Dynamical Poroplasticity Model with mixed boundary conditions – theory for $\mathcal{LM}$-type nonlinearity

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Abstract

We investigate the existence theory to the non-coercive fully dynamic model of poroplasticity with non-homogeneous mixed boundary condition and constitutive equation which belongs to the class $\mathcal{LM}$. Existence of the solution to this model is proved by using the coercive and Yosida approximations under the lowest possible assumptions about $\mathcal{LM}$-type nonlinearity of non-gradient type. Under higher assumptions about the constitutive equation and boundary conditions (see Section 7) we obtain uniqueness and higher regularity of the solutions.

Keywords: Yosida approximation, coercive approximation, energy method, inelastic deformation theory, monotone operator, poroplasticity

1. Introduction

In this paper we discuss the theory of existence and uniqueness of the weak-type solutions of the non-coercive model describing dynamics of inelastic deformations within the porous media (also known as the dynamical model of poroplasticity). The origins of this model are dated back to the 1940s, to the paper of M. Biot [4]. This model was used to describe the soil consolidation.

We assume that the porous media with the material density $\rho > 0$ lies within the subset $\Omega \subset \mathbb{R}^3$. Let $T_e > 0$ be the end time, i.e. the time until we seek the solution. We are interested in finding the following functions

- the displacement field $u : \Omega \times [0, T_e] \to \mathbb{R}^3$,
- the pore pressure of the fluid $p : \Omega \times [0, T_e] \to \mathbb{R}$,
- the inelastic deformation tensor $\varepsilon^p : \Omega \times [0, T_e] \to \mathcal{S}(3) = \mathbb{R}^{3\times 3}_{sym}$,
- the Cauchy stress tensor $T : \Omega \times [0, T_e] \to \mathcal{S}(3)$,

satisfying the system of equations:

$$\begin{aligned} \rho u_{tt}(x,t) - \mathrm{div}_x T(x,t) + \alpha \nabla_x p(x,t) &= F(x,t), \\ c_0 p_t(x,t) - c\Delta_x p(x,t) + \alpha \mathrm{div}_x u_t(x,t) &= f(x,t), \\ T(x,t) &= \mathcal{D}(\varepsilon(u(x,t)) - \varepsilon^p(x,t)), \\ \varepsilon(u(x,t)) &= \frac{1}{2}\left(\nabla u(x,t) + \nabla^T u(x,t)\right), \\ \varepsilon^p_t(x,t) &= A(T(x,t)). \end{aligned} \tag{1.1}$$

In this model $\mathcal{D} : \mathcal{S}(3) \to \mathcal{S}(3)$ is the elasticity tensor (linear, symmetric and positive-definite), $A : \mathcal{S}(3) \to \mathcal{S}(3)$ is the inelastic constitutive function, $F : \Omega \times [0, T_e] \to \mathbb{R}^3$, $f : \Omega \times [0, T_e] \to \mathbb{R}$ describe the densities of the external forces (acting on a body and a fluid respectively), ρ, α, c, c_0 are the material constants (for details see [15]).

The first equation of (1.1) is the balance of momentum coupled with the generalized Hooke's law (the third equation). The Cauchy stress tensor depends only on the elastic part of the deformation tensor,

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whereas the whole deformation tensor is described by the fourth equation. The inelastic part of the deformation tensor is given by the fifth equation (we call it *the inelastic constitutive equation*). The second equation follows from the Darcy's law combined with the mass conservation law for a fluid.

The system (1.1) is complemented with the mixed boundary conditions:

$$\begin{aligned} u(x,t) &= g_D(x,t), \quad && x \in \Gamma_D,\, t \geqslant 0, \\ (T(x,t) - \alpha p(x,t)\mathbb{I})n(x) &= g_N(x,t), \quad && x \in \Gamma_N,\, t \geqslant 0, \\ p(x,t) &= g_P(x,t), \quad && x \in \Gamma_P,\, t \geqslant 0, \\ c\frac{\partial p}{\partial n}(x,t) &= g_V(x,t), \quad && x \in \Gamma_V,\, t \geqslant 0, \end{aligned} \tag{1.2}$$

where $n(x)$ is the outward normal vector at $x \in \partial\Omega$, Γ_D, Γ_N, Γ_P, Γ_V are the open subsets of $\partial\Omega$ with the positive two-dimensional Hausdorff measure and such that

$$\partial\Omega = \overline{\Gamma}_D \cup \overline{\Gamma}_N = \overline{\Gamma}_P \cup \overline{\Gamma}_V, \quad \Gamma_D \cap \Gamma_N = \Gamma_P \cap \Gamma_V = \emptyset.$$

We also impose the following initial conditions for $x \in \Omega$:

$$\begin{aligned} u(x,0) &= u_0(x), \\ u_t(x,0) &= u_1(x), \\ p(x,0) &= p_0(x), \\ \varepsilon^p(x,0) &= \varepsilon_0^p(x). \end{aligned} \tag{1.3}$$

We assume that the Ω is open, bounded and smooth subset of $\mathbb{R}^3$ and the inelastic constitutive function A belongs to the $\mathcal{LM}$ class, i.e. A is a sum of the globally Lipschitz map $l : \mathcal{S}(3) \to \mathcal{S}(3)$ with Lipschitz constant L and the continuous, maximal monotone map $m : \mathcal{S}(3) \to \mathcal{S}(3)$. Moreover, we assume that for the sufficiently large $|T|$ the map m satisfy one of the following growth conditions:

- $$|m(T)| \leqslant C|T|^{\omega} \quad \text{and} \quad \frac{1}{C}|T|^{\omega+1} \leqslant m(T)T \quad \text{for} \quad \omega > 1, \tag{1.4}$$
- $$|m(T)| \leqslant C|T|^{\omega} \quad \text{for} \quad \omega = 1. \tag{1.5}$$

Without the loss of generality one can assume that $A(0) = m(0) = 0$.

In this paper we do not assume that A is the function of a gradient type, but under sufficiently strong assumptions about regularity of data, we were able to prove existence of a solution of (1.1)-(1.3) where equation $(1.1)_5$ is satisfied in a sense of Young measures (see Theorem 7.6). Moreover, under higher assumptions about the constitutive function A, i.e. A is deviatoric and monotone, there exists a unique solution of (1.1)-(1.3) such that $(1.1)_5$ is satisfied almost everywhere (see Theorem 7.9, Lemma 7.10).

2. Literature review

According to the authors' knowledge, the poroplasticity models with the mixed boundary conditions which have been considered so far were only partially dynamical (with $c_0 p_t \approx 0$) and equipped with the non-monotone constitutive equation (see [14]). Unfortunately, the non-monotone constitutive equation considered in that paper resulted with the low regularity of the solutions. Moreover, despite the method used (a coercive approximation with models of the monotone-type), existence, uniqueness and regularity of the solutions in a case of the non-coercive models of the monotone-type were left open for the further discussion. Papers [13] and [12] discuss the quasistatic poroplasticity model (with $\rho u_{tt} \approx 0$) with the Dirichlet type boundary conditions. It is worth mentioning that in [12] author considers a gradient-type model (the constitutive equation is a gradient of a differentiable convex function). It turns out that is the sufficient condition to obtain the solution without the Young measures theory. In this paper however we do not assume that the model is of a gradient-type and consequently we need additional assumptions on the constitutive equations to obtain this kind of result.

It is worth the attention, that there is an obvious similarity between poroplasticity model and the theoretical models of the inelastic deformation for solids, which have been extensively studied by K. Chełmiński and P. Gwiazda in [5], [8], [6], [7] (the monotone case). It is clear, based on the research related to the inelastic deformations in solids, that the essential step to understanding the model is a meticulous analysis of the monotone models, which may serve as an approximation tool for the non-monotone models. This motivation underlies our approach.

3. Structure of the article

In the fourth section, using the Galerkin method and the Banach Fixed Point Theorem, we prove an existence and uniqueness of the weak solutions of (1.1) in a case when A is a globally Lipschitz function. In the fifth section we introduce the coercive approximation and the Yosida approximation. Then we state and prove the energetic estimate independent of the Yosida approximation step. This estimate allows one to prove the existence of the unique weak solutions for the coercive approximation. In the sixth section we prove that under better assumptions on the boundary data (and so called safe-load conditions) there exists a solution (in a sense similar to introduced in [14]) of the approximated model. The final section is about improving regularity of the solution obtained in the section six. This becomes possible with an assumption of even higher regularity of the boundary data. Further discussion is based on the higher assumption on A (the monotonicity and the deviatoricity) which allows one to dispose of the Young measure that is present in the previously obtained solution. Moreover, one can prove the uniqueness of such solution. These results underlie further research related to the non-monotone models, for which the non-monotone constitutive equation may be approximated by the ones discussed in this paper.

4. Globally Lipschitz Constitutive Equation

We begin the analysis of (1.1) with the globally Lipschitz function A. Therefore we consider the following system of equations:

$$\begin{aligned} \rho u_{tt}(x,t) - \mathrm{div}_x T(x,t) + \alpha \nabla_x p(x,t) &= F(x,t), \\ c_0 p_t(x,t) - c\Delta_x p(x,t) + \alpha\, \mathrm{div}_x u_t(x,t) &= f(x,t), \\ T(x,t) &= \mathcal{D}\left(a\,\varepsilon(u(x,t)) - \varepsilon^p(x,t)\right), \\ \varepsilon(u(x,t)) &= \frac{1}{2}\left(\nabla_x u(x,t) + \nabla_x^T u(x,t)\right), \end{aligned} \tag{4.1}$$

with the assumption that $\varepsilon^p : \Omega \times [0, T_e] \to \mathcal{S}^3$ is given. Here $a > 0$ is the auxiliary constant, which will prove itself useful in the next section.

The model (4.1) is considered with the conditions (1.2) and (1.3) (except the initial condition for ε^p).

We assume the following data regularity

- For the external forces:

$$\begin{aligned} F \in H^1(0,T_e; L^2(\Omega;\mathbb{R}^3)), \quad f \in H^1(0,T_e; L^2(\Omega;\mathbb{R})), \\ F(0) \in L^2(\Omega;\mathbb{R}^3), \quad f(0) \in L^2(\Omega;\mathbb{R}), \end{aligned} \tag{4.2}$$

- For the boundary conditions:

$$\begin{aligned} g_D \in H^3(0,T_e; H^{\frac{1}{2}}(\Gamma_D;\mathbb{R}^3)) \cap H^2(0,T_e; H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^3)), \\ g_D(0),\, g_{D,t}(0) \in H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^3), \quad g_{D,tt}(0) \in H^{\frac{1}{2}}(\Gamma_D;\mathbb{R}^3), \\ g_N \in H^2(0,T_e; H^{-\frac{1}{2}}(\Gamma_N;\mathbb{R}^3)), \quad g_N(0),\, g_{N,t}(0) \in H^{-\frac{1}{2}}(\Gamma_N;\mathbb{R}^3), \\ g_P \in H^1(0,T_e; H^{\frac{3}{2}}(\Gamma_P;\mathbb{R})) \cap H^2(0,T_e; H^{\frac{1}{2}}(\Gamma_P;\mathbb{R})), \quad g_V \in H^1(0,T_e; H^{-\frac{1}{2}}(\Gamma_V;\mathbb{R})), \\ g_P(0) \in H^{\frac{3}{2}}(\Gamma_P;\mathbb{R}), \quad g_{P,t}(0) \in H^{\frac{1}{2}}(\Gamma_P;\mathbb{R}), \quad g_V(0) \in H^{-\frac{1}{2}}(\Gamma_V;\mathbb{R}), \end{aligned} \tag{4.3}$$

- For the initial conditions:

$$u_0,\, u_1 \in H^1(\Omega;\mathbb{R}^3), \quad \mathrm{div}(\varepsilon(u_0)) \in L^2(\Omega;\mathbb{R}^3), \quad p_0 \in H^1(\Omega;\mathbb{R}), \quad \Delta p_0 \in L^2(\Omega;\mathbb{R}). \tag{4.4}$$

For (4.1) we assume that the inelastic deformation tensor ε^p satisfies:

$$\varepsilon^p \in H^2(0,T_e; L^2(\Omega;\mathcal{S}(3))), \quad \varepsilon^p(0) \in L^2_{\mathrm{div}}(\Omega;\mathcal{S}(3)). \tag{4.5}$$

Additionally, we require the *compatibility conditions* of the form:

$$\begin{aligned} u_0(x) &= g_D(x,0), & u_1(x) &= g_{D,t}(x,0), & x &\in \Gamma_D \\ & & p_0(x) &= g_P(x,0), & x &\in \Gamma_P \\ & & (\mathcal{D}(a\,\varepsilon(u_0(x)) - \varepsilon^p(x,0)) - \alpha p_0(x)\mathbb{I})\, n(x) &= g_N(x,0), & x &\in \Gamma_N \\ & & c\frac{\partial p_0}{\partial n}(x) &= g_V(x,0), & x &\in \Gamma_V \end{aligned} \tag{4.6}$$

The Trace Theorem implies that there exist functions $\widetilde{g}_D : \Omega \times [0, T_e] \to \mathbb{R}^3$, $\widetilde{g}_P : \Omega \times [0, T_e] \to \mathbb{R}$ such that $\gamma(\widetilde{g}_D)_{|\Gamma_D} = g_D$, $\gamma(\widetilde{g}_P)_{|\Gamma_P} = g_P$ for $t \in [0, T_e]$, where γ is the trace operator.

Due to the linearity of the equations with respect to u and p one can put $\widetilde{u}(x,t) = u(x,t) - \widetilde{g}_D(x,t)$, $\widetilde{p}(x,t) = p(x,t) - \widetilde{g}_P(x,t)$ and pass to the following equations

$$\begin{aligned} \rho\widetilde{u}_{tt} - \mathrm{div}\widetilde{T} + \alpha\nabla\widetilde{p} &= \widetilde{F} := F - \rho\widetilde{g}_{D,tt} - \alpha\nabla\widetilde{g}_P + a\,\mathrm{div}\left(\mathcal{D}(\varepsilon(\widetilde{g}_D))\right), \\ c_0\widetilde{p}_t - c\Delta\widetilde{p} + \alpha\,\mathrm{div}\widetilde{u}_t &= \widetilde{f} := f - c_0\widetilde{g}_{P,t} - \alpha\,\mathrm{div}(\widetilde{g}_{D,t}) + c\Delta\widetilde{g}_P, \\ \widetilde{T} &= \mathcal{D}\left(a\,\varepsilon(\widetilde{u}) - \varepsilon^p\right). \end{aligned} \tag{4.7}$$

with the initial-boundary data:

$$\begin{array}{rcll} \widetilde{u}(x,0) &=& \widetilde{u}_0(x) := u_0(x) - \widetilde{g}_D(x,0), & x \in \Omega, \\ \widetilde{u}_t(x,0) &=& \widetilde{u}_1(x) := u_1(x) - \widetilde{g}_{D,t}(x,0), & x \in \Omega, \\ \widetilde{p}(x,0) &=& \widetilde{p}_0(x) := p_0(x) - \widetilde{g}_P(x,0), & x \in \Omega, \\ \widetilde{u}(x,t) &=& 0, & x \in \Gamma_D, \quad t \geq 0, \\ \left(\widetilde{T}(x,t) - \alpha\widetilde{p}(x,t)\mathbb{I}\right) n(x) &=& \widetilde{g}_N(x,t), & x \in \Gamma_N, \quad t \geq 0, \\ \widetilde{p}(x,t) &=& 0, & x \in \Gamma_P, \quad t \geq 0, \\ c\frac{\partial\widetilde{p}}{\partial n}(x,t) &=& \widetilde{g}_V(x,t) := g_V(x,t) - c\frac{\partial\widetilde{g}_P}{\partial n}(x,t), & x \in \Gamma_V, \quad t \geq 0, \end{array}$$

where $\widetilde{g}_N(x,t) := g_N(x,t) - (a\,\mathcal{D}(\varepsilon(\widetilde{g}_D(x,t))) - \alpha\widetilde{g}_P(x,t)\mathbb{I})n(x)$. Prior to the definition of the weak solution to (4.1) with the conditions (1.2) and (1.3) (except the initial condition for ε^p) we define the following subspaces of H^1:

$$\mathbb{V} := \left\{v \in H^1(\Omega;\mathbb{R}^3) : v = 0 \text{ on } \Gamma_D\right\}, \quad \mathbb{W} := \left\{w \in H^1(\Omega;\mathbb{R}) : v = 0 \text{ on } \Gamma_P\right\}.$$

Definition 4.1 (Weak solution). We say that the pair (u,p) is the weak solution of (4.1) with the initial-boundary conditions (1.2)-(1.3) (except $(1.3)_4$) if the pair $(\widetilde{u},\widetilde{p})$ such that

$$u = \widetilde{u} + \widetilde{g}_D, \quad p = \widetilde{p} + \widetilde{g}_P$$

$$(\widetilde{u},\widetilde{p}) \in W^{1,\infty}(0,T_e;H^1(\Omega;\mathbb{R}^3)) \times H^1(0,T_e;H^1(\Omega;\mathbb{R})), \quad \widetilde{u}_{tt} \in L^\infty(0,T_e;L^2(\Omega;\mathbb{R}^3)),$$

satisfy for a.e. $t \in [0,T_e]$ the following equations

$$\rho\int_\Omega \widetilde{u}_{tt} v\mathrm{d}x + \int_\Omega \mathcal{D}(a\varepsilon(\widetilde{u}) - \varepsilon^p)\varepsilon(v)\mathrm{d}x - \alpha\int_\Omega \widetilde{p}\,\mathrm{div}_x v\mathrm{d}x = \int_\Omega \widetilde{F} v\mathrm{d}x + \int_{\Gamma_N} \widetilde{g}_N v\mathrm{d}S(x), \ \forall\, v \in \mathbb{V}, \tag{4.8}$$

$$c_0\int_\Omega \widetilde{p}_t w\mathrm{d}x + c\int_\Omega \nabla_x\widetilde{p}\nabla_x w\mathrm{d}x + \alpha\int_\Omega \mathrm{div}\widetilde{u}_t w\mathrm{d}x = \int_\Omega \widetilde{f} w\mathrm{d}x + \int_{\Gamma_V} \widetilde{g}_V w\mathrm{d}S(x), \ \forall\, w \in \mathbb{W} \tag{4.9}$$

and $\widetilde{u}(x,0) = \widetilde{u}_0(x)$, $\quad \widetilde{u}_t(x,0) = \widetilde{u}_1(x)$, $\quad \widetilde{p}(x,0) = \widetilde{p}_0(x)$.

In the Definition 4.1 we formally write the boundary integrals which should be understood as the duality pairings between spaces $H^{\frac{1}{2}}$ and $H^{-\frac{1}{2}}$.

In the proceeding discussion the wave symbols are omitted for simplicity.

Due to the separability of the spaces $\mathbb{V}$ and $\mathbb{W}$ one can find the bases $\{v^k\} \subset \mathbb{V}$ and $\{w^k\} \subset \mathbb{W}$ such that both of them are orthonormal in L^2. Furthermore one can assume that v^k and w^k are smooth in the interior of Ω.

We look for the approximated solutions

$$u^m : [0,T] \to \mathbb{V}_m := \mathrm{Span}\{v^1,\ldots,v^m\}; \qquad u^m(t) := \sum_{k=1}^m g_m^k(t)v^k,$$

$$p^m : [0,T] \to \mathbb{W}_m := \mathrm{Span}\{w^1,\ldots,w^m\}; \qquad p^m(t) := \sum_{k=1}^m \widetilde{g}_m^k(t)w^k.$$

Fix the initial conditions $u^m(0)$, $u_t^m(0) \in \mathbb{V}_m$, $p^m(0) \in \mathbb{W}_m$ such that:

- $u^m(0) \to u_0$ for $m \to \infty$ in $H^1(\Omega;\mathbb{R}^3)$,

- $u_t^m(0) \to u_1$ for $m \to \infty$ in $H^1(\Omega;\mathbb{R}^3)$,
- $p^m(0) \to p_0$ for $m \to \infty$ in $H^1(\Omega;\mathbb{R})$,
- $\operatorname{div}(\varepsilon(u^m(0))) \to \operatorname{div}(\varepsilon(u_0))$ for $m \to \infty$ in $L^2(\Omega;\mathbb{R}^3)$,
- $\Delta p^m(0) \to \Delta p_0$ for $m \to \infty$ in $L^2(\Omega;\mathbb{R})$.

We now pick the sequences g_N^m, g_V^m such that

- $\{g_N^m\}_{m=1}^{\infty} \subset W^{2,\infty}(0,T_e;H^{-\frac{1}{2}}(\Gamma_N;\mathbb{R}^3))$ satisfies the compatibility condition
$$g_N^m(x,0) = (\mathcal{D}(a\,\varepsilon(u^m(x,0)) - \varepsilon^p(x,0)) - \alpha p^m(x,0)\mathbb{I})\,n(x)$$
and $g_N^m \to g_N$ for $m \to \infty$ in $W^{2,\infty}(0,T_e;H^{-\frac{1}{2}}(\Gamma_N;\mathbb{R}^3))$.
- $\{g_V^m\}_{m=1}^{\infty} \subset H^1(0,T_e;H^{-\frac{1}{2}}(\Gamma_V;\mathbb{R}))$ satisfies the compatibility condition:
$$g_V^m(x,0) = c\frac{\partial p^m}{\partial n}(x,0)$$
and $g_V^m \to g_V$ for $m \to \infty$ in $H^1(0,T_e;H^{-\frac{1}{2}}(\Gamma_V;\mathbb{R}))$.

We require that u^m, p^m satisfy for $k = 1,2,\dots,m$ and a.e. $t \in [0,T_e]$ the following equations

$$\rho\int_\Omega u_{tt}^m v^k \mathrm{d}x + \int_\Omega \mathcal{D}(a\,\varepsilon(u^m) - \varepsilon^p)\varepsilon(v^k)\mathrm{d}x - \alpha\int_\Omega p^m \operatorname{div}v^k \mathrm{d}x = \int_\Omega F v^k \mathrm{d}x + \int_{\Gamma_N} g_N^m v^k \mathrm{dS}(x), \tag{4.10}$$

$$c_0\int_\Omega p_t^m w^k \mathrm{d}x + c\int_\Omega \nabla p^m \nabla w^k \mathrm{d}x + \alpha\int_\Omega \operatorname{div}u_t^m\, w^k \mathrm{d}x = \int_\Omega f w^k \mathrm{d}x + \int_{\Gamma_V} g_V^m w^k \mathrm{dS}(x). \tag{4.11}$$

The pair (u^m, p^m) will be called the *approximated solution to (4.8)-(4.9).*

Theorem 4.2. *Suppose that the conditions (4.2)-(4.6) are satisfied. Then there exists the approximated solution to (4.8)-(4.9) for every $m \in \mathbb{N}$.*

Proof. Plugging u^m and p^m to the equations (4.10)-(4.11) yields the following system of the ODEs:

$$\rho\left(g_m^k\right)''(t) + \sum_{j=1}^m \left(g_m^j(t)\alpha_{k,j} - \widetilde{g}_m^j(t)\beta_{k,j}\right) = F_k^m(t), \tag{4.12}$$

$$c_0\left(\widetilde{g}_m^k\right)'(t) + \sum_{j=1}^m \left(\widetilde{g}_m^j(t)\gamma_{k,j} + \left(g_m^j\right)'(t)\beta_{j,k}\right) = f_k^m(t). \tag{4.13}$$

where

$$\alpha_{k,j} = a\left(\mathcal{D}(\varepsilon(v^j)),\varepsilon(v^k)\right), \quad \beta_{k,j} = \alpha\left(\operatorname{div}v^k, w^j\right), \quad \gamma_{k,j} = c\left(\nabla w^j, \nabla w^k\right)$$
$$F_k^m(t) = (F(t), v^k) + (\varepsilon^p(t),\varepsilon(v^k)) + \int_{\Gamma_N} g_N^m(t) v^k \mathrm{dS}(x), \quad f_k^m(t) = (f(t), w^k) + \int_{\Gamma_V} g_V^m(t) w^k \mathrm{dS}(x).$$

The parentheses $(\cdot,\cdot)$ denote the standard inner product in $L^2(\Omega)$ with the values either in $\mathbb{R}$ or $\mathbb{R}^3$ (depending on the context). Due to the Carathéodory's theorem one obtains the local solution to the system (4.12)-(4.13), i.e. for the fixed m and $k = 1,\dots,m$ functions $g_m^k(t)$, $\widetilde{g}_m^k(t)$ are defined on some interval $[0,T_m)$.

Next we prove that one can extend these functions up to the time T_e, i.e. $T_m = T_e$. It is sufficient to prove that there is no $k_0 \in \{1,2,\dots,m\}$ such that

$$\left|g_m^{k_0}\right| \to \infty, \qquad \text{for } t \to T_m. \tag{4.14}$$

The condition (4.14) simply means that the solution does not explode. The similar results apply to $\left(g_m^k\right)'(t)$ and $\widetilde{g}_m^k(t)$ as well.

Multiplying the kth equation of (4.12) by $\left(g_m^k\right)'(t)$ and adding them up for $k=1,\dots,m$ yields

$$\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|u_t^m\right\|_{L^2}^2+\frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{D}(\varepsilon(u^m)),\varepsilon(u^m)\right)-\alpha(\mathrm{div}u_t^m,p^m)=(F,u_t^m)+(\mathcal{D}(\varepsilon^p),\varepsilon(u_t^m))+\int\limits_{\Gamma_N}g_N^m u_t^m\mathrm{dS}(x).$$

Similarly, multiplying the kth equation of (4.13) by $\widetilde{g}_m^k(t)$ and adding them up for $k=1,\dots,m$ yields

$$\frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|p^m\right\|_{L^2}^2+c\left\|\nabla p^m\right\|_{L^2}^2+\alpha(\mathrm{div}u_t^m,p^m)=(f,p^m)+\int\limits_{\Gamma_V}g_V^m p^m\mathrm{dS}(x).$$

We now sum above equations and integrate them over $(0,t)$ for $0<t<T_m$

$$\begin{aligned}
&\rho\left\|u_t^m(t)\right\|_{L^2}^2+c_0\left\|p^m(t)\right\|_{L^2}^2+a\left(\mathcal{D}(\varepsilon(u^m(t))),\varepsilon(u^m(t))\right)+2c\int\limits_0^t\left\|\nabla p^m\right\|_{L^2}^2\mathrm{d}\tau\\
&=\rho\left\|u_t^m(0)\right\|_{L^2}^2+c_0\left\|p^m(0)\right\|_{L^2}^2+a\left(\mathcal{D}(\varepsilon(u^m(0))),\varepsilon(u^m(0))\right)+2\int\limits_0^t(F,u_t^m)\mathrm{d}\tau \qquad (4.15)\\
&+2\int\limits_0^t(f,p^m)\mathrm{d}\tau+2\int\limits_0^t(\mathcal{D}(\varepsilon^p),\varepsilon(u_t^m))\mathrm{d}\tau+2\int\limits_0^t\int\limits_{\Gamma_N}g_N^m u_t^m\mathrm{dS}(x)\mathrm{d}\tau+2\int\limits_0^t\int\limits_{\Gamma_V}g_V^m p^m\mathrm{dS}(x)\mathrm{d}\tau.
\end{aligned}$$

By using: the properties of the operator $\mathcal{D}$, the convergence of $u^m(0)\to u_0$, $u_t^m(0)\to u_1$ in $H^1(\Omega;\mathbb{R}^3)$ and $p^m(0)\to p_0$ in $H^1(\Omega;\mathbb{R})$ one obtains the following estimate (independently of m).

$$\rho\left\|u_t^m(0)\right\|_{L^2}^2+c_0\left\|p^m(0)\right\|_{L^2}^2+a\left(\mathcal{D}(\varepsilon(u^m(0))),\varepsilon(u^m(0))\right)\leqslant C.$$

Furthermore

$$\begin{aligned}
\int\limits_0^t\int\limits_{\Gamma_N}g_N^m u_t^m\mathrm{dS}(x)\mathrm{d}\tau&=-\int\limits_0^t\int\limits_{\Gamma_N}g_{N,t}^m u^m\mathrm{dS}(x)\mathrm{d}\tau+\int\limits_{\Gamma_N}g_N^m(t)u^m(t)\mathrm{dS}(x)-\int\limits_{\Gamma_N}g_N^m(0)u^m(0)\mathrm{dS}(x)\\
&\leqslant\int\limits_0^t\left\|g_{N,t}^m\right\|_{H^{-\frac12}}\left\|u^m\right\|_{H^{\frac12}}\mathrm{d}\tau+\left\|g_N^m(t)\right\|_{H^{-\frac12}}\left\|u^m(t)\right\|_{H^{\frac12}}+\left\|g_N^m(0)\right\|_{H^{-\frac12}}\left\|u^m(0)\right\|_{H^{\frac12}}.
\end{aligned}$$

Using the properties of g_N^m, the trace operator in H^1 and the standard inequalities, one can obtain the following estimate

$$\int\limits_0^t\int\limits_{\Gamma_N}g_N^m u_t^m\mathrm{dS}(x)\mathrm{d}\tau\leqslant\nu\left\|u^m\right\|_{L^\infty(H^1)}^2+C(\nu)\left(1+\left\|g_{N,t}\right\|_{L^1(H^{-\frac12})}^2+\left\|g_N\right\|_{L^\infty(H^{-\frac12})}^2\right).$$

Similarly, integrating by parts in the third integral on the right side (with respect to the time), applying the properties of the operator $\mathcal{D}$ and the standard inequalities provide the estimate

$$\int\limits_0^t\int\limits_\Omega\mathcal{D}(\varepsilon^p)\varepsilon(u_t^m)\mathrm{d}x\mathrm{d}\tau\leqslant\nu\left\|u^m\right\|_{L^\infty(H^1)}^2+C(\nu)\left(1+\left\|\varepsilon_t^p\right\|_{L^1(L^2)}^2+\left\|\varepsilon^p\right\|_{L^\infty(L^2)}^2\right).$$

To estimate the left side of (4.15) we use the properties of $\mathcal{D}$, $u^m_{|\Gamma_D}=u^m_{|\Gamma_P}=0$ and the fact that $N(u)=\left\|\varepsilon(u)\right\|_2+\int\limits_{\Gamma_D}|u|\,\mathrm{dS}(x)$ is a norm in H^1 equivalent to the standard norm in this space (see [16]). Using the weighted Schwarz and Young's inequalities (with the weight ν from the approximated solution) one obtains

$$\|u_t^m(t)\|_{L^2}^2 + \|p^m(t)\|_{L^2}^2 + \|u^m(t)\|_{H^1}^2 + \int_0^t \|p^m\|_{H^1}^2 \,\mathrm{d}\tau \le \nu\Big(\|u_t^m\|_{L^\infty(L^2)}^2 + \|p^m\|_{L^\infty(L^2)}^2$$

$$+ \|u^m\|_{L^\infty(H^1)}^2 + \int_0^t \|p^m\|_{H^1}^2 \,\mathrm{d}\tau \Big) + C(\nu)\Big(1 + \|F\|_{L^1(L^2)}^2 + \|f\|_{L^1(L^2)}^2 + \|\varepsilon^p\|_{W^{1,\infty}(L^2)}^2$$

$$+ \|g_N\|_{W^{1,\infty}(H^{-\frac{1}{2}})}^2 + \|g_V\|_{L^2(H^{-\frac{1}{2}})}^2 \Big).$$

Taking the supremum of each summand on the left side of the inequality above and putting $\nu = \frac{1}{8}$ one obtains the final estimate independent of m

$$\|u_t^m(t)\|_{L^2}^2 + \|p^m(t)\|_{L^2}^2 + \|u^m(t)\|_{H^1}^2 + \int_0^t \|p^m\|_{H^1}^2 \,\mathrm{d}\tau \leqslant C(T_e) \quad \text{for} \quad t \in [0, T_m]\,.$$

From the estimate above one obtains for every $k \in \{1, 2 \ldots, m\}$

$$\left|g_m^k(t)\right|^2 \leqslant \sum_{j=1}^m \left|g_m^j(t)\right|^2 = \|u^m(t)\|_{L^2}^2 \leqslant C(T_e) \quad \text{for} \quad t \in [0, T_m]\,.$$

One can analogously obtain the similar estimates for $(g_m^k)'(t)$ and $\widetilde{g}_m^k(t)$. □

Following the steps form the proof of the Theorem 4.2 one can obtain the first energetic estimate

$$\|u_t^m(t)\|_{L^2}^2 + \|u^m(t)\|_{H^1}^2 + \|p^m(t)\|_{L^2}^2 + \int_0^t \|p^m\|_{H^1}^2 \,\mathrm{d}\tau \leqslant C(T_e) \quad \text{for} \;\; t \in [0, T_e]\,. \tag{4.16}$$

We still require some information about u_{tt}^m, p_t^m and $\mathrm{div} u_t^m$.

Lemma 4.3 (Second Energetic Inequality). *Under the assumptions of the Theorem 4.2 the following estimate (independent of m) holds*

$$\|u_{tt}^m(t)\|_{L^2}^2 + \|u_t^m(t)\|_{H^1}^2 + \|p_t^m(t)\|_{L^2}^2 + \int_0^t \|p_t^m\|_{H^1}^2 \,\mathrm{d}\tau \leqslant C(T_e) \quad \textit{for} \;\; t \in [0, T_e]\,. \tag{4.17}$$

Proof. Firstly we differentiate (4.12)-(4.13) and multiply them by $(g_m^k)''(t)$ and $(\widetilde{g}_m^k)'(t)$ respectively. Summing up the equations for $k = 1, 2, \ldots, m$ yields

$$\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u_{tt}^m\|_{L^2}^2 + \frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{D}(\varepsilon(u_t^m)), \varepsilon(u_t^m)) + \frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|p_t^m\|_{L^2}^2 + c\,\|\nabla p_t^m\|_{L^2}^2 = (F_t, u_{tt}^m) + (f_t, p_t^m)$$

$$+ (\mathcal{D}(\varepsilon_t^p), \varepsilon(u_{tt}^m)) + \int_{\Gamma_N} g_{N,t}^m u_{tt}^m \mathrm{dS}(x) + \int_{\Gamma_V} g_{V,t}^m p_t^m \mathrm{dS}(x).$$

We integrate the inequality above over time $(0, t)$ $(0 < t < T_e)$ and do the similar estimates as in the proof of the Theorem 4.2

$$\|u_{tt}^m(t)\|_{L^2}^2 + \|u_t^m(t)\|_{H^1}^2 + \|p_t^m(t)\|_{L^2}^2 + \int_0^t \|p_t^m\|_{H^1}^2 \,\mathrm{d}\tau \leqslant C\left(\|u_{tt}^m(0)\|_{L^2}^2 + \|p_t^m(0)\|_{L^2}^2\right)$$

$$+ C(\nu)\Big(1 + \|F_t\|_{L^1(L^2)}^2 + \|f_t\|_{L^1(L^2)}^2 + \|\varepsilon_t^p\|_{L^\infty(L^2)}^2 + \|\varepsilon_{tt}^p\|_{L^1(L^2)}^2 + \|g_{N,t}\|_{L^\infty(H^{-\frac{1}{2}})}^2$$

$$+ \|g_{N,tt}\|_{L^1(H^{-\frac{1}{2}})}^2 + \|g_{V,t}\|_{L^2(H^{-\frac{1}{2}})}^2\Big) + \nu\Big(\|u_{tt}^m\|_{L^\infty(L^2)}^2 + \|p_t^m\|_{L^\infty(L^2)}^2 + \|u_t^m\|_{L^\infty(H^1)}^2$$

$$+ \int_0^t \|p_t^m\|_{H^1}^2 \,\mathrm{d}\tau\Big).$$

Choosing the sufficiently small ν and taking the supremum of the inequality above leads to the following estimate

$$\|u_{tt}^m(t)\|_{L^2}^2 + \|u_t^m(t)\|_{H^1}^2 + \|p_t^m(t)\|_{L^2}^2 + \int_0^t \|p_t^m\|_{H^1}^2 \,\mathrm{d}\tau \leqslant C(T_e) + C\left(\|u_{tt}^m(0)\|_{L^2}^2 + \|p_t^m(0)\|_{L^2}^2\right)$$

for $t \in [0, T_e]$. In order to finish the proof one has to estimate the expression $\|u_{tt}^m(0)\|_{L^2}$ and $\|p_t^m(0)\|_{L^2}$ independently of m. To estimate $\|u_{tt}^m(0)\|_{L^2}$ we multiply the kth equation of (4.12) by $(g_m^k(t))''(t)$, sum over $k = 1, 2, \ldots, m$ and plug in $t = 0$.

$$\begin{aligned}\rho\,\|u_{tt}^m(0)\|_{L^2}^2 = &\int_\Omega F(0)u_{tt}^m(0)\mathrm{d}x - \int_\Omega \mathcal{D}(a\varepsilon(u^m(0)) - \varepsilon^p(0))\varepsilon(u_{tt}^m(0))\mathrm{d}x + \alpha\int_\Omega p^m(0)\mathrm{div}u_{tt}^m(0)\mathrm{d}x\\ &+ \int_{\Gamma_N} g_N^m(0)u_{tt}^m(0)\mathrm{dS}(x).\end{aligned}$$

An integration by parts gives

$$\begin{aligned}-&\int_\Omega \mathcal{D}(a\varepsilon(u^m(0)) - \varepsilon^p(0))\varepsilon(u_{tt}^m(0))\mathrm{d}x + \alpha\int_\Omega p^m(0)\mathrm{div}u_{tt}^m(0)\mathrm{d}x\\ &= \int_\Omega \mathrm{div}\left(\mathcal{D}(a\varepsilon(u^m(0)) - \varepsilon^p(0))\right)u_{tt}^m(0)\mathrm{d}x - \alpha\int_\Omega \nabla p^m(0)u_{tt}^m(0)\mathrm{d}x\\ &\quad - \int_{\Gamma N} \underbrace{\left(\mathcal{D}(a\varepsilon(u^m(0)) - \varepsilon^p(0)) - \alpha p^m(0)\mathbb{I}\right)n}_{g_N^m(0)}\, u_{tt}^m(0)\mathrm{dS}(x).\end{aligned}$$

Hence

$$\begin{aligned}\rho\,\|u_{tt}^m(0)\|_{L^2}^2 \leqslant{}& \|F(0)\|_{L^2}\|u_{tt}^m(0)\|_{L^2} + \|\mathrm{div}\left(\mathcal{D}(a\varepsilon(u^m(0)) - \varepsilon^p(0))\right)\|_{L^2}\|u_{tt}^m(0)\|_{L^2}\\ &+ \alpha\|\nabla p^m(0)\|_{L^2}\|u_{tt}^m(0)\|_{L^2} \leqslant C\|u_{tt}^m(0)\|_{L^2}.\end{aligned}$$

Thus we obtain the estimate $\|u_{tt}^m(0)\|_{L^2} \leqslant C$ independent of m.
In order to estimate $\|p_t^m(0)\|_{L^2}$ we multiply the kth equation of (4.13) by $(\widetilde{g}_m^k(t))'(t)$ and sum up for $k = 1, 2, \ldots, m$

$$\begin{aligned}c_0\|p_t^m(0)\|_{L^2}^2 = &\int_\Omega f(0)p_t^m(0)\mathrm{d}x - c\int_\Omega \nabla p^m(0)\nabla p_t^m(0)\mathrm{d}x - \alpha\int_\Omega p_t^m(0)\mathrm{div}u_t^m(0)\mathrm{d}x\\ &+ \int_{\Gamma_V} g_V^m(0)p_t^m(0)\mathrm{dS}(x).\end{aligned}$$

Following the same procedure (an integration by parts on the right side) one discards the boundary integral over Γ_V. Hence we obtain

$$\begin{aligned}c_0\|p_t^m(0)\|_{L^2}^2 &\leqslant \|f(0)\|_{L^2}\|p_t^m(0)\|_{L^2} + c\|\Delta p^m(0)\|_{L^2}\|p_t^m(0)\|_{L^2} + \alpha\|\mathrm{div}u_t^m(0)\|_{L^2}\|p_t^m(0)\|_{L^2}\\ &\leqslant C\|p_t^m(0)\|_{L^2}.\end{aligned}$$

It implies that the estimate for $\|p_t^m(0)\|_{L^2}$ is independent of m and ends the proof of the lemma. □

We now have a sufficient information to conclude the existence of the solutions to (4.8)-(4.9).

Theorem 4.4 (Existence and uniqueness of the solution).
Suppose that the conditions (4.2)-(4.6) are satisfied. Then there exists the unique weak solution (u, p) *of (4.1) with the initial-boundary conditions (1.2)-(1.3) (except* $(1.3)_4$*) such that*

$$(u, p) \in W^{1,\infty}(0, T_e; H^1(\Omega; \mathbb{R}^3)) \times H^1(0, T_e; H^1(\Omega; \mathbb{R})),$$
$$u_{tt} \in L^\infty(0, T_e; L^2(\Omega; \mathbb{R}^3)), \quad p \in W^{1,\infty}(0, T_e; L^2(\Omega; \mathbb{R})).$$

Proof. In virtue of the energetic inequalities one can conclude that there exist the subsequences $\{\widetilde{u}^{m_k}\}$, $\{\widetilde{p}^{m_k}\}$ such that

- $\widetilde{u}^{m_k} \rightharpoonup \widetilde{u}$ in $H^1(0,T_e;H^1(\Omega;\mathbb{R}^3))$,
- $\widetilde{u}^{m_k}_{tt} \rightharpoonup \widetilde{u}_{tt}$ in $L^2(0,T_e;L^2(\Omega;\mathbb{R}^3))$,
- $\widetilde{p}^{m_k} \rightharpoonup \widetilde{p}$ in $H^1(0,T_e;H^1(\Omega;\mathbb{R}))$.

Combined with an information about the sequences $\widetilde{g}^m_N$ and $\widetilde{g}^m_V$ it allows passing to the limit in the equations (4.10)-(4.11). Due to the standard procedure one obtains that the pair $(\widetilde{u},\widetilde{p})$ satisfy the equations (4.8)-(4.9), thus the translation $(\widetilde{u}+\widetilde{g}_D,\widetilde{p}+\widetilde{g}_P)$ is the weak solution of (4.1).
The energetic inequalities provide us with the expected regularity of the solutions.

It remains to prove the uniqueness. Suppose that there exist two weak solutions, (u^1,p^1) and (u^2,p^2). Denote $u:=u^1-u^2$ and $p:=p^1-p^2$.
We fix $t\in[0,T_e]$ such that $u_t(t)\in\mathbb{V}$ and $p(t)\in\mathbb{W}$ and test the weak formulation (4.8)-(4.9) with $v=u_t(t)$ and $w=p(t)$ respectively. Due to the regularity of the solutions one can do it for a.e. $t\in[0,T_e]$

Due to the linearity we can subtract equations related to each solution and obtain

$$\begin{aligned}\rho(u_{tt},u_t)+a(\mathcal{D}(\varepsilon(u)),\varepsilon(u_t))-\alpha(p,\mathrm{div}u_t)&=0,\\ c_0(p_t,p)+c\left\|\nabla p\right\|^2_{L^2}+\alpha(\mathrm{div}u_t,p)&=0.\end{aligned}$$

Adding up the equations above and integrating over time $(0,t)$ yields

$$\rho\left\|u_t(t)\right\|^2_{L^2}+a(\mathcal{D}(\varepsilon(u(t))),\varepsilon(u(t)))+c_0\left\|p(t)\right\|^2_{L^2}+2c\int_0^t\left\|\nabla p\right\|^2_{L^2}\,\mathrm{d}\tau=0.$$

Hence $\|u\|_{H^1}=\|p\|_{L^2}=0$ for a.e. $t\in[0,T_e]$. Therefore $u\equiv p\equiv 0$ a.e. in $\Omega\times[0,T_e]$. □

Henceforth we focus on the globally Lipschitz constitutive equation. Namely

$$\begin{aligned}\rho u_{tt}(x,t)-\mathrm{div}_x T(x,t)+\alpha\nabla_x p(x,t)&=F(x,t),\\ c_0p_t(x,t)-c\Delta_x p(x,t)+\alpha\,\mathrm{div}_x u_t(x,t)&=f(x,t),\\ T(x,t)&=\mathcal{D}\left(a\,\varepsilon(u(x,t))-\varepsilon^p(x,t)\right),\\ \widehat{T}(x,t)&=\mathcal{D}\left(\varepsilon(u(x,t))-\varepsilon^p(x,t)\right),\\ \varepsilon^p_t(x,t)&=G\left(\widehat{T}(x,t)\right),\end{aligned}\tag{4.18}$$

where $G:\mathcal{S}(3)\to\mathcal{S}(3)$ is a globally Lipschitz constitutive function, i.e. there exists $L>0$ such that for all T_1, $T_2\in\mathcal{S}(3)$ the following inequality holds

$$|G(T_1)-G(T_2)|\leqslant L\,|T_1-T_2|\,.$$

The model (4.18) is equipped with the same initial-boundary conditions as (1.1).

Definition 4.5 (Weak solution). We say that (u,p,ε^p) is the weak solution of (4.18) with the initial-boundary conditions (1.2)-(1.3) if (u,p) is the weak solution of $(4.18)_{1-3}$ in virtue of the Definition 4.1 and $\varepsilon^p\in W^{1,\infty}(0,T_e;L^2(\Omega;\mathcal{S}(3)))$ satisfies the equation $(4.18)_4$ with the initial condition ε^p_0.

Theorem 4.6 (Existence and uniqueness of the solution to (4.18)).
Suppose that conditions (4.2)-(4.4), (4.6) and $\varepsilon^p_0\in L^2_{\mathrm{div}}(\Omega;\mathcal{S}(3))$ are satisfied. Then there exists a unique weak solution (u,p,ε^p) of (4.18) with initial-boundary conditions (1.2)-(1.3) such that:

$$\begin{gathered}(u,p)\in W^{1,\infty}(0,T_e;H^1(\Omega;\mathbb{R}^3))\times H^1(0,T_e;H^1(\Omega;\mathbb{R})),\quad \varepsilon^p\in H^2(0,T_e;L^2(\Omega;\mathcal{S}(3)))\\ u_{tt}\in L^\infty(0,T_e;L^2(\Omega;\mathbb{R}^3)),\quad p\in W^{1,\infty}(0,T_e;L^2(\Omega;\mathbb{R})).\end{gathered}$$

Proof. The nonlinearity in our setting is globally Lipschitz, therefore it is a natural attempt to apply the Banach Fixed Point Theorem to prove the existence and uniqueness of the solution. We begin with a definition of a space $X:=\left\{\varphi\in H^2(0,T_e;L^2(\Omega;\mathcal{S}(3))):\varphi(x,0)=\varepsilon^p_0(x)\right\}$ and an operator $S:X\to X$ given as follows. For $z\in X$ we consider the system

$$\begin{aligned}\rho u^z_{tt}(x,t)-\mathrm{div}_x\mathcal{D}\left(a\,\varepsilon(u^z(x,t))-z(x,t)\right)+\alpha\nabla_x p^z(x,t)&=F(x,t),\\ c_0p^z_t(x,t)-c\Delta_x p^z(x,t)+\alpha\,\mathrm{div}_x u^z_t(x,t)&=f(x,t).\end{aligned}\tag{4.19}$$

Due to the Theorem 4.4 there exists a unique weak solution to (4.19). Now consider the ODE

$$\begin{cases} \varepsilon_t^{p,z}(x,t) = G(\mathcal{D}\left(\varepsilon(u^z(x,t)) - \varepsilon^{p,z}(x,t)\right)), \\ \varepsilon^{p,z}(x,0) = \varepsilon_0^p(x). \end{cases} \tag{4.20}$$

Therefore we define the operator S as $Sz = \varepsilon^{p,z}$
Obviously the equation (4.20) has a unique solution $\varepsilon^{p,z}$ in the space $C^1([0,T_e]\,;L^2(\Omega;\mathcal{S}(3)))$. Furthermore $(Sz)_t$ is a globally Lipschitz map with respect to the variable t. Indeed, we take $t_1, t_2 \in [0,t]$ for $t \in [0,T_e]$

$$\begin{aligned} \|(Sz)_t(t_1) - (Sz)_t(t_2)\|_{L^2}^2 &= \int_\Omega |G(\mathcal{D}\left(\varepsilon(u^z(t_1)) - Sz(t_1)\right)) - G(\mathcal{D}\left(\varepsilon(u^z(t_2)) - Sz(t_2)\right))|^2 \,\mathrm{d}x \\ &\leqslant C \int_\Omega |\varepsilon(u^z(t_1) - u^z(t_2))|^2 + |Sz(t_1) - Sz(t_2)|^2 \,\mathrm{d}x \\ &\leqslant C \left(\|\varepsilon(u_t^z)\|_{L^\infty(0,t;L^2)}^2 + \|(Sz)_t\|_{L^\infty(0,t;L^2)}^2 \right) |t_1 - t_2|^2 . \end{aligned}$$

Thanks to the Rademacher's Theorem the function $(Sz)_t$ is differentiable almost everywhere in $(0,t)$. Moreover the Lipschitz constant is global on this time interval so

$$\|(Sz)_{tt}\|_{L^\infty(0,t;L^2)}^2 \leqslant C_L \left(\|\varepsilon(u_t^z)\|_{L^\infty(0,t;L^2)}^2 + \|(Sz)_t\|_{L^\infty(0,t;L^2)}^2 \right). \tag{4.21}$$

Therefore $Sz \in X$.

Now, observe that (u,p,ε^p) is a weak solution to (4.18) with the regularity $\varepsilon^p \in H^2(0,T_e;L^2(\Omega;\mathcal{S}(3)))$ if and only if ε^p is a fixed point for the operator S.
Unfortunately we cannot prove that the operator S is a contraction in the standard norm of $H^2(L^2)$ so the Banach Fixed Point Theorem cannot be applied directly. Nevertheless this issue can be resolved. Firstly we prove that if $0 < T \leq T_e$ is sufficiently small then for the arbitrary z, $w \in X$ we have

$$\|Sz - Sw\|_{C^1([0,T];L^2)} \leqslant \frac{1}{2} \|z - w\|_{C^1([0,T];L^2)} .$$

Indeed

$$\begin{aligned} \|Sz - Sw\|_{C^1([0,T];L^2)} &= \sup_{[0,T]} \left\| \int_0^t G(\mathcal{D}\left(\varepsilon(u^z) - Sz\right)) - G(\mathcal{D}\left(\varepsilon(u^w) - Sw\right))\mathrm{d}\tau \right\|_{L^2} \\ &+ \sup_{[0,T]} \|G(\mathcal{D}\left(\varepsilon(u^z) - Sz\right)) - G(\mathcal{D}\left(\varepsilon(u^w) - Sw\right))\|_{L^2} \\ &\leqslant C \int_0^T \|\varepsilon(u^z - u^w)\|_{L^2} + \|Sz - Sw\|_{L^2} \,\mathrm{d}\tau + C \sup_{[0,T]} \left(\|\varepsilon(u^z - u^w)\|_{L^2} + \|Sz - Sw\|_{L^2}\right) \\ & C\left(T_e\right) \sup_{[0,T]} \|\varepsilon(u^z - u^w)\|_{L^2} + C\left(T_e\right) T \|Sz - Sw\|_{C^1([0,T];L^2)} . \end{aligned}$$

The constant $C\left(T_e\right)$ is independent of T.
To obtain this inequality we used the Poincaré inequality in the form

$$\|Sz(t) - Sw(t)\|_{L^2} \leqslant T \|(Sz)_t(t) - (Sw)_t(t)\|_{L^2} , \quad \forall t \in [0,T]$$

which holds since Sz, $Sw \in X$ and $Sz(0) = Sw(0) = \varepsilon_0^p$.
To end this part of the proof it is enough to show that

$$\sup_{[0,T]} \|\varepsilon(u^z) - \varepsilon(u^w)\|_{L^2} \leqslant \widetilde{C} T \|z - w\|_{C^1([0,T];L^2)} , \quad \text{for some } \widetilde{C} > 1, \text{ independently of } T. \tag{4.22}$$

In order to get that we write the weak formulation for $\varepsilon(u^z)$ and $\varepsilon(u^w)$. Obviously $u^z - u^w = \widetilde{u}^z - \widetilde{u}^w$, so henceforth we shall skip tildas for clarity. We have

$$\rho(u_{tt}^i, v) + a(\mathcal{D}(\varepsilon(u^i)), \varepsilon(v)) - \alpha(p^i, \mathrm{div} v) = (F, v) + \int_{\Gamma_N} g_N v \mathrm{dS}(x) + (\mathcal{D}(i), \varepsilon(v)), \quad \forall v \in \mathbb{V},$$

$$c_0(p_t^i, w) + c(\nabla p^i, \nabla w) + \alpha(\mathrm{div} u_t^i, w) = (f, w) + \int_{\Gamma_V} g_V w \mathrm{dS}(x), \quad \forall w \in \mathbb{W},$$

where $i \in \{z, w\}$. Subtracting equations for $i = z$ and $i = w$, taking $v = (u^z - u^w)_t$, $w = p^z - p^w$ and then summing them up gives

$$\frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|(u^z - u^w)_t\right\|_{L^2}^2 + \frac{a}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\mathcal{D}(\varepsilon(u^z - u^w)), \varepsilon(u^z - u^w)) + \frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|p^z - p^w\right\|_{L^2}^2 \\ + c\left\|\nabla(p^z - p^w)\right\|_{L^2}^2 = (\mathcal{D}(z - w), \varepsilon(u_t^z - u_t^w)).$$

Integrating over time $(0, t)$ for $t \in (0, T)$ and using properties of the operator $\mathcal{D}$ yields

$$\left\|\varepsilon(u^z(t) - u^w(t))\right\|_{L^2}^2 \leqslant C\int_0^t (\mathcal{D}(z - w), \varepsilon(u_t^z - u_t^w))\mathrm{d}\tau = C(\mathcal{D}(z(t) - w(t)), \varepsilon(u^z(t) - u^w(t))) \\ -C\int_0^t (\mathcal{D}(z_t - w_t), \varepsilon(u^z - u^w))\mathrm{d}\tau \leqslant C\left(\sup_{[0,T]}\left\|z - w\right\|_{L^2}^2 + T^2\sup_{[0,T]}\left\|z_t - w_t\right\|_{L^2}^2\right) \\ +\frac{1}{2}\sup_{[0,T]}\left\|\varepsilon(u^z - u^w)\right\|_{L^2}^2 \leqslant CT(T_e + 1)\left\|z - w\right\|_{C^1([0,T];L^2)}^2 + \frac{1}{2}\sup_{[0,T]}\left\|\varepsilon(u^z - u^w)\right\|_{L^2}^2$$

Where, once again we used the Poincaré Inequality:

$$\left\|z(t) - w(t)\right\|_{L^2} \leqslant T\left\|z_t(t) - w_t(t)\right\|_{L^2}, \quad \forall t \in [0, T]$$

which is valid since z, $w \in X$ and $z(0) = w(0) = \varepsilon_0^p$.
After taking a supremum we obtain the expected inequality (4.22). Finally

$$\left\|Sz - Sw\right\|_{C^1([0,T];L^2(\Omega;\mathcal{S}(3)))} \leqslant C\,(T_e)\,\widetilde{C}T\left(\left\|z - w\right\|_{C^1([0,T];L^2)} + \left\|Sz - Sw\right\|_{C^1([0,T];L^2)}\right).$$

Now for every $0 < T \leqslant T_0 := \frac{1}{3C(T_e)\widetilde{C}}$ we obtain

$$\left\|Sz - Sw\right\|_{C^1([0,T];L^2)} \leqslant \frac{1}{2}\left\|z - w\right\|_{C^1([0,T];L^2)}. \tag{4.23}$$

In the second step of the proof we will define a sequence $\{z_n\}_{n=0}^{\infty} \subset X$ such that (up to a subsequence):

$$z_n \to \zeta \text{ in space } C^1([0, T_0]; L^2(\Omega; \mathcal{S}(3))), \quad z_n \rightharpoonup \zeta \text{ in } H^2(0, T_0; L^2(\Omega; \mathcal{S}(3))),$$

We define a sequence z_n in a standard way by taking the arbitrary $z_0 \in X$ and putting $z_{n+1} := Sz_n$. As in the proof of the Banach Fixed Point Theorem it is easy to obtain from (4.23) that

$$\left\|z_{n+1} - z_n\right\|_{C^1([0,T_0];L^2)} \leqslant \left(\frac{1}{2}\right)^n\left\|z_1 - z_0\right\|_{C^1([0,T_0];L^2)}.$$

Hence z_n is a Cauchy sequence in $C^1([0, T_0]; L^2)$ and has a limit ζ in this space.
Now we show that $\left\|z_{n,tt}\right\|_{L^2(0,T_0;L^2)} \leqslant C$ independently of n. Then (up to a subsequence) we obtain that $z_n \rightharpoonup \zeta$ in $H^2(0, T_0; L^2(\Omega; \mathcal{S}(3)))$ (so ζ is also an element of this space).
Since z_n is a Cauchy sequence then it is bounded by some constant C_0, i.e.

$$\left\|Sz_n\right\|_{C^1([0,T_0];L^2)} = \left\|z_{n+1}\right\|_{C^1([0,T_0];L^2)} \leqslant C_0, \tag{4.24}$$

where $C_0 > 0$ is independent of n (but it depends on whole sequence z_n). Using (4.21) we obtain

$$\left\|(Sz_n)_{tt}\right\|_{L^\infty(0,t;L^2)}^2 \leqslant C_L C_0 + C_L\left\|\varepsilon(u_t^{z_n})\right\|_{L^\infty(0,t;L^2)}^2, \quad \forall t \in [0, T_0]. \tag{4.25}$$

The main goal now is to obtain the inequality of the form

$$\left\|\varepsilon(u_t^{z_n})\right\|_{L^\infty(0,t;L^2)}^2 \leqslant C(C_0) + C(C_0)\int_0^t \left\|z_{n,tt}\right\|_{L^2}^2\mathrm{d}\tau, \quad \forall t \in [0, T_0], \tag{4.26}$$

where $C(C_0)$ depends only on data and C_0. Let us assume for a moment that (4.26) holds.
Then from (4.25) and (4.26) one can obtain for every $n \in \mathbb{N}$

$$\left\|z_{n+1,tt}\right\|_{L^\infty(0,t;L^2)}^2 \leqslant C_1 + C_1\int_0^t \left\|z_{n,tt}\right\|_{L^2}^2\mathrm{d}\tau, \quad \forall t \in [0, T_0]. \tag{4.27}$$

From (4.27) we also get (for every $\widetilde{T}_0 \in [0, T_0]$)

$$\int_0^t \|z_{n+1,tt}\|_{L^2}^2 \, \mathrm{d}\tau \leqslant \int_0^t C_1 + C_1 \int_0^\tau \|z_{n,tt}\|_{L^2}^2 \, \mathrm{d}s\mathrm{d}\tau \leqslant C_1\widetilde{T}_0 + C_1\widetilde{T}_0 \int_0^t \|z_{n,tt}\|_{L^2}^2 \, \mathrm{d}s, \quad \forall t \in [0, \widetilde{T}_0].$$

In the same manner one can obtain the inequality for every $n \geq 1$:

$$\|z_{n,tt}\|_{L^\infty(0,t;L^2)}^2 \leqslant \frac{1}{\widetilde{T}_0} \left[\sum_{i=1}^n \left(C_1\widetilde{T}_0\right)^i + \left(C_1\widetilde{T}_0\right)^n \int_0^{T_e} \|z_{0,tt}\|_{L^2}^2 \, \mathrm{d}\tau \right], \quad \forall t \in [0, \widetilde{T}_0]. \tag{4.28}$$

It suffices to take $\widetilde{T}_0 := \min\left\{T_0, \frac{1}{2C_1}\right\}$. For such $\widetilde{T}_0$ we obtain from (4.28) that

$$\|z_{n,tt}\|_{L^\infty(0,\widetilde{T}_0;L^2)}^2 \leqslant \frac{1}{\widetilde{T}_0} \left[1 + \|z_{0,tt}\|_{L^2(0,T_e;L^2)}^2\right] =: \frac{P}{\widetilde{T}_0}.$$

If $\widetilde{T}_0 = T_0$ then it is what had to be proven. If $\widetilde{T}_0 < T_0$ then from (4.27) one can easily derive the following inequality

$$\|z_{n+1,tt}\|_{L^\infty(\widetilde{T}_0,t;L^2)}^2 \leqslant (1+P)\, C_1 + C_1 \int_{\widetilde{T}_0}^t \|z_{n,tt}\|_{L^2}^2 \, \mathrm{d}\tau, \quad \forall t \in \left[\widetilde{T}_0, \max\left\{2\widetilde{T}_0, T_0\right\}\right] \tag{4.29}$$

which leads to

$$\begin{aligned}\|z_{n,tt}\|_{L^\infty\left(\widetilde{T}_0,\max\left\{2\widetilde{T}_0,T_0\right\};L^2\right)}^2 &\leqslant \frac{1}{\widetilde{T}_0} \left[(1+P) \sum_{i=1}^n \left(C_1\widetilde{T}_0\right)^i + \left(C_1\widetilde{T}_0\right)^n \int_0^{T_e} \|z_{0,tt}\|_{L^2}^2 \, \mathrm{d}\tau \right] \\ &\leqslant \frac{1}{\widetilde{T}_0} \left[1 + P + \int_0^{T_e} \|z_{0,tt}\|_{L^2}^2 \, \mathrm{d}\tau \right] \leqslant \frac{2P}{\widetilde{T}_0},\end{aligned}$$

After finitely many steps one can obtain that $\|z_{n,tt}\|_{L^\infty(0,T_0;L^2)}$ is bounded.

The only thing left in this step is to prove the inequality (4.26).
For the clarity, henceforth we identify $u^n = u^{z_n}$ and $p^n = p^{z_n}$. For almost every $t \in (0, T_e)$ one can write the weak formulation for (u^n, p^n). We take such t and denote:

$$\left(u_h^n(t), p_h^n(t), z_{n_h}(t)\right) := \left(u^n(t+h), p^n(t+h), z_n(t+h)\right),$$

where $h > 0$ is a fixed, sufficiently small constant such that the weak formulation for (u_h^n, p_h^n) can be written. We also apply the similar notation for functions $F_h, f_h, g_{N_h}, g_{V_h}$.
Note that

$$\|\varepsilon\left(u_t^n\right)\|_{L^\infty(0,t;L^2)}^2 \leqslant C\, \|\widetilde{u}_t^n\|_{L^\infty(0,t;H^1)}^2 + \|\varepsilon\left(\widetilde{g}_{D,t}\right)\|_{L^\infty(0,T_e;L^2)}^2. \tag{4.30}$$

Because of (4.30) to prove the inequality (4.26) it suffices to get

$$\|\widetilde{u}_t^n\|_{L^\infty(0,t;H^1)}^2 \leqslant C + C \int_0^t \|z_{n,tt}\|_{L^2}^2 \, \mathrm{d}\tau, \quad \forall t \in [0, T_0]. \tag{4.31}$$

Once again we shall skip tildas in the weak formulation for the clarity.
We write the weak formulations of (4.19) with $z = z_n$ in the time $t + h$ and the time t, by substracting them we obtain

$$\begin{aligned}\rho\left(u_{h,tt}^n - u_{tt}^n, v\right) &+ a\left(\mathcal{D}\left(\varepsilon\left(u_h^n - u^n\right)\right), \varepsilon(v)\right) - \alpha\left(p_h^n - p^n, \mathrm{div} v\right) \\ &= \left(F_h - F, v\right) + \int_{\Gamma_N} \left(g_{N_h} - g_N\right) v \mathrm{dS}(x) + \left(\mathcal{D}(z_{n_h} - z_n), \varepsilon(v)\right),\end{aligned}$$

$$c_0\left(p_{h,t}^n - p_t^n, w\right) + c\left(\nabla\left(p_h^n - p^n\right), \nabla w\right) + \alpha\left(\mathrm{div}\left(u_{h,t}^n - u_t^n\right), w\right) = \left(f_h - f, w\right) + \int_{\Gamma_V} \left(g_{V_h} - g_V\right) w \mathrm{dS}(x),$$

Taking $v = u_{h,t}^n - u_t^n$ and $w = p_h^n - p^n$ and summing them up gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\rho}{2}\left\|u_{h,t}^n - u_t^n\right\|_{L^2}^2 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{a}{2}\left(\mathcal{D}\left(\varepsilon\left(u_h^n - u^n\right)\right), \varepsilon\left(u_h^n - u^n\right)\right) + \frac{\mathrm{d}}{\mathrm{d}t}\frac{c_0}{2}\left\|p_h^n - p^n\right\|_{L^2}^2 + c\left\|\nabla\left(p_h^n - p^n\right)\right\|_{L^2}^2$$
$$= \left(F_h - F, u_{h,t}^n - u_t^n\right) + \left(f_h - f, p_h^n - p^n\right) + \int\limits_{\Gamma_N}\left(g_{N_h} - g_N\right)\left(u_{h,t}^n - u_t^n\right)\mathrm{dS}(x)$$
$$+ \left(\mathcal{D}(z_{n_h} - z_n), \varepsilon(u_{h,t}^n - u_t^n)\right) + \int\limits_{\Gamma_V}\left(g_{V_h} - g_V\right)\left(p_h^n - p^n\right)\mathrm{dS}(x),$$

This equality holds for almost every time in the interval $[0, T_e - h]$. By integrating it over $(0,t)$, where $t \in [0, T_e - h]$, using properties of the operator $\mathcal{D}$ and the standard inequalities and the equivalence of norms one obtains

$$\left\|u_{h,t}^n(t) - u_t^n(t)\ \right\|_{L^2}^2 + \left\|u_h^n(t) - u^n(t)\right\|_{H^1}^2 + \left\|p_h^n(t) - p^n(t)\right\|_{L^2}^2 + \int\limits_0^t \left\|\nabla\left(p_h^n - p^n\right)\right\|_{H^1}^2 \mathrm{d}\tau$$
$$\leqslant C\Big(\left\|u_{h,t}^n(0) - u_t^n(0)\right\|_{L^2}^2 + \left\|u_h^n(0) - u^n(0)\right\|_{H^1}^2 + \left\|p_h^n(0) - p^n(0)\right\|_{L^2}^2$$
$$+ \int\limits_0^t \left\|F_h - F\right\|_{L^2}\left\|u_{h,t}^n - u_t^n\right\|_{L^2}\mathrm{d}\tau + \int\limits_0^t \left\|f_h - f\right\|_{L^2}\left\|p_h^n - p^n\right\|_{L^2}\mathrm{d}\tau$$
$$+ \int\limits_0^t \left\|g_{V_h} - g_V\right\|_{H^{-\frac{1}{2}}}\left\|p_h^n - p^n\right\|_{H^1}\mathrm{d}\tau + \int\limits_0^t \left\|g_{N_h,t} - g_{N,t}\right\|_{H^{-\frac{1}{2}}}\left\|u_h^n - u^n\right\|_{H^1}\mathrm{d}\tau$$
$$+ \left\|g_{N_h}(t) - g_N(t)\right\|_{H^{-\frac{1}{2}}}\left\|u_h^n(t) - u^n(t)\right\|_{H^1} + \left\|g_{N_h}(0) - g_N(0)\right\|_{H^{-\frac{1}{2}}}\left\|u_h^n(0) - u^n(0)\right\|_{H^1}$$
$$+ \int\limits_0^t \left\|z_{n_h,t} - z_{n,t}\right\|_{L^2}\left\|u_h^n - u^n\right\|_{H^1}\mathrm{d}\tau + \left\|z_{n_h}(t) - z_n(t)\right\|_{L^2}\left\|u_h^n(t) - u^n(t)\right\|_{H^1}$$
$$+ \left\|z_{n_h}(0) - z_n(0)\right\|_{L^2}\left\|u_h^n(0) - u^n(0)\right\|_{H^1}\Big).$$

Dividing by h^2 and passing to the limit with $h \to 0^+$ yields

$$\left\|u_{tt}^n(t)\right\|_{L^2}^2 + \left\|u_t^n(t)\right\|_{H^1}^2 + \left\|p_t^n(t)\right\|_{L^2}^2 + \int\limits_0^t \left\|\nabla p_t^n\right\|_{H^1}^2 \mathrm{d}\tau \leqslant C\Big(\left\|u_{tt}^n(0)\right\|_{L^2}^2 + \left\|u_1\right\|_{H^1}^2 + \left\|p_t^n(0)\right\|_{L^2}^2$$
$$+ \int\limits_0^t \left\|F_t\right\|_{L^2}\left\|u_{tt}^n\right\|_{L^2}\mathrm{d}\tau + \int\limits_0^t \left\|f_t\right\|_{L^2}\left\|p_t^n\right\|_{L^2}\mathrm{d}\tau + \int\limits_0^t \left\|g_{V,t}\right\|_{H^{-\frac{1}{2}}}\left\|p_t^n\right\|_{H^1}\mathrm{d}\tau + \int\limits_0^t \left\|g_{N,tt}\right\|_{H^{-\frac{1}{2}}}\left\|u_t^n\right\|_{H^1}\mathrm{d}\tau$$
$$+ \left\|g_{N,t}(t)\right\|_{H^{-\frac{1}{2}}}\left\|u_t^n(t)\right\|_{H^1} + \left\|g_{N,t}(0)\right\|_{H^{-\frac{1}{2}}}\left\|u_1\right\|_{H^1} + \int\limits_0^t \left\|z_{n,tt}\right\|_{L^2}\left\|u_t^n\right\|_{H^1}\mathrm{d}\tau + \left\|z_{n,t}(t)\right\|_{L^2}\left\|u_t^n(t)\right\|_{H^1}$$
$$+ \left\|z_{n,t}(0)\right\|_{L^2}\left\|u_1\right\|_{L^2}\Big).$$

In order to estimate $\left\|u_{tt}^n(0)\right\|_{L^2}^2$ and $\left\|p_t^n(0)\right\|_{L^2}^2$ independently of n one must recall the proof of the Lemma 4.3 where the Galerkin approximation of $\left\|u_{tt}^n(0)\right\|_{L^2}$ and $\left\|p_t^n(0)\right\|_{L^2}$ satisfies estimates in the form

$$\left\|u_{tt}^{n,m}(0)\right\|_{L^2}^2 + \left\|p_t^{n,m}(0)\right\|_{L^2}^2 \leqslant C(\text{initial conditions}).$$

Which gives us the same estimate (independent of n) for the limit function $u_{tt}^n(0), p_t^n(0)$.
Now, as in the proof of the Lemma 4.3 by using standard inequalities one obtains

$$\left\|u_{tt}^n(t)\right\|_{L^2}^2 + \left\|u_t^n(t)\right\|_{H^1}^2 + \left\|p_t^n(t)\right\|_{L^2}^2 + \int\limits_0^t \left\|\nabla p_t^n\right\|_{H^1}^2 \mathrm{d}\tau \leqslant C(\nu)\left(1 + \int\limits_0^t \left\|z_{n,tt}\right\|_{L^2}^2 \mathrm{d}\tau + \left\|z_n\right\|_{C^1([0,T_1];L^2)}^2\right)$$
$$+\nu\left(\left\|u_{tt}^n\right\|_{L^\infty(0,t;L^2)}^2 + \left\|p_t^n\right\|_{L^\infty(0,t;L^2)}^2 + \int\limits_0^t \left\|p_t^n\right\|_{H^1}^2 \mathrm{d}\tau + \left\|u_t^n\right\|_{L^\infty(0,t;H^1)}^2\right),$$

where $C(\nu)$ depends only on the data, initial conditions and $\nu > 0$. Taking $\nu = \frac{1}{2}$, using (4.24) and taking a supremum over time $(0,t)$ one obtains (4.31) for $t \in [0, T_0]$. Which is the end of second part of the proof.

Since $\zeta \in H^2(0,T_0; L^2(\Omega; \mathcal{S}(3)))$ we can show in a standard way that it is a *fixed point* of the operator S (only on the time interval $[0,T_0]$). Obviously $\zeta = \lim_{n\to\infty} z_{n+1} = \lim_{n\to\infty} Sz_n$. But

$$\lim_{n\to\infty} \|Sz_n - S\zeta\|_{C^1([0,T_0];L^2)} \leqslant \frac{1}{2} \lim_{n\to\infty} \|z_n - \zeta\|_{C^1([0,T_0];L^2)} = 0.$$

Hence $S\zeta = \zeta$ for a.e. $(x,t) \in \Omega \times [0, T_0]$.
The proof of the uniqueness of a *fixed point* is also standard. We assume that $\zeta_1, \zeta_2 \in X$ are *fixed points* of S then

$$\|\zeta_1 - \zeta_2\|_{C^1([0,T_0];L^2)} = \|S\zeta_1 - S\zeta_2\|_{C^1([0,T_0];L^2)} \leqslant \frac{1}{2} \|\zeta_1 - \zeta_2\|_{C^1([0,T_0];L^2)},$$

which is true only for $\zeta_1 = \zeta_2$ for a.e. $(x,t) \in \Omega \times [0,T_0]$.

Under previous considerations one concludes that the operator S has a unique fixed point in $H^2(0,T_0; L^2(\Omega; \mathcal{S}(3)))$. If $T_0 < T_e$ then in a similar way one can extend the solution about T_0 in a unique manner by considering the space $X := \{\varphi \in H^2(0,T_e; L^2(\Omega; \mathcal{S}(3))) : \varphi(x,t) = \zeta(x,t) \text{ for } t \in [0,T_0]\}$. After the finite number of repetitions one obtains the unique fixed point for the operator S in the space $\{\varphi \in H^2(0,T_e; L^2(\Omega; \mathcal{S}(3))) : \varphi(x,0) = \varepsilon_0^p(x)\}$. This ends the proof. □

5. Coercive Approximation

The significant obstacle for the further analysis of our model (especially in the proof of the Lemma 6.3) is a presence of the non-homogeneous Dirichlet type boundary condition for the pressure p on Γ_P. To avoid this problem we consider the formal translation of (1.1) such that the pressure $p \equiv 0$ on Γ_P.

Following the notion from the previous section, $\widetilde{g}_P : \Omega \times [0,T_e] \to \mathbb{R}$ denotes a function such that

$$\gamma(\widetilde{g}_P)_{|\Gamma_P} = g_P \quad \text{for} \quad t \in [0,T_e].$$

In lieu of the initial model we, as before, discuss the following system

$$\begin{aligned} \rho u_{tt} - \mathrm{div} T + \alpha \nabla \widetilde{p} &= \overline{F} := F - \alpha \nabla \widetilde{g}_P, \\ c_0 \widetilde{p}_t - c\Delta \widetilde{p} + \alpha\, \mathrm{div} u_t &= \overline{f} := f - c_0 \widetilde{g}_{P,t} + c\Delta \widetilde{g}_P, \\ T &= \mathcal{D}\left(\varepsilon(u) - \varepsilon^p\right), \\ \varepsilon^p_t &= A(T). \end{aligned} \tag{5.1}$$

with the initial-boundary conditions

$$\begin{array}{rcll} u(x,0) &=& u_0(x), & x \in \Omega, \\ u_t(x,0) &=& u_1(x), & x \in \Omega, \\ \widetilde{p}(x,0) &=& \widetilde{p}_0(x) := p_0(x) - \widetilde{g}_P(x,0), & x \in \Omega, \\ \varepsilon^p(x,0) &=& \varepsilon_0^p(x), & x \in \Omega, \\ u(x,t) &=& g_D(x,t), & x \in \Gamma_D, \quad t \geqslant 0, \\ (T(x,t) - \alpha \widetilde{p}(x,t)\mathbb{I})\, n(x) &=& \overline{g}_N(x,t), & x \in \Gamma_N, \quad t \geqslant 0, \\ \widetilde{p}(x,t) &=& 0, & x \in \Gamma_P, \quad t \geqslant 0, \\ c\frac{\partial \widetilde{p}}{\partial n}(x,t) &=& \overline{g}_V(x,t), & x \in \Gamma_V, \quad t \geqslant 0, \end{array}$$

where $\overline{g}_N(x,t) := g_N(x,t) + \alpha \widetilde{g}_P(x,t)\mathbb{I}n(x)$, $\quad \overline{g}_V(x,t) := g_V(x,t) - c\frac{\partial \widetilde{g}_P}{\partial n}(x,t)$.

Provided that we find the solution $(u, \widetilde{p}, \varepsilon^p)$ of the system (5.1), then $(u, \widetilde{p} + \widetilde{g}_P, \varepsilon^p)$ will be the solution of (1.1). The inverse result is also true – it means that our goal is to prove the existence of the solution to the system given above.

Observe that the free energy of (5.1) is given by

$$\rho\psi(\varepsilon, \varepsilon^p)(t) = \frac{1}{2}\mathcal{D}(\varepsilon - \varepsilon^p)(\varepsilon - \varepsilon^p).$$

The energy is only a positive semi–definite quadratic form and therefore our system is *non-coercive* (for details see [1]). The lack of coercivity significantly hinders the analysis; in particular one cannot directly obtain the suitable energy estimates for the mixed-boundary case.

As a remedy we introduce a standard idea of the coercive approximation (cf. [5], [8], [12]) of (5.1) as follows

$$
\begin{aligned}
\rho u^k_{tt}(x,t) - \mathrm{div}_x T^k(x,t) + \alpha \nabla_x \widetilde{p}^k(x,t) &= \overline{F}(x,t),\\
c_0 \widetilde{p}^k_t(x,t) - c\Delta_x \widetilde{p}^k(x,t) + \alpha\, \mathrm{div}_x u^k_t(x,t) &= \overline{f}(x,t),\\
T^k(x,t) &= \mathcal{D}\left(\left(1+\frac{1}{k}\right)\varepsilon(u^k(x,t)) - \varepsilon^{p,k}(x,t)\right),\\
\widehat{T}^k(x,t) &= T^k(x,t) - \frac{1}{k}\mathcal{D}(\varepsilon(u^k(x,t))),\\
\varepsilon^{p,k}_t(x,t) &= A(\widehat{T}^k(x,t)),
\end{aligned}
\tag{5.2}
$$

where $k \geqslant 1$.

To begin, we fix k and write the free energy of (5.2)

$$
\rho\psi^k\left(\varepsilon^k, \varepsilon^{p,k}\right)(t) = \frac{1}{2}\mathcal{D}\left(\varepsilon^k - \varepsilon^{p,k}\right)\left(\varepsilon^k - \varepsilon^{p,k}\right) + \frac{1}{2k}\mathcal{D}\left(\varepsilon^k\right)\varepsilon^k.
$$

One can see that now the energy is a positive-definite quadratic form. Models with that type of energy are called *coercive*.

We intend to prove the existence of the solution to (5.2). To do that we use the results obtained in the previous section. Obviously, in this case the constitutive equation does not have to be given by the globally Lipschitz vector field. Hence, we use the Yosida approximation to A in order to work with the globally Lipschitz constitutive equation. Observe that it is sufficient to approximate the maximal-monotone operator $m(\cdot)$, i.e.

$$
A^\lambda(T) := m^\lambda(T) + l(T),
$$

where m^λ denotes the Yosida approximation of the operator m. This approximation is globally Lipschitz with $\mathrm{Lip}(m^\lambda) = 1/\lambda$ and maximal-monotone (for details see [2]).
Hence, for the fixed $k \geqslant 1$ we introduce the approximation of (5.2) as follows

$$
\begin{aligned}
\rho u^{k,\lambda}_{tt}(x,t) - \mathrm{div}_x T^{k,\lambda}(x,t) + \alpha \nabla_x \widetilde{p}^{k,\lambda}(x,t) &= \overline{F}(x,t),\\
c_0 \widetilde{p}^{k,\lambda}_t(x,t) - c\Delta_x \widetilde{p}^{k,\lambda}(x,t) + \alpha\, \mathrm{div}_x u^{k,\lambda}_t(x,t) &= \overline{f}(x,t),\\
T^{k,\lambda}(x,t) &= \mathcal{D}\left(\left(1+\frac{1}{k}\right)\varepsilon(u^{k,\lambda}(x,t)) - \varepsilon^{p,k,\lambda}(x,t)\right),\\
\widehat{T}^{k,\lambda}(x,t) &= T^{k,\lambda}(x,t) - \frac{1}{k}\mathcal{D}(\varepsilon(u^{k,\lambda}(x,t))),\\
\varepsilon^{p,k,\lambda}_t(x,t) &= A^\lambda(\widehat{T}^{k,\lambda}(x,t)),
\end{aligned}
\tag{5.3}
$$

where $\lambda > 0$.

The system (5.3) is equipped with the same conditions as in the Theorem 4.6.

Therefore, for $k \geqslant 1$ and $\lambda > 0$ in virtue of the Theorem 4.6 there exists a unique weak solution $(u^{k,\lambda}, \widetilde{p}^{k,\lambda}, \varepsilon^{p,k,\lambda})$ of (5.3) with the initial-boundary conditions (1.2)-(1.3) satisfying

$$
\begin{gathered}
(u^{k,\lambda}, \widetilde{p}^{k,\lambda}) \in W^{1,\infty}(0,T_e; H^1(\Omega;\mathbb{R}^3)) \times H^1(0,T_e; H^1(\Omega;\mathbb{R})), \quad \widetilde{p}^{k,\lambda} \in W^{1,\infty}(0,T_e; L^2(\Omega;\mathbb{R})),\\
u^{k,\lambda}_{tt} \in L^\infty(0,T_e; L^2(\Omega;\mathbb{R}^3)), \quad \varepsilon^{p,k,\lambda} \in C^1([0,T_e]; L^2(\Omega;\mathcal{S}(3))).
\end{gathered}
$$

Henceforth $k \geqslant 1$ is fixed. We want to pass to the limit with $\lambda \to 0^+$ (up to the subsequence) and prove that this limit is the unique solution to (5.2).

To reach this goal we need to obtain several energetic inequalities. It is important to note that systems (5.2) and (5.3) admit the same energy function.

The total energy of the discussed model is in the form

$$
\mathcal{E}^k(u^{k,\lambda}_t, \varepsilon^{k,\lambda}, \varepsilon^{p,k,\lambda})(t) = \frac{\rho}{2}\int_\Omega \left|u^{k,\lambda}_t(x,t)\right|^2 \mathrm{d}x + \int_\Omega \rho\psi^k\left(\varepsilon^{k,\lambda}(x,t), \varepsilon^{p,k,\lambda}(x,t)\right)\mathrm{d}x.
$$

Remark 5.1. For the sake of simplicity, henceforth we omit k, unless it leads to the confusion.

To obtain the sufficient estimates we require stronger data regularity assumptions than in the Lipschitzian case. Namely:

• Regularities of the external forces remain as before:

$$F \in H^1(0,T_e;L^2(\Omega;\mathbb{R}^3)), \quad f \in H^1(0,T_e;L^2(\Omega;\mathbb{R})), \quad F(0) \in L^2(\Omega;\mathbb{R}^3), \quad f(0) \in L^2(\Omega;\mathbb{R}), \tag{5.4}$$

• For the boundary conditions we need stronger assumption with respect to time:

$$\begin{gathered} g_D \in W^{2,\infty}(0,T_e;H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^3)) \cap W^{3,\infty}(0,T_e;H^{\frac{1}{2}}(\Gamma_D;\mathbb{R}^3)), \quad g_N \in W^{2,\infty}(0,T_e;H^{-\frac{1}{2}}(\Gamma_N;\mathbb{R}^3)), \\ g_P \in W^{2,\infty}(0,T_e;H^{\frac{3}{2}}(\Gamma_P;\mathbb{R})), \quad g_V \in W^{2,\infty}(0,T_e;H^{-\frac{1}{2}}(\Gamma_V;\mathbb{R})), \\ g_D(0),\, g_{D,t}(0) \in H^{\frac{3}{2}}(\Gamma_D;\mathbb{R}^3), \quad g_{D,tt}(0) \in H^{\frac{1}{2}}(\Gamma_D;\mathbb{R}^3), \quad g_N(0),\, g_{N,t}(0) \in H^{-\frac{1}{2}}(\Gamma_N;\mathbb{R}^3), \\ g_P(0),\, g_{P,t}(0) \in H^{\frac{3}{2}}(\Gamma_P;\mathbb{R}), \qquad g_V(0) \in H^{-\frac{1}{2}}(\Gamma_V;\mathbb{R}), \end{gathered} \tag{5.5}$$

• Regularities for the initial conditions remain the same as in the Theorem 4.6:

$$u_0, u_1 \in H^1(\Omega;\mathbb{R}^3),\ \mathrm{div}(\varepsilon(u_0)) \in L^2(\Omega;\mathbb{R}^3),\ p_0 \in H^1(\Omega;\mathbb{R}),\ \Delta p_0 \in L^2(\Omega;\mathbb{R}),\ \varepsilon_0^p \in L^2_{\mathrm{div}}(\Omega;\mathcal{S}(3)). \tag{5.6}$$

Furthermore, we assume the compatibility conditions (4.6) and also

$$m(\widehat{T}(0)) = m(\mathcal{D}(\varepsilon(u_0) - \varepsilon_0^p)) \in L^2(\Omega;\mathcal{S}(3)). \tag{5.7}$$

Theorem 5.2 (Energetic estimate). *Under the assumptions (4.6), (5.4)-(5.7) there exists a constant $C(T_e,k) > 0$ independent of λ such that*

$$\mathcal{E}^k\left(u_{tt}^\lambda, \varepsilon_t^\lambda, \varepsilon_t^{p,\lambda}\right)(t) + \int_0^t \|\widetilde{p}_t^\lambda\|_{H^1}^2\,\mathrm{d}\tau + \|\widetilde{p}_t^\lambda(t)\|_{L^2}^2 \leqslant C(T_e,k) \quad \text{for a.e. } t \in [0,T_e],$$

where $(u^\lambda, p^\lambda, \varepsilon^{p,\lambda})$ is a weak solution of (5.3) with the initial–boundary conditions (1.2)-(1.3).

Proof. Let us denote $v^\lambda(x,t) := u_t^\lambda(x,t)$ and as in the proof of Theorem 4.6 we will use a special notation for translated in time functions i.e. $(v_h^\lambda(t), \varepsilon_h^\lambda(t), \varepsilon_h^{p,\lambda}(t)) := (v^\lambda(t+h), \varepsilon^\lambda(t+h), \varepsilon^{p,\lambda}(t+h))$ where $h > 0$ is the fixed, sufficiently small constant. We also apply the similar notation for functions T_h, $\widehat{T}_h$, F_h, f_h, $\widetilde{p}_h$, g_{N_h}, g_{D_h}.

We look at the energy change between the solution translated in time and the non-translated one.

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}&\left(\mathcal{E}^k\left(v_h^\lambda - v^\lambda, \varepsilon_h^\lambda - \varepsilon^\lambda, \varepsilon_h^{p,\lambda} - \varepsilon^{p,\lambda}\right)(t)\right) = \int_\Omega \left(\widehat{T}_h^\lambda - \widehat{T}^\lambda\right)\left(\varepsilon_{h,t}^\lambda - \varepsilon_t^\lambda - \left(\varepsilon_{h,t}^{p,\lambda} - \varepsilon_t^{p,\lambda}\right)\right)\mathrm{d}x \\ &+ \int_\Omega \left(\left(T_h^\lambda - T^\lambda\right) - \left(\widehat{T}_h^\lambda - \widehat{T}^\lambda\right)\right)\left(\varepsilon_{h,t}^\lambda - \varepsilon_t^\lambda\right)\mathrm{d}x + \rho\int_\Omega (v_h^\lambda - v^\lambda)(v_{h,t}^\lambda - v_t^\lambda)\mathrm{d}x \\ &= -\int_\Omega \left(\widehat{T}_h^\lambda - \widehat{T}^\lambda\right)\left(\varepsilon_{h,t}^{p,\lambda} - \varepsilon_t^{p,\lambda}\right)\mathrm{d}x + \int_\Omega \left(T_h^\lambda - T^\lambda\right)\left(\varepsilon_{h,t}^\lambda - \varepsilon_t^\lambda\right)\mathrm{d}x + \rho\int_\Omega (v_h^\lambda - v^\lambda)(v_{h,t}^\lambda - v_t^\lambda)\mathrm{d}x. \end{aligned}$$

By the monotonicity of m^λ we have

$$-\int_\Omega \left(\widehat{T}_h^\lambda - \widehat{T}^\lambda\right)\left(\varepsilon_{h,t}^{p,\lambda} - \varepsilon_t^{p,\lambda}\right)\mathrm{d}x \leqslant L\int_\Omega \left|\widehat{T}_h^\lambda - \widehat{T}^\lambda\right|^2\mathrm{d}x.$$

Using the given system of equations

$$\begin{aligned} \rho\int_\Omega (v_h^\lambda - v^\lambda)(v_{h,t}^\lambda - v_t^\lambda)\mathrm{d}x = &\int_\Omega \left(\overline{F}_h - \overline{F}\right)\left(v_h^\lambda - v^\lambda\right)\mathrm{d}x + \int_\Omega \mathrm{div}\left(T_h^\lambda - T^\lambda\right)\left(v_h^\lambda - v^\lambda\right)\mathrm{d}x \\ &- \alpha\int_\Omega \nabla(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\left(v_h^\lambda - v^\lambda\right)\mathrm{d}x. \end{aligned}$$

Integration by parts yields

$$\int_\Omega \left(T_h^\lambda - T^\lambda\right)\left(\varepsilon_{h,t}^\lambda - \varepsilon_t^\lambda\right) \mathrm{d}x - \alpha \int_\Omega \nabla(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\left(v_h^\lambda - v^\lambda\right) \mathrm{d}x$$
$$= -\int_\Omega \operatorname{div}\left(T_h^\lambda - T^\lambda\right)\left(v_h^\lambda - v^\lambda\right) \mathrm{d}x + \alpha \int_\Omega (\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\operatorname{div}\left(v_h^\lambda - v^\lambda\right) \mathrm{d}x$$
$$+ \int_{\partial\Omega} \left(\left(T_h^\lambda - \alpha\mathbb{I}\widetilde{p}_h^\lambda\right) - \left(T^\lambda - \alpha\mathbb{I}\widetilde{p}^\lambda\right)\right) n \left(v_h^\lambda - v^\lambda\right) \mathrm{dS}(x).$$

Using the given system of equations again gives

$$\alpha \int_\Omega (\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\operatorname{div}\left(v_h^\lambda - v^\lambda\right) \mathrm{d}x = \int_\Omega (\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\left(\overline{f}_h - \overline{f}\right) \mathrm{d}x - \frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|\widetilde{p}_h^\lambda - \widetilde{p}^\lambda\right|^2 \mathrm{d}x$$
$$- c\int_\Omega \left|\nabla(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\right|^2 \mathrm{d}x + c\int_{\partial\Omega} (\widetilde{p}_h^\lambda - \widetilde{p}^\lambda) n \nabla(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda) \mathrm{dS}(x).$$

Combining the results above and using the boundary conditions of (5.3) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^k\left(v_h^\lambda - v^\lambda, \varepsilon_h^\lambda - \varepsilon^\lambda, \varepsilon_h^{p,\lambda} - \varepsilon^{p,\lambda}\right) + \frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left|\widetilde{p}_h^\lambda - \widetilde{p}^\lambda\right|^2 \mathrm{d}x + c\int_\Omega \left|\nabla(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\right|^2 \mathrm{d}x$$
$$\leqslant L\int_\Omega \left|\widehat{T}_h^\lambda - \widehat{T}^\lambda\right|^2 \mathrm{d}x + \int_\Omega \left(\overline{F}_h - \overline{F}\right)\left(v_h^\lambda - v^\lambda\right) \mathrm{d}x + \int_\Omega \left(\overline{f}_h - \overline{f}\right)(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda) \mathrm{d}x$$
$$+ \int_{\Gamma_D} \left(T_h^\lambda - T^\lambda - \alpha\mathbb{I}\left(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda\right)\right) n \left(g_{D_h,t} - g_{D,t}\right) \mathrm{dS}(x) + \int_{\Gamma_N} \left(\overline{g}_{N_h} - \overline{g}_N\right)\left(v_h^\lambda - v^\lambda\right) \mathrm{dS}(x)$$
$$+ \int_{\Gamma_V} (\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\left(\overline{g}_{V_h} - \overline{g}_V\right) \mathrm{dS}(x).$$

Observe that the boundary integrals are well-defined. In particular, from the given system of equations and the regularity of the solution we have $\left(T^\lambda - \alpha\mathbb{I}\widetilde{p}^\lambda\right) \in L^2_{\mathrm{div}}(\Omega)$, hence the trace of this function in the direction given by the outward pointing normal on the boundary $\partial\Omega$ is well-defined in $H^{-\frac{1}{2}}(\partial\Omega;\mathbb{R}^3)$.

Integrating the inequality above over $(0,t)$ and dividing both sides by h^2 gives

$$\begin{aligned}
&\frac{1}{h^2}\mathcal{E}^k\left(v_h^\lambda - v^\lambda, \varepsilon_h^\lambda - \varepsilon^\lambda, \varepsilon_h^{p,\lambda} - \varepsilon^{p,\lambda}\right)(t) + \frac{c_0}{2h^2}\left\|\widetilde{p}_h^\lambda(t) - \widetilde{p}^\lambda(t)\right\|_{L^2}^2 + \frac{c}{h^2}\int_0^t\int_\Omega \left|\nabla(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\right|^2 \mathrm{d}x\mathrm{d}\tau \\
&\leqslant \frac{1}{h^2}\mathcal{E}^k\left(v_h^\lambda - v^\lambda, \varepsilon_h^\lambda - \varepsilon^\lambda, \varepsilon_h^{p,\lambda} - \varepsilon^{p,\lambda}\right)(0) + \frac{c_0}{2h^2}\left\|\widetilde{p}_h^\lambda(0) - \widetilde{p}^\lambda(0)\right\|_{L^2}^2 + \frac{L}{h^2}\int_0^t \left\|\widehat{T}_h^\lambda - \widehat{T}^\lambda\right\|_{L^2}^2 \mathrm{d}\tau \\
&+ \frac{1}{h^2}\int_0^t\int_\Omega \left(\overline{F}_h - \overline{F}\right)\left(v_h^\lambda - v^\lambda\right) \mathrm{d}x\mathrm{d}\tau + \frac{1}{h^2}\int_0^t\int_\Omega \left(\overline{f}_h - \overline{f}\right)(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda) \mathrm{d}x\mathrm{d}\tau \\
&+ \frac{1}{h^2}\int_0^t\int_{\Gamma_V} (\widetilde{p}_h^\lambda - \widetilde{p}^\lambda)\left(\overline{g}_{V_h} - \overline{g}_V\right) \mathrm{dS}(x)\mathrm{d}\tau + \frac{1}{h^2}\int_0^t\int_{\Gamma_N} \left(\overline{g}_{N_h} - \overline{g}_N\right)\left(v_h^\lambda - v^\lambda\right) \mathrm{dS}(x)\mathrm{d}\tau \\
&+ \frac{1}{h^2}\int_0^t\int_{\Gamma_D} \left(T_h^\lambda - T^\lambda - \alpha\mathbb{I}\left(\widetilde{p}_h^\lambda - \widetilde{p}^\lambda\right)\right) n \left(g_{D_h,t} - g_{D,t}\right) \mathrm{dS}(x)\mathrm{d}\tau.
\end{aligned} \tag{5.8}$$

We now want to pass to the limit in the differential quotients. Applying the Dominated Convergence Theorem and using the regularity of the data we can pass to the limit with $h \to 0^+$ with every integral except the last two. We cannot pass to the limit with these boundary integrals in a straightforward manner, because:

- we cannot control v_t^λ in $L^2(0,T_e;H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3))$,
- we cannot control $\left(T_t^\lambda - \alpha\mathbb{I}p_t^\lambda\right)n$ in $L^2(0,T_e;H^{-\frac{1}{2}}(\partial\Omega;\mathcal{S}(3)))$.

Hence, we shift differential quotients from the unknown functions to the data as in [14] and obtain the following inequalities

$$\limsup_{h\to 0^+}\frac{1}{h^2}\int_0^t\int_{\Gamma_N}\left(\overline{g}_{N_h}-\overline{g}_N\right)\left(v_h^\lambda - v^\lambda\right)\mathrm{dS}(x)\mathrm{d}\tau$$

$$\leqslant C(T_e)\sup_{(0,t)}\|v^\lambda\|_{H^{\frac{1}{2}}(\partial\Omega,\mathbb{R}^3)}\left(\|\overline{g}_{N,tt}\|_{L^\infty(H^{-\frac{1}{2}})}+\|\overline{g}_{N,t}\|_{L^\infty(H^{-\frac{1}{2}})}\right),$$

$$\limsup_{h\to 0^+}\frac{1}{h^2}\int_0^t\int_{\Gamma_D}\left(T_h^\lambda - T^\lambda - \alpha\mathbb{I}\left(\widetilde{p}_h^\lambda-\widetilde{p}^\lambda\right)\right)n\left(g_{D_h,t}-g_{D,t}\right)\mathrm{dS}(x)\mathrm{d}\tau$$

$$\leqslant C(T_e)\sup_{(0,t)}\left\|\left(T^\lambda-\alpha\mathbb{I}\widetilde{p}^\lambda\right)n\right\|_{H^{-\frac{1}{2}}}\left(\|g_{D,ttt}\|_{L^\infty(H^{\frac{1}{2}})}+\|g_{D,tt}\|_{L^\infty(H^{\frac{1}{2}})}\right).$$

Hence, in virtue of the previous steps, after passing to the limit in (5.8) we obtain:

$$\begin{aligned}
&\mathcal{E}^k\left(v_t^\lambda,\varepsilon_t^\lambda,\varepsilon_t^{p,\lambda}\right)(t)+\frac{c_0}{2}\int_\Omega\left|\widetilde{p}_t^\lambda(t)\right|^2\mathrm{d}x + c\int_0^t\int_\Omega\left|\nabla\widetilde{p}_t^\lambda\right|^2\mathrm{d}x\mathrm{d}\tau \leqslant \mathcal{E}^k\left(v_t^\lambda,\varepsilon_t^\lambda,\varepsilon_t^{p,\lambda}\right)(0)\\
&+\frac{c_0}{2}\int_0^t\left|\widetilde{p}_t^\lambda(0)\right|^2\mathrm{d}x\mathrm{d}\tau + L\int_0^t\left\|\widehat{T}_t^\lambda\right\|_{L^2}^2\mathrm{d}\tau+\int_0^t\int_\Omega\overline{F}_t v_t^\lambda\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega\overline{f}_t\widetilde{p}_t^\lambda\mathrm{d}x\mathrm{d}\tau\\
&+C(T_e)\left(\sup_{(0,t)}\|v^\lambda\|_{H^{\frac{1}{2}}}+\sup_{(0,t)}\left\|\left(T^\lambda-\alpha\mathbb{I}\widetilde{p}^\lambda\right)n\right\|_{H^{-\frac{1}{2}}}\right)+\int_0^t\int_{\Gamma_V}\widetilde{p}_t^\lambda\overline{g}_{V,t}\mathrm{dS}(x)\mathrm{d}\tau.
\end{aligned}\tag{5.9}$$

We need the estimates independent of λ for the right hand side of the inequality above. We proceed as follows:

- $$\mathcal{E}^k\left(v_t^\lambda,\varepsilon_t^\lambda,\varepsilon_t^{p,\lambda}\right)(0)\leqslant C\left(\|u_1\|_{H^1}^2+\left\|\varepsilon_t^{p,\lambda}(0)\right\|_{L^2}^2+\left\|u_{tt}^\lambda(0)\right\|_{L^2}^2\right),$$

We now need to prove that the last two expressions are estimated independently of λ. To bound the first one we use the properties of the Yosida approximation and the assumption (5.7)

$$\begin{aligned}
\left\|\varepsilon_t^{p,\lambda}(0)\right\|_{L^2}&\leqslant\left\|m^\lambda\left(\widehat{T}^\lambda(0)\right)\right\|_{L^2}+\left\|l\left(\widehat{T}^\lambda(0)\right)\right\|_{L^2}\\
&\leqslant\left\{\begin{array}{c}\widehat{T}(0)\in L^2(\Omega;\mathcal{S}(3)),\\ \text{the Lipschitz condition } l\end{array}\right\}\leqslant C\left(\left\|m\left(\widehat{T}(0)\right)\right\|_{L^2}+\left\|\widehat{T}(0)\right\|_{L^2}+1\right)\leqslant C.
\end{aligned}$$

In order to estimate $u_{tt}^\lambda(0)$ observe that due to the steps form the proof of (5.3), this function must satisfy the same estimate as its Galerkin approximation. Namely

$$\rho\left\|u_{tt}^\lambda(0)\right\|_{L^2}\leqslant\left\|\widetilde{F}^k(0)\right\|_{L^2}+\|\mathrm{div}\left(\mathcal{D}(2\varepsilon(u^m(0))-\varepsilon^p(0))\right)\|_{L^2}+\alpha\left\|\nabla\widetilde{p}^m(0)\right\|_{L^2}\leqslant C.$$

We emphasize here that the function $\widetilde{F}^k$ depends on k (this dependency is hidden in the constant a introduced in a model from the section four). Observe that the following convergence is true in the $L^2(\Omega)$ norm:

$$F(0)-\rho\widetilde{g}_{D,tt}(0)-\alpha\nabla\widetilde{g}_P(0)+\left(1+\frac{1}{k}\right)\mathrm{div}\left(\mathcal{D}\left(\varepsilon\left(\widetilde{g}_D(0)\right)\right)\right)=\widetilde{F}^k(0)\xrightarrow{k\to\infty}\widetilde{F}(0),$$

where $\widetilde{F}(0):=F(0)-\rho\widetilde{g}_{D,tt}(0)-\alpha\nabla\widetilde{g}_P(0)+\mathrm{div}\left(\mathcal{D}\left(\varepsilon\left(\widetilde{g}_D(0)\right)\right)\right)$.
Because of that we can obtain that $\left\|u_{tt}^\lambda(0)\right\|_{L^2}$ is estimated independently of λ and k.

Similarly we estimate the second expression on the right hand side of (5.9):

$$c_0 \left\|\widetilde{p}^\lambda_t(0)\right\|_{L^2} \leqslant \left\|\widetilde{f}(0)\right\|_{L^2} + c\left\|\Delta\widetilde{p}^m(0)\right\|_{L^2} + \alpha\left\|\mathrm{div} u^m_t(0)\right\|_{L^2} \leqslant C.$$

- $$\begin{aligned}\left\|\left(T^\lambda(t)-\alpha\mathbb{I}\widetilde{p}^\lambda(t)\right)\, n\right\|_{H^{-\frac12}} &\leqslant C\left(\left\|T^\lambda(t)-\alpha\mathbb{I}\widetilde{p}^\lambda(t)\right\|_{L^2} + \left\|\mathrm{div}\left(T^\lambda(t)-\alpha\mathbb{I}\widetilde{p}^\lambda(t)\right)\right\|_{L^2}\right)\\ &\leqslant C\left(\left\|T^\lambda(0)-\alpha\mathbb{I}\widetilde{p}^\lambda(0)\right\|_{L^2} + \int_0^t \left\|T^\lambda_t-\alpha\mathbb{I}\widetilde{p}^\lambda_t\right\|_{L^2}\mathrm{d}\tau + \left\|\overline{F}(t)\right\|_{L^2} + \left\|v^\lambda_t(t)\right\|_{L^2}\right)\\ &\leqslant C(T_e,\nu) + \nu\sup_{(0,t)}\mathcal{E}^k(v^\lambda_t,\varepsilon^\lambda_t,\varepsilon^{p,\lambda}_t)(t) + \nu\sup_{(0,t)}\left\|\widetilde{p}^\lambda_t(t)\right\|^2_{L^2},\end{aligned}$$

where we used the Young inequality with the small weight ν.

- $$\begin{aligned}\left\|v^\lambda\right\|_{H^{\frac12}(\partial\Omega,\mathbb{R}^3)} &\leqslant C\left\|v^\lambda\right\|_{H^1(\Omega;\mathbb{R}^3)} \leqslant C\left\|\varepsilon^\lambda_t\right\|_{L^2(\Omega;\mathcal{S}(3))}\\ &\leqslant Ck\sqrt{\mathcal{E}^k(v^\lambda_t,\varepsilon^\lambda_t,\varepsilon^{p,\lambda}_t)(t)} \leqslant \nu\mathcal{E}^k(v^\lambda_t,\varepsilon^\lambda_t,\varepsilon^{p,\lambda}_t)(t) + C(T_e,\nu,k).\end{aligned}$$

- $$\int_0^t\int_{\Gamma_V}\widetilde{p}^\lambda_t\overline{g}_{V,t}\mathrm{dS}(x)\mathrm{d}\tau \leqslant \int_0^t\left\|\widetilde{p}^\lambda_t\right\|_{H^{\frac12}(\partial\Omega;\mathbb{R})}\left\|\overline{g}_{V,t}\right\|_{H^{-\frac12}(\Gamma_V;\mathbb{R})}\mathrm{d}\tau \leqslant \nu\int_0^t\left\|\nabla\widetilde{p}^\lambda_t\right\|^2_{L^2}\mathrm{d}\tau + C(\nu)\int_0^t\left\|\overline{g}_{V,t}\right\|_{H^{-\frac12}}\mathrm{d}\tau.$$

In the remaining expressions we use the Schwarz inequality and then use the weighted Young inequality with the small weight ν. In virtue of (5.9) one obtains

$$\begin{aligned}\mathcal{E}^k\left(v^\lambda_t,\varepsilon^\lambda_t,\varepsilon^{p,\lambda}_t\right)(t) + \frac{c_0}{2}\left\|\widetilde{p}^\lambda_t(t)\right\|^2_{L^2} + c\int_0^t\left\|\nabla\widetilde{p}^\lambda_t\right\|^2_{L^2}\mathrm{d}\tau &\leqslant C(T_e,\nu,k) + C\int_0^t\mathcal{E}^k\left(v^\lambda_t,\varepsilon^\lambda_t,\varepsilon^{p,\lambda}_t\right)\mathrm{d}\tau\\ &+\nu\sup_{(0,t)}\mathcal{E}^k(v^\lambda_t,\varepsilon^\lambda_t,\varepsilon^{p,\lambda}_t)(t) + \nu\sup_{(0,t)}\left\|\widetilde{p}^\lambda_t(t)\right\|^2_{L^2} + \nu\int_0^t\left\|\nabla\widetilde{p}^\lambda_t\right\|^2_{L^2}\mathrm{d}\tau.\end{aligned}$$

Taking the supremum over $(0,t)$ of the every summand on the left hand side of the inequality above, fixing a sufficiently small ν and applying the Gronwall inequality finishes the proof. □

Theorem 5.3 (Existence and uniqueness of the solution to coercive approximation). *Assume that the initial conditions u_0, u_1, p_0 ε^p_0 and the given functions F, f, g_D, g_N, g_P, g_V have the regularity given by (5.4)-(5.7) and satisfy the compatibility conditions (4.6). Then there exists a unique weak solution $(u^k,\widetilde{p}^k,\varepsilon^{p,k})$ of (5.2) with the initial–boundary conditions (1.2)-(1.3) and satisfy*

$$\begin{gathered}(u^k,\widetilde{p}^k)\in W^{1,\infty}(0,T_e;H^1(\Omega;\mathbb{R}^3))\times H^1(0,T_e;H^1(\Omega;\mathbb{R})),\quad \varepsilon^{p,k}\in W^{1,\infty}(0,T_e;L^2(\Omega;\mathcal{S}(3))),\\ \widetilde{p}^k\in W^{1,\infty}(0,T_e;L^2(\Omega;\mathbb{R})),\quad u^k_{tt}\in L^\infty(0,T_e;L^2(\Omega;\mathbb{R}^3)).\end{gathered}$$

Proof. In virtue of the Theorem 5.2 one obtains that the following sequences are bounded: $\{u^\lambda_{tt}\}$, $\{\varepsilon^\lambda_t\}$, $\left\{\varepsilon^{p,\lambda}_t\right\}$ and $\{\widetilde{p}^\lambda_t\}$ in $L^\infty(L^2)$ and $\{\widetilde{p}^\lambda_t\}$ in $L^2(H^1)$.

Due to the given regularity, the discussed functions are absolutely continuous with respect to t, hence using the equality

$$f(t) = f(0) + \int_0^t f_t\mathrm{d}\tau$$

and the Korn inequality yields

- $\{\widetilde{p}^\lambda\}$ is bounded in $H^1(0,T_e;H^1(\Omega;\mathbb{R}))\cap W^{1,\infty}(0,T_e;L^2(\Omega;\mathbb{R}))$ independently of λ,
- $\{u^\lambda\}$ is bounded in $W^{1,\infty}(0,T_e;H^1(\Omega;\mathbb{R}^3))\cap W^{2,\infty}(0,T_e;L^2(\Omega;\mathbb{R}^3))$ independently of λ,
- $\{\varepsilon^{p,\lambda}\}$ is bounded in $W^{1,\infty}(0,T_e;L^2(\Omega;\mathcal{S}(3)))$ independently of λ.

Hence one can select the subsequences (denoted by the same symbol as original sequence) such that for $\lambda\to 0^+$

- $u^\lambda \rightharpoonup u$ in $H^1(0,T_e;H^1(\Omega;\mathbb{R}^3))$, $u^\lambda_{tt} \rightharpoonup u_{tt}$ in $L^2(0,T_e;L^2(\Omega;\mathbb{R}^3))$,
- $\widetilde{p}^\lambda \rightharpoonup \widetilde{p}$ in $H^1(0,T_e;H^1(\Omega;\mathbb{R}))$,
- $\varepsilon^{p,\lambda} \rightharpoonup \varepsilon^p$ in $H^1(0,T_e;L^2(\Omega;\mathcal{S}(3)))$.

We now want to pass to the limit in the weak formulation of (5.3) and prove that $(u,\widetilde{p},\varepsilon^p)$ is the weak solution of (5.2). Note that the type of a convergence of the sequence u^λ remains the same as $\widetilde{u}^\lambda$ since these functions are only shifted by the function $\widetilde{g}_D$. $(u^\lambda,\widetilde{p}^\lambda,\varepsilon^{p,\lambda})$ is a weak solution of (5.3) if $(\widetilde{u}^\lambda,\widetilde{p}^\lambda,\varepsilon^{p,\lambda})$ satisfies for a.e. $t\in(0,T_e)$, all $v\in\mathbb{V}$ and all $w\in\mathbb{W}$ the following equations

$$\rho\int_\Omega \widetilde{u}^\lambda_{tt} v\mathrm{d}x + \int_\Omega \mathcal{D}\left(\frac{k+1}{k}\,\varepsilon(\widetilde{u}^\lambda)-\varepsilon^{p,\lambda}\right)\varepsilon(v)\mathrm{d}x - \alpha\int_\Omega \widetilde{p}^\lambda\,\mathrm{div}v\mathrm{d}x = \int_\Omega \widetilde{F}v\mathrm{d}x + \int_{\Gamma_N}\widetilde{g}_N v\mathrm{dS}(x), \tag{5.10}$$

$$c_0\int_\Omega \widetilde{p}^\lambda_t w\mathrm{d}x + c\int_\Omega \nabla\widetilde{p}^\lambda\nabla w\mathrm{d}x + \alpha\int_\Omega \mathrm{div}\widetilde{u}^\lambda_t\,w\mathrm{d}x = \int_\Omega \widetilde{f}w\mathrm{d}x + \int_{\Gamma_V}\widetilde{g}_V w\mathrm{dS}(x), \tag{5.11}$$

and $\varepsilon^{p,\lambda}$ is the solution to the differential equation $\begin{cases}\varepsilon^{p,\lambda}_t(x,t) = A^\lambda(\widehat{T}^\lambda(x,t)),\\ \varepsilon^{p,\lambda}(x,0)=\varepsilon^p_0(x).\end{cases}$

We can pass to the limit with $\lambda\to 0^+$ in (5.10)-(5.11) as a consequence of the convergence obtained above:

$$\rho\int_\Omega \widetilde{u}_{tt} v\mathrm{d}x + \int_\Omega \mathcal{D}\left(\frac{k+1}{k}\,\varepsilon(\widetilde{u})-\varepsilon^{p}\right)\varepsilon(v)\mathrm{d}x - \alpha\int_\Omega \widetilde{p}\,\mathrm{div}v\mathrm{d}x = \int_\Omega \widetilde{F}v\mathrm{d}x + \int_{\Gamma_N}\widetilde{g}_N v\mathrm{dS}(x), \quad \forall v\in\mathbb{V},$$

$$c_0\int_\Omega \widetilde{p}_t w\mathrm{d}x + c\int_\Omega \nabla\widetilde{p}\nabla w\mathrm{d}x + \alpha\int_\Omega \mathrm{div}\widetilde{u}_t\,w\mathrm{d}x = \int_\Omega \widetilde{f}w\mathrm{d}x + \int_{\Gamma_V}\widetilde{g}_V w\mathrm{dS}(x), \quad \forall w\in\mathbb{W}$$

for a.e. $t\in(0,T_e)$. Furthermore

$$\varepsilon^p_t(x,t) = \mathrm{w}-\lim_{\lambda\to 0^+}\varepsilon^{p,\lambda}_t(x,t) = \mathrm{w}-\lim_{\lambda\to 0^+} A^\lambda\left(\widehat{T}^\lambda(x,t)\right) = \chi(x,t).$$

Hence, if we prove that $\chi(x,t) = A(\widehat{T}(x,t))$ for a.e. $(x,t)\in\Omega\times[0,T_e]$ then, due to the equalities above, we will obtain that $(u,\widetilde{p},\varepsilon^p)$ is a weak solution of (5.2). It will finish the proof of the existence.

We now prove that $\left\{\widehat{T}^\lambda\right\}$ is a Cauchy sequence in $L^2(0,T_e;L^2(\Omega;\mathcal{S}(3)))$.

Lemma 5.4 (Refined convergence of $\{\widehat{T}^\lambda\}$). *Under the assumptions of the Theorem 5.3 for every λ, $\mu>0$ and for almost every $t\in[0,T_e]$ the following estimate holds*

$$\mathcal{E}^k(u^\lambda_t-u^\mu_t,\varepsilon^\lambda-\varepsilon^\mu,\varepsilon^{p,\lambda}-\varepsilon^{p,\mu})(t) + \int_0^t \left\|\nabla(\widetilde{p}^\lambda-\widetilde{p}^\mu)\right\|^2_{L^2}\mathrm{d}\tau + \left\|\widetilde{p}^\lambda(t)-\widetilde{p}^\mu(t)\right\|^2_{L^2} \leqslant C(T_e,k)(\lambda+\mu),$$

where $C(T_e,k)$ is independent of λ, μ.

Proof. We now look at the energy change for the difference between solutions of (5.3) (with the initial–boundary conditions (1.2)-(1.3)) considered with a parameter $\lambda:=\lambda$ and with $\lambda:=\mu$. By similar calculation as in the proof of Theorem 5.2

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}^k\left(u^\lambda_t-u^\mu_t,\varepsilon^\lambda-\varepsilon^\mu,\varepsilon^{p,\lambda}-\varepsilon^{p,\mu}\right)(t)\right) &= -\int_\Omega\left(\widehat{T}^\lambda-\widehat{T}^\mu\right)\left(\varepsilon^{p,\lambda}_t-\varepsilon^{p,\mu}_t\right)\mathrm{d}x\\ &\quad+\int_\Omega\left(T^\lambda-T^\mu\right)\left(\varepsilon^\lambda_t-\varepsilon^\mu_t\right)\mathrm{d}x + \rho\int_\Omega(u^\lambda_t-u^\mu_t)(u^\lambda_{tt}-u^\mu_{tt})\mathrm{d}x.\end{aligned}$$

Let us observe

$$-\int_\Omega\left(\widehat{T}^\lambda-\widehat{T}^\mu\right)\left(\varepsilon^{p,\lambda}_t-\varepsilon^{p,\mu}_t\right)\mathrm{d}x \leqslant -\int_\Omega\left(\widehat{T}^\lambda-\widehat{T}^\mu\right)\left(m^\lambda\left(\widehat{T}^\lambda\right)-m^\mu\left(\widehat{T}^\mu\right)\right)\mathrm{d}x + L\left\|\widehat{T}^\lambda_h-\widehat{T}^\lambda\right\|^2_{L^2}.$$

Due to the properties of the Yosida approximation of a maximal monotone operator $m^\lambda(\omega) = m\left(J_\lambda(\omega)\right)$. Moreover, by the definition of a resolvent ($J_\lambda\left(\omega\right) = u$, where $u + \lambda m(u) = \omega$) we obtain the equation

$$\widehat{T}^\lambda = J_\lambda\left(\widehat{T}^\lambda\right) + \lambda m\left(J_\lambda\left(\widehat{T}^\lambda\right)\right)$$

Hence, by monotonicity of m

$$\begin{aligned}
&-\left(\widehat{T}^\lambda - \widehat{T}^\mu\right)\left(m^\lambda\left(\widehat{T}^\lambda\right) - m^\mu\left(\widehat{T}^\mu\right)\right)\\
&\qquad \leqslant (\lambda+\mu) m\left(J_\lambda\left(\widehat{T}^\lambda\right)\right) m\left(J_\mu\left(\widehat{T}^\mu\right)\right) - \lambda\left(m\left(J_\lambda\left(\widehat{T}^\lambda\right)\right)\right)^2 - \mu\left(m\left(J_\mu\left(\widehat{T}^\mu\right)\right)\right)^2\\
&\qquad \leqslant \left\{(\lambda+\mu)ab \leqslant \left(\lambda + \frac{\mu}{4}\right)a^2 + \left(\mu + \frac{\lambda}{4}\right)b^2\right\} \leqslant \frac{\lambda+\mu}{4}\left(\left(m\left(J_\lambda\left(\widehat{T}^\lambda\right)\right)\right)^2 + \left(m\left(J_\mu\left(\widehat{T}^\mu\right)\right)\right)^2\right).
\end{aligned}$$

Similar arguments as in the proof of Theorem 5.2 lead to

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^k\left(u_t^\lambda - u_t^\lambda, \varepsilon^\lambda - \varepsilon^\mu, \varepsilon^{p,\lambda} - \varepsilon^{p,\mu}\right) + \frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\widetilde{p}^\lambda - \widetilde{p}^\mu\right\|_{L^2}^2 + c\left\|\nabla(\widetilde{p}^\lambda - \widetilde{p}^\mu)\right\|_{L^2}^2\\
&\qquad \leqslant \frac{\lambda+\mu}{4}\int_\Omega \left(m\left(J_\lambda\left(\widehat{T}^\lambda\right)\right)\right)^2 + \left(m\left(J_\mu\left(\widehat{T}^\mu\right)\right)\right)^2 \mathrm{d}x + L\left\|\widehat{T}^\lambda - \widehat{T}^\mu\right\|_{L^2}^2.
\end{aligned} \tag{5.12}$$

We now integrate the inequality (5.12) over time $(0,t)$ and obtain

$$\begin{aligned}
&\mathcal{E}^k\left(u_t^\lambda - u_t^\lambda, \varepsilon^\lambda - \varepsilon^\mu, \varepsilon^{p,\lambda} - \varepsilon^{p,\mu}\right)(t) + \frac{c_0}{2}\left\|\widetilde{p}^\lambda(t) - \widetilde{p}^\mu(t)\right\|_{L^2}^2 + c\int_0^t \left\|\nabla(\widetilde{p}^\lambda - \widetilde{p}^\mu)\right\|_{L^2}^2 \mathrm{d}\tau\\
&\qquad \leqslant \frac{\lambda+\mu}{4}\left(\left\|\varepsilon_t^{p,\lambda}\right\|_{L^2(L^2)}^2 + \|\varepsilon_t^{p,\mu}\|_{L^2(L^2)}^2\right) + L\int_0^t \left\|\widehat{T}^\lambda - \widehat{T}^\mu\right\|_{L^2}^2 \mathrm{d}\tau\\
&\qquad \leqslant (\lambda+\mu)C(T_e,k) + C\int_0^t \mathcal{E}^k\left(u_t^\lambda - u_t^\lambda, \varepsilon^\lambda - \varepsilon^\mu, \varepsilon^{p,\lambda} - \varepsilon^{p,\mu}\right)\mathrm{d}\tau.
\end{aligned}$$

The last inequality follows from Theorem 5.2. Applying the Gronwall inequality finishes the proof. □

As a corollary of the Lemma 5.4, $\left\{\widehat{T}^\lambda\right\}_{\lambda>0}$ is a Cauchy sequence in $L^2(0,T_e;L^2(\Omega;\mathcal{S}(3)))$.
Observe that $\left(J_\lambda\left(\widehat{T}^\lambda\right), m^\lambda\left(\widehat{T}^\lambda\right)\right)$ belongs to the graph of the operator m and therefore converges strongly–weakly in L^2. Due to the strong–weak closedness of the graph of a maximal–monotone operator m and the properties of the resolvent one obtains

$$\left(J_\lambda\left(\widehat{T}^\lambda\right), m^\lambda\left(\widehat{T}^\lambda\right)\right) \xrightarrow{\lambda\to 0^+} \left(\widehat{T}, m\left(\widehat{T}\right)\right).$$

Therefore we have $\mathrm{w}-\lim_{\lambda\to 0^+} A^\lambda\left(\widehat{T}^\lambda(x,t)\right) = A\left(\widehat{T}(x,t)\right)$ for a.e. $(x,t)\in\Omega\times[0,T_e]$.
This ends the proof of the existence of the solution to (5.2).

It remains to prove the uniqueness of the solution. In order to do that we consider the change of a total energy in time for the difference of the two distinct solutions $(u_1, \widetilde{p}_1, \varepsilon_1^p)$ and $(u_2, \widetilde{p}_2, \varepsilon_2^p)$.
Performing the similar computations as before we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^k\left(u_{1,t} - u_{2,t}, \varepsilon_1 - \varepsilon_2, \varepsilon_1^p - \varepsilon_2^p\right) + \frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega |\widetilde{p}_1 - \widetilde{p}_2|^2\,\mathrm{d}x + c\int_\Omega |\nabla(\widetilde{p}_1 - \widetilde{p}_2)|^2\,\mathrm{d}x \leqslant L\left\|\widehat{T}_1 - \widehat{T}_2\right\|_{L^2}^2.$$

Integration over time and the Gronwall inequality yield $\left\|\widehat{T}_1 - \widehat{T}_2\right\|_{L^2}^2 = 0$.
Hence $\widetilde{p}_1 \equiv \widetilde{p}_2$, $\varepsilon_1^p \equiv \varepsilon_2^p$ and $u_{1,t} \equiv u_{2,t}$ a.e. Additionally u_1 may differ from u_2 at most by a constant (in time) vector: $u_1(x,t) - u_2(x,t) = u_1(x,0) - u_2(x,0) = 0$ for a.e. $(x,t)\in\Omega\times[0,T_e]$. □

6. Existence of the solutions

In this section we prove the existence of solutions to (5.1) with initial-boundary conditions (1.2)-(1.3). Due to the remarks from the previous section this will be equivalent to the existence of solutions to (1.1)-(1.3).

Our goal is to pass to the limit with the sequences of solutions $(u^k, p^k, \varepsilon^{p,k})$. Unfortunately, without the coercivity, the energy estimates do not provide sufficient information about the sequences $\{\varepsilon(u^k)\}$, $\{\varepsilon^{p,k}\}$ but only about the sequence of differences $\{\varepsilon(u^k) - \varepsilon^{p,k}\}$. The additional information is obtained due to the growth conditions imposed on the constitutive equation.
The unpleasant regularity of the solutions requires further weakening of the definition of a weak solution in the case of a system (1.1)-(1.3). The culprit here is the expression $\mathrm{div} u_t$.

To proceed with the weaker definition we introduce the following space

$$\mathcal{W} = \left\{\phi \in C^1(\overline{\Omega} \times [0,T_e];\mathbb{R}) : \phi(x,T_e) = 0 \ \wedge \ \phi(x,t) = 0 \quad \text{for} \quad (x,t) \in \Gamma_P \times [0,T_e]\right\}$$

and denote by $\langle\langle \cdot,\cdot \rangle\rangle$ the duality pairing between $\mathcal{W}$ and $\mathcal{W}^*$. Also, recall that by $\langle \cdot,\cdot\rangle$ we denote the duality pairing between $\mathbb{V}$ and $\mathbb{V}^*$. We are now ready to define the solution to (1.1) - (1.3).

Definition 6.1 (Solution of (1.1) - (1.3)). We say that the quadruple $(u, p, \varepsilon^p, T = \mathcal{D}(\varepsilon(u) - \varepsilon^p))$ is a solution to (1.1) - (1.3) if

1. For a.e. $t \in [0,T_e]$ the triple $(\widetilde{u}, \widetilde{p}, \varepsilon^p)$ satisfies the system of equations:

$$\rho \langle \widetilde{u}_{tt}, v\rangle + \int_\Omega \mathcal{D}(\varepsilon(\widetilde{u}) - \varepsilon^p)\varepsilon(v)\mathrm{d}x - \alpha \int_\Omega \widetilde{p}\,\mathrm{div} v \mathrm{d}x = \int_\Omega \widetilde{F} v \mathrm{d}x + \int_{\Gamma_N} \widetilde{g}_N v \mathrm{dS}(x), \quad \forall v \in \mathbb{V}, \tag{6.1}$$

   for a.e. $t \in (0,T_e)$ and

$$\begin{aligned} c_0 \langle\langle \widetilde{p}_t, \phi\rangle\rangle + c\int_0^{T_e}\int_\Omega \nabla\widetilde{p}\nabla\phi \mathrm{d}x\mathrm{d}\tau - \alpha\int_0^{T_e}\int_\Omega \mathrm{div}\widetilde{u}\,\phi_t \mathrm{d}x\mathrm{d}\tau + \alpha\int_\Omega \mathrm{div}\widetilde{u}_0\,\phi(0)\mathrm{d}x \\ = \int_0^{T_e}\int_\Omega \widetilde{f}\phi\mathrm{d}x\mathrm{d}\tau + \int_0^{T_e}\int_{\Gamma_V} \widetilde{g}_V\,\phi\mathrm{dS}(x)\mathrm{d}\tau, \qquad \forall \phi \in \mathcal{W}, \end{aligned} \tag{6.2}$$

2. The fifth equation of (1.1) is satisfied in a sense of Young measures, i.e.

$$\varepsilon^p_t(x,t) = \int_{\mathcal{S}^3} A(S)\mathrm{d}\nu_{(x,t)}(S),$$

   where $\{\nu_{(x,t)}\}$ is a Young measure satisfying

$$T(x,t) = \int_{\mathcal{S}^3} S\mathrm{d}\nu_{(x,t)}(S) \quad \text{for a.e.} \quad (x,t) \in \Omega \times (0,T_e).$$

3. $\varepsilon^p(x,0) = \varepsilon^p_0(x), \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad p(x,0) = p_0(x).$

Furthermore, the following regularities are required

$$\begin{aligned} u \in W^{1,\infty}(0,T_e;L^2(\Omega;\mathbb{R}^3)) \cap L^\infty(0,T_e;W^{1,1+\frac{1}{\omega}}(\Omega;\mathbb{R}^3)), \quad u_{tt} \in L^2(0,T_e;\left(H^1(\Omega;\mathbb{R}^3)\right)^*), \\ p \in L^2(0,T_e;H^1(\Omega;\mathbb{R})) \cap L^\infty(0,T_e;L^2(\Omega;\mathbb{R})), \quad p_t \in \mathcal{W}^*, \\ \varepsilon^p \in L^\infty(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}^3)) \cap W^{1,1+\frac{1}{\omega}}(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}^3)), \quad T \in L^\infty(0,T_e;L^2(\Omega;\mathcal{S}^3)). \end{aligned} \tag{6.3}$$

The similar definitions appear in the papers addressing the non-monotone problems (see [14]). Therefore the proof of the existence of the solution in the sense of the Definition 6.1 is a natural step in our discussion in the case of the $\mathcal{LM}$-type constitutive equation.

To obtain the proper energetic estimates we need to restrict the class of the given boundary data. These additional assumptions are known as *the safe-load conditions* (see [17] for more details). From the physical point of view, the safe-load conditions say that the boundary data must be well–tolerated by the considered material. However, from the mathematical point of view, the safe-load conditions provide an essential information to the proof of a priori estimates for the coercive approximation (see the Lemma 6.3).

Definition 6.2 (Safe-load conditions). We say that the functions g_D, g_N satisfy *the safe-load conditions* with the regularity $1+\omega$, if there exist the initial conditions $u_0^*, u_1^* \in H^1(\Omega;\mathbb{R}^3)$ and the function $F^* \in H^1(0,T_e;L^2(\Omega;\mathbb{R}^3))$ such that there exists a solution (u^*,T^*) of the linear system

$$\begin{aligned}\rho u^*_{tt}(x,t) - \mathrm{div}_x T^*(x,t) &= F^*(x,t),\\ T^*(x,t) &= \mathcal{D}(\varepsilon(u^*(x,t))),\end{aligned}$$

with the initial–boundary conditions

$$\begin{array}{rcll} u^*(x,0) &=& u_0^*(x) & \text{for } x\in\Omega,\\ u_t^*(x,0) &=& u_1^*(x) & \text{for } x\in\Omega,\\ u^*(x,t) &=& g_D(x,t) & \text{for } x\in\Gamma_D, \quad t\geqslant 0,\\ T^*(x,t)n(x) &=& g_N(x,t) & \text{for } x\in\Gamma_N, \quad t\geqslant 0,\end{array}$$

and the regularity

$$u^*\in W^{1,\infty}(0,T_e;H^1(\Omega;\mathbb{R}^3)), \quad T^*\in L^{1+\omega}(0,T_e;L^{1+\omega}(\Omega;\mathcal{S}^3)), \quad u^*_{tt}\in L^\infty(0,T_e;L^2(\Omega;\mathcal{S}^3)).$$

We now proceed with the energetic estimate of $\left\{(u^k,\widetilde{p}^k,\varepsilon^{p,k},T^k)\right\}_{k=1}^\infty$. This result is a main step to prove the existence of the solution (according to the Definition 6.1).

Lemma 6.3 (Energetic estimate independent of k). *Assume the same as in the Theorem 5.3 and suppose that g_D, $\overline{g}_N$ satisfy the safe-load conditions with the regularity $1+\omega$. Then for almost every $t\in[0,T_e]$ the following inequality holds*

$$\mathcal{E}^k\left(u^k_t-u^*_t,\varepsilon^k-\varepsilon^*,\varepsilon^{p,k}\right)(t)+\left\|\widetilde{p}^k(t)\right\|^2_{L^2}+\int_0^t\left\|\widetilde{p}^k\right\|^2_{H^1}\mathrm{d}\tau+\int_0^t\int_\Omega m\left(\widehat{T}^k\right)\widehat{T}^k\mathrm{d}x\mathrm{d}\tau\leqslant C(T_e),$$

where $(u^k,\widetilde{p}^k,\varepsilon^{p,k})$ is the solution of (5.2) with the conditions (1.2)-(1.3), u^* is the function from the definition of the safe-load conditions and $\varepsilon^*=\varepsilon(u^*)$. Furthermore the constant $C(T_e)>0$ is independent of k.

Proof. Again we compute the difference of the energy in time. In order to dispose of the boundary integrals we shift the function u^k by u^*.

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}^k\left(u^k_t-u^*_t,\varepsilon^k-\varepsilon^*,\varepsilon^{p,k}\right)(t)\right)\rho\int_\Omega\left(u^k_t-u^*_t\right)\left(u^k_{tt}-u^*_{tt}\right)\mathrm{d}x-\int_\Omega\widehat{T}^k\varepsilon^{p,k}_t\mathrm{d}x+\int_\Omega T^*\varepsilon^{p,k}_t\mathrm{d}x\\ -\left(1+\frac{1}{k}\right)\int_\Omega T^*\left(\varepsilon^k_t-\varepsilon^*_t\right)\mathrm{d}x+\int_\Omega T^k\left(\varepsilon^k_t-\varepsilon^*_t\right)\mathrm{d}x.\end{aligned}$$

Observe that

$$-\int_\Omega T^*\left(\varepsilon^k_t-\varepsilon^*_t\right)\mathrm{d}x=\int_\Omega \mathrm{div}T^*\left(u^k_t-u^*_t\right)\mathrm{d}x-\int_{\Gamma_N}\overline{g}_N\left(u^k_t-u^*_t\right)\mathrm{dS}(x).$$

Similar computation as in the proofs of Theorem 5.2 and Lemma 5.4 gives

$$\begin{aligned}&\mathcal{E}^k\left(u^k_t-u^*_t,\varepsilon^k-\varepsilon^*,\varepsilon^{p,k}\right)(t)+\frac{c_0}{2}\left\|\widetilde{p}^k(t)\right\|^2_{L^2}+c\int_0^t\left\|\nabla\widetilde{p}^k(t)\right\|^2_{L^2}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega m\left(\widehat{T}^k\right)\widehat{T}^k\mathrm{d}x\mathrm{d}\tau\\ &\leqslant\mathcal{E}^k\left(u^k_t-u^*_t,\varepsilon^k-\varepsilon^*,\varepsilon^{p,k}\right)(0)+\frac{c_0}{2}\left\|\widetilde{p}_0\right\|^2_{L^2}+C\int_0^t\mathcal{E}^k\left(u^k_t-u^*_t,\varepsilon^k-\varepsilon^*,\varepsilon^{p,k}\right)(\tau)\mathrm{d}\tau\\ &+\int_0^t\int_\Omega\left(u^k_t-u^*_t\right)\left(\overline{F}-F^*\right)\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega\overline{f}\widetilde{p}^k\mathrm{d}x\mathrm{d}\tau+\frac{1}{k}\int_0^t\int_\Omega\mathrm{div}T^*\left(u^k_t-u^*_t\right)\mathrm{d}x\mathrm{d}\tau\\ &-\alpha\int_0^t\int_\Omega\mathrm{div}u^*_t\widetilde{p}^k\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_\Omega T^*\varepsilon^{p,k}_t\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\Gamma_V}\overline{g}_V\widetilde{p}^k\mathrm{dS}(x)\mathrm{d}\tau-\frac{1}{k}\int_0^t\int_{\Gamma_N}\overline{g}_N\left(u^k_t-u^*_t\right)\mathrm{dS}(x)\mathrm{d}\tau.\end{aligned}\tag{6.4}$$

We now have to estimate the summands on the right hand side of the inequality above.
• Obviously, first two terms can be estimate by constant which depends only on u_0, u_0^*, ε_0^p, $\widetilde{p}_0$ and $\mathcal{D}$.
• Applying the Young inequality gives

$$\int_0^t \int_\Omega T^* \varepsilon_t^{p,k} \mathrm{d}x\mathrm{d}\tau \leqslant C(\omega,\nu) \int_0^t \|T^*\|_{L^{1+\omega}}^{1+\omega} \mathrm{d}\tau + \nu \int_0^t \left\|\varepsilon_t^{p,k}\right\|_{L^{1+\frac{1}{\omega}}}^{1+\frac{1}{\omega}} \mathrm{d}\tau.$$

By the property of A, there exists a constant $D > 0$ such that for $\left|\widehat{T}^k\right| > D$ the following estimate holds

$$\begin{aligned} \int_0^t \left\|\varepsilon_t^{p,k}\right\|_{L^{1+\frac{1}{\omega}}}^{1+\frac{1}{\omega}} \mathrm{d}\tau &\leqslant C \int_0^t \int_\Omega \left|\widehat{T}^k\right|^{\omega+1} \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C \int_0^t \int_\Omega m\left(\widehat{T}^k\right) \widehat{T}^k \mathrm{d}x\mathrm{d}\tau + C T_e \sup_{(0,t)} \mathcal{E}^k\left(u_t^k - u_t^*, \varepsilon^k - \varepsilon^*, \varepsilon^{p,k}\right). \end{aligned} \tag{6.5}$$

If $\left|\widehat{T}^k\right| \leqslant D$, then from the continuity of A it follows that $\int_0^{T_e} \int_\Omega \left|A\left(\widehat{T}^k\right)\right|^{1+\frac{1}{\omega}} \mathrm{d}x\mathrm{d}\tau \leqslant C(T_e)$.
• To estimate last term we proceed as follows

$$\begin{aligned} \int_0^t \int_{\Gamma_N} \overline{g}_N \left(u_t^k - u_t^*\right) \mathrm{dS}(x)\mathrm{d}\tau = &\int_{\Gamma_N} \overline{g}_N(t) \left(u^k(t) - u^*(t)\right) \mathrm{dS}(x) - \int_{\Gamma_N} \overline{g}_N(0) \left(u^k(0) - u^*(0)\right) \mathrm{dS}(x)\mathrm{d}\tau \\ &- \int_0^t \int_{\Gamma_N} \overline{g}_{N,t} \left(u^k - u^*\right) \mathrm{dS}(x)\mathrm{d}\tau. \end{aligned}$$

Applying the Korn inequality yields

$$-\frac{1}{k} \int_0^t \int_{\Gamma_N} \overline{g}_N \left(u_t^k - u_t^*\right) \mathrm{dS}(x)\mathrm{d}\tau \leqslant \frac{\nu}{k} \sup_{(0,t)} \left\|\varepsilon^k - \varepsilon^*\right\|_{L^2}^2 + C(T_e,\nu) \left\|\overline{g}_N\right\|_{W^{1,\infty}(H^{-\frac{1}{2}})}^2 + C.$$

Using elementary estimates for the rest terms in (6.4) gives

$$\begin{aligned} &\mathcal{E}^k\left(u_t^k - u_t^*, \varepsilon^k - \varepsilon^*, \varepsilon^{p,k}\right)(t) + \frac{c_0}{2} \left\|\widetilde{p}^k(t)\right\|_{L^2}^2 + c \int_0^t \left\|\nabla \widetilde{p}^k(t)\right\|_{L^2}^2 \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_\Omega m\left(\widehat{T}^k\right) \widehat{T}^k \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C(T_e,\nu) + C \int_0^t \mathcal{E}^k\left(u_t^k - u_t^*, \varepsilon^k - \varepsilon^*, \varepsilon^{p,k}\right)(\tau)\mathrm{d}\tau + C\left(T_e\right) \nu \sup_{(0,t)} \mathcal{E}^k\left(u_t^k - u_t^*, \varepsilon^k - \varepsilon^*, \varepsilon^{p,k}\right) \\ &+ \nu \sup_{(0,t)} \left\|\widetilde{p}^k\right\|_{L^2}^2 + 2\nu \int_0^t \left\|\nabla \widetilde{p}^k\right\|_{L^2}^2 \mathrm{d}\tau + C\nu \int_0^t \int_\Omega m\left(\widehat{T}^k\right) \widehat{T}^k \mathrm{d}x\mathrm{d}\tau. \end{aligned}$$

The constants on the right hand side are independent of k. Therefore, we can take the sufficiently small ν then take the supremum over $(0,t)$ and finally apply the Gronwall inequality to finish the proof. □

Theorem 6.4 (Existence of the solutions). *Suppose that the initial conditions u_0, u_1, p_0, ε_0^p and the given functions F, f, g_D, g_N, g_P, g_V satisfy the compatibility conditions (4.6), admit the regularity (5.4)-(5.7) and g_D, $\overline{g}_N$ satisfy the safe-load conditions with the regularity $1+\omega$. Then there exists the solution $(u, \widetilde{p}, \varepsilon^p, T)$ of (5.1) with the conditions (1.2), (1.3) (understood as in the Definition 6.1).*

Proof. By the Lemma 6.3 one concludes that

- the sequence $\{u^k\}_{k=1}^\infty$ (as well as $\{\widetilde{u}^k\}_{k=1}^\infty$) is bounded in $W^{1,\infty}(0,T_e;L^2(\Omega;\mathbb{R}^3))$,
- the sequence $\left\{\widehat{T}^k\right\}_{k=1}^\infty$ is bounded in $L^\infty(0,T_e;L^2(\Omega;\mathcal{S}^3))$,

- the sequence $\left\{\frac{1}{\sqrt{k}}\varepsilon^k\right\}_{k=1}^{\infty}$ is bounded in $L^\infty(0,T_e;L^2(\Omega;\mathcal{S}^3))$,
- the sequence $\left\{\widetilde{p}^k\right\}_{k=1}^{\infty}$ is bounded in $L^2(0,T_e;H^1(\Omega;\mathbb{R}))\cap L^\infty(0,T_e;L^2(\Omega;\mathbb{R}))$,
- the sequence $\left\{T^k\right\}_{k=1}^{\infty}$ is bounded in $L^\infty(0,T_e;L^2(\Omega;\mathcal{S}^3))$, indeed: $T^k=\widehat{T}^k+\frac{1}{k}\mathcal{D}\left(\varepsilon^k\right)$,
- in virtue of (6.5) the sequence $\left\{\varepsilon_t^{p,k}\right\}_{k=1}^{\infty}$ is bounded in $L^{1+\frac{1}{\omega}}(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}^3))$,
- from the absolute continuity of $\varepsilon^{p,k}$ for every k, we obtain that the sequence $\left\{\varepsilon^{p,k}\right\}_{k=1}^{\infty}$ is bounded in $L^\infty(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}^3))$,
- the sequence $\left\{\varepsilon(u^k)\right\}_{k=1}^{\infty}$ is bounded in $L^\infty(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}^3))$,
- the sequence $\left\{\mathrm{div}u^k\right\}_{k=1}^{\infty}=\left\{\mathrm{tr}\,\varepsilon(u^k)\right\}_{k=1}^{\infty}$ is bounded $L^\infty(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}^3))$,
- due to the higher integrability of $\varepsilon_t^{p,k}$ and the Dunford-Pettis theorem we conclude that it is weakly relatively compact in $L^1(0,T_e;L^1(\Omega;\mathcal{S}^3))$.

Passing to the limit (with $k\to\infty$) in the weak formulation of the approximated system requires more information about the sequences $\left\{\widetilde{u}_{tt}^k\right\}_{k=1}^{\infty}$ and $\left\{\widetilde{p}_t^k\right\}$. We will prove that they are bounded in $L^2(0,T_e;\mathbb{V}^*)$ and $\mathcal{W}^*$ respectively. In order to estimate the first sequence we choose any $v\in\mathbb{V}$ such that $\|v\|_V\leqslant 1$. Hence

$$\rho\int\limits_\Omega \widetilde{u}_{tt}^k v\mathrm{d}x+\int\limits_\Omega\mathcal{D}\left(\varepsilon(\widetilde{u}^k)-\varepsilon^{p,k}+\frac{1}{k}\varepsilon(\widetilde{u}^k)\right)\varepsilon(v)\mathrm{d}x-\alpha\int\limits_\Omega\widetilde{p}^k\,\mathrm{div}v\mathrm{d}x=\int\limits_\Omega\widetilde{F}^k v\mathrm{d}x+\int\limits_{\Gamma_N}\widetilde{g}_N^k v\mathrm{dS}(x),$$

for a.e. $(x,t)\in\Omega\times[0,T_e]$.

Remark 6.5. Note that the right hand side of the equation above depends on k. It was not precisely stated during the previous calculations, however one can bound the sequences of those functions independently of k due to their convergence. For example

$$\widetilde{F}^k:=F-\rho\widetilde{g}_{D,tt}-\alpha\nabla\widetilde{g}_P+\left(1+\frac{1}{k}\right)\mathrm{div}\left(\mathcal{D}(\varepsilon(\widetilde{g}_D))\right)\xrightarrow{k\to\infty}F-\rho\widetilde{g}_{D,tt}-\alpha\nabla\widetilde{g}_P+\mathrm{div}\left(\mathcal{D}(\varepsilon(\widetilde{g}_D))\right)=:\widetilde{F}.$$

This convergence holds in $H^1(0,T_e;L^2(\Omega;\mathbb{R}^3))$. Similarly one can obtain $\widetilde{g}_N^k\xrightarrow{k\to\infty}\widetilde{g}_N$ in $W^{2,\infty}(0,T_e;H^{-\frac{1}{2}}(\Gamma_N;\mathbb{R}^3))$. Hence

$$\begin{aligned}\rho\int\limits_0^{T_e}\left\langle\widetilde{u}_{tt}^k,v\right\rangle^2\mathrm{d}\tau\leqslant{}&C\int\limits_0^{T_e}\left(\int\limits_\Omega\left|\widetilde{F}^k v\right|\mathrm{d}x\right)^2\mathrm{d}\tau+C\int\limits_0^{T_e}\left(\int\limits_{\Gamma_N}\left|\widetilde{g}_N^k v\right|\mathrm{dS}(x)\right)^2\mathrm{d}\tau\\&+C\int\limits_0^{T_e}\left(\int\limits_\Omega\left|\mathcal{D}\left(\varepsilon(\widetilde{u}^k)-\varepsilon^{p,k}+\frac{1}{k}\varepsilon(\widetilde{u}^k)\right)\varepsilon(v)\right|\mathrm{d}x\right)^2\mathrm{d}\tau+C\int\limits_0^{T_e}\left(\int\limits_\Omega\left|\widetilde{p}^k\,\mathrm{div}v\right|\mathrm{d}x\right)^2\mathrm{d}\tau.\end{aligned}$$

Using the estimates from the Lemma 6.3 and remark above we obtain the estimate independent of k and v

$$\rho\int\limits_0^{T_e}\left\langle\widetilde{u}_{tt}^k,v\right\rangle^2\mathrm{d}\tau\leqslant C(T_e).$$

Taking the supremum over v such that $\|v\|_{\mathbb{V}}\leqslant 1$ gives $\int\limits_0^{T_e}\left\|\widetilde{u}_{tt}^k\right\|_{\mathbb{V}^*}^2\mathrm{d}\tau\leqslant C(T_e)$. Proceeding analogously one can prove that the sequence $\left\{\widetilde{p}_t^k\right\}$ is bounded in $\mathcal{W}^*$ independently of k. In virtue of the discussion above we can extract the subsequence of u^k (denoted by the same symbol) such that

- $u^k\rightharpoonup u$ in $H^1(0,T_e;L^2(\Omega;\mathbb{R}^3))$, $u_{tt}^k\rightharpoonup u_{tt}$ in $L^2(0,T_e;\mathbb{V}^*)$,
- $\widetilde{p}^k\rightharpoonup\widetilde{p}$ in $L^2(0,T_e;H^1(\Omega;\mathbb{R}))$, $\widetilde{p}_t^k\rightharpoonup\widetilde{p}_t$ in $\mathcal{W}^*$,

- $\widehat{T}^k \rightharpoonup T$ in $L^2(0,T_e;L^2(\Omega;\mathcal{S}^3))$,
- $\frac{1}{k}\varepsilon(\widetilde{u}^k) \rightharpoonup 0$ in $L^2(0,T_e;L^2(\Omega;\mathcal{S}^3))$,
- $\varepsilon^{p,k} \rightharpoonup \varepsilon^p$ in $W^{1,1+\frac{1}{\omega}}(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}^3))$,
- $\mathrm{div} u^k \rightharpoonup \mathrm{div} u$ in $L^2(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathbb{R}))$.

It remains to pass to the limit. Suppose that $(u^k, \widetilde{p}^k, \varepsilon^{p,k})$ are the solutions of (5.2). Hence, for all $v \in \mathbb{V}$ and all $\phi \in \mathcal{W}$, the following system is satisfied

$$\rho \int_\Omega \widetilde{u}^k_{tt} v \mathrm{d}x + \int_\Omega \mathcal{D}\left(\varepsilon(\widetilde{u}^k) - \varepsilon^{p,k} + \frac{1}{k}\varepsilon(\widetilde{u}^k)\right)\varepsilon(v)\mathrm{d}x - \alpha \int_\Omega \widetilde{p}^k \,\mathrm{div} v \mathrm{d}x = \int_\Omega \widetilde{F}^k v \mathrm{d}x + \int_{\Gamma_N} \widetilde{g}^k_N v \mathrm{dS}(x),$$

$$c_0 \int_0^{T_e}\int_\Omega \widetilde{p}^k_t \phi \mathrm{d}x\mathrm{d}\tau + c\int_0^{T_e}\int_\Omega \nabla\widetilde{p}^k \nabla\phi \mathrm{d}x\mathrm{d}\tau + \alpha\int_0^{T_e}\int_\Omega \mathrm{div}\widetilde{u}^k_t\, \phi \mathrm{d}x\mathrm{d}\tau = \int_0^{T_e}\int_\Omega \widetilde{f}\phi\mathrm{d}x\mathrm{d}\tau + \int_0^{T_e}\int_{\Gamma_V} \widetilde{g}_V \phi \mathrm{dS}(x)\mathrm{d}\tau,$$

$$\varepsilon^{p,k}_t = A\left(\mathcal{D}\left(\varepsilon(u^k) - \varepsilon^{p,k}\right)\right) = A\left(\widehat{T}^k\right).$$

We integrate by parts (with respect to the time) in the third summand of the second equation. The standard procedure yields

$$\rho\,\langle \widetilde{u}_{tt}, v\rangle + \int_\Omega \mathcal{D}\left(\varepsilon(\widetilde{u}) - \varepsilon^p\right)\varepsilon(v)\mathrm{d}x - \alpha\int_\Omega \widetilde{p}\,\mathrm{div} v\mathrm{d}x = \int_\Omega \widetilde{F} v\mathrm{d}x + \int_{\Gamma_N} \widetilde{g}_N v \mathrm{dS}(x), \quad \forall v \in \mathbb{V},$$

$$\begin{aligned} c_0\,\langle\langle \widetilde{p}_t, \phi\rangle\rangle + c\int_0^{T_e}\int_\Omega \nabla\widetilde{p}\nabla\phi\mathrm{d}x\mathrm{d}\tau - \alpha\int_0^{T_e}\int_\Omega \mathrm{div}\widetilde{u}\,\phi_t \mathrm{d}x\mathrm{d}\tau + \alpha\int_\Omega \mathrm{div}\widetilde{u}_0\,\phi(0)\mathrm{d}x \\ = \int_0^{T_e}\int_\Omega \widetilde{f}\phi\mathrm{d}x\mathrm{d}\tau + \int_0^{T_e}\int_{\Gamma_V} \widetilde{g}_V\phi\mathrm{dS}(x)\mathrm{d}\tau, \quad \forall \phi \in \mathcal{W}, \end{aligned}$$

$$\varepsilon^p_t = \mathrm{w} - \lim_{k\to\infty} \varepsilon^{p,k}_t = \mathrm{w} - \lim_{k\to\infty} A\left(\widehat{T}^k\right) = \widehat{\chi}.$$

In virtue of the Fundamental Theorem for the Young Measures (for the proof see [3]) there exists the Young measure $\nu_{(x,t)}$ generated by the sequence $\left\{\widehat{T}^k\right\}_{k=1}^\infty$ such that $\widehat{T}(x,t) = \int_{\mathcal{S}^3} S \mathrm{d}\nu_{(x,t)}(S)$. Furthermore, the weak limit $\widehat{\chi}$ is in the form

$$\widehat{\chi}(x,t) = \int_{\mathcal{S}^3} A(S)\mathrm{d}\nu_{(x,t)}(S).$$

This characterisation of the non-linearity ends the proof, since $(u, \widetilde{p}, \varepsilon^p, T)$ is the solution to (5.1) with the initial–boundary conditions (1.2)-(1.3) according to the Definition 6.1. ☐

Remark 6.6. Due to the remarks before, the Theorem 5.3 is a final step of the existence problem for (1.1)-(1.3). This is true, since if $(u, \widetilde{p}, \varepsilon^p, T)$ is the solution to (5.1) with (1.2)-(1.3), then $(u, p = \widetilde{p} + \widetilde{g}_P, \varepsilon^p, T)$ is the solution to (1.1)-(1.3).

7. Higher regularity of the solutions

The regularity of the weak solution (as in Definition 6.1) obtained in the previous section is not satisfactory. The constitutive equation is satisfied only in terms of Young measures and there is no information of its uniqueness whatsoever.

In this section we improve the previous definition of the safe-load conditions by demanding higher regularity of the solution to the auxiliary linear problem and we redefine the weak solution accordingly. Consequently, we obtain an additional information about the regularity and the uniqueness.

We begin with the definition of the improved safe-load conditions.

Definition 7.1 (Improved safe-load conditions). We say that g_D, g_N satisfy the *improved safe-load conditions* with the regularity $1+\omega$ if g_D, g_N satisfy safe-load conditions with regularity $1+\omega$ (Definition 6.2) and u^*, T^* have the following regularity

$$u^* \in W^{2,\infty}(0,T_e;H^1(\Omega;\mathbb{R}^3)) \cap W^{3,\infty}(0,T_e;L^2(\Omega;\mathbb{R}^3)), \quad T^* \in W^{2,\infty}(0,T_e;L^{1+\omega}(\Omega;\mathcal{S}(3))).$$

We can now proceed with the energy estimate for time derivatives of the solutions to coercive approximation (5.2) which is independent of k.

Lemma 7.2 (Energy estimate of time derivatives). *Under the assumptions of the Theorem 5.3 and g_D, $\overline{g}_N$ satisfying the improved safe–load conditions with the regularity $1+\omega$, the following estimate holds*

$$\int\limits_0^t \mathcal{E}^k\left(u^k_{\tau\tau} - u^*_{\tau\tau}, \varepsilon^k_\tau - \varepsilon^*_\tau, \varepsilon^{p,k}_\tau\right)(\tau)\mathrm{d}\tau + \int\limits_o^t \left\|\widetilde{p}^k_\tau(\tau)\right\|^2_{L^2}\mathrm{d}\tau + \int\limits_0^t\int\limits_0^s \left\|\widetilde{p}^k_\tau(\tau)\right\|^2_{H^1}\mathrm{d}\tau\mathrm{d}s \leqslant C(T_e),$$

for $t \in [0,T_e]$, where $(u^k, \widetilde{p}^k, \varepsilon^{p,k})$ is a solution to (5.2) with the initial-boundary conditions (1.2)-(1.3), u^* is a function from the Definition 7.1, $\varepsilon^* = \varepsilon(u^*)$. Moreover, the constant $C(T_e) > 0$ is independent of k and t.

Proof. Throughout the proof we follow the notation introduced in the proof of the Theorem 5.2 and also denote $v^*(x,t) := u^*_t$ and $v^*_h(x,t) := v^*(x,t+h)$, $F^*_h(x,t) := F^*(x,t+h)$. Let $k \geqslant 1$ be fixed. Proceeding analogously to the proof of the Theorem 5.2 one obtains

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathcal{E}^k\left(v^k_h - v^k - (v^*_h - v^*), \varepsilon^k_h - \varepsilon^k - (\varepsilon^*_h - \varepsilon^*), \varepsilon^{p,k}_h - \varepsilon^{p,k}\right)(t)\right) = \int\limits_\Omega (T^*_h - T^*)\left(\varepsilon^{p,k}_{h,t} - \varepsilon^{p,k}_t\right)\mathrm{d}x \\
&- \int\limits_\Omega \left(\widehat{T}^k_h - \widehat{T}^k\right)\left(\varepsilon^{p,k}_{h,t} - \varepsilon^{p,k}_t\right)\mathrm{d}x - \left(1 + \frac{1}{k}\right)\int\limits_\Omega (T^*_h - T^*)\left(\varepsilon^k_{h,t} - \varepsilon^k_t - (\varepsilon^*_{h,t} - \varepsilon^*_t)\right)\mathrm{d}x \\
&+ \int\limits_\Omega \left(T^k_h - T^k\right)\left(\varepsilon^k_{h,t} - \varepsilon^k_t - (\varepsilon^*_{h,t} - \varepsilon^*_t)\right)\mathrm{d}x + \rho\int\limits_\Omega (v^k_h - v^k - (v^*_h - v^*))(v^k_{h,t} - v^k_t - (v^*_{h,t} - v^*_t))\mathrm{d}x.
\end{aligned}$$

Similar calculation as in the proof of Theorem 5.2 and Lemma 6.3 gives

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}^k\left(v^k_h - v^k - (v^*_h - v^*), \varepsilon^k_h - \varepsilon^k - (\varepsilon^*_h - \varepsilon^*), \varepsilon^{p,k}_h - \varepsilon^{p,k}\right) + \frac{c_0}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|\tilde{p}^k_h - \tilde{p}^k\right\|^2_{L^2} + c\left\|\nabla(\tilde{p}^k_h - \tilde{p}^k)\right\|^2_{L^2} \\
\leqslant& L\left\|\widehat{T}^k_h - \widehat{T}^k\right\|^2_{L^2} + \frac{1}{k}\int\limits_\Omega \operatorname{div}(T^*_h - T^*)\left(v^k_h - v^k - (v^*_h - v^*)\right)\mathrm{d}x + \int\limits_\Omega \left(\overline{f}_h - \overline{f}\right)\left(\tilde{p}^k_h - \tilde{p}^k\right)\mathrm{d}x \\
&- \alpha\int\limits_\Omega \operatorname{div}(v^*_h - v^*)\left(\tilde{p}^k_h - \tilde{p}^k\right)\mathrm{d}x + \int\limits_\Omega \left(\overline{F}_h - \overline{F} - (F^*_h - F^*)\right)\left(v^k_h - v^k - (v^*_h - v^*)\right)\mathrm{d}x \\
&+ \int\limits_{\Gamma_V} \left(\overline{g}_{V_h} - \overline{g}_V\right)\left(\tilde{p}^k_h - \tilde{p}^k\right)\mathrm{dS}(x) + \int\limits_\Omega (T^*_h - T^*)\left(\varepsilon^{p,k}_{h,t} - \varepsilon^{p,k}_t\right)\mathrm{d}x \\
&- \frac{1}{k}\int\limits_{\Gamma_N} \left(\overline{g}_{N_h} - \overline{g}_N\right)\left(v^k_h - v^k - (v^*_h - v^*)\right)\mathrm{dS}(x).
\end{aligned}$$

We now multiply the result above by $\frac{1}{h^2}$ and integrate twice over time: the first one over $(0,t)$ and the second one over t in range $(0,b)$ where $b\in[0,T_e-h]$. Hence

$$
\begin{aligned}
&\frac{1}{h^2}\int_0^b \mathcal{E}^k\left(v_h^k-v^k-(v_h^*-v^*),\varepsilon_h^k-\varepsilon^k-(\varepsilon_h^*-\varepsilon^*),\varepsilon_h^{p,k}-\varepsilon^{p,k}\right)(t)\mathrm{d}t\\
&+\frac{c_0}{2h^2}\int_0^b \left\|\tilde{p}_h^k(t)-\tilde{p}^k(t)\right\|_{L^2}^2\mathrm{d}t+\frac{c}{h^2}\int_0^b\int_0^t \left\|\nabla(\tilde{p}_h^k-\tilde{p}^k)\right\|_{L^2}^2\mathrm{d}\tau\mathrm{d}t\\
&\leqslant \frac{b}{h^2}\mathcal{E}^k\left(v_h^k-v^k-(v_h^*-v^*),\varepsilon_h^k-\varepsilon^k-(\varepsilon_h^*-\varepsilon^*),\varepsilon_h^{p,k}-\varepsilon^{p,k}\right)(0)+\frac{bc_0}{2h^2}\left\|\tilde{p}_h^k(0)-\tilde{p}^k(0)\right\|_{L^2}^2\\
&+\frac{L}{h^2}\int_0^b\int_0^t\left\|\widehat{T}_h^k-\widehat{T}^k\right\|_{L^2}^2\mathrm{d}\tau\mathrm{d}t+\frac{1}{kh^2}\int_0^b\int_0^t\int_\Omega \mathrm{div}\left(T_h^*-T^*\right)\left(v_h^k-v^k-(v_h^*-v^*)\right)\mathrm{d}x\mathrm{d}\tau\mathrm{d}t\\
&+\frac{1}{h^2}\int_0^b\int_0^t\int_\Omega\left(\overline{f}_h-\overline{f}\right)\left(\tilde{p}_h^k-\tilde{p}^k\right)\mathrm{d}x\mathrm{d}\tau\mathrm{d}t-\frac{\alpha}{h^2}\int_0^b\int_0^t\int_\Omega \mathrm{div}\left(v_h^*-v^*\right)\left(\tilde{p}_h^k-\tilde{p}^k\right)\mathrm{d}x\mathrm{d}\tau\mathrm{d}t\\
&+\frac{1}{h^2}\int_0^b\int_0^t\int_\Omega\left(\overline{F}_h-\overline{F}-(F_h^*-F^*)\right)\left(v_h^k-v^k-(v_h^*-v^*)\right)\mathrm{d}x\mathrm{d}\tau\mathrm{d}t\\
&+\frac{1}{h^2}\int_0^b\int_0^t\int_{\Gamma_V}\left(\overline{g}_{V_h}-\overline{g}_V\right)\left(\tilde{p}_h^k-\tilde{p}^k\right)\mathrm{dS}(x)\mathrm{d}\tau\mathrm{d}t+\frac{1}{h^2}\int_0^b\int_0^t\int_\Omega\left(T_h^*-T^*\right)\left(\varepsilon_{h,t}^{p,k}-\varepsilon_t^{p,k}\right)\mathrm{d}x\mathrm{d}\tau\mathrm{d}t\\
&-\frac{1}{kh^2}\int_0^b\int_0^t\int_{\Gamma_N}\left(\overline{g}_{N_h}-\overline{g}_N\right)\left(v_h^k-v^k-(v_h^*-v^*)\right)\mathrm{dS}(x)\mathrm{d}\tau\mathrm{d}t.
\end{aligned}
\tag{7.1}
$$

Due to the regularity of the data as in the Theorem 5.2 and the Lebesgue's Dominated Convergence Theorem one can pass to the limits with $h\to 0^+$ in every integral except the last two. This is due to the fact that, as in the proof of the Theorem 5.2, we do not control the difference $v_t^k-v^*$ in $L^2(0,T_e;H^{\frac{1}{2}}(\partial\Omega;\mathbb{R}^3))$. Moreover, we do not have any information of $\varepsilon_{tt}^{p,k}$. Therefore, for those two integrals we apply a similar reasoning.

As an example we consider one of these integrals. Firstly, observe that for a fixed k integrating by parts yields for a.e. t

$$
\begin{aligned}
\int_0^t\int_\Omega\left(T_h^*-T^*\right)\left(\varepsilon_{h,t}^{p,k}-\varepsilon_t^{p,k}\right)\mathrm{d}x\mathrm{d}\tau&=-\int_0^t\int_\Omega\left(T_{h,t}^*-T_t^*\right)\left(\varepsilon_h^{p,k}-\varepsilon^{p,k}\right)\mathrm{d}x\mathrm{d}\tau\\
&+\int_\Omega\left(T_h^*(t)-T^*(t)\right)\left(\varepsilon_h^{p,k}(t)-\varepsilon^{p,k}(t)\right)\mathrm{d}x-\int_\Omega\left(T_h^*(0)-T^*(0)\right)\left(\varepsilon_h^{p,k}(0)-\varepsilon^{p,k}(0)\right)\mathrm{d}x.
\end{aligned}
$$

Hence, the Lebesgue's Dominated Convergence Theorem gives

$$
\begin{aligned}
\frac{1}{h^2}\int_0^b\int_0^t\int_\Omega\left(T_h^*-T^*\right)\left(\varepsilon_{h,t}^{p,k}-\varepsilon_t^{p,k}\right)\mathrm{d}x\mathrm{d}\tau\mathrm{d}t\xrightarrow{h\to 0^+}&\int_0^b\int_\Omega T_t^*(t)\varepsilon_t^{p,k}(t)\mathrm{d}x\mathrm{d}t-b\int_\Omega T_t^*(0)\varepsilon_t^{p,k}(0)\mathrm{d}x\\
&-\int_0^b\int_0^t\int_\Omega T_{tt}^*\varepsilon_t^{p,k}\mathrm{d}x\mathrm{d}\tau\mathrm{d}t.
\end{aligned}
$$

Proceeding similarly one obtains

$$\frac{1}{h^2}\int_0^b\int_0^t\int_{\Gamma_N}\left(\overline{g}_{N_h}-\overline{g}_N\right)\left(v_h^k-v^k-(v_h^*-v^*)\right)\mathrm{dS}(x)\mathrm{d}\tau\mathrm{d}t \xrightarrow{h\to 0^+} \int_0^b\int_{\Gamma_N}\overline{g}_{N,t}(t)\left(v^k(t)-v^*(t)\right)\mathrm{dS}(x)\mathrm{d}t$$

$$-b\int_{\Gamma_N}\overline{g}_{N,t}(0)\left(u_1-u_1^*\right)\mathrm{dS}(x)-\int_0^b\int_0^t\int_{\Gamma_N}\overline{g}_{N,tt}\left(v^k-v^*\right)\mathrm{dS}(x)\mathrm{d}\tau\mathrm{d}t.$$

Combining these results we obtain from the inequality (7.1) the estimate

$$\begin{aligned}
&\int_0^b\mathcal{E}^k\left(v_t^k-v_t^*,\varepsilon_t^k-\varepsilon_t^*,\varepsilon_t^{p,k}\right)(t)\mathrm{d}t+\frac{c_0}{2}\int_0^b\left\|\tilde{p}_t^k(t)\right\|_{L^2}^2\mathrm{d}t+c\int_0^b\int_0^t\left\|\nabla\tilde{p}_t^k\right\|_{L^2}^2\mathrm{d}\tau\mathrm{d}t\\
&\leqslant b\mathcal{E}^k\left(v_t^k-v_t^*,\varepsilon_t^k-\varepsilon_t^*,\varepsilon_t^{p,k}\right)(0)+\frac{bc_0}{2}\left\|\tilde{p}_t^k(0)\right\|_{L^2}^2+L\int_0^b\int_0^t\left\|\widehat{T}_t^k\right\|_{L^2}^2\mathrm{d}\tau\mathrm{d}t\\
&+\frac{1}{k}\int_0^b\int_0^t\int_\Omega\mathrm{div}T_t^*\left(v_t^k-v_t^*\right)\mathrm{d}x\mathrm{d}\tau\mathrm{d}t+\int_0^b\int_0^t\int_\Omega\overline{f}_t\tilde{p}_t^k\mathrm{d}x\mathrm{d}\tau\mathrm{d}t-\alpha\int_0^b\int_0^t\int_\Omega\mathrm{div}\left(v_t^*\right)\tilde{p}_t^k\mathrm{d}x\mathrm{d}\tau\mathrm{d}t\\
&+\int_0^b\int_0^t\int_\Omega\left(\overline{F}_t-F_t^*\right)\left(v_t^k-v_t^*\right)\mathrm{d}x\mathrm{d}\tau\mathrm{d}t+\int_0^b\int_0^t\int_{\Gamma_V}\overline{g}_{V,t}\tilde{p}_t^k\mathrm{dS}(x)\mathrm{d}\tau\mathrm{d}t+\int_0^b\int_\Omega T_t^*(t)\varepsilon_t^{p,k}(t)\mathrm{d}x\mathrm{d}t\\
&-b\int_\Omega T_t^*(0)\varepsilon_t^{p,k}(0)\mathrm{d}x-\int_0^b\int_0^t\int_\Omega T_{tt}^*\varepsilon_t^{p,k}\mathrm{d}x\mathrm{d}\tau\mathrm{d}t-\frac{1}{k}\int_0^b\int_{\Gamma_N}\overline{g}_{N,t}(t)\left(v^k(t)-v^*(t)\right)\mathrm{dS}(x)\mathrm{d}t\\
&+\frac{b}{k}\int_{\Gamma_N}\overline{g}_{N,t}(0)\left(u_1-u_1^*\right)\mathrm{dS}(x)+\frac{1}{k}\int_0^b\int_0^t\int_{\Gamma_N}\overline{g}_{N,tt}\left(v^k-v^*\right)\mathrm{dS}(x)\mathrm{d}\tau\mathrm{d}t.
\end{aligned}\tag{7.2}$$

Now we proceed with estimates of the expressions on the right hand side of (7.2).

- First two integrals can be estimate by initial conditions i.e.

$$b\mathcal{E}^k\left(v_t^k-v_t^*,\ \varepsilon_t^k-\varepsilon_t^*,\varepsilon_t^{p,k}\right)(0)+\frac{bc_0}{2}\int_\Omega\left|\tilde{p}_t^k(0)\right|^2\mathrm{d}x$$

$$\leqslant C(T_e)\left(\left\|u_{tt}^k(0)\right\|_{L^2}^2+\left\|u_{tt}^*(0)\right\|_{L^2}^2+\left\|\varepsilon(u_1)\right\|_{L^2}^2\left\|\varepsilon(u_1^*)\right\|_{L^2}^2+\left\|\varepsilon_t^{p,k}(0)\right\|_{L^2}^2+\left\|\tilde{p}_t^k(0)\right\|_{L^2}^2\right)$$

To obtain this estimates independent of k one can use a similar argument as in the proof of Theorem 5.2.

- $$\frac{1}{k}\int_0^b\int_0^t\int_{\Gamma_N}\overline{g}_{N,tt}\left(v^k-v^*\right)\mathrm{dS}(x)\mathrm{d}\tau\mathrm{d}t-\frac{1}{k}\int_0^b\int_{\Gamma_N}\overline{g}_{N,t}(t)\left(v^*(t)-v^k(t)\right)\mathrm{dS}(x)\mathrm{d}t$$

$$\leqslant\frac{Cb}{k}\int_0^b\left\|\overline{g}_{N,tt}\right\|_{H^{-\frac12}(\Gamma_N)}\left\|\varepsilon_t^k-\varepsilon_t^*\right\|_{L^2(\Omega)}\mathrm{d}\tau+\frac{C}{k}\int_0^b\left\|\overline{g}_{N,t}\right\|_{H^{-\frac12}(\Gamma_N)}\left\|\varepsilon_t^k-\varepsilon_t^*\right\|_{L^2(\Omega)}\mathrm{d}t$$

$$\leqslant 2\nu\int_0^b\mathcal{E}^k\left(v_t^k-v_t^*,\varepsilon_t^k-\varepsilon_t^*,\varepsilon_t^{p,k}\right)(\tau)\mathrm{d}\tau+C(T_e,\nu)\left(\left\|\overline{g}_{N,tt}\right\|_{L^2(H^{-\frac12})}^2+\left\|\overline{g}_{N,t}\right\|_{L^2(H^{-\frac12})}^2\right).$$

- Using the Lemma 6.3 we get

$$\int_0^b\int_\Omega T_t^*(t)\varepsilon_t^{p,k}(t)\mathrm{d}x\mathrm{d}t-\int_0^b\int_0^t\int_\Omega T_{tt}^*\varepsilon_t^{p,k}\mathrm{d}x\mathrm{d}\tau\mathrm{d}t$$

$$\leqslant\left(\left\|T_t^*\right\|_{L^{1+\omega}(L^{1+\omega})}+b\left\|T_{tt}^*\right\|_{L^{1+\omega}(L^{1+\omega})}\right)\left\|\varepsilon_t^{p,k}\right\|_{L^{1+\frac{1}{\omega}}(L^{1+\frac{1}{\omega}})}\leqslant C(T_e),$$

By using standard estimates for the rest terms in (7.2) one obtains

$$\begin{aligned}\int_0^b \mathcal{E}^k\left(v_t^k - v_t^*,\ \varepsilon_t^k - \varepsilon_t^*, \varepsilon_t^{p,k}\right)(t)\mathrm{d}t + \frac{c_0}{2}\int_0^b \left\|\widetilde{p}_t^k(t)\right\|_{L^2}^2 \mathrm{d}t + c\int_0^b\int_0^t \left\|\nabla\widetilde{p}_t^k\right\|_{L^2}^2 \mathrm{d}\tau\mathrm{d}t \\ \leqslant C(T_e,\nu,\rho) + CL\int_0^b\int_0^t \mathcal{E}^k\left(v_t^k - v_t^*, \varepsilon_t^k - \varepsilon_t^*, \varepsilon_t^{p,k}\right)(\tau)\mathrm{d}\tau\mathrm{d}t + 2\nu\int_0^b \left\|\widetilde{p}_t^k(t)\right\|_{L^2}^2 \mathrm{d}t \\ + 4\nu\int_0^b \mathcal{E}^k\left(v_t^k - v_t^*, \varepsilon_t^k - \varepsilon_t^*, \varepsilon_t^{p,k}\right)(t)\mathrm{d}t + \nu\int_0^b\int_0^t \left\|\nabla\widetilde{p}_t^k(\tau)\right\|_{L^2}^2 \mathrm{d}\tau\mathrm{d}t.\end{aligned} \tag{7.3}$$

Taking a sufficiently small ν gives, for all $b \in [0, T_e)$, the following estimate

$$\begin{aligned}\int_0^b \mathcal{E}^k\left(v_t^k - v_t^*, \varepsilon_t^k - \varepsilon_t^*, \varepsilon_t^{p,k}\right)(t)\mathrm{d}t + \int_0^b \left\|\widetilde{p}_t^k(t)\right\|_{L^2}^2 \mathrm{d}t + \int_0^b\int_0^t \left\|\nabla\widetilde{p}_t^k\right\|_{L^2}^2 \mathrm{d}\tau\mathrm{d}t \\ \leqslant C(T_e) + CL\int_0^b\int_0^t \mathcal{E}^k\left(v_t^k - v_t^*, \varepsilon_t^k - \varepsilon_t^*, \varepsilon_t^{p,k}\right)(\tau)\mathrm{d}\tau\mathrm{d}t,\end{aligned} \tag{7.4}$$

where the constants $C(T_e)$ and C are independent of k.

Remark 7.3. The time interval can be closed, as the reasoning can be done (with some minor amendments) for $b = T_e - h$.

Applying the Gronwall's inequality to (7.4) ends the proof. □

The estimate from the Lemma 7.2 allows us to raise the regularity of a solution. Note that in the special case when one considers the Dirichlet boundary conditions instead of the mixed boundary conditions it is possible to obtain a better estimate without the safe–load conditions, i.e.

Proposition 7.4 (Time-derivatives estimate for the Dirichlet-type problem). *Consider (1.1)-(1.3) only with the Dirichlet-type boundary conditions for u and p (i.e. $\Gamma_D = \Gamma_P = \partial\Omega$ in (1.2)). Then, under the assumptions of the Theorem 5.2, the following estimate holds*

$$\mathcal{E}^k\left(u_{tt}^k, \varepsilon_t^k, \varepsilon_t^{p,k}\right)(t) + \int_0^t \left\|\widetilde{p}_t^k\right\|_{H^1}^2 \mathrm{d}\tau + \left\|\widetilde{p}_t^k(t)\right\|_{L^2}^2 \leqslant C(T_e), \qquad \textit{for a.e. } t \in [0, T_e],$$

where $(u^k, \widetilde{p}^k, \varepsilon^{p,k})$ is the weak solution of (5.2) with the initial-boundary conditions (1.2)-(1.3). Furthermore $C(T_e)$ is independent of k.

To prove this proposition one has to follow the proof of the Theorem 5.2. We omit this procedure here.

Let us introduce the new space of test functions, i.e.

$$\mathbb{W}^k := \left\{w \in W^{1,k}(\Omega;\mathbb{R}) : v = 0 \text{ on } \Gamma_P\right\}.$$

Recall that $\mathbb{W} = \mathbb{W}^2$ and $\mathbb{W}^k \subset \mathbb{W}$ for $k \geqslant 2$. We can now define the higher regularity solutions to (1.1)-(1.3).

Definition 7.5 (Higher regularity solution of (1.1)-(1.3)). We say that the quadruple $(u, p, \varepsilon^p, T = \mathcal{D}(\varepsilon(u) - \varepsilon^p))$ is a solution to (1.1)-(1.3) with the higher regularity, if

1. For a.e. $t \in [0, T_e]$ the triple $(\widetilde{u}, \widetilde{p}, \varepsilon^p)$ satisfies the system of equations:

$$\rho\int_\Omega \widetilde{u}_{tt} v\mathrm{d}x + \int_\Omega \mathcal{D}(\varepsilon(\widetilde{u}) - \varepsilon^p)\varepsilon(v)\mathrm{d}x - \alpha\int_\Omega \widetilde{p}\,\mathrm{div}v\mathrm{d}x = \int_\Omega \widetilde{F}v\mathrm{d}x + \int_{\Gamma_N} \widetilde{g}_N v\mathrm{dS}(x), \quad \forall v \in \mathbb{V}, \tag{7.5}$$

$$c_0\int_\Omega \widetilde{p}_t w\mathrm{d}x + c\int_\Omega \nabla\widetilde{p}\nabla w\mathrm{d}x + \alpha\int_\Omega \mathrm{div}\widetilde{u}_t\, w\mathrm{d}x = \int_\Omega \widetilde{f}w\mathrm{d}x + \int_{\Gamma_V} \widetilde{g}_V w\mathrm{dS}(x), \quad \forall w \in \mathbb{W}^{1+\omega}. \tag{7.6}$$

2. The fifth equation of (1.1) is satisfied in a sense of the Young measures, i.e.
$$\varepsilon^p_t(x,t) = \int\limits_{\mathcal{S}(3)} A(S)\mathrm{d}\nu_{(x,t)}(S),$$
where $\{\nu_{(x,t)}\}$ is the Young measure.
3. $\varepsilon^p(x,0) = \varepsilon^p_0(x), \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad p(x,0) = p_0(x).$

Furthermore, the following regularities are required
$$\begin{gathered} u \in H^2(0,T_e;L^2(\Omega;\mathbb{R}^3)) \cap W^{1,1+\frac{1}{\omega}}(0,T_e;(W^{1,1+\frac{1}{\omega}}(\Omega;\mathbb{R}^3))), \\ p \in L^2(0,T_e;H^1(\Omega;\mathbb{R})) \cap H^1(0,T_e;L^2(\Omega;\mathbb{R})), \quad \varepsilon^p \in W^{1,1+\frac{1}{\omega}}(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}(3))), \\ T \in H^1(0,T_e;L^2(\Omega;\mathcal{S}(3))) \cap L^{1+\omega}(0,T_e;L^{1+\omega}(\Omega;\mathcal{S}(3))). \end{gathered} \tag{7.7}$$

It appears that with the sufficiently high regularity of the data one can prove the existence of such solution.

Theorem 7.6 (Existence of the higher regularity solutions). *Assume that the initial values u_0, u_1, p_0, ε^p_0 and given functions F, f, g_D, g_N, g_P, g_V satisfy the compatibility conditions (4.6), have the regularities (5.4)-(5.7) and g_D, $\overline{g}_N$ satisfy the improved safe–load conditions with the regularity $1+\omega$. Then, there exists a solution (u,p,ε^p,T) of (1.1)-(1.3) with the higher regularity according to the Definition 7.5.*

Proof. Firstly, we prove the existence of a solution $(u,\widetilde{p},\varepsilon^p,T)$ with the higher regularity (according to the Definition 7.5) in a case of (5.1), then by the previous remarks the existence of the solution in a case (1.1) will immediately follow.

By the Lemma 6.3 and the Lemma 7.2 we can extract the subsequences of k (denoted with the same symbol) such that

- $u^k \rightharpoonup u$ in $H^2(0,T_e;L^2(\Omega;\mathbb{R}^3)) \cap W^{1,1+\frac{1}{\omega}}(0,T_e;W^{1,1+\frac{1}{\omega}}(\Omega;\mathbb{R}^3))$,
- $\widetilde{p}^k \rightharpoonup \widetilde{p}$ in $L^2(0,T_e;H^1(\Omega;\mathbb{R})) \cap H^1(0,T_e;L^2(\Omega;\mathbb{R}))$,
- $\widehat{T}^k \rightharpoonup T$ in $H^1(0,T_e;L^2(\Omega;\mathcal{S}(3))) \cap L^{1+\omega}(0,T_e;L^{1+\omega}(\Omega;\mathcal{S}(3)))$,
- $\frac{1}{k}\varepsilon(\widetilde{u}^k) \rightharpoonup 0$ in $H^1(0,T_e;L^2(\Omega;\mathcal{S}(3)))$,
- $\varepsilon^{p,k} \rightharpoonup \varepsilon^p$ in $W^{1,1+\frac{1}{\omega}}(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}(3)))$,
- $\mathrm{div}u^k \rightharpoonup \mathrm{div}u$ in $W^{1,1+\frac{1}{\omega}}(0,T_e;L^{1+\frac{1}{\omega}}(\Omega;\mathcal{S}(3)))$.

Let $(u^k,\widetilde{p}^k,\varepsilon^{p,k})$ be the solution of (5.2). The following system is satisfied for a.e. $t \in [0,T_e]$ and for all $v \in \mathbb{V}$, $w \in \mathbb{W} \cap C^\infty(\Omega)$
$$\rho\int\limits_\Omega \widetilde{u}^k_{tt}v\mathrm{d}x + \int\limits_\Omega \mathcal{D}\left(\varepsilon(\widetilde{u}^k) - \varepsilon^{p,k} + \frac{1}{k}\varepsilon(\widetilde{u}^k)\right)\varepsilon(v)\mathrm{d}x - \alpha\int\limits_\Omega \widetilde{p}^k\,\mathrm{div}v\mathrm{d}x = \int\limits_\Omega \widetilde{F}^k v\mathrm{d}x + \int\limits_{\Gamma_N} \widetilde{g}^k_N v\mathrm{dS}(x),$$
$$c_0\int\limits_\Omega \widetilde{p}^k_t w\mathrm{d}x + c\int\limits_\Omega \nabla\widetilde{p}^k\nabla w\mathrm{d}x + \alpha\int\limits_\Omega \mathrm{div}\widetilde{u}^k_t\, w\mathrm{d}x = \int\limits_\Omega \widetilde{f}w\mathrm{d}x + \int\limits_{\Gamma_V} \widetilde{g}_V w\mathrm{dS}(x),$$
$$\varepsilon^{p,k}_t(x,t) = A\left(\widehat{T}^k(x,t)\right), \qquad \text{for a.e. } (x,t) \in \Omega\times[0,T_e].$$

We multiply the first and the second equation by $\varphi_1,\varphi_2 \in C^\infty([0,T_e])$ (respectively) and integrate over time. Now, due to the weak convergences stated above we can pass to the limit with k:
$$\int\limits_0^{T_e}\left(\rho\int\limits_\Omega \widetilde{u}_{tt}v\mathrm{d}x + \int\limits_\Omega \mathcal{D}\left(\varepsilon(\widetilde{u}) - \varepsilon^p\right)\varepsilon(v)\mathrm{d}x - \alpha\int\limits_\Omega \widetilde{p}\,\mathrm{div}v\mathrm{d}x - \int\limits_\Omega \widetilde{F}v\mathrm{d}x - \int\limits_{\Gamma_N} \widetilde{g}_N v\mathrm{dS}(x)\right)\varphi_1\mathrm{d}\tau = 0,$$
$$\int\limits_0^{T_e}\left(c_0\int\limits_\Omega \widetilde{p}_t w\mathrm{d}x + c\int\limits_\Omega \nabla\widetilde{p}\nabla w\mathrm{d}x + \alpha\int\limits_\Omega \mathrm{div}\widetilde{u}_t\, w\mathrm{d}x - \int\limits_\Omega \widetilde{f}w\mathrm{d}x - \int\limits_{\Gamma_V} \widetilde{g}_V w\mathrm{dS}(x)\right)\varphi_2\mathrm{d}\tau = 0,$$

where the first equation is satisfied for every $v \in \mathbb{V}$ and the second equation for every $w \in \mathbb{W} \cap C^\infty(\Omega)$. Due to the density argument and the regularity of the limit functions we can write the second equation for every $w \in \mathbb{W}^{1+\omega}$. By the du Bois-Reymond Lemma one can omit the time integrals. The rest of the proof is as in the Theorem 6.4. □

Proposition 7.7. *Considering only the Dirichlet-type boundary conditions one does not require the compatibility conditions to prove the Theorem 7.6. Moreover, one can obtain the following regularity of the solution*

$$u \in W^{2,\infty}(0,T_e;L^2(\Omega;\mathbb{R}^3)), \quad p \in W^{1,\infty}(0,T_e;L^2(\Omega;\mathbb{R})), \quad T \in W^{1,\infty}(0,T_e;L^2(\Omega;\mathcal{S}(3))).$$

Unfortunately, even redefining the meaning of the solution (see the Definition 7.5) is not satisfactory from the applications point of view, since the constitutive equation is satisfied merely in the sense of the Young measures. This is also the main culprit responsible for the handicap with uniqueness theorems for the higher regularity solutions. Therefore our main goal is to dispose of the Young measures and prove the existence of the `regular` solutions to (1.1)-(1.3).

Definition 7.8 (`Regular` solution to (1.1)-(1.3)). We say that the quadruple (u, p, ε^p, T) is a `regular` solution of (1.1)-(1.3), if it is the solution according to the Definition 7.5 and the fifth equation of (1.1) is satisfied a.e. in $\Omega \times [0, T_e]$, i.e.

$$\varepsilon^p_t(x,t) = A(T(x,t)) \quad \text{for a.e.} \quad (x,t) \in \Omega \times (0,T_e).$$

One can prove the existence of such solutions assuming more about A, namely the monotonicity plays crucial role. Unfortunately, the monotonicity is still not sufficient to proceed with the desired operations in the proof. We wish to do the integration by parts in the expressions of the form:

$$\int_0^t \int_\Omega (u_t - u_t^*)\, \nabla\widetilde{p} \mathrm{d}x\mathrm{d}\tau, \quad \int_0^t \int_\Omega \widetilde{p}\Delta\widetilde{p} \mathrm{d}x\mathrm{d}\tau.$$

In order to make this kind of operations properly justified, we assume the following property of a field A

$$A : \mathcal{S}(3) \to P\mathcal{S}(3), \tag{7.8}$$

where P is projection onto the deviatoric part of the tensor given by the formula $PT = T - \frac{1}{3}\mathrm{tr}(T) \cdot \mathbb{I}$. The condition 7.8 allows one to raise the regularity of $\mathrm{div} u_t$ and $\Delta\widetilde{p}$ up to $L^2(L^2)$ by the reasoning:

$$\mathrm{div} u_t = \mathrm{tr}\varepsilon(u_t) = \mathrm{tr}\mathcal{D}^{-1}(T_t) + \mathrm{tr}\varepsilon^p_t = \mathrm{tr}\mathcal{D}^{-1}(T_t) \in L^2(0,T_e;L^2(\Omega;\mathbb{R})),$$

Hence, the weak Laplacian of the pressure is equal to

$$-c\Delta\widetilde{p} = \overline{f} - c_0\widetilde{p}_t - \alpha \mathrm{div} u_t \in L^2(0,T_e;L^2(\Omega;\mathbb{R})).$$

Moreover, upon the remark above (assuming the condition 7.8) the equation (7.6) from the Definition 7.5 and from the Definition 7.8 can be tested by a larger class of functions, namely $\mathbb{W}$. The condition 7.8 is natural, since it is common to assume that the inelastic deformation tensor is deviatoric (see [8], [12]).

Theorem 7.9 (Existence of `regular` solutions of (1.1)-(1.3)). *Assume the same as in the Theorem 7.6 and let A be monotone, satisfying the condition 7.8. Then there exists the `regular` solution (u, p, ε^p, T) of (1.1)-(1.3).*

Proof. In the proof we exploit the Minty's Monotone Trick (see [11]). We take the subsequence of solutions to the coercive model (the same as in the proof of the Theorem 7.6). We know that the weak limit of this subsequence is the solution with the higher regularity. We only need to prove that the constitutive equation is satisfied almost everywhere.

Proceeding similarly as in the beginning of the proof of the Lemma 6.3 we obtain

$$\begin{aligned}
&\int_0^t\int_\Omega A\left(\widehat{T}^k\right)\widehat{T}^k \mathrm{d}x\mathrm{d}\tau = -\mathcal{E}^k\left(u_t^k-u_t^*,\varepsilon^k-\varepsilon^*,\varepsilon^{p,k}\right)(t) - \frac{c_0}{2}\left\|\widetilde{p}^k(t)\right\|_{L^2}^2 - c\int_0^t\left\|\nabla\widetilde{p}^k(t)\right\|_{L^2}^2 \mathrm{d}x\mathrm{d}\tau\\
&+\mathcal{E}^k\left(u_t^k-u_t^*,\varepsilon^k-\varepsilon^*,\varepsilon^{p,k}\right)(0) + \frac{c_0}{2}\left\|\widetilde{p}_0\right\|_{L^2}^2 + \int_0^t\int_\Omega \left(u_t^k-u_t^*\right)\left(\overline{F}-F^*\right)\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Omega \overline{f}\widetilde{p}^k\mathrm{d}x\mathrm{d}\tau\\
&+\frac{1}{k}\int_0^t\int_\Omega \mathrm{div}T^*\left(u_t^k-u_t^*\right)\mathrm{d}x\mathrm{d}\tau - \alpha\int_0^t\int_\Omega \mathrm{div}u_t^*\widetilde{p}^k\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Omega T^*\varepsilon_t^{p,k}\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{\Gamma_V}\overline{g}_V\widetilde{p}^k\mathrm{dS}(x)\mathrm{d}\tau\\
&-\frac{1}{k}\int_0^t\int_{\Gamma_N}\overline{g}_N\left(u_t^k-u_t^*\right)\mathrm{dS}(x)\mathrm{d}\tau.
\end{aligned}\tag{7.9}$$

Observe that $f(S) := \mathcal{D}(S)S$ is weakly lower semi-continuous, since it is convex and continuous. The convexity may not be clear, however

$$f(\mu S_1 + (1-\mu)S_2) = \mu^2\mathcal{D}(S_1)S_1 + \mu(1-\mu)(\mathcal{D}(S_1)S_2 + \mathcal{D}(S_2)S_1) + (1-\mu)^2\mathcal{D}(S_2)S_2,$$

Thus, by the linearity and the positive-definiteness one has

$$0 \leqslant \mathcal{D}(S_1-S_2)(S_1-S_2) = \mathcal{D}(S_1)S_1 + \mathcal{D}(S_2)S_2 - \mathcal{D}(S_1)S_2 - \mathcal{D}(S_2)S_1.$$

Combining the expressions above yields

$$f(\mu S_1 + (1-\mu)S_2) = \mu\mathcal{D}(S_1)S_1 + (1-\mu)\mathcal{D}(S_2)S_2 = \mu f(S_1) + (1-\mu)f(S_2).$$

We now take the lower limit as $k\to\infty$ of the equation (7.9), use the weak lower semi-continuity of the norm and the Fatou's lemma leads to the inequality

$$\begin{aligned}
&\liminf_{k\to\infty}\int_0^t\int_\Omega A\left(\widehat{T}^k\right)\widehat{T}^k\mathrm{d}x\mathrm{d}\tau \leqslant -\mathcal{E}\left(u_t-u_t^*,\varepsilon-\varepsilon^*,\varepsilon^p\right)(t) - \frac{c_0}{2}\left\|\widetilde{p}(t)\right\|_{L^2}^2 - c\int_0^t\left\|\nabla\widetilde{p}(t)\right\|_{L^2}^2\mathrm{d}x\mathrm{d}\tau\\
&+\mathcal{E}\left(u_t-u_t^*,\varepsilon-\varepsilon^*,\varepsilon^p\right)(0) + \frac{c_0}{2}\left\|\widetilde{p}_0\right\|_{L^2}^2 + \int_0^t\int_\Omega\left(u_t-u_t^*\right)\left(\overline{F}-F^*\right)\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Omega\overline{f}\widetilde{p}\mathrm{d}x\mathrm{d}\tau\\
&-\alpha\int_0^t\int_\Omega\mathrm{div}u_t^*\widetilde{p}\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Omega T^*\varepsilon_t^p\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_{\Gamma_V}\overline{g}_V\widetilde{p}\mathrm{dS}(x)\mathrm{d}\tau.
\end{aligned}$$

Due to the result of the Theorem 7.6 the weak limit $(u,\widetilde{p},\varepsilon^p,T)$ is the solution of (5.1) with the initial-boundary conditions (1.2)-(1.3) (according to the Definition 7.5). The regularity of such solutions is large enough to allow the first two equations of (5.1) to hold a.e. in $\Omega\times[0,T_e]$. Because of that we can proceed similarly as before with $(u,\widetilde{p},\varepsilon^p,T)$.

$$\begin{aligned}
&\mathcal{E}\left(u_t-u_t^*,\varepsilon-\varepsilon^*,\varepsilon^p\right)(t) - \mathcal{E}\left(u_t-u_t^*,\varepsilon-\varepsilon^*,\varepsilon^p\right)(0)\\
&=\rho\int_0^t\int_\Omega\left(u_t-u_t^*\right)\left(u_{tt}-u_{tt}^*\right)\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Omega\left(T-T^*\right)\left(\varepsilon_t-\varepsilon_t^*-\varepsilon_t^p\right)\mathrm{d}x\mathrm{d}\tau\\
&=\int_0^t\int_\Omega\left(u_t-u_t^*\right)\mathrm{div}\left(T-T^*\right)\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Omega\left(u_t-u_t^*\right)\left(\overline{F}-F^*\right)\mathrm{d}x\mathrm{d}\tau - \alpha\int_0^t\int_\Omega\left(u_t-u_t^*\right)\nabla\widetilde{p}\mathrm{d}x\mathrm{d}\tau\\
&\quad+\int_0^t\int_\Omega\left(T-T^*\right)\left(\varepsilon_t-\varepsilon_t^*\right)\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_\Omega T^*\varepsilon_t^p\mathrm{d}x\mathrm{d}\tau - \int_0^t\int_\Omega T\varepsilon_t^p\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Similar calculation as in the proof of Lemma 7.2 gives

$$\int_0^t\int_\Omega (T-T^*)\left(\varepsilon_t-\varepsilon_t^*\right)\mathrm{d}x\mathrm{d}\tau-\alpha\int_0^t\int_\Omega \left(u_t-u_t^*\right)\nabla\widetilde{p}\mathrm{d}x\mathrm{d}\tau$$
$$=\int_0^t\int_\Omega \overline{f}\widetilde{p}\mathrm{d}x\mathrm{d}\tau-c\int_0^t\|\nabla\widetilde{p}(t)\|^2_{L^2}\mathrm{d}x\mathrm{d}\tau+\int_0^t\int_{\Gamma_V}\overline{g}_V\widetilde{p}\mathrm{dS}(x)\mathrm{d}\tau-\frac{c_0}{2}\|\widetilde{p}(t)\|^2_{L^2}$$
$$+\frac{c_0}{2}\|\widetilde{p}_0\|^2_{L^2}-\alpha\int_0^t\int_\Omega \mathrm{div}u_t^*\widetilde{p}\mathrm{d}x\mathrm{d}\tau-\int_0^t\int_\Omega \mathrm{div}\left(T-T^*\right)\left(u_t-u_t^*\right)\mathrm{d}x\mathrm{d}\tau.$$

Our discussion leads to the following inequality

$$\liminf_{k\to\infty}\int_0^t\int_\Omega A\left(\widehat{T}^k\right)\widehat{T}^k\mathrm{d}x\mathrm{d}\tau\leqslant\int_0^t\int_\Omega \varepsilon_t^pT\mathrm{d}x\mathrm{d}\tau. \tag{7.10}$$

To end the proof we use the Minty's Monotone Trick. Take any $S\in L^{1+\omega}(0,T_e;L^{1+\omega}(\Omega;\mathcal{S}(3)))$, due to the monotonicity of A we have

$$\int_0^t\int_\Omega \left(A(\widehat{T}^k)-A(S)\right)\left(\widehat{T}^k-S\right)\mathrm{d}x\mathrm{d}\tau\geqslant 0.$$

By splitting the integral above into four parts, using the convergence results and (7.10) one obtain

$$\int_0^t\int_\Omega \left(\varepsilon_t^p-A(S)\right)(T-S)\,\mathrm{d}x\mathrm{d}\tau\geqslant 0.$$

Now put $S=T-\mu W$, where $W\in L^{1+\omega}(0,T_e;L^{1+\omega}(\Omega;\mathcal{S}(3)))$, $\mu>0$ are arbitrary. It gives

$$\int_0^t\int_\Omega \left(\varepsilon_t^p-A(T-\mu W)\right)W\mathrm{d}x\mathrm{d}\tau\geqslant 0.$$

Using the linear growth we can dominate the integrand by the integrable function. Indeed, $(\varepsilon_t^p-A(T-\mu W))\in L^{1+\frac{1}{\omega}}(L^{1+\frac{1}{\omega}})$. Due to the continuity of the operator A

$$\int_0^t\int_\Omega \left(\varepsilon_t^p-A(T)\right)W\mathrm{d}x\mathrm{d}\tau\geqslant 0.$$

Because the sign of W is arbitrary, we can replace the inequality above with the equality. Moreover, this result is true for a.e. $t\in[0,T_e]$. Thus, we take the sequence of such *proper* times $t\to T_e$ and obtain

$$\int_0^{T_e}\int_\Omega \left(\varepsilon_t^p-A(T)\right)W\mathrm{d}x\mathrm{d}\tau=0,\qquad \forall W\in L^{1+\omega}(0,T_e;L^{1+\omega}(\Omega;\mathcal{S}(3)).$$

Since W is arbitrary, we obtain $\varepsilon_t^p(x,t)=A(T(x,t))$ for a.e. $(x,t)\in\Omega\times[0,T_e]$. □

It appears that the `regular` solutions are unique, although the proof of this fact involves, as before, the integration by parts. Thus we must additionally assume the condition 7.8.

Lemma 7.10 (Uniqueness of `regular` solutions to (1.1)-(1.3)). *Let A satisfy 7.8 and let $(u_1,p_1,\varepsilon_1^p,T_1)$ and $(u_2,p_2,\varepsilon_2^p,T_2)$ be the two distinct* `regular` *solutions of (1.1)-(1.3). Then*

$$(u_1,p_1,\varepsilon_1^p,T_1)=(u_2,p_2,\varepsilon_2^p,T_2).$$

Remark 7.11. In the Lemma 7.10 we do not assume the monotonicity of A (unlike in the previous lemma).

Proof. The similar calculations as in the proof of Theorem 7.9 yield

$$\mathcal{E}\left(u_{1,t}-u_{2,t},\ \varepsilon_1-\varepsilon_2,\varepsilon_1^p-\varepsilon_2^p\right)(t) = -\int_0^t\int_\Omega (T_1-T_2)\left(\varepsilon_{1,t}^p-\varepsilon_{2,t}^p\right)\mathrm{d}x\mathrm{d}\tau - \frac{c_0}{2}\|p_1(t)-p_2(t)\|_{L^2}^2 - c\int_0^t \|\nabla(p_1-p_2)\|_{L^2}^2\,\mathrm{d}\tau.$$

Using the properties of the functions m, l gives

$$\mathcal{E}\left(u_{1,t}-u_{2,t},\varepsilon_1-\varepsilon_2,\varepsilon_1^p-\varepsilon_2^p\right)(t) + c\int_0^t \|\nabla(p_1-p_2)\|_{L^2}^2\,\mathrm{d}\tau + \frac{c_0}{2}\|p_1(t)-p_2(t)\|_{L^2}^2 \leqslant CL\int_0^t \mathcal{E}\left(u_{1,t}-u_{2,t},\varepsilon_1-\varepsilon_2,\varepsilon_1^p-\varepsilon_2^p\right)(\tau)\mathrm{d}\tau.$$

By the Gronwall's inequality for a.e. $t\in[0,T_e]$ we obtain

$$\mathcal{E}\left(u_{1,t}-u_{2,t},\varepsilon_1-\varepsilon_2,\varepsilon_1^p-\varepsilon_2^p\right)(t)=0.$$

Hence $p_1=p_2$ and $u_1=u_2$ a.e. thus

$$\frac{1}{2}\int_\Omega \mathcal{D}\left(\varepsilon_1^p-\varepsilon_2^p\right)\left(\varepsilon_1^p-\varepsilon_2^p\right)\mathrm{d}x = \mathcal{E}\left(u_{1,t}-u_{2,t},\varepsilon_1-\varepsilon_2,\varepsilon_1^p-\varepsilon_2^p\right)(t)=0.$$

Due to the positive-definiteness of $\mathcal{D}$ the proof of the uniqueness is finished. □

Remark 7.12. Under the assumptions of the Theorem 7.9, there exists the unique `regular` solution of (1.1)-(1.3).

One can easily replace the condition 7.8 in the Theorem 7.9 and in the Lemma 7.10 by the following

$$\omega\leqslant 5 \quad \text{in growth conditions} \quad (1.4). \tag{7.11}$$

The condition 7.11 combined with the Sobolev Embedding Theorem leads to $\widetilde{p}\in L^{1+\omega}(L^{1+\omega})$ which allows one to integrate by parts and so to repeat steps from the previous proofs.

An open question remains: is this result true, provided that one omits the conditions 7.8 or 7.11? It is worth noticing that these assumptions can be replaced by the assumption: $p\in L^{1+\omega}(L^{1+\omega})$. Hence, the promising approach is either an attempt to raise the regularity of the pressure p or usage of the more sophisticated methods, to obtain the result above.

Remark 7.13. One can obtain similar results (with the lesser regularity of p) with the partially dynamic model (provided that $c_0p_t\approx 0$). This requires very similar arguments, hence we omit details.

8. References

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