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An argument for positive nominal interest[1]

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Abstract

In a dynamic economy, money provides liquidity as a medium of exchange. A central bank that sets the nominal rate of interest and distributes its profit to shareholders as dividends is traded in the asset market. A nominal rates of interest that tend to zero, but do not vanish, eliminate equilibrium allocations that do not converge to a Pareto optimal allocation.

Key words: nominal rate of interest; dynamic efficiency.

JEL classification numbers: D-60; E-10.

1 Introduction

The Pareto optimality of competitive equilibrium allocations is a major tenet of classical welfare economics and the main argument in favor of competitive markets for the allocation of resources. Deviations from the classical paradigm sever the link between Pareto optimal and competitive equilibrium allocations, with repercussions both for the theory and practice of economic policy.

Competitive equilibrium allocations may fall short of Pareto optimality in two distinct, if related, situations: (i) in economies that extend over an infinite horizon with a demographic structure of overlapping generations (Gale (1973), Samuelson (1958)) and (ii) in economies with an operative transactions technology with money that provides liquidity services as a medium of exchange (Clower (1967), Lucas and Stokey (1987))[1].

The dynamic failure of optimality in economies of overlapping generations is well understood: competitive prices that attain market clearing may fail to provide consistent accounting over infinite streams of output. Long-lived productive assets, with streams of output that extend to the infinite future, when traded in asset markets, guarantee that equilibrium prices provide consistent intertemporal valuation and restore the optimality of competitive allocations (Wilson (1981), Santos and Woodford (1997)).

When money serves as a medium of exchange, the nominal rate of interest does not allow competitive prices to exhaust the static gains from trade. Vanishing nominal rates of interest or, equivalently, the payment of interest on money balances on par with the rate of return on alternative stores of value eliminates the suboptimality of monetary equilibria (Friedman (1969)).

The argument here is that low, but not vanishing, nominal rates of interest shield the economy from intertemporal suboptimality at the cost of some static inefficiency. Differently from other arguments for a positive nominal interest, the argument does not appeal to nominal rigidities, imperfect competition or any other imperfection or incompleteness of financial markets.

In an economy of overlapping generations with cash-in-advance constraints, a central bank issues balances in exchange for bonds and distributes its profits, seignorage, as dividends to shareholders Bloise, Drèze, and Polemarchakis (2005), Nakajima and Polemarchakis (2005)). Importantly, shares to the bank are traded in the asset market and the bank is, initially, owned by a finite number of individuals, most simply among those active at the

[1] Search theoretic models of monetary economies (Diamond (1984) or Kiyotaki and Wright (1989)) are, evidently, more satisfactory, but the simple cash-in-advance formulation here, as in much of the literature, offers analytical tractability and does not play an otherwise important role in the argument argument.

starting date of economic activity. Even if not common practice, trades in shares of the centrak bank is not without precedent: shares of the Bank of England were traded until 1946. And it is correct and explicit accounting for the market in shares (Drèze and Polemarchakis (2000)) that has allowed recent formulations to resolve a conundrum (Hahn (1965)) and establish the existence of monetary equilibria even over a finite horizon.

At equilibrium, the market value of the bank is at least equal to the present value of seignorage. Seignorage corresponds to the intertemporal value of net transactions, which is, thus, finite. A condition of intra-generational heterogeneity ensures gains to trade even at intergenerational autarky, which guarantees that, provided that the nominal interest is small enough, some commodities are non-negligibly traded over the entire infinite horizon. As net transactions are finitely valued, so is the aggregate endowment of non-negligibly traded commodities. And, as a consequence, the aggregate endowment is finitely valued at equilibrium, for, otherwise, the relative prices of negligibly traded to non-negligibly traded commodities would explode across periods of trade.

As long as the nominal rate of interest is arbitrarily low, but bounded away from zero, the static inefficiency associated with non-vanishing nominal rates remains but is essentially negligible; more importantly, with the stream of seignorage bounded away from zero, the bank substitutes for the long-lived productive assets that guarantee intertemporal optimality.

In Bloise and Polemarchakis (2006) we gave an argument for the very special case of a simple economy of overlapping generations.

The connection between costly transactions and intertemporal efficiency was recognized in Weiss (1980); the argument there, however, was restricted to steady-state allocations and relied on real balances entering directly the utility functions of individuals with a positive marginal utility everywhere. The argument identified debt with money balances and, more importantly, it did not ensure dynamic efficiency of non-stationary equilibrium allocation.

We organize the development of the argument as follows: In section 2, we give simple examples that illustrate the argument. In section 3, we present the argument in abstract terms, at a level of generality that is comparable to that of Wilson (1981). This only requires the modification of budget constraints of individuals that is inherited from a primitive description of sequential trades through cash-in-advance constraints. We prove the result under a hypothesis of gains to trade that we show (in section 4) to be generically satisfied in standard stationary economies of overlapping generations with intra-generational heterogeneity. In section 4, we describe a monetary economy of overlapping generations with cash-in-advance constraints where a central bank, whose ownership is sequentially traded in the stock market, pegs the nominal rate of interest, accommodates the demand for balances and distributes the seignorage to shareholders as dividends. Not surprisingly, a canonical intertemporal consolidation of sequential budget constraints reveals that relevant equilibrium restrictions of this sequential economy are exactly those in the abstract analysis. We conclude with some remarks. [2]

2 Examples

Simple, stationary economies of overlapping generations illustrate the argument.[3] Dates or periods of trade are $\mathcal{T} = \{0, 1, 2, \ldots, t, \ldots\}$. Each non-initial generation has a life span of two periods and consists of two individuals, $i \in \mathcal{I} = \{a, b\}$. An initially old generation is active at $t = 0$.

2.1

One commodity is exchanged and consumed at each date; the commodity is perishable.

The intertemporal utility function of an individual is

$$u^i(x^i, z^i) = x^i + \ln z^i,$$

where x^i is the excess consumptions of the individual when young, while z^i is the consumption when old.

The endowment of an individual when old is $e^i > 0$ — with quasi linear preferences, it is not necessary to specify the endowment when the individual is young, when a sufficiently large endowment guarantees positive consumption.

The spot price of the commodity is p_t.

Nominal bonds, b_t, of one period maturity, serve to transfer revenue across dates.

The nominal rate of interest is $r_t \geq 0$.

Balances, m_t, provide liquidity services; they also serve as a store of value, but they are dominated as such by bonds.

At each date, a central bank or monetary authority issues bonds in exchange of balances, with

$$\frac{1}{1 + r_t} b_t + m_t = 0,$$

[2] A reader might prefer to reverse the order of presentation we chose by reading section 4 before section 3. This creates no difficulty, after a preliminary reading of the beginning of section 3 for the notation we use.

[3] Minor changes of notation from the abstract argument that follows facilitate the exposition

that it redeems at the following date, earning seignorage

$$b_t + m_t = r_t m_t$$

at date $t+1$, that it distributes as dividend to shareholders.

Shares to the bank, their number normalised to one, are traded at each date and serve as a store of value. In the absence of uncertainty, no-arbitrage requires that the returns to bonds and shares coincide, and as a consequence, the *cum dividend* price of shares, v_t, satisfies

$$v_t = \frac{r_t}{1+r_t} m_t + \frac{1}{1+r_t} v_{t+1};$$

if it is finite,

$$v_0 = \frac{r_0}{1+r_0} m_0 + \sum_{t=1}^{\infty} \frac{1}{1+r^{t-1}} (\frac{r_t}{1+r_t} m_t),$$

where

$$(1+r^t) = (1+r_0) \times \ldots \times (1+r_t), \quad t = 1, \ldots.$$

The rate of inflation is $\pi_{t+1} = (p_{t+1}/p_t) - 1$, and the real rate of interest is $\rho_{t+1} = [(1+r_{t+1})/(1+\pi_{t+1})] - 1$; real balances are $\mu_t = m_t/p_t$.

An individual, young at t, faces the budget constraints

$$p_t x_t^i + \tfrac{1}{1+r_t} b_t^i + m_t^i \leq 0,$$

$$p_{t+1} z_{t+1}^i \leq b_t^i + m_t^i + p_{t+1} e^i$$

and the cash in advance constraint [4]

$$m_t^i \geq p_t x_t^{i-}, \quad m_t^i \geq 0.$$

Equivalently, an individual faces the intertemporal budget constraint

$$x_t^i + \frac{r_t}{1+r_t} x_t^{i-} + \frac{1}{1+\rho_t} (z_{t+1}^i - e^i) \leq 0,$$

with

$$\mu_t^i \geq x_t^{i-}, \quad \mu_t^i \geq 0$$

the associated holdings of real balances.

Similarly, the cum dividend price of shares in real terms φ_t, satisfies

$$\varphi_t = \frac{r_t}{1+r_t} \mu_t + \frac{1}{1+\rho_t} \varphi_{t+1};$$

[4] $x^- i$ is the negative part of x.

if it is finite,

$$\varphi_0 = \frac{r_0}{1+r_0}\mu_0 + \sum_{t=1}^{\infty} \frac{1}{1+\rho^{t-1}} (\frac{r_t}{1+r_t}\mu_t),$$

where

$$(1+\rho^t) = (1+\rho_0) \times \ldots \times (1+\rho_t), \quad t = 1, \ldots.$$

Since shares and bonds are perfect substitutes, it is not necessary either to introduce shares explicitly in the intertemporal optimization of individuals or to distinguish between the initial value of the bank, v_0, and debt held by the initially old.

With $e^a \ll e^b$, along any equilibrium path, $x_t^a < 0$, while $x_t^b > 0$.

The solutions to the optimization problems of individuals are

$$x_t^a(\rho_t) = \tfrac{1}{(1+\rho_t)(1-\theta_t)} e^a - 1 \leq 0, \quad z_{t+1}^a(\rho_t) = (1+\rho_t)(1-\theta_t),$$

$$\mu_t^a(\rho_t) = -x_t^a,$$

and

$$x_t^b(\rho_t) = \tfrac{1}{(1+\rho_t)} e^b - 1 \geq 0, \quad z_{t+1}^b(\rho_t) = (1+\rho_t),$$

$$\mu_t^b(\rho_t) = 0,$$

where, $\theta_t = [r_t/(1+r_t)] < 1$.

Along an equilibrium path,

$$x_t^a + x_t^b + z_t^a + z_t^b = e,$$

where $e = e^a + e^b$ is the aggregate endowment of individuals when old.

With $r_t = r \geq 0$, and, as a consequence, $\theta_t = \theta$, an equilibrium path of real rates of interest satisfies

$$\rho_{t+1} = \frac{e(1-\theta) + \theta e^a}{(1-\theta)(e-(2-\theta)\rho_t + \theta)} - 1,$$

where

$$e = e^a + e^b < 2$$

is the aggregate endowment of individuals at the second date in their life spans.

If $r = 0$, there exist two steady-states, one with $\rho^* = 0$ and another with $\bar{\rho} = (e/2) - 1 < 0$; in addition, there is a continuum of non-stationary paths indexed by the initial real rate of interest, $\rho_0 \in (\bar{\rho}, \rho^*)$. The steady-state path with $\rho^* = 0$, the golden rule, supports a Pareto optimal allocation, while all other equilibrium paths support suboptimal and Pareto ranked allocations;

$\bar{\rho} = (e/2) - 1 < 0$ support intergenerational autarky. Note that, at the autarkic equilibrium, $\bar{x}^a = (4e^a - e^2)/(2e) < 0$, and, as a consequence, the associated real balances that support the equilibrium are $\bar{\mu} = -\bar{x}^a > 0$.

If $r > 0$, but sufficiently small, there is a steady-state equilibrium path with

$$\rho^*(r) = \frac{(2+e) + \sqrt{(2+e)^2 - 4(2-\theta)(e + \frac{\theta}{1-\theta}e^a)}}{2(2-\theta)} - 1 > 0.$$

By a standard argument, there is no equilibrium path with $\rho_0 \notin [\bar{\rho}(r), \rho^*(r)]$, where $\bar{\rho}(r) = [(2+e) - \sqrt{(2+e)^2 - 4(2-\theta)(e + \theta/(1-\theta)e^a)}]/[2(2-\theta)] - 1 < 0$.

For $\rho(t) \in [\bar{\rho}(r), \rho^*(r)]$, real balances are bounded below by $\mu^a(\bar{\rho}(r)) > 0$ and, as a consequence, the value of the bank is well defined and, in particular finite, only if $\rho(t) > 0$.

Since $\rho(t) \to \bar{\rho}(r) < 0$ if $\rho(t) \in [\bar{\rho}(r), \rho^*(r))$, the steady-state at $\rho^*(r)$ is the unique equilibrium path.

Importantly,

$$\lim_{r \to 0} \rho^*(r) = \rho^*;$$

as the nominal rate of interest tends to 0, the unique, steady-state real rate of interest tends to the golden rule and the associated allocation to a Pareto optimum.

The argument fails in the absence of intragenerational heterogeneity, when real balances need not be bounded away from zero as the economy tends to autarky.

Alternatively, Weiss (1980) allows real balances to enter directly the intertemporal utility function of a representative individual, $u(x, z, \mu)$, and he writes the intertemporal budget constraint as

$$x_t + \frac{1}{1+\rho}(z_{t+1} - e) + \frac{r_t - \pi_t}{1+r_t}\mu_t \leq 0,$$

which follows from the hypothesis that changes in the supply of balances are distributed as lump-sum transfers to individuals when old.

At a steady-state, optimization requires that

$$\frac{u_\mu}{u_x} = \frac{r}{1+r},$$

while market clearing requires that

$$\frac{r-\pi}{1+r}\mu^i = \frac{r-\pi}{1+r}(z-e);$$

the outstanding debt is

$$b = (z - e) + \mu.$$

At equilibria with debt, $r = \pi$ and $\rho = 0$. As a consequence, the liquidity services that balances, distinct from debt, provide, do not shield the economy from intertemporal inefficiency.

Alternatively, without debt (or, equivalently, if debt provides liquidity services and is not distinguishable from money), $\mu = z - e$ and, with $\pi = 0$ the real rate of interest is necessarily positive, $\rho = r > 0$, which, indeed, guarantees intertemporal efficiency. The hypothesis of non-vanishing marginal utility for money balances plays a role similar to that of intragenerational heterogeneity in our construction, but the logic of the arguments is different.

2.2

Two perishable commodities, $l \in \mathcal{N} = \{a, b\}$, are exchanged and consumed at each date. Individual i only consumes commodity i, but is endowed with one unit of the other commodity, $-i$, when young and nothing when old. The intertemporal utility function of an individual i is

$$u^i\left(x^i, z^i\right) = x^i + 2z^i,$$

where x^i and z^i are the consumptions of the individual in commodity i, respectively, when young and when old.

The price of commodity i at date t in present value terms is p_t^i. The constant nominal rate of interest is $r \geq 0$. An individual faces the single budget constraint

$$p_t^i x_t^i + p_{t+1}^i z_{t+1}^i \leq \left(\frac{1}{1+r}\right) p_t^{-i},$$

which reflects an underlying cash-in-advance constraint. In addition, the budget constraint of an initially old individual is

$$p_0^i z_0^i \leq \mu^i \left(\frac{r}{1+r}\right) \sum_t \left(p_t^a + p_t^b\right),$$

which reflects the hypothesis that the individual is entitled to a share $\mu^i \geq 0$ in intertemporal seignorage, so that $\mu^a + \mu^b = 1$.

Market clearing simply requires that

$$x_t^i + z_t^i = 1.$$

At equilibrium, sequential Walras Law implies

$$\left(\frac{r}{1+r}\right) \sum_i p_t^i + \sum_i p_{t+1}^i z_{t+1}^i = \sum_i p_t^i z_t^i.$$

This completes the description of the economy.

Let $\theta = [r/(1+r)] \leq 1$, for $r \geq 0$. Reinterpreting terms, one might suppose that every individual i with only $1-\theta$ units of commodity $-i$ when young and nothing when old. In addition, a real productive asset i, initially owned by old individuals, deliver ϵ units of commodity i at every date.

We consider equilibria in two distinct cases.

First, $r = 0$. From the budget constraints of initially old individuals, it follows that $z_0^i = 0$ and, so, exploiting sequential Walras Law, that $x_t^i = 1$ and $z_t^i = 0$ for every t. This requires $p_{t+1}^i \geq 2p_t^i$ for every t. The equilibrium allocation clearly fails Pareto optimality.

Alternatively, $r > 0$. From the budget constraints of initially old individuals, it follows that

$$p_0^i z_0^i = \mu^i \left(\frac{r}{1+r}\right) \sum_t \left(p_t^a + p_t^b\right),$$

and, as a consequence, that $\sum_t \left(p_t^a + p_t^b\right)$ is finite. By a canonical argument, the equilibrium allocation achieves Pareto optimality. We show that a steady state equilibrium exists under an equal distribution of seignorage, $\mu^a = \mu^b$.

Assume that $x_t^i = 0$ and $z_t^i = 1$ for every t. To obtain equilibrium prices, observe that, from the budget constraints of young individuals,

$$p_{t+1}^i = \left(\frac{1}{1+r}\right) p_t^{-i},$$

while, from the budget constraints of initially old individuals, $p_0 = p_0^a = p_0^b$; it follows that

$$p_t = p_t^a = p_t^b = \left(\frac{1}{1+r}\right)^t p_0,$$

at every date t.

For an arbitrary distribution of seignorage, a steady state equilibrium might not exist. To verify this, observe that, at a stationary equilibrium, $x_t^i = x^i$ and $z_t^i = z^i$ for every t, with $x^i + z^i = 1$. If $x^i > 0$, by utility maximization, $p_{t+1}^i \geq 2p_t^i$, which would violate the fact that $\sum_t \left(p_t^a + p_t^b\right)$ is finite. Hence, $x^i = 0$, which implies, by the budget constraint of a young individual and utility maximization,

$$2p_t^i \geq p_{t+1}^i = \left(\frac{1}{1+r}\right) p_t^{-i}.$$

In addition, an initial condition requires

$$p_0^i = \mu^i \left(\frac{r}{1+r}\right) \sum_t \left(p_t^a + p_t^b\right).$$

From both conditions, it follows that

$$2\mu^b \geq \left(\frac{1}{1+r}\right)\mu^a$$

and

$$2\mu^a \geq \left(\frac{1}{1+r}\right)\mu^b.$$

Hence, a stationary equilibrium might not exist for an arbitrary distribution of shares — well known for (stationary) economies of overlapping generations with multiple individuals in each generation and multiple commodities.

This example is designed to deliver an extremely clear conclusion about efficiency at equilibrium. In particular, a simplifying assumption, that each individual is endowed only with the commodity that he does not consume, eliminates price distortions due to cash-in-advance constraints, which only operates through pure wealth effects. Within each generation, there are actually infinite gains to trade, as young individuals are clearly better off by exchanging their endowments. Intergenerational trade allows for a further increase in welfare.

3 The Abstract Argument

There is a countable set of individuals, $\mathcal{I} = \{\ldots, i, \ldots\}$, a countable set of periods of trade, $\mathcal{T} = \{0, 1, 2, \ldots, t, \ldots\}$, and a finite set of physical commodities in every period of trade, $\mathcal{N}$. The commodity space is $\boldsymbol{L} = \mathbb{R}^{\mathcal{L}}$, where $\mathcal{L} = \mathcal{T} \times \mathcal{N}$.[5]

The consumption space of an individual is $\boldsymbol{L}^+$, the positive cone of the commodity space, and an element, $\boldsymbol{x}^i$, of the consumption space is a consumption plan. An individual is characterized by a preference relation, $\succeq^i$,

[5]The set of all real valued maps on $\mathcal{L}$ is $\boldsymbol{L} = \mathbb{R}^{\mathcal{L}}$. An element $\boldsymbol{x}$ of $\boldsymbol{L}$ is said to be positive if $\boldsymbol{x}(l) \geq 0$ for every l in $\mathcal{L}$; negligible if $\boldsymbol{x}(l) = 0$ for all but finitely many l in $\mathcal{L}$. For an element $\boldsymbol{x}$ of $\boldsymbol{L}$, $\boldsymbol{x}^+$ and $\boldsymbol{x}^-$ are, respectively, its positive part and its negative part, so that $\boldsymbol{x} = \boldsymbol{x}^+ - \boldsymbol{x}^-$ and $|\boldsymbol{x}| = \boldsymbol{x}^+ + \boldsymbol{x}^-$. The positive cone, $\boldsymbol{L}^+$, of $\boldsymbol{L}$ consists of all positive elements of $\boldsymbol{L}$. Also, $\boldsymbol{L}_0$ is the vector space consisting of all negligible elements of $\boldsymbol{L}$. Finally, for every element $\boldsymbol{x}$ of $\boldsymbol{L}$,

$$\boldsymbol{L}(\boldsymbol{x}) = \{\boldsymbol{v} \in \boldsymbol{L} : |\boldsymbol{v}| \leq \lambda\,|\boldsymbol{x}|\,, \text{ for some } \lambda > 0\}$$

is a principal ideal of $\boldsymbol{L}$. Unless otherwise stated, every topological property on $\boldsymbol{L}$ refers to the traditional product topology. We remark that, throughout the paper, the term 'positive' is used to mean 'greater than or equal to zero'.

on the consumption space and an endowment, $\boldsymbol{e}^i$, of commodities, an element of the consumption space itself. He is also entitled to a share $\mu^i \geq 0$ of aggregate revenue, so that, across individuals, $\sum_i \mu^i = 1$.

Fundamentals, $(\ldots, (\succeq^i, \boldsymbol{e}^i, \mu^i), \ldots)$, are restricted by canonical assumptions, so that every single individual is negligible. The aggregate endowment is $\sum_i \boldsymbol{e}^i$, which is understood to be a limit in the product topology.

Assumption 1 (Preferences)**.** The preference relations of individuals are convex, continuous, weakly monotone and locally non-satiated.

Assumption 2 (Endowments)**.** The endowments of individuals are positive, negligible elements of the commodity space.

Assumption 3 (Aggregate Endowment)**.** The aggregate endowment is a positive element of the commodity space.

An allocation, $\boldsymbol{x} = (\ldots, \boldsymbol{x}^i, \ldots)$, is a collection of consumption plans. It is balanced whenever $\sum_i \boldsymbol{x}^i = \sum_i \boldsymbol{e}^i$. It is feasible whenever $\sum_i \boldsymbol{x}^i \leq \sum_i \boldsymbol{e}^i$. It is individually rational whenever, for every individual i, $\boldsymbol{x}^i \succeq^i \boldsymbol{e}^i$. For a feasible allocation $\boldsymbol{x}$, aggregate consumption, $\sum_i \boldsymbol{x}^i$, is an element of $\boldsymbol{L}(\boldsymbol{e})$, where $\boldsymbol{e} = \sum_i \boldsymbol{e}^i$ is the aggregate endowment.

Trade occurs intertemporally subject to transaction costs. In an abstract formulation, it simplifies matters to assume that individuals can only trade if they deliver a value that is proportional to the value of their net transactions. Such revenues from transactions accrue to a central authority that redistributes them to individuals as lump-sum transfers, according to given shares. This abstraction corresponds to the description of a sequential monetary economy under a complete asset market and a central bank that pegs a constant nominal rate of interest and accommodates the demand for balances. In addition, the central bank, whose ownership is sequentially trade on the asset market, redistributes its profit (seignorage) as dividends to shareholders.

Prices of commodities $\boldsymbol{p}$ are also an element of $\boldsymbol{L}^+$. These are, in a sense, discounted or Arrow-Debreu prices. The duality operation on $\boldsymbol{L}^+ \times \boldsymbol{L}^+$ is defined by

$$\boldsymbol{p} \cdot \boldsymbol{v} = \sup \{\boldsymbol{p} \cdot \boldsymbol{v}_0 : \boldsymbol{v}_0 \in [\boldsymbol{0}, \boldsymbol{v}] \cap \boldsymbol{L}_0\},$$

that may be infinite.

The budget constraint of an individual is

$$\left(\frac{r}{1+r}\right) \boldsymbol{p} \cdot \left(\boldsymbol{x}^i - \boldsymbol{e}^i\right)^- + \boldsymbol{p} \cdot \left(\boldsymbol{x}^i - \boldsymbol{e}^i\right) \leq \mu^i w,$$

where $\mu^i \geq 0$ is the share of the individual i in the aggregate positive transfer w.

For given a positive nominal rate of interest, r, an *(abstract) r-equilibrium* consists of a balanced allocation, $\boldsymbol{x}$, prices, $\boldsymbol{p}$, and a aggregate positive transfer, w, such that

$$\left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\sum_i\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^- \leq w$$

and, for every individual,

$$\left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^- + \boldsymbol{p}\cdot\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right) \leq \mu^i w$$

and

$$\boldsymbol{z}^i \succ^i \boldsymbol{x}^i \quad \Longrightarrow \quad \left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\left(\boldsymbol{z}^i-\boldsymbol{e}^i\right)^- + \boldsymbol{p}\cdot\left(\boldsymbol{z}^i-\boldsymbol{e}^i\right) > \mu^i w.$$

An (abstract) r-equilibrium involves a speculative bubble if

$$b = w - \left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\sum_i\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^- > 0.$$

Notice that an (abstract) 0-equilibrium is what the literature traditionally refers to as an equilibrium with (possibly) positive outside money, or with (possibly) a positive speculative bubble.

Lemma 1. *The value of net transaction, $\boldsymbol{p}\cdot\sum_i\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^-$, is finite at every r-equilibrium with $r > 0$.*

Proof. Obvious. **Q.E.D.**

Allocation $\boldsymbol{z}$ Pareto dominates allocation $\boldsymbol{x}$ if, for every individual, $\boldsymbol{z}^i \succeq^i \boldsymbol{x}^i$ with $\boldsymbol{z}^i \succ^i \boldsymbol{x}^i$ for some. Allocation $\boldsymbol{z}$ Malinvaud dominates allocation $\boldsymbol{x}$ if $\boldsymbol{z}$ Pareto dominates $\boldsymbol{x}$, while $\boldsymbol{z}^i = \boldsymbol{x}^i$ for all but finitely many individuals (Malinvaud (1953)).

For a given positive nominal rate of interest, r, an allocation, $\boldsymbol{x}$, is *Pareto (Malinvaud) r-undominated* if it is not Pareto (Malinvaud) dominated by an alternative allocation, $\boldsymbol{z}$, that satisfies

$$\left(\frac{r}{1+r}\right)\sum_i\left(\boldsymbol{z}^i-\boldsymbol{e}^i\right)^- + \sum_i \boldsymbol{z}^i \leq \left(\frac{r}{1+r}\right)\sum_i\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^- + \sum_i \boldsymbol{x}^i.$$

Evidently, a Pareto (Malinvaud) 0-undominated allocation coincides with a standard Pareto (Malinvaud) efficient allocation.

Lemma 2. *Every r-equilibrium allocation is a Malinvaud r-undominated allocation.*

Proof. If not, there is an allocation $\boldsymbol{z}$ that Malinvaud dominates allocation $\boldsymbol{x}$ and satisfies

$$\left(\frac{r}{1+r}\right)\sum_i\left(\boldsymbol{z}^i-\boldsymbol{e}^i\right)^-+\sum_i\boldsymbol{z}^i\leq\left(\frac{r}{1+r}\right)\sum_i\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^-+\sum_i\boldsymbol{x}^i.$$

Thus, for every individual,

$$\left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\left(\boldsymbol{z}^i-\boldsymbol{e}^i\right)^-+\boldsymbol{p}\cdot\boldsymbol{z}^i\geq\left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^-+\boldsymbol{p}\cdot\boldsymbol{x}^i,$$

with at least one strict inequality. Since the allocation $\boldsymbol{z}$ coincides with the allocation $\boldsymbol{x}$ for all but finitely many individuals, aggregation across individuals yields a contradiction. **Q.E.D.**

An allocation, $\boldsymbol{x}$, involves *uniform trade* if there is a decomposition $\boldsymbol{L}_f\oplus\boldsymbol{L}_b$ of the (reduced) commodity space $\boldsymbol{L}(\boldsymbol{e})$, with

$$\boldsymbol{L}_f\subseteq\left\{\boldsymbol{v}\in\boldsymbol{L}:|\boldsymbol{v}|\leq\lambda\sum_i\left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^-,\ \text{for some }\lambda>0\right\},$$

and an allocation $\boldsymbol{v}$ such that $\sum_i\boldsymbol{v}^i$ belongs to $\boldsymbol{L}(\boldsymbol{e})$ and, for some $\lambda>0$ small enough, $\boldsymbol{x}^i-\lambda\boldsymbol{x}_b^i+\boldsymbol{v}_f^i\succeq^i\boldsymbol{x}^i$ for every individual. This requires that the set of commodities can be partitioned into commodities that are traded in some uniformly strictly positive amount and commodities that are not, in such a way that all individuals can increase their welfare by a large enough increase in consumption in the former set of commodities, even when consumption in the latter set of commodities is slightly reduced.

Assumption 4 (Gains to Trade)**.** Every individually rational balanced Malinvaud r-undominated allocation, with $r>0$ sufficiently small, involves uniform trade.

The gains to trade hypothesis extends the condition Bloise, Drèze and Polemarchakis (2004) and Dubey and Geanakoplos (2005). It has as consequence that the value of the aggregate endowment is finite at equilibrium.

Lemma 3. *The value of the aggregate endowment, $\boldsymbol{p}\cdot\sum_i\boldsymbol{e}^i$, is finite at every r-equilibrium with $r>0$ sufficiently small.*

Proof. Consider the decomposition of the (reduced) commodity space $\boldsymbol{L}_f \oplus \boldsymbol{L}_b = \boldsymbol{L}(\boldsymbol{e}) \subseteq \boldsymbol{L}$ in the hypothesis of a uniform trade. Clearly, $\boldsymbol{p}$ defines a positive σ-additive linear functional on $\boldsymbol{L}_f$. Thus, $\boldsymbol{p} \cdot \sum_i \boldsymbol{e}^i$ is unbounded only if $\boldsymbol{p} \cdot \sum_i \boldsymbol{e}_b^i$ is unbounded and, hence, only if $\boldsymbol{p} \cdot \sum_i \boldsymbol{x}_b^i$ is unbounded. Also, $\boldsymbol{p} \cdot \sum_i \boldsymbol{v}_f^i$ is finite, where $\boldsymbol{v}$ is the allocation mentioned in the definition of uniform trade.

For every individual, $\boldsymbol{z}^i \succeq^i \boldsymbol{x}^i$ implies

$$\left(\frac{r}{1+r}\right) \boldsymbol{p} \cdot \left(\boldsymbol{z}^i - \boldsymbol{e}^i\right)^- + \boldsymbol{p} \cdot \boldsymbol{z}^i \geq \left(\frac{r}{1+r}\right) \boldsymbol{p} \cdot \left(\boldsymbol{x}^i - \boldsymbol{e}^i\right)^- + \boldsymbol{p} \cdot \boldsymbol{x}^i.$$

Since $(\boldsymbol{z}^i - \boldsymbol{x}^i)^- + (\boldsymbol{x}^i - \boldsymbol{e}^i)^- \geq (\boldsymbol{z}^i - \boldsymbol{e}^i)^-$, it follows that

$$\boldsymbol{p} \cdot \left(\boldsymbol{z}^i - \boldsymbol{x}^i\right)^+ \geq \left(\frac{1}{1+r}\right) \boldsymbol{p} \cdot \left(\boldsymbol{z}^i - \boldsymbol{x}^i\right)^- .$$

As $\boldsymbol{x}^i - \lambda \boldsymbol{x}_b^i + \boldsymbol{v}_f^i \succeq^i \boldsymbol{x}^i$ for some $\lambda > 0$ sufficiently small, using the previous argument, with $\boldsymbol{z}^i = \boldsymbol{x}^i - \lambda \boldsymbol{x}_b^i + \boldsymbol{v}_f^i$ implies that

$$\boldsymbol{p} \cdot \boldsymbol{v}_f^i \geq \left(\frac{1}{1+r}\right) \lambda \boldsymbol{p} \cdot \boldsymbol{x}_b^i.$$

Aggregation across individuals yields a contradiction. **Q.E.D.**

As the aggregate endowment is finitely valued at equilibrium, canonical conclusions about efficiency and the absence of speculative bubbles can be drawn.

Proposition 1 (Almost Pareto Optimality)**.** *No r-equilibrium allocation, $\boldsymbol{x}$, with $r > 0$ sufficiently small, is Pareto dominated by an alternative allocation, $\boldsymbol{z}$, that satisfies*

$$\sum_i \boldsymbol{z}^i \leq \left(\frac{1}{1+r}\right) \sum_i \boldsymbol{e}^i.$$

Proof. As the aggregate endowment is finitely valued, it is clear that every r-equilibrium allocation, with $r > 0$ small enough, is a Pareto r-undominated allocation (the proof is just an adaptation of the proof of lemma 2). So, in order to prove that the statement in the proposition holds true, suppose not. It follows that $\boldsymbol{x}$ is Pareto dominated by an alternative allocation $\boldsymbol{z}$ that satisfies

$$
\begin{aligned}
\left(\frac{r}{1+r}\right)\sum_i \left(\boldsymbol{z}^i-\boldsymbol{e}^i\right)^- + \sum_i \boldsymbol{z}^i &\leq \\
\left(\frac{r}{1+r}\right)\sum_i \boldsymbol{e}^i + \sum_i \boldsymbol{z}^i &\leq \sum_i \boldsymbol{e}^i \\
&\leq \left(\frac{r}{1+r}\right)\sum_i \left(\boldsymbol{x}^i-\boldsymbol{e}^i\right)^- + \sum_i \boldsymbol{x}^i.
\end{aligned}
$$

This contradicts Pareto r-undomination. **Q.E.D.**

Proposition 2 (No Speculative Bubbles)**.** *No r-equilibrium, with $r > 0$ sufficiently small, involves a speculative bubble.*

Proof. As the aggregate endowment is finitely valued, the result follows from the aggregation of budget constraints across individuals. **Q.E.D.**

It remains to understand the restrictions implied by the gains to trade hypothesis (assumption 4).

4 Gains to Trade in a Stationary Economy

The hypothesis on gains to trade (assumption 4) is generically satisfied in a standard stationary economy of identical overlapping generations of heterogenous individuals. We shall simply provide the core argument, as details are straightforward but heavy in terms of notation.

The set of individuals is $\mathcal{I} = \mathcal{J} \times \mathcal{T}$, where $\mathcal{T} = \{0, 1, 2, \ldots, t, \ldots\}$ are dates or periods of trade and $\mathcal{J}$ is a finite set of individuals within a generation: for every t in $\mathcal{T}$, $\mathcal{I}^t = \{(j, t) : j \in \mathcal{J}\}$ is generation t. All generations $\mathcal{I}^{t+1}$ are identical and have life spans $\mathcal{T}^{t+1} = \{t, t+1\} \subseteq \mathcal{T}$. The initial generation $\mathcal{I}^0$ has life span $\mathcal{T}^0 = \{0\} \subseteq \mathcal{T}$.

Preferences are strictly monotone over the life span of an individual: for an individual in generation t in $\mathcal{T}$, preferences are strictly monotone on the positive cone of $\boldsymbol{L}^t = \mathbb{R}^{\mathcal{L}^t} \subseteq \mathbb{R}^{\mathcal{L}} = \boldsymbol{L}$, where $\mathcal{L}^t = \mathcal{T}^t \times \mathcal{N}$.

Endow the (reduced) commodity space $\boldsymbol{L}(\boldsymbol{e})$ with the supremum norm

$$\|\boldsymbol{v}\|_\infty = \sup\{\lambda > 0 : |\boldsymbol{v}| \leq \lambda \boldsymbol{e}\}.$$

As the economy is stationary, this involves no loss of generality. Suppose that there is $\epsilon > 0$ such that, for every individually rational, balanced, Malinvaud efficient allocation, $\boldsymbol{x}$, the aggregate net trade of every generation t in $\mathcal{T}$ is ϵ-bounded away from autarky, that is,

$$\left\| \sum_{i \in \mathcal{I}^t} \left(\boldsymbol{x}^i - \boldsymbol{e}^i \right)^+ \right\|_\infty > \epsilon.$$

In a stationary economy of identical overlapping generations, this is a rather weak requirement when there are at least two individuals in each generation.[6] It follows that there is $\epsilon > 0$ such that, provided that $r > 0$ is small enough, for every individually rational, balanced, Malinvaud r-undominated allocation, $\boldsymbol{x}$, the aggregate net trade of every generation t in $\mathcal{T}$ is ϵ-bounded away from autarky.

This is evident.

Let $\boldsymbol{e}_l$ be the aggregate endowment of commodity l in $\mathcal{L}$ (regarded as an element of the commodity space $\boldsymbol{L}$). For a generation t in $\mathcal{T}$, let $g(t)$ in $\mathcal{L}$ be a commodity such that

$$\boldsymbol{e}_{g(t)} \leq \frac{1}{\epsilon} \sum_{i \in \mathcal{I}^t} \left(\boldsymbol{x}^i - \boldsymbol{e}^i \right)^+ .$$

Such a commodity exists because net trades are uniformly bounded away (in the sup norm) from zero by $\epsilon > 0$. Decompose the aggregate endowment as $\boldsymbol{e} = \boldsymbol{e}_f + \boldsymbol{e}_b$, where

$$\boldsymbol{e}_f = \sum_{l \in g(\mathcal{T})} \boldsymbol{e}_l,$$

and

$$\boldsymbol{e}_b = \sum_{l \notin g(\mathcal{T})} \boldsymbol{e}_l.$$

Clearly, $\boldsymbol{L}_f = \boldsymbol{L}(\boldsymbol{e}_f)$ and $\boldsymbol{L}_b = \boldsymbol{L}(\boldsymbol{e}_b)$ are such that $\boldsymbol{L}(\boldsymbol{e}) = \boldsymbol{L}_f \oplus \boldsymbol{L}_b$. In addition,

$$\boldsymbol{e}_f \leq$$

$$\textstyle\sum_{t \in \mathcal{T}} \boldsymbol{e}_{g(t)} \leq \frac{1}{\epsilon} \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}^t} \left(\boldsymbol{x}^i - \boldsymbol{e}^i \right)^+ =$$

$$\textstyle\frac{1}{\epsilon} \sum_i \left(\boldsymbol{x}^i - \boldsymbol{e}^i \right)^+ = \frac{1}{\epsilon} \sum_i \left(\boldsymbol{x}^i - \boldsymbol{e}^i \right)^- ,$$

[6] As the allocation is Malinvaud efficient, it is Pareto efficient within every generation. As the allocation is individually rational and preferences are strictly monotone, if positive net trades vanish within a generation, so do negative net trades. Thus, using the fact that all generations are identical, $\epsilon > 0$ above does not exist only if no-trade is a Pareto efficient allocation within a typical generation. This does not occur generically in preferences and endowments.

so that

$$\boldsymbol{L}_f \subseteq \left\{ \boldsymbol{v} \in \boldsymbol{L} : |\boldsymbol{v}| \leq \lambda \sum_i \left(\boldsymbol{x}^i - \boldsymbol{e}^i \right)^- , \text{ for some } \lambda > 0 \right\}.$$

For an individual i in generation t in $\mathcal{T}$, let $\boldsymbol{v}^i = \boldsymbol{v}^i_f = \boldsymbol{e}_{g(t)}$. Taking into account multiplicities and using the fact that generations overlap for at most two periods, it is easily verified that

$$\sum_i \boldsymbol{v}^i = \sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{I}^t} \boldsymbol{e}_{g(t)} \leq (\#\mathcal{J}) \sum_{t \in \mathcal{T}} \boldsymbol{e}_{g(t)} \leq 2 (\#\mathcal{J}) \sum_{l \in g(\mathcal{T})} \boldsymbol{e}_l = 2 (\#\mathcal{J}) \, \boldsymbol{e}_f.$$

Using stationarity hypotheses and the strict monotonicity of preferences over relevant consumption spaces, it is simple to show that there is $1 > \lambda > 0$ such that, for every individual,

$$\boldsymbol{x}^i - \lambda \boldsymbol{x}^i_b + \boldsymbol{v}^i_f \succeq^i \boldsymbol{x}^i.$$

The gains to trade hypothesis (assumption 4) is satisfied.

5 Sequential Trade

The abstract framework accommodates a sequential economy of overlapping generations. We here present the classical arguments for the consolidation of budget constraints that are implied by a sequentially complete asset market.

5.1 Prices and Markets

In every period of trade, there are markets for commodities, balances and assets. Balances are the numéraire at every date. A constant positive nominal rate of interest, r, is pegged by the monetary authority.[7]

The asset structure consists of a one-period nominally risk-free bond and an infinitely-lived security that pays off nominal dividends in every period.

[7]As far as individuals and commodities are concerned, notation is as in section 3. In particular, an element $\boldsymbol{x}$ of $\boldsymbol{L} = \mathbb{R}^{\mathcal{T} \times \mathcal{N}}$ decomposes, across periods of trade, as

$$\boldsymbol{x} = (x_0, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots),$$

where each x_t is an element of $\mathbb{R}^{\mathcal{N}}$; an element $\boldsymbol{x}$ of $\boldsymbol{E} = \mathbb{R}^{\mathcal{T}}$ decomposes, across periods of trade, as

$$\boldsymbol{x} = (x_0, \ldots, x_{t-1}, x_t, x_{t+1}, \ldots),$$

where each x_t is an element of $\mathbb{R}$.

Short sales are allowed on bonds, but not on the security. Prices of the security $\boldsymbol{q}$ are a positive element of $\boldsymbol{E} = \mathbb{R}^{\mathcal{T}}$. These are spot prices. Dividends of the security $\boldsymbol{y}$ are a positive element of $\boldsymbol{E}$. This security is in positive net supply and, to simplify, the supply is normalized to the unity.

Discount factors, $\boldsymbol{a}$, a positive element of $\boldsymbol{E}$, are obtained by setting

$$a_t = \left(\frac{1}{1+r}\right)^t.$$

No arbitrage, jointly with the fact that the security cannot be dominated by bonds at equilibrium, implies that, in every period of trade,

$$a_t q_t = a_t y_t + a_{t+1} q_{t+1}.$$

This condition reflects the innocuous assumption that the security is priced *cum dividend.* As far as the intertemporal transfer of wealth is concerned, bonds and the security are perfect substitutes under this no-arbitrage pricing.

A standard argument implies that, in every period of trade,

$$q_t \geq \frac{1}{a_t} \sum_{s \geq t} a_s y_s.$$

That is, the price of the security is at equal to or greater than its fundamental value. The displacement of the market value of the security from its fundamental value is the speculative bubble.

Prices of commodities $\boldsymbol{p}$ are a positive element of $\boldsymbol{L}$. To avoid an excess of notation, we interpret $\boldsymbol{p}$ as present value prices of commodities. So, current (or spot) prices of commodities are

$$\left(\frac{1}{a_0} p_0, \ldots, \frac{1}{a_{t-1}} p_{t-1}, \frac{1}{a_t} p_t, \frac{1}{a_{t+1}} p_{t+1}, \ldots\right).$$

5.2 Sequential Budget Constraints

Sequential constraints are canonical. Individual i formulates a consumption plan, $\boldsymbol{x}^i$, a positive element of $\boldsymbol{L}$, and a financial plan, $(\boldsymbol{m}^i, \boldsymbol{z}^i, \boldsymbol{v}^i)$, consisting of holdings of balances, $\boldsymbol{m}^i$, a positive element of $\boldsymbol{E}$, of the security, $\boldsymbol{z}^i$, a positive element of $\boldsymbol{E}$, and of short-term bonds, $\boldsymbol{v}^i$, an element of $\boldsymbol{L}$. Individual i enters period of trade t with some accumulated nominal wealth, w_t^i; he trades in assets and balances according to the budget constraint

$$m_t^i + (q_t - y_t)\, z_t^i + \left(\frac{1}{1+r}\right) v_t^i \leq w_t^i;$$

he uses balances for the purchase of commodities, as prescribed by a cash-in-advance constraint,

$$\frac{1}{a_t} p_t \cdot \left(x_t^i - e_t^i \right)^- \leq m_t^i;$$

receives balances from the sale of commodities and he enters the following period of trade $t+1$ with nominal wealth

$$w_{t+1}^i = m_t^i + q_{t+1} z_t^i + v_t^i - \frac{1}{a_t} p_t \cdot \left(x_t^i - e_t^i \right).$$

In addition, a wealth constraint of the form

$$-\frac{1}{a_{t+1}} \sum_{s \geq t+1} p_s \cdot e_s^i \leq w_{t+1}^i$$

is imposed in order to avoid Ponzi schemes. Finally, the initial nominal wealth is given by the initial price of the security, $w_0^i = \mu^i q_0$, where $\mu^i \geq 0$ is the initial share of individual i into the security.

If a consumption plan, $\boldsymbol{x}^i$, and a financial plan, $(\boldsymbol{m}^i, \boldsymbol{z}^i, \boldsymbol{v}^i)$, satisfy all the above described restrictions at all periods of trade, we say that financial plan $(\boldsymbol{m}^i, \boldsymbol{z}^i, \boldsymbol{v}^i)$ finances consumption plan $\boldsymbol{x}^i$ (equivalently, consumption plan $\boldsymbol{x}^i$ is financed by financial plan $(\boldsymbol{m}^i, \boldsymbol{z}^i, \boldsymbol{v}^i)$). The sequential budget constraint of individual i is the set of all consumption plans, $\boldsymbol{x}^i$, that are financed by some financial plan.

Literally interpreted, our sequential budget constraint might appear contradicting the hypothesis of overlapping generations of individuals. Indeed, it can be argued that an individual might not be active at some date and, so, it is meaningless to assume that consumptions and wealth accumulation of such an individual are restricted by the entire sequence of constrains. Observe, however, that an individual should be regarded as not being active at some date only if he has no endowment of commodities and his utility is unaffected by the consumption of commodities at that date. These are joint assumptions of preferences and endowments. Letting the individual trade when he should be regarded as not being active adds redundant constraints without altering the substance. A skeptical reader might assume that an individual i is characterized by a time horizon $\mathcal{T}^i \subseteq \mathcal{T}$ of consecutive periods of trade. Both the consumption plan and the financial plan can be assumed to be zero out of the given time horizon. In the same spirit of the above observation, one might be willing to assume that the initial share into the security is strictly positive only for individuals that are active in the initial period of trade.

5.3 Intertemporal Budget Constraints

By a canonical consolidation, provided that there are no arbitrage opportunities, sequential budget constraint reduces to a single intertemporal budget constraint of the form

$$\left(\frac{r}{1+r}\right)\sum_t p_t \cdot \left(x_t^i - e_t^i\right)^- + \sum_t p_t \cdot \left(x_t^i - e_t^i\right) \leq \mu^i q_0.$$

The underlying demand of balances satisfies, in every period of trade t,

$$m_t^i \geq \frac{1}{a_t} p_t \cdot \left(x_t^i - e_t^i\right)^-,$$

with the equality whenever $r > 0$. Also, the holding of bonds and of the security, witch are perfect substitutes as far as intertemporal transfers of wealth are concerned, can be assumed to satisfy, in every period of trade t,

$$m_t^i + q_{t+1} z_t^i + v_t^i = \left(\frac{r}{1+r}\right)\frac{1}{a_t}\sum_{s \geq t+1}\left(x_s^i - e_s^i\right)^- + \frac{1}{a_t}\sum_{s \geq t} p_s \cdot \left(x_s^i - e_s^i\right).$$

As a matter of mere fact, using a more compact notation, a consumption plan, $\boldsymbol{x}^i$, is restrict by a single intertemporal budget constraint of the form

$$\left(\frac{r}{1+r}\right)\boldsymbol{p} \cdot \left(\boldsymbol{x}^i - \boldsymbol{e}^i\right)^- + \boldsymbol{p} \cdot \left(\boldsymbol{x}^i - \boldsymbol{e}^i\right) \leq \mu^i q_0.$$

The financial plan, $(\boldsymbol{m}^i, \boldsymbol{z}^i, \boldsymbol{v}^i)$, that finances an intertemporally budget feasible consumption plan, $\boldsymbol{x}^i$, can be recovered, up to an intrinsic multiplicity due to redundant assets.

5.4 The Monetary Authority

The security is backed by the ownership of a central bank, which issues balances against bonds and distributes its profit as a divided to shareholders. A plan, $(\boldsymbol{m}, \boldsymbol{v}, \boldsymbol{y})$, of the monetary authority consists of a supply of balances, $\boldsymbol{m}$, a positive element of $\boldsymbol{E}$, a demand of short-term bonds, $\boldsymbol{v}$, an element of $\boldsymbol{E}$, and dividends to shareholders, $\boldsymbol{y}$, a positive element of $\boldsymbol{E}$. A sequential budget constraint imposes

$$\boldsymbol{m} - \left(\frac{1}{1+r}\right)\boldsymbol{v} = \boldsymbol{y}.$$

The monetary authority accommodates the demand for balances (that is, $\boldsymbol{m} = \sum_i \boldsymbol{m}^i$) and runs balanced accounts (that is, $\boldsymbol{m} = \boldsymbol{v}$), so that

$$\boldsymbol{y} = \left(\frac{r}{1+r}\right)\sum_i \boldsymbol{m}^i.$$

5.5 Sequential Equilibrium

Equilibrium requires market clearing only for commodities and assets, as the demand of balances is accommodated by the monetary authority. Given a positive nominal rate of interest, r, a *sequential r-equilibrium* consists of a collection of plans for individuals,

$$\left(\ldots, \left(\boldsymbol{x}^i, \left(\boldsymbol{m}^i, \boldsymbol{z}^i, \boldsymbol{v}^i\right)\right), \ldots\right),$$

a plan for the monetary authority, $(\boldsymbol{m}, \boldsymbol{v}, \boldsymbol{y})$, prices, $\boldsymbol{p}$, and security prices, $\boldsymbol{q}$, such that the following conditions are satisfied.

(a) For every individual i, consumption plan $\boldsymbol{x}^i$ is $\succeq^i$-optimal, subject to sequential budget constraint, and is financed by financial plan $(\boldsymbol{m}^i, \boldsymbol{z}^i, \boldsymbol{v}^i)$.

(b) The monetary authority accommodates the demand for balances and runs a balanced budget or

$$\begin{aligned} \boldsymbol{m} &= \sum_i \boldsymbol{m}^i, \\ \boldsymbol{v} &= \boldsymbol{m}, \\ \boldsymbol{y} &= \left(\frac{r}{1+r}\right) \boldsymbol{m}. \end{aligned}$$

(c) Markets for commodities and assets clear or

$$\begin{aligned} \sum_i \boldsymbol{x}^i &= \sum_i \boldsymbol{e}^i, \\ \sum_i \boldsymbol{z}^i &= \mathbf{1}, \\ \sum_i \boldsymbol{v}^i &= \boldsymbol{v}. \end{aligned}$$

Clearly, at a sequential equilibrium, security prices involve no arbitrage opportunities and, in addition, the security is not dominated by bonds.

5.6 Abstraction

At equilibrium,

$$\left(\frac{r}{1+r}\right) m_t = \left(\frac{r}{1+r}\right) \frac{1}{a_t} p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^+ = \left(\frac{r}{1+r}\right) \frac{1}{a_t} p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^-$$

and, as a consequence,

$$\left(\frac{r}{1+r}\right)\sum_t p_t \cdot \sum_i \left(x_t^i - e_t^i\right)^- = \sum_t a_t y_t \leq q_0.$$

Thus, using consolidation of sequential budget constraints, at a (sequential) r-equilibrium, it follows that

$$\left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\sum_i \left(\boldsymbol{x}^i - \boldsymbol{e}^i\right)^- \leq q_0$$

and, for every individual i,

$$\left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\left(\boldsymbol{x}^i - \boldsymbol{e}^i\right)^- + \boldsymbol{p}\cdot\left(\boldsymbol{x}^i - \boldsymbol{e}^i\right) \leq \mu^i q_0$$

and

$$\boldsymbol{z}^i \succ^i \boldsymbol{x}^i \text{ implies } \left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\left(\boldsymbol{z}^i - \boldsymbol{e}^i\right)^- + \boldsymbol{p}\cdot\left(\boldsymbol{z}^i - \boldsymbol{e}^i\right) > \mu^i q_0.$$

These are the only substantial equilibrium restrictions, as market clearing for bonds and the security can be verified to hold. As a conclusion, a sequential r-equilibrium coincides with an abstract r-equilibrium.

6 Concluding Remarks

In the abstract formulation, every individual i is subject to a single budget constraint of the form

$$\left(\frac{r}{1+r}\right)\boldsymbol{p}\cdot\left(\boldsymbol{x}^i - \boldsymbol{e}^i\right)^- + \boldsymbol{p}\cdot\left(\boldsymbol{x}^i - \boldsymbol{e}^i\right) \leq w^i,$$

where w^i would be interpreted, depending on the particular institutional framework, as the value of initial asset holdings plus possibly transfers in present value terms. Thus, Walras Law imposes

$$f + b = \left(\frac{r}{1+r}\right)\sum_i \boldsymbol{p}\cdot\left(\boldsymbol{x}^i - \boldsymbol{e}^i\right)^- + \sum_i \boldsymbol{p}\cdot\left(\boldsymbol{x}^i - \boldsymbol{e}^i\right) = \sum_i w^i = w,$$

where w, f and b are understood to be (possibly non-finite) limits.[8] The argument for almost Pareto optimality moves from the observation that the

[8]The discussion here is only suggestive, so that we avoid details on conditions for well-defined, though not finite, limits.

value of net transactions is finite at equilibrium. As long as nominal rate of interest is strictly positive, $r > 0$, this occurs whenever f is finite. In addition, by local non-satiation of preferences, w is finite if at least one individual is entitled to a positive share of it (that is, $w^i = \alpha^i w$, with $\alpha^i > 0$, for some individual i).

If w is finite, then

$$w \geq \left(\frac{r}{1+r}\right) \boldsymbol{p} \cdot \sum_i \left(\boldsymbol{x}^i - \boldsymbol{e}^i\right)^- = f$$

suffices to argue that f is finite. Incidentally, the above inequality rules out a negative speculative bubble, $w - f = b \geq 0$, but the crucial point is only that it guarantees a finite value of f. Sequential trades and, in particular, a central bank quoted on the stock market serve to interpret w as the initial market value of the central bank and f as the initial fundamental value of the central bank. Thus, $w \geq f$, with w finite, is inherited by a primitive description of sequential trades under the assumption of free disposal on long-term securities, so as to rule out a negative market value of the central bank. Could the same conclusion be drawn in other institutional frameworks?

In Bloise, Drèze and Polemarchakis (2004), a central bank trades balances for bonds and runs a balanced account by redistributing its profit to shareholders. This basically requires $f = w$, which by itself does not ensure a finite value of f. However, if this redistribution of the profit is interpreted as occurring intertemporally (that is, shares are into the intertemporal value of seignorage w), w would be finite and conclusions would be equivalent.

Alternatively, in the spirit of the fiscal theory of price determination (Woodford (1994)), one interprets w as a given stock on public debt, which is, thus, finite. *A priori*, it does not follow that $w \geq f$, which incidentally shows that the price level might still be indeterminate (in that context, $f = w$ implies an intertemporally balanced public budget). However, if one assumes that public debt cannot be negative, with ambiguous implications for sequential public budget constrains, then $b \geq 0$ and, so, $w \geq f$, thus leading to analogous conclusions.

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