London School of Economics and Political Science

Equity and Power in a Cooperative Trial-and-Error Game

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A thesis submitted to the Department of Mathematics of the London School of Economics for the degree of Doctor of Philosophy, London, 2015

Declaration

I certify that the thesis I presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is entirely based on my own work.

Lemma 3.9 is related to a proof in Nax (2010) that was a result of joint efforts of Peyton Young, Harald Nax and myself.

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Abstract

General solution concepts in cooperative game theory are static, e.g., the core, the Shapley value and the Nash bargaining solution. Dynamic implementation procedures have been proposed in order to support these static solution concepts. This thesis studies an N-dimensional Markov chain motivated by a dynamic interactive trial-and-error learning model. The state space of the Markov chain is based on a cooperative game (v, N) whose characteristic function v is superadditive and monotone, with conditions on v ensuring non-emptiness of the core. Agents repeatedly bargain over a cooperative surplus by submitting their demand for their share. Each round the *payable* coalition is chosen, the feasible coalition with the maximum sum of demands. Players in the payable coalition receive their demands as payoffs, the other players receive no payoff. Players adjust their demands according to the following rule: In an efficient state (where the demand sum of all players equals the total surplus, 1) one player is chosen uniformly at random and increases his demand by ε. If demands sum to $1+\varepsilon$, one player not in the payable coalition is then chosen to reduce her demand with probability proportional to the size of her demand.

An individual's demand update decision in the learning model is based solely on the observation of his last payoff. Individual updates are in the tradition of reinforcement learning, aspiration adaption, and fictitious play. Selten (1972) found empirical evidence for an inherent equity principle in many outcomes of experimental cooperative bargaining games. By construction, the dynamic learning model presented in this thesis also has an inherent equity principle. The model is a simple modification (and the limit process) of a model introduced by Nax (2010). To our knowledge, this thesis presents the first general results of such a dynamic learning model for general 3-player games and all interesting cases of 4-player games.

The transition probabilities of the Markov process studied in this thesis are the transition probabilities between efficient states, obtained by the two steps from an efficient state to a state with demand sum $1+\varepsilon$ and back, of the described trial and error process. The process is a biased random walk on the simplex of efficient states, of which the polytope formed by the grid of core points forms the subset of particular interest. For general N-player games we introduce a coalition structure that exhibits an asymmetry of power

between its members: the *asymmetric coalition set*. We believe the concept of an asymmetric coalition set to be both novel and relevant to the study of dynamic learning models with incremental demand updates for general cooperative games. Along a face of the core polytope generated by an asymmetric coalition set, the *asymmetric face*, the bias of the process is determined by the interplay between two dynamics: the inherent equity bias, which "drags" the process towards equity, and the *asymmetric power*, which "drags" the process away from equity. If the core polytope does not contain an asymmetric face, the equity bias of the random walk determines the expected movement along the faces of the polytope. The process can only leave the core polytope from a state on an asymmetric face.

We study a special Markov chain in dimension N derived from the N-player bargaining game, where no coalitional constraints are present. Then the bias of the random walk is solely determined by the inherent equity principle: the random walk drifts towards equity, and the equilibrium distribution is concentrated around the equal split, the most equitable allocation.

For $N = 3$, no asymmetric coalition set exists. We show that the set of recurrent states of the Markov chain is the "core polygon", formed by the grid points in the core. The *cooperative outcome* **co** is the unique vector in the core with smallest $\mathbf{L}^2$-distance from the equal split. At every state of the core polygon outside a small ball around **co**, the random walk moves in expectation over one time step towards **co**. The equilibrium distribution of the Markov chain is concentrated around the vector **co**. For 3-player games this vector equals the egalitarian allocation, a concept developed by Dutta and Ray (1989).

For $N \geq 4$, games (v,N) can contain an asymmetric coalition set. For $N = 4$ the only possible asymmetric coalition set is formed by two distinct two player coalitions. We give three example games $(v,4)$ with combinatorially isomorphic core. Each of the example games has an asymmetric edge in the core. Along the asymmetric edge the inherent equity bias creates a drift dynamic "down" the asymmetric edge, and the asymmetric power creates a drift dynamic "up" the asymmetric edge. In each example game the asymmetric power is extreme, zero or moderate respectively: the equilibrium distribution of the process is concentrated at the "upper" endpoint, the "lower" endpoint (which is **co**) or around a demand vector in the interior of the asymmetric edge. Furthermore we give simulation results, which indicate that the concept of asymmetric power can be generalized to other dynamic learning processes.

Coupling is a powerful and elegant probabilistic tool with which one is often able to calculate tight bounds on the speed of convergence to equilibrium of Markov chains. We believe this technique to be novel to the study of dynamic stochastic learning processes in evolutionary game theory and hence present a general introduction to the technique. We use coupling arguments to show rapid mixing for the cooperative game process for the N-player bargaining game and for general 3-player games.

Acknowledgements

Foremost I would like to thank my PhD supervisor Graham Brightwell. I am deeply indebted to him for outstanding academic guidance and infinite patience. Without him, the proofs and concepts developed in this thesis would have 'almost surely' remained vague ideas restricted to my mind. Without him this thesis would have never materialized.

I would also like to thank Malwina Luczak. It is through the courses she taught at the LSE that my interest in probability theory developed. I am deeply indebted to her for her guidance on probability theory and in particular the introduction to Markov chain mixing and coupling.

I am indebted to Heinrich (formerly Harald) Nax for the introduction to evolutionary cooperative game theory. Our initial collaboration let to my pursuing research in this interesting branch of mathematics and the fruitful discussions helped in developing the concepts and proofs in this thesis. I am grateful to Peyton Young for introducing me to the concepts of adaptive learning processes and specific advice for solutions, such as, using the drift in the variance over mean ratio to show convergence to equilibrium.

I would like to thank all the members of the mathematics department, in particular Bernhard von Stengel, Jan van den Heuvel and Rebecca Lumb, whose support has been way beyond the expected in all aspects of life during my time at the LSE. Furthermore, I would like to thank Robert Simon for his fruitful comments on related literature.

Finally, I would like to thank my family for their unconditional support, love and encouragement.

All errors remain my own.

Chapter 1

Literature Review

Many interesting static solution concepts have been introduced in cooperative game theory. A brief summary is presented in the second part of Section 1.1. However, somewhat less successful have been the attempts to explain why, and how, particular cooperative outcomes come about. The challenge has been to find a model that dynamically supports cooperative solutions, and for which individual strategies are simple enough to model strategic play by a whole population in a realistic way.

One solution to this challenge is to find a learning model that is simple in its update rule and requires little understanding by the individual agents other than their own payoffs. Nax (2010) introduces such a model. In this evolutionary model of incremental demand updates, individuals do not have a strategic model at all. They experiment rarely by increasing their demands when "satisfied" with their payoff, and decreasing their demands when not satisfied. A detailed analysis is given in Chapter 2 Section 2.1. The dynamic learning model studied in this thesis is a modification of the Nax model amenable to analysis. In Chapter 3 Section 3.3 we show that the dynamic learning model studied in this thesis is the limit of the Nax model when the "rate of experimentation" tends to zero.

In this chapter we survey the literature on dynamic learning, forming the background to, and giving motivations for, the particular model we study. A comparison of the presented literature with the dynamic learning model studied in this thesis is given in Chapter 2 in Section 2.7.

In the model an individual's demand update decision is based solely on the observation of her last payoff. Such a learning rule is completely uncoupled. Earlier models of uncoupled learning dynamics are described in Section 1.5.

The incremental demand updates of the learning process are motivated by models in behavioral psychology such as reinforcement learning (described in Section 1.1) and aspiration adaption (described in Section 1.3).

The most significant new concept developed in this thesis is that of the *asymmetric coalition set* which gives *asymmetric power* to the players in its intersection. We believe this power to be present naturally in dynamic learning models on cooperative games based on reinforcement learning or aspiration adaption. Section 1.2 describes early concepts in social power situations and motivates our concept of asymmetric power in cooperative learning processes.

By construction, the learning process in this thesis has an inherent equity principle, which is motivated by the results of experimental games, which are described in Section 1.2.

One dynamic learning model, which has been successful in "finding" many equilibria of noncooperative games, and some of cooperative games, is adaptive play described in Section 1.4. Another branch of dynamic learning models is based on Bayesian decision theory and described at the end of Section 1.4.

1.1 Reinforcement Learning and Static Cooperative Solution Concepts

Estes (1950) develops a statistical theory of learning. He attempts to derive a statistical interpretation to the concept of stimulus and response. He explains behavior in terms of "experimentally manipulable variables" and uses mathematical modeling to describe the consequences of changes in assumptions to summarize well established empirical relationships in a statistical sense. "It is proposed that the theory be evaluated solely by its fruitfulness in generating quantitative functions relating various phenomena of learning and discrimination."

He defines quantitative laws that well describe simple behavior systems. Behavior samples that share common quantitative properties are called dependent variables, independent variables relate to environmental events. In a situation where an organism has initial probabilities of various responses, a model based on Estes theory predicts changes in these probabilities as a function of changes in the independent variables. In this sense learning is defined mathematically as the "transfer of probability between certain response classes". Estes theory of statistical learning does not intend to replicate real behavior and mechanisms within organisms or species but rather develops mathematical summaries or statistics of common behavior that is, predictions of changes in the relative probabilities of certain behaviors.

Bush and Mosteller (1955) introduce a general model of stochastic learning where "reinforcement (reward) increases the probability of the rewarded response". The model focuses on "how much a reward increments response probability". In order to model rewards and their effect on a learning process they created a linear operator model. Each reinforcement increments the response probability by a constant fraction of the difference between the current probability and the maximum probability (one). Non-reinforcement reduces the response probability by a fraction of the difference between the current response probability and the minimum (zero) probability.

The distinct multipliers associated with response and non-response depend on different variables such as deprivation level, reward size and reinforcement history. The model is completely deterministic with respect to the changes in the response probability (although it is probabilistic with respect to the response probability itself). One can derive deterministic predictions for average acquisition and extinction curves as the linearity of the model allows to aggregate across individuals. The Bush-Mosteller model predicts exponential form for the extinction and acquisition curves. Empirical verification or evidence has been mixed especially with respect to paths and variability between different asymptotic states. However the fact that the model has been able to account for asymptotic or equilibrium performance has proved highly valuable.

The model has since been extended, for example by Rescorla and Wagner (1972). There have been other more recent reinforcement learning models in research on experimental games. One such notable model is the "payoff sum model" of Roth and Erev (1995, 1998). The model describes automatized routine behavior where reinforcement learning occurs in men as well as animals of relatively low complexity. In a simple version of the model, whenever a payoff for a decision alternative has been received, the payoff sum for that alternative is increased. The probability of choosing a decision alternative is proportional to its payoff sum. An interesting and strong feature of the model is that all information needed by participants is in their own payoff sums of their decision alternatives.

Neumann (1947) models situations where “adjustments are made in a population of agents playing a cooperative game”. The concept of the core dates back to Edgeworth (1881). In game theory it is famously studied by Gillies (1959), Bondareva (1963), and Shapley (1967) who show that the set of games with nonempty core is the set of “balanced” games. All games analyzed in this thesis are balanced. Dutta and Ray (1989) introduce a solution concept called the *egalitarian allocation* and they show that for all 3-person games where the egalitarian allocation exists and the core is non-empty the egalitarian allocation lies in the core.

1.2 Equal Share Analysis and Power

Selten (1972) analyzes the empirical outcomes of cooperative bargaining. He develops theories that aim to explain many experimental allocation results. Equal share analysis is based on three hypotheses that should hold for payoff allocations in characteristic function games:

1. “no union of coalitions which have been formed could have secured a greater collective payoff
2. no alternative coalition could have been formed by giving each of its members the same amount and more than he received in the end
3. within a coalition the stronger player does not receive less than a weaker player”.

A vector satisfying the above conditions is in the equal division core. Strong players are players whose cooperation is essential to form coalitions with high equal share. He finds remarkable empirical evidence in conducted experiments with different groups of people: allocations in the equal division core are formed most of the time.

Selten later refined this concept. One such refinement is the equal division payoff bounds predicting outcomes for experimental three player superadditive games. The first experimental cooperative games were introduced in 1954 with "some experimental n-person games" by Kalish, Milnor, Nash and Nering (1954).

Harsanyi (1962a) develops concepts to measure social power and opportunity costs in the two-player bargaining setting. Furthermore Harsanyi (1962b) extends the measurement of social power to *N*-player reciprocal power situations. In "reciprocal" or "bilateral" power situations "not only can A exert pressure on B in order to get him to adopt certain specific policies, but B can do do the same to A." In these situations the extent of compliant behavior and extent of net incentives are bargained over. When "A's power over B is based on A's ability to set up rewards and or punishments for B conditional upon B's behavior" the weaker player can usually exert pressure on the stronger player by "withholding his compliance" or by raising the cost of conflict for the strong player.

In Harsanyi's (1977) "Blackmailer's fallacy" a would-be-blackmailer can cause a damage of 1000 dollars to a person. He then reasons (prematurely) that he can extract any ransom just short of 1000 dollars. However, the person he threatens to damage could argue in the same way that the blackmailer would be happy with just 1 dollar. Both arguments are not conclusive, the only real conclusion to be drawn is that the ransom must lie somewhere between zero and 1000 dollars. Harsanyi calls these limits the "concession limits". Within the concession limits the so-called "Nash-Zeuthen" theory predicts that the value of the ransom is mainly determined by the attitude towards risk of both parties involved. Furthermore, if both follow the Nash-Zeuthen theory of two-person bargaining and the situation is a bilateral power situation, then "the amount of A's power over B with respect to some action X tends to be equal to half the net strength of A's power over B with respect to the same action X". The net strength is the difference between the "gross relative strength of A's power over B with respect to action X and the gross relative strength of B's power over A with respect to some complementary action X*".

In *N*-player conflict situations the definition of the strength and amount of power are less obvious. The amount of a person's power "is a measure of the probability of his being able to achieve adoption of joint policies agreeing with his own preferences". "In this situation it is natural to define the amount of individual i's power over the joint policy of all n individuals as the probability p_1 of his being able to get his favorite joint policy X_i adopted by all individuals."

A more satisfactory power is Harsanyi's *vector measure* for i's power which states the whole probability vector $p = (p_1, p_2, ...p_n)$ giving the probabilities for the adoption of each of the alternative policies $X_1, ..., X_n$. The strength of a person's power measures the incentives he can provide to the other players to agree to his proposals, and "more general the strength of his bargaining position against the other participants".

Other definitions and descriptions of power have been given such as power in a committee system proposed by Shapley and Shubik (1954) and their measures are special cases of Harsanyi's game theoretical measure for power in n-person situations.

1.3 Bounded Rationality, Aspiration Adaption and Experimental Games

Simons (1954a,1954b) created the theory of bounded rationality to account for the fact that individuals are not fully rational. "The fully rational man is a mythical hero who knows the solutions of all mathematical problems and can immediately perform all computations." Furthermore participants are often driven by psychological or emotional factors that are not modeled by "fully rational Bayesian maximizers of utility". Experimental evidence that agents can divert strongly from fully Bayesian rationality in their decision making is found for example by Kahneman and Tversky (1982). Simons modeled decision making as a "search process guided by aspiration levels". He defines goal variables such as profit for a firm. An aspiration level is a value of a goal variable that is to be achieved by a viable decision. A search process finds possible decision alternatives, it ends once the aspiration level is satisfied. Simmons termed the notion of "satisficing" for this search. Then, once satisfied, aspiration levels are adapted dynamically and the search for a new alternative decision starts anew.

Sauermann and Selten (1962) develop an aspiration adaption theory ("Anspruchsanpassungstheorie") for the firm based on the ideas of Simons. Selten (1998) translates the theory into English and presents the theory in a more formal way. He discusses possible modifications in the light of recent experimental evidence. Selten elaborates further on the early aspiration adaption with bounded rationality by Simons. For Selten the decision maker has several goal variables, for each of which he prefers more to less. “Goal incomparibility” is if neither of two different vectors of values for goal variables dominates the other, i.e. has a higher or equal value for each goal variable. An aspiration level is a vector with values for each goal variable. The aspiration levels vary in discrete steps and form a grid in the space of the goal variables. Aspiration adaption is then done by adjusting the aspiration levels: An upward adjustment is an increase in one partial aspiration level of one goal variable to a neighboring point on the aspiration grid, a downward adjustment an equivalent decrease. A ranking of the goal variables in terms of urgency is given. Furthermore every grid point is assigned a retreat variable that is to be reduced in case of downward adjustments at this particular grid point. An aspiration scheme is a combination of aspiration grid, retreat variable specified for every point on the grid and an urgency order on the goal variables. It is important to note that both urgency order and retreat variable are ”local procedural preferences”. ”An upward adjustment step is ”feasible” if it leads to a feasible aspiration level.

Selten (1978) analyzes power in economic behavior and develops and finds evidence of an inherent equity principle in many bargaining situations. He compares normative results of Bayesian decision theory and empirically observed human behavior with bounded rationality. He looks at resulting allocations with respect to their equity properties and their “symmetry of power”. He defines power as “the capability to secure more than one’s equitable share”. All experiments are conducted on an individual basis, populations are anonymous. He concluded that superior power could lead to more. Selten finds strong empirical evidence for equity allocations as long as the differences in power are not too great: “it is not advisable to follow the natural inclination of a game theorist to concentrate attention on power explanations”. Results of characteristic function game experiments with face to face coalition bargaining agree surprisingly well with a rather simple theory called “equal share analysis”. He develops the “proportional equity rule” which gives the “same reward for every unit of achievement”. An equitable reward combination can be described as a combination which allocates the same number of reward units to every weight unit where the weighting is based on power. However the tendency to find equitable allocations has bounds. “One may conjecture that the influence of social norms is diminished in the face of substantial monetary incentives”.

Selten (1986) develops a learning theory approach based on the analysis of sequences of finite prisoner's dilemma supergames with respect to end behavior. He analyzes the effect that experience has on the play of repeated prisoner's dilemma games. His analysis contains both theoretical and experimantal results. For the experimental results 35 participants played finitely repeated prisoner's dilemma games where the number of rounds was fixed at 10 periods. Cooperation was common until close to the last rounds.

Albers and Albers (1983) analyze the "perception of degree of roundedness of numbers". An example of a recently more accurate but even more complex theory is that of Hertwig, Hoffrage, Martiguon (1999) who find that estimates are often restricted to some much smaller set of "prominent numbers". Becker and Leopold (1996) develop an "interesting experimentally based theory of expectation formation in an environment in which a subject predicts the next value of a univariate time series on the basis of past observations". The more previous local maxima are surpassed, the less likely is a continuation of an increase.

Nash (2008) develops an agencies method for modeling coalitions and cooperations in games. He studies the "evolution of cooperation among robot players through a coalition formation game with a non-cooperative procedure of acceptance of another player." Based on this theoretical study Nash, Nagel, Ockenfels and Selten (2012) conduct a laboratory experiment based on Nash's theoretical work. In the first round human players can accept a transfer of power to another player called agent. This procedure can end in either no agent at all being accepted or an agent being accepted by exactly one or both other players. No agent being accepted leads to zero payoff to all players. If several agents are accepted one is selected at random. In the second round the agent can then distribute payoffs as long as the distribution satisfies the coalitional constraints for the coalition that he is representing as agent.

Players have different strength according to their fixed position in different three-player cooperative games. The results are represented with focus on two factors: coalition building and payoff distribution. Cooperation is very successful. More than 90 percent of rounds end in the cooperation of all players, represented by the formation of the grand coalition. "Although neither cooperative nor non-cooperative game theory predicts how a grand coalition can emerge, one might speculate that the key to successful cooperation is a commonly accepted, stable agency". "The symmetry of the voting procedure induces a balance of power: Selfish agents tend to be voted out of their agency and are disciplined by reciprocal behavior".

With respect to payoff allocation the results are very interesting: In more than 50 percent of all rounds the equal split in the grand coalition is implemented. In the groups where the equal split is not prominent, focusing on divisions to the strong player, the Shapley value is the best descriptor. Allocations close to the nucleolus are also quite prominent. "Reciprocity explains the strong prominence of the equal split in the aggregate, ...both gifts and demands are highly correlated between players." The strong player on average receives more than the other players. However he receives less than his power in the cooperative game would suggest.

It is to the joint effort of the group to form the grand coalition. "However, one problem is that the core can exist of many points without distinguishing a preferred point.". He pays attention to the Shapley value as a measure of power in the game. However he only considers three player games without singleton coalitions, "...the nucleolus serves the most dissatisfied player most". One of the interests into the research was trying to understand why humans interact in a cooperative way. "The evolution of human altruism and cooperation is a puzzle. Unlike other animals, people frequently cooperate even absent of any material or reputational incentive to do so." This research highlights that "efficiency requires people's willingness to accept the agency of others".

1.4 Adaptive Play and Rational Learning

Brown (1951) introduced fictitious play as a learning process for zero-sum games. In the course of the game players assume that the average play of the historical choices of the opponents is the best prediction for the players' future play and hence choose a best response in return. In fictitious play all players base their decisions on the entire history of the game. The distribution of strategy choices is stationary. After a large number of plays it quickly becomes unreasonable to assume that players "remember" the complete past and that players are able to base their best response on the entire historic distribution of the other players' actions.

In "evolution of conventions" Young (1993) develops adaptive play which extends the model of fictitious play by assuming that players base their decision on a sample of the recent history. "One way to think about the sampling procedure is to ask around, to find out how the game was played in recent rounds" or they "randomly hear about certain precedents" that have occurred recently. If an equilibrium has been played long enough, it becomes a convention: people will keep playing it. Eventually some equilibrium will be played with probability one. "Finite memory allows past mis-coordination to be forgotten eventually." For weakly acyclic games such as coordination games or common interest games adaptive play will converge to a Nash equilibrium if players make no mistakes and histories to sample from are sufficiently long relative to the sample size. The main convergence Theorem relies on the underlying game to be weakly acyclic. "A game is weakly acyclic if and only if from every strategy tuple there exists a finite sequence of best replies by one agent at a time that ends in a strict pure Nash equilibrium". A necessary assumption for adaptive play is that any possible sample of specific length of the recent history has positive probability. Young deliberately models players with little sophistication. "I have deliberately chosen to focus on the case where agents do not learn in order to show that convergence to equilibrium can occur with no common knowledge and with only a minimum degree of rationality on the parts of the agents. Society can learn even when its members do not."

People make errors for a variety of reasons. Young (1993) introduces random "errors" or "experiments" where sometimes a player randomly chooses any strategy instead of his best response. In such a random error model all equilibria will be played some of the time. One can say which equilibrium is played most of the time when the rate of experimentation is small enough. To differentiate between the different equilibria Young introduces the concept of *stochastically stable states*: A state is stochastically stable if its limit exists when the error goes to zero, and when the state has positive measure under the invariant distribution. The concept of stochastic stability can be described as robustness against perturbations.

The process of strategy updates is modeled as a Markov chain. One individual is chosen to update his best reply from each group of players in each round. Transition between states are from sets of histories to the successor set of histories, where a new history is added and the oldest one deleted. Transition probabilities are between successor states, only successor states have positive probability for which all strategies chosen in the new history are best replies to some subset of the previous set of histories. A state of histories of a fixed length T is an absorbing state if it consists of T identical pure Nash equilibria in a row. "If any other response than a pure Nash equilibrium has left the collective memory of a set of groups, they will continue to behave according to the conventions of their own group".

Young (1998) introduces a variety of "simple adaptive learning processes". These concern individual participants, that have only partial knowledge of the economic situation they are in. However, over time, these simple learning rules can "converge to complex equilibrium patterns of behavior". He finds that for quite a large number of classical solutions in game theory, there exist induced random processes, which under their equilibrium measure, will be close to the classical solution most of the time. "Indeed a surprising number of classical solution concepts in game theory can be recovered via this route". Other models with evolutionary stable strategies are Axelrod (1984) and Fudenberg and Maskin (1990).

For processes based on transferable utility games the dynamics are determined by two features: coalition formation and bargaining about the appropriate split of the surplus within each coalition. *Dynamic learning models* provide a framework for analyzing the path that the process takes from a starting state into the core or to a more refined solution concept. Arnold (1990) develops a dynamic learning model with local interaction and player mobility. Players move freely between locations, choosing a location and an action for the game. A theoretical paper on coalition formation is Shenoy (1979).

Packel (1981) develops a model of endogenous coalition formation. His process is a Markov chain on the set of payoff allocations. Transitions between two states depend on the number of coalitions that prefer a new state to the old one. He shows that for games with non empty core the process moves with probability one into the set of core allocations. The strong core is a singleton set of un-dominated states to which there is a path of positive probability from every other state. If the strong core is non empty, the process will settle in it with probability one.

In Konishi and Ray's (2003) model the transition probabilities between coalition structures are determined by a Markov chain. Players maximize their discounted expected future payoffs conditional on the transition probabilities between coalition structures. For any initial state there is a value function for each player that describes the respective players expected discounted payoff. A state in this model consists of a coalition structure and a respective vector of value functions. Transitions between two states have only positive probability if there is a coalition such that all players in that coalition prefer (not necessarily strictly) the new state to the old. Furthermore there is no other state that this coalition strictly prefers. If there is a new coalition structure for which all members of one coalition are strictly better off then the process will have zero probability of remaining in the old state. However players in this model are farsighted, myopic players only exist for an extreme discount factor of zero. Given a discount factor that is large enough, Konishi and Ray prove an equivalence relation between the core and a unique limit state.

Based on Peyton Young's (1993a) adaptive play Agastya (1997) develops a version of adaptive play on the state space derived from cooperative games. In this model of social learning with coalition formation and allocation, conventions are core allocations. Agastya provides sufficient conditions for global convergence to the core.

In the model there are N classes of players and a discrete demand space where all demands are multiples of a "smallest money unit". The updating process is directly taken from Peyton Young: One player is chosen from each class in each round to play a cooperative game. To decide on a strategy for that round each player inspects or "samples" k demand vectors from the last T. This models a situation where a player finds out what has happened in the recent past and forms beliefs about his opponents behavior. He then plays a best response to the beliefs generated by the sample of recent history he has observed. If a player's sample consists of a single strategy of his opponents repeated k times, he assumes that the same strategy will be played in the next period. Again it is essential that all possibilities of sampling of the recent past have positive probability. Demands are submitted simultaneously and players stay committed to their demand for that period. At the end of the period the surplus is divided amongst the players taking part in that particular round.

The resulting process is a Markov chain on a discrete state space. The assumption of a "smallest money unit" is made to avoid an infinite dimensional strategy space and ensures that a best response exists for every vector of histories. Additionally it ensures that demands can be increased to exactly the value of a coalition. Transitions between states depend on the likelihood for particular samples of size k being drawn given the sample history of size T. The joint transition probabilities are the product of the marginal (individual) transition probabilities and so independence of the belief formation of the individual players is assumed. Players use the same k and T to form beliefs.

The process is a model of social, not individual, learning. In each round, one player from a group of players is chosen. The players belief formation are independent of each other and Agastya assumes weak acyclicity of the process. Weak acyclicity is the main feature of the model that allows to obtain global stability (in probabilistic sense) of core allocations. The assumption of convexity of the characteristic function of the cooperative game simplifies the proof of acyclicity. Assuming convexity is equivalent to assuming non decreasing returns to scale in economic terms and hence is a reasonable assumption for many real world situations.

Further assumptions on the underlying cooperative game are that no dummy player exists and the core is non-empty. If the core is empty it is not hard to construct examples in which the space is not weakly acyclic. Agastya's assumptions impose little restriction on institutional details but strong restrictions with convexity of characteristic function. The rule for coalition formation is maximal in terms of set inclusion. The decision on the final coalition may depend on the sequence in which demands are processed. If it is possible to satisfy the demands of a coalition without hurting any other player, then their demands will be met for sure. Amongst others Agastya gives an example game with a convention that is not in the core and an example game with empty core where no convention exists.

Agastya (1999) adds the feature of random perturbations to his model on adaptive play for cooperative games (1997). The main result of Agastya is that an allocation is stochastically stable if and only if it maximizes a certain real valued function on the domain of core allocations. Furthermore he finds that this maximum always exists. If there are four or more players multiple stochastically stable allocations are possible. He again adapts Peyton Young's methods from evolution of conventions. The evolution of play is modeled as a Markov chain with the set of all histories of length T as state space. The transition probabilities are induced by the behavior rules. The updating rule in each round for each player is identical to the unperturbed dynamics apart from a small probability of error with which the player adopts a strategy that is not a best reply to the k histories sampled from the size T set of histories that players can remember. The non best reply strategy is chosen arbitrarily from the set of feasible strategies. Such a strategy is called a "mistake". The probabilities of mistakes for different players are independent of each other.

Restricting the behavior rules of the players to be time independent makes the process stationary. The assumptions are adapted from the unperturbed case: the core is discrete and restricted to multiples of the smallest money unit, demands must be strictly individually rational. When players err with positive probability, conventions cannot be established. Nevertheless, if the probability of errors is small, the process continues to be attracted towards conventions without actually settling down. The process will spend more time in some conventions than in others.

To prove which conventions are stochastically stable Agastya adapts a proof technique from Young (1993a). Given a convention x, an x-tree is a directed graph with the set of conventions as its vertices such that from every convention other than x there is a unique path directed to x and there are no cycles. Resistance is defined as the minimum number of mistakes necessary for the one period transition from a state s to s'. For any two states s and s' $r(s,s')$ is the minimum number of mistakes required to reach s' from s through a sequence of one period transitions. The resistance of an x-tree is defined as the resistance summed over all of its edges.

The stochastic potential of a convention x is the resistance of the tree that has the least resistance among all x-trees. Agastya's main theorem states that a state s is stochastically stable if and only if it is a convention and has the least stochastic potential among all conventions. Exact results for allocations under the stochastically stable allocation can only be given for players that get more than in the Nash bargaining solution. For these players exact limits exist. However, for the remaining players, these limits do not exist.

Arnold and Schwalbe (2000) develop a model of dynamic coalition formation where a player switches coalition only if his expected payoff in new the coalition is higher. Given feasibility he demands as much as possible. They analyze the path of the process into the core.

An allocation is feasible if the demands to players are at most equal to the maximum sum available to all under the optimal coalition structure. In each round, each player submits a choice of demand and coalition. Each player can only choose to join one coalition. Players can stay in their present coalition, join any other existing coalition or can form a singleton. If the demands in one coalition are feasible then all players in that coalition receive their demand. Each round a player is chosen at random according to the binomial distribution to update his choice of coalition and demand. The player chosen to update expects the current coalition structure and the demands of the other players to remain unchanged in the next round. These assumptions are close to reality as one can choose the binomial distribution in such a way that the probability of two players simultaneously updating their demands is very small.

Players have a “reservation payoff” that they receive if their chosen coalition does not form or if they choose the singleton coalition. Players’ strategy is to maximize their payoff myopically: they choose the coalition which will give the player the highest possible payoff. Further conditions are that his demand is the maximum he can receive whilst the chosen coalition remains just feasible given the previous demands of the other members of that coalition. If there is more than one coalition that would yield him the maximum payoff, he chooses one randomly. However a player would only switch a coalition if it yielded him a strictly higher payoff. A real world interpretation of the coalition formation process might be that people come together at certain meeting points or different locations, and so a player that wants to change to a new coalition has to go to the meeting point of that coalition.

This strategy profile of the players results in a Markov chain for which each state of the state space consists of a coalition structure and a set of demands for all players. The probability to move to a new state from a given state is the product of the individual transition probabilities multiplied by the probability of other players not adjusting demands between these two states. As players can only choose existing coalitions, the process can get stuck in suboptimal strategies if the current choice of coalition is the best reply for each player. So potentially blocking coalitions can exist. As an example of a cycle for this process the three player majority game is presented: Coalitions with cardinality of at least two have worth ten, all other coalitions’ worth is zero. The process based on this majority game will get stuck in a dominated allocation or cycle.

A perturbed version of the model in adaption to Young and Agastya is introduced. With small probability, players are allowed to change to strategies that are myopically suboptimal. However - different to Young and Agastya - only if they are members of a potentially blocking coalition. So a player accepts a temporarily lower payoff by experimenting in order to break a suboptimal coalition structure. Experimenting occurs deliberately, conditional on the current situation. For the process with experimentation and for superadditive transferable utility games Arnold and Schwalbe show that the set of demand vectors associated with an absorbing state of the best-reply-process coincides with the set of core allocations. The set of stochastically stable states is a subset of the set of absorbing states for the model with no noise.

Newton (2012) develops another adaptive play process in the tradition of Young and Agastya, with some redefinition of what generates a best response. He shows convergence for any superadditive characteristic function with non-empty core. The process selects the stochastically stable states within core allocations, where the stability of a core allocation increases with the the wealth of the poorest player. Players randomly sample from the recent histories, and play a best response. Randomly (from some distribution with full support over all strategies) they commit an error and choose another strategy. States consist of a demand and a set of players to form a coalition with. Players will form larger coalitions rather than smaller ones. The change in actions from period to period can then be modeled as a Markov chain where the actions of opponents are correlated. A related process was introduced in Karandikar et al (1998).

Bayesian Updating or Rational dynamic learning models have been successful in dynamically "finding" equilibria in games. In Kalai and Lehrer's (1993) N-player game model players form subjective beliefs about the opponents strategies according to which they choose their own strategies. The subjective beliefs are updated according to a Bayesian rule. The main results are the following: If players know only their own payoff matrices (and discount factor) and play to maximize expected utility they end up eventually playing an approximate (ε-)Nash equilibrium of the repeated game. However the play of a repeated game, in the rational learning model of Kalai and Lehrer, must not necessarily eventually resemble play of exact equilibria. Levy (2014) shows that play can remain distant from the play of any equilibrium of the original game. Levy shows further that the same holds true in Bayesian games.

Kalai and Lehrer's model does not require full knowledge of each player's strategies, and neither requires knowledge of the full prior distributions (subjective beliefs) of the other players but relies on the "grain of truth" assumption: Each player's "belief distribution on play paths" cannot rule out events with positive probability in the game; absolute continuity of the beliefs distribution with respect to the distribution of possible paths is required. However it is important to note that the belief distribution must not assign zero probability to any possible path.

The model exhibits path dependence on the assumed prior distributions of players. "Players who hold optimistic prior probabilities will follow a cooperative path while pessimistic players must eventually follow a non-cooperative path. Thus, in the case of multiple equilibria, initial prior beliefs determine the final choice." A player in Kalai and Lehrer's model will experiment when he believes that this will benefit his final present value of expected payoff. The subjective belief distribution converges to the distribution of actual paths, eventually being contained in any ε neighborhood of the real probability distribution of the path of the game for any small ε. Amongst others the model is applied to the infinite prisoner's dilemma and the chicken game.

One major difference to adaptive play is that the Bayesian updating rule of Kalai and Lehrer needs perfect recall in all payoff matrices for the complete history of the game. This results from the reliance on Kuhn's Theorem which enables the authors to replace a probability distribution over many strategies (mixed strategy) by a single behavior strategy. "Thus, a strategy specifies how a player randomizes over his choices of actions after every history". The "rational learning" requires intellect and memory of the players and it is rather complicated. Experimentation depends on the understanding and intellect of the player. This highlights one weakness of these models: the reliance on the maximization of future expected payoffs. It can be very difficult to determine the best strategy for maximizing expected future payoff in non trivial games.

Earlier approaches of learning in a similar way have been studied by Harsanyi (1967), and Aumann and Maschler (1967).

1.5 Uncoupled Stochastic Learning Dynamics

In an *uncoupled* process the adjustment of a player's strategy does not depend on the payoff functions of the other players. The strategy adjustment may however depend on the player's own past payoffs or the other players' strategies.

Hart and Mas-Colell (2006) showed the impossibility of stationary deterministic uncoupled dynamics to converge to Nash equilibria for bounded recall for all games. Bounded recall implies that there is a finite integer T, so that each player bases its play only on the last T rounds of play. For stochastic uncoupled dynamics convergence results have been shown.

Foster and Young's (2003) regret testing is a process that learns to play a Nash equilibrium. The response rule of a player depends solely on his own payoffs, neither on other players past or present actions nor on their payoffs. Foster and Young define the concept of *radically uncoupled*. In the subsequent literature this is known as completely uncoupled. In regret testing periods of play are grouped into finite sets of periods called rounds. During each round each player commits to play a mixed strategy profile according to which he chooses an action each period. At the end of each period players receive a payoff. At random a player sometimes chooses an action drawn from a different mixed strategy profile. At the end of each round each player compares the average payoff received from the strategy committed to play to each of the payoffs he received when experimenting with a random strategy. If at least one of the payoffs from a random play is higher by a fixed threshold (the "regret") than the average payoff over the mixed strategy committed to play, the player chooses a new mixed strategy profile uniformly at random from the set of all mixed strategies for the next round. Otherwise he continues to play the same mixed strategy profile for the next set. A payoff resulting from a random diversion in play is a statistical estimate of the average payoff the player could have achieved if he committed to that strategy for the whole set of periods. Hence the difference to the average payoff over the chosen mixed strategy profile is the regret of not having committed to the other strategy.

Foster and Young show that for finite two-player games "players" period-by-period behaviors are close to equilibrium with high probability" after a sufficiently long time. They prove convergence in probability to the set of Nash equilibria by "annealing" the parameters. More and more refined parameters are used over new rounds. The proof shows convergence of period-by-period behavior which is a stronger result than showing only convergence in "time average behaviors" such as in regret matching.

Even close to the equilibrium players can occasionally choose other strategies and hence the process can in theory move away again from equilibrium. One essential argument in the proof of Foster and Young is that "the expected time it takes to get close to equilibrium is much shorter than the expected time it takes to move away again". The model is very flexible in the amount of information players need to have. It is not necessary that players all use the same fixed number of periods before revision, nor is it essential that they all use the same random distribution. None of these relaxations of assumptions will change the nature of the proof structure significantly. The one essential feature that needs to be preserved is that players, when they experiment, assign positive probability to each possible mixed strategy profile, that is, they need to use a measure that is absolutely continuous with respect to the uniform measure. In comparison to the Bayesian updating rules players beliefs and strategies need not be aligned before the game and the required "sophistication" of players is relatively small.

Germano and Lugosi (2004) create a variant of Foster and Young's Regret testing for which they manage to show almost sure convergence to Nash equilibria by annealing the search procedure via a procedure of localization. They show the existence of a globally convergent learning rule. Similarly to Foster and Young's regret testing players observe their own payoffs for sufficiently long periods of time. Time is divided into periods of some fixed length T. At the beginning of each period, each player chooses a mixed strategy profile at random to which he remains committed during the duration of T rounds of play. If a strategy was close to optimal over the previous period (i.e. he could not have performed much better with any other strategy), a player will repeat that strategy profile in the next period, otherwise the player will choose a new mixed strategy profile at random. "The procedure thus implements a kind of exhaustive search with agents separately testing their own actions through summary statistics of past payoffs".

The method developed here guarantees convergence to just one Nash equilibrium, the limiting equilibrium may depend on the actual random realization of the sequence of plays. The main proof uses Doeblin's condition which assumes that one can change from any strategy to any other strategy in the state space in one adjustment of the chosen mixed strategy. So changes in behavior are not incremental but can be complete. Any measure absolutely continuous to the uniform distribution will work, the essential feature is again that each strategy profile is assigned non-zero probability. The procedure is some probabilistic trial and error search over all possible strategies.

Germano and Lugosi introduce a version of persistent randomness or permutation to the model: with some small probability players change their strategy even in the case when their strategy is close to optimal with respect to the past payoffs. The process of mixed action profiles taken at the beginning of each period is an irreducible Markov chain. The basic idea of the proofs is to anneal experimental regret testing: First one set of parameters for a fixed number of periods is used, then for the next set of periods the parameters are "refined" where for example the length of the period is increased. This ensures that not an infinite sequence of periods where strategies are far away exists. It is essential for the proof that after each change of parameter the search is localized such that each player limits its choice to a small neighborhood of the mixed action played right before. Another challenge is that the values of the parameters of the procedure (sequences) do not depend on parameters of the game since the game is uncoupled.

After some short searching over of a set of periods, by chance a mixed action profile will be played that will be ε close to a Nash equilibrium. Once in a close neighborhood of a Nash equilibrium players will have a small expected regret: the process continues with this value for a much longer time than the search period. It is essential that the length of time spent in the search period is negligible compared to the time spent ε-close to a Nash equilibria. This procedure is uncoupled as well, i.e. the actions of each player only depend on the players own past payoffs and not on the payoffs of the other players. However players require more coordination than in Foster and Young's regret testing: All players use the same parameters and the intervals of playing a fixed strategy over a period must be synchronized. The proof is very complex and the model is less applicable to real world phenomena.

As with Foster and Young's regret testing, the model has a variant which extends to the unknown game model: players observe their own past realized payoffs but not the actions of the other players. The unknown actions of each player can depend only on own past realized payoffs, without seeing the actions taken by the rest of the players.

The speed of convergence is rather slow. As the length of play is MT until a mixed action profile is ε-close to a Nash equilibrium with probability at least $1-\varepsilon$ of the time, the bound on the mixing time is $O\left(\left(\frac{1}{\varepsilon}\right)^{C}\right)$ where C can be large. The speed of convergence is at least exponentially slow as a function of the number of players and the number of actions of each player.

A similar model with faster mixing time is for example Cesa-Bianchi and Lugosi (2003). They show that there exists an uncoupled way of play such that after $O\left(\varepsilon^{-2}\log C\right)$ convergence has occurred where C is some constant.

Young (2008) introduces trial-and-error model with a completely uncoupled learning rule. In his “interactive trial and error learning” players randomly experiment with new strategies, keeping the new strategy if it leads to a higher payoff. Furthermore if a player experiences a lower payoff due to a strategy change of another person he will start a search for a new strategy.

This process models situations where “people interact, but they do not know how their interactions affect their payoffs. They are engaged in a game but they do not know what the game is or who the other players are. ” As examples of such situations Young gives commuters on the road or a market with many competing firms, where no single firm has exact knowledge of the other firms’ strategies. In a similar way, traders in a financial market cannot observe strategies of the other participants. However their actions can have beneficial or detrimental effects on each other. The “search procedures are triggered by different psychological states or moods” where mood changes depend on both the player’s current payoffs and his payoff expectations. The different mood states can lead to a more directed search if the player is content or a more random, or mutational, search if the player is not content. Before changing their strategies players can wait for a specified period of rounds. A “mood” is for Young “a state variable that determines how an agent responds to the recent payoff history given the agent’s current expectations”.

Two phases in the process are prominent: combined they lead to the play of a Nash equilibrium with a high proportion of the time. In one phase cautious amendment and adaption of strategies leads to higher and higher payoffs until a Nash equilibrium is reached. In the other phase dissatisfied players implement a random search which can then lead to other players also starting a random search which enables the process “find” other Nash equilibria that are far away from the previous states.

In Nax, Pradelski & Young (2013) a similar process is studied. It is interesting that the rule for the class of assignment games leads to the least core (another refinement of the core).

Chapter 2

Summary of Results

The most significant new concept developed in this thesis is that of *asymmetric power* explained in Section 2.5. The dynamic of asymmetric power is present along the face generated by an *asymmetric coalition set* (explained with example on page 51). We believe this concept to be of general interest to the study of dynamic learning models on cooperative games for which incremental updates of the dependent variables are in the tradition of reinforcement learning and aspiration adaption processes.

Section 2.4 explains the analysis of equity in 3-player games. In 3-player games no asymmetric coalition set exists. We show that the grid points in the core form the set of recurrent states of the Markov chain. We prove that for all balanced games $(v,3)$, where v is superadditive, the equilibrium distribution of the Markov chain is concentrated around **co**, the vector in the core with smallest $\mathbf{L}^2$-distance from the equal split.

For $N = 4$ the situation is very different. In Section 2.5 we show that the core of a 4-player game has an asymmetric edge if and only if the core polytope contains a face generated by two distinct 2-player coalitions. We introduce three example games with combinatorially isomorphic core, all containing an asymmetric edge. We calculate the drift along the asymmetric edge and show that the equilibrium distribution of the Markov chain can be concentrated at a state far away from **co**. To argue for the general applicability of the concept of asymmetric power, we show further, with a different method, that the asymmetric power exists even when the Markov chain is restricted to the core.

The main methodological contribution of this thesis is to introduce the coupling technique to bound the speed of convergence to the equilibrium measure of the cooperative game process. We believe the introduction of coupling to be novel to the study of speed of convergence to equilibrium of stochastic learning processes. The concept of coupling is explained in Section 2.6.

The *coalition's surplus* is the demand sum of players in a coalition in excess of the worth of the coalition. In Section 2.3 we show that the coalition's surplus is a birth-and-death chain that returns frequently to the 0-state if and only if the worth of the coalition divided by the worth of the grand coalition is strictly larger than the proportion of players in that coalition. This transformation is useful to determine which facets of the core are "relevant" to the dynamics of the process.

2.1 The Original Process

The Markov chain studied in this thesis is a modified version of the completely uncoupled chain introduced by Nax in his PhD thesis (2011). In his thesis Nax introduces a learning rule in which players experiment rarely when "satisfied" by increasing demands with a fixed, small probability called the "rate of experimentation" e. A player adapts his strategies quickly when "unsatisfied" by reducing demands with a different probability proportional to the magnitude of his demand.

The process of incremental demand updates is in the tradition of Selten's (1998) aspiration adaption theory; the process transitions only to states in a local neighborhood. Furthermore the demand decreases proportional to the magnitude of the demand are in the tradition of reinforcement learning. Small adjustments to the demands cause small changes of the probabilities of the response. Selten (1972) found evidence for an inherent equity principle in many bargaining situations. The combination of uniform demand increases and proportional demand decreases results in an "inherent equity principle" in the process: All else being equal, players with higher demands reduce demands more frequently. This gives the process a tendency to "search" for equitable allocation.

The process has features that make it a model for many real world bargaining situations: the process is simple and generic and requires very little sophistication of the players. It is well suited to model social learning, where players are assumed to be individuals from a certain group or party. The simple rules and little sophistication required by the players make the process suited as a theoretical model for real word experimental research with groups of players.

From now onwards the chain studied in this thesis will be referred to as the “modified” process and the chain introduced by Nax as the “original” process.

The behavior of the original process is determined by:

- how coalitions form,
- how demands are updated.

The main coalition formation rule introduced by Nax is the following:

In each round the *payable* coalition, the feasible coalition with the maximum sum of demands is chosen. Ties are broken by a simple rule. The payable coalition can be thought of as selected by nature or an administration. Players in the payable coalition will be paid their demand and possibly will experiment in the next round for a greater share by incrementally increasing their demand. Players not in the payable coalition tend to reduce demands in an attempt to adjust their unsuccessful demand bid.

The demand updating rule introduced by Nax is the following: Each round one player is selected uniformly at random to update his demand. All other players' demands remain unchanged.

The player selected to update his demand:

1. increases his demand by a positive constant amount ε with small constant probability e if he forms part of the payable coalition (a player knows that he forms part of the payable coalition from the fact that he receives his demand as payoff),
2. decreases his demand by ε with positive probability proportional to the value of his demand if he does not form part of the payable coalition (a player knows that he is not part of the payable coalition if he does not receive his demand as payoff),
3. does not change his demand with the remaining probability.

The constant ε can be seen as the smallest money unit, in an adaption from Agastya (1997). For this process this smallest unit ε is assumed to be very small, and the value function of the coalitions are restricted to multiples of ε. The rate of experimentation

e is thought to be small too. A player who receives his demand as payoff, experiments only rarely in order to gain a higher payoff. A player who receives no payoff, will react faster and decrease his demand with higher probability, if his demand is not close to zero. An interpretation of the coalition formation process is that nature or a governing body decides on the payable coalition. Different algorithms for choosing the payable coalition are possible. The player's decision is totally based on what he receives, therefore the original process is a completely uncoupled game process. The process requires very little sophistication of the players compared to adaptive learning, Bayesian updating or other stochastic learning models.

The process is a Markov chain that moves moving on an N-dimensional ε-mesh. The state dependent factors that determine the transition probabilities of the players are:

- which players form the payable coalition (these players increase demands with equal probability),
- the magnitude of the demands of all players not in the payable coalition (those players will reduce demands with probability proportional to the size of their demand).

The 1-step transition probabilities depend on the payable coalition at the current state. The difference between the sum of demands of players in a coalition and the worth of a coalition is the *coalition's surplus*. In the core, a coalition is feasible if the coalition's surplus is zero. A coalition with zero surplus is called a *binding* coalition. The n-step transition probabilities depend on the set of coalitions that can become the payable coalition over the next n steps. Informally, the process is "locally impacted" only by such coalitions, whose coalition's surplus is negative or close to the worth of the coalition (close meaning some small constant number of ε steps).

Consider the following example game: the worth of the grand coalition is 1, the worth of C_{12} is 0.9, the worth of C_{13} is 0.7 and all other coalitions have worth 0.

The state $(0.7, 0.2, 0.1)$ is an efficient state as the demands sum up to one. The state is in the core since $d^1 + d^2 \geq 0.9$ and $d^1 + d^3 \geq 0.7$. C_{12} is binding since $d^1 + d^2 = 0.9$. If player one increases his demand to state $\mathbf{d}_1 = (0.7 + \varepsilon, 0.2, 0.1)$, then at the new state no coalition is feasible any more and so no player can be in the payable coalition. The process will remain in that state until one of the three players decreases his demand and the process is in a new efficient state. If player one decreases his demand (with probability $\frac{0.7+\varepsilon}{1+\varepsilon}$) the next efficient state is again $(0.7, 0.2, 0.1)$. If player two decreases his demand (with probability $\frac{0.2}{1+\varepsilon}$) the next efficient state is $(0.7+\varepsilon, 0.2-\varepsilon, 0.1)$ and if player three decreases his demand (with probability $\frac{0.1}{1+\varepsilon}$) the next efficient state is $(0.7+\varepsilon, 0.2, 0.1-\varepsilon)$.

At a particular state the original process can move away from the set of efficient states by at most $(k+1)\varepsilon$ steps where k is the size of the set of binding coalitions at that state. For example, consider the following possible transition from the efficient state $\mathbf{d}_0 = (0.7, 0.2, 0.1)$. In round one, player three increases his demand and the process is in state $\mathbf{d}_1 = (0.7, 0.2, 0.1+\varepsilon)$. Now coalition C_{12} is still binding and so in round two, if chosen to update, player one (or two) can increase his demand so that the process is in state $\mathbf{d}_2 = (0.7+\varepsilon, 0.2, 0.1+\varepsilon)$, where the sum of demands is $1+2\varepsilon$. However, observe that if e is small relative to the size of player three's demand, it is much more likely that player three will reduce his demand again (with probability $\frac{0.1+\varepsilon}{1+\varepsilon}$) taking the process back to the original state before player one (or two) increases his demand (with probability e). So most of the time the process will evolve in a way that a player increases his demand and in the next round a player decreases her demand.

The original process moves along the set of efficient states, making short excursions away from this set. The process started in an efficient state can (in theory) move "far" away from the starting state until it reaches the next efficient state. Consider for example the state $\mathbf{d}_2 = (0.7+\varepsilon, 0.2, 0.1+\varepsilon)$ from the previous example. In round three now player two can reduce demands as no coalition is feasible and the process can reach state $\mathbf{d}_3 = (0.7+\varepsilon, 0.2-\varepsilon, 0.1+\varepsilon)$. The transitions of round two and three can repeat now p-times to take the process to the state in $\mathbf{d}_{p+1} = (0.7+p\varepsilon, 0.2-p\varepsilon, 0.1+\varepsilon)$, if now player three decreases demands the process is in the next efficient state $\mathbf{d}_{p+2} = (0.7+p\varepsilon, 0.2-p\varepsilon, 0.1)$. Such a combination of transitions occurs with probability of order $O(e^p)$ and so a "far away" transition to the next efficient state is very unlikely.

The rate of experimentation e is assumed to be small. One could restrict players' demands to some global minimum level below which they would not reduce demands by introducing a global outside option and then choose e small relative to this global minimum demand. For any results in this thesis this is not required.

Although transitions to new efficient states far away from a starting efficient state happen rarely, the calculation of all possible transition paths is very complicated. Consider for example the state $\mathbf{d}_0 = (0.7, 0.2, 0.1)$ in the previous example. Let's assume that $\varepsilon = 0.00001$. Then in theory the process could transition to the state $\mathbf{d}_T = (0.79, 0.11, 0.1)$ and any other state between $\mathbf{d}_0$ and $\mathbf{d}_T$ with $0.7 \leq d^1 \leq 0.79$, $d^2 = 0.9 - d^1$ and $d^3 = 0.1$. The probability of such paths are very small but positive, to calculate all these probabilities is tedious and complicated. The analysis of the stopped process, which moves only on the set of efficient states, is hence not easily tractable for the original process.

2.2 The Modified Process

In this thesis a modified version of the original Markov chain, which is amenable to analysis, is studied. A *neighbor* $\mathbf{d}(i, j)$ of state $\mathbf{d}$ is the efficient state that can be reached from $\mathbf{d}$ by player i increasing his demand, and player j decreasing her demand. The modified process, in an efficient state, is restricted to have zero probability to move to states that are not neighbors of this efficient state.

Compared to the original process, the coalition formation rule and principles 2. and 3. of the demand update rule, introduced on page 34, remain the same for the modified process, only demand update rule 1. is changed. We state again the complete coalition formation and demand update rule; the part that has changed compared to the original process is in bold.

- Each round the *payable* coalition, the feasible coalition with the maximum sum of demands, is chosen.
- Each round one player is selected uniformly at random to update his demand. All other players' demands remain unchanged.

The player selected to update his demand

1. increases his demand by a positive constant amount ε with small constant probability e **if the payable coalition is the grand coalition** (a player does not know that he forms part of the grand coalition from the payoff received),
2. decreases his demand by ε with positive probability proportional to the value of his demand if he does not form part of the payable coalition (a player knows that he is not part of the payable coalition if he does not receive his demand as payoff),

3. not change his demand with the remaining probability.

This modified process is not completely uncoupled. A player now requires information about the total sum of demands of all players. However it is shown in Chapter Section 3.3 that the modified process is the limit of the original completely uncoupled process when the rate of experimentation e tends to zero. We show that the transition probability between efficient states of the original process converges to the transition probability of the modified process. Then we show that the unique equilibrium distribution of the modified process is the limit of the equilibrium distribution of the original process when the rate of experimentation goes to zero. Hence any results in this thesis hold in the limit as well for the completely uncoupled original process. Intuitively, the modified process, that moves on the set of efficient states, is the embedded original process, for which all paths that have probability of order e^2 or smaller are ignored. Simulations are presented in Chapter 5 where even for $e = 0.001$ and $e = 0.01$ the modified process is a very good approximation to the original process.

One step of the random walk implies one player increasing demands and one player decreasing demands. The state space of the modified process is the $N-1$-dimensional simplex formed by the efficient states in the convex hull of the selfish splits, $(1,0,...,0)$, $(0,1,...,0)$,...,$(0,0,...,1)$, the states where one player demands the whole surplus and all other players demand nothing. The well-known *core* is the set of states (or demand vectors) at which all linear coalition constraints are satisfied simultaneously. At a state in the core the demand sum of each coalition is at least the worth of the coalition. Geometrically, the grid of core points forms a polytope. Facets, representing the states where exactly one coalition is binding, have dimension $N-2$. Faces representing the set of states where exactly the same k-coalitions are binding, have dimension $N-k-1$. The *boundary* of the core is the set of all states that are on at least one face of the core. Each state can have at most N^2-N neighbors, however coalitional constraints reduce the number of neighbors, if the state is on a face of the core polytope.

The modified process is a biased random walk on the core polytope. In the interior of the core, the bias of the random walk depends only on the magnitude of the demand of the players. For example, suppose the state $\mathbf{d}_i = (0.6, 0.15, 0.14, 0.11)$ is in the interior of the core. The most likely transitions from $\mathbf{d}_i$ are to neighbors $\mathbf{d}_i(2,1) = (0.6-\varepsilon, 0.15+\varepsilon, 0.14, 0.11)$, $\mathbf{d}_i(3,1)$ and $\mathbf{d}_i(4,1)$, since player one has the largest demand and hence reduces with highest probability in the interior of the core. The least likely transitions are to neighbors $\mathbf{d}_i(1,4)$, $\mathbf{d}_i(2,4)$ and $\mathbf{d}_i(3,4)$ since player three has the smallest demand. In the interior of the core the modified process has hence an inherent equity bias (demand reductions are proportional to the magnitude of the demand of a player). Unless the equal split state, where all players demand $\frac{1}{N}$, forms part of the core, the modified process will drift in the interior of the core until it reaches the boundary of the core.

	3-**player game**	
structure	**dimension**	**name**
core	2	*polygon*
$N-2$-*facet*	1	*edge*
$N-3$-*face*	0	vertex

For three player games the $N-2$-dimensional facets of the core form edges. $N-3$-dimensional faces of the core are sets of states where exactly two coalitions are binding simultaneously. For three player games they are vertices, and hence each vertex is unique state.

	4-**player game**	
structure	**dimension**	**name**
core	3	*polyhedron*
$N-2$-*facet*	2	*polygon*
$N-3$-*face*	1	edge
$N-4$-*face*	0	vertex

$N-3$-dimensional facets of the core of 4-player games are edges. We call the edge where, e.g., the coalitions C_{12} and C_{13} are binding the edge generated by coalitions C_{12} and C_{13}.

On a face of the core, the bias of the random walk depends on the magnitude of the demand of the players and on the membership of the players to the coalitions that generate the face. Suppose the same state $\mathbf{d}_b = (0.6, 0.15, 0.14, 0.11)$, in a different game, is now on the boundary of the core. Let's assume that for this different game $v(C_{12}) = 0.75$ and $v(C_{13}) = 0.74$. The state $\mathbf{d}_b$ is now on the edge generated by the coalitions C_{12} and C_{13}. If player two increases his demand and the modified process is now in the intermediate inefficient state $\mathbf{d}_b(2) = (0.6, 0.15+\varepsilon, 0.14, 0.11)$. Coalition C_{13} is still binding since $d^1 + d^3 = 0.6 + 0.14 = v(C_{13}) = 0.74$. Hence from $\mathbf{d}_b(2)$ player one cannot reduce his demand. The random walk has zero probability to transition from state $\mathbf{d}_b = (0.6, 0.15, 0.14, 0.11)$ to states $\mathbf{d}_b(2,1)$, $\mathbf{d}_b(3,1)$ and $\mathbf{d}_b(4,1)$.

In the modified process all players increase demands with equal probability. The state dependent factors determining the transition probabilities of the random walk are:

- which players form part of a feasible coalition at the efficient state (these players can possibly form part of the payable coalition once a player has increased demands),
- the magnitude of the demands of all players.

Of particular interest is the movement of the modified process along edges of the core. In Section 2.5 page 51 we introduce a coalition structure that exhibits an asymmetry of power between its members: the *asymmetric coalition set*. Along a face of the core polytope generated by an asymmetric coalition set, the *asymmetric face*, the demand sum of the players in the union of the coalition set is not constant. For example consider again the state $\mathbf{d}_b = (0.6, 0.15, 0.14, 0.11)$ in the second example game, where $v(C_{12}) = 0.75$ and $v(C_{13}) = 0.74$. The state $\mathbf{d}_{b2} = (0.6+\varepsilon, 0.15-\varepsilon, 0.14-\varepsilon, 0.11+\varepsilon)$ is as well on the edge generated by $v(C_{12})$ and $v(C_{13})$. The demand sum for the players in the union of C_{12} and C_{13} is 0.89 at $\mathbf{d}_b$, and $0.89-\varepsilon$ at $\mathbf{d}_{b2}$. When the player in the intersection of C_{12} and C_{13}, player one, increases his demand, both players two and three need to decrease their demand to reach the new efficient state $\mathbf{d}_{b2}$ on the edge. No new state on the edge can be reached by a simple rearrangement of demands of players within the coalition set. The symmetry of the coalition set is broken and the movement between states on the edge cannot be explained solely by the equity bias anymore.

The bias of the modified process is determined by the interplay between two dynamics: the inherent equity bias, which “drags” the process towards equity, and the *asymmetric power*, which “drags” the process away from equity. If the core polytope does not contain an asymmetric face, the equity bias of the random walk determines the expected movement along the faces of the polytope. A detailed explanation of the movement of the modified process along different symmetric and asymmetric faces of the core is given in Section 2.5 on pages 50–52.

The modification of the process makes the analysis more tractable. It enables us to solve large classes of games by calculating the one-step drift of the random walk and to analyze general games locally by applying techniques and standard results from the theory of random walks and birth and death chains. The transformation that enables the use of these techniques is explained in Section 2.3.

2.3 The Cooperative Game Process for Simple N-Player Games

The Cooperative Game Process for the N-Player Bargaining Game

In Chapter 3 Section 3.4, the state space of the process is derived from a version of Harsanyi's N-player bargaining game. The worth of the grand coalition is one, all other coalitions have zero worth. The core for this game is the set of efficient states.

The vertex states on the boundary of the core are the selfish splits. All transitions to neighbors on the simplex are possible. The bias of the random walk at a particular state depends on the differences in demands only. Transitions to states happen with probability proportional to the size of the demand of the player decreasing demands.

The process is called a *cooperative game process* if $v(C)$ is a multiple of ε for all coalitions C.

The cooperative outcome is the state with smallest Euclidean norm in the intersection of the core of the underlying game and the states on the ε-mesh forming the simplex. For the cooperative game process for the N-player bargaining game, this state is the *equal split* where each player demands $\frac{1}{N}$.

The solution concept we introduce and apply in this thesis is the Markovian cooperative equilibrium. If it exists, it is the unique state d^*, such that for all $\alpha > 0$ the equilibrium distribution of the cooperative game process on all states that have a greater L^2-distance than α from d^*, tends to zero as ε tends to zero. Intuitively, by refining the ε-mesh enough the process will have limit mass one in any α-neighborhood of the Markovian cooperative equilibrium.

It is proved in Chapter 3 Section 3.4 that the equal split is a Markovian cooperative equilibrium for any $N \geq 2$. We show that, for any $\alpha > 0$, the equilibrium distribution on all states with L^2-distance of at least α from equal split is $\frac{\varepsilon}{N\alpha}$.

The main concepts of the proof are the following: We analyze the expected change over one step of the biased random walk's L^2-distance from equal split, called the 1-step drift. We show that the cooperative game process for the N-player bargaining game has a strong negative 1-step drift at all states outside an α- neighborhood around the cooperative outcome.

From the definition of the equilibrium distribution of an irreducible Markov chain it follows that the average drift over all states in equilibrium must be zero. It then follows that the (few) states with (bounded) positive drift, which must lie within the α-neighborhood around the cooperative equilibrium, must have very high mass under the equilibrium distribution to compensate the very many states with strong negative drift outside the α-neighborhood. Since the neighborhood around equal split is chosen arbitrarily, it follows that equal split is a Markovian cooperative equilibrium.

The negative 1-step drift of a state is proportional to the L^2-distance at that state. Hence, the further the cooperative game process is away from equal split, the stronger it drifts towards it. This behavior is "caused" by the inherent equity bias of the process, which is implemented via the rule that demand reductions are proportional to the size of the demands of the players in the demand update rule. If demand reductions were uniform (as demand increases are), the equilibrium distribution of the random walk would not be concentrated around a single state in general.

Expected Return Times to Facets of the Core

In an extension to the N-player bargaining set-up, we analyze a "well behaved" family of N-player cooperative game processes. For a major coalition, the average payout in the coalition is higher than the average payout at equal split ($\frac{v(C)}{|C|} > \frac{1}{N}$).

In the major coalition game set-up, all coalitions, of which the major coalition is not a subset, have zero worth. We show that the cooperative game process returns frequently to the facet generated by the major coalition C^0 if and only if $\frac{v(C^0)}{|C^0|} > \frac{1}{N}$. This is very interesting: on its "journey through the core" for general balanced N-player games it is of interest to which facets of the core the Markov process returns frequently. Facets, for which $\frac{v(C^0)}{|C^0|} < \frac{1}{N}$, are of little relevance for the understanding of the long term behavior of the process in the core. Although these results are shown in the major coalition set-up only, in a general balanced cooperative game, the cooperative game process' "behavior" along any facet is identical to the behavior along a major coalition as long as the process is reasonably far away from another face of the core.

We give sharp bounds for the expected return time to the facet generated by the major coalition as a function of the worth of the major coalition. The more the coalition is worth, the faster the cooperative game process will return to the facet. The expected return time satisfies $\mathbb{E}(T_1) < \frac{1}{1-\phi}$ where $\phi = \frac{|C^0|(1-v(C))}{(N-|C^0|)v(C)}$, where C^0 is the major coalition. Furthermore we show that $\mathbb{P}\left(\frac{CS^{C^0}(V_t)}{\varepsilon} > K\right) < \frac{\frac{1}{1-\phi}}{K}$ for all $K > 0$ and so we derive a probabilistic bound on how far the process can move away from the facet generated by the major coalition.

In order to show these results, we apply a trick: Recall that the *coalition's surplus* is the demand sum in that coalition in excess of the worth of the coalition. When the cooperative game process moves in the core, the demand sum of a coalition is in fact a birth and death chain, where the zero state corresponds to the cooperative game process being in a state on the facet generated by that coalition. This trick, or better understanding, enables us to apply standard results from birth-and-death chains and random walks to the calculation of the expected return time of the cooperative game process to facets of the core.

Consider for example the four player game where the worth of coalition C_{123} is 0.9, the grand coalition C_{1234} is worth 1 and all other coalitions have worth 0. So $\phi = \frac{3(0.1)}{0.9} = \frac{1}{3}$. Then the expected return time of the cooperative game process to the set of states where coalition C_{123} is binding is less than $\frac{1}{1-\frac{1}{3}}$ and hence is less than 1.5. The process moves in very close proximity to the facet generated by coalition C_{123}. If the worth of coalition C_{123} is 0.76 instead, the expected return time is less than 19 time steps of the cooperative game process, so the process still moves in close proximity of the facet, however its excursions away take longer time and hence the process on average moves further away from the facet before returning. If the worth of C_{123} is 0.7 the process will not return frequently to the set of states where C_{123} is binding, in fact such a major coalition game is very much like the N-player bargaining game and the process will settle at equal split. The facet of the core generated by a coalition with $\frac{v(C^0)}{|C^0|} < \frac{1}{N}$ is not really relevant for the dynamics of the cooperative game process since the cooperative game process will drift away from the facet.

We introduce the major coalition set-up to show how to analyze the cooperative game process in a more generic way than through the analysis of 1-step drifts. By construction (demand reductions proportional to the magnitude of the demand) a general cooperative game process for N-player games moves along coalition structures that form some subset of the "boundary" of the core. There are specific faces of the core along which the process moves, that is to which the process returns frequently. We make an important and interesting connection in this chapter between the worth of a coalition (set) and the local return behavior of the process to the face generated by this coalition (set). We show in this chapter that a random walk is a very accurate approximation of the local behavior of the process in the neighborhood of states where exactly one coalition is binding, a facet of the core. By the use of this transformation we can precisely answer the question, if the process returns frequently to the face generated by that coalition structure, and if so, how likely far away transitions are.

The following reasons make this analysis attractive and very useful:

- one step drift analysis is tedious as each state has up to $N^2 - N$ neighbors and a 1-step drift towards cooperative outcome does not always exist, for an example of a set of states which do not exhibit a 1-step drift towards the cooperative outcome see Chapter 5
- in contrast to other dynamic learning processes (such as adaptive learning or Bayesian updating) the original and the modified processes studied in this thesis are only impacted by "local" constraints and hence analyzing only "relevant" faces of the core polytope simplifies the analysis. Determinining which faces of the core are "recurrent" and calculating the "drift" of the process along these faces is a promising route for a generic N-player analysis of the cooperative game process and possibly of similar dynamic learning processes with incremental demand updates or aspiration adaption.

Given a coalition C, we can calculate at each state $\mathbf{d}$ the coalition's surplus $CS^C(\mathbf{d})$. For example if the worth of a coalition is 0.9 and the sum of demands of all players in that coalition is $0.9+5\varepsilon$ the coalition's surplus is 5ε. To express it differently, the birth and death chain formed by $\frac{CS^C(\mathbf{d})}{\varepsilon}$ is in state 5.

We show that the process $\frac{CS^C(V_t)}{\varepsilon}$ for the family of games defined in this section is in fact a birth and death chain. We bound the expected return time of the chain by defining a random walk that is a "pessimistic" version of the birth-and-death chain. Then "locally" the coalition's surplus $\frac{CS^C(V_t)}{\varepsilon}$ behaves very much like the birth and death chain defined here. Using the bound of the random walk we show, under which conditions the process stays in close neighborhood of the face generated by the major coalition.

We make a conjecture about the return behavior of the cooperative game process to the face generated by two coalitions, say C_1 and C_2. The process $\left(\frac{CS^{C^1}(\mathbf{d})}{\varepsilon}, \frac{CS^{C^2}(\mathbf{d})}{\varepsilon}\right)$ behaves much like a process whose coordinates are two dependent random walks. Suppose that $\frac{v(C^1)}{|C^1|} > \frac{v(C^2)}{|C^2|} > \frac{1}{N}$, then we say that C^1 is the *leading coalition* of the two. How will the process move along the face generated by coalitions C^1 and C^2? We calculate the proportions of time when the leading coalition's surplus is zero and non-zero. The return behavior of coalition C^2 is dependent on the proportion of times that the leading coalition, C^1, is in the 0-state; and the proportion of time that the leading coalition is in the non-zero states. The average transition probability of C^2's coalition's surplus to increase by 1 is taken over the proportion of times that C^1's coalition's surplus is in the 0-state or in the non-0-states. The conjecture states: If the average probability of $\frac{CS^{C^2}}{\varepsilon}$ to increase by 1 is smaller than the average probability of $\frac{CS^{C^2}}{\varepsilon}$ to decrease by 1, then the process $\left(\frac{CS^{C^1}(\mathbf{d})}{\varepsilon}, \frac{CS^{C^2}(\mathbf{d})}{\varepsilon}\right)$ will return frequently to the face generated by C^1 and C^2. We have conducted Monte Carlo simulations for three and four player games that strongly support this conjecture. A mathematical proof is not trivial. However we believe this analysis to be very interesting for the general understanding of a game. Given a game, one can focus the analysis on those faces of the core that are "recurrent". We believe this analysis to be relevant as well for faces generated by more than two coalitions (although for three and four player games that is not needed). By knowing which faces are "recurrent" one can focus on calculating the drift of the cooperative game process along all these faces. This simplifies the analysis enormously compared to analyzing the drift at each step of the state space. If the game does not contain an asymmetric coalition set (expalined in Section 2.5), then one can apply symmetry arguments to show that the drift along the face is directed towards equity.

We show further that at each state there is a 1-step drift towards the cooperative outcome in the major coalition set-up. The Markovian cooperative equilibrium is **co**. This last part is a simple adaption of the drift analysis in the N-player bargaining game.

2.4 Equity in General 3-Player Games

In Chapter 4 we analyze the cooperative game process for balanced superadditive 3-player games. The main theorem of this chapter is a global convergence result. The cooperative outcome, the most equitable state in the core, is a Markovian cooperative equilibrium for any three player balanced superadditive game.

We summarize the set of intermediate results that are used to prove the global convergence result.

We prove that all three player games satisfying condition 4.4 are balanced and hence have a non-empty core. This proof closely follows Gilles (2010) and is added for completeness.

In order to prove convergence to the cooperative equilibrium for each three player game, We first need to derive an algorithm that finds the cooperative outcome for each game. A candidate cooperative outcome is determined: the state with smallest Euclidean norm in the core of the 'reduced' three-player game, where the worths of the singleton coalitions are set to zero. The core of this reduced game is called the **2**-core.

If this candidate cooperative outcome is a member of the core of the original game then it is the cooperative outcome as the core of the original game is a subset of the **2**-core. Otherwise, one singleton coalition must have worth more than assigned to its corresponding player under the candidate cooperative outcome. In this case the cooperative outcome is the state with smallest norm in the intersection of the core and the hyperplane corresponding to the singleton coalition of that player.

Then we show that the set of recurrent states for all balanced superadditive three player games is the core. So the process will eventually reach the core, never leave it again and have positive mass under the equilibrium distribution for every state in the core (albeit very small for most of them).

We first show that the process, once in the core, will never leave the core again. An elegant proof of this fact is given in Chapter 5 where we define a certain coalition structure and show that the process can only leave the core at a state where both coalitions in that particular coalition structure are binding. Since the coalition structure does not exist in the three player setting, a cooperative game process for the three-player game, once in the core, will not leave the core again. For a sketch of the proof, we refer the reader here to Chapter 5.

Then we show that from any state outside the core there is a path of positive probability into the core. We first show that there is a path of positive probability into the **2**-core and then from any state in the **2**-core there is a path of positive probability into the core.

To show connectedness of the core, we show that all interior states, states where no coalition is binding, are connected. Then we show that if an interior state exists, from any state there is a positive probability of transitioning to an interior state and each state can be reached from at least one interior state. The connectedness of the core follows.

To calculate the bound on the drifts for all states outside an α-neighborhood of the cooperative outcome is not trivial. There are many cases to be considered. We partition the core into three sets of states and derive a bound on the drift for each set. It is not in general true for N-player cooperative game processes that at every state a drift towards the cooperative outcome **co** exists, as shown ,e.g., in Chapter 5.

The first set in the partition of the core is the set of states where no coalition is binding. We calculate the drift for these states. The calculation of the drift for these states is straight forward and follows the same principle as in the N-player bargaining game.

Given a game $(v,3)$ let $\mathcal{C}^{\mathbf{co}}(v,3)$ be the set of coalitions that are binding at **co**. For the second set of states in the partition of the core the only binding coalitions are coalitions not in $\mathcal{C}^{\mathbf{co}}(v,3)$. We apply a trick: we compare the drift at a state in this set for a particular game with the drift at the same state of the equivalent game with the worths of all coalitions not in $\mathcal{C}^{\mathbf{co}}(v,3)$ set to zero. The trick can be applied as the cooperative outcome does not change if the worth of a coalition not in $\mathcal{C}^{\mathbf{co}}(v,3)$ is altered. This 'trick' reduces the amount of cases significantly, still some special cases remain. The principle used is the following: we split the drift at a state into the drift components arising from each of the three players increasing demands at that state. Then we compare each of the three drift components with the equivalent drift component in the game where the binding coalitions are removed. Consider the following example games $(3, v)$ and $(3, v^{co})$:

For both games the grand coalition has worth one, and C_{12} has worth 0.9. The worth of C_{13} is 0.5 for $(3, v)$ and zero for $(3, v^{co})$. The cooperative outcome (for both games) is the state $(0.45, 0.45, 0.1)$. The coalition C_{12} is binding at the cooperative outcome whereas the coalition C_{13} (for game $(3, v)$) is not (as $0.45 + 0.1 \neq 0.5$).

We want to compare the drift at state $\mathbf{d} = (0.425, 0.5, 0.0725)$ for both games. For game $(3, v)$ at state $\mathbf{d}$ only coalition C_{13} is binding. So if either player one or three increase demands the component drifts for $(3, v)$ and $(3, v^{co})$ are identical. If player two increases his demand for game $(3, v)$ C_{13} is the payable coalition and so player two has to decrease demands again and the drift component is zero. However if player two increases his demand for game $(3, v^{co})$ players one and three can decrease demands as well. Both player one and three have demands below their cooperative outcome at state $\mathbf{d}$ and so reducing their demands 'moves' the process away from the cooperative outcome. The drift towards the cooperative outcome at $\mathbf{d}$ is stronger for the game $(3, v)$ where players one and three cannot reduce demands.

We show that for all states where drift components are not identical, the drift component is stronger (more negative) for the game $(3, v)$. Intuitively speaking players that cannot reduce demands because they are in the payable coalition that forms not part of the set $\mathcal{C}^{\mathbf{co}}(v, 3)$, have on average demands that are below the cooperative outcome, and letting them reduce demands would on average 'move' the process away instead of towards the cooperative outcome.

Finally, we calculate the drift for all states in the core, where at least a coalition in the set $\mathcal{C}^{\mathbf{co}}(v, 3)$ is binding. There are few cases left to be considered, however the above tricks have reduced the possible combinations considerably.

Combining the drift on all three sets of the partition, we deduce a global bound on the drift for a cooperative game process $(v, 3, \varepsilon)$ as $\frac{-2\varepsilon}{9(1+\varepsilon)} D(\mathbf{d}) + 2\varepsilon^2$.

We apply the same argument as for the N-player bargaining game that the average drift in equilibrium is zero over all states. Given the drift outside a small α neighborhood is at most $\frac{-2\varepsilon}{9(1+\varepsilon)} D(\mathbf{d}) + 2\varepsilon^2$ we deduce the mass on the states in the neighborhood of the cooperative outcome must be very high. Since this holds for all $\alpha > 0$ this completes the proof that the cooperative outcome is a Markovian cooperative equilibrium for all balanced superadditive three player games.

2.5 Power in 4-Player Games

In this chapter, we introduce the main contribution of this thesis, the analysis of a new concept of power for evolutionary cooperative game processes, which arises in games with more than three players. According to Harsanyi, "Power is the ability to generate more than the equal share." The cooperative game process is the result of a very simplistic and generic updating rule that involves a minimum of intellect. Individuals can be seen as quasi-robots executing behavior according to a simple rule. In such a setting power will not arise through intellect or clever decision choices but can only arise through the given situation or position in the game.

Selten (1972) found evidence that a strong equity principle is present in many social situations. The main a priori feature of the cooperative game process is that this equity principle is modeled via the probability of demand reductions being proportional to the size of the demand. All else being equal, players with larger demands will reduce demands more often. In the bargaining setting the process spends most of the time in the allocation where complete symmetry of power exists, the equal split. In the three player game setting the players can have superior power but the power is not stronger than the equity bias: the 'force' that establishes power is the core; all coalition structures in the three-player setting have an inherent symmetry of power and so the equity principle prevails. In equilibrium the process spends most of the time around the state in the core with least power to the strong player(s). For four players (and for all $N > 4$) a specific coalition structure, the 'asymmetric coalition structure' exists that gives rise to a new concept of power. That is, despite the implicit equity bias, the cooperative game process drifts away from the cooperative outcome, which is the state in the core where the 'strong' player has least power. The Markovian cooperative equilibrium can be a state with superior power to the strong player, 'far' away from the most egalitarian allocation in the core.

The asymmetric coalition structure exists only for $N \geq 4$. The inherent power in the coalition structure, the 'asymmetric power', can be stronger than the inherent equity bias of the process. This asymmetric power is present in the trial and error setting, where strategies or demands are updated incrementally in the tradition of reinforcement learning and aspiration adaption. It would be interesting to conduct experiments or analyze more general processes where the state space is based on games with asymmetric coalition structures.

Recall from Section 2.3 that specific coalition structures exist to which the process returns frequently on its "journey through the core". In order to understand the concept of asymmetric power, we first present two coalition structures which exhibit symmetry of power of its members. Then we will introduce one coalition structure, where the members have asymmetric power, and where the cooperative game process can drift away from the most equal allocation in the core. We analyze the dynamics which lead to superior power of the strong player in detail, and we explain why, and how, asymmetric power can arise.

In an adaption to Harsanyi's (1962b) definitions of power we define the limit power of a player for a specific game. We split the limit power then into the sum of the player's 'core power' and 'asymmetric power'. Then we analyze three example games with respect to the inherent equity properties and their 'symmetry of power' and calculate the power to each player in the three example games.

Consider the following example game:

example game						
	$v(C_{1234})$	$v(C_{123})$	$v(C_{124})$	$v(C_{134})$	$v(C_{12})$	$v(C_{13})$
1	1	0.88	0.79	0.78	0.75	0.74

Table 2.1 $v(C)$ for all coalitions C with $v(C) \neq 0$

The state $\mathbf{d}_{s1} = (0.75, 0.02, 0.21, 0.02)$ is on the face of the core polytope generated by the coalitional constraint of C_{124}. This face is an example of a symmetric coalition structure. For sufficiently small ε, in the neighborhood of $\mathbf{d}_{s1}$, all states are from one of two 'types': states where coalition C_{124} is binding, or states where no coalition is binding. Since $\frac{v(C)}{|C|} > \frac{1}{N}$ or ($\frac{0.79}{3} > 0.25$) the random walk frequently returns to states where coalition C_{124} is binding. (In fact, from Section 2.3 we know that in this neighborhood the expected return time to states where C_{124} is binding is less than 5, as $\phi_{C_{124}} = 3\frac{1-0.79}{0.79}$ and $\mathbb{E}T \leq \frac{1}{1-\phi}$). At each state where C_{124} is binding, player three's demand is 0.21, and so while the random walk moves along the recurrent structure where coalition C_{124} is binding, player one's, two's and four's demands will change materially. Player one has much larger demands than the other players and hence will reduce demands more frequently than the other players in states where no coalition is binding. If C_{124} is binding, none of the players in C_{124} can reduce demands. So, on average, player one reduces demands, and players two and four, on average, increase demands until the random walk reaches a neighborhood where a new coalition structure is present.

In the neighborhood around the state $\mathbf{d}_{s2} = (0.71, 0.04, 0.21, 0.04)$ both coalitions C_{124} and coalition C_{12} are binding. We call the set of states where two coalitions are binding an 'edge'. There are four types of states: states where both C_{124} and C_{12} are binding (states on the edge), two types of states where exactly one of the two coalitions is binding, and states where no coalition is binding. In all states on the edge, demands of players three and four are fixed at 0.21 and 0.04 respectively. The only 'free' players within this coalition structure are players one and two. Since both 'free' players are members of C_{12} and C_{124}, the potentially binding coalitions in this local neighborhood, player one and two reduce demands proportional to their magnitude of demands while the random walk 'drifts along the edge' and so player one's demand reduces on average and player two's demand increases on average until the random walk reaches a neighborhood where a new coalition structure exists.

In the neighborhoods around $\mathbf{d}_{s1}$ and $\mathbf{d}_{s2}$, the 'free' players, are symmetric in the sense that the only difference in the transition dynamics of their demands is the magnitude of their demands. They are member of the same coalitions relevant in the neighborhood. So the equity bias inherent in the process prevails, and the cooperative game process drifts along the coalition structure "towards equity".

The new power concept is only prevalent in the neighborhood of a special coalition set.

An asymmetric coalition set satisfies the following conditions:

1. No coalition is a subset of another coalition in the set.

2. The intersection of all coalitions is non-empty,

3. The union of all coalitions is not the grand coalition.

The point is that the face generated by the asymmetric coalition set is asymmetric: Along the face, the demand sum of the players in the union of the coalitions in the asymmetric coalition set is not constant: If the player in the intersection increases his demand, for each coalition in the asymmetric coalition set one player not in the intersection needs to reduce her demand so that a new state on the asymmetric face can be reached. Thus in order to reach the next state on the asymmetric face the total demand sum in the union of the asymmetric coalition set will go down, if the player in the intersection increases his demand, and up, if the player in the intersection reduces his demand. Or alternatively, the demand of player(s) in the complement of the asymmetric coalition set varies along that face.

The player in the intersection is called the “strong” player. The only way that the inherent equity bias can be countered by a stronger dynamic, is if there exists an asymmetry in the coalition structure, making the strong player a member of more coalitions than the weaker players in the structure. So the intersection needs to be non-empty for a strong player to exist.

For all coalitions in the asymmetric coalition structure to be binding simultaneously at different states along the asymmetric structure, an additional player is needed to compensate for the asymmetry: When player one increases his demand, both weaker players decrease their demands. In order for states in this structure to be in the core (demands summing to one) there needs to be the complement player who ‘copies’ the strong players’ demand updates.

We believe that if a coalition structure does not satisfy all the three conditions for an asymmetric coalition structure, then the process moves along that structure driven by the inherent equity bias. This implies that the cooperative game process should settle in the most equal allocation for all games where no asymmetric coalition structure exists. However this is not proven in this thesis and is an interesting avenue for further research.

In the neighborhood around the next example state an asymmetric coalition structure exists, and so the asymmetric power concept is relevant here. In the neighborhood around the state $(0.55, 0.2, 0.19, 0.04)$ both coalitions C_{12} and coalition C_{13} (and no other coalitions) can be binding. Player one is the player in the intersection, the ‘strong’ player, players two and three are the ‘weak’ players that are both only member of one coalition, player four is the player in the complement of the union, and so is the complement player. Observe that a complement player is not a dummy player. Without the complement player the asymmetric power does not exist. In the neighborhood around $(0.55, 0.2, 0.19, 0.04)$

there are four types of states in the core: states where both C_{12} and C_{13} are binding (states on the 'asymmetric' edge), states where exactly one of the two coalitions is binding, and states where no coalition is binding. Furthermore there are states outside the core (where the sum of demands of players in coalition C_{13} is less than 0.74. The existence of states outside the core is not necessary for prevalence of asymmetric power, see Chapter (5.4)).

Each state on the asymmetric edge is uniquely determined by the demand of player one. The 'free' players are now players one, two and three. To reach the 'next' state on the edge, player one increases (or decreases) his demand by ε, and players two and three each decrease (or increase) their demand respectively. In order for the state to be efficient, the complement player four needs to increase (or decrease) his demand by ε as well. The important point is that player one is member of both coalitions C_{12} and C_{13} whereas both players two and three are only member of one of the coalitions each. The complement player four is not member of any coalition. The coalition structure is 'asymmetric': Player one can only reduce his demand in states where no coalition is binding, whereas players two and three can reduce their demand in other states. Between returns to the asymmetric edge there are two 'opposing forces' impacting the drift of the random walk:

1. Inherent equity bias:

   Player one has higher demands than players two and three and hence reduces demands more frequently in states where no coalition is binding.

2. power through asymmetric coalition structure (asymmetric power):

   Player one is more often member of a binding (or strictly feasible) coalition than players two and three. So he will not reduce demands on the set of states where at least one coalition is binding (or strictly feasible), and so he will hence reduce demands less frequently than players two and three.

Will the process drift 'up' or 'down' the asymmetric edge? This depends on which of the two effects described above is stronger: The equity bias or the asymmetric power of the coalition structure. To determine whether the process drifts up or down the asymmetric edge requires calculating the probability (or bounds on the probability) of the process being in the different kind of states, and it requires to calculate the drift (or bounds on the drift) for each of these states on a trajectory of the process between two states on the asymmetric edge. Before we give a summary of the techniques to calculate the drift along the asymmetric edge, we define power for the cooperative game process.

Harsanyi (1962b) defines the amount of power of a player with respect to a preferred strategy as the probability with which the player can “enforce” that strategy. In the cooperative game process players cannot “enforce” strategies. Harsanyi’s definition of a vector measure of the amount of power ranks a players’ strategies according to preference. The amount of power for each strategy is then given by the vector of probabilities with which the player can enforce the respective strategies.

In similar fashion, states with non zero equilibrium measure for the cooperative game process are ranked by the demand to a player, and then the vector of the amount of power is the respective equilibrium distribution of each state sorted by preference (magnitude of demand).

However the state space for the cooperative game process can be large, the smaller ε is taken, the larger the state space is. So we define power to a player in the cooperative game process as the limit power when ε tends to zero, giving measure one to the Markovian cooperative outcome and zero to all other states. The limit strength of power for a cooperative game process is then the sum of the following two components:

- The *core power* of a player in a cooperative game process is the difference between a player’s demand at the cooperative outcome, the state i the core with closed distance to equal split, and at equal split. This power is adapted from Harsanyi’s (1962b) or Selten’s concept of power as the ‘capability to secure more than the equitable share’. The ‘force’ generating the core power to a player is the constraints of the core.

- The *asymmetric power* of a player in a cooperative game process is the difference between a player’s demand at the Markovian cooperative equilibrium and at the cooperative outcome. The asymmetric power is the ‘capability to secure more than at the most equitable core allocation’. The ‘force’ generating the asymmetric power are the dynamics inherent in the asymmetric coalition structure. We believe this concept of power to be very interesting and novel to the study of (cooperative) game theory and stochastic learning processes.

The *limit power* or (simply) power to a player in a cooperative game process is the sum of the core power and the asymmetric power.

We discuss three specific example games. These games will all have an asymmetric edge and we give a heuristic analysis backed up by simulations to show, that each game has a Markovian cooperative equilibrium on the asymmetric edge but they appear on different parts of the edge. We will analyze for each example game for each player the core power and the asymmetric power of the players.

We now specify the three example games and describe their geometric structure. The first example game has already been introduced as example before, we repeat it here for ease of comparison with the other games.

example game						
	$v(C_{1234})$	$v(C_{123})$	$v(C_{124})$	$v(C_{134})$	$v(C_{12})$	$v(C_{13})$
1	1	0.88	0.79	0.78	0.75	0.74
2	1	0.88	0.86	0.85	0.75	0.74
3	1	0.95	0.79	0.78	0.75	0.74

Table 2.2 $v(C)$ for all coalitions C with $v(C) \neq 0$ for example games 1, 2 and 3

In each case $\{C_{12}, C_{13}\}$ is an asymmetric coalition structure. Concession limits are the extreme outcomes, between which the outcome must fall in bargaining between rational players, Harsanyi (1962a, 1962b). Hence the two extreme states on the asymmetric edge where player one has largest demand and where player one has smallest demand are named the ‘upper concession limit’ and the ‘lower concession limit’ respectively. The Markovian cooperative equilibrium must lie between (inclusive) the upper and lower concession limits. In all three example games the cooperative outcome corresponds to the lower concession limit.

1. For game one the upper concession limit is $(0.61, 0.14, 0.13, 0.12)$, the lower concession limit is $(0.53, 0.22, 0.21, 0.04)$.

   In the neighborhood around the lower concession limit, there is a strong positive drift in the demand of player one based on the probability distribution “in equilibrium” over the 2-dimensional localized chain. The random walk cooperative game process drifts up the asymmetric edge.

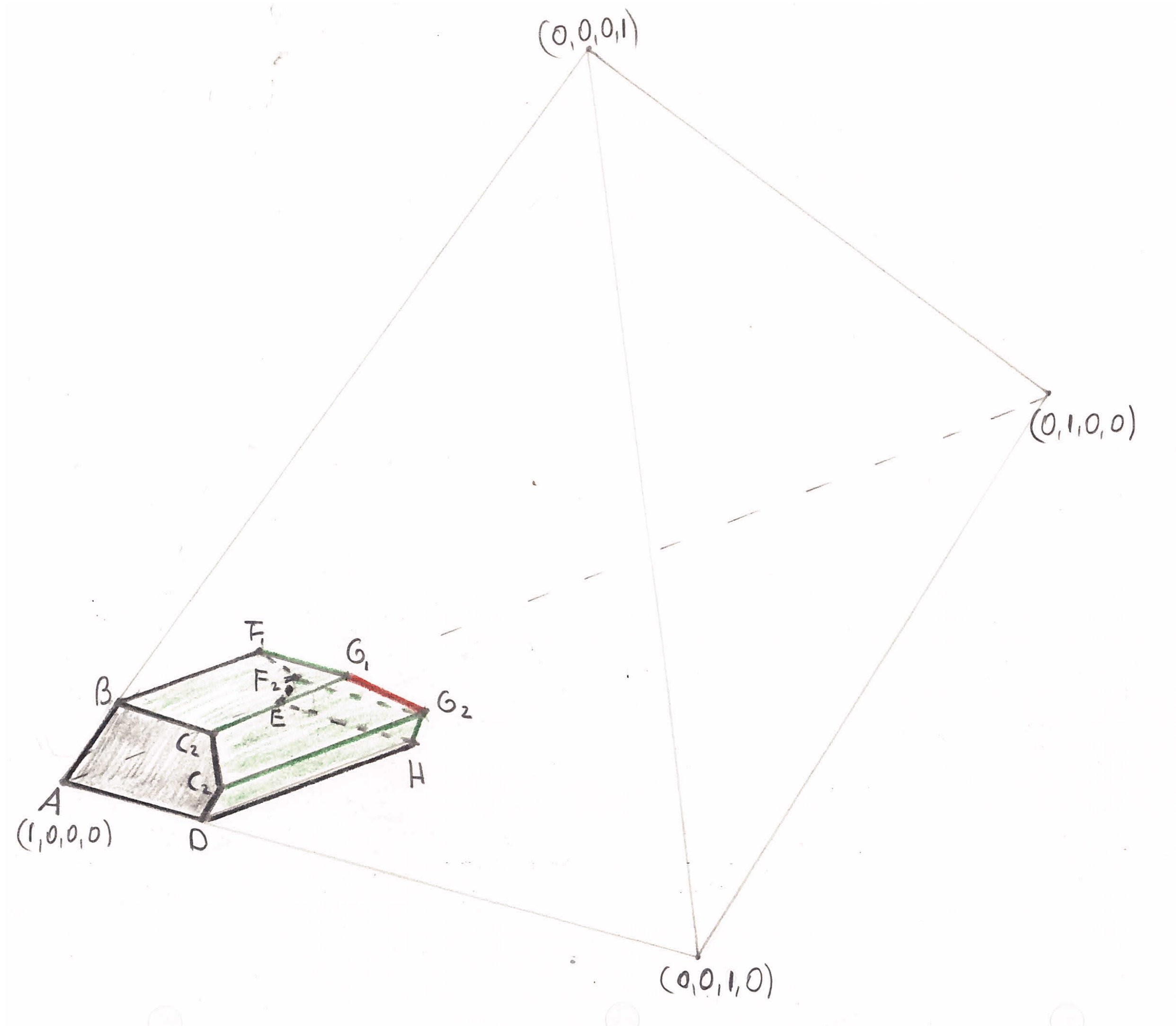


Figure 2.1 Graphical representation of the core for example game 1. The asymmetric edge is sketched in red with G_1 and G_2 representing the upper and lower concession limits respectively.

In the neighborhood around the upper concession limit, there is a strong negative drift in the demand of player one based on the probability distribution "in equilibrium" over the 2-dimensional localized chain. The random walk cooperative game process drifts "down" the asymmetric edge. For game one the Markovian cooperative equilibrium is the state on the asymmetric edge where the drift in d^1 is zero. The drift in d^1 for the restricted localized chain is zero at

$(0.583586, 0.166414, 0.156414, 0.093586)$ and so the Markovian cooperative equilibrium is in the "interior" of the asymmetric edge.

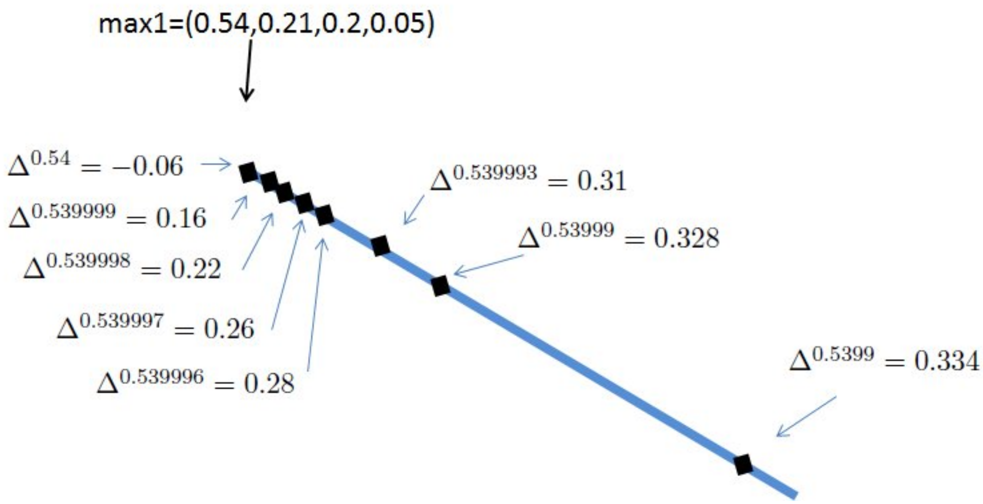


Figure 2.2 Sketch of the asymmetric edge of example game 3 with simulated values of the drift along the edge for different starting states in the immediate neighborhood of the upper concession limit (max1). The process drifts "up" the whole asymmetric edge and so the Markovian cooperative equilibrium equals the upper concession limit in this game.

The vector of the core power of the players is given by the difference between the cooperative outcome and equal split and equals $(0.28, -0.03, -0.04, -0.21)$. The vector of the asymmetric power is given by the difference between the Markovian cooperative outcome and the cooperative outcome and is given approximately by $(0.054, -0.054, -0.054, 0.054)$. The total power to each player is then given by the sum of the core power and the asymmetric power and equals

$(0.334, -0.084, -0.094, -0.156)$. Observe that the complement player four has the same asymmetric power as player one. He basically gets a "free power ride" from player one. A natural bound to the strong player's asymmetric power is the amount of free ride that the complement player can get: the complement player can never have more total power than the weak players.

2. For game two the upper concession limit is $(0.61, 0.14, 0.13, 0.12)$, the lower concession limit is $(0.6, 0.15, 0.14, 0.11)$. There is a strong negative drift at each state of the asymmetric edge (apart from states in close proximity to the lower concession limit) and so the Markovian cooperative equilibrium is the lower concession limit (the cooperative outcome). The core power of each player is given by

$(0.35, -0.1, -0.11, -0.14)$. Since the Markovian cooperative equilibrium is the cooperative outcome the asymmetric power is zero for all players.

3. For game three the upper concession limit is given by $(0.54, 0.21, 0.2, 0.05)$, the lower concession limit is given by $(0.53, 0.22, 0.21, 0.04)$ There is a strong positive drift at each state of the asymmetric edge (apart from states in close proximity of the upper concession limit) and so the Markovian cooperative equilibrium is the upper concession limit.

   The core power of each player is given by $(0.29, -0.04, -0.05, -0.2)$. The asymmetric power is given by $(0.01, -0.01, -0.01, 0.01)$ and so the total power to each player is given by $(0.3, -0.05, -0.06, -0.19)$.

   To understand the behavior of the cooperative game process in the neighborhood of the upper concession limit, we simulated the chain for example game 3 and calculated the drift for states in the close neighborhood of the upper concession limit. In figure (2.2) the drifts are sketched for different states where $\varepsilon = 0.000001$. We see that the drift becomes smaller but stays positive even very close to the upper concession limit.

To calculate the drift of the process along the asymmetric coalition structure, we apply a localization technique (or trick). Demands on an excursion between two states on the asymmetric edge are assumed to be constant. If ε is sufficiently small this is a reasonable assumption. The below table shows an excursion of the random walk between two states on the asymmetric edge, for each state on the excursion it shows which coalitions are binding, and it depicts the "localized" demands used in the calculation of transition probabilities.

time index	*state*	*binding coalitions*	*localized demands*	*state of localized chain*
t_0	$(0.55, 0.2, 0.19, 0.04)$	C_{12}, C_{13}	$(0.55, 0.2, 0.19, 0.04)$	$(0,0)$
t_1	$(0.55+\varepsilon, 0.2, 0.19-\varepsilon, 0.04)$	C_{13}	$(0.55, 0.2, 0.19, 0.04)$	$(1,0)$
t_2	$(0.55+2\varepsilon, 0.2-\varepsilon, 0.19-\varepsilon, 0.04)$	$\emptyset$	$(0.55, 0.2, 0.19, 0.04)$	$(1,1)$
t_3	$(0.55+\varepsilon, 0.2-\varepsilon, 0.19-\varepsilon, 0.04+\varepsilon)$	C_{12}, C_{13}		$(0,0)$

Between successive returns to the asymmetric edge, the process visits different kinds of states. At t_0, the random walk starts on the asymmetric edge, and in subsequent steps, it leaves the asymmetric edge. At t_1, player one has increased demands, and player three has decreased demands, now only coalition C_{13} is binding. At t_2, player one has increased demands again, and player two reduced demands, now no coalition is binding. At t_3 player four has increased demands and player one decreased demands and the random walk is back on the asymmetric edge.

On these excursions the random walk may leave the core. At states on the asymmetric edge, and states not in the core, the process has a positive drift in d^1 over one step of the random walk process. We show in Lemma (5.8), that at all states not on the asymmetric edge, the drift in d^1 is negative.

We introduce the localized 2-dimensional chain, the joint chain of the coalition's surplus for C_{12} and for C_{13}. For example at t_0 the localized 2-dimensional chain is in the state $(0,0)$, this corresponds to the state on the asymmetric edge. At t_1, where only coalition C_{13} is binding, the equivalent state in the localized chain is $(1,0)$, at t_2, where no coalition is binding, the equivalent state of the localized chain is $(1,1)$ as at $(0.55+2\varepsilon, 0.2-\varepsilon, 0.19-\varepsilon, 0.04)$ the coalition's surplus for C_{12} is ε, and the coalition's surplus of C_{13} is ε. At t_3, the 4-dimensional cooperative game process is back in a new state on the asymmetric edge, and the 2-dimensional local chain is back in the state $(0,0)$. As a further example, a state where the coalition's surplus of C_{12} is 3ε and the coalition's surplus of C_{13} is 2ε corresponds to state $(3,2)$ in the localized 2-dimensional chain.

To understand what the overall drift is over an excursion from the asymmetric edge to another state on the asymmetric edge, we approximate the probability, "in equilibrium", that the chain lies on the asymmetric edge or outside the core. In this thesis two techniques are introduced:

- The first technique restricts the state space of the localized 2-dimensional chain to some small fixed number of steps away from the state $(0,0)$ for both coordinates. This leads to a (small) finite Markov chain, that is solved with traditional Markov chain techniques using the 'localized' demands and drifts. A sketch of an illustrative example of a localized chain restricted to move at most two steps away from $(0,0)$ is given in figure (2.3). For actual calculations much larger restricted chains were used to approximate the drift at different states on the asymmetric edge. Simulations confirm that these are accurate approximations of drift along the asymmetric edge. The coalition's surplus of C_{12} can never be negative and hence the first coordinate of the localized chain is non-negative. In figure (2.3) red and orange states

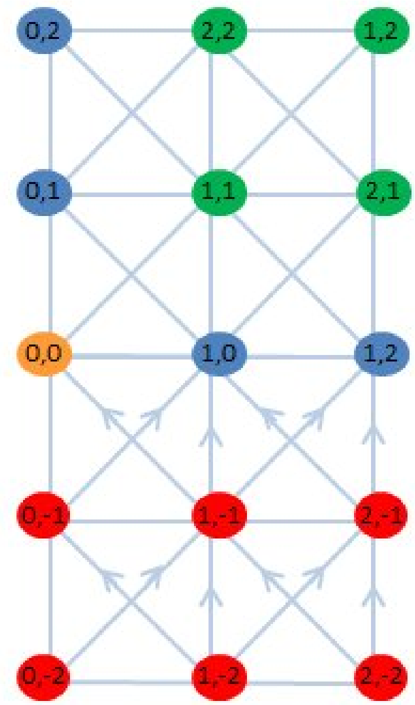


Figure 2.3 Sketch of restricted localized 2-dimensional chain associated with asymmetric coalition structure $\{C_{12}, C_{13}\}$

correspond to states with strong positive drift in d^1. Green states correspond to states with strong negative drift in d^1.

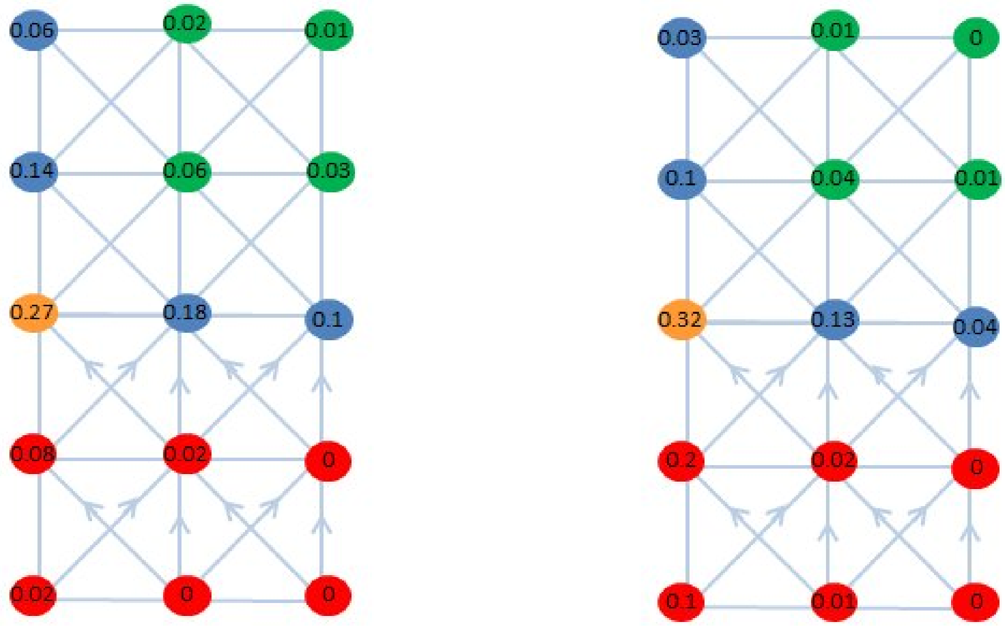
In figure (2.4) two localized 2-dimensional chains with equilibrium distribution at each state are sketched, the difference between them is the "location" on the asymmetric edge: the first chain "starts" in the state on the asymmetric edge where the demand of player one is 0.6 ($(0.6, 0.15, 0.14, 0.11)$); the second chain "starts" in the state on the asymmetric edge where the demand of player one is 0.54

($(0.54, 0.21, 0.2, 0.05)$). The equilibrium distribution of the localized restricted chain on states with strong drift in d^1 (orange and red states) is higher for the chain where the demand of player one is lower (0.54). There is a positive drift in d^1 ("up" the asymmetric edge) at state $(0.54, 0.21, 0.2, 0.05)$. The equilibrium distribution on states with strong drift in d^1 is lower for the chain where the demand of player one is higher (0.6). There is a negative drift in d^1 ("down" the asymmetric edge) at state $(0.6, 0.15, 0.14, 0.11)$.

- The second technique restricts the process to stay in the core. Then there are only four types of states for which the "in equilibrium" probability is calculated, states where both C_{12} and C_{13} are binding, then two sets of states where exactly one of them is binding, and states where none of them is binding. For a specific example game, we define “pessimistic” random walks, that always have a higher probability of increasing by one and a lower probability of decreasing by one than the surpluses of coalitions C_{12} and C_{13}. The point is here, that it is very difficult to exactly analyze the dependence of the coalition's surplus for C_{12} and C_{13}. However, if the pessimistic random walks associated with C_{12} and C_{13} are in the zero states with high enough probability so that the sum is greater than one, we know a minimum probability (lower bound) that the localized 2-dimensional chain must be in the $(0,0)$-state - and with which the cooperative game process must be in a state on the asymmetric edge.

The positive drift in d^1 is strong on the asymmetric edge (d^1 cannot decrease on the edge) and it turns out that this lower bound on the probability of being on the asymmetric edge is high enough to prove a positive drift in d^1 all along the asymmetric edge for a specific example game defined in Chapter 5.4.

Since the demand of player one uniquely defines a state on the asymmetric edge, the Markovian cooperative equilibrium is the state on the asymmetric edge where the expected change in the demand of player one between two successive returns to the asymmetric edge is zero.



$\pi_{\mathbf{2}}^{0.6}$ $\pi_{\mathbf{2}}^{0.54}$

Figure 2.4 Sketch of the equilibrium distributions of two versions of the localized 2-dimensional restricted chain "started" at states on the asymmetric edge corresponding to a demand for player one of 0.6 and 0.54 respectively.

2.6 Speed of Convergence Analysis via Coupling

The main methodological contribution of the thesis is to show how to use coupling to demonstrate speed of convergence for stochastic learning processes. In several branches of probability theory the coupling technique has been one of the methods of choice to analyze the speed of convergence of Markov chains. Levin, Peres and Wilmer (2009) is a good introduction to different applications of couplings with the main focus on analyzing the speed of convergence to equilibrium of Markov chains.

Young (1998) applies a coupling to the analysis of symmetric 2-person coordination games. Liggett (1985) applies couplings to particle systems in the context of statistical physics, biology and economics. However, we believe this technique has not yet become a general tool in the study of stochastic learning processes. Recently speed of convergence analysis has become a topic of general interest for stochastic learning processes but, somewhat strangely, the coupling tool seems not to be one tool of choice for the analysis of speed of convergence of stochastic learning processes. Hence we present our analysis in general form.

Coupling is a powerful and elegant tool with which one is often able to calculate tight bounds on the mixing time and to reduce the complexity or length of calculations compared to other techniques. There are general principles in coupling which we summarize hereafter. However finding a good coupling is more like an art and a good coupling is often specific to the inherent dynamics of the process.

The main purpose of Coupling is to find bounds on the mixing time of stochastic processes. There are a variety of particular cases for which the Coupling method can be used.

- general purpose of the Coupling method:

  Finding bounds on the speed of convergence of stochastic processes

- particular applications of the Coupling method:

  - proving existence of stationary measure,
  - bounding return times or return probabilities,
  - proving limit theorems,
  - deriving inequalities,
  - obtaining approximations...

To construct a coupling one creates a joint distribution of two Markov chains that preserves the marginal distribution of each chain while moving the two chains 'closer to each other' wherever possible. If the two chains are identical chains on the same probability space then once they have coalesced one can move the chains in sync ever after. A famous theorem by Aldous (see, e.g., Levin, Peres and Wilmer (2009), Chapter 4.2) states that one can bound the probabilistic distance between two measures by the probability that any coupling has not coalesced. Hence finding a good coupling that coalesces fast with high probability finds a sharp bound on speed of convergence for a Markov chain or stochastic learning process.

There always exists the independent coupling, where both random variables are independent of each other. However, the aim of a coupling is to find a joint distribution that "forces" the two random variables to move towards each other as often as possible whilst preserving the marginal distribution of each.

To analyze the speed of convergence of Markov chains to the equilibrium distribution one usually compares either two distributions, say μ and ν, that are "opposite" or far away from each other. (ν can be taken as the equilibrium distribution). One then calculates after how many steps the distributions are "close" to each other.

It follows from Aldous theorem, mentioned above, that one can couple two versions of a Markov chain started at "opposite locations" of the state space. If one can show that after t time steps the probability that the two have not met is very low, one knows that the distance between the two distributions after t steps is very small.

In Chapter 6 we define a coupling on the cooperative game process and use it to show that the cooperative game process converges fast to equilibrium. In particular, we show for the N-player bargaining set-up that the chain (V_t) started from any efficient state is close to equal split after at most about $C\varepsilon^N$ steps, for a suitable constant C. For the three-player cooperative game process, as introduced in Chapter 4, we show that the chain (V_t) started from any state in the core is close to the cooperative outcome after at most about $C\varepsilon^N$ steps.

One needs a measure of distance between the two random variables in the coupling in order to determine if they have coalesced. For the cooperative game process we will use the L^1 distance. One trick is to define (find) a coupling such that the distance between the two random variables in the coupling will decrease in expectation at each time step. In the case of the N-player bargaining set-up we define the joint movement of two chains sitting anywhere on the set of efficient states in such a way that the probability of them moving further apart is zero and the probability of the distance between the two random variables to decrease by 2ε is proportional to the L^1 -distance between the coupled versions of the cooperative game process. The instantaneous speed of convergence is faster when the two chains are further apart.

We define the joint movement by picking the same player in both chains to increase demands. Then the players that have smaller demands than their corresponding player in the other chain will mimic the other chains player whenever he reduces demands. If the same player in both chains reduces demands, the distance does not change. If two different players reduce demands, the player that reduces demands in either chain has higher demands than the respective player in the other chain. So the distance decreases by 2ε. By this coupling we show that the probability that two chains started anywhere on the state space will have coalesced, increases rapidly with time. The cooperative game process for the N-player bargaining game converges to equilibrium rapidly.

In Theorem (6.3) we show that the coupling based on the cooperative game process for the three player game decreases in expectation by a constant over each time step.

In the three player cooperative game setting, we apply the same coupling rule as in the N-player bargaining set-up: the same player increases his demand in both chains and the same player decreases demands in both chains whenever possible. It is not true anymore that the chains have zero probability to move further apart since in the three player set-up the probability of demand reductions depends on the coalition structure at a state: it is not necessarily true anymore that the player with higher demands has a higher probability of reducing demands. The distance between the two chains forming the coupling reduces by 2ε if both players reducing demands have larger demands than the equivalent players in the other chain. The distance increases by 2ε if the players reducing demands both have smaller demands than the equivalent players in the other chains, and the distance stays zero otherwise. The proof shows case by case that, at each possible joint state, where the same player has increased demands in both chains, the difference in probability between decreases and increases of 2ε is positive and proportional to the L^1-distance between the states. So in expectation the distance reduces fast and we can conclude rapid mixing of the cooperative game process for three player games.

2.7 Comparison with Other Learning Processes

The process studied in this thesis is a fully dynamic model for learning in cooperative games, suited to be used to model situations with many players. Players base their incremental demand adaptations solely on how well they did in the past. They do not have a strategic model. Information about others is limited and players do not need to perform mathematical calculations, or follow complex strategies dependent on other players' behavior, in order to update their demand.

There are three major elements to the incremental demand updates of the process:

1. Players occasionally experiment to discover whether alternative actions could lead to higher payoffs.

2. Upward and downward adjustments are made locally and in small increments.

3. The higher the loss that a player experiences during cooperative failure, the more likely this player is to reduce his demand.

In the long run, core outcomes with high levels of equity are favored as long as no asymmetric face forms part of the core. Along the asymmetric face, there is an interplay between the asymmetric power and the equity bias of the process.

Estes (1950) introduces a reinforcement learning model to describe a situation where an organism has initial probabilities of various responses. A model based on Estes' theory predicts changes in these probabilities as a function of changes in the independent variables. Estes defined learning mathematically as the "transfer of probability between certain response classes". By increasing and decreasing their demands, players in our model change the probabilities of the dependent variables, the demands.

In Sauermann and Selten's (1962) model of aspiration adaption, players adapt aspiration levels incrementally in order to achieve success with respect to some goal variables. The incremental updates of demands in response to the payoff received in our model is in tradition of their aspiration adaption model.

Bush and Mosteller (1955) develop a model where each reinforcement increments the response probability by a constant fraction of the difference between the current probability and the maximum probability. Non-reinforcement reduces the response probability by a fraction of the difference between the current response probability and the minimum probability. This model is a precursor to completely uncoupled learning rules. A learning procedure is completely uncoupled if the behavior of a player depends only on the players own history. When players do not understand the whole behavioral interplay, they do not need to play a best response to other players' actions. Each player only observes how well he did in the past before taking a decision. When there is a large number of agents, and there is no common knowledge in repeated complex situations, modeling behavior as an uncoupled process makes a lot of sense.

The completely uncoupled demand updating rule is the crucial feature of the dynamic learning model introduced by Nax (2010) and modified in this thesis. Since the response is uncoupled, no noncooperative structure or players' strategies need to be specified. Interactions can be modeled dynamically by adaptations of demands. This simple set-up is ideal for many cooperative situations, including those with a large number of players, or situations where each player is not a single individual but a set of many individuals sharing the same constraints (e.g. members of classes, types of workers or voters, etc.)

The regret testing models of Foster and Young (2003), and Germano and Lugosi (2004) are completely uncoupled, however a much higher sophistication of players is assumed. The main convergence proof in Germano and Lugosi's variant of regret testing uses Doeblin's condition which assumes that one can change from any strategy to any other strategy in the state space in one adjustment of the mixed strategy. So changes in behavior are not incremental. In a sense the regret testing proof depends on the fact that the procedure is a random trial and error search over all possible strategies. The regret testing method guarantees convergence to just one Nash equilibrium, where the limiting equilibrium may depend on the actual random realization of the sequence of plays. Although Germano and Lugosi show almost sure convergence, which is a stronger result than the convergence that we obtain in this thesis, the proof structure with "annealing regret testing" feels somewhat constructed. The cooperative game process was constructed as adaptive incremental demand (or aspiration) update process with an inherent equity bias. The nature of the process has not been amended in order to be able to prove results.

The incremental demand update rule is a major difference to adaptive play, where a player can choose any other strategy from the space of strategies. The adaptive play processes from Young (1993), Agastya (1999) or Newton (2010) can "jump" far away in one step. Furthermore only one person is chosen to update demands each round whereas in adaptive play one player from each group revises strategies or demands. A similarity to Newton's model of stochastic stability is that more egalitarian conventions may be more stable. The easier ways to leave a convention involve richer players responding to the errors of poorer players.

The Bayesian updating model of Kalai and Lehrer (1993) needs perfect recall in all payoff matrices for the complete history of the game. This results from the reliance on Kuhn's Theorem which allows to replace a probability distribution over many strategies (mixed strategy) by a single behavior strategy. Bayesian updating or rational learning requires intellect and memory of the players and it is rather complicated. This stands in stark contrast to the completely uncoupled game model of the original process introduced by Nax where only a minimum requirement on the knowledge and sophistication of players is made. Another difference of the Kalai Lehrer approach is the dependence of the paths (and results) of play on the assumed prior distributions of players. The cooperative game process (on superadditive balanced games) is an ergodic Markov chain with a single irreducible class and hence converges to a unique equilibrium distribution. It is independent of any starting state (at least for all models with a unique connected set of recurrent states studied in this thesis). Experimenting in Kalai Lehrer is different as well: A player decides rationally when to experiment whereas in adaptive play and the cooperative game process experimenting is a random event in the tradition of mutations in evolutionary biology.

Chapter 3

The Cooperative Game Process for Simple N-Player Games

3.1 Definition of the Cooperative Game Process

The set-up is that of an N-person cooperative game. Subsets of $\{1,2,...,N\}$ are called *coalitions*. Let C^G be the grand coalition of all players $1,..,N$, $C^G = \{1,...,N\}$. Let $\mathcal{P}\left(C^G\right)$ be the power set of C^G. Each coalition $C \in \mathcal{P}\left(C^G\right)$ has a *worth* $v(C)$, where v is a function from $\mathcal{P}\left(C^G\right)$ to $[0,1]$ and $v\left(C^G\right) = 1$. The pair (v,N) is called a *game*.

Let $\varepsilon = \frac{1}{M}$ for some $M > N, M \in \mathbb{N}$. A *demand vector* $\mathbf{d}$ is an N-tuple $\left(d^1, d^2, ..., d^N\right)$, where each d^i is a multiple of ε between 0 and $1+\varepsilon$ inclusive.

We define $\Omega_\varepsilon = \{\mathbf{d} \mid d^i \in \{0, \varepsilon, ..., 1+\varepsilon\}, \forall i \in \{1,2,...,N\}\}$, the set of demand vectors, and Ω^E to be the set of demand vectors $\mathbf{d}$ such that $\sum_i d^i = 1$. An *efficient state* is a demand vector in Ω^E.

The *core* $\Omega^C = \Omega^C(v,N)$ is the set of efficient states $\mathbf{d} \in \Omega^E$ such that $\forall C \in \mathcal{P}\left(C^G\right)$,

$$\sum_{j \in C} d^j \geq v(C). \tag{3.1}$$

Let $\mu = \frac{\mathbf{1}}{\mathbf{N}} = \left(\frac{1}{N}, ..., \frac{1}{N}, \frac{1}{N}\right)$. The *cooperative outcome* $\mathbf{co} = \mathbf{co}(v,N)$ is the vector in the core that has the smallest $\mathbf{L}^2$ distance from μ, i.e. the 'most equal' allocation in the core. If the core is empty, $\mathbf{co}$ is not defined.

Lemma 3.1. *If the core is non empty, the cooperative outcome is the state* $\mathbf{d}$ *in the core minimizing* $\sum_{i\in\{1,\dots,N\}}\left(d^i\right)^2$.

Proof: By assumption, the cooperative outcome minimizes $\sum_{i\in\{1,\dots,N\}}\left(d^i-\frac{1}{N}\right)^2$. Multiplying out yields $\sum_{i\in\{1,\dots,N\}}\left(d^i-\frac{1}{N}\right)^2=\sum_{i\in\{1,\dots,N\}}\left(d^i\right)^2-\frac{2}{N}\sum_{i\in\{1,\dots,N\}}\left(d^i\right)+\frac{1}{N}$. Since $\mathbf{co}$ is in the core, $\sum_{i\in\{1,\dots,N\}}d^i=1$. So $\sum_{i\in\{1,\dots,N\}}\left(d^i-\frac{1}{N}\right)^2=\sum_{i\in\{1,\dots,N\}}\left(d^i\right)^2+\frac{1}{N}-\frac{2}{N}$.

So for states in the core $\sum_{i\in\{1,\dots,N\}}\left(d^i-\frac{1}{N}\right)^2=\sum_{i\in\{1,\dots,N\}}\left(d^i\right)^2-\frac{1}{N}$ and $\mathbf{co}$ is the most equal allocation. □

We assume *superadditivity* of the worth function v, that is, that

$$v\left(C^1\cup C^2\right)\geq v\left(C^1\right)+v\left(C^2\right) \tag{3.2}$$

whenever $C^1,C^2\subseteq C^G$ and $C^1\cap C^2=\emptyset$.

A *superadditive game* is a game (v,N) where v is superadditive. Observe that superadditivity of the worth function implies *monotonicity* of the worth function, that is, if $C^1\subset C^2$, then

$$v\left(C^1\right)\leq v\left(C^2\right) \tag{3.3}$$

since $v\left(C^2\right)\geq v\left(C^1\right)+v\left(C^2\setminus C^1\right)\geq v\left(C^1\right)$.

In this thesis we study a random process, whereby each player iteratively makes a demand and nature generates payoffs depending on the set of demands. Players adjust their demand according to a simple rule in response to the distribution of payoffs in the previous round. The aim is to show that, with some conditions, in the long run the vector of demands is usually close to $\mathbf{co}$.

Let the demand of player i at discrete time $n\in\mathbb{N}$ be given by d^i_n. Each player iteratively makes a demand and nature generates payoffs depending on the set of demands. The process starts from any efficient state. Each period, each agent submits a demand. Then nature distributes payoffs according to the following rule:

The *payable coalition* is the coalition C for which the cumulative sum of demands is the largest one whilst not exceeding the coalitional worth $v(C)$. If there is more than one coalition achieving this maximum, preference is first by number of players in that coalition. If two coalitions have the same sum of demands and number of players then the coalition is chosen which includes the player with smallest index number not included in the other coalition. Players in the payable coalition receive their demand as payoff; players outside that coalition receive 0. If, for all coalitions C in $\mathcal{P}\left(C^G\right)$, the sum of demands of the players in that coalition exceeds the coalitional worths, the payable coalition is the empty set. The payable coalition can be thought of as the chosen allocation by nature or an administrative body. If players are not chosen for the payable coalition their only possible action when chosen to update demands is to reduce demands. They have an incentive to do so as they want to increase their chances of being in the payable coalition.

For the next period, a player is selected uniformly at random, and this player then changes his demand according to the following rule: If the demands sum to at most 1, he will increase his demand by ε. Otherwise, if the player is in the payable coalition he will not change his demand. If neither of the previous two conditions hold the player will decrease his demand by ε with probability proportional to the size of his demand and else will not change his demand.

The point is that the learning rule is very simple and can be applied even if the players know only their own payoff and whether or not the sum of demands is 1. They know nothing about the history of the game before the previous step. Yet in some games the vector of demands, in the long run, is close to at least one notion of optimality for the game.

For distinct $i, j...k \in \{1,2,...,N\}$, let $C_{ij...k}$ be the coalition of players i,j,...,k.

For $\mathbf{d} \in \Omega_\varepsilon$, a coalition C is *feasible* if $\sum_{j\in C} d^j \leq v(C)$ and *strictly feasible* if $\sum_{j\in C} d^j < v(C)$. A coalition C is *binding* at $\mathbf{d} \in \Omega_\varepsilon$ if $\sum_{j\in C} d^j = v(C)$. For $\mathbf{d} \in \Omega_\varepsilon$, let $C' = C'(\mathbf{d})$ be the feasible coalition with the maximum sum of demands, i.e. the *payable coalition*. If no coalition is feasible, then $C'(\mathbf{d}) = \emptyset$.

Given an efficient state $\mathbf{d} = \left(d^1, d^2, ..., d^N\right)$, let $\mathbf{d}(i,j)$ be the efficient state where $d^i(i,j) = d^i + \varepsilon, d^j(i,j) = d^j - \varepsilon$ and $d^k(i,j) = d^k$ for $k \neq i,j$. We call $\mathbf{d}(i,j)$ a *neighbor* of $\mathbf{d}$.

We define the Markov chain $(X_n)_{n=0}^{\infty} = (\mathbf{d}_n)_{n=0}^{\infty} = \left(d_n^1, d_n^2, ..., d_n^N\right)_{n=0}^{\infty}$ in discrete time n, on the finite state space Ω_ε.

Let $(n_t)_{t=0}^{\infty}$ be the sequence of times n when X_n is in Ω^E. Set $V_t = X_{n_t}$, for $t = 0,1...$. One transition of (V_t) from $\mathbf{d} \in \Omega^E$ involves two changes of state of the chain (X_n). One player i increases his demand, and then the chain (X_n) is in an intermediate inefficient state $\mathbf{d}(i)$, where $d(i)^j = d^j$ if $j \neq i$ and $d(i)^i = d^i + \varepsilon$ and where $\sum_{j=1}^{N} d(i)^j = 1 + \varepsilon$; then one player j (with possibly $j = i$) decreases demands so that the chain (V_t) is in state $\mathbf{d}(i,j) \in \Omega^E$.

We define $p_{\mathbf{d}}^{i,j}$ to be the probability to go from an efficient state $\mathbf{d}$ to $\mathbf{d}(i,j)$ over one step of (V_t). The transition probability $p_{\mathbf{d}}^{i,j}$ depends only on the demands of the players that are not in the payable coalition at $\mathbf{d}(i)$. So $p^{i,j} = \frac{1}{N}\frac{d^j}{\sum_{l \notin C'(\mathbf{d}(i))} d^l + \varepsilon}\mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}}$ if $i \neq j$ and $p^{i,j} = \frac{1}{N}\frac{d^j + \varepsilon}{\sum_{l \notin C'(\mathbf{d}(i))} d^l + \varepsilon}\mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}}$ if $i = j$.

We work with the chain (V_t), as its chain dynamics are more tractable.

The chain $(X_n)_{n=0}^{\infty} = (\mathbf{d}_n)_{n=0}^{\infty} = \left(d_n^1, d_n^2, \ldots, d_n^N\right)_{n=0}^{\infty}$ is introduced for the first time in this thesis. It is however motivated, and closely related, to a Markov chain introduced by Nax (2010) called the original process in this thesis. The definition of the original process is given in Section 3.2.

The incremental demand update ε is in general assumed to be very small. The next definitions ensure that the worth function of the cooperative game is consistent with the ε-grid restricting the possible values for the worth of a coalition to multiples of ε.

Let (v,N) be a superadditive game. Then $M \in \mathbb{N}$ is v-compatible if $Mv(C)$ is an integer for all C in $\mathcal{P}(N)$. We call the Markov chain (V_t) a *cooperative game process* (v,N,ε) if $\varepsilon = \frac{1}{M}$ for some v-compatible M.

For a coalition C, let the *coalition's surplus* $CS^C(\mathbf{d})$ of C be $v(C) - \sum_{i \in C} d^i$. So a coalition is feasible at $\mathbf{d}$ if $CS^C(\mathbf{d}) \geq 0$, and binding at $\mathbf{d}$ if $CS^C(\mathbf{d}) = 0$. A state $\mathbf{d}$ is in the core if $CS^C(\mathbf{d}) \leq 0$ for all $C \in \mathcal{P}\left(C^G\right)$.

Given a game (v,N), for $1 < s \leq N$, let the *s-simplified game* (v_s,N) be the game where $v_s(C) = 0$ if $|C| < s$ and $v_s(C) = v(C)$ otherwise.

Given a game (v,N), let the *s-core* $\Omega^s(v,N)$ and the *s-cooperative outcome* $\mathbf{co}^s(v,N)$ for $s \leq N-1$ be the core $\Omega(v_s,N)$ and the cooperative outcome $\mathbf{co}(v_s,N)$ respectively for the *s-simplified game* (v_s,N).

Lemma 3.2. *Given a game (v,N), for $1 \leq s_1 < s_2 < N$ the s_1-core is a subset of the s_2-core. Therefore the core $\Omega(v,N)$ is the 1-core $\Omega^1(v,N)$ and $\mathbf{co} = \mathbf{co}^1$.*

Given a coalition C^* let $\Omega^{C^*} = \{\mathbf{d} \mid \sum_{i \in C^*} d^i \geq v(C^*)\}$.

Proof: The $(N-1)$-core Ω^{N-1} is the intersection of all Ω^{C^*} for $|C^*| = N-1$.

The $(N-s)$-core is the intersection of all Ω^s for $s \geq N-s$. So the s_1-core is a subset of the s_2-core for $1 \leq s_1 < s_2 < N$. It follows that the core is the 1-core and so $\mathbf{co} = \mathbf{co}^1$. □

Lemma 3.3. *Given a game (v,N), if $\mathbf{co}^{s_2} \in \Omega^{s_1}$ with $s_1 < s_2$, then $\mathbf{co}^{s_1} = \mathbf{co}^{s_2}$. So, if $\mathbf{co}^s \in \Omega^C$ for some $1 \leq s < N$, then $\mathbf{co} = \mathbf{co}^s$.*

Proof: Suppose that there was another state $\mathbf{co}^{s_1}$ in the s_1-core with smaller $\mathbf{L}^2$-distance from $\left(\frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}\right)$ than $\mathbf{co}^{s_2}$. Since the s_1-core is a subset of the s_2-core, $\mathbf{co}^{s_1}$ is in the s_2-core and the fact that it has smaller $\mathbf{L}^2$ distance from $\left(\frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}\right)$ than $\mathbf{co}^{s_2}$ which by assumption is the state in the s_2-core with smallest $\mathbf{L}^2$-distance from $\left(\frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}\right)$ is a contradiction. So we conclude that $\mathbf{co}^{s_1} = \mathbf{co}^{s_2}$ if $s_2 > s_1$ and $\mathbf{co}^{s_2}$ is in the s_1-core. □

For $\mathbf{d}$ a demand vector, and $\mathbf{co}$ the vector of cooperative outcomes, define

$$D(\mathbf{d}) = \sum_{k=1}^{N} \left(d^k - co_k\right)^2 \tag{3.4}$$

the square of the $\mathbf{L}^2$ distance between $\mathbf{d}$ and $\mathbf{co}$.

The *drift* $Dr_V(\mathbf{co}, \mathbf{d})$ of a given state $\mathbf{d}$ is defined as the expected change in $D(\mathbf{d})$ over one step of V_t.

$$Dr_V(\mathbf{co}, V_t) = \mathbb{E}[D(V_{t+1}) - D(V_t) \mid V_t = \mathbf{d}]. \tag{3.5}$$

Let π_V be the equilibrium distribution of the chain (V_t), and $\pi_{V,S_\alpha} = \mathbb{P}_{\pi_V}(S_\alpha)$ where $S_\alpha = \{\mathbf{d} | D(\mathbf{d}) > \alpha\}$.

A *solution concept* is a formal rule for describing which strategies will be adopted by the players. A *solution vector* for a given solution concept is a demand vector that is identified through the application of the solution concept.

Markovian cooperative equilibrium

We now define the *Markovian cooperative equilibrium*, a solution concept for a cooperative game process as defined in the previous section.

Given a $\left(v, N, \frac{1}{M}\right)$-cooperative game process, if there exists a unique state $\mathbf{d}^* \in \Omega^C$ such that, for all $\alpha > 0$, $\mathbb{P}_{\pi_{V,M}}(||\mathbf{d} - \mathbf{d}^*||_2 > \alpha) \to 0$ as $M \to \infty$ then $\mathbf{d}^* = \mathbf{mce}$, the *Markovian cooperative equilibrium.*

The Markovian cooperative equilibrium is related to Young's (2009) concept of *closeness most of the time* where a Nash equilibrium will be played a *high proportion of the time*.

We will see in the following sections that, under certain conditions, the Markovian cooperative equilibrium coincides with **co**, however in many general cases it does not coincide with any solution concept we are aware of.

Given a game (v,N), for $1 < k \leq N$, let the *k-simplified game* (v_k,N) be the game where $v_k(C) = 0$ if $|C| < k$ and $v_k(C) = v(C)$ otherwise.

Given a game (v,N), let the *k-core* $\Omega^k(v,N)$ and the *k-cooperative outcome* $\mathbf{co}^k(v,N)$ for $k \leq N-1$ be the core $\Omega(v_k,N)$ and the cooperative outcome $\mathbf{co}(v_k,N)$ respectively for the *k-simplified game* (v_k,N).

Lemma 3.4. *Given a game (v,N), for $1 \leq k_1 < k_2 < N$ the k_1-core is a subset of the k_2-core. Therefore the core $\Omega(v,N)$ is the 1-core $\Omega^1(v,N)$ and $\mathbf{co} = \mathbf{co}^1$.*

Given a coalition C^* let $\Omega^{C^*} = \{\mathbf{d} \mid \sum_{i \in C^*} d^i \geq v(C^*)\}$.

Proof: The $(N-1)$-core Ω^{N-1} is the intersection of all Ω^{C^*} for $|C^*| = N-1$.

The $(N-k)$-core is the intersection of all Ω^k for $k \geq N-k$. So the k_1-core is a subset of the k_2-core for $1 \leq k_1 < k_2 < N$. It follows that the core is the 1-core and so $\mathbf{co} = \mathbf{co}^1$.

□

Lemma 3.5. *Given a game (v,N), if $\mathbf{co}^{k_2} \in \Omega^{k_1}$ with $k_1 < k_2$, then $\mathbf{co}^{k_1} = \mathbf{co}^{k_2}$. So, if $\mathbf{co}^k \in \Omega^C$ for some $1 \leq k < N$, then $\mathbf{co} = \mathbf{co}^k$.*

Proof: Suppose that there was another state $\mathbf{co}^{k_1}$ in the k_1-core with smaller $\mathbf{L}^2$-distance from $\left(\frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}\right)$ than $\mathbf{co}^{k_2}$. Since the k_1-core is a subset of the k_2-core, $\mathbf{co}^{k_1}$ is in the k_2-core and the fact that it has smaller $\mathbf{L}^2$ distance from $\left(\frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}\right)$ than $\mathbf{co}^{k_2}$ which by assumption is the state in the k_2-core with smallest $\mathbf{L}^2$-distance from $\left(\frac{1}{N}, \frac{1}{N}, \dots \frac{1}{N}\right)$ is a contradiction. So we conclude that $\mathbf{co}^{k_1} = \mathbf{co}^{k_2}$ if $k_2 > k_1$ and $\mathbf{co}^{k_2}$ is in the k_1-core. □

3.2 The 'Original' O-Cooperative Game Process

We now introduce the chain (O_n^e), the original chain introduced by Nax (2010), that is closely related to the chain (V_t).

Recall the Markov process $(X_n)_{n=0}^{\infty} = (\mathbf{d}_n)_{n=0}^{\infty} = \left(d_n^1, d_n^2, ..., d_n^N\right)_{n=0}^{\infty}$ in discrete time n, on the finite state space Ω_ε, introduced in 3.1.

The state space for (O_n^e) is Ω^ε and the chain is similar to the chain $(X_n)_{n=0}^{\infty} = (\mathbf{d}_n)_{n=0}^{\infty} = \left(d_n^1, d_n^2, ..., d_n^N\right)_{n=0}^{\infty}$.

Let the demand of player i at discrete time $n \in \mathbb{N}$ be given by d_n^i. The transition probabilities differ for the chain (O_n^e). We now describe the rule for updating demands for the chain (O_n^e).

Each player iteratively makes a demand and nature generates payoffs depending on the set of demands. Players adjust their demands according to a simple rule in response to the distribution of payoffs in the previous round.

The process starts from any state $\mathbf{d} \in \Omega^\varepsilon$. Each period, each agent submits a demand. Then nature distributes payoffs according to the following rule: If a player is in the payable coalition, he receives his demand as payoff. Otherwise the payoff to that player is 0.

Each period, one player i is selected uniformly at random and updates his demand. For all players $j \neq i$ who are not selected, $d_n^j = d_{n-1}^j$. The player will update his demand according to the following rule:

If the player is in the payable coalition he increases his demand by ε with fixed probability e for $0 < e \leq 1$ and the player will not change his demands with probability $1-e$. Otherwise, the player decreases his demand with probability $\frac{d_{n-1}^i}{1+\varepsilon}$ and with probability $1 - \frac{d_{n-1}^i}{1+\varepsilon}$ he will not change his demand.

Let $(n_t)_{t=0}^{\infty}$ be the sequence of times n when (O_n) is in Ω^E and consider the Markov chain $\left(O_{n_t}^e\right)_{t=0}^{\infty}$. One transition of $\left(O_{n_t}^e\right)$ from $\mathbf{d} \in \Omega^E$ can involve multiple changes of states of the chain (O_n^e).

For the chain (V_t), for a state $\mathbf{d} \in \Omega^E$ the set of new efficient states, that can be reached in one time step t, is the set of neighbors $\mathbf{d}(i,j)$ for $i, j \in \{1,2,3,4\}$. For the chain $\left(O_{n_t}^e\right)$ the set of new efficient states that can be reached over one time step t is much larger and the analysis of transition dynamics for the chain $\left(O_{n_t}^e\right)$ is complicated. We introduced the chain (V_t) as a version of $\left(O_{n_t}^e\right)$ that is amenable to analysis. We can view (V_t) as a limit to $\left(O_{n_t}^e\right)$ as e approaches 0. Suppose the chain (V_t) is in an inefficient state where demands sum to $1+\varepsilon$. Eventually one player will reduce his demand and the chain (V_t) is again in an efficient state. Suppose the chain (O_n^e) is in an inefficient state. If e is small, the probability that a player in the payable coalition increases his demand before any player not in the payable coalition reduces demands is small. So the chains (V_t) and the chain $\left(O_{n_t}^e\right)$ are closely related. For the N-player bargaining game, where the payable coalition is either the grand coalition or the empty set, the chains (V_t) and $\left(O_{n_t}^e\right)$ are identical.

Let (v,N) be a superadditive game. We call the Markov chain $\left(O_{n_t}^e\right)$ a *O-cooperative game process with rate of experimentation e* $O(v,N,\varepsilon,e)$ if $\varepsilon = \frac{1}{M}$ for some v-compatible M and for some constant scalar value e. The rate of experimentation is motivated by mutations in evolutionary biology and generally assumed to be small.

Given a $O\left(v,N,\frac{1}{M},e\right)$-cooperative game process, if there exists a unique state $\mathbf{d}^* \in \Omega^C$ such that, for all $\alpha > 0$, $\mathbb{P}_{\pi_{V,M}}\left(||\mathbf{d}-\mathbf{d}^*||_2 > \alpha\right) \to 0$ as $M \to \infty$ then $\mathbf{d}^* = Z\mathbf{mce}$, the *$O^e$-Markovian cooperative equilibrium.*

We show via simulation that the chain (V_t) has similar behavior to the chain (O_{n_t}). We give simulation results for the chain (O_{n_t}) for example games $1-3$ introduced on page 55 and show that the O^e-Markovian cooperative equilibrium is the same as the Markovian cooperative equilibrium for games $2-3$ and that the O^e-Markovian cooperative equilibrium is in the interior of the asymmetric edge for game 1 and its location depends on e.

In this section we give simulation results for the chain $\left(O_{n_t}^e\right)$ for different values of e and the chain (V_t) for example games $1-4$. We show via simulation that the behavior of the chain (V_t) is similar to the behavior of the chain $\left(O_{n_t}^e\right)$. The arguments can easily be extended to a wider class of games and we believe that the chain (V_t) is very useful in understanding the behavior of the chain $\left(O_{n_t}^e\right)$. The analysis of the chain $\left(O_{n_t}^e\right)$ is very complicated, although still complicated analysis of the chain (V_t) is more tractable.

We now define an estimator for the Markovian cooperative equilibrium and the O^e-Markovian cooperative equilibrium and then compare the simulated results of the different chains. The main interest for example games $1-4$ is, where the simulated estimator of the Markovian cooperative equilibrium is located.

For a specific cooperative game process $V_t(v,N,\varepsilon)$ with a Markovian cooperative equilibrium and a starting state $\mathbf{d}$, let $aV_T = \frac{1}{T}\sum_{t=1}^{T} V_t$. Since the Markov chain (V_t) is ergodic, we expect that $\lim_{\varepsilon\to 0}\lim_{r\to\infty} aV_{T,\varepsilon}$ almost surely exists and is equal to $\mathbf{mce}$. We simulate sample paths of length T for large $T = 10^8$ for the chain (V_t) started from any efficient state **d** in the core.

Let $aO^e_{T,\varepsilon}$ be defined accordingly for the chain $O^e_{n_t}$ for a given value of e.

Table 3.1 depicts the cooperative outcome **co**, the upper concession limit **ucl1** and the lower concession limit **lcl1** and simulated results for $aV_{T,\varepsilon}$ and $aO^e_{T,\varepsilon}$ for $T = 10^8, \varepsilon = 10^{-6}$ and $e = 0.1$ and $e = 0.01$.

	example game 1	*example game* 2	*example game* 3
co_1	0.53	0.6	0.53
$lcl1_1$	0.53	0.6	0.53
$ucl1_1$	0.61	0.61	0.54
$aV_{10^8,10^{-6}}$	0.583586	0.600001	
$aO^{0.1}_{10^8,10^{-6}}$	0.5655	0.600001	0.54
$aO^{0.01}_{10^8,10^{-6}}$	0.581919	0.600002	0.540001

Table 3.1 Simulated estimators for the Markovian cooperative equilibrium of the chain V_t ($aV_{10^8,10^{-6}}$) for 10^8 simulations and $\varepsilon = 10^{-6}$ and the estimator for the Markovian cooperative equilibrium for the original chain ($aO^{0.1}_{10^8,10^{-6}}$,$aO^{0.01}_{10^8,10^{-6}}$) with rates of experimentation 0.1 and 0.01 respectively for example games $1-3$. The demand of player one (d^1) is given, uniquely identifying a state on the asymmetric edge

For $e = 0.1$ and example game 1 $aO^{0.1}_{10^8,10^{-6}}$ is located in the interior of the asymmetric edge. It is located significantly lower than $aV_{10^8,10^{-6}}$. However, $aO^{0.01}_{10^8,10^{-6}}$ is located very close to $aV_{10^8,10^{-6}}$. It seems that for small e the Markovian cooperative equilibrium is indeed a very good approximation for the O-Markovian cooperative equilibrium.

More simulation results for games with $N > 4$ and different probabilistic assumptions are available. All confirm the here presented results.

3.3 The Cooperative Game Process is the Limit of the Original Process

In this section we analyze in more detail the relationship between the original and the modified process. We show that the equilibrium distribution of the modified process is the limit of the equilibrium distribution of the original process when the rate of experimentation tends to zero.

Let M_0 be the transition matrix for the modified cooperative game process, and let M_e be the transition matrix between efficient states of the original cooperative game process with rate of experimentation e. The state space of efficient states (and hence the size of the transition matrix) is identical for both chains. The size of both transition matrices is of order $\left(\frac{1}{\varepsilon}\right)^{N-1} \times \left(\frac{1}{\varepsilon}\right)^{N-1}$. For the modified process only transitions to neighbors have non-zero probabilities. Recall that a neighbor $\mathbf{d}(i,j)$ of $\mathbf{d}$ is the new efficient state where player i has increased demands and player j has decreased demands by ε. For the N-player bargaining game the transition matrices for the original embedded process and the modified process are equal. However this is not true in general. For the 3-player game, the transition matrices are not equal any more. Examples of transitions that have zero probability for the modified process but positive probabilities for the original process have been introduced in Chapter 1 in Section (2). The rate of experimentation e is assumed to be small. In fact in this section we will analyze the limit behavior when e tends to zero.

The following example is re-introduced to illustrate the analysis in the remainder of this section. We are comparing possible transitions from the efficient state $\mathbf{d}_0 = (0.7, 0.2, 0.1)$ for both the modified and the original process for the following example game: The worth of the grand coalition is one, the worth of C_{12} is 0.9, the worth of C_{13} is 0.7 and all other coalitions have worth zero.

For the modified process, the only transitions from $\mathbf{d}_0 = (0.7, 0.2, 0.1)$ to new efficient states with non-zero transition probabilities are the transitions to states $\mathbf{d}_0(1,2) = (0.7+\varepsilon, 0.2-\varepsilon, 0.1)$, $\mathbf{d}_0(1,3)$, $\mathbf{d}_0(2,1)$ and $\mathbf{d}_0(2,3)$.

For the original process, many more efficient states can be reached with positive (albeit very small) probability. Consider for example the following transition to the new efficient state $\mathbf{d}_* = (0.7+2\varepsilon, 0.2-2\varepsilon, 0.1)$: In round one player three increases his demand and the process is in state $\mathbf{d}_1 = (0.7, 0.2, 0.1+\varepsilon)$. Now coalition C_{12} is still binding. In round two, if chosen to update, player one can increase his demand so that the process is in state $\mathbf{d}_2 = (0.7+\varepsilon, 0.2, 0.1+\varepsilon)$, where the sum of demands is $1+2\varepsilon$. Suppose, in round three, player two is chosen to decrease demands and the process transitions to state $\mathbf{d}_3 = (0.7+\varepsilon, 0.2-\varepsilon, 0.1+\varepsilon)$. After another increase by player one ($\mathbf{d}_4 = (0.7+2\varepsilon, 0.2-\varepsilon, 0.1+\varepsilon)$) and decrease by player two ($\mathbf{d}_5 = (0.7+2\varepsilon, 0.2-2\varepsilon, 0.1+\varepsilon)$), finally player three decreases demands in round six and the process is in the new efficient state $d_* = (0.7+2\varepsilon, 0.2-2\varepsilon, 0.1)$. Observe that for the transition from $\mathbf{d}_0$ to $\mathbf{d}_*$ three times a player has increased his demand. This transition happens hence with probability of order e^3 whereas any transition to a state that can be reached by the modified process happens with probability of order e as only one player needs to be chosen to increase demands. When e tends to zero, transition probabilities for the original process that have zero probability for the modified process tend to zero very fast since they are of order at least e^2. Far away transitions are extremely unlikely.

The transition probabilities from a state $\mathbf{d}$ of the modified process (ignoring the case when $i = j$) are

$$p^{i,j} = \frac{1}{N} \frac{d^j}{\sum_{l \notin C'(\mathbf{d}(i))} d^l + \varepsilon} \mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}}$$

if $i \neq j$.

Recall from Section 3.2 that, for the original chain, if a player is in the payable coalition, he increases his demand by ε with fixed probability e, for $0 < e \leq 1$, and the player will not change his demand with probability $1-e$. Otherwise, the player decreases his demand with probability $\frac{d^i_{n-1}}{1+\varepsilon}$ and with probability $1 - \frac{d^i_{n-1}}{1+\varepsilon}$ he will not change his demand.

Recall that $(n_t)_{t=0}^{\infty}$ is the sequence of times n when (O^e_n) is in Ω^E and the Markov chain $\left(O^e_{n_t}\right)_{t=0}^{\infty}$. One transition of $\left(O^e_{n_t}\right)$ from $\mathbf{d} \in \Omega^E$ can involve multiple changes of states of the chain (O^e_n).

For the original process, it is important to differentiate between the transition probabilities of the embedded chain $\left(O^e_{n_t}\right)$ that only moves between the efficient states and the 1-step chain (O^e_n) that moves with transition probabilities as specified by the trial and error rule on a state space including inefficient states. The time indices are different.

Let $p_{\mathbf{d}(i),e}^{\mathbf{d}^*}$ be the sum of the probabilities of all paths of the original chain (O_n^e) from $\mathbf{d}(i)$ to the efficient state $\mathbf{d}^*$. Let the transition probability of the embedded original chain $\left(O_{n_t}^e\right)$ between two efficient states $\mathbf{d}$ and $\mathbf{d}^*$ be $p_{\mathbf{d},e}^{\mathbf{d}^*} = \sum_{i=1}^N p_{\mathbf{d}(i),e}^{\mathbf{d}^*}$. This definition of the transition probability prevents the transition probabilities of the embedded original chain to tend to zero with e. The embedded original chain $\left(O_{n_t}^e\right)$ moves faster than the original chain (O_n^e). The original chain is a lazy version of the embedded chain.

Lemma 3.6. *Let (v,N) be a superadditive N-player cooperative game and let ε be equal to $\frac{1}{M}$ for some $M \in \mathbb{N}$. Let (V_t) be the (v,N,ε) modified cooperative game process. Let $\left(O_{n_t}^e\right)$ be the embedded original $O(v,N,\varepsilon,e)$ cooperative game process with rate of experimentation e. Let M_0 be the transition matrix of the modified cooperative game process and M_e be the transition matrix of the original cooperative game process. Then when e tends to zero each entry of M_e tends to the respective entry of M_0.*

Proof:

Let $p_{\mathbf{d}(i),e}^{i,j}$ be the probability of the path of the original chain (O_n^e) from state $\mathbf{d}(i)$ to the efficient neighbor $\mathbf{d}(i,j)$ of $\mathbf{d}$, where the only inefficient state on the path is $\mathbf{d}(i)$. For such a path player j reduces his demand directly from $\mathbf{d}(i)$ and no player in the payable coalition at $\mathbf{d}(i)$ increases his demand along that path. The probability of such a path is of constant order and does not involve a factor e.

Let $p_{\mathbf{d}(i),e}^{i,2+}$ be the sum of the probabilities of all those paths of the original chain (O_n^e) from state $\mathbf{d}(i)$ to a new efficient state, which include (along the path) at least one inefficient state other than $\mathbf{d}(i)$ (and hence each path contains at least one more demand increase from state $\mathbf{d}(i)$). The summands of $p_{\mathbf{d}(i),e}^{i,2+}$ are either probabilities of paths to states which are not neighbors of $\mathbf{d}$ or probabilities of paths from $\mathbf{d}(i)$ to neighbors of $\mathbf{d}$ where on the path the chain (O_n^e) visits intermediate inefficient states other than $\mathbf{d}(i)$ before reaching the efficient neighbor. The probability of each summand of $p_{\mathbf{d}(i),e}^{i,2+}$ (each individual path) is of order e^h for $h \geq 1$ from $\mathbf{d}(i)$. Observe that $\sum_{j\in N} p_{\mathbf{d}(i),e}^{i,j} + p_{\mathbf{d}(i),e}^{i,2+} = 1$.

Observe that only a player not in the payable coalition at $\mathbf{d}(i)$ can decrease his demand at $\mathbf{d}(i)$. If chosen to update at $\mathbf{d}(i)$, a player in the payable coalition will increase his demand with probability e. Hence $p_{\mathbf{d}(i),e}^{i,j} = \frac{d^j}{\sum_{l\notin C'(\mathbf{d}(i))} d^l + \varepsilon + p_{\mathbf{d}(i),e}^{i,2+}} \mathbf{1}_{\{j\notin C'(\mathbf{d}(i))\}}$. Since each individual summand of $p_{\mathbf{d},e}^{i,2+}$ involves at least one further demand increase, which happens with probability e, we can rewrite the above expression as

$$p_{\mathbf{d}(i),e}^{i,j} = \frac{d^j}{\sum_{l \notin C'(\mathbf{d}(i))} d^l + \varepsilon + O(e)} \mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}}$$

where $O(e)$ is an expression of order at least e and not to be confused with the original cooperative game process O_n^e. The transition probability of the embedded chain from $\mathbf{d}$ to the efficient neighbor $\mathbf{d}(i,j)$ is then

$$p_{\mathbf{d},e}^{i,j} = \frac{1}{N} \frac{d^j}{\sum_{l \notin C'(\mathbf{d}(i))} d^l + \varepsilon + O(e)} \mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}}$$

. Now when e tends to zero, the transition probability $p_{\mathbf{d},e}^{i,j}$ tends to the transition probability of the modified cooperative game process

$$p_{\mathbf{d}}^{i,j} = \frac{1}{N} \frac{d^j}{\sum_{l \notin C'(\mathbf{d}(i))} d^l + \varepsilon} \mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}}$$

.

So each transition probability between efficient states of the transition matrix M_e for the embedded original cooperative game process with rate of experimentation e tends to the respective entry of the transition matrix M_0 of the modified cooperative game process when e tends to zero. □

Let the equilibrium distribution of M_e and M_0 be μ_e and μ_0 respectively. We now show that μ_e tends to μ_0 when e tends to zero.

Theorem 3.7. *Let (v,N) be a superadditive N-player cooperative game and suppose ε is equal to $\frac{1}{M}$ for some $M \in \mathbb{N}$. Let (V_t) be the (v,N,ε) modified cooperative game process with a unique communicating class of non transient states. Let $\left(O_{n_t}^e\right)$ be the original $O(v,N,\varepsilon,e)$ cooperative game process defined on the same set of efficient states. Then, if the rate of experimentation e tends to zero, the equilibrium distribution of the original embedded cooperative game process tends to μ_0.*

Proof:

By assumption, (V_t) has a unique communicating class of non-transient states. By elementary results on finite Markov chain theory, see for example Grimmett and Stirzacker (2001), there exists a unique stationary distribution μ_0 for (V_t).

By Lemma (3.6), when e tends to zero, the transition matrix of the original cooperative game process defined on the set of efficient states M_e tends to M_0, the transition matrix for the modified cooperative game process.

By stationarity of μ_e, it holds that $M_e\mu_e = \mu_e$, so $(M_e - I)\mu_e = 0$ where I is the identity matrix. In the same way $M_0\mu_0 = \mu_0$ and so $(M_0 - I)\mu_0 = 0$.

We know that $||\mu_e||_1 = 1$ and $\mu_e \geq 0$ for each rate of experimentation e. So the space of vectors μ_e is compact. Let $(M_{e_k}) = M_{e_1}, M_{e_2}, M_{e_3}...$ be a sequence of matrices for different decreasing values of $(e_k) = e_1, e_2, e_3, ...$ tending to zero.

Since $\mu_{e_1}, \mu_{e_2}, \mu_{e_3}...$ live in a compact space, there is some subsequence $\mu_{e_{k_i}}$ of the sequence μ_{e_k} converging to a μ^*. Then taking the limit

$$0 = \lim_{i\to\infty}\left(M_{e_{k_i}} - I\right)\mu_{e_{k_i}} = (M_0 - I)\mu^*.$$

It follows now from $(M_0 - I)\mu^* = 0$ and $||\mu^*||_1 = 1$ and $\mu^* \geq 0$ that μ^* is an equilibrium distribution for M_0. Since M_0 has a unique equilibrium distribution, it follows that $\mu^* = \mu_0$.

Suppose μ_{e_k} does not tend to μ_0. Then by compactness there exists a subsequence μ_{e_n} that converges to some limit μ^* other than μ_0. We have shown that this is a contradiction and hence we conclude that μ_{e_k} tends to μ_0 when e_k tends to zero. □

For the cooperative game process for the N-player bargaining game, the unique communicating class with non-transient states is the set of efficient states. In Chapter 4 Section 4.3 and in Chapter 5, it is shown that the modified cooperative game process has a unique communicating class of non-transient states: the core for the cooperative game process for balanced superadditive three player games and balanced superadditive four player games without asymmetric coalition structure, and the extended core for the cooperative game process for general balanced superadditive four player games.

Hence, for all processes studied in this thesis, the equilibrium distribution of the modified cooperative game process is the limit of the equilibrium distribution of the embedded original cooperative game process when the rate of experimentation tends to zero.

3.4 Equity in the N-Player Bargaining Game

Let the N-player bargaining game be the game (v,N) where $v(C)=0$ for all $C \in \mathcal{P}\left(C^G\right), C \neq C^G$ and $v\left(C^G\right)=1$. The core then consists of all efficient states and $\mathbf{co}=\mu$. In this section, we first look at the special case where (v,N) is the N-player bargaining game. I show that $\mathbf{co}$ is a Markovian cooperative equilibrium for this game, for every $N \geq 2$.

The Theorem is given by

Theorem 3.8. *If (v,N) is the N-player bargaining game and ε is equal to $\frac{1}{M}$ for some $M \in \mathbb{N}$. Let (V_t) be the (v,N,ε) cooperative game process. Then, for $\forall \alpha > 0$,*

$$\mathbb{P}_{\pi_{v,\varepsilon}}(D(\mathbf{d}) > \alpha) \leq \frac{\varepsilon}{N\alpha}.$$

and proved at the end of this section.

In this particular case $\mathbf{co}=\left(\frac{1}{N},...,\frac{1}{N}\right)$ and $D(\mathbf{d}) = ||\mathbf{co}-\mathbf{d}||$. Hence in this case the Markovian cooperative equilibrium equals the Nash Bargaining solution of Peleg et al (2007).

The following lemma shows that, ignoring the second order terms, the expected change in the value of the $\mathbf{L}^2$-distance between $\mathbf{co}$ and the chain (V_t) over one step of (V_t) is negative and proportional to D:

In the N-player bargaining set-up the transition probabilities for the cooperative game process are given by $p^{i,j}=\frac{d^j}{N(1+\varepsilon)}$ if $i \neq j$ and $p^{i,j}=\frac{d^j+\varepsilon}{N(1+\varepsilon)}$ if $i=j$.

Lemma 3.9. $Dr(\mathbf{co},V_t) = \frac{-2\varepsilon}{1+\varepsilon}\left(D(\mathbf{d})+\varepsilon\frac{N-1}{N^2}\right)$.

Proof: The change in $D(\mathbf{d})$ when $i \neq j$ is given by

$$D(\mathbf{d}(i,j)) - D(\mathbf{d}) = \frac{1}{N}\left(2\varepsilon^2 + 2\varepsilon(\left(d^i-\mu\right)-\left(d^j-\mu\right)\right) \tag{3.6}$$

.

Since $\max_{i,j\in[1,...N]} \mid d^i - d^j \mid = 1$ the maximum change in $D(\mathbf{d})$ is given by $\frac{1}{N}\left(2\varepsilon^2+2\varepsilon\right) < \frac{3\varepsilon}{N} \leq \varepsilon$ for $N \geq 3$.

$$
\begin{aligned}
D(\mathbf{d}) &= \\
&= \frac{2\varepsilon}{(1+\varepsilon)N^2}\sum_{i=1}^{N}\sum_{j\neq i} d^j\left(\left(d^i-\mu\right)-\left(d^j-\mu\right)+\varepsilon\right) \\
&= \frac{2\varepsilon}{(1+\varepsilon)N^2}[\sum_{i=1}^{N}\sum_{j\neq i} d^j\left(d^i-\mu\right)-\sum_{i=1}^{N}\sum_{j\neq i} d^j\left(d^j-\mu\right)+\varepsilon\sum_{i=1}^{N}\sum_{j\neq i} d^j] \\
&= \frac{2\varepsilon}{(1+\varepsilon)N^2}[\sum_{i=1}^{N}\left(1-d^i\right)\left(d^i-\mu\right)-\sum_{i=1}^{N}\sum_{j\neq i}\left(d^j-\mu+\mu\right)\left(d^j-\mu\right)+\varepsilon\sum_{i=1}^{N}\sum_{j\neq i} d^j] \\
&= \frac{2\varepsilon}{(1+\varepsilon)N^2}[\sum_{i=1}^{N}\left(d^i-\mu\right)-\sum_{i=1}^{N} d^i\left(d^i-\mu\right)-\sum_{i=1}^{N}\sum_{j\neq i}\left(d^j-\mu\right)^2-\sum_{i=1}^{N}\sum_{j\neq i}\mu\left(d^j-\mu\right)+\varepsilon\sum_{i=1}^{N}\sum_{j\neq i} d^j] \\
&= \frac{2\varepsilon}{(1+\varepsilon)N^2}[0-\sum_{i=1}^{N}\left(d^i-\mu\right)^2-\sum_{i=1}^{N}\mu\left(d^i-\mu\right)-(N-1)\sum_{i=1}^{N}\left(d^i-\mu\right)^2-(N-1)0+\varepsilon\sum_{i=1}^{N}\left(1-d^i\right)] \\
&= \frac{2\varepsilon}{(1+\varepsilon)N^2}[-\sum_{i=1}^{N}\left(d^i-\mu\right)^2-0-(N-1)\sum_{i=1}^{N}\left(d^i-\mu\right)^2+(N-1)\varepsilon] \\
&= \frac{2\varepsilon}{(1+\varepsilon)N^2}\left[-N\sum_{i=1}^{N}\left(d^i-\mu\right)^2+(N-1)\varepsilon\right]
\end{aligned}
$$

□

Let π be the equilibrium distribution and $\pi_{S_\alpha}=\mathbb{P}_\pi(S_\alpha)$ where $S_\alpha=\{\mathbf{d}|D(\mathbf{d})>\alpha\}$.

Proof: of Theorem (3.8)

We know from Theorem (3.9) that

$f(\mathbf{d})=\mathbb{E}[D(V_{t+1})-D(V_t)|V_t=\mathbf{d}]=\frac{2\varepsilon}{1+\varepsilon}\left(-D(\mathbf{d})+\varepsilon\frac{N-1}{N^2}\right)$.

Now in equilibrium $0=\sum_{\mathbf{d}\in\Omega^E}\pi_{\mathbf{d}}\mathbb{E}[D(V_{t+1})-D(V_t)|V_t=\mathbf{d}]$.

So $0<\frac{2\varepsilon}{(1+\varepsilon)}\left(\pi_{S_\alpha}\left(-\alpha+\frac{\varepsilon}{N}\right)+(1-\pi_{S_\alpha})\frac{\varepsilon}{N}\right)=\frac{2\varepsilon}{(1+\varepsilon)}\left(-\alpha\pi_{S_\alpha}+\frac{\varepsilon}{N}\right)$ so $\pi_{S_\alpha}<\frac{\varepsilon}{N\alpha}$. □

3.5 Major Coalition Games and Expected Return Times to Facets of the Core

In this section we analyze a 'well behaved' family of N-player cooperative game processes: one "major" coalition determines the dynamics of the game. We prove that the cooperative game process returns freqeuntly to a coalition exactly if the average payout to a player in that coalition is larger than what he would receive at equal spilt. The main contribution of this chapter is to show that the process $CS^{C}(V_t)$, the surplus of a coalition, is in fact a birth and death chain. This transformation enables the application of standard calculations and results on birth and death chains and random walks to the calculation of the return time of the cooperative game process.

In Theorem (3.10) we calculate a bound on the expected return time of the chain (V_t) to the set of states where the major coalition is binding. In Theorem (3.11) we provide a probabilistic bound on the number of ε-steps that the process can move away from the set of states where the major coalition is binding.

In the proof of the bound of the expected return time we use the fact that a random walk can be defined that is a 'pessimistic' version of the birth and death chain. We like to point out that, although the proofs here are only for major coalition games, the behavior of the cooperative game process for a general N-player game is very similar to that of a major coalition game, as long as the chain (V_t) is 'far away' from states in the intersection of two hyperplanes $\mathcal{H}\left(C^1\right) \cap \mathcal{H}\left(C^2\right)$. Then 'locally' the chain $CS^{C}(V_t)$ behaves very much like the birth and death chain defined here.

We will see in Chapter 5 that it is not in general true for N-player cooperative game processes that at every state $\mathbf{d} \in \Omega^{C}$ a drift in $D(\mathbf{d})$ towards the cooperative outcome $\mathbf{co}$ exists. Having a local bound on the return time becomes a useful tool when the core contains states that do not have a drift towards $\mathbf{co}$.

In Lemma (3.8) we show that at each state $\mathbf{d} \in \Omega^{C}$ there is a drift in $D(\mathbf{d})$ towards the cooperative outcome $\mathbf{co}$ and we show that the Markovian cooperative equilibrium is $\mathbf{co}$. Theorem (3.13) then follows from Lemma (3.8) as in the N-player bargaining case.

For $C \in \mathcal{P}\left(C^{G}\right)$, let $\mathcal{H}(C) = \{\mathbf{d} \in \Omega^{C} \mid \sum_{i \in C} d^i = v(C)\}$.

Let C^0 be a specific coalition with $|C^0| \geq 2$ and $C^0 \neq C^G$, the grand coalition. Suppose also that

1. $v\left(C^0\right) > \frac{|C_0|}{N}$

2. $v(C) = 1$ if $C = C^G$; $v(C) = v\left(C^0\right)$ if $C^0 \subset C$ and $C \neq C^G$, $v(C) = 0$ otherwise.

In a game (v, N), with v satisfying $(1.)$ and $(2.)$ we call C^0 the *major coalition* and the game (v, N) a *major coalition game*. Observe that a major coalition game is superadditive.

Let $\Omega^B = \mathcal{H}\left(C^0\right)$, that is $\Omega^B = \{\mathbf{d} \mid CS^{C^0}(\mathbf{d}) = 0\}$, the set of all states where the major coalition is binding. The set Ω^B is a $N-2$-dimensional subset of the $N-1$-dimensional set Ω^E. Let $T_0, T_1, \ldots$ be the sequence of random times defined recursively with $T_0 = 0$ and for $j \geq 1$, $T_j = \min\{t > T_{j-1} \mid V_t \in \Omega^B\}$.

We now discuss the chain (V_t) for a major coalition game cooperative process. The core is the the set $\{\mathbf{d} \in \Omega^E \mid \sum_{i \in C^0} d^i \geq v\left(C^0\right)\} = \{\mathbf{d} \in \Omega^E \mid CS^{C^0}(\mathbf{d}) \geq 0\}$. We analyze the behavior of the chain (V_t) started in a state on Ω^B. Once in the core the chain (V_t) will not leave the core again.

The next Theorem describes the expected return time of the chain (V_t) over excursions away from the hyperplane. We show that the chain (V_t) stays in close neighborhood of the hyperplane corresponding to states, where the major coalition C^0 is binding.

Theorem 3.10. *Let the chain (V_t) start in any state in Ω^B.*

Then the cooperative game process will return "fast" to Ω^B exactly for coalitions where the equal payout to a player in that coalition is higher than the payout to the player at equal split. Furthermore the "speed of return" is proportional to the worth of the coalition with the expected return time of (V_t) to Ω^B satisfying $\mathbb{E}(T_1) < \frac{1}{1-\phi}$ where $\phi = \frac{|C^0|(1-v(C))}{\left(N-|C^0|\right)v(C)}$.

This theorem shows that the chain (V_t) returns fast to states where C^0 is binding. Before we prove Theorem (3.10) we now make the direct link between the chain $\frac{CS_t^{C^0}}{\varepsilon}$ and a birth and death chain. This transformation is not obvious however it provides us with a set of tools from the theory of birth and death chains and then, random walks, that we believe is very relevant and useful to analyze general cooperative game processes, not just in the major coalition setting.

We now discuss the chain $\frac{CS_t^{C^0}}{\varepsilon}$. The chain increases by ε if (V_t) transitions from a state $\mathbf{d}$ to a neighbor $\mathbf{d}(i,j)$ for $\imath, j \in \{1,...,N\}$ where i is in C^0 and j is in the complement of C^0. Observe that the chain increases by ε with $\frac{|C^0|\sum_{j\notin C^0} d^j}{N(1+\varepsilon)}$ where $\frac{|C^0|}{N}$ is the probability that a player in C^0 is chosen to increase his demand and $\frac{\sum_{j\notin C^0} d^j}{N(1+\varepsilon)}$ is the probability that a player $j \notin C^0$ decreases demands from $\mathbf{d}(i)$. Similarly, the probability that the chain $\frac{CS_t^{C^0}}{\varepsilon}$ decreases by ε from an efficient state $\mathbf{d} \notin \Omega^B$ is given by $\frac{\left(N-|C^0|\right)\sum_{i\in C^0} d^i}{N(1+\varepsilon)}$. Observe further that if $\frac{CS_t^{C^0}}{\varepsilon} = i$ for some $i \in \{0,..., \frac{1-v\left(C^0\right)}{\varepsilon}\}$ then the sum of the demands in coalition C^0 is given by $v\left(C^0\right) + i\varepsilon$ and the sum of the demands of the players in the complement of C^0 is given by $1 - v\left(C^0\right) - i\varepsilon$. Now we observe that in effect the Markov chain $\frac{CS_t^{C^0}}{\varepsilon}$ follows a birth and death chain where state i of the birth and death chain corresponds to the state where $CS^{C^0}(\mathbf{d}) = i\varepsilon$. The birth and death chain increases by 1, if a player in C^0 increases his demand and a player in the complement of C^0 decreases demands. The birth and death chain decreases by 1, if a player in the complement of C^0 increases his demand and a player in C^0 decreases demands. The maximum sum of demands of the players in the complement of C^0 is $1 - v\left(C^0\right) + \varepsilon$ and the minimum sum of demands in the complement of C^0 is 0. Similarly the maximum the players in the coalition C^0 can demand together is $1+\varepsilon$ and the minimum $v\left(C^0\right)$.

Now we prove Theorem (3.10).

Proof: As just discussed the Markov chain $\frac{CS_t^{C^0}}{\varepsilon}$ follows a birth and death chain BD^{C^0} where state i of the birth and death chain corresponds to the state where $CS^{C^0}(\mathbf{d}) = i\varepsilon$. The transition probabilities at state i are given by $p_{i,i+1} = \max\{\frac{|C^0|}{N}\frac{1-v\left(C^0\right)-i\varepsilon}{(1+\varepsilon)}, 0\}$ and $p_{i,i-1} = \min\{\frac{N-|C^0|}{N}\frac{v\left(C^0\right)+i\varepsilon}{(1+\varepsilon)}, 1\}$.

Let π_0^{BD} be the equilibrium probability of the 0-state of BD^{C^0} and let π_0^{RW} be the equilibrium distribution of the 0-state of a random walk with transition probabilities

$p_{i,i+1} = \frac{|C^0|}{N}\frac{1-v\left(C^0\right)}{(1+\varepsilon)}$, $p_{i,i-1} = \frac{N-|C^0|}{N}\frac{v\left(C^0\right)}{(1+\varepsilon)}$, $p_{0,-1} = 0$ and $p_{\frac{1-v(C^0)}{\varepsilon}, \frac{1-v(C^0)}{\varepsilon}+1} = 0$.

We want to show that π_0^{RW} is a lower bound for π_0^{RW}.

From Grimmett and Stirzacker (2001) we know that the equilibrium distribution of the 0-state of a birth and death chain is given by

$$\pi_0^{BD} = \frac{1}{\sum_{k=1}^{\infty} \frac{p_{0,1}}{p_{1,0}} \frac{p_{1,2}}{p_{2,1}} \cdots \frac{p_{k-1,k}}{p_{k,k-1}}}$$

.

Observe that $\frac{p_{i,i+1}}{p_{i+1,i}} = \frac{|C^0|}{N-|C^0|} \frac{1-v\left(C^0\right)-i\varepsilon}{v\left(C^0\right)+i\varepsilon}$ and

$\frac{p_{i,i+1}}{p_{i+1,i}} < \frac{|C^0|}{N-|C^0|} \frac{1-v\left(C^0\right)}{v\left(C^0\right)}$ and so $\sum_{k=1}^{\infty} \frac{p_{0,1}}{p_{1,0}} \frac{p_{1,2}}{p_{2,1}} \cdots \frac{p_{k-1,k}}{p_{k,k-1}} < \sum_{k=1}^{\infty} \phi^k = \frac{1}{1-\phi}$ where $\phi = \frac{|C^0|}{N-|C^0|} \frac{1-v\left(C^0\right)}{v\left(C^0\right)}$.
As $\pi_0^{BD} = \frac{1}{\sum_{k=1}^{\infty} \frac{p_{0,1}}{p_{1,0}} \frac{p_{1,2}}{p_{2,1}} \cdots \frac{p_{k-1,k}}{p_{k,k-1}}}$ we know that $\pi_0^{BD} > 1-\phi = \pi_0^{RW} = \frac{1}{\sum_{k=1}^{\infty} \phi^k}$.

From Grimmett and Stirzacker (2001) we know that the expected return time to a state i of an irreducible aperiodic Markov chain is given by $\frac{1}{\pi_i}$ where π_i is the equilibrium distribution of state i.

Hence the expected return time of the chain (V_t) to $\mathcal{H}\left(C^0\right)$ is less than $\frac{1}{1-\phi}$.

□

We like to make a quick excursion to general N-player cooperative game processes. If a general cooperative game process (V_t) is 'close' to an intersection of two or more hyperplanes $\mathcal{H}(C)$ for different coalitions $C \in \mathcal{P}\left(C^G\right)$ then the behavior of the process is similar to a dependent Markov chain where each coordinate corresponds to one birth and death chain BD^C.

The next theorem shows that, if C^0 has been binding once, the probability to go far away from states where C^0 is binding, is very small.

Let τ_{C^0} be the first time that the chain (V_t) is in $\mathcal{H}\left(C^0\right)$.

The next theorem shows that, if C^0 has been binding once, the probability to go far away from states, where C^0 is binding, is very small.

Let τ_{C^0} be the first time that the chain (V_t) is in $\mathcal{H}\left(C^0\right)$.

Theorem 3.11. *Let the chain (V_t) start in any state in Ω^B. It holds that*

$$\mathbb{P}\left(\frac{CS^{C^0}(V_t)}{\varepsilon} > K\right) < \frac{\frac{1}{1-\phi}}{K} \tag{3.7}$$

for all $K > 0$.

Proof: We use Markov's inequality to prove the bound.

□

Observe that the $co_i = \frac{v(C^0)}{|C^0|}$ for $i \in C^0$ and $co_j = \frac{1-v(C^0)}{N-|C^0|}$ for $j \notin C^0$.

We analyze the drift over one time step of the chain (V_t) show that at each state $\mathbf{d} \in \Omega^C$ there is a drift in $D(\mathbf{d})$ towards $\mathbf{co}$. .

Theorem 3.12. *Let the chain (V_t) start in any state on Ω^C. Then*

$$Dr(V_t, \mathbf{co}) \leq \frac{-\varepsilon D(\mathbf{d})}{N(1+\varepsilon)} + 2\varepsilon^2. \tag{3.8}$$

Proof: Let $v(C^0) = c_0$. The drift is given by

$$\sum_{i=1}^{N} \sum_{j=1}^{N} p_{\mathbf{d}}^{i,j} \left(-D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(i,j))\right). \tag{3.9}$$

If C^0 is not binding at $\mathbf{d}(i)$, the payable coalition at $\mathbf{d}(i)$ is the empty set and so $p_{\mathbf{d}}^{i,j} = \frac{d^j}{N(1+\varepsilon)}$ if $i \neq j$ and $p_{\mathbf{d}}^{i,j} = \frac{d^j+\varepsilon}{N(1+\varepsilon)}$ otherwise. So

$$\begin{aligned}
Dr(V_t, \mathbf{co}) &\leq \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{d^j}{1+\varepsilon} \left(-\left(co_i - d^i\right)^2 - \left(co_j - d^j\right)^2 + \left(co_i - \left(d^i + \varepsilon\right)\right)^2 + \left(co_j - \left(d^j - \varepsilon\right)\right)^2\right) \\
&\quad + 2\varepsilon^2 \\
Dr(V_t, \mathbf{co}) &= \sum_{i=1}^{N} \sum_{j=1}^{N} \frac{d^j}{1+\varepsilon} \left[-2\varepsilon\left(co_i - d^i\right) + 2\varepsilon\left(co_j - d^j\right) + 2\varepsilon^2\right] + 2\varepsilon^2 \\
Dr(V_t, \mathbf{co}) &= \frac{2\varepsilon}{N(1+\varepsilon)} \left[-\sum_{i=1}^{N} \sum_{j=1}^{N} d^j\left(co_i - d^i\right) + \sum_{i=1}^{N} \sum_{j=1}^{N} d^j\left(co_j - d^j\right)\right] + \frac{2\varepsilon^2}{1+\varepsilon} + 2\varepsilon^2 \\
Dr(V_t, \mathbf{co}) &= \frac{2\varepsilon}{(1+\varepsilon)} \sum_{j=1}^{N} \left[\frac{1}{2}\left(d^j - co_j\right)\left(co_j - d^j\right) + \frac{1}{2}\left(d^j + co_j\right)\left(co_j - d^j\right)\right] + \frac{2\varepsilon^2}{1+\varepsilon} + 2\varepsilon^2
\end{aligned}$$

and so

$$Dr(V_t, \mathbf{co}) = -\frac{\varepsilon}{(1+\varepsilon)} \sum_{j=1}^{N} \left(d^j - co_j\right)^2 + \frac{\varepsilon}{1+\varepsilon} \left[\sum_{j=1}^{N} \left(co_j\right)^2 - \sum_{j=1}^{N} \left(d^j\right)^2\right] + \frac{2\varepsilon^2}{1+\varepsilon} + 2\varepsilon^2 \tag{3.10a}$$

It follows that

$$Dr(V_t, \mathbf{d}) < \frac{2\varepsilon}{N(1+\varepsilon)} \left(-D(\mathbf{d})\right) + 2\varepsilon^2. \tag{3.11}$$

Suppose at $\mathbf{d}$, C^0 is binding. If a player in C^0 increases his demand, the payable coalition is the empty set and so $p_{\mathbf{d}}^{i,j} = \frac{d^j}{N(1+\varepsilon)}$ if $i \neq j$ and $p_{\mathbf{d}}^{i,j} = \frac{d^j+\varepsilon}{N(1+\varepsilon)}$ otherwise. If player k in the complement of C^0 increases his demand, C^0 is the payable coalition and so $p_{\mathbf{d}}^{k,j} = \frac{d^j}{N(1-c_0+\varepsilon)}$ if $k \neq j$ and $j \notin C^0$ and $p_{\mathbf{d}}^{k,j} = 0$ otherwise.

The drift is given by

$$Dr(V_t, \mathbf{co})_{C^0} + Dr(V_t, \mathbf{co})_{C^G \setminus C^0} \tag{3.12}$$

where

$$Dr(V_t, \mathbf{co})_{C^0} = \sum_{i \in C^0} \sum_{j \in C^G} p_{\mathbf{d}}^{i,j} \left(-D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(i,j))\right) \tag{3.13}$$

and

$$Dr(V_t, \mathbf{co})_{C^G \setminus C^0} = \sum_{k \notin C^0} \sum_{l \notin C^0} p_{\mathbf{d}}^{k,l} \left(-D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(k,l))\right). \tag{3.14}$$

The drift is given by

$$\begin{aligned}
Dr(V_t, \mathbf{co})_{C^G \setminus C^0} &= \sum_{k \notin C^0} \sum_{l \notin C^0} \frac{d^l}{1-c_0+\varepsilon} \left(-\left(co_k - d^k\right)^2 - \left(co_l - d^l\right)^2\right. \\
&\quad \left. + \left(co_k - \left(d^k + \varepsilon\right)\right)^2 + \left(co_l - \left(d^l - \varepsilon\right)\right)^2\right. \\
Dr(V_t, \mathbf{co})_{C^G \setminus C^0} &= \sum_{k \in C^0} \sum_{l \in C^0} \frac{d^l}{1-c_0+\varepsilon} \left[-2\varepsilon\left(co_k - d^k\right) + 2\varepsilon\left(co_l - d^l\right) + 2\varepsilon^2\right] \\
Dr(V_t, \mathbf{co})_{C^G \setminus C^0} &= \frac{2\varepsilon}{|C^G \setminus C^0|(1-c_0+\varepsilon)} \left[\sum_{k \notin C^0} \sum_{l \notin C^0} d^l\left(co_k - d^k\right) + \sum_{k \notin C^0} \sum_{l \notin C^0} d^l\left(co_l - d^l\right)\right] \\
&\quad + \frac{2\varepsilon^2}{1-c_0+\varepsilon}
\end{aligned}$$

Observe that $\sum_{k\notin C^0}\left(co_k - d^k\right) = 0$ and so $-\sum_{k\notin C^0}\sum_{l\notin C^0} d^l\left(co_k - d^k\right) = 0$

$$
\begin{aligned}
Dr\left(V_t,\mathbf{co}\right)_{C^G\setminus C^0} &= \frac{2\varepsilon}{(1-c_0+\varepsilon)}\sum_{l\notin C^0}\left[\frac{1}{2}\left(d^j - co_j\right)\left(co_j - d^j\right) + \frac{1}{2}\left(d^j + co_j\right)\left(co_j - d^j\right)\right] \\
&\quad + \frac{2\varepsilon^2}{1-c_0+\varepsilon} \\
Dr\left(V_t,\mathbf{co}\right)_{C^G\setminus C^0} &= -\frac{\varepsilon}{(1-c_0+\varepsilon)}\sum_{l\notin C^0}\left(d^j - co_j\right)^2 + \frac{\varepsilon}{1-c_0+\varepsilon}\left[\sum_{l\notin C^0}\left(co_j\right)^2 - \sum_{l\notin C^0}\left(d^j\right)^2\right] \\
&\quad + \frac{2\varepsilon^2}{1-c_0+\varepsilon}
\end{aligned}
\tag{3.16a}
$$

□

We use Lemma 4.20 to show that **co** is a Markovian cooperative equilibrium for all N-player major coalition games.

Theorem 3.13. *Suppose (v,N) is a major coalition game, and ε is equal to $\frac{1}{M}$ for some v-compatible $M \in \mathbb{N}$. Let (V_t) be the (v,N,ε) cooperative game process. Then, for $\alpha > 0$,*

$$
\mathbb{P}_{\pi_v,\varepsilon}\left(D\left(\mathbf{d}\right) > \alpha\right) \leq \frac{\varepsilon}{N\alpha}.
$$

Chapter 4

Equity in General 3-Player Games

In this chapter we analyze 3-player cooperative game processes. The chapter is structured as followed:

In 4.1 we introduce the concept of a *balanced* coalition, we state the Bondareva-Shapley Theorem and apply it to prove that the core of a 3-player game is non empty if and only if $v(C_{12}) + v(C_{13}) + v(C_{23}) \leq 2$. This subsection directly follows Gilles (2010). In the remainder of this chapter on 3-player games we assume that the core is non-empty.

In 4.2 we describe an algorithm that finds **co** for each 3-player cooperative game. Recall that the cooperative outcome $\mathbf{co} = (co_1, co_2, co_3)$ is defined as the state in the core that minimizes $\sum_{i\in\{1,2,3\}} \left(d^i - \frac{1}{3}\right)^2$. From Lemma (3.1) we know that the *cooperative outcome* **co** is the most equal allocation in the core, that is, the allocation in the core minimizing $\sum_{i\in\{1,2,3\}} \left(d^i\right)^2$. The algorithm first finds $\mathbf{co}^2$, the cooperative outcome for the 2-core, then it checks if $\mathbf{co}^2$ is a member of the core. If so, $\mathbf{co} = \mathbf{co}^2$. Otherwise exactly one inequality $s_i > co_i^2$ holds, and **co** must lie on the hyperplane $\{\mathbf{d} \mid d^i = s_i\}$. We give graphical representations of the geometric structure of different 3-player games and the precise location of the cooperative outcome in the core.

In 4.3 in Lemma (4.7) we show that all states outside the core are transient and then, in Lemma (4.8) that the set of recurrent states is indeed the core. For this, we show, and then use the fact, that the core is connected. In the beginning of 4.3 we first state and prove a set of lemmas, that we frequently apply in the main proofs of this subsection. Finally, we state and prove Lemma (4.9), that we apply repeatedly in the proofs in the succeeding subsections.

In 4.4 we show that, at every state $\mathbf{d} \in \Omega^C$ there is a drift proportional to $D(\mathbf{d})$. So for all states not in the close neighborhood of **co**, there is a negative drift. Given a game $(v,3)$ let $\mathcal{C}^{\mathbf{co}}(v,3)$ be the set of coalitions that are binding at **co**. We partition the core. The first set in the partition is the set of states where no coalition is binding. We calculate the drift for these states. The second set of states is the set of states, where coalitions can be binding that are not in $\mathcal{C}^{\mathbf{co}}(v,3)$. We compare the drift at a state in this set for a particular game with the drift of a game, where all coalitions not in $\mathcal{C}^{\mathbf{co}}(v,3)$ are removed. We hence deduce a bound on the drift for the second set of states. Finally, we calculate the drift for all states in the core, where at least a coalition in the set $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding.

In Lemma (4.20) the drift on all three sets of the partition is combined and a global bound on the drift for a cooperative game process $(v,3,\varepsilon)$ is calculated. In Theorem (4.21) we use Lemma (4.20) to show that **co** is a Markovian cooperative equilibrium for all superadditive 3-player games satisfying (4.4).

4.1 Non-Emptiness of the Core

The definitions and proofs in this subsection directly follow from Chapter 2 of Gilles (2010).

Let $\mathcal{B} \subset \mathcal{P}\left(C^G\right) \setminus \emptyset$ be a set of non-empty coalitions called a *collection*. The collection $\mathcal{B}$ is *balanced* if there exist numbers $\lambda_C > 0$ for $C \in \mathcal{B}$ such that for each player $i \in \{1,2,3\}$

$$\sum_{C \in \mathcal{B} | i \in C} \lambda_C = 1. \tag{4.1}$$

Let the members of $\{\lambda_C \mid C \in \mathcal{B}\}$ be called *balancing coefficients* of the collection $\mathcal{B}$. A *minimal balanced collection* is a balanced collection $\mathcal{B}$ that does not contain a proper balanced subcollection. Observe that any partition of $\{1,2,3\}$ is balanced as long as the individual coalitions C in the partition have balancing coefficient $\lambda_C = 1$.

The next theorem is modified from Theorem (2.10) from Gilles (2010) and derives some basic properties of balanced collections.

Theorem 4.1. *Let* $\{1,2,3\}$ *be the set of players for a game* $(v,3)$. *Then*

1. *the union of balanced collections on* $\{1,2,3\}$ *is balanced,*

2. *a balanced collection is minimal if and only if it has a unique set of balancing coefficients and*

3. *any balanced collection is the union of minimal balanced collections.*

The next lemma follows from Theorem (4.1.2).

Lemma 4.2. *A minimal balanced collection consists of at most* 3 *coalitions.*

We now state the Bondareva-Shapley Theorem in terms of minimal balanced coalitions.

Theorem 4.3. *(Bondareva-Shapley) Let* (v,N) *be a game. Then the core of* (v,N) *is non-empty if and only if for every minimal balanced collection* $\mathcal{B} \subset \mathcal{P}\left(C^G\right)$ *with balancing coefficients* $\{\lambda_C \mid C \in \mathcal{B}\}$ *it holds that*

$$\sum_{C\in\mathcal{B}} \lambda_C v(C) \leq v\left(C^G\right). \tag{4.2}$$

We set $v(C_{123}) = 1$, $v(C_{12}) = a, v(C_{13}) = b, v(C_{23}) = c$ and $v(C_i) = s_i$ for $i = 1,2,3$. We order the players in such a way that

$$a \geq b \geq c. \tag{4.3}$$

Using the above version of the Bondareva-Shapley Theorem, the 3-player version reduces to

Lemma 4.4. *Let* $(v,3)$ *be superadditive. Then the core of* $(v,3)$ *is empty if and only if*

$$a + b + c \leq 2. \tag{4.4}$$

Proof: If the balanced collection is trivial and forms a partition of $\{1,2,3\}$ then the lemma follows from superadditivity. The only non-trivial minimal balanced collection is $\mathcal{B} = \{C_{12}, C_{13}, C_{23}\}$ with balancing coefficients $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$. The results follows immediately after applying Theorem (4.3). □

In the remainder of the chapter on 3-player games we assume that condition (4.4) holds. So from now on we assume that any 3-player game $(v,3)$ has a non-empty core.

4.2 Finding the Most Equitable Allocation in the Core

Recall from Chapter (3.1) that the 2-simplified game $(v_2, 3)$ of game $(v, 3)$ is the game where $v_2(C) = 0$ if $|C| < 2$ and $v_2(C) = v(C)$ otherwise. Recall further that the 2-core Ω^2 and $\mathbf{co}^2$ are the core and cooperative outcome, respectively, of the 2-simplified game.

Let $\mathcal{H}(C)$ be the set of states in the core where $\sum_i d^i = v(C_i)$, the set of states where the surplus of C is zero. Observe that in the 3-player set-up for $C \neq \emptyset, C \neq C^G$, $\mathcal{H}(C)$ is a set

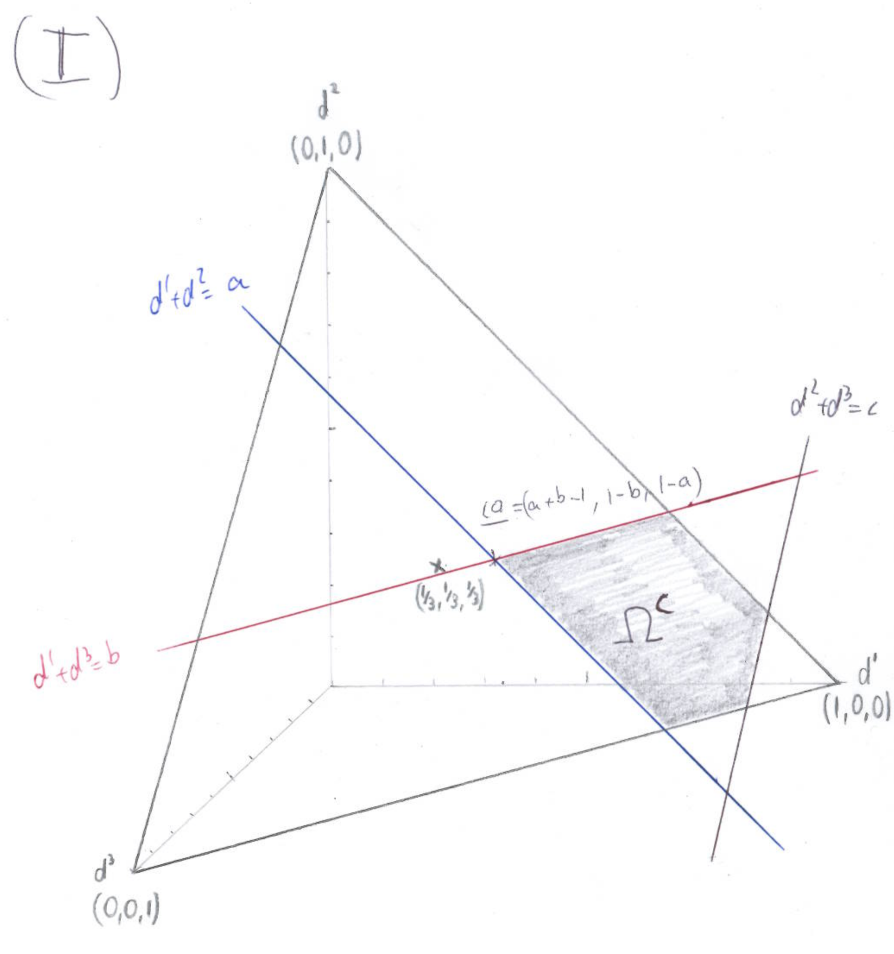


Figure 4.1 Graphical representation of core and outcome I in table (4.5), where $v(C_{23}) = c$ is small.

In each Figure 4.1, 4.2, 6.1, 6.2, 4.5 and 4.6 the cores of different games $(v, 3)$ are depicted, each corresponding to a different set of functions v, for v satisfying certain conditions. The location of the cooperative outcome in the core depends on these conditions on v. For example, in Figure 4.1, $\frac{a}{2} \geq 1 - b$ whereas in Figure 4.2, $\frac{a}{2} < 1 - b$. If

$\frac{a}{2} < 1-b$ then the state $\left(\frac{a}{2}, \frac{a}{2}, 1-a\right)$ is in the 2-core and so the cooperative outcome **co** lies in the interior of the hyperplane $\mathcal{H}(C_{12})$, depicted as the blue line $\{\mathbf{d} \mid d^1 + d^2 = a\}$ in Figure 4.2. However, in Figure 4.1, the state $\left(\frac{a}{2}, \frac{a}{2}, 1-a\right)$ is not in the 2-core, and so the cooperative outcome is in the intersection of $\mathcal{H}(C_{12})$ and $\mathcal{H}(C_{13})$ and given by $(a+b-1, 1-b, 1-a)$.

In Figure 6.1, like in Figure 4.2, the state $\left(\frac{a}{2}, \frac{a}{2}, 1-a\right)$ is in the 2-core. However, unlike in Figures 4.1 and 4.2, the 2-core strictly contains the core and $\left(\frac{a}{2}, \frac{a}{2}, 1-a\right) \in \Omega^2 \setminus \Omega^C$. Neither $\left(\frac{a}{2}, \frac{a}{2}, 1-a\right)$ nor any state, where C_{13} is binding, is in the core. Since $s_1 > \frac{a}{2}$ the hyperplane $\mathcal{H}(C_1)$ forms a boundary of the core and the cooperative outcome is the uniq

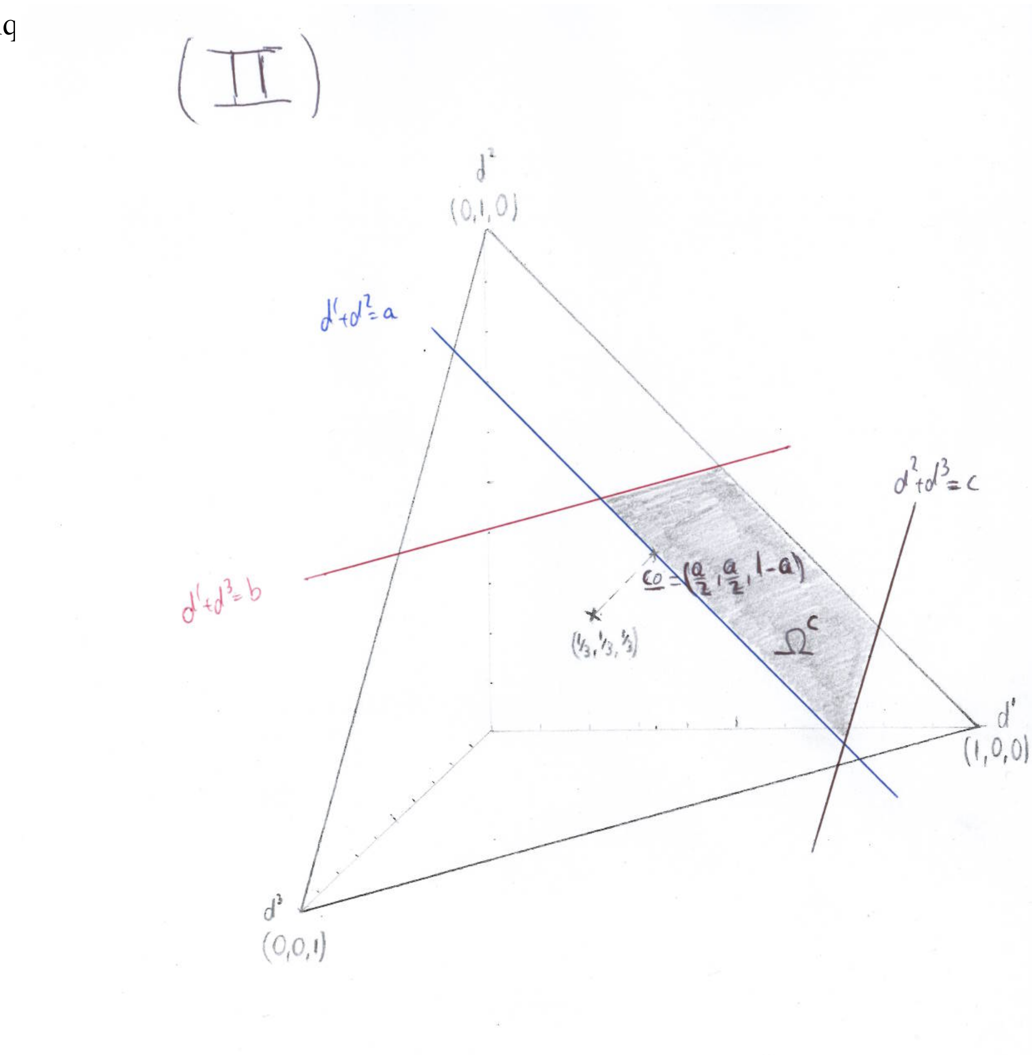


Figure 4.2 Graphical representation of core and outcome `II` in table (4.5).

The following algorithm finds **co** for each 3-player cooperative game. It first finds $\mathbf{co}^2$, then it checks if $\mathbf{co}^2$ is a member of the core. If so, $\mathbf{co} = \mathbf{co}^2$. Otherwise the core is strictly contained in the 2-core and so the algorithm calculates **co** taking into account that exactly one of $s_1 > co_1^2$, $s_2 > co_2^2$ or $s_3 > co_3^2$ can hold, say s_i and that **co** is then the state in the core with smallest Euclidean distance of the set of the set of states on the line $\mathcal{H}(C_i)$.

1. If $v(C_{12}) = a \leq \frac{2}{3}$, then

   (a) $\mathbf{co}^2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Otherwise, set $co_3^2 = 1 - a$. If $v(C_{13}) = b \leq 1 - \frac{a}{2}$ then

   (b) $\mathbf{co}^2 = \left(\frac{a}{2}, \frac{a}{2}, 1 - a\right)$. Otherwise

   (c) $\mathbf{co}^2 = (a + b - 1, 1 - b, 1 - a)$.

   Now, that the algorithm has identified $\mathbf{co}^2$, it finds **co** by checking whether one of the singleton coalitions, say C_i, is strictly feasible at $\mathbf{co}^2$. If so, **co** must be on $\mathcal{H}(C_i)$.

2. If $s_i < co_i^2$ for all $i \in \{1, 2, 3\}$, then $\mathbf{co} = \mathbf{co}^2$. This corresponds to cases $\mathbf{co} = (a + b - 1, 1 - b, 1 - a)$, $\mathbf{co} = \left(\frac{a}{2}, \frac{a}{2}, 1 - a\right)$ and $\mathbf{co} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ which correspond to outcomes `I`, `II` and `III` in table (4.5) respectively.

   If $s_1 > \mathbf{co}_1^2$, then $co_1 = s_1$. If $2a \geq 1 + s_1$, $co_2 = a - s_1$ this is case `VII` in table (4.5). Else $co_2 = co_3 = \frac{1 - s_1}{2}$ and this is outcome `IV` in table (4.5).

   If $s_2 > \mathbf{co}_2^2$, then $co_2 = s_2$ and this is outcome `V` in table (4.5). If $2a \geq 1 + s_2$, $co_1 = a - s_2$ else $co_1 = co_3 = \frac{1 - s_1}{2}$ and **co** is given by outcome `VI` in table (4.5).

   If $s_3 > \mathbf{co}_3^2$, then $\mathbf{co} = \left(\frac{1 - s_3}{3}, \frac{1 - s_3}{3}, s_3\right)$ and is given by outcome `VIII`.

All possible combinations of **co** are given in the table below:

	I	II	III	VII
co_1 co_2 co_3	$a+b-1$ $1-b$ $1-a$	$\frac{a}{2}$ $\frac{a}{2}$ $1-a$	$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$	s_1 $a-s_1$ $1-a$
conditions	$\frac{a}{2} \geq 1-b$ $s_1 \leq a+b-1$	$\frac{a}{2} \leq 1-b$ $s_1 \leq \frac{a}{2}$ $s_2 \leq \frac{a}{2}$	$a \leq \frac{2}{3}$ $s_1, s_2, s_3 \leq \frac{1}{3}$	$s_1 \geq a+b-1$ $s_1 \geq \frac{a}{2}$ $2a \geq 1+s_1$
	IV	V	VI	VIII
co_1 co_2 co_3	s_1 $\frac{1-s_1}{2}$ $\frac{1-s_1}{2}$	$a-s_2$ s_2 $1-a$	$\frac{1-s_2}{2}$ s_2 $\frac{1-s_2}{2}$	$\frac{1-s_3}{2}$ $\frac{1-s_3}{2}$ s_3
conditions	$s_1 \geq a+b-1$ $s_1 \geq \frac{a}{2}$ $2a \leq 1+s_1$	$\frac{a}{2} \leq 1-b$ $s_2 \geq \frac{a}{2}$ $2a \geq 1+s_2$	$\frac{a}{2} \leq 1-b$ $s_2 \geq \frac{a}{2}$ $2a \leq 1+s_2$	$a \leq \frac{2}{3}$ $s_3 \geq \frac{1}{3}$

(4.5)

Lemma 4.5. *The algorithm described above finds* **co***, the most equal allocation in the core.*

Proof: We know from Lemma (3.5) that if $\mathbf{co}^2$ is in the core, then $\mathbf{co}^2 = \mathbf{co}$.

By assumption, $a \geq b \geq c$. If $1-a \geq \frac{1}{3}$ then $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right) \in \Omega^2$ and so $\mathbf{co}^2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Otherwise, if $1-a < \frac{1}{3}$ then any state with $d^3 < 1-a$ is not in Ω^2. So the state with smallest $\mathbf{L}^2$-distance from $\mathbf{\frac{1}{3}}$ is on the hyperplane $\mathcal{H}(C_{12}) = \{\mathbf{d} \mid d^1 + d^2 = a\}$ and so $co_3 = 1-a$. If $1-b \geq \frac{a}{2}$, then $\left(\frac{a}{2}, \frac{a}{2}, 1-a\right)$ is on $\mathcal{H}(C_{12})$ and so $\mathbf{co}^2 = \left(\frac{a}{2}, \frac{a}{2}, 1-a\right)$. However, if $1-b < \frac{a}{2}$, then $\left(\frac{a}{2}, \frac{a}{2}, 1-a\right)$ is not on $\mathcal{H}(C_{12})$ and $\mathbf{co^2}$ will be in the intersection of the hyperplanes $\mathcal{H}(C_{12})$ and $\mathcal{H}(C_{13}) = \{\mathbf{d} \mid d^1 + d^3 = b\}$ and hence is $(a+b-1, 1-b, 1-a)$.

We now show by contradiction that at most one singleton coalition can be strictly feasible at $\mathbf{co}^2$. Given singleton coalitions C_i, C_j, by superadditivity $v(C_i) + v\left(C_j\right) \leq v\left(C_{ij}\right)$. Suppose there were two singleton coalitions with $v(C_i) > co_i^2$ and $v\left(C_j\right) > co_j^2$

but then $v(C_{ij}) > co_i^2 + co_j^2$ which contradicts that $\mathbf{co}^2 \in \Omega^2$. So at most one singleton coalition, say C_i, can have $v(C_i) > co_i^2$.

Suppose $i = 1$. Then $\mathbf{co}$ has to be on the hyperplane $\mathcal{H}(C_1)$ and so $co_1 = s_1$. If $2a \leq 1 + s_1$ then $\left(s_1, \frac{1-s_1}{2}, \frac{1-s_1}{2}\right) \in \Omega^2$. We now show that it is not possible that $s_2 > \frac{1-s_1}{2}$. Since $2a \leq 1 + s_1$, it holds that $\frac{1-s_1}{2} + s_1 = \frac{1+s_1}{2} > a$. Suppose $s_2 > \frac{1-s_1}{2}$. Then $s_2 + s_1 > a$ which is a contradiction. Hence it is not possible that $s_2 > \frac{1-s_1}{2}$. Since $a \geq b$ we conclude that $s_3 > \frac{1-s_1}{2}$ cannot hold. So $\left(s_1, \frac{1-s_1}{2}, \frac{1-s_1}{2}\right) = \mathbf{co}$.

If $2a > 1 + s_1$, $\mathbf{co}$ is in the intersection of the hyperplanes $\mathcal{H}(C_1)$ and $\mathcal{H}(C_{12})$ and so is $(s_1, a - s_1, 1 - a)$.

If $i = 2$ the identical argument holds with players 1 and 2 exchanged.

Suppose $i = 3$. The condition $s_3 > co_3^2$ can only hold if $co_3 < 1 - a$ and so $\mathbf{co}^2 = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$. Then $\mathbf{co}$ is on the hyperplane $\mathcal{H}(C_3)$ and so $co_3 = s_3$. We now show by contradiction, that, for $i \in \{1, 2\}$, $co_i > \frac{1-s_3}{2}$ cannot hold. Observe that, if $s_3 > \frac{1}{3}$, then $s_3 + \frac{1-s_3}{2} > \frac{2}{3}$. Suppose now $\mathbf{co}$ is in the intersection of $\mathcal{H}(C_3)$ and another hyperplane corresponding to a singleton coalition, say $\mathcal{H}(C_k)$. But then, by superadditivity, the worth of coalition C_{k3} is at least $s_3 + s_k$. This contradicts that $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ is in Ω^2. So we conclude that no other player k can have $s_k > \frac{1-s_3}{2}$ and so $\mathbf{co} = \left(\frac{1-s_3}{2}, \frac{1-s_3}{2}, s_3\right)$. □

One can show that the algorithm to find $\mathbf{co}$ in all cases when $a > \frac{2}{3}$ is summarized below:

For the 3-player game the cooperative outcome is given by

$$\begin{aligned}
co_1 = &\ \min[\max\left(a + b - 1, \tfrac{a}{2}, s_1\right), \max\left(a - s_2, \tfrac{1-s_2}{2}\right)] \\
co_2 = &\ \max[\min\left(1 - b, \tfrac{a}{2}, \max[a - s_1, \tfrac{1-s_1}{2}]\right), s_2] \\
co_3 = &\ \min\left(1 - a, \tfrac{1-s_1}{2}, \tfrac{1-s_2}{2}\right).
\end{aligned} \tag{4.6}$$

4.3 The Core is the Set of Recurrent States

Before we prove, that states outside the core are transient, and that the set of recurrent states is the core, we state and prove the following conditions, that are used repeatedly in the proofs of Lemma (4.7) and Theorem (4.13).

Let a player's cooperative distance $P^i(\mathbf{d}) = P^i$ be $d^i - co_i$ and let a coalition's surplus $CS^C(\mathbf{d}) = CS^C = \sum_{k \in C} d^k - v(C)$.

Let $\mathcal{C}^2$ be the set of 2-player coalitions.

Lemma 4.6. *1. For a state $\mathbf{d} \in \Omega^C$ feasibility of a coalition implies that the coalition is binding.*

2. *For $i \in \{1,2,3\}$ and for $\mathbf{d} \in \Omega^C$ it holds that $C'(\mathbf{d}(i))$ must be binding both at $\mathbf{d}$ and at $\mathbf{d}(i)$. Furthermore it holds that a player i cannot be a member of a coalition that is binding at $\mathbf{d}(i)$ and hence cannot be member of $C'(\mathbf{d}(i))$.*

3. *For a state $\mathbf{d} \in \Omega^C$, if C_i and C_j are both binding, then so is C_{ij}. Furthermore, for any state $\mathbf{d} \in \Omega^E$, if two singleton coalitions C_i and C_j are feasible, with at least one of them strictly feasible, then C_{ij} is strictly feasible.*

4. *If $C \in \mathcal{P}\left(C^G\right)$ is strictly feasible at $\mathbf{d}$ (or $\mathbf{d}(i)$), then $C^G \setminus C$ cannot be feasible at $\mathbf{d}$ (or $\mathbf{d}(i)$).*

5. *Suppose some 2-player coalition is strictly feasible at $\mathbf{d} \in \Omega^C$ or $\mathbf{d}(i)$, then $C'(\mathbf{d})$ or $C'(\mathbf{d}(i))$ cannot be a singleton coalition.*

In the succeeding enumeration the conditions of Lemma (4.6) are proved.

Proof:

1. A feasible coalition has $CS^C(\mathbf{d}) \leq 0$ and for a state in the core it holds that $CS^C(\mathbf{d}) \geq 0$. So feasibility of a coalition C in the core implies that coalition C is binding, that is $CS^C(\mathbf{d}) = 0$.

2. $C'(\mathbf{d})$ and $C'(\mathbf{d}(i))$ are by definition feasible coalitions, so if $\mathbf{d}$ is in the core, and so is $\mathbf{d}(i)$, they are binding coalitions. A coalition C, that is binding at $\mathbf{d}(i)$ and has i as member, must have $CS^C(\mathbf{d}) > CS^C(\mathbf{d}(i)) = 0$ which contradicts that $\mathbf{d} \in \Omega^C$.

3. Superadditivity of v implies that $v(C_i) + v\left(C_j\right) \leq v\left(C_{ij}\right)$. If C_i and C_j are binding then $d^i = v(C_i)$ and $d^j = v\left(C_j\right)$. So $d^i + d^j \leq v\left(C_{ij}\right)$ and, in the core, $d^i + d^j \geq v\left(C_{ij}\right)$ so $d^i + d^j = v\left(C_{ij}\right)$. The other statement is proved similarly.

4. We will proof this by contradiction. Suppose that $C \in \mathcal{P}\left(C^G\right)$ is strictly feasible at $\mathbf{d} \in \Omega^E$, that is $\sum_{i \in C} d^i < v(C)$, and suppose that $C^G \setminus C$ is feasible as well, that is $\sum_{j \notin C} d^j \leq v\left(C^G \setminus C\right)$. But then at $\mathbf{d}$, by superadditivity of v, it holds that $\sum_{i \in C} d^i + \sum_{j \notin C} d^j < v(C) + v\left(C^G \setminus C\right) \leq v\left(C^G\right) = 1$, contradicting the fact that $\mathbf{d} \in \Omega^E$. We

conclude that, at an efficient state $\mathbf{d}$, a coalition cannot be strictly feasible when its complement is feasible. At an intermediate inefficient state $\sum_{j\in\{1,2,3\}} d^j(i) = 1+\varepsilon$, and so the same argument applies to intermediate inefficient states $\mathbf{d}(i)$.

5. From item (4) it follows directly that the singleton coalition must be a subset of C. But in the ordering for C' a coalition is preferred to its subsets. So if a 2-player coalition is strictly feasible at $\mathbf{d}$ or $\mathbf{d}(i)$ then $C'(\mathbf{d})$ or $C'(\mathbf{d}(i))$ cannot be a singleton coalition.

□

In Figure 4.3 we depict a subset of the set of efficient states Ω^E. The set $\mathcal{H}(C_{12})$ is the set of states on the thick blue line, $\mathcal{H}(C_{13})$ is the set of states on the thick red line and $\mathcal{H}(C_{23})$ is the set of states that are on the thick black line. In this example a sketch of the cooperative outcome is the intersection of $\mathcal{H}(C_{12})$ and $\mathcal{H}(C_{13})$.

We depict three different states, $\mathbf{co}$, $\mathbf{d}$ and $\mathbf{d}^*$ and their neighbors. Neighbors, that the chain (V_t) can reach with positive probability, are depicted green. Neighbors, that cannot be reached, are grey. At $\mathbf{co}$, $C'(\mathbf{co}(2)) = C_{13}$, $C'(\mathbf{co}(3)) = C_{12}$ and $C'(\mathbf{co}(1)) = \emptyset$, so the only 'green neighbors' are states $\mathbf{d}$ where $d^1 = co_1 + \varepsilon$. At $\mathbf{d}$, for $i \in \{1,2,3\}$, $C'(\mathbf{d}(i)) = \emptyset$ and so all neighbors are green. At $\mathbf{d}^*$, $C'(\mathbf{d}^*(2)) = C_{13}$ and so $p^{2,1}_{\mathbf{d}^*} = p^{2,3}_{\mathbf{d}^*} = 0$ neighbors $\mathbf{d}^*(2,1)$ and $\mathbf{d}^*(2,3)$ are grey.

Lemma 4.7. *Any state $\mathbf{d} \in \Omega^E \setminus \Omega^C$ is transient.*

In the first part of the proof we show that once in Ω^C, the chain V_t cannot leave Ω^C.

In the second part of the proof we show that from any state $\mathbf{d} \in \Omega^E \setminus \Omega^C$ the chain V_t has a path of positive probability to Ω^C.

Proof: Suppose $V_t \in \Omega^C$. If $V_{t+1} \notin \Omega^C$, then there must exist a pair of players i, j, and a coalition C^* that is binding at $\mathbf{d}$ and $\mathbf{d}(i)$, that has $CS^{C^*}(\mathbf{d}(i,j)) < 0$ with $j \in C^*$ and with $p^{i,j}_{\mathbf{d}} > 0$. For $p^{i,j}_{\mathbf{d}} > 0$ and $CS^{C^*}(\mathbf{d}(i,j)) < 0$ to hold, player i cannot be member of C^* and $C^* \neq C'(\mathbf{d}(i))$.

Let $l \in \{1,2,3\}, l \neq i, l \neq j$. Since by Lemma (4.6). (2) it holds that $i \notin C'$, and $j \notin C'$ by assumption, it follows that $C' = C_l$.

By Lemma (4.6.2) it holds that $i \notin C^*$, and since $C^* \neq C'(\mathbf{d}(i))$ it holds that $v(C^*) \leq v(C_l)$. So C^* can only be C_j. But since C_j and C_l are binding at $\mathbf{d}(i)$, by Lemma (4.6.3)

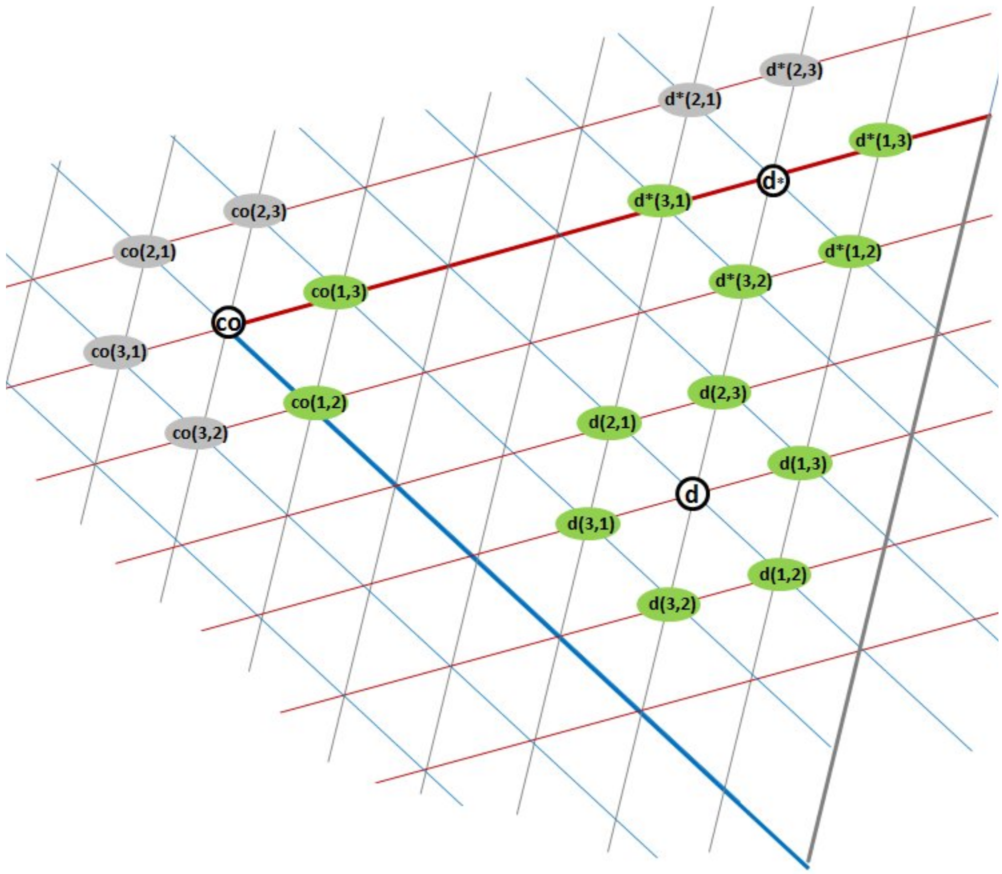


Figure 4.3 Sketch of a subset of Ω^E for a cooperative game process for a 3-player game with different states $\mathbf{d}$ and their neighbors $\mathbf{d}(i,j)$ for $i,j \in \{1,2,3\}$.

that implies that C_{jl} is binding at $\mathbf{d}(i)$ and so $C' = C_{jl}$. Since j,l,i were chosen arbitrarily, that implies that the chain V_t cannot leave Ω^C.

Now we show that the chain (V_t) will move from any state in Ω^E into Ω^C. We first show that from any state not in the 2-core, there is a path of positive probability into the 2-core. Then we show that from any state in the 2-core, there is a path of positive probability into the core.

Suppose at $\mathbf{d} \in \Omega^E$ there exists a strictly feasible 2-player coalition $C^A = C_{ij}$. We need to show that there exists a state $\mathbf{d}(r,k)$ for $r \in \{i,j\}$, such that k is not member of a binding 2-player coalition at $\mathbf{d}(r)$ and $p^{r,k} > 0$. Suppose no other two player coalition is binding at $\mathbf{d}$. Then by Lemma (4.6.4) C_k cannot be binding at $\mathbf{d}(r)$ and so $p^{i,k} > 0$ and $p^{j,k} > 0$. Suppose one other 2-player coalition is binding at $\mathbf{d}$, say C_{ik}. Then for $r = i$ it holds that C_{rk} is not binding at $\mathbf{d}(r)$, and so $CS^{C_{rk}}(\mathbf{d}(r,k)) \geq 0$, and $CS^{C_{rj}}(\mathbf{d}(r,k)) > CS^{C_{rj}}(\mathbf{d})$. Again C_k cannot be binding at $\mathbf{d}(r)$ by Lemma (4.6.4) and so $p^{r,k} > 0$.

Suppose that C_{jk} is binding at $\mathbf{d}$ simultaneously while C_{ij} is strictly feasible and C_{jk} is binding. We will show that this contradicts (4.4) which we assumed to hold to guarantee that the core is non-empty. If $d^i + d^j < v\left(C_{ij}\right)$, $d^i + d^k = v(C_{ik})$ and $d^j + d^k = v\left(C_{jk}\right)$ all hold simultaneously, then $2\left(d^i + d^j + d^k\right) < v\left(C_{ij}\right) + v(C_{ik}) + v\left(C_{jk}\right)$. But since $\mathbf{d} \in \Omega^E$ it follows that $2\left(d^i + d^j + d^k\right) = 2 < v\left(C_{ij}\right) + v(C_{ik}) + v\left(C_{jk}\right)$ and that contradicts (4.4). So C_{jk} cannot be binding at $\mathbf{d}$ simultaneously while C_{ij} is strictly feasible and C_{jk} is binding. We conclude that there is a path of positive probability into the 2-core.

Now we will show that from any state in the 2-core, there is a path of positive probability into the core. By Lemma (4.6.3) we know that if a singleton coalition is strictly feasible at $\mathbf{d}$, but no 2-player coalition is strictly feasible, then no other singleton coalition can be feasible at $\mathbf{d}$. The same holds directly for $\mathbf{d}(i)$ for $i \in \{1,2,3\}$. Let C_i be the strictly feasible singleton coalition at $\mathbf{d}$. We know from Lemma (4.6.4) that C_{jk} cannot be binding at $\mathbf{d}(i)$. We know as well from Lemma (4.6.2) that i cannot be member of a binding coalition at $\mathbf{d}(i)$. So there is no 2-player coalition that can be binding at $\mathbf{d}(i)$ and no other singleton coalition and so $p^{i,s} > 0$ for all $s \in \{1,2,3\}$ and so there is a path of positive probability from the 2-core into the core.

So there is a path of positive probability from any state into the 2-core and from any state in the 2-core, there is a path of positive probability into the core. From any state in the core the chain (V_t) cannot leave the core. Hence each efficient state outside the core is transient.

□

Let an *interior state* be a state where no coalition is binding and let a *boundary state* be a state in the core where at least one coalition is binding.

The core is a union of interior states and boundary states. Boundary states of the core are classified as *vertex states*, where at least two coalitions are binding, or *edge* states, where exactly one coalition is binding.

We first describe two special cases of a game $(v,3)$ where the core does not contain any interior states.

For a 3-player game $(v,3)$, if there exists a state $\mathbf{d}$ where all 2-player coalitions are binding, then $\Omega^C = \mathbf{d}$. By Lemma (4.6.3), if for a game $(v,3)$ there exists at a state $\mathbf{d}$, where all three singleton coalitions are binding, then at $\mathbf{d}$ all 2-player coalitions are binding and again the core consists of a single point.

If $v\left(C_{ij}\right) = 1 - v\left(C^G \setminus C_{ij}\right)$ and $s_i + s_j < v\left(C_{ij}\right)$ then the core consists of a set of states that can be joined by a line.

We will first show that any two interior states are connected through a path of interior states. Since for interior states all $p^{i,j} > 0$ for any combination of i,j with $i,j \in [1,2,3]$, this implies that there is a path of positive probability between any two interior states.

Figure 4.4 depicts the interior of the core for a cooperative game process for a general 3-player game. To show that the interior of the core is connected, we show that any two interior states are connected. To show that states $\mathbf{d}^*$ and $\mathbf{d}^{**}$ in the interior of the core are connected, we show that the state $\mathbf{d}^{***}$ has to lie in the interior of the core as well. Intuitively, in Lemma (4.8) we argue, that any hyperplane representing a binding coalition, that intersects either of the green lines in 4.4, implies that $\mathbf{d}^{**}$ is not in the interior of the core, which contradicts the assumption that both $\mathbf{d}^*$ and $\mathbf{d}^{**}$ are in the interior of the core.

Then we will show that for any two states on the boundary there is a path of positive probability in either direction between them.

Finally we show that there is a state on the boundary that has positive probability of going to an interior state. The transition dynamics of interior states do not change until a state on the boundary is reached, so there exists always an interior state which has positive probability of going to a state on the boundary.

Lemma 4.8. *The set of recurrent states for all games $(v,3)$ is the core.*

Proof: Let d^* and d^{**} be any two interior states of the core.

Suppose now that $d^{1*} = d^{1**} + k_1\varepsilon$, $d^{2*} = d^{2**} + k_2\varepsilon$ and $d^{3*} = d^{3**} + k_3\varepsilon$, for $i \in \{1,2,3\}$ let $k_i \in \mathbb{Z}$ and $k_1 + k_2 + k_3 = 0$. Wlog we assume that $k_1 > 0$ and $k_2 < 0$ and $k_3 < 0$. We claim that the state $\mathbf{d}^{***} = \left(d^{1**} + (k_1 - k_2)\varepsilon, d^{2**}, d^{3**} + k_3\varepsilon\right)$ is in the interior of the core. Then, by convexity of the core, any state on the line joining d^* and d^{***} and any state on the line joining d^{**} and d^{***} is in the interior of the core.

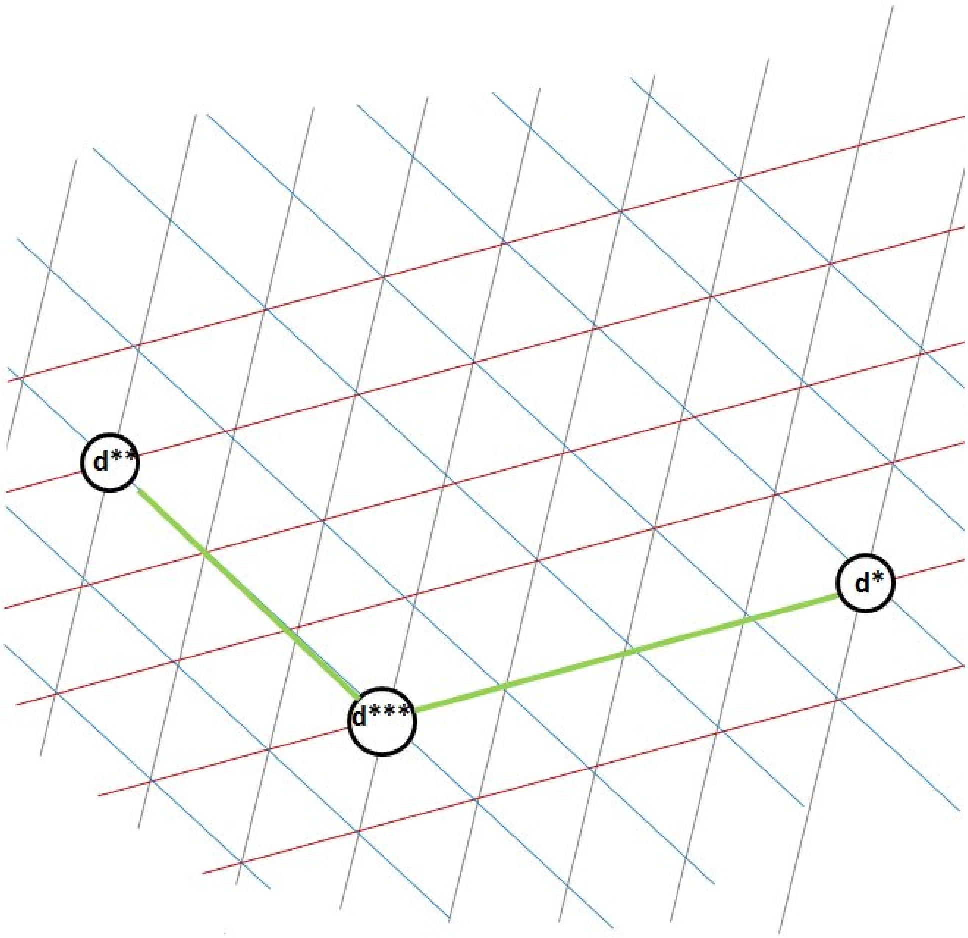


Figure 4.4 Sketch of the interior of the core for a cooperative game process for a 3-player game with different states $\mathbf{d}^{*}$, $\mathbf{d}^{**}$ and $\mathbf{d}^{***}$.

We now prove by contradiction that $\mathbf{d}^{***}$ is in the interior of the core. Suppose that $\mathbf{d}^{***}$ is not in the interior of the core. Then there must be a coalition C such that $\sum_{i \in C} d^i \leq v(C)$. Observe that $C \neq C_1$ since $d^{1**} < d^{1***}$ and $\mathbf{d}^{**}$ is by assumption in the interior of the core. Furthermore $C \neq C_{13}$ since $d^{1**} + d^{3**} < d^{1***} + d^{3***}$. Observe that for no other coalition the sum of demands decreases between $\mathbf{d}^{*}$ and $\mathbf{d}^{***}$. As well, $d^{2***} < d^{2*} < 1$ and so the state $\mathbf{d}^{***}$ lies in the interior of the core. Since the choice of players was arbitrary, this shows that all states in the interior of the core are connected.

Now we show that for a game $(v, 3)$ there is a path of positive probability along the boundary states of the core.

We first deal with the case where the core is a set of states that can be joined by a line. Suppose it holds that $v\left(C_{ij}\right) = 1 - v\left(C^G \setminus C_{ij}\right)$ where $s_i + s_j < v\left(C_{ij}\right)$. Then, if $\mathbf{d}$ is an edge state where $d^i + d^j = v\left(C_{ij}\right)$ and no other coalition apart from $C^G \setminus C_{ij}$ is binding, states $\mathbf{d}(i,j)$ and $\mathbf{d}(j,i)$ are boundary states. It holds that $p^{i,j} = \frac{1}{3}\frac{d^j}{d^i+d^j+\varepsilon} > 0$ and $p^{j,i} = \frac{1}{3}\frac{d^j}{d^i+d^j+\varepsilon} > 0$ and so both $\mathbf{d}(j,i)$ and $\mathbf{d}(i,j)$ can be reached with positive probability. Suppose now that the chain (V_t) is in a vertex state where $d^i + d^k = v(C_{ik})$. Then $p^{j,i} = 0$ and $p^{i,j} = \frac{1}{3}\frac{d^j}{d^i+d^j+\varepsilon} > 0$. If additionally $d^i = s_i$, since by assumption $d^k = s_k$, it holds as well that $d^i + d^k = v(C_{ik})$ and the argument is the same. For other vertex states very similar arguments apply.

Suppose that the core has at least one interior point. We want to show that there is a path of positive probability in either direction between any two boundary states.

Suppose the chain (V_t) is in an edge state. Observe that in any edge state, if a member of the binding coalition increases his demand, the payable coalition is the empty set and so $p_{\mathbf{d}}^{i,j} > 0$ for all $i,j \in \{1,2,3\}$.

Suppose the chain (V_t) is in a vertex state. The two edges, that the vertex state connects, are either two edges corresponding to a binding singleton coalition, two edges corresponding to binding 2-player coalitions or the vertex could connect an edge corresponding to a binding singleton coalition and an edge corresponding to a binding 2-player coalition.

Suppose the vertex state links two 2-player coalitions. If the player in the intersection of the two coalitions increases his demand, no coalition is feasible any longer and so both remaining players in either coalition can reduce demands and so the chain can move from the vertex states to either of the two neighboring edge states. These dynamics are not changed if the singleton coalition of the player in the intersection is binding.

Suppose the vertex state links two edges where singleton coalitions, C_i and C_j respectively, are binding. The respective neighboring edge states are $\mathbf{d}(i,k)$ and $\mathbf{d}(j,k)$ respectively. $C'(\mathbf{d}(i)) = C_j$ and so $p_{\mathbf{d}}^{i,k} > 0$. Similarly $C'(\mathbf{d}(j)) = C_i$ and so $p_{\mathbf{d}}^{j,k} > 0$. So with positive probability the chain (V_t) moves to either of the adjoint edge states.

Suppose the vertex state links two edges, one where a singleton coalition is binding, and one where a 2-player coalition is binding. We are now analyzing the case where the core has a non-empty set of interior states. This implies that the singleton coalition is not the complement of the 2-player coalition and so must be a subset. If the player in the singleton coalition increases his demand, the payable coalition is the empty set and so all both, the remaining player in the 2-player coalition or the third player, that is not in the 2-player coalition, reduce demands with positive probability . Hence the chain can move from the vertex state to both neighboring edge states.

From any edge state, if a member of the coalition increases his demand, then $C' = \emptyset$ and so each player can reduce demands and hence the chain will move to an interior state with positive probability. For any interior state $\mathbf{d}$ adjoint to an edge state, transitions to $\mathbf{d}(i, j)$ for all $i, j \in \{1,2,3\}$ are possible and so the chain can move with positive probability to a state on the boundary.

We know from Lemma (4.7), that any state outside the core is transient. We conclude that the set of recurrent states is the core. □

The next lemma states that the player with largest demands at a state in the core must have demands that are at least equal to his cooperative outcome value. We use this lemma repeatedly in subsection 4.4 and Chapter 6 Section 6.2. Intuitively, a player with demands above his cooperative outcome adds a negative contribution to the drift when decreasing demands and a positive contribution when increasing demands. In this lemma we conduct a simple case analysis of the different possible cooperative outcomes given in (4.5) in Section 4.2.

Lemma 4.9. *Let* $\mathbf{d}$ *be any state in the core. Let player* l *be the player with largest demands at* $\mathbf{d}$*. Then* $d^l \geq co_l$ *and for any other player* k *with* $d^k \leq co_k$ *it holds that* $co_l \geq co_k$*.*

Proof: By our ordering of the players, we assumed in (4.3) that $a \geq b \geq c$. If $a \leq \frac{2}{3}$, by (4.5), this implies that $\mathbf{co} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$ or $\mathbf{co} = \left(\frac{1-s_3}{2}, \frac{1-s_3}{2}, s_3\right)$. For the former, the result follows trivially. In the latter if the player with largest demands is player 3, the result follows from the fact that $\mathbf{d} \in \Omega^C$. If $l = 1$ or $l = 2$, the result follows directly from the fact, that players 1 and 2 have the same cooperative outcome value.

Suppose now that $a > \frac{2}{3}$. For any state $\mathbf{d} \in \Omega^C$ it holds that $d^1 + d^2 \geq a$ and $d^1 + d^2 + d^3 = 1$. So for $\mathbf{d} \in \Omega^C$ it holds that $d^3 < \frac{1}{3}$ and so l, the player with largest demands, has to be either player 1 or player 2.

Let player m be the player in C_{12} with the higher cooperative outcome. From 4.2, we see that m can have

$$
\begin{array}{llcll}
(\texttt{I}) & co_m & = & s_m & \texttt{where} \quad s_m > \frac{a}{2} \\
(\texttt{II}) & co_m & = & \frac{a}{2} & \\
(\texttt{III}) & co_m & = & a+b-1 & \texttt{with} \quad \mathbf{co} = \quad (a+b-1, 1-b, 1-a)
\end{array}
$$

In case $(\texttt{I})$ it holds that $d^m \geq s_m$ as $\mathbf{d} \in \Omega^C$.

In case $(\texttt{II})$ if $d^m > d^j$, $m, j \in [1,2]$ then $d^m > \frac{a}{2}$ since $d^1 + d^2 \geq a$.

In case $(\texttt{III})$ it must be that $m = 1$. Let $x \in \mathbf{Z}$ and set $d^1 = a + b - 1 - x\varepsilon$. Suppose now that $d^1 < co_1$, that is, suppose $x > 0$. But then $d^2 \geq 1 - b + x\varepsilon$ for $d^1 + d^2 \geq a$ to hold and $d^3 \geq 1 - a + x\varepsilon$ for $d^1 + d^3 \geq b$ to hold but then $d^1 + d^2 + d^3 \geq a + b - 1 - x\varepsilon + 1 - b + x\varepsilon + 1 - a + x\varepsilon \geq 1 + x\varepsilon$ which is a contradiction since for states in the core $\sum_i d^i = 1$. Hence $x \leq 0$ and so $d^m \geq co_m$ holds for all $\mathbf{d} \in \Omega^C$.

So we have shown that m, the player with the largest cooperative outcome co_m, must have $d^m \geq co_m$. If there is any other player, say $k \in [1,2,3]$ with $d^k > d^m$ then $d^k > co_k$ as $co_m > co_k$ by assumption. □

Lemma 4.10. *Given a game $(v, 3)$, let m be a player such that $co_m \geq co_i$ for $i \in \{1,2,3\}$. Then at any state in the core it holds that $d^m \geq co_m$.*

Proof: The result has been proved in the proof of Lemma (4.9). □

4.4 The Markovian Cooperative Equilibrium is the Most Equitable Allocation in the Core

We define, for $i = 1,2,3$,

$$
Dr(V, \mathbf{d})_i = \frac{1}{3} \mathbb{E}[D(V_{t+1}) - D(V_t) \mid V_t = \mathbf{d}, X_{n_t+1} = \mathbf{d}(i)]. \tag{4.7}
$$

The quantity $Dr(V, \mathbf{d})_i$ is the contribution to the drift from the case when player i increases his demand. Observe that $\sum_{i=1}^{3} Dr(V, \mathbf{d})_i = Dr(V, \mathbf{d})$.

Lemma 4.11. *For any state $\mathbf{d} \in \Omega^E$, and for any player i, j in $\{1,2,3\}$ it holds that $\mid d^i - co_i \mid^2 + \mid d^j - co_j \mid^2 \geq \frac{1}{3} D(\mathbf{d})$.*

Proof: Let z be the other player. Observe that $\mid d^i - co_i \mid + \mid d^j - co_j \mid \geq \mid d^z - co_z \mid$ since $\sum_{k=1}^{3} d^k - co_k = 0$. So

$$2\left(\mid d^i - co_i \mid^2 + \mid d^j - co_j \mid^2\right) \geq \mid d^i - co_i \mid^2 + \mid d^j - co_j \mid^2 + 2 \mid d^i - co_i \mid \mid d^j - co_j \mid$$

and $\mid d^i - co_i \mid^2 + \mid d^j - co_j \mid^2 + 2 \mid d^i - co_i \mid \mid d^j - co_j \mid = \left(\mid d^i - co_i \mid^2 + \mid d^j - co_j \mid^2\right)$. Observe further that $\left(\mid d^i - co_i \mid^2 + \mid d^j - co_j \mid^2\right) \geq \mid d^z - co_z \mid^2$ and so

$$3\left(\mid d^i - co_i \mid^2 + \mid d^j - co_j \mid^2\right) \geq D(\mathbf{d}) \text{ and hence } \mid d^i - co_i \mid^2 + \mid d^j - co_j \mid^2 \geq \tfrac{1}{3} D(\mathbf{d}).$$

□

To analyze the contribution to the drift at $\mathbf{d}$ from a transition to a particular neighbor $\mathbf{d}(i,j)$ there are contributions from both player i that increases his demand and from player j that decreases demands. If $d^i < co_i$ and $d^j > co_j$, both contributions to the drift are negative. The following Lemma bounds the drift $Dr(V,\mathbf{d})_i$ if $d^i \leq co_i$, $d^j \geq co_j$ and $d^z \geq co_z$.

Lemma 4.12. *For any state $\mathbf{d} \in \Omega^C$ where $d^i \leq co_i$, $d^j \geq co_j$ and $d^z \geq co_z$ it holds that $Dr(V_t,\mathbf{d})_i \leq \frac{-2\varepsilon}{9(1+\varepsilon)} D(\mathbf{d}) + 2\varepsilon^2$. Furthermore, if $d^i \leq co_i$ and $d^j \leq co_j$ and $d^z \geq co_z$ and only transitions to new efficient states $\mathbf{d}(i,z)$ and $\mathbf{d}(j,z)$ are possible, then $Dr(V_t,\mathbf{d}) \leq \frac{-2\varepsilon}{9(1+\varepsilon)} D(\mathbf{d}) + 2\varepsilon^2$.*

Proof: Observe that $Dr(V_t,\mathbf{d})_i = \frac{2\varepsilon}{3}\left(p^{i,j}\left(d^i - co_i - \left(d^j - co_j\right)\right) + p^{i,z}\left(\left(d^i - co_i\right) - \left(d^j - co_j\right)\right)\right) + 2\varepsilon^2 \leq \frac{-2\varepsilon}{3(1+\varepsilon)}\left(d^j \mid d^j - co_j \mid + d^z \mid d^z - co_z \mid\right) + 2\varepsilon^2$.

As $d^j \geq co_j$ it holds that $d^j \geq \left(d^j - co_j\right)$ and as $d^z \geq co_z$ it holds that $d^z \geq (d^z - co_z)$. So $Dr(V_t,\mathbf{d})_i \leq \frac{-2\varepsilon}{3(1+\varepsilon)}\left(\left(d^j - co_j\right)^2 + (d^z - co_z)^2\right) + 2\varepsilon^2$.

Now from Lemma (4.11) it follows that $Dr(V_t,\mathbf{d})_i \leq \frac{-2\varepsilon}{3(1+\varepsilon)} \frac{1}{3} D(\mathbf{d}) + 2\varepsilon^2$ and so $Dr(V_t,\mathbf{d})_i \leq \frac{-2\varepsilon}{9(1+\varepsilon)} D(\mathbf{d}) + 2\varepsilon^2$.

The second part of the lemma is proved similarly.

□

Let $\mathbf{co}(v,3)$ be the cooperative outcome of game $(v,3)$.

We have shown that all states not in Ω^C are transient. Now we show that the drift for all $\mathbf{d}$ in Ω^C is at most $-\delta + 2\varepsilon^2$ where $\delta(\mathbf{d},\mathbf{co}) > 0$ is proportional to the $\mathbf{L}^2$ distance of $\mathbf{d}$ and $\mathbf{co}$.

Theorem 4.13. *For any state* $\mathbf{d} \in \Omega^C$ *it holds that* $Dr(\mathbf{d}) \leq -\frac{2\varepsilon}{9(1+\varepsilon)} D(\mathbf{d}) + 2\varepsilon^2$.

Given a game $(v,3)$ let $\mathcal{C}^{\mathbf{co}}(v,3)$ be the set of coalitions that are binding at **co**. Let the $\mathcal{C}^{\mathbf{co}}$-simplified game $(v^{\mathbf{co}},3)$ be the game where $v^{\mathbf{co}}(C) = 0$ if $C \notin \mathcal{C}^{\mathbf{co}}(v,3)$ and $v^{\mathbf{co}}(C) = v(C)$ otherwise. Observe that the $\mathcal{C}^{\mathbf{co}}$-simplified game $(v^{\mathbf{co}},3)$ is not necessarily superadditive.

Lemma 4.14. *The cooperative outcome of the game* $(v^{\mathbf{co}},3)$ *is* **co***, the cooperative outcome of the original game* $(v,3)$.

Proof: We will show this by contradiction. Let $\Omega^{\mathbf{co}}$ be the core of $(v^{\mathbf{co}},3)$. Suppose that there is a state $\mathbf{co}'$ in $\Omega^{\mathbf{co}}$ with smaller $\mathbf{L}^2$-distance from $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ than **co**. Let S be the line between **co** and $\mathbf{co}'$ and let Ω^P be the set of states in Ω^E that lie on S. Observe that by construction each state in Ω^P has smaller $\mathbf{L}^2$-distance from $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ than **co**.

When constructing $\Omega^{\mathbf{co}}$ from Ω^C, we remove finitely many hyperplanes. Let K_{C^*} be the hyperplane that cuts S closest to **co**. By assumption, C^* is in $\mathcal{C}^{\mathbf{co}}$ and so K_{C^*} cannot include **co**. But then, for all sufficiently small ε, there is a point in the intersection of Ω^C and Ω^P and so is closer to $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ than **co**, contradicting the assumption that **co** is the closest point to $\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$ in Ω^C. We conclude that $\mathbf{co}' = \mathbf{co}$. □

Lemma 4.15. *If at* $\mathbf{d} \in \Omega^C$ *there are at least* 2 *binding coalitions* $C \in \mathcal{C}^{\mathbf{co}}(v,3)$*, then* $\mathbf{d} = \mathbf{co}$.

Proof: Observe that for a game $(v,3)$ the set of states, where a coalition is binding, is a set of points that can be joined by a line. The intersection of two such sets of states is at most one state. So if at $\mathbf{d} \in \Omega^C$ there are at least 2 binding coalitions $C \in \mathcal{C}^{\mathbf{co}}(v,3)$, then $\mathbf{d} = \mathbf{co}$. □

We are to prove that at each state $\mathbf{d} \in \Omega^C$ there is a drift that is proportional to the distance $D(\mathbf{d})$.

The first lemma states that this is true if no coalition is binding at **d**.

The second lemma shows that this is true for states where no coalition in $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding at **d**.

Finally the third lemma shows that this is true for states where one coalition in $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding.

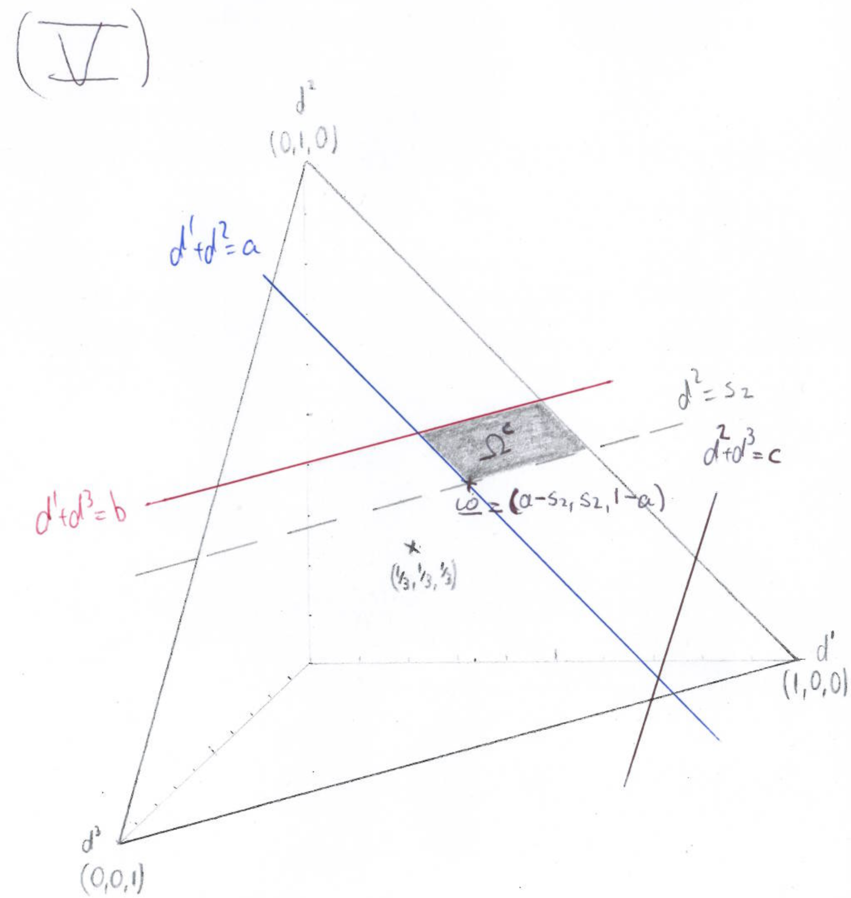


Figure 4.5 Graphical representation of core and outcome V in table (4.5).

Lemma 4.16. *Suppose at* $\mathbf{d} \in \Omega^C$ *no coalition is binding. Then*

$$Dr(V_t, \mathbf{d}) < \frac{2\varepsilon}{3(1+\varepsilon)}(-D(\mathbf{d})) + 2\varepsilon^2. \tag{4.8}$$

Proof: The drift is given by

$$\sum_{i=1}^{3}\sum_{j=1}^{3} p_{\mathbf{d}}^{i,j}\left(-D(\mathbf{co},\mathbf{d}) + D(\mathbf{co},\mathbf{d}(i,j))\right). \tag{4.9}$$

If no coalition is binding, the payable coalition is the empty set and so $p_{\mathbf{d}}^{i,j} = \frac{d^j}{3(1+\varepsilon)}$ if $i \neq j$ and $p_{\mathbf{d}}^{i,j} = \frac{d^j+\varepsilon}{3(1+\varepsilon)}$ otherwise. So

$$Dr\left(V_t,\mathbf{co}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{d^j}{3\left(1+\varepsilon\right)}\left(-\left(co_i-d^i\right)^2-\left(co_j-d^j\right)^2+\left(co_i-\left(d^i+\varepsilon\right)\right)^2+\left(co_j-\left(d^j-\varepsilon\right)\right)^2\right)$$
$$+\sum_{i=1}^{3}\frac{\varepsilon}{3\left(1+\varepsilon\right)}\left(-\left(co_i-d^i\right)^2-\left(co_i-d^i\right)^2+\left(co_i-\left(d^i+\varepsilon\right)\right)^2+\left(co_i-\left(d^i-\varepsilon\right)\right)^2\right)$$
$$Dr\left(V_t,\mathbf{co}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3}\frac{d^j}{3\left(1+\varepsilon\right)}\left[-2\varepsilon\left(co_i-d^i\right)+2\varepsilon\left(co_j-d^j\right)+2\varepsilon^2\right]+\sum_{i=1}^{3}\frac{\varepsilon}{3\left(1+\varepsilon\right)}2\varepsilon^2$$
$$Dr\left(V_t,\mathbf{co}\right) = \frac{2\varepsilon}{3\left(1+\varepsilon\right)}\left[-\sum_{i=1}^{3}\sum_{j=1}^{3}d^j\left(co_i-d^i\right)+\sum_{i=1}^{3}\sum_{j=1}^{3}d^j\left(co_j-d^j\right)\right]+2\varepsilon^2$$
$$Dr\left(V_t,\mathbf{co}\right) = \frac{2\varepsilon}{3\left(1+\varepsilon\right)}\sum_{j=1}^{3}\left[\frac{1}{2}\left(d^j-co_j\right)\left(co_j-d^j\right)+\frac{1}{2}\left(d^j+co_j\right)\left(co_j-d^j\right)\right]+2\varepsilon^2$$

it follows that

$$Dr\left(V_t,\mathbf{co}\right) = -\frac{\varepsilon}{\left(1+\varepsilon\right)}\sum_{j=1}^{3}\left(d^j-co_j\right)^2+\frac{\varepsilon}{1+\varepsilon}\left[\sum_{j=1}^{3}\left(co_j\right)^2-\sum_{j=1}^{3}\left(d^j\right)^2\right]+2\varepsilon^2 \tag{4.10a}$$

So

$$Dr\left(V_t,\mathbf{d}\right) < \frac{2\varepsilon}{3\left(1+\varepsilon\right)}\left(-D\left(\mathbf{d}\right)\right)+2\varepsilon^2. \tag{4.11}$$

□

Let $Dr^{\mathbf{co}}\left(\mathbf{d},\mathbf{co}\right)$ be the drift at state $\mathbf{d}$ for the $\mathcal{C}^{\mathbf{co}}$-simplified game.

We will compare the drift of a state $\mathbf{d} \in \Omega^C$ where no coalition in $\mathcal{C}^{\mathbf{co}}\left(v,3\right)$ is binding with the drift at $\mathbf{d}$ for the game $\left(v^{\mathbf{co}},3\right)$. We use Lemma (4.14) to deduce that $\mathbf{co}\left(v,3\right) = \mathbf{co}\left(v^{\mathbf{co}},3\right)$. The $p_{\mathbf{d}}^{i,j}$ differ between the two drifts calculations. The basic idea is, that if at $\mathbf{d}\left(i\right)$ a coalition is the payable coalition in $\left(v,3,\varepsilon\right)$ but the payable coalition in $\mathbf{d}\left(i\right)$ is the empty set for $\left(v^{\mathbf{co}},3,\varepsilon\right)$, then players in that coalition in the payable coalition will not reduce demands in $\left(v,3,\varepsilon\right)$ and we will show that the drift will be smaller for the cooperative game process $\left(v,3,\varepsilon\right)$ than for the cooperative game process $\left(v^{\mathbf{co}},3,\varepsilon\right)$. Since we know a bound on the drift for $\left(v^{\mathbf{co}},3,\varepsilon\right)$ from Lemma (4.16), we know that the same bound holds for $\left(v,3,\varepsilon\right)$. There are some special cases, and we analyze them as well in the following lemma.

Lemma 4.17. *Suppose at* $\mathbf{d} \in \Omega^C$ *no coalition in* $\mathcal{C}^{\mathbf{co}}(v,3)$ *is binding. Then*

$$Dr(V_t,\mathbf{d}) < \frac{2\varepsilon}{9(1+\varepsilon)}(-D(\mathbf{d})) + 2\varepsilon^2. \tag{4.12}$$

Proof: By Lemma (4.16), if no coalition is binding at $\mathbf{d}$, then $Dr(V_t,\mathbf{d}) < \frac{2\varepsilon}{3(1+\varepsilon)}(-D(\mathbf{d})) + 2\varepsilon^2$.

Suppose at $\mathbf{d}$ only a 2-player coalition $C \notin \mathcal{C}^{\mathbf{co}}$ is binding. Let i be the player not in C and j,k be the remaining players. Then, if j or k increase demands, no coalition is feasible and the transition probabilities for v and $v^{\mathbf{co}}$ are identical and so $Dr_j^{\mathbf{co}}(V_t,\mathbf{co}) = Dr_j(V_t,\mathbf{co})$ and $Dr_k^{\mathbf{co}}(V_t,\mathbf{co}) = Dr_k(V_t,\mathbf{co})$.

If i increases his demand however, the transition probabilities differ and now we show that $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) > Dr_i(V_t,\mathbf{co})$.

For $(v,3)$, the probability for i to decrease is 1, $p_{\mathbf{d}}^{i,i} = 1$, and so $Dr_i(V_t,\mathbf{co}) = 0$. Since $C_{jk} \notin \mathcal{C}^{\mathbf{co}}(v,3)$ it holds that $d^i > co_i$ and so $d^j + d^k < co_j + co_k$. So either $d^j < co_j$ and $d^k < co_k$ or $d^j < co_j$ and $d^k > co_k$ with $\mid d^j - co_j \mid = \mid d^i - co_i + d^k - co_k \mid$ must hold.

If $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) \geq 0 = Dr_i(V_t,\mathbf{co})$ it follows that $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) > Dr_i(V_t,\mathbf{co})$ and so $Dr(V_t,\mathbf{co}) \leq Dr^{\mathbf{co}}(V_t,\mathbf{co}) < \frac{2\varepsilon}{3(1+\varepsilon)}(-D(\mathbf{d})) + 2\varepsilon^2$.

Now we show that

$$Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) = p_{\mathbf{d}}^{i,j}(-D(\mathbf{co},\mathbf{d}) + D(\mathbf{co},\mathbf{d}(i,j))) + p_{\mathbf{d}}^{i,k}(-D(\mathbf{co},\mathbf{d}) + D(\mathbf{co},\mathbf{d}(i,k))) \geq 0$$

.

If $d^i > co_i$, $d^j < co_j$ and $d^k < co_k$ then both

$-D(\mathbf{co},\mathbf{d}) + D(\mathbf{co},\mathbf{d}(i,j)) > 0$ and $-D(\mathbf{co},\mathbf{d}) + D(\mathbf{co},\mathbf{d}(i,k)) > 0$ and so the lemma holds for this particular case.

So, the next case we analyze is if $d^k > co_k$ with $\mid d^j - co_j \mid = \mid d^i - co_i + d^k - co_k \mid$ and $\mid d^j - co_j \mid > \mid d^k - co_k \mid$.

As $Dr^{\mathbf{co}}(V_t,\mathbf{co})_i = -\frac{2\varepsilon}{3}\left(\frac{d^j}{1+\varepsilon}(co_i - d^i + d^j - co_j) + \frac{d^k}{1+\varepsilon}(co_i - d^i + d^k - co_k)\right) + 2\varepsilon^2$ and since since $co_j - d^j > d^k - co_k$ it holds that $Dr^{\mathbf{co}}(V_t,\mathbf{co})_i$ can only be negative if $d^k > d^j$. Otherwise, if $d^k \leq d^j$ we know the lemma holds so for the next case we assume that $d^k > d^j$. When $d^k > d^j$ we cannot show that $Dr^{\mathbf{co}}(V_t,\mathbf{co})_i > 0$. However our strategy is now to show that $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) > Dr_j^{\mathbf{co}}(V_t,\mathbf{co}) + Dr_k^{\mathbf{co}}(V_t,\mathbf{co})$, since

$Dr(V_t,\mathbf{co}) = Dr(V_t,\mathbf{co})_i + Dr(V_t,\mathbf{co})_j + Dr(V_t,\mathbf{co})_k = Dr^{\mathbf{co}}(V_t,\mathbf{co})_j + Dr^{\mathbf{co}}(V_t,\mathbf{co})_k$ we can then conclude that as long as $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) > Dr_j^{\mathbf{co}}(V_t,\mathbf{co}) + Dr_k^{\mathbf{co}}(V_t,\mathbf{co})$ it holds that $Dr(V_t,\mathbf{co}) < \frac{1}{2}Dr^{\mathbf{co}}(V_t,\mathbf{co})$ which satisfies the bound in the lemma.

The following equations show that $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) > Dr_j^{\mathbf{co}}(V_t,\mathbf{co}) + Dr_k^{\mathbf{co}}(V_t,\mathbf{co})$.

$$Dr^{\mathbf{co}}(V_t,\mathbf{co})_j = -\frac{2\varepsilon}{3}\left(\frac{d^i}{1+\varepsilon}\left(co_j - d^j + d^i - co_i\right) + \frac{d^k}{1+\varepsilon}\left(co_j - d^j + d^k - co_k\right)\right) + 2\varepsilon^2.$$

$$Dr^{\mathbf{co}}(V_t,\mathbf{co})_k = -\frac{2\varepsilon}{3}\left(\frac{d^i}{1+\varepsilon}\left(co_k - d^k + d^i - co_i\right) + \frac{d^j}{1+\varepsilon}\left(co_k - d^k + d^j - co_j\right)\right) + 2\varepsilon^2.$$

$$Dr^{\mathbf{co}}(V_t,\mathbf{co})_i = -\frac{2\varepsilon}{3}\left(\frac{d^j}{1+\varepsilon}\left(co_i - d^i + d^j - co_j\right) + \frac{d^k}{1+\varepsilon}\left(co_i - d^i + d^k - co_k\right)\right) + 2\varepsilon^2.$$

Now let $Dr_{com} = Dr_j^{\mathbf{co}}(V_t,\mathbf{co}) + Dr_k^{\mathbf{co}}(V_t,\mathbf{co}) - Dr_i^{\mathbf{co}}(V_t,\mathbf{co})$. First we cancel the common terms $d^j\left(d^j - co_j\right)$ and $d^k\left(d^k - co_k\right)$. Then we observe that $\left(co_j - d^j\right) + \left(co_k - d^k\right) > 0$. Since $d^k > d^j$ by assumption, $d^k\left(co_j - d^j\right) + d^j\left(co_k - d^k\right) > 0$. So combining these yields

$$Dr(V_t,\mathbf{co})_{com} \leq -\frac{2\varepsilon}{3}\left(\frac{d^i}{1+\varepsilon}2\left(d^i - co_i\right) - \frac{d^j + d^k}{1+\varepsilon}\left(co_i - d^i\right)\right) + 2\varepsilon^2.$$

Since $co_i - d^i < 0$ we conclude that $Dr_{com} < 2\varepsilon^2$ and so $Dr(V_t,\mathbf{co}) < \frac{1}{2}Dr^{\mathbf{co}}(V_t,\mathbf{co})$.

(4.13a)

So since $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) > Dr_j^{\mathbf{co}}(V_t,\mathbf{co}) + Dr_k^{\mathbf{co}}(V_t,\mathbf{co})$ we conclude that for this particular case $Dr(V_t,\mathbf{co}) < \frac{1}{2}Dr^{\mathbf{co}}(V_t,\mathbf{co})$ for all previous cases it was true that $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) \geq 0$ and so that $Dr(V_t,\mathbf{co}) < Dr^{\mathbf{co}}(V_t,\mathbf{co})$.

Now we analyze the case when at $\mathbf{d}$ only a singleton coalition $C \notin \mathcal{C}^{\mathbf{co}}$ is binding. Let i be the player in C and j,k be the remaining players. Then, if i increases his demand, no coalition is feasible and the transition probabilities for both games are identical and so $Dr_i^{\mathbf{co}}(V_t,\mathbf{co}) = Dr_i(V_t,\mathbf{co})$.

If j or k increase however, the transition probabilities differ and $p_{\mathbf{d}}^{j,i} = p_{\mathbf{d}}^{k,i} = 0$ for $(v,3)$. Since $C_i \notin \mathcal{C}^{\mathbf{co}}(v,3)$ it holds that $d^i < co_i$. So at least one of $d^k > co_k$ or $d^j > co_j$ has to hold. We want to show that $Dr(v,\mathbf{co})_j + Dr(v,\mathbf{co})_k \leq Dr^{\mathbf{co}}(v,\mathbf{co})_j + Dr^{\mathbf{co}}(v,\mathbf{co})_k$, because then we can conclude that $Dr(v,\mathbf{co}) < Dr^{\mathbf{co}}(v,\mathbf{co})$.

Let $Dr_{com} = [Dr^{\mathbf{co}}(v,\mathbf{co})_j + Dr^{\mathbf{co}}(v,\mathbf{co})_k - Dr(v,\mathbf{co})_j - Dr(v,\mathbf{co})_k]$. Then, if $Dr_{com} \geq 0$ we know that $Dr(v,\mathbf{co})_j + Dr(v,\mathbf{co})_k \leq Dr^{\mathbf{co}}(v,\mathbf{co})_j + Dr^{\mathbf{co}}(v,\mathbf{co})_k$ and so $Dr(v,\mathbf{co}) \leq$

$Dr^{\mathbf{co}}(v,\mathbf{co})$ and the bound on $Dr^{\mathbf{co}}(v,\mathbf{co})$ from Lemma 4.16 for $Dr^{\mathbf{co}}(v,\mathbf{co})$ holds as well for $Dr(v,\mathbf{co})$.

So we show that $Dr_{com} \geq 0$. For the cooperative game process $(v,3,\varepsilon)$ at $\mathbf{d}(k)$ and at $\mathbf{d}(j)$ the payable coalition is C_i and $p_{\mathbf{d}}^{j,k} = \frac{d^k}{d^k+d^j+\varepsilon}$ and $p_{\mathbf{d}}^{k,j} = \frac{d^j}{d^k+d^j+\varepsilon}$. Since $d^i = 1-d^j-d^k$ we can rewrite the transition probabilities as $p_{\mathbf{d}}^{j,k} = \frac{d^k}{1+\varepsilon} + d^i \frac{d^k}{d^j+d^k+\varepsilon}$ and $p_{\mathbf{d}}^{k,j} = \frac{d^j}{1+\varepsilon} + d^i \frac{d^j}{d^j+d^k+\varepsilon}$. Then, in $Dr^{\mathbf{co}}(v,\mathbf{co})_j + Dr^{\mathbf{co}}(v,\mathbf{co})_k - Dr(v,\mathbf{co})_j - Dr(v,\mathbf{co})_k$ the terms $\frac{2\varepsilon}{3(1+\varepsilon)}[(co_j - d^j) + (d^k - co_k)$ and $\frac{2\varepsilon}{3(1+\varepsilon)}[(co_k - d^k) + (d^j - co_j)$ cancel out and

$$
\begin{aligned}
Dr_{com} = &-\frac{2\varepsilon}{3(1+\varepsilon)}[(co_j - d^j) + (d^i - co_i) + \left(co_k - d^k\right) + (d^i - co_i)] + 2\varepsilon^2 \\
&+ \frac{2\varepsilon}{3}[\frac{d^k}{d^k+d^j+\varepsilon}\left((co_j - d^j) + \left(d^k - co_k\right)\right) + \frac{d^j}{d^j+d^k+\varepsilon}\left(\left(co_k - d^k\right) + (d^j - co_j)\right)] \\
&+ 2\varepsilon^2 \\
Dr_{com} = &-\frac{2\varepsilon}{3(1+\varepsilon)}[3\left(d^i - co_i\right) + 2\varepsilon^2 \\
&+ \frac{2\varepsilon}{3(1+)}[\frac{d^k}{d^k+d^j+\varepsilon}\left((co_j - d^j) + \left(d^k - co_k\right)\right) + \frac{d^j}{d^j+d^k+\varepsilon}\left(\left(co_k - d^k\right) + (d^j - co_j)\right)] \\
&+ 2\varepsilon^2 \\
Dr_{com} = &-\frac{2\varepsilon}{3(1+\varepsilon)}[3\left(d^i - co_i\right) + 2\varepsilon^2 \\
&+ \frac{2\varepsilon}{3}[\frac{d^j - d^k}{d^k+d^j+\varepsilon}\left(\left(co_k - d^k\right) + (d^j - co_j)\right)] + 2\varepsilon^2
\end{aligned}
\tag{4.14a}
$$

Suppose $d^k > co_k$ and $d^j > co_j$ hold. Since $d^i - co_i = co_j - d^j + co_k - d^k$, it follows that $d^i - co_i < co_j - d^j + d^k - co_k$ and $d^i - co_i < co_k - d^k + d^j - co_j$. Since $\frac{d^k}{d^k+d^j+\varepsilon} < 1$ and $\frac{d^k}{d^k+d^j+\varepsilon} < 1$ we conclude that $Dr_{com} > 0$ and so $Dr(v,\mathbf{co})_j + Dr(v,\mathbf{co})_k \leq Dr^{\mathbf{co}}(v,\mathbf{co})_j + Dr^{\mathbf{co}}(v,\mathbf{co})_k$, and so $Dr(v,\mathbf{co}) < Dr^{\mathbf{co}}(v,\mathbf{co})$.

Suppose $d^k < co_k$ and $d^j > co_j$ hold. From Lemma (4.9) we know that this implies that player j is the player with largest demands and so $d^j > d^k$. Then $Dr_{com} > 0$ and so $Dr(v,\mathbf{co}) < Dr^{\mathbf{co}}(v,\mathbf{co})$. Now we have concluded the case analysis in cases that a singleton coalition or a 2-player coalition are binding.

Suppose now at $\mathbf{d}$ more than one coalition is binding.

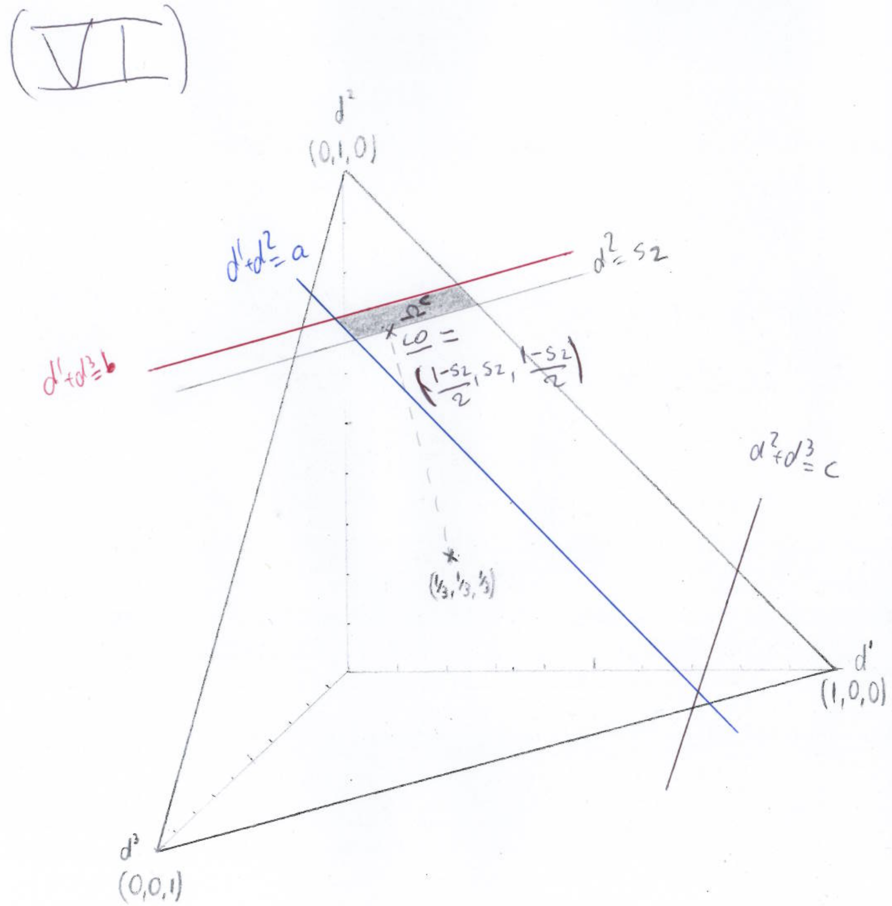


Figure 4.6 Graphical representation of core and outcome VI in table (4.5).

Suppose at $\mathbf{d}$ two 2-player coalition $C_{ij} \notin \mathcal{C}^{\mathbf{co}}$ and $C_{ik} \notin \mathcal{C}^{\mathbf{co}}$ are binding. Then $p_{\mathbf{d}}^{j,i} = p_{\mathbf{d}}^{j,k} = p_{\mathbf{d}}^{k,i} = p_{\mathbf{d}}^{k,j} = 0$ and furthermore $d^i < co_i$, $d^j > co_j$ and $d^k > co_k$. We use Lemma (4.12) to conclude that $Dr^{\mathbf{co}}(V_t, \mathbf{co}) \leq \frac{-2\varepsilon}{9(1+\varepsilon)} D(\mathbf{d}) + 2\varepsilon^2$.

Observe that if at $\mathbf{d}$ a 2-player coalition $C_{ij} \notin \mathcal{C}^{\mathbf{co}}$ and a singleton coalition that is not in $\mathcal{C}^{\mathbf{co}}$ are binding, that the singleton coalition cannot be the complement of C_{ij} as neither of them is in $\mathcal{C}^{\mathbf{co}}$.

So suppose that at $\mathbf{d}$ a 2-player coalition $C_{ij} \notin \mathcal{C}^{\mathbf{co}}$ and two singleton coalitions C_i and C_j both not in $\mathcal{C}^{\mathbf{co}}$ are binding. Then observe that $p_{\mathbf{d}}^{k,j} = 0$, $p_{\mathbf{d}}^{k,i} = 0$, $p_{\mathbf{d}}^{j,i} = 0$ and $p_{\mathbf{d}}^{i,j} = 0$. The only possible transitions of the chain (V_t) to a new efficient state are to $\mathbf{d}(i,k)$ and $\mathbf{d}(j,k)$. Since it holds that $d^i < co_i$, and $d^j < co_j$ since both C_i and C_j are binding and not in $\mathcal{C}^{\mathbf{co}}$ it holds that $d^k > co_k$. So we apply Lemma (3.8) to conclude that $Dr(V_t, \mathbf{d}) \leq \frac{-2\varepsilon}{9(1+\varepsilon)} D(\mathbf{d}) + 2\varepsilon^2$.

So suppose now that at $\mathbf{d}$ a 2-player coalition $C_{ij} \notin \mathcal{C}^{\mathbf{co}}$ and one singleton coalition $C_i \notin \mathcal{C}^{\mathbf{co}}$ are binding. We will analyze the drift. Observe that $p_{\mathbf{d}}^{j,k} > 0$, $p_{\mathbf{d}}^{i,j} > 0$and $p_{\mathbf{d}}^{i,k} > 0$, all other transitions to neighbors have zero probability. So the drift simplifies to

$$Dr^{\mathbf{co}}\left(V_t,\mathbf{co}\right)_j = -\frac{2\varepsilon}{3}\left(\frac{d^k}{d^j+d^k+\varepsilon}\left(co_j - d^j + d^k - co_k\right)\right) + 2\varepsilon^2.$$
$$Dr^{\mathbf{co}}\left(V_t,\mathbf{co}\right)_i = -\frac{2\varepsilon}{3}\left(\frac{d^j}{1+\varepsilon}\left(co_i - d^i + d^j - co_j\right) + \frac{d^k}{1+\varepsilon}\left(co_i - d^i + d^k - co_k\right)\right) + 2\varepsilon^2.$$

We know that $d^i < co_i$ since C_i is binding and not in $\mathcal{C}^{\mathbf{co}}$ and similarly that $d^k > co_k$ since C_{ij} is binding and $C_{ij} \notin \mathcal{C}^{\mathbf{co}}$. So suppose first that $d^j < co_j$. Then we know by Lemma (4.9) that player k is the player with largest demands at $\mathbf{d}$ and so $\frac{d^k}{d^j+d^k+\varepsilon}\left(co_j - d^j + d^k - co_k\right) - \frac{d^j}{1+\varepsilon}\left(co_i - d^i + d^j - co_j\right) > 0$ and so

$$\begin{aligned}
Dr\left(V_t,\mathbf{co}\right)_{com} &\leq -\frac{2\varepsilon}{3}\frac{d^k}{1+\varepsilon}\left(co_i - d^i + d^k - co_k\right) + 2\varepsilon^2. \\
&\leq -\frac{2\varepsilon}{3}[\left(co_i - d^i\right)^2 + \left(d^k - co_k\right)^2] + 2\varepsilon^2. \\
&\leq -\frac{2\varepsilon}{9}[D\left(\mathbf{d}\right)] + 2\varepsilon^2.
\end{aligned} \tag{4.15a}$$

We conclude that it only remains to show that, if $d^j > co_j$ and $d^k > co_k$ and $d^i < co_i$ at a state $\mathbf{d}$ where coalitions C_i and C_{ij} are binding and neither is in the set $\mathcal{C}^{\mathbf{co}}$. We will compare the drift at this state, with the drift of the game where we have analyzed the drift before, when only coalition C_{12} is binding but is not in $\mathcal{C}^{\mathbf{co}}$. Observe that if player i increases, the transition dynamics of the chain (V_t) to neighbors are identical for both games since in both cases the payable coalition is the empty set. If player k increases his demand, again the transition dynamics of the chain (V_t) to neighbors are identical for both games since in both cases the payable coalition is C_{ij}. So the only difference arises if player j increases his demand as in the game where two coalitions are binding, the payable coalition is C_i whereas in the game where only C_{ij} is binding, the payable coalition is the empty set. Let the game with two coalitions C_{ij} be $(v,3)$ and the other game with only C_{ij} be referred to as $(v,3)^*$ and its drift as $Dr^*(V_t,\mathbf{co})$. We want to show that the drift for $(v,3)$ is less than for $(v,3)^*$.

$$\begin{aligned} Dr^*(V_t,\mathbf{co}) - Dr(V_t,\mathbf{co}) = & -\frac{2\varepsilon}{3}[\left(\frac{d^i}{1+\varepsilon}\left(co_j - d^j + d^i - co_i\right) + \frac{d^k}{1+\varepsilon}\left(co_j - d^j + d^k - co_k\right)\right)] \\ & + 2\varepsilon^2 \\ & + \frac{2\varepsilon}{3}\left(\frac{d^k}{d^j+d^k+\varepsilon}\left(co_j - d^j + d^k - co_k\right)\right) + 2\varepsilon^2. \end{aligned} \tag{4.16a}$$

First assume that $d^j - co_j < d^k - co_k$. Then $\left(co_j - d^j + d^k - co_k\right) > 0$ and $\left(co_j - d^j + d^i - co_i\right) < 0$ and since $\frac{d^k}{d^k+d^j+\varepsilon} > \frac{d^k}{1+\varepsilon}$ it follows that the drift for $(v,3)$ is less than for $(v,3)^*$.

Now we assume that $d^j - co_j > d^k - co_k$. Then

$$\left(co_j - d^j + d^i - co_i\right) < \left(co_j - d^j + d^k - co_k\right) < 0.$$

If we can show that $\frac{d^i+d^k}{1+\varepsilon} > \frac{d^k}{d^k+d^j+\varepsilon}$ then we know that $Dr^*(V_t,\mathbf{co}) - Dr(V_t,\mathbf{co}) > 0$ and then we know that the bound on the drift for $(v,3)^*$ holds as well for $(v,3)$. But $d^j + d^k + \varepsilon = 1 - d^i + \varepsilon$ and so we have, ignoring the ε terms,

$$\frac{\left(1-d^i\right)\left(d^i+d^k\right)}{d^j+d^k+\varepsilon} = \frac{d^i\left(1-d^i-d^k\right)+d^k}{d^j+d^k+\varepsilon} > \frac{d^k}{d^j+d^k+\varepsilon}$$

and so it follows that the drift for $(v,3)$ is less than for $(v,3)^*$. Since we showed before that the drift for $(v,3)^*$ is less than the drift for $(v^{\mathbf{co}},3)$ we have shown that the bound from Lemma (3.8) holds.

Now we have concluded the analysis of all cases. Over all the cases the worst bound is $Dr(V_t,\mathbf{d}) \leq \frac{2\varepsilon}{9(1+\varepsilon)}(-D(\mathbf{d})) + 2\varepsilon^2$ and this concludes the proof of this lemma. □

Lemma 4.18. *Let* $\mathbf{d}$ *be any state in the core. Let player* l *be the player with largest cooperative outcome at* $\mathbf{d}$. *Suppose that* $d^l = co_l$. *Let player* m *be the player with the second largest cooperative outcome in the core, if the second and thirds player have the same cooperative outcome then let* d^l *be the player with second largest demands. Then* $d^m \geq co_m$.

Proof: If $\mathbf{co} = \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, then $d^1 = d^2 = d^3 = \frac{1}{3}$ and Lemma (4.18) holds trivially.

Otherwise, suppose that $d^m < co_m$. From table ... describing the cooperative outcome for all case where $\mathbf{co} \neq \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$, we see that in cases I, II, III, V, this implies that $d^1 + d^2 < a$. However this contradicts the fact that $\mathbf{d} \in \Omega^C$ and so we conclude that in these cases $d^m \geq co_m$ has to hold.

In cases IV and VI the players with second and third largest demands have the same cooperative outcome and so (4.18) holds trivially. □

Lemma 4.19. *Suppose at $\mathbf{d} \in \Omega^C$ one coalition C in $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding. Then*

$$Dr(V_t, \mathbf{d}) < \frac{2\varepsilon}{3(1+\varepsilon)}(-D(\mathbf{d})) + 2\varepsilon^2. \tag{4.17}$$

Proof: If C is a singleton coalition, let $C = C_i$ and if C is a 2-player coalition, then let $C = C_{jk}$. Then in either case $d^i = co_i$ by assumption and so $d^j - co_j = co_k - d^k$. Let $d^j > d^k$. We know from (4.9) that the player with largest demands has demands at least equal to his cooperative outcome value. Furthermore we know from Lemma (4.18) that, if the player with largest demands has demands equal to his cooperative outcome value, the player with second largest demands has demands at least equal to his cooperative outcome value. We conclude that $d^j \geq co_j$ and as long as $\mathbf{d} \neq \mathbf{co}$ it holds that $d^j > co_j$.

Suppose now that no other coalition but C is binding at $\mathbf{d}$. Then players j,k are either both member of the coalition in case $C = C_{jk}$ or they are both not member of the coalition in case $C = C_i$ so in either case $C'(\mathbf{d}(j)) = C'(\mathbf{d}(k))$. Furthermore $\left(d^j - co_j\right) + \left(d^k - co_k\right) = 0$ and $\left(d^i - co_i\right) = 0$ and so it follows that

$$-D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(j,i)) - D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(k,i)) = 0.$$

Furthermore

$$p_{\mathbf{d}}^{i,j}(-D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(i,j))) + p_{\mathbf{d}}^{i,k}(-D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(i,k))) = \left(p_{\mathbf{d}}^{i,j} - p_{\mathbf{d}}^{i,k}\right)\left(d^j - co_j\right)$$

and

$$p_{\mathbf{d}}^{k,j}(-D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(k,j))) + p_{\mathbf{d}}^{j,k}(-D(\mathbf{co}, \mathbf{d}) + D(\mathbf{co}, \mathbf{d}(j,k))) = \left(p_{\mathbf{d}}^{k,j} - p_{\mathbf{d}}^{j,k}\right) 2\left(d^j - co_j\right)$$

since $\left(d^j - co_j\right) = \left(co_k - d^k\right)$.

We know that $d^j > d^k$, $d^j > co_j$ and $d^j - co_j = co_k - d^k$. From Lemma (4.18) as well that $co_j \geq co_k$ so we conclude that $d^j - d^k \geq \left(d^j - co_j\right) + \left(co_k - d^k\right) = 2\left(d^j - co_j\right)$.

In general,

$$Dr\left(V_t, \mathbf{co}\right) = \sum_{i=1}^{3}\sum_{j=1}^{3} p_{\mathbf{d}}^{i,j}\left(-\left(co_i - d^i\right)^2 - \left(co_j - d^j\right)^2 + \left(co_i - \left(d^i + \varepsilon\right)\right)^2 + \left(co_j - \left(d^j - \varepsilon\right)\right)^2\right)$$

in this particular case

$$Dr\left(V_t, \mathbf{co}\right) \leq -2\varepsilon[\left(p_{\mathbf{d}}^{i,j} - p_{\mathbf{d}}^{i,k}\right)\left(d^j - co_j\right) + \left(p_{\mathbf{d}}^{k,j} - p_{\mathbf{d}}^{j,k}\right) 2\left(d^j - co_j\right)] + 2\varepsilon^2$$

If $C = C_i$ then $p_{\mathbf{d}}^{i,j} - p_{\mathbf{d}}^{i,k} = \frac{d^j - d^k}{1+}$ and $p_{\mathbf{d}}^{k,j} - p_{\mathbf{d}}^{j,k} = \frac{d^j - d^k}{d^j - d^k + \varepsilon}$, if $C = C_{jk}$ then $p_{\mathbf{d}}^{i,j} - p_{\mathbf{d}}^{i,k} = 0$ and $p_{\mathbf{d}}^{k,j} - p_{\mathbf{d}}^{j,k} = \frac{d^j - d^k}{1+\varepsilon}$ so in either case

$$Dr\left(V_t, \mathbf{co}\right) \leq -2\varepsilon\frac{d^j - d^k}{1+\varepsilon} 2\left(d^j - co_j\right) + 2\varepsilon^2$$

and so, finally,

$$\begin{aligned} Dr\left(V_t, \mathbf{co}\right) &\leq -2\varepsilon[\left(d^j - co_j\right)^2 + \left(d^k - co_k\right)^2 + 2\varepsilon^2 \\ &\leq -\frac{2\varepsilon}{3} D\left(\mathbf{d}\right) + 2\varepsilon^2. \end{aligned} \tag{4.18a}$$

So we have concluded that, if only one coalition C in $\mathcal{C}^{\mathbf{co}}\left(v, 3\right)$ is binding, the lemma holds. Now we will analyze what other coalitions $C^* \notin \mathcal{C}^{\mathbf{co}}\left(v, 3\right)$ can be binding, and what that implies for the drift. Since $d^j > co_j$ the singleton coalition C_j cannot be binding and so $C^* \neq C_j$. Since $d^i = co_i$ and $d^j > co_j$, C_{ij} cannot be binding and so $C^* \neq C_{ij}$. In case that $C = C_i$ coalition C_{jk} cannot be binding, because, by assumption, it is not in the set $\mathcal{C}^{\mathbf{co}}\left(v, 3\right)$ and so $C^* \neq C_{jk}$. Similarly, in case that $C = C_{jk}$ $C^* \neq C_i$. So the only two coalitions that could be binding at $\mathbf{d}$ and are not in $\mathcal{C}^{\mathbf{co}}\left(v, 3\right)$ are C_{ik} and C_k.

Now we will analyze, how it impacts the drift if $C^* = C_{ik}$ is binding as well. The dynamics only differ at $\mathbf{d}\left(j\right)$. If player j increases his demand, he will decrease demands again. Since $d^{>} co_j$ this will not increase the drift. So we conclude that the bound $Dr\left(V_t, \mathbf{co}\right) \leq -\frac{2\varepsilon}{3} D\left(\mathbf{d}\right) + 2\varepsilon^2$ holds as well in this case.

Now we analyze, how is the drift impacted, if $C^* = C_k$, that is, if C_k is binding as well. First we analyze how $Dr\left(V_t, \mathbf{co}\right)_i$ changes. Now it holds that $p_{\mathbf{d}}^{i,k} = 0$ and $p_{\mathbf{d}}^{i,j} = \frac{d^j}{d^i + d^j + \varepsilon} \geq$

$\frac{d^j}{1+\varepsilon}$. So compared to the case where only a coalition in $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding, $p^{i,k}$ decreases and $p^{i,j}$ increases and so $Dr(V_t,\mathbf{co})_i$ cannot increase.

Now we look at $Dr(V_t,\mathbf{co})_k$. Observe that if $C=C_i$ then by Lemma (4.6.3), C_{ik} is binding as well and the analysis is the same as when $C^*=C_{ik}$ and so $Dr(V_t,\mathbf{co})_k$ does not increase. Now assume that $C=C_{jk}$. If k increases now, player j will not decrease and player k decreases with higher probability given by $\frac{d^k+\varepsilon}{d^k+d^i+\varepsilon}$ compared to the case when only a coalition in $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding, in which case he decreases with $\frac{d^k+\varepsilon}{1+\varepsilon}$. So again, $Dr(V_t,\mathbf{co})_k$ does not increase and so we conclude that the bounds on the drift, calculated for the case when only a coalition in $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding, holds as well when $C^*=C_k$ or $C^*=C_{ik}$ are binding as well at $\mathbf{d}$. Since no other coalition can be C^*, we conclude that $Dr(V_t,\mathbf{co})\leq -\frac{2\varepsilon}{3}D(\mathbf{d})+2\varepsilon^2$ whenever a coalition in $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding.

□

Lemma 4.20. *For any state* $\mathbf{d}\in\Omega^C$

$$Dr(V_t,\mathbf{d}) < \frac{-2\varepsilon}{9(1+\varepsilon)}D(\mathbf{d})+2\varepsilon^2. \tag{4.19}$$

Proof: The sets of states, where no coalition is binding, no coalition in $\mathcal{C}^{\mathbf{co}}(v,3)$ and the set where a coalition in $\mathcal{C}^{\mathbf{co}}(v,3)$ is binding, form a partition of the core. So combining the results for Lemma (4.19), Lemma (4.17) and Lemma (4.16) proves this lemma. □

We use Lemma (4.20) to show that **co** is a Markovian cooperative equilibrium for all superadditive 3-player games satisfying (4.4).

Theorem 4.21. *Suppose* $(v,3)$ *is a superadditive* 3*-player game satisfying* (4.4)*, and* ε *is equal to* $\frac{1}{M}$ *for some v-compatible* $M\in\mathbb{N}$*. Let* (V_t) *be the* $(v,3,\varepsilon)$ *cooperative game process. Then, for* $\alpha>0$*,*

$$\mathbb{P}_{\pi_v,\varepsilon}(D(\mathbf{d})>\alpha)\leq\frac{\varepsilon}{9\alpha}.$$

Chapter 5

Power in 4-Player Games

In Section 5.1 we define asymmetric coalition structures. In Theorem (5.1) we show that once in the core a cooperative game process can only leave the core via a state where two coalitions in an asymmetric coalition structure are binding. In Lemma (5.2) we show that a set of coalitions in a four player game is an asymmetric coalition structure if and only if the set consists of two distinct two-player coalitions. This is an important result: if no asymmetric coalition set is present the inherent equity bias is the decisive dynamic and the cooperative game process drifts towards equity. We give an extensive analysis of the behavior along the asymmetric edge generated by two distinct two player coalitions in Section 5.2. Hence all "interesting" cases for 4-player games are analyzed in this chapter. Finally we make the conjecture (5.4) that for a general N-player balanced superadditive game every state outside the core is transient if the game does not have a state in the core where two distinct coalitions in one asymmetric coalition set are binding.

In Section 5.2 we first introduce three particular example games which all have an asymmetric coalition structure. We analyze in detail the geometric structure of Ω^C for these example games, focusing in general on the sets of states, where one or more coalitions are binding, and in particular on the unique set of states where coalitions in the asymmetric coalition structure are binding, this set is the asymmetric edge. We prove that at any state not in the asymmetric edge there exists a drift towards the asymmetric edge for each of the three example games.

We then introduce the concepts of core power and asymmetric power for cooperative game processes. Core power is an adaption of the power concepts introduced by Harsany and Selten. The core power of a player corresponds to the demand of a player at the state in the core with the smallest distance to equal split. The player in the intersection of the asymmetric coalition structure has *asymmetric power* because he is member of more binding coalitions than the other players in the asymmetric coalition structure. When the cooperative game process moves along the face of the core polytope generated by the asymmetric coalition structure (two distinct two-player coalitions for four player games) the player in the intersection can only reduce demands if the cooperative game process is in the interior of the polytope. On states where at least one of the coalitions generating the face of the asymmetric coalition structure is binding he does not reduce demands. On the excursions of the cooperative game process away from the asymmetric edge and back there is an interplay between the inherent equity bias of the process that “drags” the cooperative game process towards equity or “down” the asymmetric edge and the dynamic of the asymmetric power which “drags” the process away from equity or “up” the asymmetric edge. We describe the particular power situations for each of three specific example games. The point is, that the geometric structure of example games $1-3$ is very similar, their cores are in fact combinatorially isomorphic. We want to highlight that it is not trivial that the power situations in the three example games are very different. These examples games are not just any three specific games. For 4-player games if no asymmetric coalition set is present then along all faces of the polytope in the core the equity bias is the only dynamic. Although a complete analysis of 4-player games is not given in this thesis, all possible cases of interest are covered with the above example games. No other coalition structure can be present in a superadditive 4-player game (and probably in any superadditive N-player game). Thus these examples anlayzed are generic, highlighting the one main principle of interest for the cooperative trial-and-error game process.

In Section 5.3 we analyze the interplay between the asymmetric power dynamic and the inherent equity bias along the asymmetric edge: we give a heuristic analysis backed up by simulations to show, how the different power situations arise. We show that each example game from Section 5.2 has a Markovian cooperative equilibrium on the asymmetric edge but they appear on different parts of the edge, namely the *upper concession limit*, where the player in the intersection has highest demands in the asymmetric edge, the *lower concession limit* where he has lowest demands in the asymmetric edge and a state in the interior of the asymmetric edge. We develop a concept to simplify and investigate the behavior of the chain (V_t) in the neighborhood of an asymmetric edge. We apply the concept and use it to calculate approximate solutions for the Markovian cooperative equilibrium for example games $1-3$.

We first introduce an "idealized version" of the chain (V_t). We fix a state $\mathbf{d}^*$ on the asymmetric edge, and then we use the idealized chain started from $\mathbf{d}^*$ to approximate the behavior of (V_t) during an excursion from and back to the asymmetric edge. The idealized version of (V_t) transitions according to a law that is calculated with transition probabilities that assumes constant demands over an excursion. The transition probabilities thus differ only on different sets of states in the neighborhood of the face: states where both coalitions generating the asymmetric edge are binding (one dimensional edge), states where one of the coalitions is binding (two-dimensional plane), the interior of the polytope where no coalition is binding and states outside the core where the two player coalition with smaller worth has a negative coalition's surplus.

We make the connection between the three-dimensional chain (V_t), that moves on the set of recurrent states, a set, that strictly contains the core, and the two-dimensional chain where each coordinate represents the coalition's surplus for the respective coalition $\left(\frac{CS^{C_{12}}(V_t)}{\varepsilon}, \frac{CS^{C_{12}}(V_t)}{\varepsilon}\right)$, that moves on a subset of the 2-dimensional Euclidean grid. We define sets of states $\mathcal{S}_{m,n}$, where m corresponds to the x-coordinate and n represents the y-coordinate of a state in the 2-dimensional Euclidean grid. On the sets $\mathcal{S}_{m,n}$, the transition dynamics of the idealized 3-dimensional chain are identical for all states in $\mathcal{S}_{m,n}$. As long as the chain (V_t) does not hit any hyperplane corresponding to a three-player coalition, the distribution on these sets of the 2-dimensional idealized chain exactly represents the distribution on these sets of the 3-dimensional idealized chain.

Along the asymmetric edge (from the lower concession limit up to the upper concession limit) each state has a unique demand for any player. A state on the asymmetric edge is uniquely determined by any one of its coordinates, so analyzing the drift in d^1 gives us information on the drift of the chain (V_t). We define the drift in d^1 for the different chains. Important is, that the drift in d^1 over a step of the idealized chains is identical on the sets $\mathcal{S}_{m,n}$. We then define the equilibrium distribution of the idealized 2-dimensional chain for a given starting state on the asymmetric edge. We use this equilibrium distribution on the sets $\mathcal{S}_{m,n}$ and the drifts in d^1 on these sets to define the drift in d^1 over an excursion from the asymmetric edge and back to the asymmetric edge. Then we restrict the state space and define a restricted idealized chain on this state space. Finally we explicitly calculate the equilibrium distribution, drift in d^1 on the different sets of states, and the drift in d^1 over an excursion of the 2-dimensional idealized chain. We use these results to explain, why the Markovian cooperative equilibrium is located differently for the example games $1-3$.

We have shown two interesting results related to asymmetric coalition structures:

- the cooperative game process can leave the core via states where the coalitions in the structure are binding
- asymmetric power arises to the player(s) in the intersection of the asymmetric coalition structure.

In Section 5.4 we prove that the former is not necessary for the latter to exist: asymmetric power does not rely on the cooperative game process to leave the core. We want to highlight that this indicates that the concept of asymmetric power is applicable to much more general settings than the coooperative game process defined in this thesis. In particular in Section 5.4 we discuss a Markov chain (W_t) that is closely related to the chain (V_t), it is in fact a version of the chain (V_t) restricted to the core. For the chain (W_t) we are able to prove a positive drift on the asymmetric edge for a specific 4-player example game. We expect that the chain (W_t) is very important for understanding the behavior of the chains (V_t) and (O_t), restricting the chain to remain in the core does not change the nature of the chain however it simplifies the analysis of the behavior of the chain between excursions from the asymmetric edge.

In Section 5.5 we conclude the analysis and give simulation results for example games $1-3$. The simulation results confirm the results of this section. Furthermore the simulation results show, that the chain (V_t) does not move 'far away' from the asymmetric edge in terms of multiples of ε. We look at the average path of the chain (V_t) to the Markovian cooperative equilibrium from different starting states and give maximum distances in each coordinate after fixed time intervals.

5.1 Asymmetric Coalition Sets

In the introduction Chapter 2 we gave examples of symmetric coalition structures and one example of an asymmetric coalition structure. The asymmetric coalition structure has two dynamics associated with it that make it very interesting for the study of the cooperative game process in particular and cooperative trial and error processes in general: In the neighborhood around the face generated by the asymmetric coalition structure (asymmetric face) a power dynamic is present that can be stronger than the inherent equity bias of the cooperative game process. Furthermore the cooperative game process can leave the core via states on the face of the polytope forming the core generated by the coalitions in the asymmetric coalition set. We will first define an asymmetric coalition set and then show the latter in Theorem (5.1). The analysis of asymmetric power is conducted in this chapter in Section 5.2.

In this chapter we analyze games $(v,4)$. The following definition however we give for N-player games as Theorem (5.1) is an important result for the cooperative game process for general N-player games (v,N) using this definition.

For a game (v,N), for $k < 2^N$ let an *asymmetric coalition set* $\{C^i,...,C^k\} = \mathcal{AC} \subset \mathcal{P}\left(C^G\right)$ be a set of coalitions satisfying the following three constraints:

1. No coalition in $\mathcal{AC}$ is a subset of another coalition in $\mathcal{AC}$.
2. The union of the complement of $\mathcal{AC}$ is non-empty that is $\left(\cup_{i=1}^k C^i\right)^c \neq \emptyset$.
3. The intersection of $\mathcal{AC}$ is non-empty that is $\cap_{i=1}^k C^i \neq \emptyset$.

The player(s) in the intersection of the asymmetric coalition set is the *strong player(s)*. The remaining players in the asymmetric coalition structure are the *weak players* and the player in the complement is the *complement player(s)*. From the definition it follows directly that there must be at least two weak players.

Chapter 3 Section 3.5 shows that the cooperative game process moves often along faces of the core polytope. In this chapter we will show that the cooperative game process exhibits interesting behavior when "drifting" along faces generated by an asymmetric coalition set. The face generated by an asymmetric coalition set is called an *asymmetric* *face*. Observe that two distinct 2-player coalitions in the 4-player setting form an asymmetric coalition set. The asymmetric face generated by two distinct two player coalitions in a game $(v, 4)$ is called an *asymmetric edge*.

Section 5.2 shows that the strong player can have extra power compared to the weak players. The strong player(s) are member of all coalitions in the structure. Under the dynamics of the cooperative game process in the neighborhood of a asymmetric face the strong player only reduces his demand in states where no coalition is binding, whereas the weak players can reduce demands in other states.

The next Theorem (5.1) shows that a cooperative game process for a superadditive N-player game can only transition out of the core via a state on an asymmetric face. On states outside the core at least one of the coalitions of the asymmetric coalition set is strictly feasible. As long as a coalition in the asymmetric coalition structure is feasible the strong player will not reduce demands. This gives power to the strong player: he can increase demands if chosen to update however he cannot reduce demands. This benefits the strong player's power as Section 5.2 shows. However the cooperative game process leaving the core is not a necessary condition for power to the strong player to exist as Section 5.4 shows.

Theorem 5.1. *Suppose for a game (v, N) and a state $\mathbf{d}$ in the core there exist $i, j \in \{1, ..., N\}$ such that the neighbor $\mathbf{d}(i, j)$ is not in the core. Then if $p_{\mathbf{d}}^{i,j} > 0$ there must be two coalitions in one asymmetric coalition set that are binding at $\mathbf{d}$ and at $\mathbf{d}(i)$.*

Proof: In order for $\mathbf{d}$ to be in the core and its neighbor $\mathbf{d}(i, j)$ to be outside the core, at $\mathbf{d}(i)$ there must be at least two binding coalitions; one of them must be $C'(d(i))$, let $C = C'(d(i))$ and the other coalition be called C^*. We show that for $p_{\mathbf{d}}^{i,j} > 0$ all three conditions below must hold.

1. Neither of the two coalitions can be a subset of the other.

2. The union of C and C^* cannot be the grand coalition.

3. The intersection of the two coalitions C and C^* cannot be empty.

We show by contradiction that all three conditions must hold.

1. Otherwise their union is $C'(d(i))$ and as no player in $C'(d(i))$ reduces demands from $\mathbf{d}(i)$ it follows that $p_{\mathbf{d}}^{i,j} = 0$.

2. As $\mathbf{d}$ is in the core, for both coalitions to be binding at $\mathbf{d}(i)$ there must exist a player i that is not a member of either of them and so is not in their union. If their union is the grand coalition C^G then there cannot exist such a player i.

3. If the intersection of the two coalitions is empty and both coalitions are binding, then their union must be binding by superadditivity and their union is $C'(d(i))$, the preferred payable coalition at $\mathbf{d}(i)$, and so player j is member of $C'(d(i))$ and cannot reduce demands at $\mathbf{d}(i)$.

Observe that the above conditions imply that C and C^* must be in the same asymmetric coalition set.

We have shown that for $p_{\mathbf{d}}^{i,j} > 0$ all of the above conditions must apply to a pair of coalitions C and C^* and states $\mathbf{d}$ in the core and its neighbor $\mathbf{d}(i,j)$ outside the core.

Observe that if at $\mathbf{d}(i)$ (and hence at $\mathbf{d}$ as well) two coalitions $C'(d(i))$ and C^* are binding. Then there must exist a $j \in C^*$ not in $C'(d(i))$ since by assumption C^* is not a subset of $C'(d(i))$. We conclude that if $\mathbf{d}$ is in the core and its neighbor $\mathbf{d}(i,j)$ is not in the core, and if $p_{\mathbf{d}}^{i,j} > 0$ then at $\mathbf{d}$ and at $\mathbf{d}(i)$ two coalitions in the same asymmetric coalition set must be binding.

□

Lemma (5.2) shows that the only asymmetric coalition set for a game $(v,4)$ can be a set including two distinct 2-player coalitions with non-empty intersection. This is an important result, it partitions the set of 4-player games $(v,4)$ into two sets: Games where no asymmetric coalition set is present for which the inherent equity bias lets the process "drift" towards equity. The second set of 4-player games contains all games of special interest, and (5.2) shows that all games in this set include an asymmetric edge generated by two distinct two player coalitions with non-empty intersection.

Lemma 5.2. *Let $\mathcal{S} \subset \mathcal{P}\left(C^G\right)$ for a game $(v,4)$. Then $\mathcal{S}$ is an asymmetric coalition structure if and only if $\mathcal{S} = \{C_{ij}, C_{ik}\}$ for two distinct 2-player coalitions C_{ij} and C_{ik} with $\{i,j,k\}$ in $\{1,2,3,4\}$.*

Proof: We first show the 'only if' part. We analyze possible pairs of coalitions for $(v,4)$ that could form an asymmetric coalition set. A singleton coalition cannot have a non-empty intersection with another coalition without being a subset at the same time. So a singleton coalition cannot be part of an asymmetric coalition set. The union of a 3-player coalition with any other coalition that is not its subset is the grand coalition C_{1234}. So a 3-player coalition cannot be part of an asymmetric coalition set.

So it follows that all coalitions must be 2-player coalitions. There can be at most two 2-player coalitions in the asymmetric structure that must have non-empty intersection, otherwise their union is the grand coalition.

Now we show the 'if' part. If there are distinct 2-player coalitions C_{ij} and C_{ik} that have a non-empty intersection then there exists a player l that is not in their union and so all conditions for an asymmetric coalition set are satisfied. □

Lemma (5.3) follows directly from Theorem (5.1) and Lemma (5.2) and shows that for all 4-player games that do not have a face of the core polytope generated by two distinct 2-player coalitions the cooperative game process cannot leave the core polytope once in the core.

Lemma 5.3. *Let $(v,4)$ be a superadditive cooperative game and $\left(4, v, \frac{1}{M}\right)$ be a cooperative game process with non-empty core. Then if no state $\mathbf{d}$ in the core exists, where two distinct 2-player coalitions are binding, and the chain is in in the core, the chain cannot leave the core.*

Proof: We show that once in the core, the chain (V_t) cannot leave the core again. From Lemma (5.1) we know that if for a state $\mathbf{d} \in \Omega^C$ it holds that a neighbor $\mathbf{d}(i,j)$ is not in the core and $p_{\mathbf{d}}^{i,j} > 0$ then the state $\mathbf{d}$ must have two binding coalitions that are in one asymmetric coalition set. By Lemma (5.2) for a game $(v,4)$ an asymmetric coalition set consists of two distinct 2-player coalitions. By assumption there is no state in the core where two distinct 2-player coalitions are binding and so the chain (V_t) cannot leave the core. □

We now make a conjecture about (V,N) superadditive games.

Conjecture 5.4. *Suppose a superadditive balanced game (v,N) does not have a state* $\mathbf{d}$ *in the core, where two distinct coalitions in one asymmetric coalition set are binding. Then every state outside the core is transient.*

We give here the outline of a proof.

Proof: We first show that once in the core, the chain (V_t) cannot leave the core again. From Lemma (5.1) we know that if for a state $\mathbf{d} \in \Omega^C$ it holds that a neighbor $\mathbf{d}(i,j)$ is not in the core and $p_{\mathbf{d}}^{i,j} > 0$ then the state $\mathbf{d}$ must have two binding coalitions that are in one asymmetric coalition set and the first part of the result follows.

Now we show that from each state outside the core, the chain (V_t) can transition to the core with positive probability. If no coalition is strictly feasible at $\mathbf{d}$, the state $\mathbf{d}$ is in the core. Otherwise let C be the preferred strictly feasible coalition at $\mathbf{d}$. Then we claim that there exists a player $i \in C$ and a player $j \notin C$ such that at $\mathbf{d}(i)$ no coalition with j as member is binding and so $p_{\mathbf{d}}^{i,j} > 0$.

If at $\mathbf{d}$ no coalition is binding, any player in C can be taken as i and any player not in C can be taken as j and so there exists a pair i,j such that at $\mathbf{d}(i)$ no coalition with j as member is binding and so $p_{\mathbf{d}}^{i,j} > 0$. Suppose at $\mathbf{d}$ a coalition C^* is binding. If C^* is a subset of C then the case is identical to when no coalition is binding at $\mathbf{d}$. Otherwise we first show that the intersection of C and C^* is non-empty. We prove this by contradiction. Suppose the intersection was empty, then the union of C and C^* would be strictly feasible by superadditivity. But then as any coalition is preferred to its subsets the union would be the preferred strictly feasible coalition contradicting the assumption that C is the preferred strictly feasible coalition. So the intersection cannot be empty. Then we choose i from the intersection and j from $C^* \setminus C$.

So we need to show that there exists not another coalition C^{**} that is binding at $\mathbf{d}$ and includes j. Again C^{**} could not be in the complement of C by superadditivity so it would need to have a non-empty intersection with C.

We need to show that the following conditions cannot be met for all players i in the complement of C at the same time. We expect them to contradict the assumption of balancedness of the game (v,N).

C is strictly feasible at $\mathbf{d}$.

Each player j in the complement of C must be member of two coalitions that have separate non-empty intersections with C and are binding at $\mathbf{d}$.

If we can show that the above statement leads to a contradiction, (presumably contradicting the assumption of balancedness of the game (v,N)) then the conjecture holds as there exists a player $i \in C$ and a player $j \notin C$ such that at $\mathbf{d}(i)$ no coalition with j as member is binding and so $p_{\mathbf{d}}^{i,j} > 0$.

□

5.2 Power in 4-Player Cooperative Games

The concept of power for the cooperative game process

Harsanyi (1962a,1962b) analyses the measurement of social power in the 2-player bargaining game and for n-Person reciprocal Power Situations. Harsanyi defines different concepts and quantitative measurements for Social Power. The amount of a person's power "is a measure of the probability of his being able to achieve adoption of joint policies agreeing with his own preferences". "In this situation it is natural to define the amount of individual i's power over the joint policy of all n individuals as the probability of his being able to get his favorite joint policy adopted by all individuals." The amount of power of a player with respect to a preferred strategy is thus defined as the probability with which the player can "enforce" that strategy. Under the dynamics of the cooperative game process players cannot "enforce" strategies.

A more satisfactory power is Harsanyi's *vector measure* for i's power which is the vector of probabilities $p = (p_1, p_2, \dots p_n)$ for the adoption of each of the alternative policies $X_1, \dots, X_n$. For the cooperative game process an equivalent definition of the amount of power is the equilibrium distribution over states where a preferred outcome of a player is achieved. Harsanyi's definition of a vector measure of the amount of power ranks a players' strategies according to preference. The amount of power for each strategy is then given by the vector of probabilities with which the player can enforce the respective strategies.

In similar fashion, states with non zero equilibrium measure for the cooperative game process are ranked by the demand to a player, and then the vector of the amount of power is the respective equilibrium distribution of each state sorted by preference (magnitude of demand). For states that the player is indifferent (same demand to that player) the equilibrium distributions are summed. The strength of a player can then be defined as the weighted sum of the demands of a player at each state, where the weighting is the equilibrium distribution for that state. This corresponds to the expected demand of a player in equilibrium.

However the state space for the cooperative game process is in general large, ε is assumed to be small and the smaller ε is taken, the larger the state space is. It is tedious to work with a measure of power taking the whole state space into account. When ε tends to zero the Markovian cooperative equilibrium is the state which has limit mass one under the equilibrium distribution. So when ε tends to zero the vector of the amount of power has entries with limits zero for all states but the Markovian cooperative equilibrium and one for the Markovian cooperative equilibrium.

We define power to a player under the dynamics of the cooperative game process as the limit power when ε tends to zero, giving measure one to the Markovian cooperative outcome and zero to all other states. The *limit strength of power* or simply *power* to a player under the dynamics of the cooperative game process is the sum of the following two components:

- The *core power* of a player in a cooperative game process is the difference between a player's demand at the cooperative outcome and at equal split. This power is adapted from Harsanyi's or Selten's concept of power as the 'capability to secure more than the equitable share'. The 'force' generating the core power to a player is the constraints of the core.
- The *asymmetric power* of a player in a cooperative game process is the difference between a player's demand at the Markovian cooperative equilibrium and at the cooperative outcome. The asymmetric power is the 'capability to secure more than at the most equitable core allocation'. The 'force' generating the asymmetric power are the dynamics inherent in the asymmetric coalition structure. We believe this concept of power to be very interesting and novel to the study of (cooperative) game theory and stochastic learning processes.

Introducing three generic example games with asymmetric edge

The three example games from Section 2.5 are re-introduced next and their geometric structure is described. These are generic examples: Section 5.1 Lemma (5.2) shows that these example games constitute the only real "interesting" cases of 4-players games, in fact they represent the only 4-player games that having an asymmetric coalition set with an associated asymmetric edge in the core. Section 5.3 gives a heuristic analysis backed up by simulations to show, that each game has a Markovian cooperative equilibrium on the asymmetric edge but they appear on different parts of the edge. In this section, first for each example game for each player the core power and the asymmetric power of the players are examined.

Remember that $\mathbf{co^3}$ is the vector in the 3-core Ω^3 with the smallest $\mathbf{L}^2$-distance from μ.

Remember that an asymmetric edge $\mathcal{AE}$ of an asymmetric coalition structure $\mathcal{AC}$ for a game $(v,4)$ is a set of states $\mathbf{d} \in \Omega^C$ where all coalitions $C \in \mathcal{AC}$ are binding. For the example games the asymmetric edge is the face of the core polytope generated by two distinct two-player coalitions. All three example games are homeomorphic. The difference between the example games is the length and location of the asymmetric edge. The asymmetric edge of game 2 is a subset of the asymmetric edge of game 1, located at the upper end, the asymmetric edge of game 3 is as well a subset of the asymmetric edge of example game 1, located at the lower end of the asymmetric edge of game 1. Hence the demand of player one is significantly larger at each state on the asymmetric edge of game 2 than at any state on the asymmetric edge of game 3.

In each case $\{C_{12}, C_{13}\}$ is an asymmetric coalition structure. Concession limits are the extreme outcomes, between which the outcome must fall in bargaining between rational players Harsanyi (1962a, 1962b). Hence the two extreme states on the asymmetric edge where player one has largest demand and where player one has smallest demand are named the *upper concession limit* and the *lower concession limit* respectively. The Markovian cooperative equilibrium must lie between (inclusive) the upper and lower concession limits. In all three example games the cooperative outcome, the state in the core with smallest Euclidean norm, corresponds to the lower concession limit.

Given an asymmetric edge $\mathcal{AE} = \{\mathcal{H}\left(C_{ij}\right) \cap \mathcal{H}\left(C_{ik}\right)\}$ let $\mathbf{max1} \in \mathcal{AE}$ be the *upper concession limit* the unique state with maximum demands for player i in $\mathcal{AE}$ and $\mathbf{min1} \in \mathcal{AE}$ be the *lower concession limit*, the unique state with minimum demands for player i in $\mathcal{AE}$.

We look at the geometric structure of example games 1, 2 and 3. We describe the sets Ω^E, Ω^3, Ω^C and the asymmetric edges. Figure 5.1 shows Ω^C for example game 1. In the figure, the asymmetric edge $\mathcal{AE}^1$ is depicted as a red line.

In figure 5.2 we depict a sketch of $\mathcal{AE}^1 = [(0.53, 0.22, 0.21, 0.04), (0.61, 0.14, 0.13, 0.12)]$. The asymmetric edge of game 2,

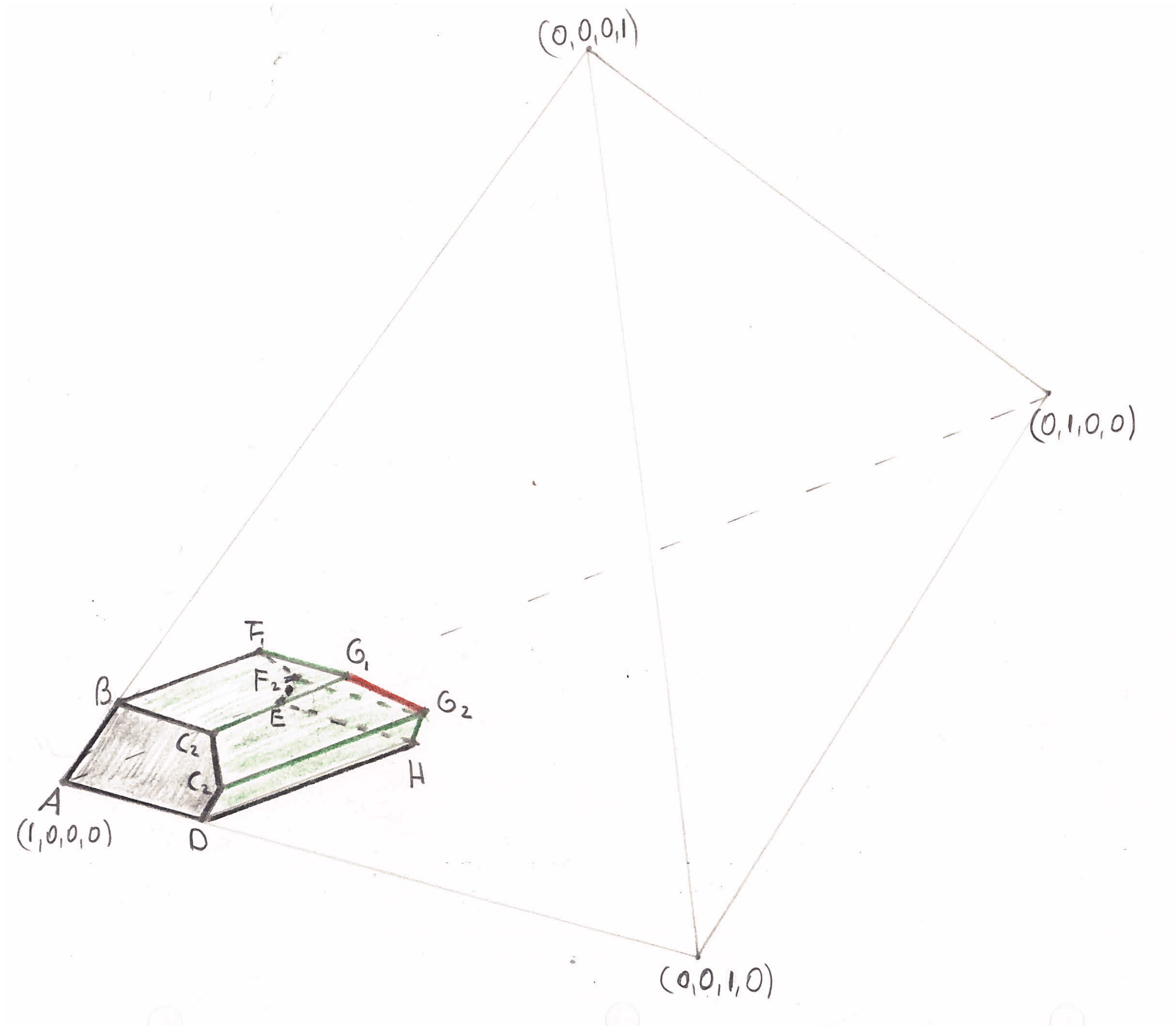


Figure 5.1 Graphical representation of Ω^C for game 1.

$\mathcal{AE}^2 = [(0.6, 0.15, 0.14, 0.11), (0.61, 0.14, 0.13, 0.12)]$, is a subset of $\mathcal{AE}^1$ and depicted in green. The asymmetric edge of game 3,

$\mathcal{AE}^3 = [(0.53, 0.22, 0.21, 0.04), (0.54, 0.21, 0.20, 0.05)]$, is as well a subset of $\mathcal{AE}^1$ and depicted in light blue.

The below description of the three example games uses the terms “negative drift” and “positive drift” down and up respectively the asymmetric edge. The introduction in Section 2.5 described two dynamics present along the asymmetric edge: The inherent equity bias of the cooperative game process, causing a drift “down” the asymmetric edge as player one has largest demands and in states where no coalition is binding he will reduce his demand more frequently than the other players. The second dynamic along

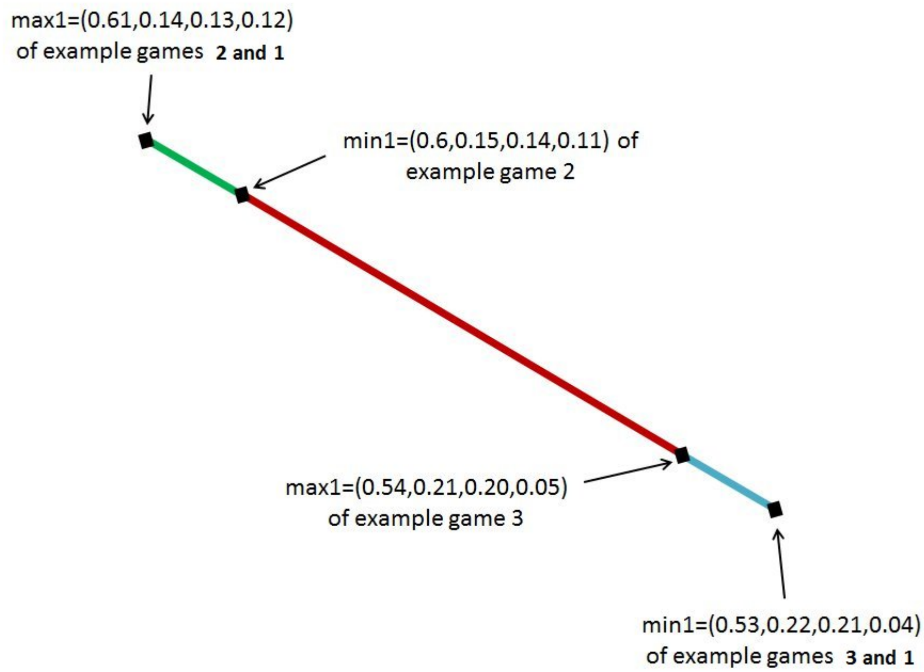


Figure 5.2 Descriptive sketch of $\mathcal{AE}^1$.

the asymmetric edge is the asymmetric power to player one: whenever any coalition generating the asymmetric edge is binding he will not reduce his demand. Thus on a trajectory of the cooperative game process from the asymmetric edge away and back to the asymmetric edge these two dynamics compete.

Between successive returns to $\mathcal{AE}$, the chain (V_t) visits different kinds of states. For the first step, (V_t) is on the asymmetric edge, and in subsequent steps it leaves the asymmetric edge and may leave the core. At states on **AE**, and states not in the core, (V_t) has a positive drift in d^1 over one step of (V_t). Lemma (5.8) shows that at all other states of the state space of games $1-3$ the drift in d^1 is negative.

To understand, what the overall drift is, over an excursion of (V_t) from the asymmetric edge to another state on the asymmetric edge, we need to approximate the probability, "in equilibrium", that the chain lies on $\mathcal{AE}$ or outside Ω^C. On the 'upper part' of $\mathcal{AE}^1$, during an excursion of the chain (V_t) the value of d^1 and d^4 are larger, and the value of d^2 and d^3 are smaller, than on the 'lower part' of $\mathcal{AE}^1$. The difference in demands implies different transition probabilities between states. As a result,"in equilibrium", the chain (V_t) spends more time on the asymmetric edge or outside the core in the 'lower part' of the asymmetric edge than in the 'upper part' and so the expected change in d^1 over an excursion from the asymmetric edge is negative for game 2 and positive for game 3.

A detailed analysis of these dynamics and mathematical solutions are presented in Section 5.3. For now the term "negative drift" (in the demand of player one) on the asymmetric edge means that the dynamic caused by the equity bias is stronger and "positive drift" means that the asymmetric power dynamic is stronger.

example game						
	$v(C_{1234})$	$v(C_{123})$	$v(C_{124})$	$v(C_{134})$	$v(C_{12})$	$v(C_{13})$
1	1	0.88	0.79	0.78	0.75	0.74
2	1	0.88	0.86	0.85	0.75	0.74
3	1	0.95	0.79	0.78	0.75	0.74

Table 5.1 $v(C)$ for all coalitions C with $v(C) \neq 0$ for example games 1, 2 and 3

1. For example game 1 the upper concession limit is $(0.61, 0.14, 0.13, 0.12)$, the lower concession limit is $(0.53, 0.22, 0.21, 0.04)$.

   In the neighborhood around the lower concession limit, there is a strong positive drift in the demand of player one based on the probability distribution "in equilibrium" over the 2-dimensional localized chain. The random walk cooperative game process drifts up the asymmetric edge.

   In the neighborhood around the upper concession limit, there is a strong negative drift in the demand of player one based on the probability distribution "in equilibrium" over the 2-dimensional localized chain. The random walk cooperative game process drifts "down" the asymmetric edge. For game one the Markovian cooperative equilibrium is the state on the asymmetric edge where the drift in d^1 is zero. The drift in d^1 for the restricted localized chain is zero at

   $(0.583586, 0.166414, 0.156414, 0.093586)$ and so the Markovian cooperative equilibrium is in the "interior" of the asymmetric edge.

   The vector of the core power of the players is given by the difference between the cooperative outcome and equal split and equals $(0.28, -0.03, -0.04, -0.21)$. The vector of the asymmetric power is given by the difference between the Markovian cooperative outcome and the cooperative outcome and is given approximately by

$(0.054, -0.054, -0.054, 0.054)$. The total power to each player is then given by the sum of the core power and the asymmetric power and equals

$(0.334, -0.084, -0.094, -0.156)$. Observe that the complement player four has the same asymmetric power as player one. He basically gets a "free power ride" from player one. A natural bound to the strong player's asymmetric power is the amount of free ride that the complement player can get: the complement player can never have more total power than the weak players.

2. For example game 2 the upper concession limit is $(0.61, 0.14, 0.13, 0.12)$, the lower concession limit is $(0.6, 0.15, 0.14, 0.11)$. There is a strong negative drift at each state of the asymmetric edge (apart from states in close proximity to the lower concession limit) and so the Markovian cooperative equilibrium is the lower concession limit (the cooperative outcome). The core power of each player is given by $(0.35, -0.1, -0.11, -0.14)$. Since the Markovian cooperative equilibrium is the cooperative outcome the asymmetric power is zero for all players.

3. For example game 3 the upper concession limit is given by $(0.54, 0.21, 0.2, 0.05)$, the lower concession limit is given by $(0.53, 0.22, 0.21, 0.04)$ There is a strong positive drift at each state of the asymmetric edge (apart from states in close proximity of the upper concession limit) and so the Markovian cooperative equilibrium is the upper concession limit.

   The core power of each player is given by $(0.29, -0.04, -0.05, -0.2)$. The asymmetric power is given by $(0.01, -0.01, -0.01, 0.01)$ and so the total power to each player is given by $(0.3, -0.05, -0.06, -0.19)$.

   To understand the behavior of the cooperative game process in the neighborhood of the upper concession limit we simulated the chain for example game 3 and calculated the drift for states in the close neighborhood of the upper concession limit. In figure (2.2) the drifts are sketched for different states where $\varepsilon = 0.000001$. We see that the drift becomes smaller but stays positive even very close to the upper concession limit.

In the neighborhood of the 'upper part' of $\mathcal{AE}^1$, the transitions of the chain (V_t) for example game 1 and 3 are identical until the chain in example game 3 hits the hyperplane $\mathcal{H}(C_{123})$; the transitions for example game 1 and 2 are identical in the neighborhood of the 'lower part' of the $\mathcal{AE}^1$ until the chain (V_t) in example game 2 hits one of the hyperplanes $\mathcal{H}(C_{124})$ or $\mathcal{H}(C_{134})$. So we focus our analysis on the asymmetric edge of example game 1.

We expect, that the Markovian cooperative equilibrium is the state on the asymmetric edge, where the drift in d^1 is zero. The results indicate that, for $z_0 \approx 0.058$, the expected change in d^1 between successive returns to the asymmetric edge is negative for states in $\mathcal{AE}^1$ with $d^1 > 0.53 + z_0$ and positive for states with $d^1 < 0.53 + z_0$. So the long term behavior of the chain (V_t) is to drifts along the asymmetric edge towards $\mathbf{min1} = \mathbf{co}$ for example game 2 and so $\mathbf{mce} = \mathbf{co}$. For example game 3, the long term behavior of the chain (V_t) is to drift along the asymmetric edge towards $\mathbf{max1} \neq \mathbf{co}$ and so $\mathbf{mce} = \mathbf{max1} \neq \mathbf{co}$. For example game 1, we expect, that the Markovian cooperative equilibrium is the state at which the expected change in d^1 between successive returns to the asymmetric edge is 0, and so $\mathbf{mce} \approx (0.53 + z_0, 0.22 - z_0, 0.21 - z_0, 0.04 + z_0)$.

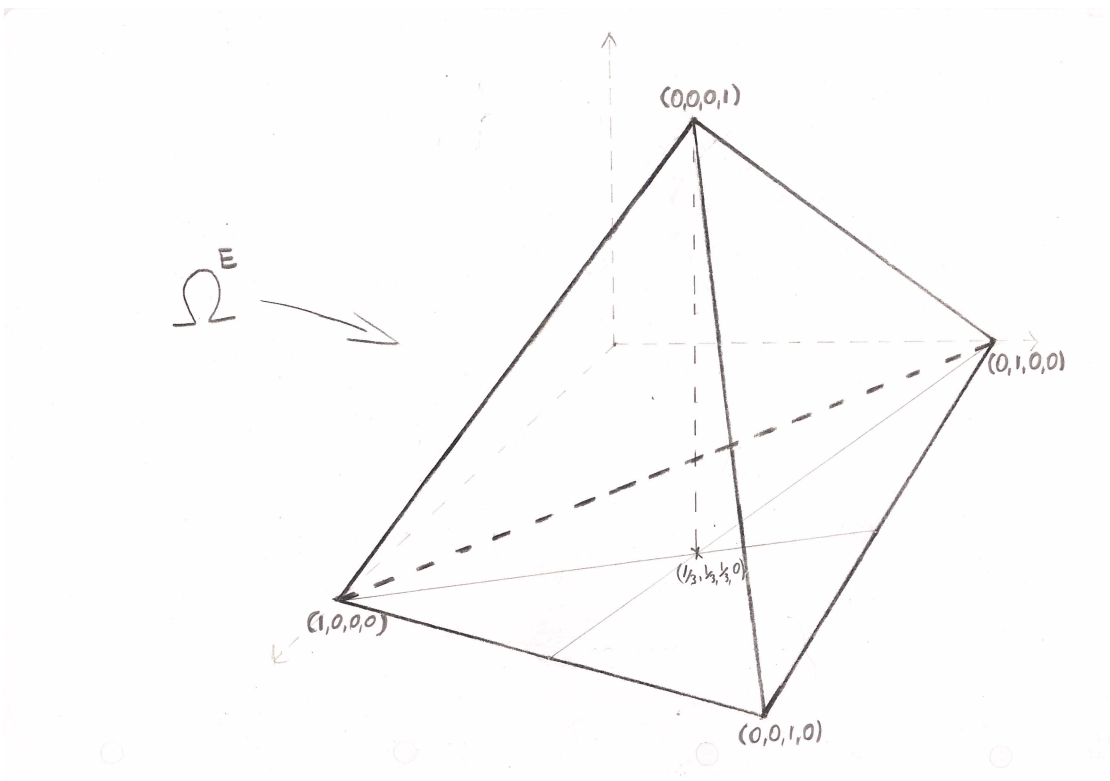


Figure 5.3 Graphical representation of Ω^E for game 1.

We look in detail at the geometric structure of example games 1, 2 and 3.

The set Ω^E of efficient states forms a 3-dimensional convex polytope in $\mathbb{R}^4$. A sketch of Ω^E is given in figure (5.3).

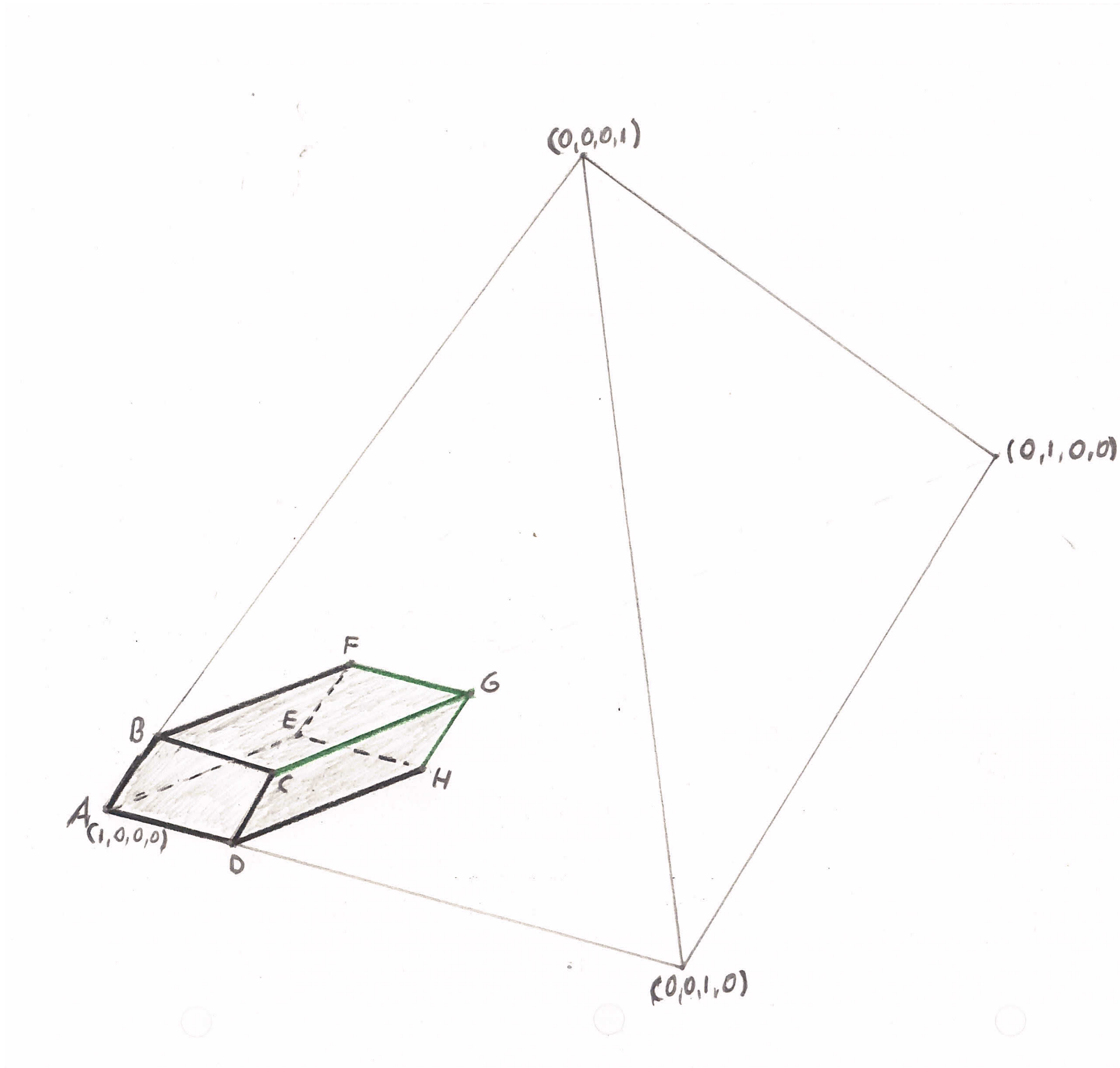


Figure 5.4 Graphical representation of Ω^3 for game 1.

The 3-core Ω^3 of example game 1 is given by

$$\{0.39+x+y+z, 0.22-x, 0.21-y, 0.2-z\}$$

for $x,y,z \in \mathbb{R}^+$ and $x \leq 0.22, y \leq 0.21, z \leq 0.2$. The set Ω^3 forms a 3-dimensional convex polytope and is a subspace of Ω^E. A sketch of Ω^3 is given in figure 5.4.

The vertices of Ω^3 are:

$A = (1,0,0,0)$, $B = (0.89,0,0,0.11)$, $C = (0.68,0,0.21,0.11)$, $D = (0.79,0,0.21,0)$, $E = (0.78,0.22,0,0)$, $F = (0.67,0.22,0,0.11)$, $G = (0.46,0.22,0.21,0.11)$ and

$H = (0.57,0.22,0.21,0)$.

In Figure 5.1 the sets $\mathcal{H}(C)$ are depicted in light green. Remember from Theorem (3.11), that, for major coalition games, the probability, that the chain (V_t) is 'far away' from the hyperplane, where a coalition is binding, is very low. Observe that in the neighborhood, where only one coalition is binding, the chain (V_t) behaves like in a major coalition game and so will not move 'far away' from states where a coalition is binding.

- $\mathcal{H}(C_{123})$ with vertices B,F_1,G_1 and C_1.
- $\mathcal{H}(C_{124})$ with vertices D,C_2,G_2 and H.
- $\mathcal{H}(C_{134})$ with vertices E,F_2,G_2 and H.
- $\mathcal{H}(C_{12})$ with vertices F_2,F_1,G_1 and G_2.
- $\mathcal{H}(C_{13})$ with vertices C_2,C_1,G_1 and G_2.

In Figure 5.5 the asymmetric edge is depicted as solid red line. All other edges are depicted as solid green lines.

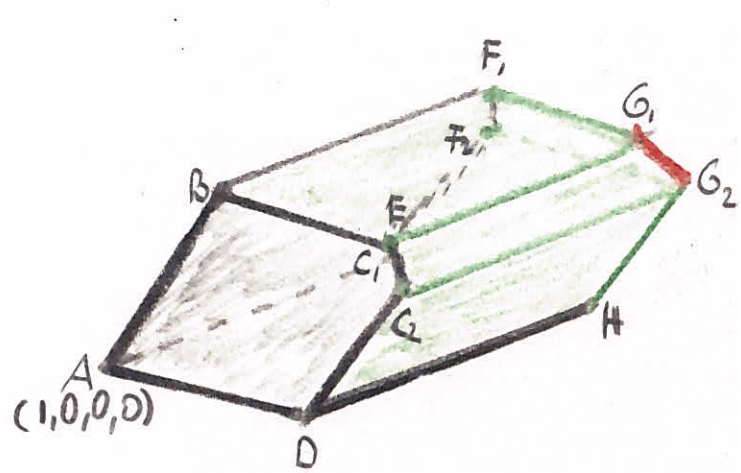


Figure 5.5 Graphical representation of Ω^C for game 2.

For C^1 and $C^2 \in \mathcal{P}(C^G)$ let $\mathcal{E}(C^1,C^2) = \{\mathcal{H}(C^1) \cap \mathcal{H}(C^2)\}$.

Example games 1,2 and 3 have 6 edges.

- $\mathcal{E}(C_{123},C_{12})$ with vertices F_1 and G_1.

- $\mathcal{E}(C_{123}, C_{13})$ with vertices C_1 and G_1.
- $\mathcal{E}(C_{124}, C_{134})$ with vertices G_2 and H.
- $\mathcal{E}(C_{124}, C_{12})$ with vertices F_2 and G_2.
- $\mathcal{E}(C_{134}, C_{13})$ with vertices C_2 and G_2.
- $\mathcal{E}(C_{12}, C_{13})$ with vertices G_1 and G_2.

The core of example game 1 is given by

$$\{0.53 + x, 0.22 - y, 0.21 - z, 0.04 + y + z - x\}$$

for $x, y, z \in \mathbb{R}^+$, $x \geq y, z$ and $x \leq 0.08$. The core forms a 3-dimensional convex polytope and is a subset of Ω^E. A sketch of the core is given in figure (5.1).

The vertices of Ω^C for game 3 are:

$A = (1,0,0,0)$, $B = (0.89,0,0,0.11)$, $C_1 = (0.75,0,0.14,0.11)$, $C_2 = (0.75,0,0.21,0.04)$, $D = (0.79,0,0.21,0)$, $E = (0.78,0.22,0,0)$, $F_1 = (0.74,0.15,0,0.11)$, $F_2 = (0.74,0.22,0,0.04)$, $G_1 = (0.6,0.15,0.14,0.11)$, $G_2 = (0.53,0.22,0.21,0.04)$ and $H = (0.57,0.22,0.21,0)$.

Drift towards the asymmetric edge for states in the extended core of example games $1-3$

This subsection first defines the extended core for games $(v,4)$ with an asymmetric edge. The extended core needs to be defined as from states on the asymmetric edge the cooperative game process can leave the core, as shown in Theorem (5.1). Then Lemma (5.6) shows that all states outside the extended core are transient for example games $1-3$. Finally Lemma (5.8) shows, that the chain (V_t) drifts towards **co** from any point in the core not on the asymmetric edge. Combined these results show that the interesting behavior of the cooperative game process for example games $1-3$ is happening around the asymmetric edge: at all other states in the set of recurrent states there is a drift towards **co** and since **co** is the lower concession limit situated on the asymmetric edge the process drifts towards the asymmetric edge. Then after these results Section 5.3 finally analyzes the behavior of the cooperative game process along the asymmetric edge.

Suppose a game $(v,4)$ has an asymmetric coalition set $\mathcal{AC} = \{C, C^*\}$ where C is preferred to C^* in the ordering introduced in Section 3.1 (that is C will be the payable coalition if both are feasible). Then let $\Omega^\pi = \Omega^C \cup \{\mathbf{d}' \in \Omega^E \mid \sum_{i \in C^*} d'^i < v(C^*)\}$ be the *extended core*.

Recall that a coalition C is strictly feasible at $\mathbf{d} \in \Omega^\varepsilon$ if $\sum_{i \in C} d^i < v(C)$.

Lemma 5.5. *Suppose at $\mathbf{d} \in \Omega^E$ a coalition C^1 is strictly feasible and another coalition C^2 is binding with $C^1 \cap C^2 = \emptyset$. Then $C^1 \cup C^2$ must be strictly feasible.*

Proof: By superadditivity it holds that $v\left(C^1 \cup C^2\right) \geq v\left(C^1\right) + v\left(C^1\right)$. Since C^1 is strictly feasible and C^2 is binding, and since C^1 and C^2 have an empty intersection it holds that $\sum_{i \in C^1} d^i + \sum_{j \in C^2} d^j = \sum_{k \in C^1 \cup C^2} d^k < v\left(C^1\right) + v\left(C^2\right) \leq v\left(C^1 \cup C^2\right)$ and the result follows. □

Lemma 5.6. *Let $(v,4)$ be example game 1,2 or 3 as defined in 5.2 subsection 2 and $\left(4, v, \frac{1}{M}\right)$ be a cooperative game process. Then all states outside the extended core are transient.*

Proof: We shall prove first that from any state in the set Ω^π the chain V_t cannot transition to any state $\Omega^E \setminus \Omega^\pi$.

A player can only reduce demands with positive probability if he is not member of the largest feasible coalition $C'(\mathbf{d})$. For the chain V_t to move out of the extended core there needs to be a state $\mathbf{d}$ in the extended core and a neighbor $\mathbf{d}(i,j)$ with $p_{\mathbf{d}}^{i,j} > 0$ such that a coalition with j as member has $CS^{\mathbf{d}(i,j)} < 0$, that is a state where the coalition's surplus of a coalition with j as member is negative.

As C_{123} has the largest coalition value with 0.88 the chain V_t cannot transition to a state with $CS^{123}(\mathbf{d}(i,j)) > 0$ from a state with $CS^{123}(\mathbf{d}) = 0$ as C_{123} will always be the coalition with the largest sum of demands on Ω^π.

Suppose V_t is in a state $\mathbf{d} \in \Omega^\pi$ such that $CS^{C_{134}}(\mathbf{d}) = 0$. We want to show that it is not possible for the chain V_t to transition to a new state $\mathbf{d}$ with $CS^{C_{134}}(\mathbf{d}(i,j)) = -\varepsilon$.

To transition to a new state $\mathbf{d}(i,j)$ with $CS^{C_{134}}(\mathbf{d}(i,j)) = -\varepsilon$, d^2 needs to increase and one of d^1, d^3, d^4 need to decrease. This implies that at $\mathbf{d}(2)$ a coalition with $v(C) > 0.78$ must be feasible. The only two coalitions with larger coalition value than 0.78 are

C_{123} and C_{124}. The fact that $\mathbf{d}$ is in Ω^{π} implies that if either of the two coalitions is feasible it must be binding. (That is $d^1+d^2+d^4=0.79$ or $d^1+d^2+d^3=0.89$.) Suppose that $CS^{C_{124}}(\mathbf{d}(2))=0$. This implies that $CS^{C_{124}}(\mathbf{d})=-\varepsilon$ which contradicts the fact that $\mathbf{d}$ is in Ω^{π}. Suppose that $CS^{C_{123}}(\mathbf{d}(2))=0$. This implies that $CS^{C_{123}}(\mathbf{d})=-\varepsilon$ which contradicts the fact that $\mathbf{d}$ is in Ω^{π} and so the chain V_t will not transition to any state with $CS^{C_{134}}(\mathbf{d}(i,j))=-\varepsilon$.

Suppose V_t is in a state $\mathbf{d}\in\Omega^{\pi}$ such that $CS^{C_{124}}(\mathbf{d}(i,j))=0$. Clearly d^3 needs to increase and one of d^1,d^2,d^4 need to decrease for this transition to happen. As only C_{123} has a larger coalition value than C_{124} at $\mathbf{d}(3)$ there can be no binding coalition with larger sum of demands and so the chain V_t will not transition to any state with $CS^{C_{124}}(\mathbf{d}(i,j))=-\varepsilon$.

Next we will show that if V_t is in a state $\mathbf{d}\in\Omega^{\pi}$ such that $CS^{C_{12}}=0$ it is is not possible for the chain to transition to a state $\mathbf{d}$ with $CS^{C_{12}}(\mathbf{d}(i,j))=-\varepsilon$.

Clearly d^3 or d^4 need to increase and one of d^1 or d^2 need to decrease for this transition to happen. So at $\mathbf{d}(3)$ or at $\mathbf{d}(4)$ a coalition with $v(C)\geq 0.75$ must be feasible. The only coalitions with $v(C)\geq 0.75$ are 3-player coalitions. As $\mathbf{d}$ is in Ω^{π} this implies that the coalition must be binding.

At $\mathbf{d}(3)$ or $\mathbf{d}(4)$ coalition C_{134} cannot be binding as otherwise $CS^{C_{134}}(\mathbf{d})=\varepsilon$ which contradicts the fact that $\mathbf{d}\in\Omega^{\pi}$.

Suppose at $\mathbf{d}(3)$ coalition C_{124} is binding. Since $C_{12}\subset C_{124}$ neither d^2 nor d^1 can decrease demands from $\mathbf{d}(3)$. Suppose at $\mathbf{d}(4)$ coalition C_{123} is binding. Since $C_{12}\subset C_{123}$ neither d^1 nor d^2 can decrease demands from $\mathbf{d}(4)$. We conclude that the chain V_t will not transition to a state with $CS^{C_{12}}=-\varepsilon$.

We have shown that from any state $\mathbf{d}$ in Ω^{π} the chain cannot transition to a state in $\Omega^{E}\setminus\Omega^{\pi}$.

Now we will show that the chain V_t will transition to a state in Ω^{π} with positive probability from any state in Ω^{E}.

Suppose that the chain V_t is in a state where C_{123} is the largest feasible coalition. This implies that $d^4\geq 0.12$. As long as C_{123} remains feasible, if d^4 increases and hence the chain transitions to the intermediate state $\mathbf{d}(4)$ the only possible transition from $\mathbf{d}(4)$ is that d^4 decreases again as $C'(\mathbf{d}(4))=C_{123}$. As long as C_{123} remains strictly feasible one of d^1, d^2, d^3 will increase and d^4 will decrease until the chain has reached a state $\mathbf{d}$ where C_{123} is binding, that is where $d^4 = 0.12$.

In such a state $\mathbf{d}$ where C_{123} is binding, if d^4 increases d^4 will have to decrease straight away again. If one of d^1, d^2, d^3 increase, C_{123} is not feasible anymore. Suppose $C'(\mathbf{d}(i)) = C_{124}$ for $i \in [1,2,3]$. So d^3 will have to decrease from $\mathbf{d}(i)$ and until the chain reaches a state with $d^4 = 0.11$ and $d^3 = 0.21$ the only possible transitions between efficient states are that d^1 or d^2 will increase and d^3 will decrease.

At a state where $d^4 = 0.12$ and $d^3 = 0.21$ both C_{123} and C_{124} are binding.

Suppose that at $C'(\mathbf{d}(i)) = C_{134}$ for $i \in [1,2]$. That implies that $d^2 \geq 0.22$. At such a state $\mathbf{d}$ it holds that $C'(\mathbf{d}(4)) = C_{123}$ and $C'(\mathbf{d}(3)) = C_{124}$. The only possible transitions between distinct efficient states are that d^1 increases and d^2 decreases. This will happen until $d^2 = 0.22$.

At a state where $d^4 = 0.12$, $d^3 = 0.21$ and $d^2 = 0.22$ it holds that $C'(\mathbf{d}(4)) = C_{123}$, $C'(\mathbf{d}(3)) = C_{124}$, $C'(\mathbf{d}(2)) = C_{134}$ and $C'(\mathbf{d}(1)) = C_{12}$. So the only possible demand transitions between efficient states are that d^1 increases and d^4 or d^3 decreases. So $p_{\mathbf{d}}^{1,4} > 0$. This is true as well for all states of the form $(0.45 + k\varepsilon, 0.22, 0.21, 0.12 - k\varepsilon)$ for $0 \leq k\varepsilon \leq 0.07$. So there is a path of positive probability to the state where $d^1 = 0.53$. Observe that $(0.53, 0.22, 0.21, 0.04)$ is in Ω^π.

Suppose that at any state $\mathbf{d} \in \Omega^E$ the largest feasible coalition is C_{124}. The chain V_t will transition in similar fashion to the above paragraphs only that at the state where $d^3 = d^4$ the largest feasible coalition will be C_{123} and the chain V_t will transition as described in the third paragraph of this proof. A similar argument holds for a state $\mathbf{d} \in \Omega^E$ where the largest feasible coalition is C_{134}.

Hence from any state $\mathbf{d} \notin \Omega^\pi$ the chain will transition to the state $(0.53, 0.22, 0.21, 0.2)$ which is in Ω^π.

□

Corollary 5.7. *The only states in Ω^C from which the chain (V_t) can transition to a state in the set $\Omega^\pi \setminus \Omega^C$ are states where C_{12} and C_{13} are binding.*

Proof: This follows directly from 5.1. □

Lemma (5.6) and Lemma (5.7) are proved only for game 1 but the proof for games 2 and 3 is identical if one exchanges the coalitional worths of the 3-player coalitions of game c) with the respective coalitional worths of the 3-player coalitions from games 2 or 3.

We now prove that at each state in the extended core not on the asymmetric edge the chain (V_t) has a negative drift in d^1.

For $\mathbf{d} \in \Omega^E$, let $\Delta(V_t, \mathbf{d}) = \mathbb{E}\left(d^1_{t+1} - d^1_t \mid \mathbf{d}\right)$ be the drift in d^1 at $\mathbf{d}$.

Lemma 5.8. *If $(v, 4)$ is the example game $1, 2$ or 3, and $\varepsilon = \frac{1}{M}$ for some v-compatible M, then at all states in the core $\mathbf{d} \in \Omega^\pi \setminus \mathcal{AE}$ where $d^1 = k$, $\Delta(V_t, \mathbf{d}) \leq \frac{1-2k}{4(1+\varepsilon)}$.*

Proof: Observe that for example games $1-3$ for a state $\mathbf{d} \notin \mathcal{AE}$ there exists always a player $i \neq 1$ such that $C'(\mathbf{d}(i)) = \emptyset$. So the transition to the neighbor $\mathbf{d}(i,1)$ has probability $p^{i,1}_{\mathbf{d}} = \frac{d^1}{4(1+\varepsilon)}$.

Furthermore, since player 1 is member of all coalitions in $\mathcal{P}\left(C^G\right)$ for example games $1-3$ it holds that at $\mathbf{d}(1)$ the payable coalition is always the empty set and so player 1 increases his demand from every state with probability $\frac{1-d^1}{4(1+\varepsilon)}$. So an upper bound for $\Delta(V_t, \mathbf{d})$ for all states not on the asymmetric edge is given by $\frac{1}{4(1+\varepsilon)}\left(1 - 2d^1\right)$ and if $d^1 = k$ at $\mathbf{d}$ it holds that $\Delta(V_t, \mathbf{d}) \leq \frac{1-2k}{4(1+\varepsilon)}$.

□

Observe that at each state $\mathbf{d} \in \Omega^C$ d^1 is strictly larger than 0.5 and so for example games $1-3$, at each state in the core, d^1 has a negative drift. The dynamics at states in the extended core that are not in the core (states where coalition C_{13} is strictly feasible) are described in the next section and covered in the behavior of the cooperative game process in the neighborhood of the asymmetric edge.

5.3 Determining the Winner: Equity Bias vs Asymmetric Power

Since **co** is in the asymmetric edge, it follows from Lemma (5.8), that the chain (V_t) drifts until it reaches the neighborhood of the asymmetric edge. Now the long-term behavior of (V_t) depends on the drifts in this neighborhood. As d^1 varies along the asymmetric edge, we investigate the drift in d^1 between successive returns of the chain (V_t) to the asymmetric edge.

Between successive returns to $\mathcal{AE}$, the chain (V_t) visits different kinds of states. For the first step, (V_t) is on the asymmetric edge, and in subsequent steps it leaves the asymmetric edge and may leave the core. At states on **AE**, and states not in the core, (V_t) has a positive drift in d^1 over one step of (V_t).

To understand, what the overall drift is, over an excursion of (V_t) from the asymmetric edge to another state on the asymmetric edge, we need to approximate the probability, "in equilibrium", that the chain lies on the different sets of states that it visits over an excursion. The difference in demands of the players implies different transition probabilities between states. As a result,"in equilibrium", the chain (V_t) spends more time on the asymmetric edge or outside the core in the 'lower part' of the asymmetric edge than in the 'upper part' and so the expected change in d^1 over an excursion from the asymmetric edge is negative for game 2 and positive for game 3.

In subsection 5.3.1 the "idealized" version of the chain (V_t) is introduuced via the definition of the "idealized" transition probabilities. To calculate the transition probabilities of the process along the asymmetric coalition structure we apply a localization technique (or trick). Demands on an excursion between two states on the asymmetric edge are assumed to be constant. If ε is sufficiently small this is a reasonable assumption.

In subsection 5.3.2 the different kinds of states are partitioned into sets according to the chain of joint coalition surplus' for coalition C_{12} and coalition C_{13}. In Chapter 3 Section 3.5 a random walk approximation to an individual coalition's surplus chain was used to bound the return times of a chain $\left(\frac{CS^C}{\varepsilon}\right)$. In Section 5.4 a similar approximation to the joint chain $\left(\frac{CS^{C_{12}}}{\varepsilon}, \frac{CS^{C_{13}}}{\varepsilon}\right)_t$ is used to calculate a bound on the probability of being in a state on the asymmetric edge. In this section since there is dependence between the two coalition's surplus chains exact probabilities are required to calculate the equilibrium distribution and so the transition probabilities of the chain $\left(\frac{CS^{C_{12}}}{\varepsilon}, \frac{CS^{C_{13}}}{\varepsilon}\right)_t$ are calculated explicitly. For example the probability of the chain to go from a state $(0,0)$ to a state $(0,1)$ or $(1,-1)$ for example are calculated. In fact in this section given a state of the two-dimensional chain $\left(\frac{CS^{C_{12}}}{\varepsilon}, \frac{CS^{C_{13}}}{\varepsilon}\right)_t$ all transition probabilities to states where either coordinate is increased or decreased by one are calculated.

The "idealized" transition probabilities of the chain (V_t) are identical on the different sets corresponding to states of the two-dimensional chain $\left(\frac{CS^{C_{12}}}{\varepsilon}, \frac{CS^{C_{13}}}{\varepsilon}\right)_t$ of the coalitions' surpluses.

Then in subsection 5.3.3 the expected change in d^1 over all sets of the state space of the idealized version of the chain $\left(\frac{CS^{C_{12}}}{\varepsilon}, \frac{CS^{C_{13}}}{\varepsilon}\right)_t$ are calculated.

In subsection 5.3.4 the state space of the two-dimensional idealized chain is restricted to some integer multiples for each coordinate. Since the extended core does not contain any state with negative coalition's surplus of C_{12} the first coordinate is at least 0. To restrict the chain at a fixed coordinate, all transition probabilities from a given state to states that would have a larger coordinate than the restriction are set to zero and their original mass is added to the probability to remain in the same state.

In subsection 5.3.5 finally the equilibrium distribution for the restricted two-dimensional idealized chain is calculated. The transition probabilities and different restricted state spaces are described in detail. Finally the drift for a given value of d^1 along the asymmetric edge is calculated, in particular for a value of d^1 corresponding to the upper and lower concession limits for example games $1-3$. For example game 1 the state in the interior of the asymmetric edge is identified (approximated) which has a drift of zero in d^1.

We now introduce the definitions and develop the concept that we use in this section to approximate the behavior of the chain (V_t) in the neighborhood of the asymmetric edge.

Suppose the chain (V_t) starts in a state $\mathbf{d}^* \in \mathcal{AE}$. Let $R_1(\mathbf{d}^*) = \min\{t > 0 | CS^{C^{12}}(V_t) = 0, CS^{C^{13}}(V_t) = 0\}$. Recall that for $\mathbf{d} \in \Omega^E$ $p_{\mathbf{d}}^{i,j} = \frac{1}{4}\frac{d^j}{\sum_{l \notin C'(\mathbf{d}(i))} d^l + \varepsilon}\mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}}$ if $i \neq j$ and $p_{\mathbf{d}}^{i,j} = \frac{1}{4}\frac{d^j + \varepsilon}{\sum_{l \notin C'(\mathbf{d}(i))} d^l + \varepsilon}\mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}}$ if $i = j$.

The 3-dimensional "idealized chain" $\left(V_t^{\mathbf{d}^*}\right)$

We now define an "idealized " Markov chain, with state space Ω^E, where the transition probabilities of the chain (V_t) during an excursion from $\mathbf{d}^*$ to another state on the asymmetric edge, are approximated by using the demands at $\mathbf{d}^*$ instead of the demands at $\mathbf{d}_t$.

For $\mathbf{d}^* \in \mathcal{AE}, \mathbf{d} \in \Omega^E$ and for $t < R_1(\mathbf{d}^*)$, let

$$p_{\mathbf{d}}^{i,j}(\mathbf{d}^*) = \frac{1}{4}\frac{d^{*j}}{\sum_{l \notin C'(\mathbf{d}(i))} d^{*l} + \varepsilon}\mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}} \tag{5.1}$$

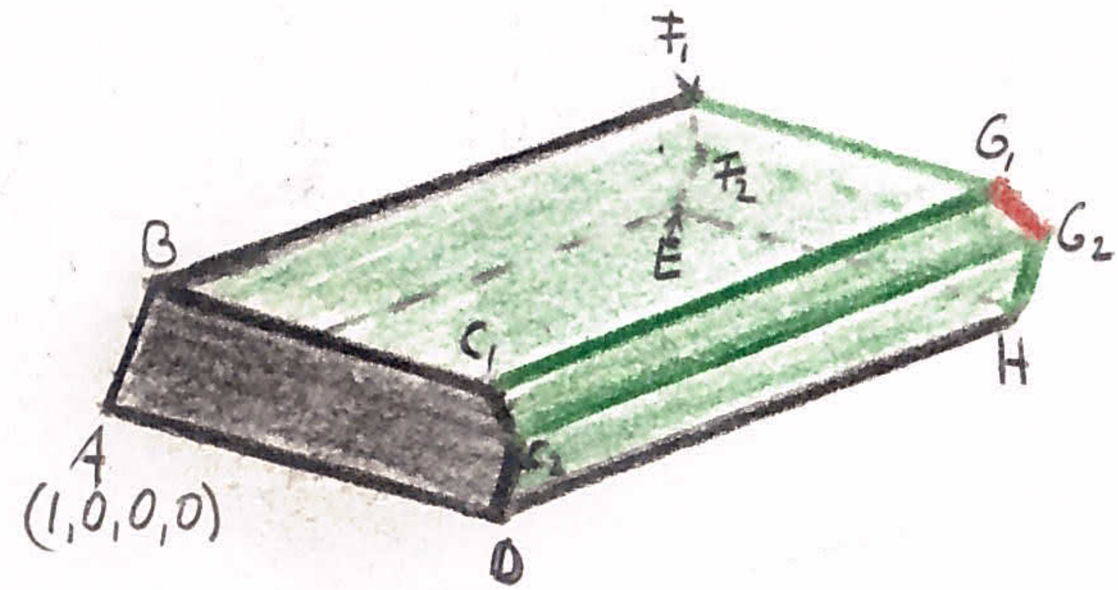


Figure 5.6 Graphical representation of Ω^C for game 3.

if $i \neq j$ and

$$p_{\mathbf{d}}^{i,j}(\mathbf{d}^*) = \frac{1}{4} \frac{d^{*j} + \varepsilon}{\sum_{l \notin C'(\mathbf{d}(i))} d^{*l} + \varepsilon} \mathbf{1}_{\{j \notin C'(\mathbf{d}(i))\}} \tag{5.2}$$

if $i = j$.

A state on the asymmetric edge is uniquely determined by the value of d^1. We can express the demands of players $1-3$ as a function of $d^{*1}, v(C_{12})$ and $v(C_{13})$. Observe that for $t < R_1(\mathbf{d}^*)$,

$$(V_t) = \left(d^{*1} + l_t^1 \varepsilon, d^{*2} + l_t^2 \varepsilon, d^{*3} + l_t^3 \varepsilon, d^{*4} + l_t^4 \varepsilon\right)$$

which can alternatively be expressed as

$$(V_t) = \left(d^{*1} + l_t^1 \varepsilon, c_{12} - d^{*1} + l_t^2 \varepsilon, c_{13} - d^{*1} + l_t^3 \varepsilon, 1 + d^{*1} - c_{12} - c_{13} + l_t^4 \varepsilon\right)$$

for some $l^1, l^2, l^3, l^4 \in \mathbb{Z}$.

So $p_{\mathbf{d}}^{i,j}(\mathbf{d}^*)$ are the approximate transition probabilities for the chain (V_t) on the trajectory for $t < R_1(\mathbf{d}^*)$, where the demands of (V_t) are replaced by the demands at $\mathbf{d}^*$, that is where all l_t^i are set to 0 for $t > 0$. For small ε, and as long as l^1, l^2, l^3, l^4 can be bounded from above, the approximation is accurate.

For $\mathbf{d}^* \in \mathcal{AE}$, and for $t < R_1(\mathbf{d}^*)$, let

$$\left(V_t^{\mathbf{d}^*}\right)$$

be the Markov chain that starts in $\mathbf{d}^*$ and moves with transition probabilities $p_{\mathbf{d}}^{i,j}(\mathbf{d}^*)$.

The 2-dimensional "idealized chain" $\left(DV_t^{\mathbf{d}^*}\right)$ and sets $\mathcal{S}_{m,n}$

Instead of analyzing the 3-dimensional chain (V_t) between successive returns to the asymmetric edge, we analyze the 2-dimensional chain

$$\left(\frac{CS^{C_{12}}(V_t)}{\varepsilon}, \frac{CS^{C_{13}}(V_t)}{\varepsilon}\right).$$

The state space is given by

$$\Omega^V = \{0, \ldots \frac{1-v(C_{12})}{\varepsilon}\} \times \{-\frac{v(C_{13})}{\varepsilon}, \ldots \frac{1-v(C_{13})}{\varepsilon}\}.$$

For $0 \leq m \leq \frac{1-v(C_{12})}{\varepsilon}$ and $\frac{-v(C_{12})}{\varepsilon} \leq n \leq \frac{1-v(C_{12})}{\varepsilon}$, let

$$\mathcal{S}_{m,n}$$

be the set of efficient states where $\frac{CS^{C_{12}}(\mathbf{d})}{\varepsilon} = m$ and $\frac{CS^{C_{13}}(\mathbf{d})}{\varepsilon} = n$ and no other coalition is feasible. Remember that the transition probability $p_{\mathbf{d}}^{i,j}$ depends only on the demands of the players that are not in the payable coalition at $\mathbf{d}(i)$.

Lemma 5.9. *For $\mathbf{d}$ and $\mathbf{d}'$ in $\mathcal{S}_{m,n}$, and $i \in \{1,2,3,4\}$, the payable coalition at the intermediate inefficient states $\mathbf{d}(i)$ and $\mathbf{d}'(i)$ is the same for all $i \in \{1,2,3,4\}$.*

Proof: Per definition, the only coalitions that can be feasible in states in $\mathcal{S}_{m,n}$, are C_{12} and C_{13} and C^G. This implies as well that the only coalitions that can be feasible in $\mathbf{d}(i)$, for $i \in \{1,2,3,4\}$ are C_{12} and C_{13}. Suppose $\mathbf{d}$ and $\mathbf{d}'$ are in the same set $\mathcal{S}_{m,n}$. If, for $i \in \{3,4\}$, C_{12} is binding at $\mathbf{d}(i)$, then it must be binding as well at $\mathbf{d}'(i)$. Similarly, for $i \in \{1,2,3,4\}$, if C_{13} is feasible at $\mathbf{d}(i)$, then it is feasible as well at $\mathbf{d}'(i)$. So $C'(\mathbf{d}(i)) = C'(\mathbf{d}'(i))$ for all $i \in \{1,2,3,4\}$ and $\mathbf{d}'$,$\mathbf{d}'$ in $\mathcal{S}_{m,n}$. □

It follows directly that

$$p_{\mathbf{d}}^{i,j}(\mathbf{d}^*) = p_{\mathbf{d}'}^{i,j}(d^*)$$

if $\mathbf{d}'$ and $\mathbf{d}$ are both in the same set $\mathcal{S}_{m,n}$.

We want to approximate the 2-dimensional chain

$$\left(\frac{CS^{C_{12}}(V_t)}{\varepsilon}, \frac{CS^{C_{13}}(V_t)}{\varepsilon}\right)$$

by using the transition $p_{\mathbf{d}}^{i,j}(\mathbf{d}^*)$ instead of $p_{\mathbf{d}}^{i,j}$.

For $\mathbf{d}$ in $\mathcal{S}_{m,n}$, $\mathbf{d}^* \in \mathcal{AE}$ and $q \in \{m-1, m, m+1\}$ and $r \in \{n-1, n, n+1\}$, let

$$p_{\mathbf{d}}^{q,r}(\mathbf{d}^*) = \sum_{i,j \in \{1,2,3,4\}} p_{\mathbf{d}}^{i,j}(\mathbf{d}^*) \mathbf{1}_{\{\mathbf{d}(i,j) \in \mathcal{S}_{q,r}\}}. \tag{5.3}$$

Now $p_{\mathbf{d}}^{q,r}(\mathbf{d}^*)$ depends on $\mathbf{d}$ only through m and n, so we use the notation

$$p_{\mathbf{d}}^{m,n}(\mathbf{d}^*)$$

as well.

For $\mathbf{d}^* \in \mathcal{AE}$, let the dependent two dimensional chain $\left(DV_t^{\mathbf{d}^*}\right)$ be the Markov chain defined on

$$\Omega^V = \{0, \ldots \frac{1-v(C_{12})}{\varepsilon}\} \times \{-\frac{v(C_{13})}{\varepsilon}, \ldots \frac{1-v(C_{13})}{\varepsilon}\}$$

that moves with transition probabilities $p_{m,n}^{q,r}(\mathbf{d}^*)$ if $m < \frac{1-v(C_{12})}{\varepsilon}$ and $\frac{-v(C_{12})}{\varepsilon} < n < \frac{1-v(C_{12})}{\varepsilon}$.

Figure 5.7 is a sketch of a subset of the state space of $\left(DV_t^{\mathbf{d}^*}\right)$. Arrows between two states imply, that the chain $\left(DV_t^{\mathbf{d}^*}\right)$ can only transition in the direction of the arrow. Simple lines between two states imply that transitions in both directions have positive probability.

We set $ub1 = \frac{1-v(C_{12})}{\varepsilon}$, the upper boundary for the first coordinate of the chain $\left(DV_t^{\mathbf{d}^*}\right)$. Similarly we set $ub2 = \frac{1-v(C_{13})}{\varepsilon}$ and $lb2 = \frac{-v(C_{13})}{\varepsilon}$, the upper boundary and lower boundary for the second coordinate of the chain $\left(DV_t^{\mathbf{d}^*}\right)$ respectively.

Then

$$p_{ub1,n}^{ub1,r}(\mathbf{d}^*) = p_{ub1-1,n}^{ub1-1,r}(\mathbf{d}^*) + \sum_{r \in \{-1,0,1\}} p_{ub1-1,n}^{ub1,r}(\mathbf{d}^*) \tag{5.4}$$

and set $p_{ub1,n}^{ub1+1,r}(\mathbf{d}^*) = 0$. Similarly, let

$$p_{m,ub2}^{q,ub2}(\mathbf{d}^*) = p_{m,ub2-1}^{q,ub2-1}(\mathbf{d}^*) + \sum_{q \in \{-1,0,1\}} p_{m,ub2-1}^{q,ub2}(\mathbf{d}^*) \tag{5.5}$$

and set $p_{m,ub2}^{q,ub2+1}(\mathbf{d}^*) = 0$. Likewise let

$$p_{m,lb2}^{q,lb2}(\mathbf{d}^*) = p_{m,lb2+1}^{q,lb2+1}(\mathbf{d}^*) + \sum_{q \in \{-1,0,1\}} p_{m,lb2+1}^{q,lb2}(\mathbf{d}^*) \tag{5.6}$$

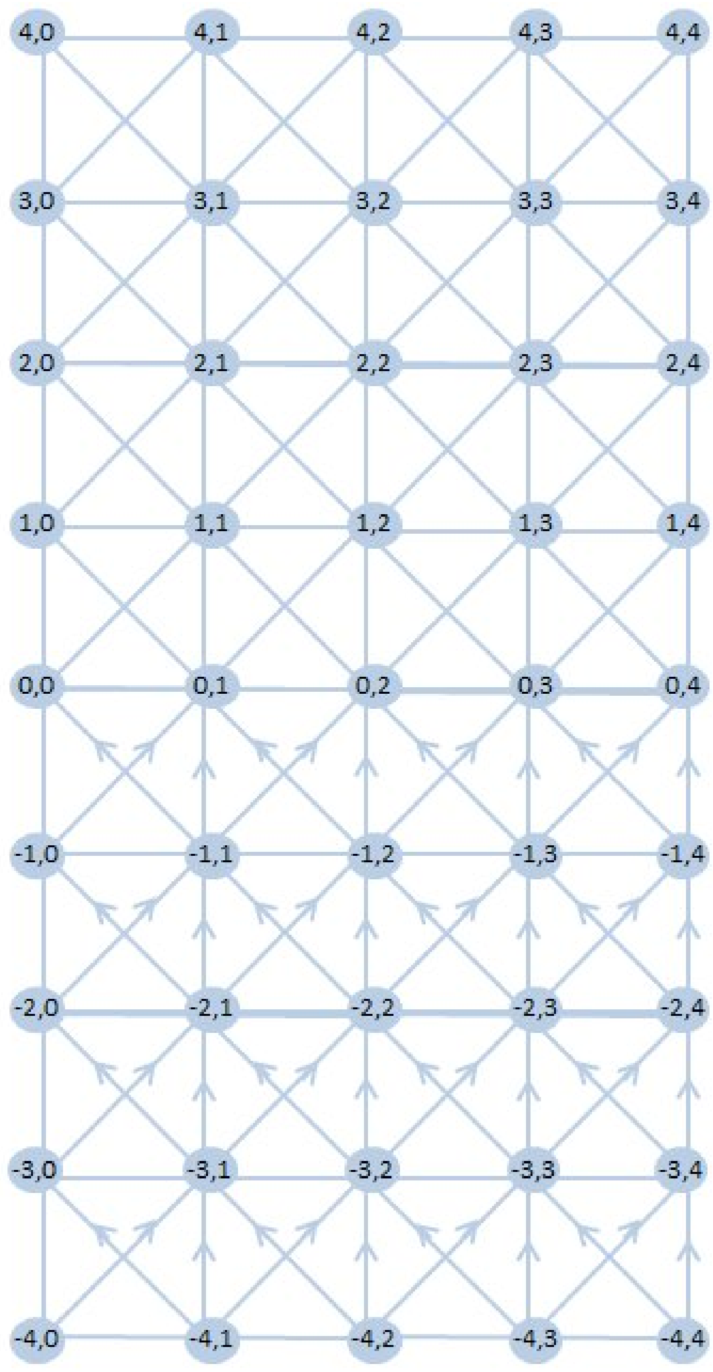


Figure 5.7 Sketch of a subset of the state space of the chain $\left(DV_t^{\mathbf{d}^*}\right)$

and set $p^{q,lb2-1}_{m,lb2}(\mathbf{d}^*) = 0$.

Observe that, if the first coordinate of $\left(DV_t^{\mathbf{d}^*}\right)$ is zero, then the chain $\left(V_t^{\mathbf{d}^*}\right)$ is in a state on the hyperplane $\mathcal{H}(C_{12})$. If the second coordinate of $\left(DV_t^{\mathbf{d}^*}\right)$ is zero, then the chain $\left(V_t^{\mathbf{d}^*}\right)$ being in a state on the hyperplane $\mathcal{H}(C_{13})$.

For $0 < m \leq ub1$ and $0 < ub2$, let

$$\mathcal{S}_0 = \left(\bigcup_{0<m\leq ub1} \mathcal{S}_{m,0}\right) \bigcup \left(\bigcup_{0<n\leq ub2} \mathcal{S}_{0,n}\right)$$

If the second coordinate of $\left(DV_t^{\mathbf{d}^*}\right)$ is less than zero, that corresponds to the (V_t) being in a state outside of the core.

For $0 \leq m \leq ub1$ and $lb1 \geq n < 0$, let

$$\mathcal{S}_- = \bigcup_{lb1\geq n<0} \mathcal{S}_{m,n}.$$

If both coordinates of $\left(DV_t^{\mathbf{d}^*}\right)$ are greater than zero, that corresponds to (V_t) being in a state in the interior of the the core. Let

$$\mathcal{S}_+ = \bigcup_{0<m\leq ub1, 0<n\leq ub2} \mathcal{S}_{m,n}.$$

Drift in d^1 and equilibrium distribution for $\left(DV_t^{\mathbf{d}^*}\right)$

We will now investigate the expected change in d^1 over the sets $\mathcal{S}_+$, $\mathcal{S}_-$, $\mathcal{S}_0$ and $\mathcal{S}_{0,0}$. First we need to define the expected change in d^1 for the chain $\left(V_t^{\mathbf{d}^*}\right)$.

For $\mathbf{d}^* \in \mathcal{AE}$, $\mathbf{d} \in \Omega^E$ and $t < R_1(\mathbf{d}^*)$, let $\Delta\left(V_t^{\mathbf{d}^*}, \mathbf{d}\right) = \mathbb{E}\left(d^1_{t+1}\left(V_{t+1}^{\mathbf{d}^*}\right) - d^1_t\left(V_t^{\mathbf{d}^*}\right) \mid \mathbf{d}\right)$ be the drift in d^1 over one step of $V_t^{\mathbf{d}^*}$ at $\mathbf{d}$. Observe that $\Delta\left(V_t^{\mathbf{d}^*}, \mathbf{d}\right)$ depends on $\mathbf{d}$ only via the set $\mathcal{S}_{m,n}$.

For $\mathbf{d}^* \in \mathcal{AE}$ and $\mathbf{d} \in \mathcal{S}_{m,n}$, let $\Delta^{\mathbf{d}^*}_{m,n} = \Delta\left(V_t^{\mathbf{d}^*}, \mathbf{d}\right)$.

We describe briefly the expected change in d^1 over different sets $\mathcal{S}_{m,n}$. If $n < 0$, then this corresponds to the chain $\left(V_t^{\mathbf{d}^*}\right)$ to be in a state $\mathbf{d}$ not in the core where C_{13} is strictly feasible. So for $i \in \{1,2,3,4\}$, player 1 will always be a member of $C^{'(\mathbf{d}(i))}$ and so player 1's demands will increase with probability $\frac{1}{4}$ and will not decrease.

If $m = 0$ and $n = 0$, then this corresponds to the chain $\left(V_t^{\mathbf{d}^*}\right)$ to be in a state $\mathbf{d} \in \mathcal{AE}$. So for $i \in \{2,3,4\}$, player 1 will always be a member of $C^{'(\mathbf{d}(i))}$ and so player 1's demands will increase with probability $\frac{1}{4}\left(1 - \mathbf{d}^{*1}\right)$ and will not decrease.

If $m = 0$ and $n > 0$, then this corresponds to the chain $\left(V_t^{\mathbf{d}^*}\right)$ to be in a state $\mathbf{d} \in \mathcal{H}(C_{12})$. So for $i \in \{3,4\}$, player 1 be a member of $C^{'(\mathbf{d}(i))}$ and so player 1's demands will increase with probability $\frac{1}{4}\left(1-\mathbf{d}^{*1}\right)$ and will decrease with probability $\frac{1}{4}\left(\mathbf{d}^{*1}\right)$. Similarly, If $m > 0$ and $n = 0$, then this corresponds to the chain $\left(V_t^{\mathbf{d}^*}\right)$ to be in a state $\mathbf{d} \in \mathcal{H}(C_{13})$. So for $i \in \{2,4\}$, player 1 be a member of $C^{'(\mathbf{d}(i))}$ and so player 1's demands will increase with probability $\frac{1}{4}\left(1-\mathbf{d}^{*1}\right)$ and will decrease with probability $\frac{1}{4}\left(\mathbf{d}^{*1}\right)$.

Finally, if $m = 0$ and $n > 0$, then this corresponds to the chain $\left(V_t^{\mathbf{d}^*}\right)$ to be an interior state of the core. So for $i \in \{1,2,3,4\}$, $C^{'(\mathbf{d}(i))}$ is the empty set and so player 1's demands will increase with probability $\frac{1}{4}\left(1-\mathbf{d}^{*1}\right)$ and will decrease with probability $\frac{3}{4}\left(\mathbf{d}^{*1}\right)$.

We can summarize the drift in d^1 over the sets $\mathcal{S}_{m,n}$.

For states $\mathbf{d} \in \mathcal{S}(m,n)$ with $n < 0$, it holds that

$$\Delta_{-}^{\mathbf{d}^*} = \frac{1}{4}.$$

On the asymmetric edge,

$$\Delta_{0,0}^{\mathbf{d}^*} = \frac{1}{4}\left(1-\mathbf{d}^{*1}\right).$$

For on either hyperplane $\mathcal{H}(C_{12})$ or $\mathcal{H}(C_{13})$, if exactly one of $n > 0$ or $m > 0$,

$$\Delta_{0}^{\mathbf{d}^*} = \frac{1}{4}[\left(1-\mathbf{d}^{*1}\right) - \mathbf{d}^{*1}] = \frac{1-2\mathbf{d}^{*1}}{4}.$$

Finally, for states in the interior of the core, where both $n > 0$ and $m > 0$,

$$\Delta_{+,+}^{\mathbf{d}^*} = \frac{1}{4}[\left(1-\mathbf{d}^{*1}\right) - 3\mathbf{d}^{*1}] = \frac{1-4\mathbf{d}^{*1}}{4}.$$

Now we will define the expected change in d^1 over one excursion from the asymmetric edge.

For $\mathbf{d}^* \in \mathcal{AE}$ let $\pi_{\mathcal{S}_{m,n}}^{\mathbf{d}^*}$ be the equilibrium distribution of $\left(DV_t^{\mathbf{d}^*}\right)$ on the set $\mathcal{S}_{m,n}$.

For $\mathbf{d}^* \in \mathcal{AE}$ let $\Delta^{\mathbf{d}^*} = \sum_{m,n} \pi_{\mathcal{S}_{m,n}}^{\mathbf{d}^*} \Delta_{m,n}^{\mathbf{d}^*}$, the approximate drift in d^1 over one excursion of the chain $\left(DV_t^{\mathbf{d}^*}\right)$ from the asymmetric edge with starting state $\mathbf{d}^*$.

The distribution of $\left(DV_t^{\mathbf{d}^*}\right)$ over the sets $\mathcal{S}_{m,n}$ is an approximation to the distribution of the chain $\left(CS^{C_{12}}(V_t), CS^{C_{12}}(V_t)\right)$ over the sets $\mathcal{S}_{m,n}$ as long as the chain (V_t) hits no hyperplane corresponding to a 3-player coalition.

Restrictions to the state space of the "idealized chain" $\left(DV_t^{\mathbf{d}^*}\right)$

We now consider restrictions of $\left(DV_t^{\mathbf{d}^*}\right)$, for which we can explicitly calculate the equilibrium distribution. We will define an upper restriction for the first coordinate of the chain, $ur1$, and an upper restriction $ur2$ and a lower restriction $lr2$ for the second coordinate of the chain. For some $k \in \mathbf{N}, k > 0$, let $\mathbf{k} = (k, k, -k)$ be a restriction vector where $ur1 = ur2 = -lr2 = k$.

For $\mathbf{d}^* \in \mathcal{AE}$, let the dependent two dimensional chain $\left(DV_t^{\mathbf{d}^*,\mathbf{k}}\right)$ be the Markov chain defined on

$$\Omega^{V^{\mathbf{k}}} = \{0, ...k\} \times \{-k, ...k\}$$

that moves with transition probabilities $p_{m,n}^{q,r}(\mathbf{d}^*)$ if $m < k$ and $-k < n < k$.

Then

$$p_{k,n}^{k,r}(\mathbf{d}^*) = p_{k-1,n}^{k-1,r}(\mathbf{d}^*) + \sum_{r \in \{-1,0,1\}} p_{k-1,n}^{k,r}(\mathbf{d}^*) \tag{5.7}$$

and set $p_{k,n}^{k+1,r}(\mathbf{d}^*) = 0$. Similarly, let

$$p_{m,k}^{q,k}(\mathbf{d}^*) = p_{m,k-1}^{q,k-1}(\mathbf{d}^*) + \sum_{q \in \{-1,0,1\}} p_{m,k-1}^{q,k}(\mathbf{d}^*) \tag{5.8}$$

and set $p_{m,k}^{q,k+1}(\mathbf{d}^*) = 0$. Likewise let

$$p_{m,-k}^{q,-k}(\mathbf{d}^*) = p_{m,-k+1}^{q,-k+1}(\mathbf{d}^*) + \sum_{q \in \{-1,0,1\}} p_{m,-k+1}^{q,-k}(\mathbf{d}^*) \tag{5.9}$$

and set $p_{m,-k}^{q,-k-1}(\mathbf{d}^*) = 0$.

Observe that the size of $\Omega^{V^{\mathbf{k}}}$ is given by $2(k+1)^2 - (k+1)$.

In figure 5.8 we sketch $\Omega^{V^{\mathbf{2}}}$. As in Figure 5.7, arrows between two states imply, that the chain $\left(DV_t^{\mathbf{d}^*,\mathbf{2}}\right)$ can only transition in the direction of the arrow. Simple lines between two states imply that transitions in both directions have positive probability. For example, from state $(1,-1)$ the chain $\left(DV_t^{\mathbf{d}^*,\mathbf{2}}\right)$ can transition to state $(1,0)$ in one time step, however the chain cannot transition directly from $(1,0)$ to $(1,-1)$. To transition from $(1,0)$ to $(1,0)$ several transitions are necessary that must include a transition from $(0,0)$ to $(0,-1)$.

States, that have drift $\Delta_{0,0}^{\mathbf{d}^*} = \frac{1}{4}\left(1 - \mathbf{d}^{*1}\right)$ are colored orange, states with drift $\Delta_{+,+} = \frac{1-4\mathbf{d}^{*1}}{4}$ are depicted green, states with drift $\Delta_0 = \frac{1-2\mathbf{d}^{*1}}{4}$ are depicted blue and, finally,

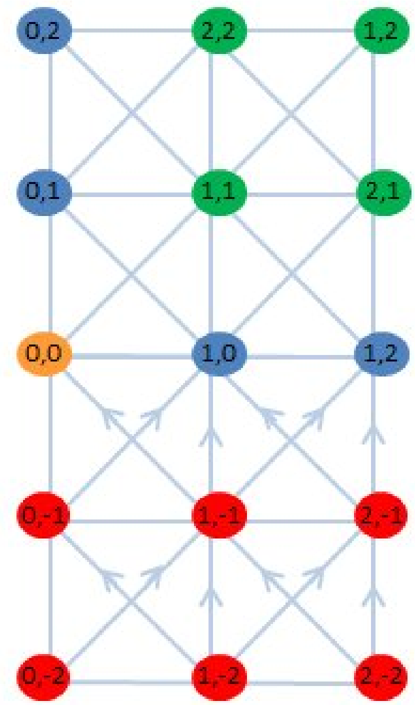


Figure 5.8 Sketch of the restricted chain $\left(DV_t^{\mathbf{d}^*,\mathbf{2}}\right)$

states with drift $\Delta_- = \frac{1}{4}$ are colored red. Red states always have a very strong positive drift, orange states always have a positive drift, green states always have a negative drift and for blue states, the sign of the drift depends on $\mathbf{d}^*$. For example games $1-3$, blue states have a slightly negative drift.

In 5.3 we calculate the equilibrium distribution and the drift in d^1 over an excursion of $\left(DV_t^{\mathbf{d}^*,\mathbf{k}}\right)$ from the asymmetric edge for different starting states $\mathbf{d}^* \in \mathcal{AE}^1$. What we are really interested in, is the distribution over the different colored sets.

Calculating the equilibrium distribution of the restricted chain

A *neighboring set* for a set $\mathcal{S}_{m,n}$ is a set $\mathcal{S}_{m+x,n+y}$ for $x,y \in [-1,0,1]$.

The transition probabilities $p_{m,n}^{m+x,n+y}(k)$ from set $\mathcal{S}_{q,r}$ to all neighboring sets $\mathcal{S}_{m+x,n+y}$, as defined in (5.11), for $x,y \in [-1,0,1]$ are given in the below tables.

The neighboring sets of set $\mathcal{S}_{m,n}$ are listed in the first column. The new efficient states, that lie in set $\mathcal{S}_{q,r}$ and can be reached from set $\mathcal{S}_{q,r}$ are depicted in the second column. The third column lists the $p_{m,n}^{q,r}(\mathbf{d}^*)$. The fourth and fifth column, respectively, list the transition probabilities for $\mathbf{d}^* = (0.54, 0.21, 0.2, 0.05)$ and $\mathbf{d}^* = (0.6, 0.15, 0.14, 0.11)$, the states **max1** for example game three and **min1** for example game two respectively.

The first table refers to transitions out of the sets $\mathcal{S}_{0,0}$ to all neighboring sets, that can be reached with positive probability. Since a state $\mathbf{d}^* \in \mathcal{AE}$ is uniquely determined by $\mathbf{d}^*$, we may use the notation $p_{0,0}^{q,r}\left(d^{1*}\right)$ as alternative to $p_{0,0}^{q,r}(\mathbf{d}^*)$.

$S_{q,r}$	new states in $S_{q,r}$	$4p_{0,0}^{q,r}(\mathbf{d}^*)$	$p_{0,0}^{q,r}(0.54)$	$p_{0,0}^{q,r}(0.60)$
$S_{1,1}$	$\mathbf{d}(1,4)$	d^{*4}	0.01	0.03
$S_{1,0}$	$\mathbf{d}(1,3), \mathbf{d}(2,4)$	$d^{*3} + \frac{d^{*4}}{d^{*2}+d^{*4}}$	0.1	0.14
$S_{0,1}$	$\mathbf{d}(1,2), \mathbf{d}(3,4)$	$d^{*2} + \frac{d^{*4}}{d^{*3}+d^{*4}}$	0.1	0.15
$S_{0,0}$	$\mathbf{d}(i,i)$ for $i \in \{1,2,3,4\}$	$d^{*1} + \frac{d^{*2}}{d^{*2}+d^{*4}} + 1$	0.59	0.54
$S_{0,-1}$	$\mathbf{d}(4,3)$	$\frac{d^{*3}}{d^{*3}+d^{*4}}$	0.2	0.14

Table 5.2 Transition probabilities from $\mathcal{S}_{0,0}$ to all neighboring sets for example game 1

Recall that states $\mathbf{d} \in \mathcal{S}_{0,0}$ are states on the asymmetric edge. We like to point the reader to the special properties of the transitions from set $\mathcal{S}_{0,0}$ to set $\mathcal{S}_{0,-1}$. Lemma (5.7) states, that the chain (V_t) can only leave the core from states on the asymmetric edge. For this to happen, player four increases his demand from a state in $\mathcal{S}_{0,0}$ and the payable coalition is then C_{12}. Player three or player four can decrease demands. If player three decreases demands, the new efficient state $\mathbf{d}(4,3)$ is in set $\mathcal{S}_{0,-1}$. The respective transition probability, to go from set $S_{0,0}$ to set $S_{0,-1}$, is given by $\frac{d^{*3}}{d^{*3}+d^{*4}} = \frac{c_{13}-d^{*1}}{1-c_{12}}$. To restrict the chain (V_t) to move only on states in the core, the only restriction to the transition dynamics would be to set $p_{0,0}^{0,-1}(\mathbf{d}^*) = 0$. Observe that at the 'lower part' of the asymmetric edge of game 1, at state $\mathbf{d}^* = (0.54, 0.21, 0.2, 0.05)$, $p_{0,0}^{0,-1}(\mathbf{d}^*) = 0.2$ whereas at state $\mathbf{d}^* = (0.6, 0.15, 0.14, 0.11)$, $p_{0,0}^{0,-1}(\mathbf{d}^*) = 0.14$. So the chain transitions more frequently to states outside the core in the 'lower part' of the asymmetric edge.

$S_{q,r}$	`new states in` $\mathcal{S}_{q,r}$	$4p^{q,r}_{m,0}(\mathbf{d}^*)$	$p^{q,r}_{0,0}(0.54)$	$p^{q,r}_{0,0}(0.60)$
$S_{m+1,1}$	$\mathbf{d}(1,4)$	d^{*4}	0.01	0.03
$S_{m+1,0}$	$\mathbf{d}(1,3), \mathbf{d}(2,4)$	$d^{*3} + \frac{d^{*4}}{d^{*4}+d^{*2}}$	0.01	0.14
$S_{m,1}$	$\mathbf{d}(1,2), \mathbf{d}(3,4)$	$d^{*2} + d^{*4}$	0.07	0.07
$S_{m,0}$	$\mathbf{d}(i,i)$ `for` $i \in \{1,2,3,4\}$	$d^{*1} + d^{*3} + 1$	0.44	0.44
$S_{m-1,1}$	$\mathbf{d}(3,2)$	d^{*2}	0.05	0.04
$S_{m-1,0}$	$\mathbf{d}(3,1), \mathbf{d}(4,2)$	$d^{*1} + \frac{d^{*4}}{d^{*4}+d^{*2}}$	0.34	0.29

Table 5.3 Transition probabilities from $\mathcal{S}_{m,0}$ for $m \geq 1$

The states $\mathbf{d} \in \mathcal{S}_{m,0}$ for $m \geq 1$ are states on the hyperplane $\mathcal{H}(C_{13})$. For all intermediate inefficient states reachable from $\mathcal{S}_{m,0}$, the payable coalition is either C_{13} or the empty set. So transitions to a set with negative value in the second coordinate, to the sets $S_{q+1,-1}$, $S_{q,-1}$ and $S_{q-1,-1}$, are impossible. If players 2 or 4 increase demands, one of the two will decrease demands. Both $\mathbf{d}(1,3)$ and $\mathbf{d}(2,4)$ are in set $\mathcal{S}_{m+1,0}$. Player 1 is chosen to increase demands with probability $\frac{1}{4}$, the payable coalition is the empty set and so player 3 reduces demands with probability $d^{*3} = c_{13} - d^{*1}$. If player 2 increases his demand, C_{13} is the payable coalition and player 4 reduces demands with $\frac{d^{*4}}{d^{*3}+d^{*4}} = \frac{1+d^{*1}-c_{12}-c_{13}}{1-c_{12}}$ so $p^{m+1,0}_{m,0}(\mathbf{d}^*) = \frac{c_{13}-d^{*1}}{+}\frac{1+d^{*1}-c_{12}-c_{13}}{4(1-c_{12})}$.

$S_{q,r}$	`new states in` $\mathcal{S}_{q,r}$	$4p^{q,r}_{m,n}(\mathbf{d}^*)$	$p^{q,r}_{m,n}(0.54)$	$p^{q,r}_{0,0}(0.62)$
$S_{m+1,n+1}$	$\mathbf{d}(1,4)$	d^{*4}	0.01	0.03
$S_{m+1,n}$	$\mathbf{d}(1,3), \mathbf{d}(2,4)$	$d^{*3} + d^{*4}$	0.06	0.06
$S_{m+1,n-1}$	$\mathbf{d}(2,3)$	d^{*3}	0.05	0.04
$S_{m,n+1}$	$\mathbf{d}(1,2), \mathbf{d}(3,4)$	$d^{*2} + d^{*4}$	0.07	0.07
$S_{m,n}$	$\mathbf{d}(i,i)$ `for` $i \in \{1,2,3,4\}$	1	0.25	0.25
$S_{m,n-1}$	$\mathbf{d}(4,3), \mathbf{d}(2,1)$	$d^{*3} + d^{*1}$	0.19	0.19
$S_{m-1,n+1}$	$\mathbf{d}(3,2)$	d^{*2}	0.05	0.04
$S_{m-1,n}$	$\mathbf{d}(4,2), \mathbf{d}(3,1)$	$d^{*2} + d^{*1}$	0.19	0.19
$S_{m-1,n-1}$	$\mathbf{d}(4,1)$	d^{*1}	0.14	0.15

Table 5.4 Transition probabilities from $\mathcal{S}_{m,n}$ for $m,n \geq 1$

States $\mathbf{d} \in \mathcal{S}_{m,n}$ for $m,n \geq 1$ are in the interior of the core and at every possible inefficient state $\mathbf{d}(i)$ the payable coalition is the empty set. So all players can reduce demands and do so proportional to their demand at $\mathbf{d}^*$, the 'reference state' on the asymmetric edge. For example, if player 4 increases his demand and player 1 reduces demands, then the chain $V_t^{\mathbf{d}^*}$ transitions to state $\mathbf{d}(4,1)$, and the chain $DV_t^{\mathbf{d}^*}$ transitions to a state in $\mathcal{S}_{m-1,n-1}$.

$S_{q,r}$	new states in $\mathcal{S}_{q,r}$	$4p_{0,n}^{q,r}(\mathbf{d}^*)$	$p_{0,n}^{q,r}(0.54)$	$p_{0,n}^{q,r}(0.6)$
$S_{1,n+1}$	$\mathbf{d}(1,4)$	d^{*4}	0.01	0.03
$S_{1,n-1}$	$\mathbf{d}(2,3)$	d^{*3}	0.05	0.04
$S_{1,n}$	$\mathbf{d}(1,3), \mathbf{d}(2,4)$	$d^{*3}+d^{*4}$	0.06	0.06
$S_{0,n+1}$	$\mathbf{d}(1,2), \mathbf{d}(3,4)$	$d^{*2}+\frac{d^{*4}}{d^{*3}+d^{*4}}$	0.1	0.15
$S_{0,n}$	$\mathbf{d}(i,i)$ for $i \in \{1,2,3,4\}$	$d^{*1}+d^{*2}+1$	0.44	0.44
$S_{0,n-1}$	$\mathbf{d}(4,3), \mathbf{d}(2,1)$	$d^{*1}+\frac{d^{*3}}{d^{*3}+d^{*4}}$	0.34	0.29

Table 5.5 Transition probabilities from $\mathcal{S}_{0,n}$ for $n \geq 1$

The states $\mathbf{d} \in \mathcal{S}_{0,n}$ for $n \geq 1$ are states on the hyperplane $\mathcal{H}(C_{12})$. The transition dynamics are very similar to the dynamics for the states on the hyperplane $\mathcal{H}(C_{12})$ described above, just the values of the demands of the players 2 and 3 differ.

$S_{q,r}$	new states in $\mathcal{S}_{q,r}$	$4p_{0,-n}^{q,r}(\mathbf{d}^*)$	$p_{0,-n}^{q,r}(0.54)$	$p_{0,-n}^{q,r}(0.6)$
$S_{1,n+1}$	$\mathbf{d}(1,4)$	$\frac{d^{*4}}{d^{*2}+d^{*4}}$	0.05	0.11
$S_{1,n}$	$\mathbf{d}(2,4)$	$\frac{d^{*4}}{d^{*2}+d^{*4}}$	0.05	0.11
$S_{0,n+1}$	$\mathbf{d}(1,2), \mathbf{d}(3,4)$	$\frac{d^{*2}}{d^{*2}+d^{*4}}+\frac{d^{*4}}{d^{*3}+d^{*4}}$	0.25	0.25
$S_{0,n}$	$\mathbf{d}(2,2), \mathbf{d}(3,3), \mathbf{d}(4,4)$	$1+\frac{d^{*2}}{d^{*2}+d^{*4}}$	0.45	0.39
$S_{0,n-1}$	$\mathbf{d}(4,3)$	$\frac{d^{*3}}{d^{*3}+d^{*4}}$	0.2	0.14

Table 5.6 Transition probabilities from $\mathcal{S}_{0,-n}$ for $n \leq 1$

The states $\mathbf{d} \in \mathcal{S}_{0,-n}$ for $n \leq 1$ are states not in the core where coalition C_{12} is binding. The transition dynamics are rather complex. At $\mathbf{d}(4)$ and $\mathbf{d}(3)$, coalition C_{12} is the payable coalition, At $\mathbf{d}(1)$ and $\mathbf{d}(2)$, coalition C_{13} is the payable coalition.

$S_{q,r}$	new states in $\mathcal{S}_{q,r}$	$4p^{q,r}_{m,-n}(\mathbf{d}^*)$	$p^{q,r}_{0,0}(0.54)$	$p^{q,r}_{0,0}(0.6)$
$S_{m+1,n+1}$	$\mathbf{d}(1,4)$	$\frac{d^{*4}}{d^{*2}+d^{*4}}$	0.05	0.11
$S_{m+1,n}$	$\mathbf{d}(2,4)$	$\frac{d^{*4}}{d^{*2}+d^{*4}}$	0.05	0.11
$S_{m,n+1}$	$\mathbf{d}(1,2), \mathbf{d}(3,4)$	$\frac{d^{*2}}{d^{*2}+d^{*4}} + d^{*4}$	0.25	0.25
$S_{m,n}$	$\mathbf{d}(2,2), \mathbf{d}(4,4)$	1	0.25	0.25
$S_{m-1,n+1}$	$\mathbf{d}(3,2)$	$\frac{d^{*2}}{d^{*2}+d^{*4}}$	0.2	0.14
$S_{m-1,n}$	$\mathbf{d}(4,2)$	$\frac{d^{*2}}{d^{*2}+d^{*4}}$	0.2	0.14

Table 5.7 Transition probabilities from $\mathcal{S}_{m,-n}$

The states $\mathbf{d} \in \mathcal{S}_{m,-n}$ for $m1$, $n \leq 1$ are states not in the core where coalition C_{12} is not binding. Now, for $i \in \{1,2,3,4\}$, coalition C_{13} is the payable coalition at $\mathbf{d}(i)$. The probabilities all have the factor $\frac{1}{1-c_{13}}$, showing that only players 2 or 4 can reduce demands.

The stationary distribution $\pi_{\mathcal{S}_{m,n}}$ of a set $\mathcal{S}_{q,r}$ under the chain $\left(DV_t^{\mathbf{d}^*}\right)$ can be expressed in terms of the stationary distributions of the neighboring sets:

$$\pi_{\mathcal{S}_{m,n}} = \sum_{x \in [-1,0,1]} \sum_{y \in [-1,0,1]} p^{m,n}_{m+x,n+y} \pi_{\mathcal{S}_{q+x,r+y}}. \tag{5.10}$$

For a restricted state space $\Omega^{V^{\mathbf{k}}}$ of size $K = 2(k+1)^2 - (k+1)$, let $A^{\mathbf{d}^*}_{\mathbf{k}}$ be the transpose of the transition matrix for the chain $\left(DV_t^{\mathbf{d}^*,\mathbf{k}}\right)$. Each of the K columns contains the, at most nine non-zero, transition probabilities from a set in $\mathcal{S}_{m,n}$ in $\Omega^{V^{\mathbf{k}}}$ to all its neighboring sets in $\Omega^{V^{\mathbf{k}}}$. For large values of k, the matrix is sparse. We will consider only small integers k for $k \in \{2,...,10\}$. Simulations in 5.5 show, that the chain $\left(DV_t^{\mathbf{d}^*,\mathbf{2}}\right)$ and $\left(DV_t^{\mathbf{d}^*,\mathbf{3}}\right)$ approximate the behavior in the neighborhood of the asymmetric edge reasonably well, and the chain $\left(DV_t^{\mathbf{d}^*,\mathbf{10}}\right)$ approximates the behavior very accurately.

Given a Matrix Q of dimension K let Q_1 be the matrix Q where the last row is replaced by the all-ones vector $(1,1,...1)$ of dimension m.

Let $\pi^{\mathbf{d}^*,\mathbf{k}}$ be the equilibrium distribution of the Markov chain $\left(DV_t^{\mathbf{d}^*,\mathbf{k}}\right)$. Then $\pi^{\mathbf{d}^*}_{\mathbf{k}}$ is the solution to the system of linear equations $(A-I)_1 .\pi^{\mathbf{d}^*}_{\mathbf{k}} = 0$ where I is the identity matrix of dimension K and . is the dot product.

Let $\Delta^{\mathbf{d}^*}_{\mathcal{S}_{m,n},\mathbf{k}}$ be the vector with entries $\Delta^{\mathbf{d}^*}_{m,n}$ for each $\mathcal{S}_{m,n} \in \Omega^{V^{\mathbf{k}}}$ where the respective entry in $\Delta^{\mathbf{d}^*}_{\mathbf{k}}$ corresponds to the respective entries in $\pi^{\mathbf{d}^*,\mathbf{k}}$. Entries $\Delta^{\mathbf{d}^*}_{m,n}$, corresponding to sets $\mathcal{S}_{m,n}$ with $m = k$ or $n = k$ or $-k$, still get assigned the general value $\Delta^{\mathbf{d}^*}_{m,n}$ for a set $\mathcal{S}_{m,n}$.

Let $\left(\pi^{\mathbf{d}^*}_{\mathbf{k}}\right)^t$ be the transpose of $\pi^{\mathbf{d}^*}_{\mathbf{k}}$. We approximate $\Delta^{\mathbf{d}^*}$ with $\Delta^{\mathbf{d}^*}_{\mathbf{k}} = \left(\pi^{\mathbf{d}^*}_{\mathbf{k}}\right)^t \Delta^{\mathbf{d}^*}_{\mathcal{S}_{m,n},\mathbf{k}}$.

We recall the geometric properties and our conjectures about the asymmetric edges of example games $1-3$.

(a) For example game 1, $\mathbf{co}^3 = (0.37, 0.22, 0.21, 0.2)$ and $\mathbf{co} = \mathbf{min1} = (0.53, 0.22, 0.21, 0.04)$ and $\mathbf{max1} = (0.61, 0.14, 0.13, 0.12)$. The asymmetric edge is given by

$\mathcal{AE}^1 = (0.53+z, 0.22-z, 0.21-z, 0.04+z)$ for z a multiple of ε and $0 \leq z \leq 0.08$. We conjecture that neither **co** nor **max1** is a Markovian cooperative equilibrium but that there is a Markovian cooperative equilibrium with

$$\mathbf{mce} \approx (0.583586, 0.166414, 0.156414, 0.093586)$$

.

(b) For example game 2, $\mathbf{co}^3 = (0.59, 0.15, 0.14, 0.12)$, $\mathbf{co} = \mathbf{min1}\,(0.6, 0.15, 0.14, 0.11)$ and $\mathbf{max1} = (0.61, 0.14, 0.13, 0.12)$. The asymmetric edge is the set of points that can be joined by the line segment between **max1** and **min1**.

The asymmetric edge is given by $\mathcal{AE}^2 = (0.6+x, 0.15-x, 0.14-x, 0.11+x)$ for x a multiple of ε and $0 \leq x \leq 0.01$. We conjecture that $\mathbf{co} = \mathbf{min1}$ is a Markovian cooperative equilibrium.

(c) For example game 3, $\mathbf{co}^3 = (0.37, 0.22, 0.21, 0.2)$, $\mathbf{co} = \mathbf{min1} = (0.53, 0.22, 0.21, 0.04)$ and $\mathbf{max1} = (0.54, 0.21, 0.2, 0.05)$. The asymmetric edge is given by

$\mathcal{AE}^3 = (0.53+y, 0.22-y, 0.21-y, 0.04+y)$ for y a multiple of ε and $0 \leq y \leq 0.01$. We conjecture that **max1** is a Markovian cooperative equilibrium.

Recall that $\mathcal{AE}^2$, the asymmetric edge of example game 2 is a subset of the 'upper part', and $\mathcal{AE}^3$, the asymmetric edge of example game 3, is a subset of the 'lower part' of $\mathcal{AE}^1$.

Since a state $\mathbf{d}^* \in \mathcal{AE}$ is uniquely determined by d^1, we use the notation $\left(\pi^{d^{*1}}_{\mathbf{k}}\right)^t$ and $\Delta^{d^{*1}}_{\mathbf{k}}$ instead of $\left(\pi^{\mathbf{d}^*}\mathbf{k}\right)^t$ and $\Delta^{\mathbf{d}^*}_{\mathbf{k}}$. We want to show that $\left(\pi^{0.6}_{\mathbf{k}}\right)^t \Delta^{0.6}_{\mathbf{k}} < 0$ and $\left(\pi^{0.54}_{\mathbf{k}}\right)^t \Delta^{0.54}_{\mathbf{k}} > 0$.

We give now, as an example, the calculations for $\mathbf{k}=2$ and $d^1=0.6$. In table 5.8, the matrix $A_{\mathbf{2}}^{0.6}$ of transition probabilities of the chain $\left(DV_t^{0.6,\mathbf{2}}\right)$ is depicted. For example, the transition probability from a state $\mathbf{d}\in\mathcal{S}_{0,0}$ to a state in the set $\mathcal{S}_{1,1}$, highlighted with blue, is given by $A_{\mathbf{2}}^{0.6}(5,1)=p_{0,0}^{1,1}(0.6,\mathbf{2})=0.03$ and taken from table 5.2. The transition probability to stay in set $\mathcal{S}_{2,2}$ over one step of $\left(DV_t^{0.6,\mathbf{2}}\right)$, highlighted in green, is $A_{\mathbf{2}}^{0.6}(13,13)=p_{2,2}^{2,2}(0.6,\mathbf{2})=1-\left(p_{2,2}^{1,1}(0.6,\mathbf{2})+p_{2,2}^{1,2}(0.6,\mathbf{2})+p_{2,2}^{2,1}(0.6,\mathbf{2})\right)=0.48$.

Observe, that only transition probabilities to neighboring states are non-zero and that the maximum number of non-zero entries in one row is nine. This holds true of any restricted chain, irrespective of $\mathbf{k}$.

	$\mathcal{S}_{0,0}$	$\mathcal{S}_{0,-1}$	$\mathcal{S}_{1,-1}$	$\mathcal{S}_{1,0}$	$\mathcal{S}_{1,1}$	$\mathcal{S}_{0,1}$	$\mathcal{S}_{0,-2}$	$\mathcal{S}_{1,-2}$	$\mathcal{S}_{2,-2}$	$\mathcal{S}_{2,-1}$	$\mathcal{S}_{2,0}$	$\mathcal{S}_{2,1}$	$\mathcal{S}_{2,2}$	$\mathcal{S}_{1,2}$	$\mathcal{S}_{0,2}$
$\mathcal{S}_{0,0}$	0.54	0.25	0.14	0.29	0.15	0.29	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\mathcal{S}_{0,-1}$	0.14	0.39	0.14	0.0	0.0	0.0	0.25	0.14	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\mathcal{S}_{1,-1}$	0.0	0.11	0.25	0.0	0.0	0.0	0.11	0.25	0.14	0.14	0.0	0.0	0.0	0.0	0.0
$\mathcal{S}_{1,0}$	0.14	0.11	0.25	0.43	0.18	0.04	0.0	0.0	0.0	0.14	0.29	0.14	0.0	0.0	0.0
$\mathcal{S}_{1,1}$	0.03	0.0	0.0	0.07	0.25	0.06	0.0	0.0	0.0	0.0	0.04	0.19	0.15	0.18	0.04
$\mathcal{S}_{0,1}$	0.15	0.0	0.0	0.04	0.19	0.44	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.15	0.29
$\mathcal{S}_{0,-2}$	0.0	0.14	0.0	0.0	0.0	0.0	0.53	0.14	0.0	0.0	0.0	0.0	0.0	0.0	0.0
$\mathcal{S}_{1,-2}$	0.0	0.0	0.0	0.0	0.0	0.0	0.11	0.25	0.14	0.0	0.0	0.0	0.0	0.0	0.0
$\mathcal{S}_{2,-2}$	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.11	0.46	0.0	0.0	0.0	0.0	0.0	0.0
$\mathcal{S}_{2,-1}$	0.0	0.0	0.11	0.0	0.0	0.0	0.0	0.11	0.25	0.46	0.0	0.0	0.0	0.0	0.0
$\mathcal{S}_{2,0}$	0.0	0.0	0.11	0.14	0.04	0.0	0.0	0.0	0.0	0.25	0.60	0.30	0.0	0.0	0.0
$\mathcal{S}_{2,1}$	0.0	0.0	0.0	0.03	0.06	0.0	0.0	0.0	0.0	0.0	0.07	0.38	0.18	0.04	0.0
$\mathcal{S}_{2,2}$	0.0	0.0	0.0	0.0	0.03	0.0	0.0	0.0	0.0	0.0	0.0	0.07	0.48	0.06	0.0
$\mathcal{S}_{1,2}$	0.0	0.0	0.0	0.0	0.07	0.03	0.0	0.0	0.0	0.0	0.0	0.04	0.19	0.38	0.06
$\mathcal{S}_{0,2}$	0.0	0.0	0.0	0.0	0.04	0.15	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.19	0.61

Table 5.8 $A_{\mathbf{2}}^{0.6}$, the matrix of transition probabilities of the chain $\left(DV_t^{0.6,\mathbf{2}}\right)$

Figure 5.9 depicts the state space $\Omega^{V^{\mathbf{2}}}$, where in the first sketch, states are replaced with the corresponding equilibrium distribution for chain $\left(DV_t^{0.6,\mathbf{2}}\right)$, and in the second sketch with the equilibrium distribution for $\left(DV_t^{0.54,\mathbf{2}}\right)$.

The chain $\left(DV_t^{0.6,\mathbf{2}}\right)$, corresponding to the starting state $(0.6,0.15,0.14,0.11)\in\mathcal{AE}^1$, has lower mass on the red states and orange states but higher mass on the green and blue states. So the chain spends more time in states with negative drift in d^1 than the chain $\left(DV_t^{0.54,\mathbf{2}}\right)$, starting at the 'lower part' of $\mathcal{AE}^1$. Restricting the state space of $\left(DV_t^{\mathbf{d}^*}\right)$ to $\Omega^{V^{\mathbf{2}}}$ is a very strong simplification. However, we can see that the mass is strongly concentrated closely around the states in $\mathcal{S}_{0,0}$.

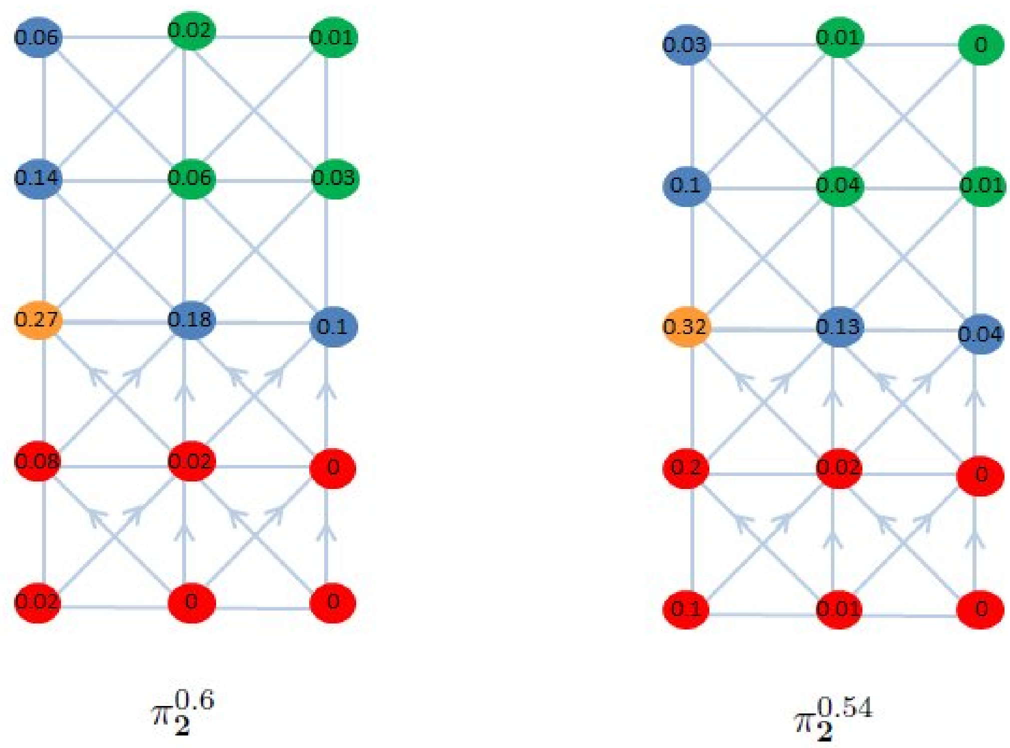


Figure 5.9 Sketch of the equilibrium distributions of chains $\left(DV_t^{0.6,\mathbf{2}}\right)$ and $\left(DV_t^{0.54,\mathbf{2}}\right)$ for example game 1

In the below table, columns $2-5$ contain the vectors $\pi_{\mathbf{2}}^{\mathbf{d}^*}$, summed over the sets $\mathcal{S}_-$, $\mathcal{S}_{+,+}$, $\mathcal{S}_0$ and $\mathcal{S}_{0,0}$ for different starting states. Columns $6-9$ contain the vectors of respective drifts $\Delta_{\mathbf{2}}^{\mathbf{d}^*}$. Numeric values are rounded to two decimal places and results are given for $d^1 = 0.6$, $d^1 = 0.61$, $d^1 = 0.53$ and $d^1 = 0.54$.

	$\pi_{\mathbf{2}}^{0.6}$	$\pi_{\mathbf{2}}^{0.61}$	$\pi_{\mathbf{2}}^{0.54}$	$\pi_{\mathbf{2}}^{0.53}$	$\Delta_{\mathbf{2}}^{0.6}$	$\Delta_{\mathbf{2}}^{0.61}$	$\Delta_{\mathbf{2}}^{0.54}$	$\Delta_{\mathbf{2}}^{0.53}$
$\mathcal{S}_-$	0.12	0.11	0.33	0.36	0.25	0.25	0.25	0.25
$\mathcal{S}_{0,0}$	0.27	0.26	0.32	0.32	0.1	0.1	0.12	0.12
$\mathcal{S}_0$	0.48	0.5	0.3	0.27	-0.05	-0.06	-0.02	-0.02
$\mathcal{S}_{+,+}$	0.12	0.13	0.06	0.05	-0.35	-0.36	-0.29	-0.28

Table 5.9 Equilibrium distribution and drift in the sets $\mathcal{S}_-$, $\mathcal{S}_{+,+}$, $\mathcal{S}_0$ and $\mathcal{S}_{0,0}$ for the restricted chain $\left(DV_t^{\mathbf{d}^*,\mathbf{2}}\right)$ for different starting states

The next table depicts, for a given $\mathbf{d}^* \in \mathcal{AE}^1$, the results for $\Delta^{\mathbf{d}^*} = \sum_{m,n} \pi_{\mathcal{S}_{m,n}}^{\mathbf{d}^*,\mathbf{k}} \Delta_{m,n}^{\mathbf{d}^*}$, the drift in d^1 over one excursion of the chain $\left(DV_t^{\mathbf{d}^*,\mathbf{k}}\right)$ from the asymmetric edge with starting state $\mathbf{d}^*$.

	$\Delta_{\mathbf{2}}^{0.6}$	$\Delta_{\mathbf{2}}^{0.61}$	$\Delta_{\mathbf{2}}^{0.54}$	$\Delta_{\mathbf{2}}^{0.53}$
	-0.01ε	-0.02ε	0.09ε	0.11ε

Table 5.10 Drift over an excursion from the asymmetric edge $\Delta_{\mathbf{2}}^{\mathbf{d}^*}$ for different starting states

At the 'lower end' of the asymmetric edge for game 1, the approximated drift is positive. At the 'upper end' of the asymmetric edge the drift is negative. Figure 5.10 gives a sketch of the asymmetric edge with the drifts over an excursion from the asymmetric edge, started in the 'lower part' at $(0.54, 0.21, 0.2, 0.05)$ and in the 'upper part' at $(0.6, 0.15, 0.14, 0.11)$. The two states on $\mathcal{AE}^1$ correspond to the states **max1** for game 3 and **min1** for game 2. Observe that, for states closer to **min1** in game 2, the set of trajectories that will hit the hyperplanes C_{124} or C_{134} before returning to a state in $\mathcal{AE}$ will increase. This impacts the drift so the approximation given here is accurate outside a small neighborhood of the state **min1**. However, this 'neighborhood' is independent of ε and so, for ε tending to zero, the only candidate for **mce** for example game 2 is **min1** $= (0.6, 0.15, 0.14, 0.11)$ and for example game 3, the only candidate for **mce** is **max1** $= (0.54, 0.21, 0.2, 0.05)$.

In order to get a more accurate estimate of the Markovian cooperative equilibrium we increase the state space of the restricted chain. We define a new restricted state space, taking into account that states $\mathcal{S}_{m,m}$ for larger m have lower mass under the equilibrium distribution than states close to a hyperplane. We set $\Omega^{V2} = \bigcup_i \mathcal{S}_{i,ri}$ for $i \in \{0,1,2,3,4,5\}$ and $-10 \le r0 \le 5, -6 \le r1 \le 4, -2 \le r2 \le 4, 0 \le r3 \le 2, 0 \le r4 \le 2$.

The restricted state space has size 41. We took into account, that states on the positive and negative diagonal do not have very high mass under the equilibrium distribution. Solving for d^{*1} where $\Delta^{\mathbf{d}^*} = 0$ yields $d^1 \approx 0.5955$ for $\left(DV_t^{\mathbf{d}^*}, \mathbf{2}\right)$. Solving for d^{*1} where $\Delta^{\mathbf{d}^*} = 0$ yields $d^1 \approx 0.5838$ for the second, more refined restricted state space.

In Section 5.5 we give Monte Carlo simulation results for the chain (V_t) started in a state $\mathbf{d}^*$ on the asymmetric edge.

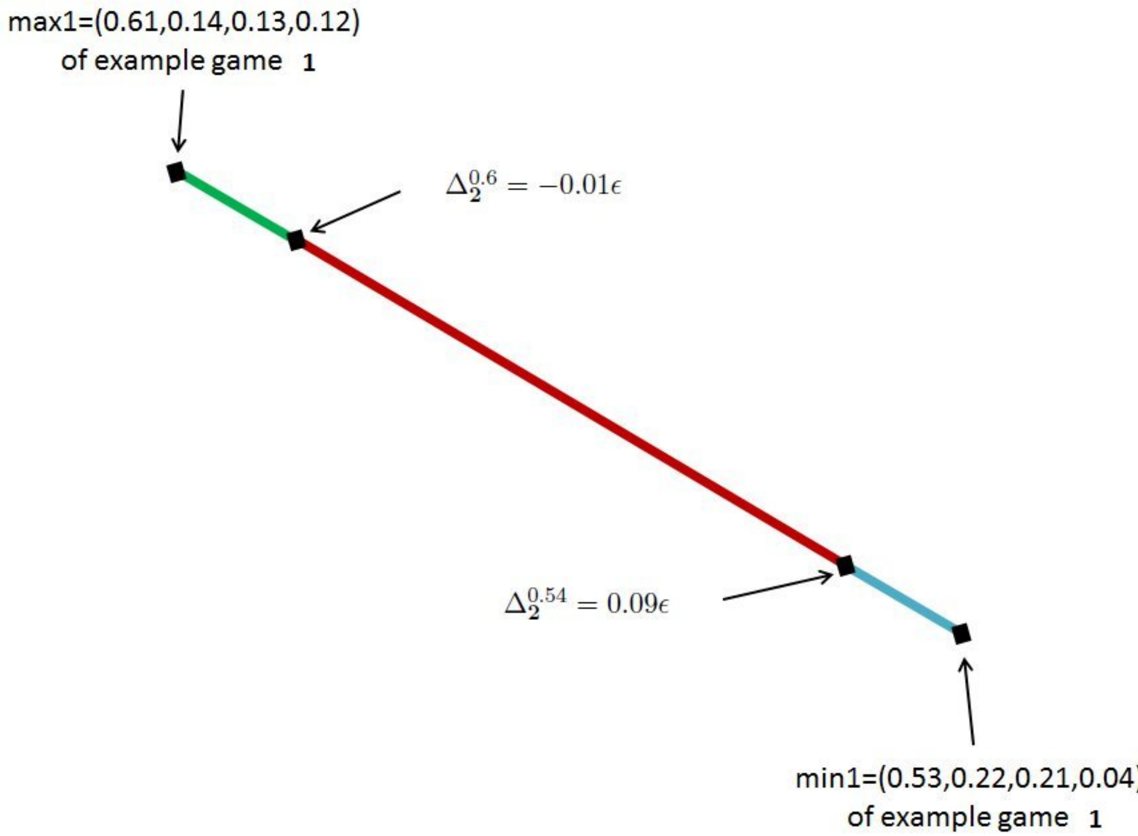


Figure 5.10 Sketch of the asymmetric edge of example game 1 with $\Delta_2^{0.6}$ and $\Delta_2^{0.54}$

5.4 Power in Core-Restricted 4-Player Game

In Section 5.4, we discuss a Markov chain (W_t) that is closely related to the chain (V_t), it is in fact a version of the chain (V_t) restricted to the core. We only define the chain (W_t) for games $(v,4)$ as we are able to prove a positive drift on the asymmetric edge for a specific 4-player example game on the chain (W_t). However one can easily extend that definition for the chain (W_t) to general N-player games and we expect that the chain (W_t) is very important for understanding the behavior of the chains (V_t) and (O_t).

In Section 5.3 we analyzed the behavior of the chain (V_t) along the asymmetric edge for 4-player games in detail. An understanding of the behavior along the asymmetric edge is essential to an understanding of where the Markovian cooperative equilibrium is located. We divided the states into sets according to their drift and the set of states outside the core had the strongest drift in d^1. On all sets of states that are in the core, the chain (W_t) has identical drift in d^1 as the chain (V_t). Transition probabilities from a state in the core not on the asymmetric edge to neighbors are identical for (V_t) and (W_t). On the asymmetric edge the chain (W_t) is restricted to 'stay in the same state', when the chain (V_t) transitions to a neighbor outside the core. As the chain (W_t) does not move on the states with the strongest positive drift and has identical transition probabilities to new states outside the core, if the chain (W_t) has a positive drift in d^1 along the asymmetric edge, we strongly expect that the chain (V_t) has a stronger drift along the asymmetric edge.

Let $(v,4)$ be a superadditive game. We call the Markov chain (W_t) a *W-cooperative game process* $W(v,N,\varepsilon)$ if $\varepsilon = \frac{1}{M}$ for some v-compatible M.

Given a $W\left(v,N,\frac{1}{M}\right)$-cooperative game process, if there exists a unique state $\mathbf{d}^* \in \Omega^C$ such that, for all $\alpha > 0$, $\mathbb{P}_{\pi_{V,M}}(||\mathbf{d}-\mathbf{d}^*||_2 > \alpha) \to 0$ as $M \to \infty$ then $\mathbf{d}^* = W\mathbf{mce}$, the *W-Markovian cooperative equilibrium*.

We introduce a specific example game, that has an asymmetric edge. We investigate the drift in d^1 between successive returns of the chain (W_t) to the asymmetric edge and we show, for the 'idealized chain' of (W_t) that this drift is positive for all starting states on the asymmetric edge. So we expect that the *W-Markovian cooperative equilibrium* is max 1 and we confirm that by simulations.

We now introduce example game 4.

						example
$v(C_{1234})$	$v(C_{123})$	$v(C_{124})$	$v(C_{134})$	$v(C_{12})$	$v(C_{13})$	
1	0.95	0.86	0.64	0.82	0.6	4

Table 5.11 $v(C)$ for all $\{C \in \mathcal{P}\left(C^G\right) \mid v(C) \neq 0\}$ for example game 1

For example game 4, $\mathbf{co}^3 = (0.45, 0.36, 0.14, 0.05)$, $\mathbf{co} = (0.46, 0.36, 0.14, 0.04)$ and $\mathbf{max1} = (0.47, 0.35, 0.13, 0.05)$, $\mathbf{min1} = (0.46, 0.36, 0.14, 0.04)$.

In this section we analyze the equilibrium distribution for an 'idealized' version of the chain (W_t). The chains (W_t) and (V_t) are very closely related.

We like to point out, that if the chain (V_t) started from a state on the asymmetric edge transitions to a new neighbor in the core, then its behavior is identical to the behavior of the chain (W_t) until it reaches another state on the asymmetric edge. Only if the chain (V_t) directly transitions to a neighbor outside the core, by player 4 increasing his demand and player 3 decreasing his demand, while the chain (W_t) 'sits' and waits on the asymmetric edge, the behavior will be different for both chains till they reach a new state on the asymmetric edge.

We define an 'idealized chain' (W_t^*) in the same way that we defined the idealized version of the chain (V_t^*) and then the 2-dimensional idealized chain (DW_t) in the same way that we defined the chain (DV_t) in the previous section.

In Figure 5.11 we depict a subset of the state spaces of the chains (DV_t) and (DW_t). The chain (W_t) cannot transition to states outside the core so the state space of the chain (DW_t) is the state space of the chain (DV_t) with the second coordinate restricted to non negative values.

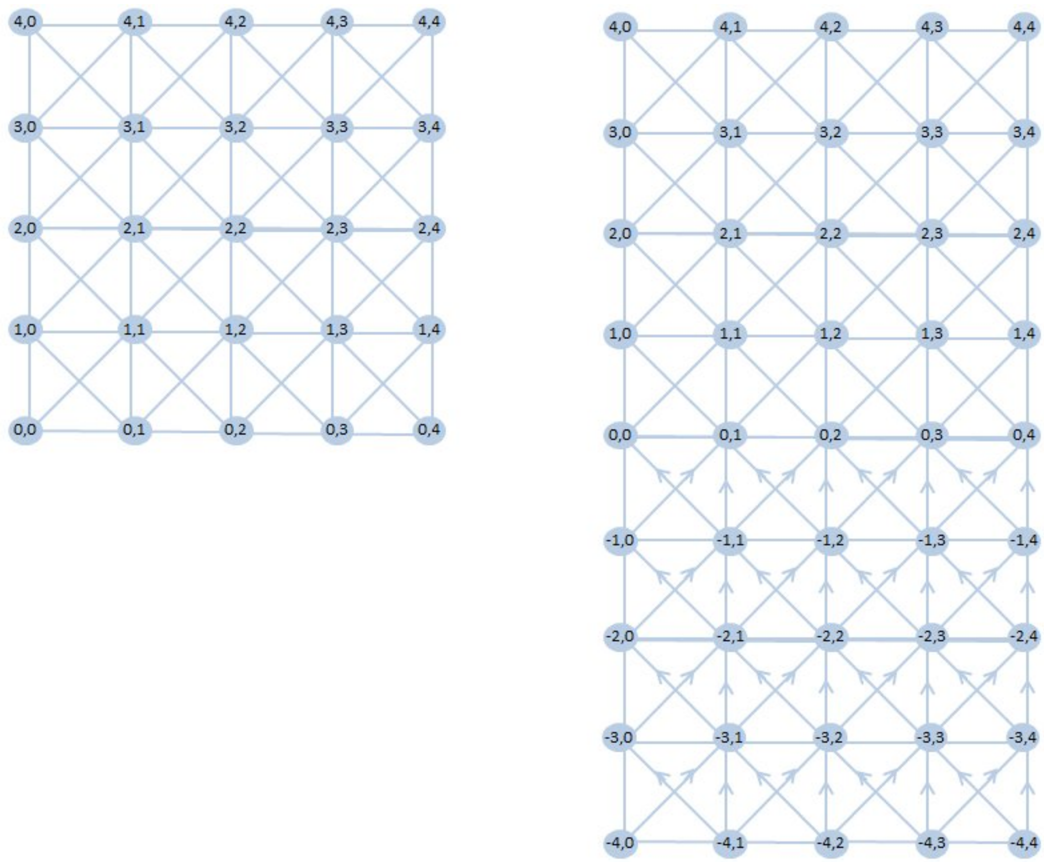


Figure 5.11 Sketch of a subset of the state space of the chains $(DV_t^{\mathbf{d}^*})$ and $(DW_t^{\mathbf{d}^*})$

Another important feature of the chain (W_t) is that it has identical drift as the chain (V_t) on all states in the core. This is depicted in Figure 5.12 where the actual drifts for the starting state on the upper end of the asymmetric edge are given.

Recall that for example game 4, **co** is in the interior of the asymmetric edge. We analyze the change in d^1 between excursions of the chain (W_t) from and back to the asymmetric edge of example game 4. Observe that for example game 4, **co** is in the interior of the asymmetric edge.

Then we show for this example game, that the expected change in d^1 in equilibrium of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ for each 'reference state' $\mathbf{d}^*$ is positive.

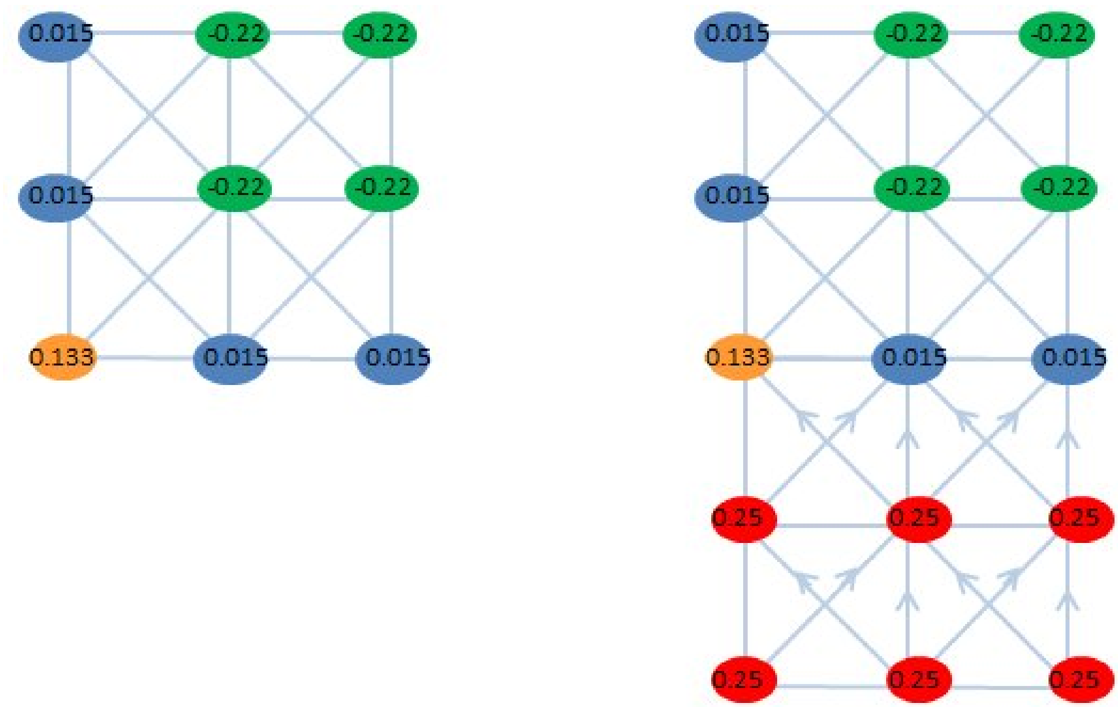


Figure 5.12 Sketch of the drifts on the subset of the state space of the chain $\left(DV_t^{\mathbf{d}^*}\right)$ and $\left(DW_t^{\mathbf{d}^*}\right)$

The following definitions are equivalent to the respective definitions in the previous section when (V_t) is exchanged with (W_t).

For all $\mathbf{d} \in \Omega^C \setminus \mathcal{AE}$, let $p_{\mathbf{d},W}^{i,j} = p_{\mathbf{d}}^{i,j}$. For $\mathbf{d} \in \mathcal{AE}$, let $p_{\mathbf{d},W}^{4,3} = 0$, $p_{\mathbf{d},W}^{4,4} = p_{\mathbf{d}}^{4,4} + p_{\mathbf{d}}^{4,3}$ and $p_{\mathbf{d},W}^{i,j} = p_{\mathbf{d}}^{i,j}$ for all other combinations of i,j with $i,j \in \{1,2,3,4\}$. The only difference between the chains (W_t) and (V_t) is that from a state on the asymmetric edge the chain cannot leave the core. Now we define the 'idealized' transition probabilities for the chain (W_t).

For $\mathbf{d}^* \in \mathcal{AE}$, for all $\mathbf{d} \in \Omega^C \setminus \mathcal{AE}$, let $p_{\mathbf{d},W}^{i,j}(\mathbf{d}^*) = p_{\mathbf{d}}^{i,j}(\mathbf{d}^*)$, and for $\mathbf{d} \in \mathcal{AE}$, let $p_{\mathbf{d},W}^{4,3}(\mathbf{d}^*) = 0$, $p_{\mathbf{d},W}^{4,4}(\mathbf{d}^*) = p_{\mathbf{d}}^{4,4}(\mathbf{d}^*) + p_{\mathbf{d}}^{4,3}(\mathbf{d}^*)$.

The 2-dimensional chain

$$\left(\frac{CS^{C_{12}}(W_t)}{\varepsilon},\frac{CS^{C_{13}}(W_t)}{\varepsilon}\right)$$

is now defined on

$$\Omega^W=\{0,...\frac{1-v(C_{12})}{\varepsilon}\}\times\{0...\frac{1-v(C_{13})}{\varepsilon}\}.$$

For $0\le m\le\frac{1-v(C_{12})}{\varepsilon}$ and $0\le n\le\frac{1-v(C_{13})}{\varepsilon}$, let $\mathcal{S}^W_{m,n}$ be the set of efficient states where $\frac{CS^{C_{12}}(\mathbf{d})}{\varepsilon}=m$ and $\frac{CS^{C_{13}}(\mathbf{d})}{\varepsilon}=n$ and no other coalition is feasible.

Again we want to approximate the 2-dimensional chain

$$\left(\frac{CS^{C_{12}}(W_t)}{\varepsilon},\frac{CS^{C_{13}}(W_t)}{\varepsilon}\right)$$

by using the transition $p^{i,j}_{\mathbf{d},W}(\mathbf{d}^*)$ instead of $p^{i,j}_{\mathbf{d},W}$.

For $\mathbf{d}$ in $\mathcal{S}^W_{m,n}$, $\mathbf{d}^*\in\mathcal{AE}$ and $q\in\{m-1,m,m+1\}$ and $r\in\{n-1,n,n+1\}$, let

$$p^{q,r}_{\mathbf{d}}(\mathbf{d}^*,W)=\sum_{i,j\in\{1,2,3,4\}}p^{i,j}_{\mathbf{d},W}(\mathbf{d}^*)\,\mathbf{1}_{\{\mathbf{d}(i,j)\in\mathcal{S}^W_{q,r}\}}. \tag{5.11}$$

As before $p^{q,r}_{\mathbf{d}}(\mathbf{d}^*,W)$ depends on $\mathbf{d}$ only via the sets $\mathcal{S}^W_{m,n}$ that is why we may use the notation $p^{q,r}_{m,n}(\mathbf{d}^*,W)$ as well.

For $\mathbf{d}^*\in\mathcal{AE}$, let the dependent two dimensional chain $\left(DW^{\mathbf{d}^*}_t\right)$ be the Markov chain defined on

$$\Omega^W=\{0,...\frac{1-v(C_{12})}{\varepsilon}\}\times\{0,...\frac{1-v(C_{13})}{\varepsilon}\}$$

that moves with transition probabilities $p^{q,r}_{m,n}(\mathbf{d}^*,W)$ if $0\le m<\frac{1-v(C_{12})}{\varepsilon}$ and $0\le n<\frac{1-v(C_{13})}{\varepsilon}$.

Let $\left(DW1^{\mathbf{d}^*}\left(DW^{\mathbf{d}^*}_t\right)_t\right)$ be the first coordinate of the chain $\left(DW^{\mathbf{d}^*}_t\right)$ and equivalently $\left(DW2^{\mathbf{d}^*}\left(DW^{\mathbf{d}^*}_t\right)_t\right)$ be the second coordinate of the chain $\left(DW^{\mathbf{d}^*}_t\right)$.

Observe that, if the first coordinate of $\left(DW^{\mathbf{d}^*}_t\right)$ is zero, then the chain $\left(W^{\mathbf{d}^*}_t\right)$ is in a state on the hyperplane $\mathcal{H}(C_{12})$. If the second coordinate of $\left(DW^{\mathbf{d}^*}_t\right)$ is zero, then the chain $\left(W^{\mathbf{d}^*}_t\right)$ is in a state on the hyperplane $\mathcal{H}(C_{13})$.

Let $\mathcal{S}_{0,+}=\bigcup_{0<m\le\frac{1-v(C_{13})}{\varepsilon}}\mathcal{S}^W_{m,0}$, let $\mathcal{S}_{+,0}=\bigcup_{0<m\le\frac{1-v(C_{12})}{\varepsilon}}\mathcal{S}^W_{m,0}$ and

$$\mathcal{S}_{+,+}=\bigcup_{0<m\le\frac{1-v(C_{13})}{\varepsilon},\mathbf{0}<n\le\frac{1-v(C_{13})}{\varepsilon}}\mathcal{S}^W_{m,n}.$$

We will now investigate the expected change in d^1 over the sets $\mathcal{S}_{+,+}$, $\mathcal{S}_{0,+}$, $\mathcal{S}_{+,0}$ and $\mathcal{S}_{0,0}$.

For $\mathbf{d}^* \in \mathcal{AE}$, $\mathbf{d} \in \Omega^E$, let $\Delta\left(W_t^{\mathbf{d}^*}, \mathbf{d}\right) = \mathbb{E}\left(d_{t+1}^1\left(W_{t+1}^{\mathbf{d}^*}\right) - d_t^1\left(W_t^{\mathbf{d}^*}\right) \mid \mathbf{d}\right)$ be the drift in d^1 over one step of $\left(W_t^{\mathbf{d}^*}\right)$ at $\mathbf{d}$.

As $\Delta\left(W_t^{\mathbf{d}^*}, \mathbf{d}\right)$ is constant on $\mathcal{S}_{m,n}^W$, for $\mathbf{d}^* \in \mathcal{AE}$ and $\mathbf{d} \in \mathcal{S}_{m,n}^W$ we use $\Delta_{m,n}^{\mathbf{d}^*,W} = \Delta\left(W_t^{\mathbf{d}^*}, \mathbf{d}\right)$ as well as notation.

On the asymmetric edge, $\Delta_{0,0}^{\mathbf{d}^*,W} = \frac{1}{4}\left(1 - \mathbf{d}^{*1}\right)$.

On either hyperplane $\mathcal{H}\left(C_{12}\right)$ or $\mathcal{H}\left(C_{13}\right)$, if exactly one of $n > 0$ or $m > 0$,

$$\Delta_{0,+}^{\mathbf{d}^*,W} = \Delta_{+,0}^{\mathbf{d}^*} = \frac{1 - 2\mathbf{d}^{*1}}{4}.$$

Finally, for states in the interior of the core, where both $n > 0$ and $m > 0$,

$$\Delta_{+,+}^{\mathbf{d}^*,W} = \frac{1}{4}\left[\left(1 - \mathbf{d}^{*1}\right) - 3\mathbf{d}^{*1}\right] = \frac{1 - 4\mathbf{d}^{*1}}{4}.$$

This is identical to the the drift on these sets for the chain $\left(V_t\right)$.

Now we will define the expected change in d^1 over one excursion from the asymmetric edge.

For $\mathbf{d}^* \in \mathcal{AE}$ let $\pi_{\mathcal{S}_{m,n}}^{\mathbf{d}^*,W}$ be the equilibrium distribution of $\left(DW_t^{\mathbf{d}^*}\right)$ on the set $\mathcal{S}_{m,n}$.

For $\mathbf{d}^* \in \mathcal{AE}$ let $\Delta^{\mathbf{d}^*,W} = \sum_{m,n} \pi_{\mathcal{S}_{m,n}}^{\mathbf{d}^*,W} \Delta_{m,n}^{\mathbf{d}^*}$ be the drift in d^1 of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ in equilibrium.

We try to find a lower bound on the probability that the chain $\left(DW_t\right)$ is in the sets $\mathcal{S}_{0,+}$, $\mathcal{S}_{+,0}$ and $\mathcal{S}_{0,0}$.

We assume that ε is small and that demands on the excursion from the asymmetric edge back to the asymmetric edge can be well approximated by the demands at the starting state at the asymmetric edge.

Observe that the transition probabilities $p_{\mathbf{d}}^{q,r}\left(\mathbf{d}^*, W\right)$ are constant for $\mathbf{d}$ in each of the sets $\mathcal{S}_{0,0}$, $\mathcal{S}_{+,+}$, $\mathcal{S}_{+,0}$ and $\mathcal{S}_{0,+}$. We will use, for example, $p_{+,+}^{q,r}\left(\mathbf{d}^*, W\right)$ as the probability $p_{\mathbf{d}}^{q,r}\left(\mathbf{d}^*, W\right)$ for all states $\mathbf{d} \in \mathcal{S}_{+,+}$.

Then, for $\phi \in \{(0,0),(0,+),(+,0),(+,+)\}$ let $p_{\phi}^{C_{12},+}\left(\mathbf{d}^*\right) = \sum_{r \in \{-1,0,1\}} p_{\phi}^{m+1,r}\left(\mathbf{d}^*, W\right)$ and $p_{\phi}^{C_{12},-}\left(\mathbf{d}^*\right) = \sum_{r \in \{-1,0,1\}} p_{\phi}^{m-1,r}\left(\mathbf{d}^*, W\right)$.

Equivalently let $p_{\phi}^{C_{13},+}(\mathbf{d}^*) = \sum_{r\in\{-1,0,1\}} p_{\phi}^{r,m+1}(\mathbf{d}^*, W)$ and

$$p_{\phi}^{C_{13},-}(\mathbf{d}^*) = \sum_{r\in\{-1,0,1\}} p_{\phi}^{r,m-1}(\mathbf{d}^*, W)$$

.

In Table 5.12 we summarize the transition probabilities for first and second coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$. $p_{+,+}^{C_{12},+}$ is the probability, that the first coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ increases by one if $\left(DW1_t^{\mathbf{d}^*}\right)$ is in the set $\mathcal{S}_{+,+}$. Observe that $p_{+,+}^{C_{12},+}(\mathbf{d}^*) = p_{0,+}^{C_{12},+}(\mathbf{d}^*)$ and so $\left(DW1_t^{\mathbf{d}^*}\right)$, the first coordinate of the state $\left(DW_t^{\mathbf{d}^*}\right)$, increases by one with the same probability no matter whether $\mathbf{d} \in \Omega^E$ is in a state on $\mathcal{H}(C_{12})$ or in a state in the interior of the core. $p_{0,0}^{C_{12},-}(\mathbf{d}^*) = p_{0,+}^{C_{12},-}(\mathbf{d}^*) = 0$, as in a state on the hyperplane $\mathcal{H}(C_{12})$ $\left(DW1_t^{\mathbf{d}^*}\right)$ cannot decrease by one. The same holds true for any state on the asymmetric edge.

$p_{+,+}^{C_{12},+}(\mathbf{d}^*) = \frac{1-C_{12}}{2}$	$p_{+,+}^{C_{12},-}(\mathbf{d}^*) = \frac{C_{12}}{2}$
$p_{+,0}^{C_{12},+}(\mathbf{d}^*) = \frac{1-C_{12}}{4} + \frac{d^{*4}}{4\left(d^{*2}+d^{*4}\right)}$	$p_{+,0}^{C_{12},-}(\mathbf{d}^*) = \frac{d^{*2}}{4\left(d^{*2}+d^{*4}\right)} + \frac{C_{12}}{4}$
$p_{0,+}^{C_{12},+}(\mathbf{d}^*) = \frac{1-C_{12}}{2}$	$p_{0,+}^{C_{12},-}(\mathbf{d}^*) = 0$
$p_{0,0}^{C_{12},+}(\mathbf{d}^*) = \frac{1-C_{12}}{4} + \frac{d^{*4}}{4\left(d^{*2}+d^{*4}\right)}$	$p_{0,0}^{C_{12},-}(\mathbf{d}^*) = 0$

Table 5.12 Transition probabilities for $\left(DW1_t^{\mathbf{d}^*}\right)$

Similarly, in Table 5.12 the transition probabilities for the second coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ are given.

$p_{+,+}^{C_{13},+}(\mathbf{d}^*) = \frac{1-C_{13}}{2}$	$p_{+,+}^{C_{13},-}(\mathbf{d}^*) = \frac{C_{13}}{2}$
$p_{+,0}^{C_{13},+}(\mathbf{d}^*) = \frac{1-C_{13}}{2}$	$p_{+,0}^{C_{13},-}(\mathbf{d}^*) = 0$
$p_{0,+}^{C_{13},+}(\mathbf{d}^*) = \frac{1-C_{13}}{4} + \frac{d^{*4}}{4\left(d^{*3}+d^{*4}\right)}$	$p_{0,+}^{C_{13},-}(\mathbf{d}^*) = \frac{d^{*3}}{4\left(d^{*3}+d^{*4}\right)} + \frac{C_{13}}{4}$
$p_{0,0}^{C_{13},+}(\mathbf{d}^*) = \frac{1-C_{13}}{4} + \frac{d^{*4}}{4\left(d^{*3}+d^{*4}\right)}$	$p_{0,0}^{C_{12},-}(\mathbf{d}^*) = 0$

Table 5.13 Transition probabilities for $\left(DW2_t^{\mathbf{d}^*}\right)$

Let $p_{max}^{C_{13},+}(\mathbf{d}^*) = \max\{p_{+,+}^{C_{13},+}(\mathbf{d}^*), p_{0,+}^{C_{13},+}(\mathbf{d}^*)\}$ and

$$p_{min}^{C_{13},-}(\mathbf{d}^*) = \min\{p_{+,+}^{C_{13},-}(\mathbf{d}^*), p_{0,+}^{C_{13},-}(\mathbf{d}^*)\}.$$

Let $p_{max}^{C_{12},+}(\mathbf{d}^*) = \max\{p_{+,+}^{C_{12},+}(\mathbf{d}^*), p_{+,0}^{C_{12},+}(\mathbf{d}^*)\}$ and

$$p_{min}^{C_{12},-}(\mathbf{d}^*) = \min\{p_{+,+}^{C_{12},-}(\mathbf{d}^*), p_{+,0}^{C_{12},-}(\mathbf{d}^*)\}$$

.

Let $\left(RW_t^{12,\mathbf{d}^*}\right)$ be the 'pessimistic' random walk defined on the state space $\{0,...,\frac{1-v(C_{12})}{\varepsilon}\}$ with transition probabilities $p_{i,i+1} = p_{max}^{C_{12},+}(\mathbf{d}^*)$ for $i \geq 0$ and $p_{i,i-1} = p_{min}^{C_{12},-}(\mathbf{d}^*)$ if $i \geq 1$ and $p_{0,-1} = 0$. Let $\left(RW_t^{13,\mathbf{d}^*}\right)$ be the second coordinate 'pessimistic' random walk defined on the state space $\{0,...,\frac{1-v(C_{13})}{\varepsilon}\}$ with transition probabilities $p_{i,i+1} = p_{max}^{C_{13},+}(\mathbf{d}^*)$ for $i \geq 0$ and $p_{i,i-1} = p_{min}^{C_{13},-}(\mathbf{d}^*)$ if $i \geq 1$ and $p_{0,-1} = 0$.

We can calculate the equilibrium probability for the 0-state for $\left(RW_t^{12,\mathbf{d}^*}\right)$ and $\left(RW_t^{13,\mathbf{d}^*}\right)$. The point is that no matter at which state the chain $\left(DW_t^{\mathbf{d}^*}\right)$ is, the pessimistic random walks have a higher probability of increasing by one than the respective coordinates of the chain, that includes states where either coordinate or both are zero. As well, no matter at which state the chain $\left(DW_t^{\mathbf{d}^*}\right)$ is, the pessimistic random walks have a lower probability of decreasing by one than the respective coordinates of the chain, apart from when the random walks are in the 0-state. So we can couple each coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ started at $(0,0)$ separately with its respective pessimistic random walk in such a way, that the random walk is always 'above' the respective coordinate of the chain. The point is here, that it is very difficult to exactly analyze the dependence of the coordinates in the chain $\left(DW_t^{\mathbf{d}^*}\right)$. However, if each coordinate is 0 with high enough probability so that the the sum is greater than one, we know a minimum probability that the coordinates must both be 0. As well, both coordinates cannot be zero with a higher probability than either of the chains alone and so we know a maximum probability that both coordinates are zero.

As we will show in Theorem (5.11) no matter how the dependence within these limits, there is always a positive drift on the asymmetric edge of example game 4 and hence the chain (W_t) drifts towards **max1** on the 'upper end' of the asymmetric edge, away from **co**.

We use these bounds to find a upper bound on the equilibrium distribution for the states $\mathbf{d} \in \mathcal{S}_{+,+}$. Then we show, using these bounds, that $\Delta^{\mathbf{d}^*,W} > 0$ for all starting states $\mathbf{d}^*$ along the asymmetric edge.

Let $\pi_0^{RW1,\mathbf{d}^*}$ and $\pi_0^{RW2,\mathbf{d}^*}$ be the equilibrium probabilities for the 0-state of the 'pessimistic ' random walks. We will use these as lower bounds for $\sum_{m\geq 0} \pi_{(0,m)}^{DW^{\mathbf{d}^*}}$ and $\sum_{n\geq 0} \pi_{(n,0)}^{DW^{\mathbf{d}^*}}$. These bounds we will use to calculate a lower bound on the drift in equilibrium of the chain $\left(DW_t^{\mathbf{d}^*}\right)$.

$p^{C_{12},+}_{+,+}(0.47) = 0.09$	$p^{C_{12},-}_{+,+}(0.47) = 0.41$
$p^{C_{12},+}_{+,0}(0.47) = 0.08$	$p^{C_{12},-}_{+,0}(0.47) = 0.42$
$p^{C_{12},+}_{0,+}(0.47) = 0.09$	$p^{C_{12},-}_{0,+}(0.47) = 0$
$p^{C_{12},+}_{0,0}(0.47) = 0.08$	$p^{C_{12},-}_{0,0}(0.47) = 0$

Table 5.14 Transition probabilities for the first coordinate of the chain $\left(DW_t^{0.47}\right)$

Similarly, in Table 5.12 the transition probabilities for the second coordinate of the chain $\left(DW_t^{0.45}\right)$ are given.

$p^{C_{13},+}_{+,+}(0.47) = 0.2$	$p^{C_{13},-}_{+,+}(0.47) = 0.3$
$p^{C_{13},+}_{+,0}(0.47) = 0.2$	$p^{C_{13},-}_{+,0}(0.47) = 0$
$p^{C_{13},+}_{0,+}(0.47) = 0.17$	$p^{C_{13},-}_{0,+}(0.47) = 0.33$
$p^{C_{13},+}_{0,0}(0.47) = 0.17$	$p^{C_{12},-}_{0,0}(0.47) = 0$

Table 5.15 Transition probabilities for the second coordinate of the chain $\left(DW_t^{0.47}\right)$

Lemma 5.10. *For all states* $\mathbf{d}^*$ *on the asymmetric edge of or example game* 4*, it holds that* $\sum_{m\geq 0}\pi^{DW^{\mathbf{d}^*}}_{(0,m)} \geq \pi_0^{RW1,\mathbf{d}^*}$ *and* $\sum_{n\geq 0}\pi^{DW^{\mathbf{d}^*}}_{(n,0)} \geq \pi_0^{RW2,\mathbf{d}^*}$.

Proof: By construction of $\left(RW1_t^{\mathbf{d}^*}\right)$, it is true that at each state, including states where either coordinate of $\left(DW_t^{\mathbf{d}^*}\right)$ is 0, the 'pessimistic' random walk has higher probability to increase by one than the first coordinate. As well at each state apart from when the first coordinate is zero, the random walk has a lower probability of decreasing than the first coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$. So we can couple the first coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ started from the $(0,0)$-state with $\left(RW1_t^{\mathbf{d}^*}\right)$ started in the 0-state in such a way that it always holds that the first coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ is less than $\left(RW1_t^{\mathbf{d}^*}\right)$. So we can conclude that $\sum_{m\geq 0}\pi^{DW^{\mathbf{d}^*}}_{(0,m)} \geq \pi_0^{RW1,\mathbf{d}^*}$. The same argument applies to the second coordinate of the chain. □

Theorem 5.11. *If* (W_t) *is the* W*-cooperative game process for example game* 4 *then* $\Delta^{\mathbf{d}^*,W} > 0$ *for all starting states* $\mathbf{d}^*$ *along the asymmetric edge.*

Proof: We know from Lemma (5.10) that for all states $\mathbf{d}^*$ on the asymmetric edge of or example game 4, it holds that $\sum_{m\geq 0}\pi^{DW^{\mathbf{d}^*}}_{(0,m)} \geq \pi_0^{RW1,\mathbf{d}^*}$ and $\sum_{n\geq 0}\pi^{DW^{\mathbf{d}^*}}_{(n,0)} \geq \pi_0^{RW2,\mathbf{d}^*}$.

We first show that $\Delta^{0.47,W} > 0$, then we show that $\Delta^{0.46,W} > 0$. So we know that at both endpoints on the asymmetric edge the theorem holds. Then we argue that the lemma holds as well for all interior states.

We first calculate a bound on $\sum_{n\geq 0}\pi_{(n,0)}^{DW^{\mathbf{d}^*}}$ by calculating $\pi_0^{RW2,\mathbf{d}^*}$ explicitly.

Observe that $p_{+,+}^{C_{13},+}(0.47) = 0.2$ and $p_{0,+}^{C_{13},+}(0.47) = 0.17$ and so $p_{max}^{C_{13},+}(0.47) = 0.2$. Furthermore $p_{min}^{C_{13},-}(0.47) = 0.3$ since $p_{+,+}^{C_{13},-}(0.47) = 0.3$ and $p_{0,+}^{C_{13},-}(\mathbf{d}^*) = 0.33$.

And $p_{max}^{C_{12},+}(\mathbf{d}^*) = \max\{p_{+,+}^{C_{12},+}(\mathbf{d}^*), p_{+,0}^{C_{12},+}(\mathbf{d}^*)\}$ and

$$p_{min}^{C_{12},-}(\mathbf{d}^*) = \min\{p_{+,+}^{C_{12},-}(\mathbf{d}^*), p_{+,0}^{C_{12},-}(\mathbf{d}^*)\}$$

.

From Grimmett, Stirzacker (2001) we know that the equilibrium distribution of a random walk is given by $1-\phi$ where ϕ is the ratio $\phi = \frac{p_{i,i+1}}{p_{i+1,i}}$ where $p_{i,i+1}$ is the probability that the random walk increases one step at a state $i \neq 0$ and $p_{i+1,i}$ is the probability that the random walk decreases one step if at $i \neq 0$. Let ϕ_1 correspond to $RW1$ and ϕ_2 correspond to $RW2$. Then $\phi_1 = \frac{0.09}{0.41} = 0.22$ and so $\pi_0^{RW1,0.47} = 0.78$ and $\phi_2 = \frac{0.2}{0.3} = \frac{2}{3}$ and so $\pi_0^{RW1,0.47} = \frac{1}{3}$. By Lemma (5.10) it follows that the first coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ is in the zero state with a probability of at least 0.78, and the second coordinate is in a zero state with probability of at least $\frac{1}{3}$. We do not know the exact dependence of the first and the second coordinate of the chain $\left(DW_t^{\mathbf{d}^*}\right)$ however we know that the chain $DW_t^{\mathbf{d}^*}$ in equilibrium is at least with probability 0.11 and at most with probability $\frac{1}{3}$ in the set $\mathcal{S}_{0,0}$.

We claim that as long as the drifts are such that $\Delta_{0,0}^{\mathbf{d}^*,W} > 0$, $\Delta_{0,+}^{\mathbf{d}^*,W} > 0$, $\Delta_{+,0}^{\mathbf{d}^*,W} > 0$ and $\Delta_{+,+}^{\mathbf{d}^*,W} < 0$ then the chain $\left(DW_t^{\mathbf{d}^*}\right)$ that is with the maximal possible probability in the set $\mathcal{S}_{0,0}$ gives a lower bound on the drift compared to all other possible such chains that are in the set $\mathcal{S}_{0,0}$ with lower probability.

Observe that the equilibrium distribution over the different sets $\mathcal{S}_\phi$ can vary between $\pi_{0,0} = \frac{1}{3} - x$, $\pi_{0,+} = 0.78 - \frac{1}{3} + x, \frac{1}{3} + x$, $\pi_{+,0} = 0 + x$ and $\pi_{+,+} = 0.22 - x$ for $0 \leq x \leq 0.22$.

If x increases both the equilibrium probability on $\mathcal{S}_{+,+}$ and on $\mathcal{S}_{0,0}$ decreases by x whereas the equilibrium probability on $\mathcal{S}_{0,+}$ and $\mathcal{S}_{+,0}$ increases by x. Since $\Delta_{0,0}^{\mathbf{d}^*,W} < |\Delta_{+,+}^{\mathbf{d}^*,W}|$ the overall change in drift will be positive for x increasing. So it suffices to calculate the drift for the maximum possible probability that the chain $DW_t^{\mathbf{d}^*}$ is in $\mathcal{S}_{0,0}$.

Observe that $\Delta_{0,0}^{0.47,W} = 0.53$, $\Delta_{+,0}^{0.47,W} = \Delta_{0,+}^{0.47,W} = 0.06$ and $\Delta_{+,+}^{0.47,W} = -0.88$ and so it holds that $\Delta^{0.47,W} = \frac{0.53}{3} + 0.06 * 0.45 + 0.06 * 0 - 0.88 * 0.22 = 0.01$.

The same calculation for the $d^{*1} = 0.46$ yields $\Delta^{0.47,W} = 0.03$.

The larger d^{*1} (representing a 'higher' location on the asymmetric edge), the smaller the drift becomes. However for all states along the asymmetric edge it holds that $\Delta^{\mathbf{d}^*,W} > 0$.

$\square$

5.5 Simulation Results and Conclusions

The code is written in visual $C++$ and it simulates sample paths of the chain (V_t). Random numbers are generated via the *boost* library uniform number generator.

We fix $\varepsilon = 0.000001$. We start the chain (V_t) in a particular state $\mathbf{d}^*$ on the asymmetric edge and run the simulation until the chain (V_t) has reached a new state on the asymmetric edge.

We run 10000 simulations per starting state $\mathbf{d}^*$ corresponding to 10000 excursions from state $\mathbf{d}^*$ on the asymmetric edge. For $1 \leq n \leq 10000$, let $(V_{t,n})$ be the n-th simulated sample path, or excursion, of the chain (V_t).

For each $(V_{t,n})$, we record the new state on the asymmetric edge $\mathbf{d}^{**}(\mathbf{d}^*, n)$, the return time $R_1(\mathbf{d}^*, n)$, as well $R(\mathcal{S}_-, n)$, $R(\mathcal{S}_{+,+}, n)$, $R(\mathcal{S}_0, n)$ and $R(\mathcal{S}_{0,0}, n)$, the times spent in each of these four sets. Recall that only states, where at most coalitions C_{12} or C_{13} are binding, are members of these sets. So $R(\mathbf{d}^*, \mathcal{S}_-, n) + R(\mathbf{d}^*, \mathcal{S}_{+,+}, n) + R(\mathbf{d}^*, \mathcal{S}_0, n) + R(\mathbf{d}^*, \mathcal{S}_{0,0}, n) = R_1(\mathbf{d}^*, n)$ if the sample path $(V_{t,n})$ does not hit any hyperplane corresponding to a three player coalition. Furthermore, we record

$$\max 12(\mathbf{d}^*, n) = \max_{t<R_1} \frac{\left(CS^{C_{12}}(V_t)\right)}{\varepsilon}$$

, $\max 13(\mathbf{d}^*, n) = \max_{t<R_1} \frac{\left(CS^{C_{13}}(V_{t,n})\right)}{\varepsilon}$, $\min 13(\mathbf{d}^*, n) = \min_{t<R_1} \frac{\left(CS^{C_{13}}(V_{t,n})\right)}{\varepsilon}$ and, finally, $\max d(\mathbf{d}^*, n) = \max_{t<R_1, i} \frac{|d_{t,n}^i - d^{i*}|}{\varepsilon}$.

We then calculate the expected return time from $\mathbf{d}^*$ as $\frac{\sum_n R_1(\mathbf{d}^*, n)}{10000}$, the drift $\Delta^{\mathbf{d}^*}$ we approximate with $\frac{\sum_n \left(d^{1**}(\mathbf{d}^*, n) - d^{1*}\right)}{10000}$, the distribution over, for example, set $\mathcal{S}_+$, we approximate with $\frac{\sum_n R(\mathcal{S}_-, n)}{\sum_n (R_1(\mathbf{d}^*, n))}$.

Let $H3(\mathbf{d}^*, n) = 1$ if the sample path $(V_{t,n})$ hits a hyperplane corresponding to a 3-player coalition and $H3(\mathbf{d}^*, n) = 0$ otherwise. We calculate $\frac{\sum_n H3(\mathbf{d}^*,n)}{10000}$, the fraction of sample paths that hit another hyperplane than $\mathcal{H}(C_{12})$ and $\mathcal{H}(C_{13})$ before returning to the asymmetric edge.

Similarly we calculate the maximum over all simulations over $\max 12(\mathbf{d}^*, n)$,

$\max 13(\mathbf{d}^*, n)$, $\max d(\mathbf{d}^*, n)$ and the minimum over all simulations of $\min 13(\mathbf{d}^*, n)$. This is to give us some indication of how 'far away' the chain $(V_{t,n})$ moves away from starting states on the asymmetric edge before returning to the edge.

The below table depicts the simulated equilibrium distribution of the chain $\left(DV_t^{\mathbf{d}^*}\right)$ over the sets $\mathcal{S}_-$, $\mathcal{S}_0$, $\mathcal{S}_{+,+}$ and $\mathcal{S}_{0,0}$. Comparing table 5.16 with the simulated results to table 5.9 with the results for the chain $\left(DV_t^{\mathbf{d}^*,\mathbf{2}}\right)$, we can see that $\left(DV_t^{\mathbf{d}^*,\mathbf{2}}\right)$ already reasonably well approximates the actual equilibrium distribution of the chain $\left(DV_t^{\mathbf{d}^*}\right)$ on these sets. We will see in table 5.17, that the chain $\left(DV_t^{\mathbf{d}^*}\right)$ does not go far away from the asymmetric edge, and that this explains, why, somewhat surprisingly, the restricted chain is already a reasonably good approximation, despite the fact that the state space is only of size 15.

	$\pi^{0.6}$	$\pi^{0.54}$	$\Delta_{\mathbf{2}}^{0.6}$	$\Delta_{\mathbf{2}}^{0.54}$
$\mathcal{S}_-$	0.11	0.36	0.25	0.25
$\mathcal{S}_{0,0}$	0.21	0.27	0.1	0.12
$\mathcal{S}_0$	0.51	0.3	−0.05	−0.02
$\mathcal{S}_{+,+}$	0.17	0.07	−0.35	−0.28

Table 5.16 Simulated equilibrium distribution in the sets $\mathcal{S}_-$, $\mathcal{S}_{+,+}$, $\mathcal{S}_0$ and $\mathcal{S}_{0,0}$ for the chain $\left(DV_t^{\mathbf{d}^*}\right)$ for different starting states

Table 5.17 displays simulation results for $\mathbb{E}(R_1)$, $\max 12(\mathbf{d}^*) = \max_n \max 12(\mathbf{d}^*, n)$, $\max 13(\mathbf{d}^*) = \max_n \max 13(\mathbf{d}^*, n)$, $\min 13(\mathbf{d}^*) = \min_n \min 13(\mathbf{d}^*, n)$ and $\max d(\mathbf{d}^*) = \max_n \max d(\mathbf{d}^*, n)$ for $n = 10000$ for the chain $\left(DV^{\mathbf{d}^*}\right)_t$ for different starting states on the asymmetric edge.

	$d^{*1} = 0.54$	$d^{*1} = 0.56$	$d^{*1} = 0.58$	$d^{*1} = 0.6$
$\mathbb{E}(R_1)$	3.7	3.7	4	4.6
$\max d$	32	17	12	17
max 12	9	10	10	10
max 13	8	8	10	13
min 13	– 12	−9	−8	−8

Table 5.17 Simulated results for the expected return time to $\mathcal{AE}^1$, $\max 12(\mathbf{d}^*)$, $\max 13(\mathbf{d}^*)$, $\min 13(\mathbf{d}^*)$ and $\max d$ for $n = 10000$ and different starting states on the asymmetric edge

The below table depicts the asymmetric edge of game 1 with the simulated drift $\Delta_{\mathbf{d}^*}$ for several starting states along the asymmetric edge. At the 'lower part' of the asymmetric edge, the drift is positive, the drift decreases the more, the more the starting state is located in the 'upper region' of the asymmetric edge. Around $(0.584, \ldots)$ the drift vanishes and then turns positive until, in the 'upper part' of the asymmetric edge, the drift is strongly negative.

In Figure 5.14 the upper part of the asymmetric edge for example game 3 is depicted. Drifts of $\left(DV_t^{\mathbf{d}^*}\right)$ for different starting states in close proximity of the Markovian cooperative equilibrium are depicted. Surprisingly, the only state with negative drift is **max1**. Already from state $(0.539999, 0.210001, 0.200001, 0.499999)$ the drift is positive. For states with $d^1 > 0.53999$, the drift decreases rapidly, the closer the starting states are to **mce**. On the asymmetric edge of example game 1, $\Delta^{\mathbf{d}^*}$ for these same states is approximately 0.33 for all of them. We highlight, that the lower drift for the states closer and closer to the Markovian cooperative equilibrium coincides with a higher and higher fraction of the sample paths (V_t, n) hitting the hyperplane $\mathcal{H}(C_{123})$ before returning to the asymmetric edge. In such close proximity to the Markovian cooperative equilibrium, the equilibrium distribution of the 2-dimensional chain $\left(DV_t^{\mathbf{d}^*}\right)$ does not accurately represent the equilibrium distribution of the 3-dimensional chain $\left(V_t^{\mathbf{d}^*}\right)$ as the sample paths that hit $\mathcal{H}(C_{123})$ include transitions from states, which are not member of any set $\mathcal{S}_{m,n}$. In the same way, the equilibrium distribution of the 2-dimensional chain $\left(DV_t^{\mathbf{d}}\right)$ does not accurately represent the equilibrium distribution of (V_t) for those sample paths. However, only for states very close to the Markovian cooperative equilibrium this seems to impact the

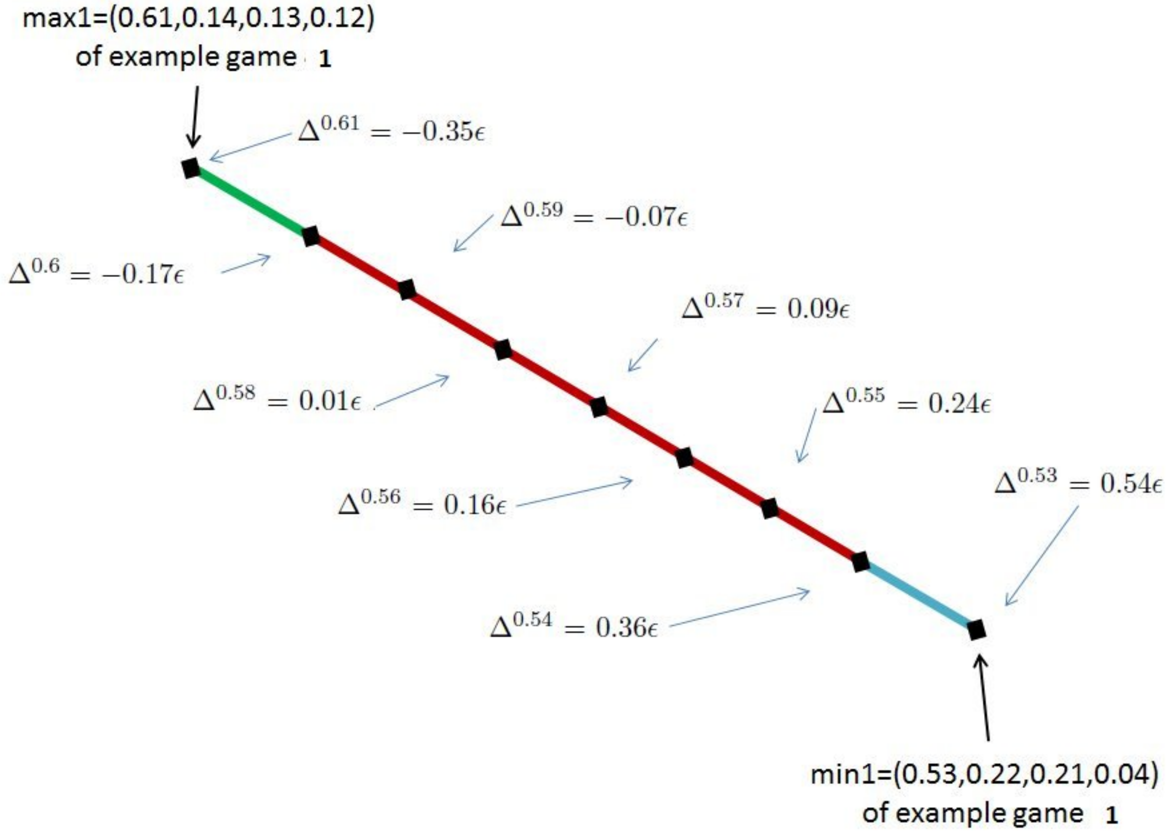


Figure 5.13 Sketch of the asymmetric edge of example game 1 with simulated values of $\Delta^{\mathbf{d}^*}$ for different starting states $\mathbf{d}^*$ along the asymmetric edge.

drift. For states, which are more than 10ε away from the Markovian cooperative equilibrium, the drift is almost identical for a state $\mathbf{d}^*$ on the asymmetric edge of example game 3 and on the asymmetric edge of example game 1.

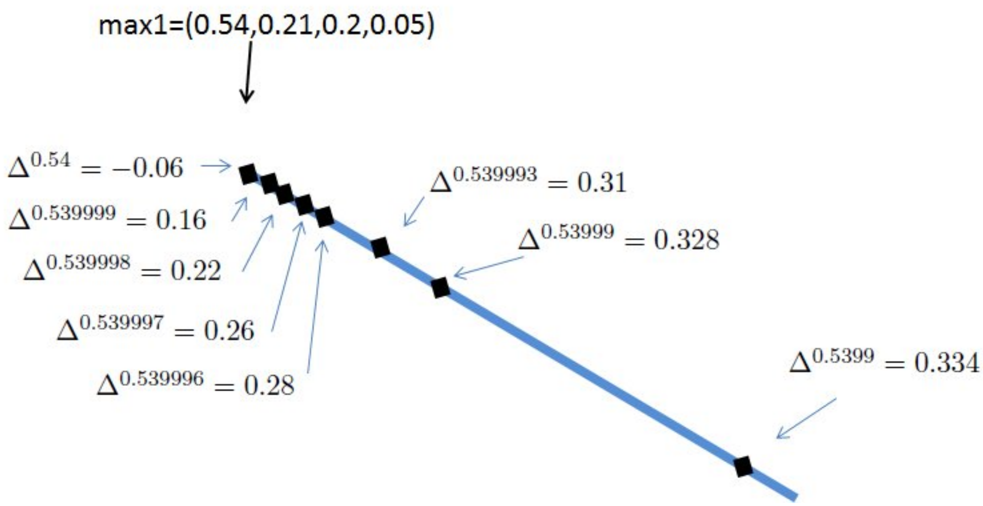


Figure 5.14 Sketch of the asymmetric edge of example game 3 with simulated values of $\frac{\Delta^{\mathbf{d}^*}}{\varepsilon}$ for different starting states $\mathbf{d}^*$ in the immediate neighborhood of $\mathbf{max1} = \mathbf{mce}$.

Chapter 6

Speed of Convergence Analysis via Coupling

6.1 Introduction to the Coupling Method

The main methodological contribution of the thesis is to show how to use coupling to demonstrate speed of convergence for stochastic learning processes. In several branches of probability theory the coupling technique has been one of the methods of choice to analyze the speed of convergence of Markov chains. Levin and Peres (2009) is a good introduction to different applications of couplings with main focus on analyzing the speed of convergence to equilibrium of Markov chains.

However we believe this technique to be novel to the study of stochastic learning processes in evolutionary game theory. That is why we present our analysis in general form. The topic of speed of convergence analysis has recently become a topic of general interest.

Coupling is a powerful and elegant tool with which one is often able to calculate tight bounds on the mixing time and to reduce the complexity or length of calculations compared to other techniques. There are general principles in coupling which we summarize hereafter. However finding a good coupling is more like an art and a good coupling is often specific to the inherent dynamics of the process.

The main purpose of Coupling is to find bounds on the mixing time of stochastic processes. There are a variety of particular cases for which the Coupling method can be used.

- general purpose of the Coupling method:

  Finding bounds on the speed of convergence of stochastic processes

- particular applications of the Coupling method:

  - proving existence of stationary measure,
  - bounding return times or return probabilities,
  - proving limit theorems,
  - deriving inequalities,
  - obtaining approximations...

How exactly does a coupling work? In order to show the speed of convergence of Markov chains one first needs to have a notion of distance between two probability distributions.

The following definitions closely follow Levin and Peres (2009). The *total variation distance* between two distributions μ and ν on Ω^E is given by

$$\parallel \mu - \nu \parallel_{TV} = \frac{1}{2} \sum_{\mathbf{d} \in \Omega^E} \mid \mu(\mathbf{d}) - \nu(\mathbf{d}) \mid . \tag{6.1}$$

A *coupling* of two probability distributions μ and ν is a pair of random variables (X,Y) defined on a single probability space such that the marginal distribution of X is μ and the marginal distribution of Y is ν. That is, a coupling (X,Y) satisfies $\mathbb{P}\{X = x\}$ and $\mathbb{P}\{Y = y\}$.

There always exists the independent coupling, where both random variables are independent of each other. However the aim of a coupling is to find a joint distribution that "forces" the two random vairables to move towards each other as often as possible whilst preserving the marginal distribution of each.

To analyze the speed of convergence of Markov chains to the equilibrium distribution one usually compares either two distributions that are "opposite" or far away from each other or any possible distribution with the equilibrium distribution. We then calculate after how many transitions the chain started "far away" from equilibrium is close to the equilibrium distribution. The following important result bounds the distance between two distributions (e.g. a particular starting distribution and the equilibrium measure) with the probabiliy that the coupling of the markov chains started in these two states (or distributions) has not coalesced. Theorem (6.1) is the main theorem linking convergence of Markov chains to equilibrium and coupling. For a proof we refer the reader to Levin and Peres (2009).

Theorem 6.1. *For any coupling of V_t and V_t' it holds that* $\| P^t(\mathbf{d},\cdot) - P^t(\mathbf{d}',\cdot) \|_{TV} \leq \mathbb{P}\{V_t \neq V_t'\}$.

So if one can couple two versions of a Markov chain started at "opposite locations" of the state space and show that after t time steps the probability that the two have not met is very low, we know that the total variation distance after t time steps is very small. In order to show that Markov chains "move closer to each other" or "meet" on a state space, a notion of distance between the two Markov chains on that state space is essential.

When is a Markov chain "close" to equilibrium? For $\mathbf{d} \in \Omega^E$ and $t \in \mathbb{N}$, let $P^t(\mathbf{d},\cdot)$ be the distribution of (V_t) conditioned on $V_0 = \mathbf{d}$. We want to bound the maximum distance of the Markov chain started in any state of the state space and the equilibrium distribution after t steps. So let $m(t) = \max_{\mathbf{d}\in\Omega^E} \| P^t(\mathbf{d},\cdot) - \pi \|$ where π is the equilibrium distribution of the chain (V_t).

Then let the *mixing time* be $t_{mix}\left(\frac{1}{4}\right) = \min\{t \mid m(t) \leq \frac{1}{4}\}$. The mixing time is generally applied to measure how quick a Markov chain is close to equilibrium. After the mixing time has elapsed, the Markov chain is close to equilibrium for any possible starting state. The choice of $\frac{1}{4}$ in the definition of mixing time is somewhat arbitrary but it is the generally applied threshold.

What makes a coupling useful to bound the mixing time? First one needs a measure of distance between the two chains in the coupling. In Section 6.2 the L^1 distance between different states is used to define a distance between two versions of the cooperative game process. The best (trick) one can hope for is to define (find) a coupling such that the distance between the two chains in the coupling will never increase and decrease by some positive probability at each time step. In Theorem (6.2) it is shown that under the coupling described Section 6.2 the distance between the two versions of the cooperative game process for the N-player bargaining game never increases and decreases over each time step with strictly positive probability proportional to the L^1 distance between the two versions of the chain.

Most of the time it is not possible to find a coupling such that the distance between the two versions never increases. The next best alternative to look for is to find a coupling for which the distance between the two chains decreases in expectation over each time step. In Theorem (6.3) it is shown that under the coupling described in Section 6.2 the distance between the two versions of the cooperative game process for superadditive balanced three-player games decreases in expectation by a factor proportional to the L^1 distance of the two versions over each time step.

6.2 The Cooperative Game Process has Rapid Mixing

In this section we define a coupling on the cooperative game process and use it to show that the cooperative game process converges fast to equilibrium. In particular we show for the N-player bargaining set-up that the chain (V_t) started from any efficient state is close to equal split fast. For the cooperative game process for three-player games as introduced in Chapter 4 we show that the chain (V_t) started from any state in the core is close to the vector **co** fast.

Let τ be the first time t that $|V_t - V_t'|_{\mathbf{1}} = 0$.

For $\mathbf{d}, \mathbf{d}' \in \Omega^E$, let $k^{i,j}_{\mathbf{d},\mathbf{d}'} = \min\left(p^{i,j}_{\mathbf{d}}, p^{i,j}_{\mathbf{d}'}\right)$.

For $\mathbf{d}, \mathbf{d}' \in \Omega^E$, let $l^{i,j}_{\mathbf{d},\mathbf{d}'} = \max\{0, p^{i,j}_{\mathbf{d}} - p^{i,j}_{\mathbf{d}'}\}$.

We will couple the chains (V_t) and (V_t') in the following way:

1. A player is chosen uniformly at random and that player, say i, increases his demand by ε in both chains (V_t) and (V_t').
2. For $j \in \{1, 2, ..., N\}$, player j decreases demands jointly in both chains with probability $k^{i,j}_{\mathbf{d},\mathbf{d}'}$. Then players decrease their demands in both chains according to their remaining marginal probabilities $l^{i,j}_{\mathbf{d},\mathbf{d}'}$ and $l^{i,j}_{\mathbf{d}',\mathbf{d}}$.

We couple two copies of V_t, say V_t and V_t', defined on the same probability space from any different starting states $\mathbf{d}$ and $\mathbf{d}^{'}$.

The below table describes the effect of the coupling $\left(V_t, V_t^{'}\right)$ in the N-player bargaining setting on $|\mathbf{d}-\mathbf{d}'|_{\mathbf{1}}$ for 3-players started from $\mathbf{d} = (0.7, 0.2, 0.1)$ and $\mathbf{d}^{'} = (0.2, 0.5.0.3)$ over one time step t.

Suppose player one is to update demands (simultaneously in both chains). The coupling is now in the joint state $\{(0.7+\varepsilon, 0.2, 0.1), (0.2+\varepsilon, 0.5.0.3)\}$. Whenever player one decreases demands in chain two (with probability $k^1\left(\mathbf{d}(1) = \frac{0.2+\varepsilon}{1+\varepsilon}\right)$ player one in chain one will decrease as well his demand. In the same way, whenever players two and three in chain one decrease demands (with probabilities $\frac{0.2}{1+\varepsilon}$ and $\frac{0.1}{1+\varepsilon}$ so will players two and three in chain two decrease demands. For all joint demand decreases for players one, two and three which happen with probabilities $k^1, k^2 and k^3$ the L^1 distance for the coupling remains the same. However player one in chain one will decrease demands with an additional probability of $l_1^1 = \frac{0.5}{1+\varepsilon}$. The subscript identifies the chain for which l^1 strictly larger than 0. Players two and three in chain two will decrease demands with an additional probability of $l_2^2 = \frac{0.3}{1+\varepsilon}$ and $l_2^3 = \frac{0.2}{1+\varepsilon}$ respectively. Observe that under the coupling if a player i reduces demands in chain one while in the chain two another player j reduces demands, player i must have larger demands in chain one than player i in chain two and player j in chain two much have larger demands than player j in chain one. Hence these joint decreases where not the same player reduces demands in both chains reduce the L^1-distance by 2ε.

If player two or three increase demands the effect on the L^1 distance is exactly the same as when player one increases his demand.

		d^1	d'^1	d^2	d'^2	d^3	d'^3
	value	0.7	0.2	0.2	0.5	0.1	0.3
		decrease by	player 1	decrease by	player 2	decrease by	player 3
increase	k^j	$\frac{0.2+\varepsilon}{1+\varepsilon}$	$\frac{0.2+\varepsilon}{1+\varepsilon}$	$\frac{0.2}{1+\varepsilon}$	$\frac{0.2}{1+\varepsilon}$	$\frac{0.1}{1+\varepsilon}$	$\frac{0.1}{1+\varepsilon}$
by	l^j	$\frac{0.5}{1+\varepsilon}$	0.0	0.0	$\frac{0.3}{1+\varepsilon}$	0.0	$\frac{0.2}{1+\varepsilon}$
player 1	$-l^j\varepsilon$	$-\frac{0.5}{1+\varepsilon}(\varepsilon)$	0	0	$-\frac{0.3}{1+\varepsilon}(\varepsilon)$	0	$-\frac{0.2}{1+\varepsilon}(\varepsilon)$
increase	k^j	$\frac{0.2}{1+\varepsilon}$	$\frac{0.2}{1+\varepsilon}$	$\frac{0.2+\varepsilon}{1+\varepsilon}$	$\frac{0.2+\varepsilon}{1+\varepsilon}$	$\frac{0.1}{1+\varepsilon}$	$\frac{0.1}{1+\varepsilon}$
by	l^j	$\frac{0.5}{1+\varepsilon}$	0.0	0.0	$\frac{0.3}{1+\varepsilon}$	0.0	$\frac{0.2}{1+\varepsilon}$
player 2	$-l^j\varepsilon$	$-\frac{0.5}{1+\varepsilon}(\varepsilon)$	0	0	$-\frac{0.3}{1+\varepsilon}(\varepsilon)$	0	$-\frac{0.2}{1+\varepsilon}(\varepsilon)$
increase	k^j	$\frac{0.2}{1+\varepsilon}$	$\frac{0.2}{1+\varepsilon}$	$\frac{0.2}{1+\varepsilon}$	$\frac{0.2}{1+\varepsilon}$	$\frac{0.1+\varepsilon}{1+\varepsilon}$	$\frac{0.1+\varepsilon}{1+\varepsilon}$
by	l^j	$\frac{0.5}{1+\varepsilon}$	0.0	0.0	$\frac{0.3}{1+\varepsilon}$	0.0	$\frac{0.2}{1+\varepsilon}$
player 3	$-l^j\varepsilon$	$-\frac{0.5}{1+\varepsilon}(\varepsilon)$	0	0	$-\frac{0.3}{1+\varepsilon}(\varepsilon)$	0	$-\frac{0.2}{1+\varepsilon}(\varepsilon)$

Theorem 6.2. *Suppose (v,N) is a N-player game satisfying the conditions on v from the N-player bargaining setting and ε is equal to $\frac{1}{M}$ for some v-compatible $M \in \mathbb{N}$. Let (V_t) and (V_t') be two versions of the (v,N,ε) cooperative game process. Then under the coupling of $\left(V_t, V_t^{'}\right)$ it holds that $\mathbb{E}\left(|\mathbf{d}_{t+1}-\mathbf{d}'_{t+1}|_\mathbf{1} - |\mathbf{d}_t-\mathbf{d}'_t|_\mathbf{1}\right) = -\frac{|\mathbf{d}_t-\mathbf{d}'_t|_\mathbf{1}}{1+\varepsilon}\varepsilon$ for all states $\mathbf{d},\mathbf{d}' \in \Omega^E$.*

Theorem 6.3. *Suppose $(v,3)$ is a superadditive 3-player game satisfying (4.4), and ε is equal to $\frac{1}{M}$ for some v-compatible $M \in \mathbb{N}$. Let (V_t) and (V_t') be two versions of the $(v,3,\varepsilon)$ cooperative game process. Then under the coupling of $\left(V_t, V_t^{'}\right)$ it holds that $\mathbb{E}\left(|\mathbf{d}_{t+1}-\mathbf{d}'_{t+1}|_\mathbf{1} - |\mathbf{d}_t-\mathbf{d}'_t|_\mathbf{1}\right) = -\frac{1}{6}\frac{|\mathbf{d}_t-\mathbf{d}'_t|_\mathbf{1}}{1+\varepsilon}\varepsilon$ for all states $\mathbf{d},\mathbf{d}' \in \Omega^C$.*

We use Theorems 6.2 and 6.3 respectively to deduce a bound on the mixing time for the cooperative game process both in the three player superadditive setting and in the N-player bargaining setting.

Lemma 6.4. *The mixing time of the chain V_t in the set-up of the N-player bargaining model and in the set-up of the three player superadditive game setting is of order $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.*

Proof: We know from Theorem (6.2) and (6.3) that

$$\mathbb{E}\left(|\mathbf{d}_{t+1}-\mathbf{d}'_{t+1}|_\mathbf{1} \mid \mathbf{d}_t,\mathbf{d}'_t\right) \leq \left(1-\frac{\varepsilon}{1+\varepsilon}\right)|\mathbf{d}_{t+1}-\mathbf{d}'_{t+1}|_\mathbf{1}. \tag{6.2}$$

Then

$$\mathbb{E}\left(|\mathbf{d}_t-\mathbf{d}'_t|_\mathbf{1} \mid \mathbf{d}_0,\mathbf{d}'_0\right) \leq \left(\frac{1}{1+\varepsilon}\right)^t |\mathbf{d}_0-\mathbf{d}'_0|_\mathbf{1}. \tag{6.3}$$

Observe that $\max_{\mathbf{d}\in\Omega^E}|\mathbf{d}-\mathbf{d}'|_\mathbf{1} = 2$ and is attained in all combination of states $(\mathbf{d},\mathbf{d}')$ where the demands of the players are such that if $d^i > 0$ then $d'^i = 0$ vice versa.

So $\mathbb{E}\left(|\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1} \mid \mathbf{d}_0, \mathbf{d}'_0\right) \leq \frac{2}{(1+\varepsilon)^t}$.

Then we use the Markov inequality to deduce that

$$\mathbb{P}\left(|\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1} \geq \varepsilon\right) \leq \frac{1}{\varepsilon}\mathbb{E}|\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1} \leq \frac{1}{\varepsilon}\frac{2}{(1+\varepsilon)^t} \tag{6.4}$$

Furthermore we know from Levin and Peres (2009) that $\parallel \mu(\mathbf{d}) - \nu(\mathbf{d}) \parallel_{TV} \leq \mathbb{P}(\mathbf{d}_t \neq \mathbf{d}'_t) = \mathbb{P}(|\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1} \geq \varepsilon) \leq \frac{2}{\varepsilon(1+\varepsilon)^t}$ and so the mixing time is of order $O\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$.

□

Proof: of Theorem 6.2

For both chains V_t and $V_t^{'}$ the same player i is chosen to increase demands by ε.

Then from $\mathbf{d}(i)$ and $\mathbf{d}^{'}(i)$ we decrease demands of player j jointly for both chains with probability $k^{i,j}$. This has no effect on $|\mathbf{d} - \mathbf{d}'|_\mathbf{1}$ as for both chains player i's demand is increased and player j's demand is decreased (with possibly $j = i$).

Then with the remaining marginal probabilities to decrease from $\mathbf{d}(i)$, whenever either d^j or $d^{'j}$ decrease with probability $l^{i,j}$ or $l^{'i,'j}$, the distance $|\mathbf{d} - \mathbf{d}'|_\mathbf{1}$ decreases by $l^{i,j}\varepsilon$ or $l^{'i,'j}\varepsilon$.

Observe that $\sum_{j\in[1,..,N]}\left(l^{i,j} + l^{'i,'j}\right) = \frac{|\mathbf{d}-\mathbf{d}'|_\mathbf{1}}{1+\varepsilon}$ and so $\mathbb{P}\left(|\mathbf{d} - \mathbf{d}'|_{\mathbf{1}_{t+1}} = |\mathbf{d} - \mathbf{d}'|_{\mathbf{1}_t} - \varepsilon\right) = \frac{|\mathbf{d}-\mathbf{d}'|_\mathbf{1}}{1+\varepsilon}$.

□

Proof: of Theorem 6.3

Observe that if the same player reduces in (V_t) and (V'_t), then $\left(|\mathbf{d}_{t+1} - \mathbf{d}'_{t+1}|_\mathbf{1} - |\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1}\right) = 0$. If a player j decreases demands under (V_t) and player k decreases demands under (V'_t) with $d^j > d^{'j}$ and $d^k < d^{'k}$, then $\left(|\mathbf{d}_{t+1} - \mathbf{d}'_{t+1}|_\mathbf{1} - |\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1}\right) = -2\varepsilon$. If If a player j decreases demands under (V_t) and player k decreases demands under (V'_t) with $d^j < d^{'j}$ and $d^k > d^{'k}$, then $\left(|\mathbf{d}_{t+1} - \mathbf{d}'_{t+1}|_\mathbf{1} - |\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1}\right) = 2\varepsilon$. Otherwise, if a player j decreases demands under (V_t) and player k decreases demands under (V'_t) with $d^j > d^{'j}$ and $d^k > d^{'k}$ or with $d^j < d^{'j}$ and $d^k < d^{'k}$, then $\left(|\mathbf{d}_{t+1} - \mathbf{d}'_{t+1}|_\mathbf{1} - |\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1}\right) = 0$.

We will show that under the coupling a change of 2ε never happens and changes of -2ε happen frequently.

A feature in the N-player bargaining set-up was that, if $d^i > d^{'i}$, then $p_\mathbf{d}^{j,i} > p_{\mathbf{d}'}^{j,i}$ for all $j \in N$ and hence the distance $|\mathbf{d}_t - \mathbf{d}'_t|_\mathbf{1}$ will always change by -2ε if not the same players decrease in V and V' and stay constant if the same players in V and V' decrease.

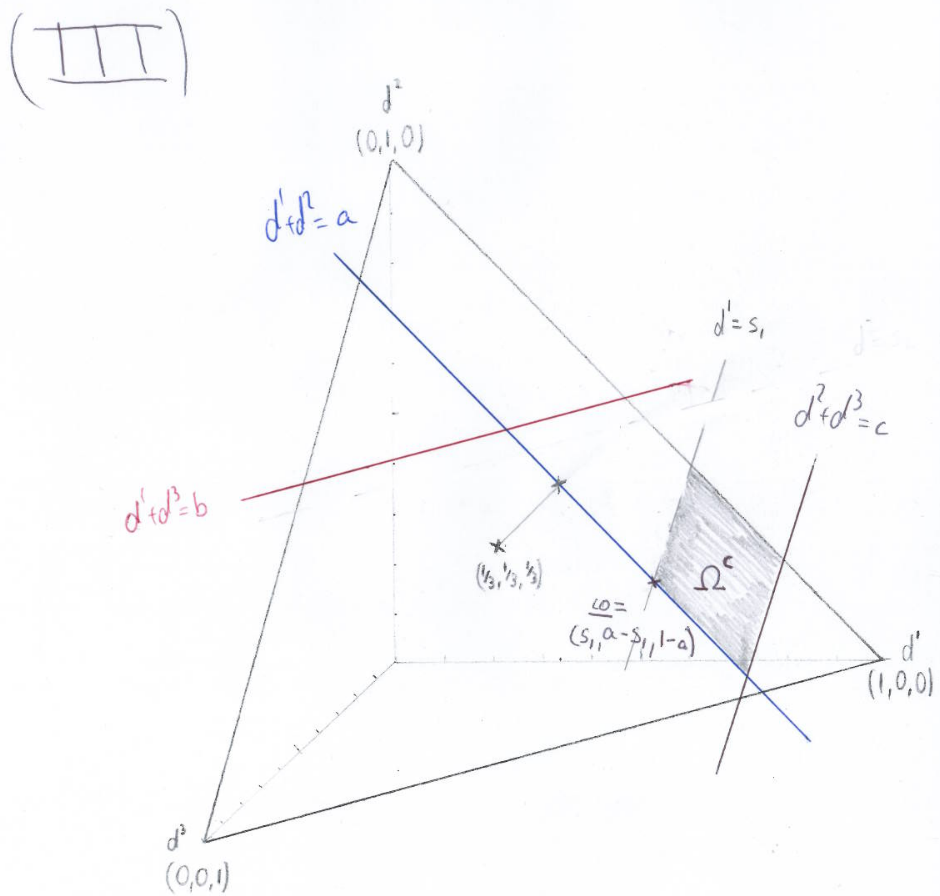


Figure 6.1 Graphical representation of core and outcome `III` in table (4.5).

Now in the 3-player bargaining set-up the only combined demand decreases for V and V' that are in effect different to the N-player bargaining case can occur if one of the two players, say i, has $d^i < d'^i$ but $p^{j,i}_{\mathbf{d}} > p^{j',i'}_{\mathbf{d}'}$ for some j in $[1,2,3]$. In that case the player with smaller demands in V, V' will decrease with higher probability and the contribution to the change in $|\mathbf{d}_t - \mathbf{d}'_t|_{\mathbf{1}}$ will be positive. Our strategy is to show that in case that it happens for a player i that $d^i < d'^i$ but $p^{j,i}_{\mathbf{d}} > p^{j,i}_{\mathbf{d}'}$ then this player i will decrease jointly with a player $k \neq i$ in $[1,2,3]$ where $d^k < d'^k$ and $p^{j,k}_{\mathbf{d}} < p^{j,k}_{\mathbf{d}'}$ and so the combined decrease of d^i, d'^j contributes at most 0 to the change in $|\mathbf{d}_t - \mathbf{d}'_t|_{\mathbf{1}}$.

The table below gives an overview of all possible combinations of states at $\mathbf{d}(i)$.

case	I	II	III	IV	V	VI	VII
$C'(\mathbf{d}(i))$	$\emptyset$	C_j	C_{jk}	C_j	C_j	C_{jk}	C_{jk}
$C'(\mathbf{d}'(i))$	$\emptyset$	$\emptyset$	$\emptyset$	C_k	C_j	C_j	C_{jk}

Table 6.1 All possible combinations for $C'(\mathbf{d}(i)), C'(\mathbf{d}'(i))$

Suppose at $(\mathbf{d}(i), \mathbf{d}'(i))$ it holds that $C'(\mathbf{d}(i)) = \emptyset$ and $C'(\mathbf{d}'(i)) = \emptyset$. Observe that this situation is identical to $\mathbf{d}(i), \mathbf{d}'(i)$ with the set-up of the 3-player bargaining setting (6.2). We deduce that the expected change in $|\mathbf{d}-\mathbf{d}'|_1$ conditioned on $\mathbf{d}(i)$ is at least $-\frac{|\mathbf{d}-\mathbf{d}'|_1}{1+\varepsilon}\varepsilon$.

Suppose at $(\mathbf{d}(i), \mathbf{d}'(i))$ it holds that $C'(\mathbf{d}(i)) = C_j$ and $C'(\mathbf{d}'(i)) = \emptyset$. Let i,k be the other two players.

By assumption the states $\mathbf{d}$ and $\mathbf{d}'$ are in the core and so it holds that $d^j < d'^j$. So $d^i + d^k > d'^i + d'^k$ and so for at most one of i,k, say i, it can hold that $d^i < d'^i$.

If both $d^i > d'^i$ and $d^k > d'^k$ the case is similar to the previous case where in both chains (V_t) and (V'_t) the payable coalition was the empty set. However joint demand decreases between different players happen more frequently since players i,j reduce demands with $\frac{d^i}{d^i+d^k+\varepsilon}$ and $\frac{d^k}{d^i+d^k+\varepsilon}$ since player j forms the payable coalition. So $p^{i,i}_{\mathbf{d}} - p^{i,i}_{\mathbf{d}'} = \frac{d^i}{d^i+d^k+\varepsilon} - \frac{d'^i}{1+\varepsilon} > \frac{d^i}{1+\varepsilon} - d'^i$, $p^{i,k} - p'^{i,'k} = \frac{d^k}{d^i+d^k+\varepsilon} - \frac{d'^k}{1+\varepsilon} > \frac{d^k}{1+\varepsilon} - \frac{d'^k}{1+\varepsilon}$ and $p'^{i,'j} - p^{i,j} = \frac{d'^j}{1+\varepsilon} > \frac{d'^j}{1+\varepsilon} - \frac{d^j}{1+\varepsilon}$. So a change in $|\mathbf{d}-\mathbf{d}'|_1$ of -2ε happens with higher probability than in the previous case, where at $\mathbf{d}$ no coalition was binding and so we deduce that the expected change in $|\mathbf{d}-\mathbf{d}'|_1$ conditioned on $\mathbf{d}(i)$ is at least $-\frac{|\mathbf{d}-\mathbf{d}'|_1}{1+\varepsilon}\varepsilon$.

Suppose that $d^i < d'^i$ and suppose that $\frac{d^i+\varepsilon}{d^i+d^k+\varepsilon} > \frac{d'^i+\varepsilon}{1+\varepsilon}$. The table below shows the possible joint demand decreases from $\mathbf{d}(i)$ for V, V'.

	d'^i	d'^j	d'^k
d^i	0	$+-$	x
d^j	x	x	x
d^k	x	$--$	0

Table 6.2 Case $C'(\mathbf{d}(i)) = C_j, C'(\mathbf{d}'(i)) = \emptyset, d^i < d'^i$ and $\frac{d^i+\varepsilon}{d^i+d^k+\varepsilon} > \frac{d'^i+\varepsilon}{1+\varepsilon}$

The third row represents joint demand decreases of d^j and d'^i, d'^j, d'^k respectively. Since $C'(\mathbf{d}(i)) = C_j$, player j will never decrease demands from $\mathbf{d}(i)$ and hence none of these is possible. This is marked with an x in the respective entries in the below table. Joint demand decreases by the same player in V, V' don't change $|\mathbf{d}-\mathbf{d}'|_1$ hence the respective entries along the diagonal are 0.

Since $p^{i,k}_{\mathbf{d}} > p^{i,k}_{\mathbf{d}'}$, d'^k decreases only jointly with player d^k. Similarly since $p^{i,i}_{\mathbf{d}} > p^{i,i}_{\mathbf{d}'}$ the probability of a joint decrease of d'^i and d^k is 0 which is again marked with an x. We see that there are only two possible joint demand decreases of different players. If d^i and d'^j decrease jointly, the decrease in d^i increases and the decrease in d'^j decreases $|\mathbf{d}-\mathbf{d}'|_1$ by ε. This is indicated in the table with a $+-$. The second possible joint decrease is of d^k and d'^j. Both decreases of demands decrease $|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ and hence the respective entry in the table is marked with a $--$. We will show that the probability of d^k and d'^j decreasing jointly is at least $\frac{1}{2}\frac{|\mathbf{d}_t-\mathbf{d}'_t|_\mathbf{1}}{1+\varepsilon}$.

The overall change of d^k and d'^j decreasing simultaneously is -2ε and this happens with probability $\frac{d^k}{d^i+d^k+\varepsilon}-\frac{d'^k}{1+\varepsilon}>d^k-d'^k$. Since by assumption $d^j<d'^j$ and $d^i<d'^i$ it holds that $d^k-d'^k=d'^j-d^j+d'^i-d^i=\frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$.

Suppose at $(\mathbf{d}(i),\mathbf{d}'(i))$ it holds that $C'(\mathbf{d}(i))=C_{jk}$ and $C'(\mathbf{d}'(i))=\emptyset$.

Then $d^i>d'^i$ and $d^j+d^k<d'^j+d'^k$. Observe that at least one of $d'^j>d^j$ or $d'^k>d^k$ has to hold. Suppose that both $d^i<d'^i$ and $d^j<d'^j$. Then the possible joint demand changes are summarized in the below table.

	d'^i	d'^j	d'^k
d^i	0	$--$	$--$
d^j	x	x	x
d^k	x	x	x

Table 6.3 Case $C'(\mathbf{d}(i))=C_{jk}$,$C'(\mathbf{d}('i))=\emptyset$, $d^j<d'^j$ and $d^k<d'^k$

Suppose now that $d^j>d'^j$ and $d^k<d'^k$. Then the possible joint demand changes are summarized in the below table.

	d'^i	d'^j	d'^k
d^i	0	$-+$	$--$
d^j	x	x	x
d^k	x	x	x

Table 6.4 Case $C'(\mathbf{d}(i))=C_{jk}$,$C'(\mathbf{d}('i))=\emptyset$, $d^j>d'^j$ and $d^k<d'^k$

Then $d'^k-d^k=d^j-d'^j+d^i-d'^i$ and so $d'^k-d^k=\frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ and so the probability that $|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ decreases by 2ε is at least $\frac{1}{2(1+\varepsilon)}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ conditional on $C'(\mathbf{d}(i))=C_{jk}$ and $C'(\mathbf{d}'(i))=\emptyset$.

Suppose at $(\mathbf{d}(i),\mathbf{d}'(i))$ it holds that $C'(\mathbf{d}(i))=C_j$ and $C'(\mathbf{d}'(i))=C_k$ with $v(C_j)>v(C_k)$.

Then $d^j<d'^j$ and $d^k>d'^k$ and furthermore $d^i+d^k<d'^i+d'^k$.

If $d^i>d'^i$ it holds that $p^{i,i}=\frac{d^i+\varepsilon}{d^i+d^k}>p'^{i,'i}=\frac{d'^i+\varepsilon}{d'^i+d'^j+\varepsilon}$ and the possible joint demand changes are summarized in the below table.

	d'^i	d'^j	d'^k
d^i	0	$--$	x
d^j	x	x	x
d^k	x	$--$	x

Table 6.5 Case $C'(\mathbf{d}(i)) = C_j, C'(\mathbf{d}('i)) = C_k$, $d^i > d'^i$

If $d^i < d'^i$ but $\frac{d^i+\varepsilon}{d^i+d^k+\varepsilon} > \frac{d'^i+\varepsilon}{d'^i+d'^j+\varepsilon}$ then the joint demand changes are summarized in the below table.

	d'^i	d'^j	d'^k
d^i	0	$+-$	x
d^j	x	x	x
d^k	x	$--$	x

Table 6.6 Case $C'(\mathbf{d}(i)) = C_j, C'(\mathbf{d}('i)) = C_k$, $d^i < d'^i$ and $\frac{d^i+\varepsilon}{d^i+d^k+\varepsilon} > \frac{d'^i+\varepsilon}{d'^i+d'^j+\varepsilon}$

In this case the expected change in $|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ conditioned on $\mathbf{d}(i)$ is given by $\frac{d^i+\varepsilon}{d^i+d^k+\varepsilon} - \frac{d'^i+\varepsilon}{d'^i+d'^j+\varepsilon} - \frac{d^k}{d^i+d^k+\varepsilon} - \frac{d'^j}{d'^i+d'^j+\varepsilon}$ which simplifies to $\frac{d^i+\varepsilon-d^k}{d^i+d^k+\varepsilon} - 1 < -\frac{d^k}{d^i+d^k+\varepsilon} < -d^k$. Now since $d^i < d'^i$, $d^j < d'^j$ and $d^k > d'^k$ it holds that $d^k - d'^k = \frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ and so $-d^k < -\frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$. So the expected change in $|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ is at least $-\frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}\varepsilon$.

So the expected change in $|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ is smaller than in the case just described.

Suppose at $(\mathbf{d}(i), \mathbf{d}'(i))$ it holds that $C'(\mathbf{d}(i)) = C_j$ and $C'(\mathbf{d}'(i)) = C_j$. Then $d^j = d'^j$ and $d^i + d^k = d'^i + d'^k$. This case is basically the same as a scaled version of the 2-player bargaining game where the denominator of $p^{i,i}_{\mathbf{d}}, p^{i,i}_{\mathbf{d}'}, p^{i,k}_{\mathbf{d}}, p^{i,k}_{\mathbf{d}'}$ is given by $d^i + d^k + \varepsilon = d'^i + d'^k + \varepsilon$ instead of by $1+\varepsilon$.

Suppose at $(\mathbf{d}(i), \mathbf{d}'(i))$ it holds that $C'(\mathbf{d}(i)) = C_{jk}$ and $C'(\mathbf{d}'(i)) = C_j$.

Then $d^j + d^k < d'^j + d'^k$ which implies $d^i > d'^i$and $d^j > d'^j$. So we conclude that $d^k < d'^k$ and $d'^k - d^k = \frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$.

The possible joint demand decreases are given in the below table.

	d'^i	d'^j	d'^k
d^i	0	x	$--$
d^j	x	x	x
d^k	x	x	x

Table 6.7 Case $C'(\mathbf{d}(i)) = C_{jk}, C'(\mathbf{d}('i)) = C_j$

If d^i and d'^k reduce jointly with probability $\frac{d'^k}{d'^k+d'^i+\varepsilon} > d'^k - d^k = \frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ the change in $\frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ is -2ε. So the expected change in $|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ is at least $-|\mathbf{d}-\mathbf{d}'|_\mathbf{1}\varepsilon$.

Suppose at $(\mathbf{d}(i), \mathbf{d}'(i))$ it holds that $C'(\mathbf{d}(i)) = C_{jk}$ and $C'(\mathbf{d}'(i)) = C_{jk}$. Then d^i and d'^i will reduce with probability 1 and no change happens to $|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$. Observe that as long as $\mathbf{d} \neq \mathbf{d}'$ this situation at $\mathbf{d}(i)$ can happen for at most one $i \in [1,2,3]$.

We have shown that in all cases apart from when $C'(\mathbf{d}(i)) = C_{jk}$ and $C'(\mathbf{d}'(i)) = C_{jk}$ it holds that the expected change in $|\mathbf{d}-\mathbf{d}'|_\mathbf{1}$ is at least $\frac{1}{2}|\mathbf{d}-\mathbf{d}'|_\mathbf{1}\varepsilon$. So we conclude that

$$\mathbb{E}\left(|\mathbf{d}_{t+1}-\mathbf{d}'_{t+1}|_\mathbf{1} - |\mathbf{d}_t-\mathbf{d}'_t|_\mathbf{1}\right) = -\frac{1}{6}\frac{|\mathbf{d}_t-\mathbf{d}'_t|_\mathbf{1}}{1+\varepsilon}\varepsilon$$

.

□

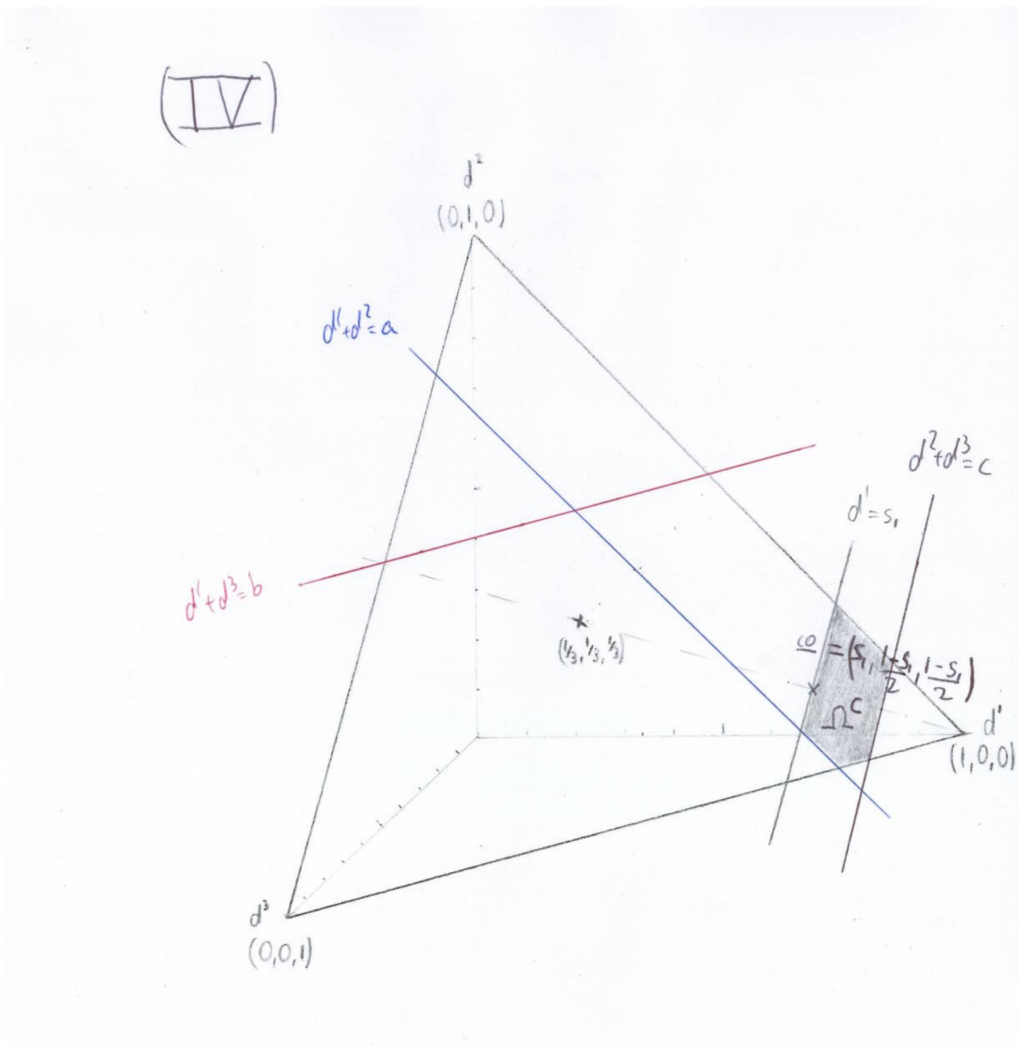


Figure 6.2 Graphical representation of core and outcome `IV` in table (4.5).

Chapter 7

Remarks and Open Problems

There are three “branches” of open problems that we find interesting and relevant. We give a short summary in the below list and then elaborate further on them in the next paragraphs.

1. The first branch is to analyze other incremental demand or aspiration adaption processes on cooperative games. We expect that asymmetric power is present in different dynamic learning processes on cooperative games. The features a dynamic learning process should possess so that the power dynamic is likely to be present along an asymmetric face of the core are: incremental demand updates and some dynamic that results in the process moving in close proximity of the faces of the core a lot of the time.

2. The second branch is to apply the coupling technique to other dynamic learning models, such as aspiration adaption, Bayesian updating, and in particular, completely uncoupled processes.

3. The third branch is to apply and extend the tools developed in this thesis to prove general results for the N-player cooperative game processes. We expect that this is complicated when the core polytope contains asymmetric faces. In all other games we expect that via symmetry arguments , as outlied further below, general proofs are feasible.

The cooperative game process moves as a biased random walk on the polytope formed by the set of efficient states in the core. The inherent equity bias lets the random walk drift towards faces of the polytope formed by the core. It is of great interest to analyze other incremental dynamic learning models where the state space is based on a cooperative game with an asymmetric face. An extreme example game is the following: $v(C_{12345678}) = 1$, $v(C_{1i}) = 0.8$ for i in $\{2,3,4,5,6\}$, with the worths of all other coalitions zero. Considering for example the state $(0.75, 0.05, 0.05, 0.05, 0.05, 0.05, 0, 0)$.

Player 1 is the strong player, players $2-6$ are the weak players and players $7-8$ are the complement players. Different incremental learning rules could be analyzed on this game. This could involve experimental games (on-line experiments) with large populations of agents playing repeated cooperative games, where some games include, and some games do not include, an asymmetric edge. It would be interesting to see for these experimental games if the equity achieved in these different games is statistically significantly lower for games with asymmetric faces. "Extreme" cases of games (such as the one given above) could be used first. Experimental games could be played where each round one coalition is determined and members paid their payoffs or demands. We strongly believe that there will be processes exhibiting asymmetric power for such games.

There seem to be a variety of open problems to apply the coupling method to dynamic learning models. First of all, previous bounds on convergence to equilibrium in the literature could be re-examined, furthermore new models can be analyzed. The applicability of the coupling method is vast, very different dynamic learning models could be analyzed since the coupling technique can be applied to many different situations. Many interesting results using couplings on a variety of Markov processes exist, some of which could well be adapted to the dynamic learning models in the literature.

Conjecture 7.1. *Suppose $(v,4)$ is a superadditive balanced game and the polytope corresponding to its core does not contain a face generated by an asymmetric coalition set and suppose $(v,4,\varepsilon)$ is the respective cooperative game process for some v-compatible $\varepsilon = \frac{1}{M}$.*

Then the Markovian cooperative equilibrium is **co** *the most equal allocation in the core.*

For 4-player games we believe the above conjecture to be true. Whether it might hold for N-player games will be interesting. There is some evidence including Theorem (5.1) that this might be true. Investigations in this direction seem promising.

An outline of the proof is given. First, all possible edges of such a game $(v,4)$ are listed.

coalitions	*fixed players*	*free players*
C_{12}, C_{123}	$4,3$	$1,2$
C_{12}, C_{134}	$2,1$	$3,4$
C_{123}, C_{124}	$3,4$	$1,2$

Table 7.1 Classification of all symmetric edges for 4-player games

Given a game $(v,4)$, the following "procedure" is an alternative way to determine the Markovian cooperative equilibrium, especially if 1-step drift analysis feels tedious.

1. Determine all recurrent edges. A recurrent edge is an edge where $\phi_{C_1} < 1$ and $\phi_{C_2} < 1$ where ϕ_{C_1} and ϕ_{C_2} are defined in equations (7.1) and (7.2).

2. For all recurrent edges of the game, determine for each edge the free players. For example for an edge generated by coalitions C_{123} and C_{124}, with $v(C_{123}) = 0.8$ and $v(C_{124}) = 0.79$ the free players are players 1 and 2 since $d^3 = 0.21$ and $d^4 = 0.2$ at each state on the edge. Along the edge, players 1 ad 2 are both members of both coalitions generating the face and hence the equity bias is the only dynamic impacting the drift along this edge. The player with larger demands will reduce his demand with higher probability than the other player and that defines the direction of the drift.

3. Repeat this analysis for all recurrent edges of the game.

4. Show that once close to a vertex state, the probability to hit a state on a new recurrent edge is high.

5. The above arguments combined show that for a particular game the path through the core is known, all recurrent edges lead to an edge with **co** as member. Once the process is on such an edge, the process drifts towards **co**.

6. generalize the procedure to cover all possible combinations of symmetric edges in 4-player games (for which the core does not contain an asymmetric edge)

We now use the definitions from Chapter 5 Section 5.4. Please refer to this Section for definitions of, e.g., $p_{+,0}^{C^1,+}(\mathbf{d}^*)$. Recall that an edge is the set of states in the intersection of two facets $\mathcal{H}\left(C^1\right)$ and $\mathcal{H}\left(C^2\right)$ of the core.

Let $T_{\mathcal{E}}$ be the first time that the chain (W_t) returns to $\mathcal{E}\left(C^1, C^2\right)$. We want to analyze what conditions the coalitions C^1 and C^2 must satisfy such that the chain (W_t) returns frequently to the edge.

Suppose the chain (W_t) starts in a state on a symmetric edge, e.g., generated by C_{123} and C_{124}, with $v(C_{123}) = 0.8$ and $v(C_{124}) = 0.79$. We define now two values ϕ_{C_1} and ϕ_{C_2} that are important in the analysis of the edges.

$$\phi_{C_1} = \frac{\pi_0^{RW2,\mathbf{d}^*} p_{+,0}^{C^1,+}(\mathbf{d}^*) + \left(1 - \pi_0^{RW2,\mathbf{d}^*}\right) p_{+,+}^{C^1,+}(\mathbf{d}^*)}{\pi_0^{RW2,\mathbf{d}^*} p_{+,0}^{C^1,-}(\mathbf{d}^*) + \left(1 - \pi_0^{RW2,\mathbf{d}^*}\right) p_{+,+}^{C^1,-}(\mathbf{d}^*)} \tag{7.1}$$

and

$$\phi_{C_2} = \frac{\pi_0^{RW1,\mathbf{d}^*} p_{+,0}^{C^2,+}(\mathbf{d}^*) + \left(1-\pi_0^{RW1,\mathbf{d}^*}\right) p_{+,+}^{C^2,+}(\mathbf{d}^*)}{\pi_0^{RW1,\mathbf{d}^*} p_{+,0}^{C^2,+}(\mathbf{d}^*) + \left(1-\pi_0^{RW1,\mathbf{d}^*}\right) p_{+,+}^{C^2,-}(\mathbf{d}^*)}. \tag{7.2}$$

We calculate the proportions of time when the leading coalition's surplus (in the example game the leading coalition is C_{123}) is zero and non-zero. The return behavior of coalition C_{124} is dependent on the proportion of times that the leading coalition, C_{123}, is in the 0-state; and the proportion of time that the leading coalition is in the non-zero states. The average transition probability of C_{124}'s coalition's surplus to increase by 1 is taken over the proportion of times that C_{123}'s coalition's surplus is in the 0-state or in the non-0-states. If the average probability of $\frac{CS^{C_{124}}}{\varepsilon}$ to increase by 1 is smaller than the average probability of $\frac{CS^{C_{124}}}{\varepsilon}$ to decrease by 1, then the process $\left(\frac{CS^{C_{123}}(\mathbf{d})}{\varepsilon}, \frac{CS^{C_{124}}(\mathbf{d})}{\varepsilon}\right)$ will return frequently to the face generated by C_{123} and C_{124}.

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