\Big(1 - e^{\beta\left(h-\frac{2}{N}\right)}\Big)},$$

which simplifies to give the desired expression. The limit $N \to \infty$ follows easily. □

For the case of a box in $\mathbb{Z}$, we consider $\Lambda_N^{(per)}$ on the vertex set $[-N, N-1] \cap \mathbb{Z}$, where we use the box with $2N$ sites as it will make the notation in the proof easier to follow.

Theorem 5.8. *Let $\Lambda_N^{(per)} = [-N, N]$ as defined in Appendix A. Then*

$$\mathbb{P}^B\big[\mathcal{L}_0 = 0\big] = \frac{\prod_{j=1}^{2N}\Big(1 - \exp\Big(\beta\Big(\cos\Big(\frac{\pi j}{N}\Big) - 1 + h\Big)\Big)\Big)}{\prod_{j=1}^{2N-1}\Big(1 - \exp\Big(\beta\Big(\cos\Big(\frac{\pi j}{2N}\Big) - 1 + h\Big)\Big)\Big)}, \tag{5.9}$$

and in the limit $N \to \infty$

$$\lim_{N\to\infty} \mathbb{P}_{\beta,h,\Lambda_N}[\mathcal{L}_0 = 0] = \sqrt{(1 - e^{\beta h})(1 - e^{\beta(h-2)})}. \tag{5.10}$$

Proof. Let $\Lambda_N^* = \Lambda_N^{(per)} \backslash \{0\}$, and note that $w_{xy}^* = w_{xy}$ for all $x, y \neq 0, \pm 1$, and

$$\begin{aligned} w_{1,2}^* = w_{1,2} = \frac{1}{2}, \qquad & \kappa_1^* = w_{1,0} = \frac{1}{2}, \\ w_{-1,-2}^* = w_{-1,-2} = \frac{1}{2}, \qquad & \kappa_{-1}^* = w_{-1,0} = \frac{1}{2}. \end{aligned}$$

We recognise these as the weights associated with the lattice box on $2N-1$ vertices with Dirichlet boundary conditions. The equation (5.9) follows on recognising the spectra as

$$\mathrm{Spec}(Q_N) = \Big\{ \cos\big(\pi \tfrac{j}{N}\big) - 1 \Big\}_{j=1}^{2N}, \qquad \mathrm{Spec}(Q_N^*) = \Big\{ \cos\big(\pi \tfrac{j}{2N}\big) - 1 \Big\}_{j=1}^{2N-1},$$

the proof for Q_N is given in Appendix A, whilst the case of the path can be found in [LPW09] Section 12.3.2. We concentrate on obtaining the limit.
We rewrite

$$\mathrm{Spec}(Q_N) = \Big\{ \cos\big(\pi \tfrac{\lfloor j/2 \rfloor}{N}\big) - 1 \Big\}_{j=1}^{2N},$$

and define the function $f(x) = 1 - \exp\big(\beta\big(\cos(\pi x) - 1 + h\big)\big)$, we have

$$\begin{aligned} \mathbb{P}^B\big[\mathcal{L}_0 = 0\big] &= \frac{\prod_{i=1}^{2N} f\Big(\frac{\lfloor i/2 \rfloor}{N}\Big)}{\prod_{j=1}^{2N-1} f\Big(\frac{j}{2N}\Big)}, \\ &= f\bigg(\frac{\lfloor 1/2 \rfloor}{N}\bigg) \prod_{j=1}^{2N-1} \frac{f\Big(\frac{\lfloor (j+1)/2 \rfloor}{N}\Big)}{f\Big(\frac{j}{2N}\Big)}, \end{aligned}$$

and note that when j is even the terms cancel so that

$$= f(0) \prod_{j=1}^{N} \frac{f\Big(\frac{2j}{2N}\Big)}{f\Big(\frac{2j-1}{2N}\Big)}.$$

To compute the limit we will take logarithms and then show that the resulting summations converge. Before doing so we note that

$$\begin{aligned} \log f(x) &= -\sum_{k=1}^{\infty} \frac{1}{k} \Big(\exp\big(\beta\big(\cos(\pi x) - 1 + h\big)\big)\Big)^k \\ &= -\sum_{k=1}^{\infty} \frac{e^{k\beta(h-1)}}{k} \exp\big(k\beta\cos(\pi x)\big), \end{aligned}$$

so that

$$\log f(x) - \log f(y) = \sum_{k=1}^{\infty} \frac{e^{k\beta(h-1)}}{k} \Big(\exp\big(k\beta\cos(\pi y)\big) - \exp\big(k\beta\cos(\pi x)\big)\Big),$$

and introducing the functions $g_k(x) = \exp(k\beta\cos(\pi x))$, this is

$$= \sum_{k=1}^{\infty} \frac{e^{k\beta(h-1)}}{k} \big(g_k(y) - g_k(x)\big).$$

Then, returning to the computation of the hole probability, and using the notation above

$$\begin{aligned}
\log \mathbb{P}^B\big[\mathcal{L}_0 = 0\big] &= \log f(0) + \sum_{j=1}^{N} \log f\left(\frac{2j}{2N}\right) - \log f\left(\frac{2j-1}{2N}\right) \\
&= \log f(0) + \sum_{j=1}^{N}\sum_{k=1}^{\infty} \frac{e^{k\beta(h-1)}}{k}\left(g_k\Big(\frac{2j-1}{2N}\Big) - g_k\Big(\frac{2j}{2N}\Big)\right) \\
&= \log f(0) + \sum_{k=1}^{\infty} \frac{e^{k\beta(h-1)}}{k} \sum_{j=1}^{N}\left(g_k\Big(\frac{2j-1}{2N}\Big) - g_k\Big(\frac{2j}{2N}\Big)\right). \qquad (5.11)
\end{aligned}$$

We fix $k \geq 1$, and consider the summation in N. Since the function g_k is continuous and differentiable, the mean value theorem asserts that there is a value $c_N(j) \in [(2j-1)/2N, 2j/2N)$ such that

$$\frac{1}{2N} g_k'\Big(c_N(j)\Big) = g_k\left(\frac{2j}{2N}\right) - g_k\left(\frac{2j-1}{2N}\right),$$

and we have

$$\sum_{j=1}^{N}\left(g_k\Big(\frac{2j-1}{2N}\Big) - g_k\Big(\frac{2j}{2N}\Big)\right) = -\frac{1}{2}\sum_{j=1}^{N} \frac{1}{N} g_k'\Big(c_N(j)\Big).$$

Since $c_N(j) \in [(j-1)/N, j/N)$, $j = 1, \dots, N$, and these intervals form a partition of $[0,1)$, the above can be interpreted as a Riemann sum, and taking the limit in $N \to \infty$

$$\begin{aligned}
\lim_{N\to\infty} \sum_{j=1}^{N}\left(g_k\Big(\frac{2j-1}{2N}\Big) - g_k\Big(\frac{2j}{2N}\Big)\right) &= -\frac{1}{2}\int_0^1 g_k'(x)\mathrm{d}x \\
&= \frac{1}{2}\big(g_k(0) - g_k(1)\big) \\
&= \frac{1}{2}\Big(e^{k\beta} - e^{-k\beta}\Big).
\end{aligned}$$

Of course we recognise the above as being $\sinh(k\beta)$, however since we now wish to take the series in $k \geq 1$, we leave it in this form. Returning to (5.11) we have

$$\begin{aligned}\log \mathbb{P}^B\big[\mathcal{L}_0 = 0\big] &= \log f(0) + \sum_{k=1}^{\infty} \frac{e^{k\beta(h-1)}}{2k}\Big(e^{k\beta} - e^{-k\beta}\Big) \\ &= \log\Big(1 - e^{\beta h}\Big) + \frac{1}{2}\sum_{k=1}^{\infty}\frac{e^{k\beta h}}{k} - \frac{1}{2}\sum_{k=1}^{\infty}\frac{e^{k\beta(h-2)}}{k} \\ &= \frac{1}{2}\Big(\log\Big(1 - e^{\beta h}\Big) + \log\Big(1 - e^{\beta(h-2)}\Big)\Big),\end{aligned}$$

from which the result follows. □

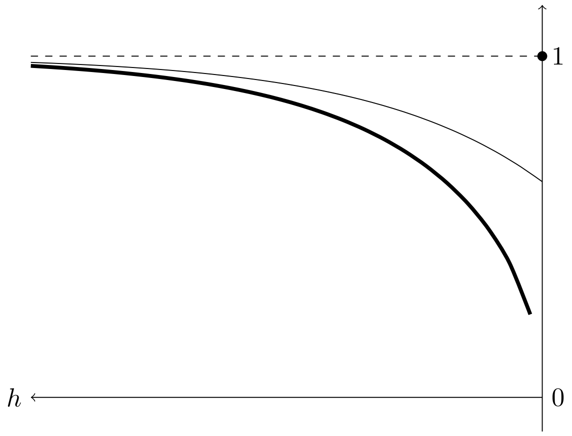

Figure 5.1: The probability that a site is a hole as a function of $h < 0$, at $\beta = 1$. The light curve corresponds to the hole probability in the limit for K_N, which is positive at $h = 0$. The bold curve corresponds to the limit of a lattice box in $\mathbb{Z}$, in this case the probability of a site being a hole converges to 0 as $h \nearrow 0$.

We anticipate that a similar method could be employed to derive a lower bound for the hole distribution for a lattice box $\Lambda_N^{(per)} = [-N, N]^d$ in $\mathbb{Z}^d$. Denoting $0 \in \Lambda_N$ for the origin, we obtain a bound on $\mathcal{L}_0$ by removing the set $A = \{x \in \Lambda_N : x_1 = 0\}$ from Λ_N, which has the effect of 'opening' the torus. The remaining graph $\Lambda \backslash A$ is now the lattice box with Dirichlet boundary conditions.
The exact form taken by the two limit formaulae in the above results is of interest in itself. Plots of the two functions are given in Figure 5.1 for varying $h < 0$. Once again we see that transience and recurrence play a role in determining the behaviour as $h \nearrow 0$. It is reasonable to expect that if the graph approaches a transient graph in the limit, then the hole probability should remain positive. Similarly, if the graph is recurrent in the limit, then the probability of any given site being a hole should converge to 0.

Appendix A

Examples of Graph Convergence

In this section we prove some statements regarding spectral convergence of graphs: most noteably we prove graph convergence of lattice boxes $\Lambda \subset \mathbb{Z}^d$. Throughout we use the notation introduced in Section 1.1, and assume that Λ is finite, loop-free, and irreducible.

The Complete Graph, K_n

For the complete graph on n vertices, denoted K_n, we identify the vertex set with $[N] \coloneqq \{1, \ldots, n\}$, and define the weight function and killing for $x, y \in [N]$, $x \neq y$ by

$$w_{xy} = \frac{1}{n}, \qquad \kappa_x = \frac{1}{n}.$$

The resulting walk $\overline{X}$ agrees with X and is the simple continuous time random walk on K_n with unit jump rate, and geometric killing. Since $\lambda \equiv 1$, and P is symmetric we immediately have that $\mathrm{Spec}(Q) \subset [-2, 0]$ from Theorem 1.6. Writing the generator in the form

$$Q = n^{-1}J - (1 + n^{-1})I,$$

where J is the $n \times n$ matrix with all entries equal to 1. We note that since J has $n-1$ repeated eigenvalues equal to 0, and a single eigenvalue equal to n, the eigenvalues of Q are

$$\begin{aligned} \eta_1 = \ldots = \eta_{n-1} &= -\frac{n+1}{n}, \\ \eta_n &= -\frac{1}{n}. \end{aligned}$$

For $n \geq 2$ the canonical distribution functions $\phi_n \colon (0,1] \to [-2,0]$ are

$$\phi_n(u) = \begin{cases} -\frac{1}{n} & \text{if } u \in (0, n^{-1}], \\ -\frac{n+1}{n} & \text{if } u \in (n^{-1}, 1]. \end{cases}$$

Clearly $\phi_n(u) \to -1$, for all $u \in (0,1]$, and K_n is a convergent graph sequence with limiting spectral measure given by the point mass $m_\infty = \delta_{-1}$.

The Cyclic Graph, C_n

We move towards more physically relevant examples, with the goal of describing cubic lattices in general dimensions. An important stepping stone will be the analysis of the symmetric walk on the cycle C_n, whose vertex set is given by the interval $\{1, \ldots, n\}$. We consider in greater generality the family of non-symmetric walks on the cycle, with drift $q \in [0,1]$, which is determined by the edge weights

$$w_{xy} = \begin{cases} q & \text{if } y = x+1, \text{ or } x = n,\, y = 1, \\ (1-q) & \text{if } y = x-1, \text{ or } x = 1,\, y = n. \\ 0 & \text{else.} \end{cases}$$

We set $\kappa \equiv 0$. The resulting random walk is the unit-rate random walk with drift on the n-cycle. In the extreme cases $q = 0,\, 1$ this corresponds to a Poisson jump process on the cycle, which is considered in Section 3.2. Since $\lambda \equiv 1$, the eigenvalues of $Q = \lambda(P - I)$ lie in the disk $\{z \colon\ |z+1| \leq 1\} \subset \mathbb{H}$. Moreover since Q is a circulant matrix, $Q = \operatorname{circ}(-1, q, 0, \ldots, 0, 1-q)$, its eigenvalues are completely determined, see Theorem B.18

$$\begin{aligned} \eta_j &= -1 + e^{-2\pi i \frac{j}{n}} + q\Big(e^{-2\pi i \frac{j}{n}} - e^{2\pi i \frac{j}{n}}\Big) \\ &= -1 + \cos\bigg(2\pi \frac{k}{n}\bigg) + i(2q-1)\sin\bigg(2\pi\frac{k}{n}\bigg). \end{aligned}$$

The canonical distribution functions are

$$\phi_n^q(u) = -1 + \cos\bigg(2\pi\frac{\lceil nu \rceil}{n}\bigg) + i(2q-1)\sin\bigg(2\pi\frac{\lceil nu \rceil}{n}\bigg), \qquad u \in (0,1]. \tag{A.1}$$

Pointwise convergence is once again immediate, with the limit

$$\phi_\infty^q(u) = -1 + \cos(2\pi u) + i(2q-1)\sin(2\pi u).$$

Note that when $q = 1/2$ the imaginary term drops out, as expected since then P is symmetric. Recalling that the functions ϕ_∞ are used only as a change of variables, they are not unique. In the case $q = 1/2$ we note that we could equally well use

$\tilde{\phi}_\infty(u) = \cos(\pi u) - 1$. To see this it is sufficient to check that $|\phi_\infty^{-1}(B)| = |\tilde{\phi}_\infty^{-1}(B)|$, for measurable B. Using symmetry of ϕ_∞ through the line $u = 1/2$, and the fact that ϕ_∞ is invertible on $(0, 1/2]$ then

$$\begin{aligned}
|\phi_\infty^{-1}(B)| &= 2\left|\frac{1}{2\pi}\cos^{-1}(1+B)\right| \\
&= \left|\frac{1}{\pi}\cos^{-1}(1+B)\right| \\
&= |\tilde{\phi}_\infty^{-1}(B)|,
\end{aligned}$$

where we use the standard notation $x + B := \{x + b \colon b \in B\}$ for the shift of a set $B \subset \mathbb{R}$ by a real $x \in \mathbb{R}$. Consequently in the following we will work with $\tilde{\phi}_\infty(u) = \cos(\pi u) - 1$, as it is notationally simpler.

The Integer Lattice, $\mathbb{Z}$

The previous calculations for C_n will now play an important role in simplifying the analysis for lattice boxes, for which explicit calculation of the eigenvalues is unavailable.

For $n \geq 1$ we identify the vertex set of a 1-dimensional lattice box as $\Lambda_n = [-n, n] \cap \mathbb{Z}$, and we assign edges to nearest neighbours in the box

$$w_{xy} = \begin{cases} \frac{1}{2} & \text{if } |x - y| = 1, \\ 0 & \text{else.} \end{cases}$$

We set the killing vector to be $\kappa_{-n} = \kappa_n = 1/2$, and zero in the interior: $\kappa_x = 0$, $-n < x < n$. The resulting random walk is unit rate simple random walk with killing on the boundary. We note that the generator matrix Q is in fact a cofactor of the larger generator Q' of the graph $C_{2(n+1)}$, with $q = 1/2$, obtained by deleting the final row and column, as seen below (the braces denoting the matrix Q).

$$\begin{pmatrix}
\left\{\begin{matrix}
-1 & \frac{1}{2} & 0 & \cdots & 0 & 0 \\
\frac{1}{2} & -1 & \frac{1}{2} & \cdots & 0 & 0 \\
0 & \frac{1}{2} & -1 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & \frac{1}{2} \\
0 & 0 & 0 & \cdots & \frac{1}{2} & -1
\end{matrix}\right\} &
\begin{matrix} \frac{1}{2} \\ 0 \\ 0 \\ \vdots \\ 0 \\ \frac{1}{2} \end{matrix} \\
\begin{matrix} \frac{1}{2} & 0 & 0 & \cdots & 0 & \frac{1}{2} \end{matrix} & -1
\end{pmatrix}$$

Denoting $\mathrm{Spec}(Q) = (\nu_j)_{j=1}^{2n+1}$ and $\mathrm{Spec}(Q') = (\eta_j)_{j=1}^{2(n+1)}$, where we assume the eigenvalues to be in decreasing order, then the eigenvalue interlacing theorem, Theorem B.17, says

$$\eta_1 \geq \nu_1 \geq \eta_2 \geq \cdots \geq \eta_{2n+1} \geq \nu_{2n+1} \geq \eta_{2(n+1)}.$$

Moreover, as in the previous section, the ν_j are known explicitly, and in decreasing order are

$$\eta_j := \cos\left(\pi \frac{\lfloor \frac{j}{2} \rfloor}{n+1}\right) - 1,$$

Let ϕ_n denote the canonical distribution for Q, then using the interlacing inequality

$$\eta_{\lceil (2n+1)u \rceil} \geq \phi_n(u) = \nu_{\lceil (2n+1)u \rceil} \geq \eta_{\lceil (2n+1)u \rceil + 1},$$

and using our knowledge of the eigenvalues η_j

$$\cos\left(\pi \frac{\lfloor \frac{\lceil 2(n+1)u \rceil}{2} \rfloor}{n+1}\right) - 1 \geq \phi_n(u) \geq \cos\left(\pi \frac{\lfloor \frac{\lceil 2(n+1)u \rceil + 1}{2} \rfloor}{n+1}\right) - 1.$$

From which we see pointwise convergence of $\phi_n(u) \to \cos(\pi u) - 1$, since both the left and right sides of the above converge.

It is unsurprising that the lattice box should converge to the same spectral distribution as the cycle, since one can interpret the cycle as being none other than a lattice box with periodic boundaries. Since our results for thermodynamic limits will be phrased in terms of the limiting spectral distribution, the above amounts to the fact that thermodynamic properties are independent of whether we take Dirichlet or periodic boundary conditions. To be complete, we show that the same is true when taking hard boundary conditions.

The lattice box $\Lambda_n = [-n, n]$ with reflecting boundary is as above, except that we set $\kappa \equiv 0$, so that there is no longer killing on the boundary; we denote Q'' for the generator. On deleting the first and last rows and columns of Q'' we arrive at the generator Q of the walk with absorbing boundaries on the smaller box $\Lambda_{n-1} = [-(n-1), (n-1)]$. As such we can once again use the eigenvalue interlacing theorem. Writing $\text{Spec}(Q) = (\nu_j)_{j=1}^{2n-1}$, $\text{Spec}(Q'') = (\eta_j)_{j=1}^{2n+1}$ for the spectra arranged in decreasing order. The interlacing relation now reads

$$\eta_j \geq \nu_j \geq \eta_{j+2}, \qquad j = 1, \ldots, 2n-1.$$

with the consequence that we can bound the eigenvalues η_j of Q'' by

$$\nu_{j-2} \geq \eta_j \geq \nu_j, \qquad j = 3, \ldots, 2n-1.$$

This bound, in conjunction with the bound in terms of the eigenvalues of Q' (for the walk on the cycle) ensures pointwise convergence of the canonical distribution functions on the open interval $(0, 1)$.

Hypercubic Lattices, $\mathbb{Z}^d$

We now obtain the more general result for graph convergence of lattice boxes in $d \geq 1$ dimensions. In line with our observation above, it suffices to consider only the case of lattice boxes with periodic boundary conditions. Let $\Lambda_n = [-n, n]^d \cap \mathbb{Z}^d$, with weights

$$w_{xy} = \begin{cases} \frac{1}{2d} & \text{if } \exists\, i \text{ st. } x_i - y_i \pmod{n} \equiv 1 \text{ or } -1, \text{ and } x_j = y_j \text{ for } i \neq j, \\ 0 & \text{else.} \end{cases}$$

The killing vector is degenerate, $\kappa \equiv 0$. Let $P = (2d)^{-1} A_d$, so that A is the adjacency matrix of the graph; the d-torus can be seen as the Cartesian product of d copies of C_{2n+1}, from which it follows that the adjacency matrix can be written as the Kronecker sum of the d-copies of the adjacency matrix A_1 of C_{2n+1}, $A_d = A_1 \oplus \cdots \oplus A_1$, see Appendix B for the definition of the Kronecker sum. Consequently, Lemma B.13, if $\mathrm{Spec}(A_1) = (\alpha_j)_{j=1}^{2n+1}$ then

$$\mathrm{Spec}(A_d) = \Big\{ \alpha_{j_1} + \cdots + \alpha_{j_d} \colon (j_1, \dots, j_d) \in \{1, \dots, (2n+1)\}^d \Big\}.$$

Writing the spectrum of Q (the generator for Λ_n) as $\mathrm{Spec}(Q) = (\eta_{\underline{j}})_{\underline{j} \in I}$, where the indices run over $\underline{j} = (j_1, \dots, j_d) \in \{1, \dots, (2n+1)\}^d$, we can write the eigenvalues explicitly thanks to the equation above (A.1)

$$\eta_{\underline{j}} = \frac{1}{d} \sum_{i=1}^{d} \cos\left(2\pi \frac{j_i}{2n+1} \right) - 1.$$

At this point, rather than working with the canonical distribution functions as we have done until now, we make use of the general definition of a spectral distribution, and work with functions $\psi_n \colon (0, 1]^d \to [-2, 0]$. Writing $\underline{u} = (u_1, \dots, u_d) \in (0, 1]^d$ we define

$$\psi_n(\underline{u}) = \eta_{\lceil (2n+1)\underline{u} \rceil}, \qquad u \in (0, 1]^d,$$

with the convention that $\lceil \underline{u} \rceil = (\lceil u_1 \rceil, \dots, \lceil u_d \rceil)$. Letting ϕ_n denote the distribution function for C_{2n+1} as in (A.1) we have that

$$\psi_n(\underline{u}) = \frac{1}{d} \sum_{i=1}^{d} \phi_n(u_i).$$

from which the pointwise convergence of $\psi_n \to \psi_\infty$ is immediate

$$\begin{aligned}\psi_\infty(\underline{u}) &= \lim_{n\to\infty} \frac{1}{d}\sum_{i=1}^{d} \phi_n(u_i) \\ &= \frac{1}{d}\sum_{i=1}^{d} \cos(2\pi u_i) - 1.\end{aligned}$$

As with the case $d = 1$, we can replace the distribution function above by the simpler

$$\psi'_\infty(u) = \frac{1}{d}\sum_{i=1}^{d} \cos(\pi u_i) - 1.$$

Appendix B

Linear Algebra

In this appendix we collate several definitions and statements from linear algebra. Throughout we consider (finite) matrices with entries in $\mathbb{C}$.

B.1 Positive Definite Matrices

A complex valued matrix $A \in \mathbb{C}^{n\times n}$ is Hermitian if it is equal to its conjugate transpose: $A = A^*$.

Definition B.1. *A Hermitian matrix $A \in \mathbb{C}^{n\times n}$ is positive-definite (resp. positive-semidefinite) if for all $x \in \mathbb{C}^n\backslash\{0\}$, x^*Ax is real and: $x^*Ax > 0$ (resp. $x^*Ax \geq 0$).*

The following is an immediate consequence of the definition.

Proposition B.2. *If $A, B \in C^{n\times n}$ are both positive-definite matrices, then $A + B$ is positive-definite. If either of A, B are allowed to be positive-semidefinite, then $A + B$ is positive-semidefinite.*

The following provides useful alternative characterisations.

Theorem B.3. *Let A be a Hermitian matrix. The following are equivalent:*

1. *A is positive-definite.*
2. *All eigenvalues of A are strictly positive.*
3. *A^{-1} exists and is positive-definite.*

See [HJ13], p.438. In the case that A is positive-semidefinite, equivalence of condition 2 to condition 1 still holds on allowing the eigenvalues to be non-negative.

Corollary B.4. *Let $A, P \in \mathbb{C}^{n\times n}$. If A is positive-definite then P^*AP is positive-semidefinite. Moreover, if P is invertible, then P^*AP is positive-definite.*

Proof. Let $x \in \mathbb{C}^n \backslash \{0\}$. Then

$$x^*(P^*AP)x = (Px)^*A(Px) \geq 0,$$

so that P^*AP is positive-semidefinite. Suppose in addition that P is invertible. Since P^*AP is positive-semidefinite, it's eigenvalues are positive, so it suffices to show that 0 is not an eigenvalue. But since A, P are both invertible $\det(A), \det(P) \neq 0$ and hence $\det(P^*AP) \neq 0$. So 0 is not an eigenvalue of P^*AP. □

A matrix can only be positive-definite if it is Hermitian, but we would like to be able to derive a similar positivity condition for eigenvalues of non-Hermitian matrices, in line with Theorem B.3. A suitable result for our purposes is the following.

Proposition B.5. *Let $A \in \mathbb{C}^{n \times n}$ with positive-semidefinite Hermitian part, $\frac{1}{2}(A + A^*)$. Then all eigenvalues $\eta \in \operatorname{Spec}(A)$ have non-negative real part: $\operatorname{Re} \eta \geq 0$.*

See [HJ94], pp.3–4.

B.2 Normal Matrices

Our intention in this section is to provide a useful description of the spectral radius of a normal matrix, the content of Corollary B.11. We first recall the singular value decomposition of a square matrix.

Theorem B.6 (Singular Value Decomposition). *Let $A \in \mathbb{C}^{n \times n}$. There exist unitary matrices $U, V \in \mathbb{C}^{n \times n}$ and a diagonal matrix $\Sigma = \operatorname{diag}(\sigma_1, \ldots, \sigma_n)$ with $\sigma_1 \geq \cdots \geq \sigma_n \geq 0$ such that $A = U\Sigma V^*$. Moreover the σ_k, $k = 1, \ldots n$ are the positive square roots of the eigenvalues of AA^*.*

See [HJ13], pp. 150–1. The values $\sigma_k = \sigma_k(A)$, $k = 1, \ldots, n$ are referred to as the singular values of A. The following generalises the Courant-Fischer formula for the eigenvalues of a square matrix, we state only the result for the largest singular value.

Theorem B.7 (Courant-Fischer). *Let $A \in \mathbb{C}^{n \times n}$, and let σ_1 denote the largest singular values of A. Then*

$$\sigma_1 = \max_{\|x\|_2 = 1} \|Ax\|_2,$$

See [HJ13], pp. 451–2. The singular values of A can be used to bound the spectral radius, which we recall is defined as the maximum modulus of the eigenvalue of A

$$\rho(A) := \max_{1 \leq k \leq n} |\lambda_k|.$$

Proposition B.8. *The function* $\|\cdot\|\colon \mathbb{C}^{n\times n} \to \mathbb{R}_+$, $A \mapsto \sigma_1(A)$ *defines a matrix norm. Furthermore*

$$0 \le \rho(A) \le \sigma_1(A).$$

See [HJ13], pp.346–7. Our intention is to refine the above proposition in the case that A is normal. We recall that a matrix $A \in \mathbb{C}^{n\times n}$ is said to be *normal* if it commutes with its conjugate transpose: $AA^* = A^*A$. Clearly any Hermitian matrix is normal. The following theorem classifies normal matrices, [HJ13] pp.150–1.

Theorem B.9. *A matrix* $A \in \mathbb{C}^{n\times n}$ *is normal if and only if it is unitarily diagonalisable: there exist* $U \in \mathbb{C}^{n\times n}$ *unitary, such that*

$$A = UDU^*,$$

where $D = \operatorname{diag}(\lambda_1, \ldots, \lambda_n)$ *and* λ_k, $k = 1, \ldots, n$ *are the eigenvalues of* A.

Corollary B.10. *For* A *normal, the singular values of* A *are the moduli of the eigenvalues:* $\sigma_k = |\lambda_k|$.

Proof. Suppose A is normal. From Theorem B.9, there is a unitary $U \in \mathbb{C}^{n\times n}$ such that $A = UDU^*$. Hence

$$AA^* = (UDU^*)(UD^*U^*) = UDD^*U^*,$$

which implies that AA^* is normal and has eigenvalues given by the diagonal of $DD^* = \operatorname{diag}(|\lambda_1|^2, \ldots, |\lambda_n|^2)$, with λ_k, $k = 1, \ldots, n$ the eigenvalues of A. Hence from Theorem B.6 the singular values of A are exactly the moduli of the eigenvalues of A. $\square$

The following is now an immediate consequence of the above corollary and Proposition B.8.

Corollary B.11. *For* A *normal*

$$\rho(A) = \sigma_1(A) = \|A\|.$$

This has all been building towards the next proposition which is required in Chapter 3 in deriving the Laplace transform of the loop occupation field under the Markov loop measure.

Proposition B.12. *Let* A, B *be normal. Then*

$$\rho(A + B) \le \rho(A) + \rho(B).$$

Proof. According to Proposition B.8 and the Courant–Fischer theorem

$$\begin{aligned}\rho(A+B) &\leq \sigma_1(A+B)\\ &= \max_{\|x\|_2=1} \|(A+B)x\|\\ &\leq \max_{\|x\|_2=1} \|Ax\| + \max_{\|x\|_2=1} \|Bx\|,\\ &= \sigma_1(A)+\sigma_1(B).\end{aligned}$$

The claim now follows from Corollary B.11. □

B.3 Kronecker Products and Sums

For matrices $A \in \mathbb{C}^{m\times n}$, $B \in \mathbb{C}^{p\times q}$, the *Kronecker product* of A with B is the matrix $A \otimes B \in \mathbb{C}^{mp\times nq}$ given in block form by

$$A \otimes B := \begin{pmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{pmatrix}.$$

We write $(A \otimes B)_{i_j k_l} = a_{ik}b_{jl}$. If $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times m}$ are square matrices, the *Kronecker sum* of A with B is the matrix $A \oplus B \in \mathbb{C}^{mn\times mn}$

$$A \oplus B := (A \otimes I_m) + (I_n \otimes B),$$

so that $(A \oplus B)_{i_j k_l} = a_{ik}\delta_{j,l} + b_{jl}\delta_{i,k}$.

Lemma B.13. *For square matrices $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{m\times m}$ with spectra denoted* $\mathrm{Spec}(A)$, $\mathrm{Spec}(B)$ *respectively*

$$\begin{aligned}\mathrm{Spec}(A\otimes B) &= \{\lambda\eta :\ \lambda \in \mathrm{Spec}(A),\, \eta \in \mathrm{Spec}(B)\},\\ \mathrm{Spec}(A\oplus B) &= \{\lambda+\eta :\ \lambda \in \mathrm{Spec}(A),\, \eta \in \mathrm{Spec}(B)\}.\end{aligned}$$

Proof. We show that if $Au = \lambda u$, $Bv = \eta v$ then $u \otimes v$ is an eigenvector of both

$A \otimes B$ and $A \oplus B$ with corresponding eigenvalues $\lambda\eta$ and $\lambda + \eta$, respectively.

$$
\begin{aligned}
(A \otimes B)(u \otimes v)_{i_j} &= \sum_{k_l} (A \otimes B)_{i_j k_l} (u \otimes v)_{k_l} \\
&= \sum_{k,l} a_{ik} b_{jl} u_k v_l \\
&= \left(\sum_k a_{ik} u_k \right) \left(\sum_l b_{jl} v_l \right) \\
&= \lambda\eta u_i v_j \\
&= \lambda\eta (u \otimes v)_{i_j}.
\end{aligned}
$$

Similarly

$$
\begin{aligned}
(A \oplus B)(u \otimes v)_{i_j} &= \sum_{k_l} (a_{ik}\delta_{j,l} + b_{jl}\delta_{i,k}) u_k v_l \\
&= v_j \sum_k a_{ik} u_k + u_i \sum_l b_{jl} v_l \\
&= v_j \lambda u_i + u_i \eta v_j \\
&= (\lambda + \eta)(u \otimes v)_{i_j}.
\end{aligned}
$$

□

Lemma B.14. *Let* $A, C \in \mathbb{C}^{n\times n}$, $B, D \in C^{m\times m}$. *Then*

1. $(A \otimes B)^* = A^* \otimes B^*$.

2. $(A \otimes B)(C \otimes D) = (AC \otimes BD)$.

These follow by similar calculations to the above; for details, see [HJ94], pp.243–4.

Proposition B.15. *If* $A \in \mathbb{C}^{n\times n}, B \in \mathbb{C}^{m\times m}$ *are both normal matrices, then* $A \otimes B$ *and* $A \oplus B$ *are normal.*

Proof. Consider first $A \otimes B$. Using Theorem B.9 we write $A = UDU^*$, $B = VEV^*$ with U, V unitary, and D, E diagonal. Then

$$
\begin{aligned}
A \otimes B &= (UDU^*) \otimes (VEV^*) \\
&= (U \otimes V)(D \otimes E)(U^* \otimes V^*),
\end{aligned}
$$

from the second claim of the preceeding lemma. Clearly $D \otimes E$ is diagonal, and again from the lemma we know

$$
= (U \otimes V)(D \otimes E)(U \otimes V)^*,
$$

so it remains to check that $U \otimes V$ is unitary. But

$$\begin{aligned}(U \otimes V)(U \otimes V)^* &= (U \otimes V)(U^* \otimes V^*) \\ &= (UU^*) \otimes (VV^*) \\ &= I_{mn}.\end{aligned}$$

where we have used the fact that $I_n \otimes I_m = I_{mn}$.
Considering now $(A \oplus B)$, we note that in general if A, B are both normal then $A + B$ is normal if and only if

$$AB^* + BA^* = A^*B + B^*A.$$

Applying this to $(A \otimes I_m)$, $(I_n \otimes B)$, by Lemma B.14

$$\begin{aligned}(A \otimes I_m)(I_n \otimes B)^* + (I_n \otimes B)(A \otimes I_m)^* &= A \otimes B^* + A^* \otimes B \\ &= (A \otimes I_m)^*(I_n \otimes B) + (I_m \otimes B)^*(A \otimes I_m).\end{aligned}$$

□

B.4 Miscellaneous Matrix Identities

In the following we denote the eigenvalues of a matrix $A \in \mathbb{C}^{n\times n}$ by $\lambda_1(A), \cdots, \lambda_n(A)$, and suppose that they are in decreasing order: $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$. We present two famous eigenvalue inequalities.

Theorem B.16 (Weyl's Inequality)**.** *Suppose $A, B \in \mathbb{C}^{n\times n}$ are Hermitian matrices, so that the eigenvalues of A, B, and $A + B$ are real. Then, the eigenvalues of $A + B$ satisfy*

$$\begin{aligned}&\lambda_i(A) + \lambda_j(B) \leq \lambda_k(A + B) \qquad \text{whenever } k \leq i + j - n, \text{ and} \\ &\lambda_i(A) + \lambda_j(B) \geq \lambda_k(A + B) \qquad \text{whenever } k \geq i + j - 1.\end{aligned}$$

In particular

$$\lambda_k(A) + \lambda_n(B) \leq \lambda_k(A + B) \leq \lambda_k(A) + \lambda_1(B).$$

See [HJ13] Theorem 4.3.1 pp.239–40.

Theorem B.17 (Cauchy's Interlacing Theorem)**.** *Let $A \in \mathbb{C}^{n\times n}$ be Hermitian, and let B be the matrix obtained from A by deleting the last m rows and columns, $1 \leq m < n$. Then*

$$\lambda_k(A) \geq \lambda_k(B) \geq \lambda_{k+m}(A), \qquad k = 1, \ldots, n - m.$$

In particular for $m = 1$

$$\lambda_k(A) \geq \lambda_k(B) \geq \lambda_{k+1}(A), \qquad k = 1, \dots, n-1.$$

See [HJ13], pp. 246–7.
Recall that a matrix $A \in \mathbb{C}^{n\times n}$ is circulant if it is of the form

$$A = \begin{pmatrix} a_0 & a_1 & a_2 & & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & a_2 & & \vdots \\ & a_{n-1} & a_0 & a_1 & \ddots & \\ \vdots & \ddots & \ddots & \ddots & & a_2 \\ & & & & & a_1 \\ a_1 & \cdots & & a_{n-1} & & a_0 \end{pmatrix},$$

we write $A = \operatorname{circ}(a_0, \dots, a_{n-1})$.

Theorem B.18. *Let $A = \operatorname{circ}(a_0, \dots, a_{n-1})$, with $a_1, \dots, a_n \in \mathbb{C}$. Then*

$$\lambda_k = \sum_{j=0}^{n-1} a_j \omega_k^j,$$

is an eigenvalue with eigenvector $(1, \omega_k, \dots, \omega_k^{n-1})$, where $\omega_k = \exp(2\pi i \frac{k}{n})$.

See [Gray06], p.186.
The following result, whilst strictly not related to matrices will be used in the context of circulant matrices.

Proposition B.19. *For $a \in \mathbb{R}$, $b > 0$, and ω_k, $k = 1, \dots, n$ the n-th roots of unity,*

$$\prod_{k=1}^{n} \big(a - b\omega_k\big) = a^n - b^n.$$

Proof. The roots of unity are defined to be the solutions to $x^n - 1 = 0$, and consequently we have: $x^n - 1 = \prod_{k=1}^{n}(x - \omega_k)$. Then

$$\prod_{k=1}^{n} \big(a - b\omega_k\big) \;=\; b^n \prod_{k=1}^{n} \Big(\frac{a}{b} - \omega_k\Big) \;=\; b^n \Big(\Big(\frac{a}{b}\Big)^n - 1\Big). \qquad \square$$

At several points we will make use of matrix power series. The following result will be used in defining the Green's function for a random walk.

Proposition B.20. *If $A \in \mathbb{C}^{n\times n}$ has spectral radius $\rho(A) < 1$, then the following* *power series exists and is equal to*

$$(I - A)^{-1} = \sum_{n=0}^{\infty} A^n.$$

See [HJ13], p.351. Recall that the matrix exponential is defined by the power series

$$e^A = \exp(A) := \sum_{n=0}^{\infty} \frac{1}{n!} A^n,$$

which exists for all $A \in \mathbb{C}^{n \times n}$. A matrix B is said to be a logarithm of A if it satisfies $A = e^B$; logarithms need not exist, and are not neccessarily unique.

Proposition B.21. *If* $\operatorname{Spec}(A) \cap \mathbb{R}_{\leq 0} = \varnothing$ *then there exists a logarithm* X *with* $\operatorname{Spec}(X) \subset \{z \in \mathbb{C} : -\pi < \operatorname{Im}(z) < \pi\}$.

See [Hig08], p.20. Under the conditions of the theorem above we refer to X as the principle logarithm of A, and write $X = \log A$.

Proposition B.22. *If* $\operatorname{Spec}(A) \cap \mathbb{R}_{\leq 0} = \varnothing$ *and* $\rho(A) < 1$*, then* $\log(A)$ *satisfies*

$$\log(I - A) = -\sum_{k=0}^{\infty} \frac{1}{k} A^k.$$

See [Hig08], p.273. Under the same assumptions we can easily confirm the trace identity for matrix logarithms: $\operatorname{Tr} \log(I - A) = \log \det(I - A)$, note

$$\begin{aligned} \operatorname{Tr} \log(I - A) &= -\sum_{j=1}^{n} \sum_{k=0}^{\infty} \frac{1}{k} A_{jj}^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k} \operatorname{Tr} A^k \\ &= \sum_{k=0}^{\infty} \frac{1}{k} \sum_{j=1}^{n} \eta_j^k \end{aligned}$$

with $\operatorname{Spec}(A) = \{\eta_j\}_{j=1}^n$. Then changing the order of summation again we obtain

$$\begin{aligned} &= \sum_{j=1}^{n} \log(1 - \eta_j) \\ &= \log\left(\prod_{j=1}^{n} (1 - \eta_j)\right). \end{aligned}$$

Appendix C

Some Analysis on $\ell_1(\mathbb{R})$

The following results all pertain to Chapter 4 where we considered large deviations principles for the cycle distribution of a loop soup. Our first aim is to prove that we can construct the cycle distribution as a measure on $\ell_1(\mathbb{R})$.

Let $\underline{m} = (m_j)_{j\geq 1}$ be a positive summable sequence, $m_j \geq 0$ and $\sum_j m_j = M < \infty$. Associated with each m_j we have the probability measure $P_j \sim \mathrm{Poi}(m_j)$, the law of the Poisson distribution with mean m_j. We construct a product measure $P_{\underline{m}} = \otimes_{i=1}^{\infty} P_{m_j}$ in two stages. We first prove that such a measure exists on the space $\ell_1(\mathbb{N})$, and then extend it to the larger space $\ell_1(\mathbb{R})$.

The space $\ell_1(\mathbb{N})$ consists of all convergent integer sequences, and as such is isomorphic to $c_0(\mathbb{N})$ the space of terminating sequences. In particular this space is countable. For $\underline{n} = (n_j)_{j\geq 1}$ define

$$\begin{aligned}\widetilde{P}_{\underline{m}}(\underline{n}) &= \prod_{j\geq 1} P_j(n_j) \\ &= \prod_{j\geq 1} \frac{e^{-m_j} m_j^{n_j}}{n_j!}.\end{aligned}$$

Lemma C.1. *$\widetilde{P}_{\underline{m}}$ defines a probability measure on $\ell_1(\mathbb{N})$.*

Proof. Since $\ell_1(\mathbb{N})$ is countable, a measure is determined by its value at each point $\underline{n} \in \ell_1(\mathbb{N})$. As such, $\widetilde{P}_{\underline{m}}$ is by default a measure. It remains to prove that $\widetilde{P}_{\underline{m}}\big(\ell_1(\mathbb{N})\big) = 1$. To this end, noting that every subset $S \subset \ell_1(\mathbb{N})$ is measurable define the sequence of sets

$$S_J := \underbrace{\mathbb{N} \times \cdots \times \mathbb{N}}_{\text{J times}} \times \{0\} \cdots$$

so that S_J is the collection of integer sequences which terminate after at most J terms, and we have $\ell_1(\mathbb{N}) = \cup_{J\geq 0} S_J$. Moreover, $S_J \subset S_{J+1}$ holds for all $J \geq 0$ and so continuity of measures assures

$$
\begin{aligned}
\widetilde{P}_{\underline{m}}\big(\ell_1(\mathbb{N})\big) &= \lim_{J\to\infty} \widetilde{P}_{\underline{m}}(S_J) \\
&= \lim_{J\to\infty} \sum_{n_1=0}^{\infty} \cdots \sum_{n_J=0}^{\infty} \prod_{j=1}^{J} \frac{e^{-m_j} m_j^{n_j}}{n_j!} \prod_{j>J} e^{-m_j},
\end{aligned}
$$

collating all the exponential terms

$$
= \lim_{J\to\infty} e^{-M} \sum_{n_1=0}^{\infty} \cdots \sum_{n_J=0}^{\infty} \prod_{j=1}^{J} \frac{e^{-m_j} m_j^{n_j}}{n_j!}
$$

and since there are only finitely many sums we can factor out the product as

$$
\begin{aligned}
&= \lim_{J\to\infty} e^{-M} \prod_{j=1}^{J} \sum_{n=0}^{\infty} \frac{m_j^n}{n!} \\
&= \lim_{J\to\infty} e^{-M} \prod_{j=1}^{J} e^{m_j} \\
&= \lim_{J\to\infty} \exp\left(-M + \sum_{j=1}^{J} m_j\right) \\
&= 1,
\end{aligned}
$$

where the final conclusion follows from continuity of the exponential. □

We conclude that we can in fact determine an equivalent measure on $\ell_1(\mathbb{R})$.

Theorem C.2. *For $B \in \mathcal{B}$ the Borel σ-algebra of $\ell_1(\mathbb{R})$, define*

$$
P_{\underline{m}}(B) := \widetilde{P}_{\underline{m}}(B \cap \ell_1(\mathbb{N})).
$$

Then $P_{\underline{m}}$ defines a probability measure on $(\ell_1(\mathbb{R}), \mathcal{B})$.

Proof. The proof is immediate. For $\varnothing \in \mathcal{B}$,

$$
P_{\underline{m}}(\varnothing) = \widetilde{P}_{\underline{m}}(\varnothing \cap \ell_1(\mathbb{N})) = \widetilde{P}_{\underline{m}}(\varnothing) = 0.
$$

Similarly since $\ell_1(\mathbb{N}) \subset \ell_1(\mathbb{R})$

$$
P_{\underline{m}}\big(\ell_1(\mathbb{R})\big) = \widetilde{P}_{\underline{m}}\big(\ell_1(\mathbb{R}) \cap \ell_1(\mathbb{N})\big) = \widetilde{P}_{\underline{m}}\big(\ell_1(\mathbb{N})\big) = 1.
$$

And if $(B_i)_{i\geq 1}$ is a collection of pairwise disjoint measurable sets, $B_i \in \mathcal{B}$, then

$$\begin{aligned} P_{\underline{m}}\big(\cup_{i=1}^{\infty} B_i\big) &= \widetilde{P}_{\underline{m}}\big(\big(\cup_{i=1}^{\infty} B_i\big) \cap \ell_1(\mathbb{N})\big) \\ &= \widetilde{P}_{\underline{m}}\big(\cup_{i=1}^{\infty}\big(B_i \cap \ell_1(\mathbb{N})\big)\big) \end{aligned}$$

and since these sets are disjoint in $\ell_1(\mathbb{N})$

$$\begin{aligned} &= \sum_{i=1}^{\infty} \widetilde{P}_{\underline{m}}(B_i \cap \ell_1(\mathbb{N})) \\ &= \sum_{i=1}^{\infty} P_{\underline{m}}(B_i). \end{aligned}$$

□

Having established that the measure $P_{\underline{m}}$ is well defined, the second technical challenge to overcome in Section 4.1 is proving exponential tightness for the measures $\mathbb{P}_\Lambda = (|\Lambda|^{-1} * P_{\underline{m}})$, where now we have assumed $m_j = \mu^B_{\beta,h,\Lambda}(\Gamma_j)$. The proof of this, Proposition 4.8, relies on the following topological result.

Lemma C.3. *Let $\underline{x} \in \ell_1(\mathbb{R})$, and define the set*

$$K_{\underline{x}} := \{y \in \ell_1(\mathbb{R}) : |y_j| \leq |x_j| \, \forall j \geq 1\}.$$

Then $K_{\underline{x}}$ is closed and bounded, and moreover is a compact subset of $\ell_1(\mathbb{R})$.

Proof. We proceed by first confirming that $K_{\underline{x}}$ is closed and bounded, which will subsequently be used to show compactness. We recall that the metric on $\ell_1(\mathbb{R})$ is given by $d(\underline{y}, \underline{z}) = \sum_{j\geq 1} |y_j - z_j|$, and that a subset $S \subset \ell_1(\mathbb{R})$ is bounded if there exists $\underline{s} \in S$, and $M \geq 0$ such that: $d(\underline{y}, \underline{s}) \leq M$ for all $\underline{y} \in S$. Denoting $\underline{0}$ for the sequence of all zeros, then $\underline{0} \in K_{\underline{x}}$, and writing $M = \sum_{j\geq 1} |x_j|$

$$d(0, \underline{y}) = \sum_{j\geq 1} |y_j| \leq \sum_{j\geq 1} |x_j| = M, \qquad \forall \underline{y} \in K_{\underline{x}},$$

which is to say that $K_{\underline{x}}$ is bounded.

Suppose that $K_{\underline{x}}$ is not closed, then there exists a sequence $\underline{y}^{(n)} \in K_{\underline{x}}$, which converges to $\underline{y} \in \ell_1(\mathbb{R}) \backslash K_{\underline{x}}$. Considering such $\underline{y} \notin K_{\underline{x}}$, there is a $k \geq 1$ such that $|y_k| > |x_k|$. Set $\varepsilon = \frac{1}{2}(|y_k| - |x_k|)$, and note that since $\underline{y}^{(n)} \to \underline{y}$, then for n sufficiently large $d(\underline{y}^{(n)}, \underline{y}) < \varepsilon$. Then

$$\begin{aligned} |y_k^{(n)} - y_k| &\leq \sum_{j\geq 1} |y_j^{(n)} - y_j| \\ &< \frac{1}{2}(|y_k| - |x_k|). \end{aligned}$$

Then using the reverse triangle inequality

$$\big||y_k^{(n)}| - |y_k|\big| \leq |y_k^{(n)} - y_k| < \frac{1}{2}(|y_k| - |x_k|),$$

from which we have

$$\begin{aligned}\big|y_k^{(n)}\big| &> |y_k| - \frac{1}{2}(|y_k| - |x_k|) \\ &= \frac{1}{2}(|y_k| + |x_k|) \\ &> |x_k|,\end{aligned}$$

contradicting that $y^{(n)} \in K_{\underline{x}}$, and so $K_{\underline{x}}$ is closed.
Turning to compactness we note that given a complete set, then if it is closed and totally bounded, then it is also compact [Sut81] p.141. In our instance, since $\ell_1(\mathbb{R})$ is a Banach space it is, by definition, complete and consequently so to is $K_{\underline{x}}$. We recall that $K_{\underline{x}}$ is totally bounded if for any $\varepsilon > 0$ we can find a finite collection of sequences $\underline{z}^{(1)}, \dots, \underline{z}^{(I)} \in K_{\underline{x}}$ such that $K_{\underline{x}} \subset \cup_{i=1}^{I} B(\underline{z}^{(i)}, \varepsilon)$, where $B(\underline{z}, \varepsilon)$ denotes the ε-open ball around $\underline{z}$; we say that the $\underline{z}^{(i)}$ form a finite ε-net.
Fix $\varepsilon > 0$, and let $N \geq 1$ be such that $\sum_{j>N} |x_j| < \varepsilon/2$, and let $K_{\underline{x}}^N \subset K_{\underline{x}}$ be the set of sequences

$$K_{\underline{x}}^N := \{y \in K_{\underline{x}} : \; y_j = 0, \, j > N\}.$$

The set $K_{\underline{x}}^N$ is isomorphic to $\big[-|x_1|, |x_1|\big] \times \cdots \big[-|x_N|, |x_N|\big] \subset \mathbb{R}^N$, which is a closed and totally bounded subset of $\mathbb{R}^N$: hence $K_{\underline{x}}^N$ is totally bounded. Hence we can find an $\varepsilon/2$-net $\underline{z}^{(1)} \dots, \underline{z}^{(I)} \in K_{\underline{x}}^N$ for $K_{\underline{x}}^N$. For $\underline{y} \in K_{\underline{x}}$ let $\underline{y}^N \in K_{\underline{x}}^N$ be the sequence which agrees with $\underline{y}$ on the first N terms, and choose $z^{(i)}$ from the $\varepsilon/2$-net of $K_{\underline{x}}^N$ such that $\underline{y}^N \in B(\underline{z}^{(i)}, \varepsilon/2)$. Then

$$\begin{aligned}d(\underline{y}, \underline{z}^{(i)}) &= \sum_{j \geq 1} |y_j - z_j^{(i)}| \\ &= \sum_{j=1}^{N} |y_j^N - z_j^{(i)}| + \sum_{j>N} |y_j^N| \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\end{aligned}$$

where the first term is bounded by $\varepsilon/2$ since $\underline{y}^N \in B(\underline{z}^{(i)})$, whilst the second is similarly bounded by the definition of N. In particular we have shown that the sequences $\underline{z}^{(1)}, \dots, \underline{z}^{(I)}$ are a finite ε-net for $K_{\underline{x}}$, and hence the set is totally bounded. □

The analysis in Chapter 4 of the rate functions associated with LDPs rely on convex analysis on $\ell_1(\mathbb{R})$. We recall that a functional $f\colon \ell_1(\mathbb{R}) \to \mathbb{R}$ is said to be convex on a domain $D \subset \ell_1(\mathbb{R})$ if for all $\underline{x}$, $\underline{y} \in D$

$$f\big(t\underline{x} + (1-t)\underline{y}\big) \geq tf(\underline{x}) + (1-t)f(\underline{y}), \qquad \forall\, t \in [0,1].$$

We say that f is strictly convex if the above holds with strict inequality when $\underline{x} \neq \underline{y}$, and $t \in (0,1)$. The following result is used implicitly when we are solving for minima of rate functions. Recall that f has a local minimum at $\underline{x}$ if there is an $\varepsilon > 0$ such that for all $\underline{y} \in \ell_1(\mathbb{R})$ such that $\underline{y} \in B_\varepsilon(\underline{x})$, $f(x) < f(y)$. In the following we write $\ell_1(\mathbb{R}_+)$ for the space of convergent positive series.

Proposition C.4. *Let $f\colon \ell_1(\mathbb{R}_+) \to \mathbb{R}$ be strictly convex, then f achieves at most one local minimum.*

Proof. Suppose that f has a local minimum at both $\underline{x}, \underline{y} \in \ell_1(\mathbb{R}_+)$, and let $f(\underline{x}) = a \geq b = f(\underline{y})$. Then for all $t \in (0,1)$

$$f\big(t\underline{x} + (1-t)\underline{y}\big) < ta + (1-t)b \leq a.$$

In particular: $f\big(t\underline{x} + (1-t)\underline{y}\big) < f(\underline{x})$ for all $t \in (0,1)$, so it suffices to show that for t sufficiently close to 1, $t\underline{x} + (1-t)\underline{y} \in B_\varepsilon(\underline{x})$. But this is apparent on writing

$$d\big(\underline{x}, t\underline{x} + (1-t)\underline{y}\big) = (1-t)\sum_{j\geq 1} |x_j - y_j|,$$

which tends to 0 as $t \to 1$. $\square$

In Section 4.1 we introduced the Gâteaux derivative of a functional $f\colon \ell_1(\mathbb{R}) \to \mathbb{R}$ via the formula

$$\mathrm{d}f(\underline{x};\underline{y}) := \frac{\mathrm{d}}{\mathrm{d}\varepsilon} f(\underline{x} + \varepsilon\underline{y})_{|\varepsilon=0},$$

when the right hand side exists we say that f is Gâteaux differentiable at $\underline{x}$, in direction $\underline{y}$. As with derivatives of functions, the Gâteaux derivative can be used to identify the extrema of functionals. This is the content of the following lemma.

Lemma C.5. *Let $f\colon \ell_1(\mathbb{R}_+) \to \mathbb{R}$ be strictly convex and Gâteaux differentiable. The point $\underline{x} \in \operatorname{int} \ell_1(\mathbb{R}_+)$ is a unique minimum for f if and only if $\mathrm{d}f(\underline{x};\underline{y}) = 0$ for all $\underline{y} \in \ell_1(\mathbb{R})$.*

Proof. Suppose that f has its unique minimum at $\underline{x} \in \operatorname{int} \ell_1(\mathbb{R}_+)$. Let $y \in \ell_1(\mathbb{R})$ then

$$df(\underline{x};\underline{y}) = \lim_{\varepsilon \to 0} \frac{f(\underline{x} + \varepsilon\underline{y}) - f(\underline{x})}{\varepsilon}$$

and for ε sufficiently small $\underline{x} + \varepsilon\underline{y} \in \ell_1(\mathbb{R}_+)$, and since $\underline{x}$ is the unique local minimum the numerator is positive, and hence: $df(\underline{x};\underline{y}) \geq 0$ for all $\underline{y} \in \ell_1(\mathbb{R})$. But now considering $-\underline{y} = (-y_1, -y_2, \ldots) \in \ell_1(\mathbb{R})$

$$\mathrm{d}f(\underline{x}; -\underline{y}) = \lim_{\varepsilon \to 0} \frac{f(\underline{x} - \varepsilon \underline{y}) - f(\underline{x})}{\varepsilon},$$

substituting $\tilde{\varepsilon} = -\varepsilon$

$$\begin{aligned} &= -\lim_{\tilde{\varepsilon} \to 0} \frac{f(\underline{x} + \tilde{\varepsilon} \underline{y}) - f(\underline{x})}{\tilde{\varepsilon}} \\ &= -\mathrm{d}f(\underline{x}; \underline{y}) \end{aligned}$$

But since $\mathrm{d}f(\underline{x}; -\underline{y}) \geq 0$, the above asserts both $\mathrm{d}f(\underline{x}; \underline{y}) \geq 0$ and$-\mathrm{d}f(\underline{x}; \underline{y}) \geq 0$, from which the claim follows.
Conversely, suppose that $\mathrm{d}f(\underline{x}; \underline{y}) = 0$, for all $y \in \ell_1(\mathbb{R})$. In particular choose $y \in \ell_1(\mathbb{R}_+)$,then

$$\begin{aligned} \mathrm{d}f(\underline{x}; \underline{y} - \underline{x}) &= \lim_{\varepsilon \to 0} \frac{f\big(\underline{x} + \varepsilon(\underline{y} - \underline{x})\big) - f(\underline{x})}{\varepsilon} \\ &= \lim_{\varepsilon \to 0} \frac{f\big((1 - \varepsilon)\underline{x} + \varepsilon \underline{y}\big) - f(\underline{x})}{\varepsilon} \end{aligned}$$

Since both terms are in $\ell_1(\mathbb{R}_+)$, using the convexity of f

$$\begin{aligned} &\leq \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\big(\varepsilon f(\underline{y}) - \varepsilon f(\underline{x})\big) \\ &= f(\underline{y}) - f(\underline{x}). \end{aligned}$$

Rearranging the above, and relying on the hypothesis that $\mathrm{d}f(\underline{x}; \underline{y}) = 0$

$$f(\underline{x}) \leq f(\underline{y}) \qquad \forall \underline{y} \in \ell_1(\mathbb{R}_+),$$

which is to say $\underline{x}$ is a local minimum. The claim follows on appealing to Proposition C.4. □

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