Inventories in General Equilibrium Dynamics

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Declaration

I hereby declare that this thesis is my own work and that, to the best of my knowledge and belief, it contains no material previously published by another person nor material that has been accepted for the award of any other degree or diploma of any university or other institute of higher education, except where due acknowledgment has been made.

Katsuyuki Shibayama

Abstract

This thesis analyses inventories empirically and theoretically. Inventories are important in understanding business cycles, not only because inventory investment accounts for a large share of GDP growth rate. This thesis also emphasises the cyclicality of inventories.

Often, business cycles are regarded as exponential decays, i.e., successive deviations from the steady state and their returning processes. In contrast, this thesis offers a battery of evidence that economic variables, such as sales and inventories, follow damping oscillations, i.e., stable sine waves. This means that a boom is the seed of the recession that follows, and vice versa. This thesis also reveals inventories' role in such endogenous cycles.

The first chapter presents empirical evidence of periodicity. VAR estimations find evidence of sine waves – namely, complex roots. Indeed, the detected complex roots seem to capture the actual business cycles; the estimated lengths of one business cycle are close to those of the post-war average in both Japan and the United States. This chapter also shows that peaks and bottoms of inventories lag behind those of production; such a time lag is called a phase shift. In addition, this chapter finds that the U.S. Federal Reserve anticipates inventory cycles, while the Bank of Japan does not.

The second chapter constructs a theoretical model with a stockout constraint and a production chain in the rational dynamic general equilibrium framework, which quantitatively satisfies stylised inventory facts. Importantly, the model successfully mimics observed inventory cycles. Moreover, working hours are more

volatile and the correlation between labour productivity and output is lower than in the standard real business cycle model.

Finally, the third chapter offers a solution algorithm for linear rational expectation models under imperfect information. Inventories are closely related to imperfect information, and inventories are often regarded as buffers against unobserved demand shocks.

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Introduction

Inventories are important in understanding business cycles. An obvious reason for this is that inventory investment is very volatile. As a result, despite its small share of GDP, inventory investment accounts for a large share of the GDP growth rate (see Fitzgerald (1997) and Blinder and Maccini (1991a), among others). However, this thesis also emphasises the cyclicality of inventories.

The main conclusion of this thesis is that business cycles are damping oscillations, not exponential decays. Most modern macroeconomic researchers perhaps recognise the concept of "business cycles" as successive deviations from the steady state and their returning processes (see Prescott (1986)); in this view, business cycles are triggered by exogenous shocks, and the endogenous mechanism in an economy generates only a monotonic convergence toward the steady state.

In contrast, this paper offers a battery of evidence that economic variables, such as sales and inventories, follow sine waves, meaning that booms and recessions tend to occur alternately. More precisely, a boom is the seed of the recession that follows, and vice versa. Moreover, this thesis reveals the role of inventories in such endogenous cycles.

The most important motivation for this thesis is to understand the so-called inventory cycles (see Figures 1.1 and 1.2 in the first chapter), which plot the year-on-year changes in production (or shipment) and inventories on the y- and x-axes, respectively. These phase diagrams exhibit clear clockwise movements, and they are observed in most periods.

Business practitioners informally explain these swirls as follows. Interestingly, it seems that they implicitly assume that (i) the target level of inventories is an increasing function of sales and (ii) the production chain is a key factor. The former means that, during a boom, demand is strong and firms want to accumulate inventories by increasing production; however, such inventories become excessive once that boom has dissipated, and they thus serve as the seed of the recession that follows. At this point, firms try to reduce their inventories by cutting production. It is important to note that production cuts imply a decrease in the demand for intermediate goods. Hence, due to the law of motion of inventories (1.2), excessive inventories continue to increase, even when production starts declining. Moreover, even when inventories return to a normal level, firms tend to cut their production further because the weak demand caused by the production reduction further pulls down the target level of inventories. Again, however, such low inventories fall short of the target level once firms bring their production back to a normal level, and they thus serve as the seed of the boom that follows. Hence, firms start recovering their reduced inventories, and this process repeats itself.

In this respect, the first chapter presents empirical evidence of periodicity. VAR estimations find evidence of sine waves – namely, complex roots. Indeed, the detected complex roots seem to capture the actual business cycles; the estimated lengths of one business cycle are close to those of the post-war average in both Japan and the United States. In addition, this chapter shows that peaks and bottoms of inventories lag behind those of production; such a time lag is called a phase shift. Note that this time lag implies that excessive inventories continue to increase, even when production starts declining, and vice versa. In addition, such a time lag is algebraically important in generating a near-circular trajectory in the phase diagrams, as illustrated in Figures 1.1 and 1.2, and it is practically useful in short-term economic forecasts. Finally, in relation to monetary policy, the first chapter shows that the U.S. Federal Reserve anticipates inventory cycles, while the Bank of Japan does not.

On the other hand, the second chapter constructs a theoretical model with a stockout constraint and a production chain in the rational dynamic general equilibrium framework. This model quantitatively satisfies two stylised inventory facts: (i) production is more volatile than sales and (ii) inventory investment is procyclical. More importantly, the model is also, to a certain extent, successful in mimicking observed inventory cycles. In addition, as a by-product, the production chain generates a slow adjustment of inventories, which is regarded as another inventory puzzle. Note that this slow adjustment insinuates, at least potentially, that excessive inventories continue to increase even when production starts declining, and vice versa. As a result, working hours are more volatile, and the correlation between labour productivity and output is lower than in the standard real business cycle (RBC) model. Intuitively, because the inventories of intermediate goods adjust quite slowly, firms cannot increase the input of intermediate goods during booms; instead, firms are forced to substitute intermediate goods with labour, and working hours hence become more volatile. Because working hours increase sharply when output is strong, labour productivity (output/working hours) does not increase very much; labour productivity, therefore, is not strongly correlated with output. In sum, our general equilibrium model with inventories not only satisfies inventory facts, but also improves the standard RBC model in terms of labour behaviour.

Finally, the third chapter discusses the effect of imperfect information in general equilibrium models. Inventories are closely related to imperfect information; for example, it is often argued that inventories are used as buffers against unobserved demand shocks. This chapter proposes the general principles of a solution algorithm for imperfect information models: (i) no endogenous variables can respond to unobserved shocks and (ii) observed shocks cannot be a source of expectation errors. It then shows that *in general* (a) the stability property of models, such as saddle-path stable, sun-spot and explosive equilibria, is invariant against changes in the information structure and (b) the direct effect of imperfect information lasts only for S periods, if the smallest information set in the expectation operator of a system of equations includes all the information up to time $t - S - 1$. In a sense, the third chapter shows that a change in information structure does not alter qualitative properties of models. This chapter does demonstrate, however, that the quantitative effect of imperfect information can be significant.

Chapter 1

Periodicity of Inventories

VAR estimations with inventory level data in this chapter detect complex roots, implying that variables, such as production and inventories, follow damping oscillations. This suggests that a boom is the seed of the following recession, and vice versa.

The main findings include: (1) the estimated cycle lengths are close to the post-war average of actual business cycle lengths; (2) inventories lag behind production by nearly one year; consequently, their contemporaneous covariance is almost zero; (3) monetary policies react sharply to demand shocks, but not to supply shocks; and (4) monetary policy is forward-looking in the U.S., but not in Japan.

1.1 Introduction

Understanding inventories enables the understanding of business cycles. At present, inventory data are invaluable in this endeavour, while the idea of inventory cycles dates back to Kitchin (1923). This chapter is motivated especially by so-called business cycles (see Figures 1.1 and 1.2), which are phase diagrams of year-on-year percentage changes in production/shipment (on the y-axis) and inventories (on the x-axis). These clockwise movements are stable in past and present data, and they are especially useful for short-run forecasts of economic conditions.

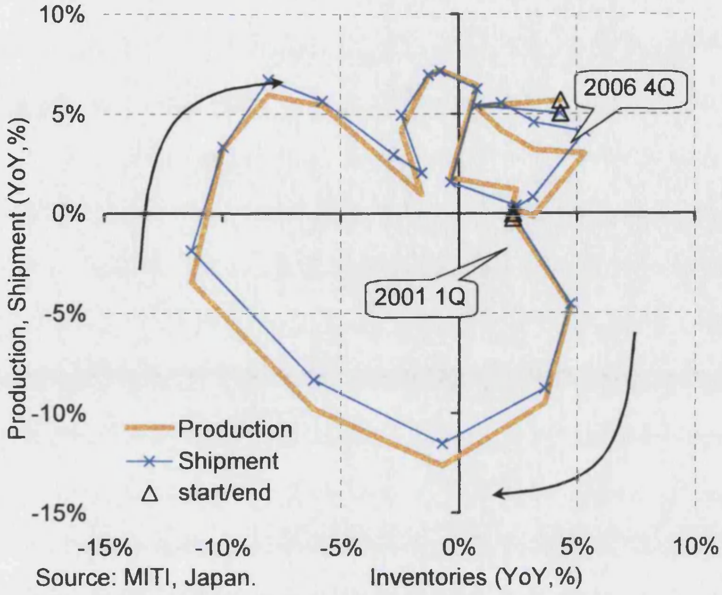

Figure 1.1: Inventory cycle in Japan.

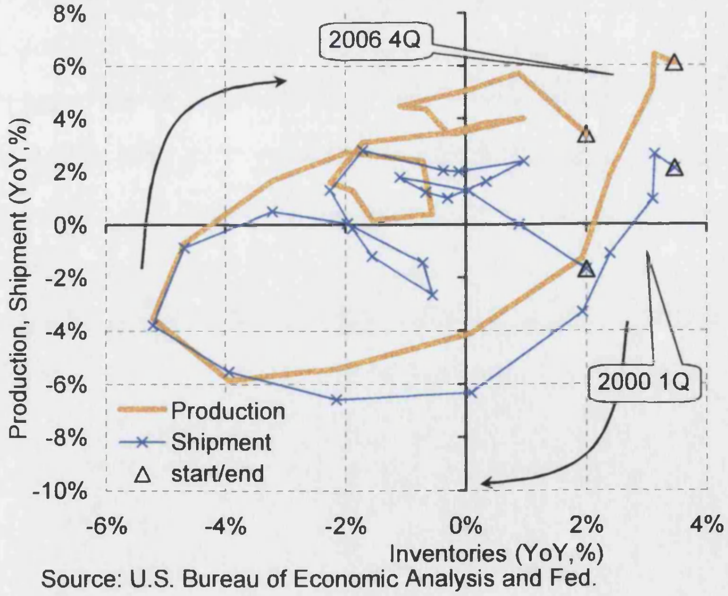

Figure 1.2: Inventory cycle in the United States.

The most important thing to note is that the cycle concept described in this chapter is a damping oscillation (stable sine curve), rather than an exponential decay. This implies that the business cycle is endogenously generated, in contrast with the view that business cycles are successive deviations from the steady state and their returning processes (see Prescott (1986)). The main thrust of this chapter is that a boom is the seed of the following recession, and a recession is the seed of the following boom.

This chapter shows the results based on two types of estimations: three- and six-variable VAR using Japanese and U.S. data.[1] Each estimation uses three types of data sets: level data, HP-filtered seasonally adjusted data (HP-s.a.), and year-on-year change (YoY) data. The purpose of the three-variable VAR is to test the existence of inventory cycles. It uses three endogenous variables: production (output), shipment (sales), and inventories data, in addition to exogenous variables (such as a constant). The six-variable VAR, which additionally includes overnight call rates and price indicators, is estimated in order to investigate the implications for monetary policy.

For each data set, the three-variable VAR finds one conjugate pair of complex roots that corresponds to the business cycle, and it seems that its existence is statistically significant because all of the trials in the bootstrapping experiments for each data set detect a stable conjugate pair of complex roots. Moreover, the implied cycle lengths are close to the actual average of post-war business cycles. For example, the implied cycle length for Japanese level data is 56 months, while the length of the average post-war business cycle is 50 months.

By construction, production, shipment and inventories exhibit the same cycle length. However, the peaks and bottoms of inventories lag behind those of production/shipment[2] by 12 to 14 months. Each detected lag is quite close to 1/4 of the estimated business cycle length. In the parlance of difference equations, the

[1]The results for the U.S. data, qualitatively not very different from those for Japanese data, are detailed in the Appendix.

[2]Production and shipment move together very closely, and hence they are interchangeable in most discussions.

phase shift (time lag) between production/shipment and inventories is around $\pi/2$ (orthogonal), because conventionally the length of one cycle is normalised to 2π.

The orthogonal phase shift reveals several important facts. First, it implies that the locus of the phase diagram in the (inventories, production/shipment) plane must have a clockwise movement with a nearly circular trajectory,[3] which is consistent with the so-called inventory cycles (Figure 1.1). Second, the contemporaneous covariance between production/shipment and inventories is almost zero (namely, orthogonal), although they are dynamically related. When complex roots are important, researchers will fail to capture the dynamic relationships among variables if they focus only on contemporaneous variances and covariances. Third, the finding that the peaks of inventories lag behind those of production by $\pi/2$ implies that the bottoms of inventories precede the peaks of production by $\pi/2$.[4] In other words, if inventories are currently bottoming out, then production is likely to reach its peak and start declining 14 to 16 months later in Japan (15 to 20 months later in the United States). This is perhaps one of the reasons why practitioners consider inventories so important; inventories are very informative for short-run economic forecasts.

Monetary policy is a main interest in the six-variable estimations. The most important observation is that monetary policy reacts sharply to a demand shock (a shock in the shipment equation), but not to a supply shock (a shock in the production equation). This is perhaps because the boom after a positive demand shock lasts longer than that after a supply shock.[5] This is consistent with the target inventory model, in which the target level of inventories is an increasing function of demand. According to the model, a positive demand shock reduces inventories and, as a result, production continues to rise to replenish inventories. On the other hand, a supply shock increases inventories, and hence firms cut their

[3] It is roughly $\pi/3$ for the U.S. data, implying that the trajectory of the inventory cycle is an ellipse with the major axis running from the northeast to the southwest (Figure 1.2).

[4] More precisely, if the phase shift and the cycle length are s and L months, respectively, the bottoms of inventories precede the peaks of production by $L/2 - s$ months.

[5] However, this is observed only in the Japanese data, but not in the U.S. data (see the Appendix).

production to adjust their inventories.

Interestingly, the phase shift between the overnight call rate and production is around 2 months in the Japanese data. Given the fact that statistics are released 1 to 3 months after the period from which they are culled, the Bank of Japan (BoJ) reacts to real variables with no time lag. In contrast, the lag for the U.S. Federal Reserve (Fed) is around -4 months! The negative lag, of course, insinuates that the Fed's monetary policy is pre-emptive/forward-looking.

Because of the use of level data, the main challenge of this chapter is the treatment of non-stationarity. Indeed, Monte Carlo experiments for the three-variable VAR suggests that the hypothesis that the system of equations has one *real* unit root cannot be rejected under some maintained hypotheses. However, the norm of the business cycle complex roots is significantly less than one. Moreover, the same Monte Carlo experiments show that the real unit root affects the estimated period length and phase shifts only negligibly. In addition, to check for robustness, VARs are estimated by using two additional data series (HP-s.a. and YoY data, as mentioned above). In these two stationary data sets, we obtain results quantitatively quite similar to those of level data.

The plan of this chapter is as follows. The next section reviews theories on how to compute the cycle length and phase shifts from VAR estimates. The results of the three- and six-variable VARs with Japanese data are discussed in Sections 1.3 and 1.4, respectively. The estimation results with the U.S. data are discussed in the Appendix, because the quality of the U.S. data set (and, as a result, its estimation performance) is not as good as the Japanese one. Though the three-variable VAR is something of a subset of the six-variable VAR, the former has its own worth; it allows for Monte Carlo experiments, and the estimation results are more precise and reliable. Section 1.5 briefly reviews old and modern thoughts on business cycles, and proposes the concept of pseudo-propagation. Section 1.6 concludes.

1.2 Preparations before Estimations

This section briefly reviews the evidence of periodicity. The key checkpoints are a conjugate pair of complex roots and phase shifts. The existence of complex roots implies that the system of equations can be represented by a sine curve (as well as some other terms).

1.2.1 Conjugate Pair of Complex Roots

This subsection briefly introduces key notations. We estimate the coefficient matrices of the following VAR.

$$y_t = z_t A + y_{t-1} B_1 + y_{t-2} B_2 + \cdots + y_{t-M} B_M + \xi_t C \tag{1.1}$$

where A, B and C are real coefficient matrices, and z_t, y_t and ξ_t are the row vectors of exogenous variables (time trend, seasonal dummies, etc.), endogenous variables and *iid* shocks, respectively.

It is known that any complex roots, if they exist, must appear in pairs – any complex root $z = a + bi$ has its conjugate $z^H = a - bi$, where $i = \sqrt{-1}$. It is also known that if there are complex roots, the solution of an endogenous variable includes a term such as

$$\alpha_{kj} \rho_{kj}^t \sin\left(\theta_{kj} t + \beta_{kj} \right)$$

where t is time and α_{kj}, β_{kj}, ρ_{kj} and θ_{kj} are parameters that are functions of elements in VAR coefficient matrices B_m and the variance-covariance matrix of the error term. The subscript kj implies that the term is in the solution of the k-th variable and is related to the j-th eigenvalue (and its conjugate).

The economic meanings of these parameters are as follows. α_{kj} is a kind of size parameter. $\rho_{kj} = \rho_j = \sqrt{a_j^2 + b_j^2}$ is the absolute value of the complex roots.[6] $\theta_{kj} = \theta_j = \arctan(b_j / a_j)$ is the frequency of the sine function, and hence the length

[6]For example, if there is a ρ_j whose absolute value is unity, then the term represents a unit root while all ρ_j must be less than 1 in absolute terms to have a stable system.

of one period is $2\pi/\theta_j$. β_{kj} is the phase, which shows the "initial state" of the k-th variable right after an shock.[7] $\left(\beta_{kj} - \beta_{lj}\right)/\theta_{kj}$ is the phase shift (in time) between the k-th and l-th variables. If it is x months, then it means that the peaks and bottoms of the k-th variables precede those of the l-th variable by x months. It can be shown that the phase shift (in angle), $s_{kl,j} = \beta_{kj} - \beta_{lj}$, is a function only of the elements in matrices B_m, although β_{kj} alone depends on past and present shocks as well.

1.2.2 Phase Shifts

Phase shifts have important implications in dynamic relationships among variables, because, intuitively, they indicate time lags among variables. This subsection briefly reviews (a) the limitation of contemporaneous covariances and (b) the empirical implication for inventories.

Limitation of Contemporaneous Covariances

A phase plane exhibits a stable spiral only if there is at least one pair of conjugate complex roots, and stable spirals can be classified into eight leading cases by phase shifts (see Figure 1.3). It is clear that, even when two variables have a close dynamic relationship with each other, the contemporaneous covariance between them is close to zero if their phase shift is near $\pm\pi/2$.

Of course, the entire story is not so simple. If the true data generating process (DGP) is very noisy, the effect of endogenous dynamic relationships, governed by matrices B_m, may be swamped by the initial effects of shocks. In such cases, contemporaneous covariances are determined mainly by matrix C in (1.1). Nonetheless, the limitation of contemporaneous second moments can be very serious. Indeed, the dynamic relationship between inventories and production is one example. They have a close dynamic relationship, but their contemporaneous covariance is close to zero, as shown in the subsequent sections.

[7] See footnotes 16 and 17 to understand the intuition of the "initial state."

Implication for Inventory Cycles

To have phase diagrams such as inventory cycles (Figure 1.1), the value of the phase shift between production/shipment and inventories must be around $\pi/2$. This value is predicted through the following two observations. First, *the phase shift must be positive*, because the direction of inventory cycles is clockwise. Second, *the phase shift should be around either* $+\pi/2$ *or* $-\pi/2$, because the contemporaneous correlation between inventories and production/sales is close to zero in the data.

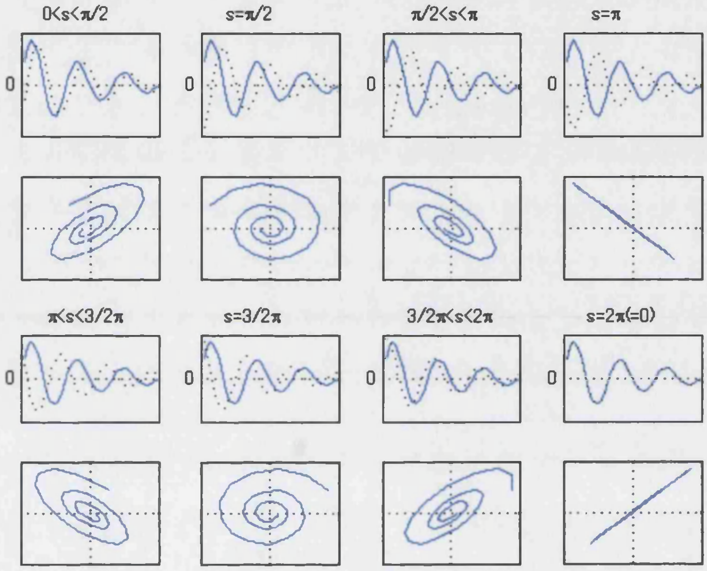

Figure 1.3: Impulse response functions and phase diagrams. s shows a phase shift. The solid and dotted lines in the IRFs correspond to the y- and x-axes in the phase diagrams, respectively.

1.2.3 Non-Stationarity

One of the challenges of this chapter is the use of level data, which almost inevitably causes the non-stationarity problem. This chapter tackles this problem in two ways: with Monte Carlo simulations and filtered data sets.[8]

[8] In addition, as preliminary tests, Johansen's (1991) trace tests indicate that there exists at least one cointegration vector at the 1% level. For these trace tests, two preliminary estimations are conducted: one includes constant and seasonal dummies, and the other additionally includes the linear time trend. These tests are conducted by using PcGive, an econometric software;

As mentioned above, Monte Carlo experiments reveal that there is one *real* unit root under the assumption that the true DGP has no time trend. However, they strongly reject the hypothesis that the absolute value (norm) of business cycle complex roots is $+1$ under most maintained hypotheses.[9] Moreover, the same Monte Carlo experiments show that the effects of the unit root on the estimated cycle length and phase shifts are quantitatively negligible.

Furthermore, to check the robustness of the estimated results, this chapter also implements two additional VARs: estimations based on (i) HP-s.a. and (ii) YoY data. Though, presumably, the estimation results based on the filtered data are also biased, most of the findings in the level data estimation are supported by the two additional estimations. This implies that the estimation bias in each data set is not serious.

Sketch of Monte Carlo Experiments

This subsection sketches the Monte Carlo Experiments conducted in this chapter. Assume that the true data generating process follows a VAR(1) process to keep exposition simple.

$$y_t = y_{t-1}B + \xi_t C$$

where ξ_t is assumed to be *iid.* Matrix $\hat{B}$ is first estimated by OLS. If there are no multiple roots, $\hat{B}$ can be decomposed by eigenvalue matrix $\hat{\Lambda}$ and eigenvector matrix $\hat{V}$.

$$\hat{B} = \hat{V}\hat{\Lambda}\hat{V}^{-1}, \qquad \hat{\Lambda} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_K \end{bmatrix}$$

where K is the number of roots (number of endogenous variables times VAR order, in general).

The idea of our Monte Carlo experiments in this chapter is as follows. For

however, the trace test with a fifth-order polynomial time trend is not conducted.

[9] In this chapter, an assumption on the true DGP is called a maintained hypothesis.

example, if the first eigenvalue is suspected to be a unit root, then the true DGP is assumed to be generated by $\check{B}$ such that

$$\check{B} = \hat{V}\check{\Lambda}\hat{V}^{-1}, \qquad \check{\Lambda} = \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & \lambda_K \end{bmatrix}$$

Keeping $\hat{V}$ unchanged, the $\check{B}$ is constructed based on $\check{\Lambda}$. Then, by generating artificial innovations $\{\check{\xi}_t^j\}_{j=0}^N$,[10] $\check{B}$ and C matrices yield artificial data sets $\{\check{y}_t^j\}_{j=1}^N$, where N is the number of trials in Monte Carlo experiments. Estimates such as period length are computed for each $\check{y}_t^j$, and their distributions are obtained by stacking such estimates for $j = 1, \cdots, N$. Though true V and Λ are unknown, presumably $\hat{V}$ and $\hat{\Lambda}$ do not vary far from them, because the Monte Carlo experiments themselves show evidence of very tight estimations. Loosely speaking, the Monte Carlo experiments are implemented in the neighbourhood of the true value.

1.2.4 Related Literature

In this subsection, we briefly review existing VAR analyses for inventories with aggregate data (see Chapter 2.2 for a more general literature review). Though our objectives is to reveal the reaction of the central banks in business cycles, most existing research investigates the reaction of inventories to monetary policy and focuses on the importance of inventory behaviour as a channel of the monetary transmission mechanism.

For the U.S., Gertler and Gilchrist (1994) show that, after a tight monetary policy shock, small firms decumulate their inventories, while large firms accumulate them. They conclude this difference derives from the difference in creditworthiness.

[10] A row vector $\check{\xi}_t^j$ is generated by resampling $\hat{\xi}_t = \left(y_t - y_{t-1}\hat{B}\right) C_{hol}^{-}$, where $\hat{B}$ is estimated by the simple OLS and C_{hol}^{-} is the inverse of the upper triangular matrix C_{hol} such that $\frac{1}{T}\sum_t \xi_t^T \xi_t = C_{hol}^T C_{hol}$. Generating $\check{\xi}_t^j$ by the standard normal distribution does not change the results quantitatively; as long as its variance is unchanged, the distribution of $\hat{\xi}_t$ has only negligible effects.

During tight monetary policy periods, large firms can finance their inventories, while small firms cannot (see also Barnanke and Gertler (1995)). Kashyap et al. (1994) also report the essentially identical results by using firm level data.

For Japan, several studies such as Yoshikawa et al. (1993) emphasise the importance of the inventory channel. A tight monetary policy first negatively affects inventory investments, and such weak inventory investments then affect real economic activity because inventories are working capital (see also Teruyama's survey (2001) for analyses in this line). Indeed, inventories are working capital in theoretical models such as the stockout avoidance model; inventories are necessary capital for successful sales activity.

These authors reveal the importance of inventories in the sense that the effects of the credit channel manifest themselves in the behaviour of inventories.

1.3 Three-Variable VAR

This section describes the results of the three-variable VAR, in which production (output), shipment (sales) and inventories as well as the exogenous seasonal dummy variables and time trend are regressed. The three-variable VARs allow us to establish valid Monte Carlo simulations. Contrarily, in the six-variable VARs, there exist several pairs of complex roots similar to each other. Such roots are mixed each other in some Monte Carlo experiments, which prevents us from tracking the behaviour of one specific pair of complex roots throughout the simulations.

1.3.1 Description of Details

Original Data

This chapter uses the data of industrial production in Japan.[11] The data estimated in three-variable VAR are (1) production (output), (2) shipment (sales) and (3)

[11]The data are available on the website of the Ministry of Economy Trade and Industry of Japan.
http://www.meti.go.jp/english/statistics/index.html

inventories. All of them are of "mining and manufacturing" (i.e. all sectors) from January 1978 to December 2006. All variables are the average of physical units of goods weighted by value-added in the baseline year. The data quality is thought to be extremely high, given the ministry's strong authority over Japanese manufacturers.

Recursiveness Assumption

To identify the coefficient matrix on shocks C in equation (1.1), this chapter adopts a recursiveness assumption. Specifically, for the three-variable VAR, shocks in the inventory equation do not affect current production or sales, and those in the production equations do not affect current sales. For the former, inventories should be affected by shocks in output and sales, because the law of motion of inventories is presumably an identity.

$$U_t = U_{t-1} + Y_t - S_t \tag{1.2}$$

where U_t represents the goods that are not sold in markets at time t (i.e., inventories), which are carried to the next period (hence, $U_t - U_{t-1}$ is the inventory investment), Y_t is production, and S_t is sales at time t. The latter means that production can respond to shocks contemporaneously. Importantly, the recursiveness assumption affects only IRFs but not other results such as phase shifts and spectrums.

Bootstrapping

The bootstrapping method is used to compute the standard deviations of estimates and confidence intervals. In addition, the standard deviations of period length and phase shifts are computed, as long as a cycle exists for all the trials in the bootstrapping.

Order Selection Criterion

For the level data (not seasonally adjusted), some information criteria suggest very long VAR orders (maximum time lag of endogenous variables), perhaps because the fixed seasonal dummy cannot perfectly eliminate the seasonality. Judging from the AIC and SIC of HP-s.a. and YoY estimations, it seems that the best VAR order is somewhere between 2 to 4. Hence, the VAR order in this chapter is always 3 to facilitate comparisons. Fortunately, the quantitative effect of changing the VAR order is not substantial for any of the following results (see below). Most estimates are quantitatively robust against changes in the VAR order.

Data Format

There are three estimations, each of which uses a different data set (different data format), though the functional form is (1.1) for all three. The first is the benchmark estimation, using the level data (before seasonal adjustment) with a polynomial time trend. The second and third ones use the HP-filtered seasonally adjusted (HP-s.a.) data and year-on-year (YoY) change data. Presumably, the level data set is subject to the non-stationarity problem, while filtered data are subject to the artificial endogeneity problem. Rather than directly tackling these problems separately, this chapter compares these three specifications to evaluate how seriously the estimates are biased. As shown below, these three estimates show results very similar to each other, supporting the view that the estimated business cycles are not strongly biased.

(I) Benchmark Estimation (with Level Data) The benchmark estimation uses non-seasonally adjusted level data. It also includes seasonal dummies and a 5th-order polynomial time trend. The former and latter are included to eliminate seasonality and trend, respectively.

Polynomial Time Trend: The benchmark estimation includes the 5th-order polynomial of time. This time trend well mimics the HP-filter with smoothing parameter $\lambda_M = 130,000$.[12] Given the HP-filter's popularity, the HP-filtered series (the original series minus the HP-trend) is preferable in detecting cycles *recognised by practitioners*. However, the HP-filter artificially causes the endogeneity problem. On the other hand, the exogenous 5th-order polynomial does not bias OLS estimates, and it eliminates almost the same trend as the HP-filter does.

However, it is important to note that the estimated cycle length is very sensitive to the specification of the time trend (see below for a detailed discussion).

Seasonal Dummy: In addition, the VAR estimation also includes the seasonal dummies. However, the fixed seasonal dummies cannot completely eliminate seasonality. Visually examining the plots of the fitted and actual data, it seems that seasonal fluctuation is growing over time.

(II) Estimation with HP-Filtered Seasonally Adjusted Data This estimation uses HP-s.a. data, which, by construction, are stationary. However, both the HP-filter and seasonal adjustment are essentially moving averages of past and future values, which implies that the residuals can be correlated to the regressors.

(III) Estimation with YoY Data This estimation uses YoY change data. If original data are $I(1)$, then YoY data are stationary. The main problem with YoY data is that they could magnify the effect of noise.

1.3.2 Roots of Coefficient Matrix

If at least one conjugate pair of imaginary roots exists, then at least potentially there is a mechanism that generates a cycle. There are 9 roots (=number of

[12]Numerical experiments, shown in the Appendix, demonstrate that the smoothing parameter for monthly data, which is equivalent to $\lambda_Q = 1600$ for quarterly data, is slightly less than $\lambda_M = 130,000$. The rule of thumb $\lambda_M = 14,400$ generates a too well-fitted HP-trend series (i.e., not smooth enough). This finding endorses the result of Ravn and Uhlig (2002).

Table 1.1: Estimated business cycle roots (three-variable VARs with Japanese data).

Panel I: Level

Roots	0.95±0.11i	0.6887	-0.35±0.45i	-0.28±0.28i	0.07±0.45i	
Norm	0.9541	0.6887	0.5723	0.4014	0.4555	
Angle	±0.0357π	0	±0.7081π	±0.7503π	±0.4502π	
Cycle length	56.05	+inf	2.82	2.67	4.44	

Panel II: HP-s.a.

Roots	0.96±0.11i	0.64807	-0.34±0.29i	-0.10±0.38i	-0.1429	0.047281
Norm	0.9704	0.64807	0.4493	0.3944	0.1429	0.047281
Angle	±0.0359π	0	±0.7783π	±0.5816π	0	0
Cycle length	55.64	+inf	2.57	3.44	+inf	+inf

Panel III: YoY

Roots	0.96±0.11i	0.82106	-0.31±0.44i	-0.23±0.23i	0.2686	-0.24244
Norm	0.9651	0.8211	0.5423	0.3228	0.2686	0.2424
Angle	±0.0347π	0	±0.6949π	±0.7526π	0	0
Cycle length	57.63	+inf	2.88	2.66	+inf	+inf

variables × number of order).

Implied Cycles

The conjugate pair $0.95 \pm 0.11i$ is evidence that the endogenous variables follow a sine curve. These complex roots imply a cycle 56.1 months long (s.d. = 2.6 months), which is near the post-war average in Japan (50.3 months).[13] It is possible to compute the standard deviation of the cycle length because no trials in the bootstrapping experiments lack these complex roots.

The other three cycles are 2.7 to 4.4 months in length. One possibility is that they are evidence that the inventories work as buffers in very high frequencies (see Section 1.3.3). However, they may simply capture high-frequency noise and seasonality that cannot be perfectly eliminated by dummy variables.[14] In any event, it is difficult to establish their statistical significance, because they are often mixed with each other in the bootstrapping, and are therefore almost impossible to distinguish.

The estimated period length does not change considerably in the other two

[13] In Japan, a governmental committee determines the business cycle dates. http://www.esri.cao.go.jp/en/stat/di/041112rdates.html

[14] In this sense, just having complex roots itself is not very interesting at all. It is important to have complex roots that correspond to the business cycle.

Table 1.2: Phase shifts (three-variable VARs with Japanese data).

	(Cycle length)	Sales	Inventories
Level	(56.1)	-0.3146 mths	12.416 mths
HP-s.a.	(55.6)	0.2110 mths	13.527 mths
YoY	(57.6)	-0.4733 mths	14.209 mths

Note: Time-lags from production.

data sets: 55.6 months (s.d. = 5.4 months) in the HP-s.a. data, and 57.6 months (s.d. = 6.7 months) in the YoY data.

Phase Shifts

With respect to the business cycle roots detected in the level data, the peaks and troughs of inventories lag behind those of production and shipment by 12.4 and 12.1 months, respectively. As expected, the phase shift between production/shipment and inventories is close to 1/4 of the period length. There is almost no time lag between production and shipment.

Table 1.3: Implied cycle lengths (three-variable VARs with Japanese data).

Time Poly. Order:	1	2	3	4	5	6	8	10
VAR(2)	206.6	109.4	98.35	75.78	59.88	60.14	58.32	53.35
VAR(3)	168.5	102.5	90.44	69.67	56.05	56.79	55.01	50.28
VAR(4)	153.4	100.3	94.71	72.78	57.11	57.59	55.25	51.00

Note: Estimation based on the level data.

Effect of Time Trend

In most specifications of the time trend, the VAR estimation detects one significant pair of business cycle complex roots. However, the estimated cycle length crucially depends on the choice of time trend, while the effect of the VAR order is not very strong. For example, the VAR(3) with a linear time trend shows that the length of one business cycle is 168.5 months (see Table 1.3). This means that the estimated cycle length with the level data is not robust against the specification changes of time trend, while the phase shift between production/shipment and inventories is almost always close to 1/4 of the business cycle's length. In addition, the specification of the time trend affects the norm of the largest *real* root (see the next subsection).

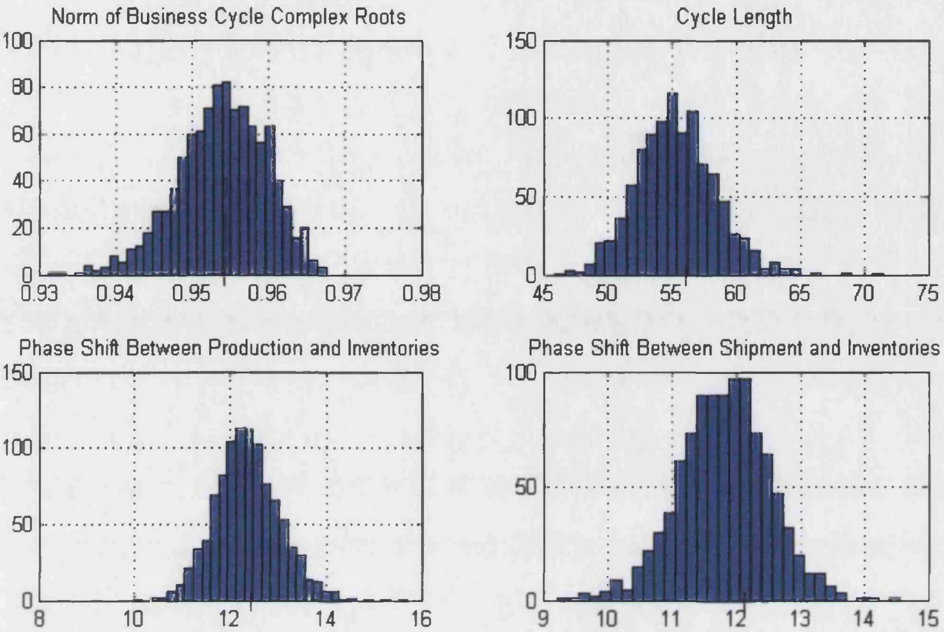

Figure 1.4: Distributions generated by 1,000 trials. H_M: There is one real unit root. Ticks on the x-axis show the true value in H_M.

Effect of Unit Root

Surprisingly, in the level data estimation, we cannot rule out the possibility that the real root (0.6887) in the level data is a unit root. Certainly, 0.6887 appears to be far from +1, but the norm of this root is strongly affected by the time trend; as the order of the time trend polynomial decreases, the norm moves towards +1. At limit, the hypothesis that there is one real unit root is not rejected under the maintained hypothesis that there is no time trend in the true DGP.

However, these Monte Carlo experiments show that the existence of the real unit root only slightly affects the cycle length and phase shifts. Figures 1.4 and 1.5 show the selected distributions under the maintained hypotheses that there is one *real* unit root and that there is one pair of unit complex roots, respectively. Both experiments assume that the true DGP has the 5th-order polynomial time

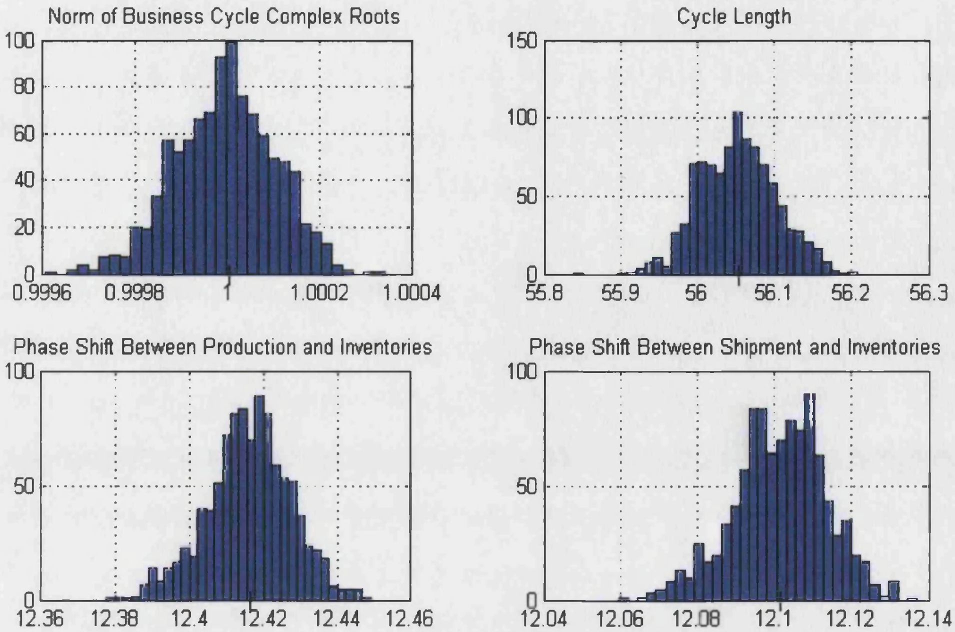

Figure 1.5: Distributions generated by 1,000 trials. H_M: There is one pair of complex unit roots. Ticks on the x-axis show the true value in H_M.

trend. These results show that the estimates are very precise and the distributions are skewed only slightly. For example, the upper-right panel of Figure 1.4 shows that the distribution of the cycle length centres on 55 months, which is very close to the true value in the DGP (56.1 months, as denoted by "|" on the x-axis). Also, the top-left panel in Figure 1.5 suggests that the absolute value (norm) of the estimated business cycle complex roots (0.9541) is far enough from +1. Even though the true DGP is assumed to have no time trend, the same exercise still suggests that the business cycle complex roots are not unit roots.

1.3.3 Impulse Response Functions

Clearly, all of the impulse response functions show the shape of sine curve fluctuations. Visually reviewing the distance between two peaks in each IRF, we can see that the length of one cycle is roughly 56 months, almost same length implied by the business cycle complex roots.

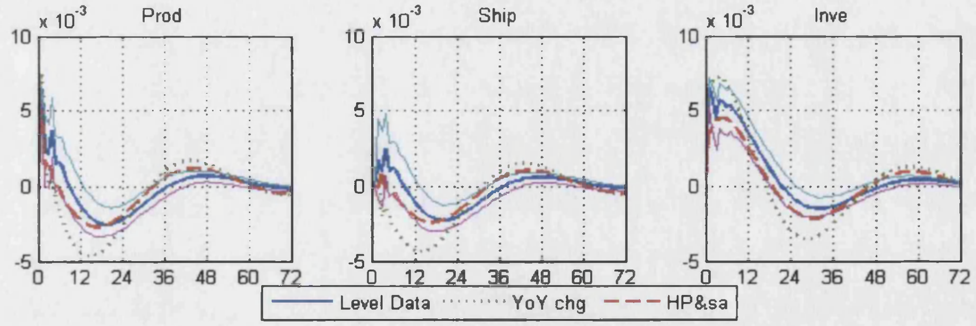

Figure 1.6: IRFs due to a positive shock in the production equation (three-variable VARs with Japanese data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

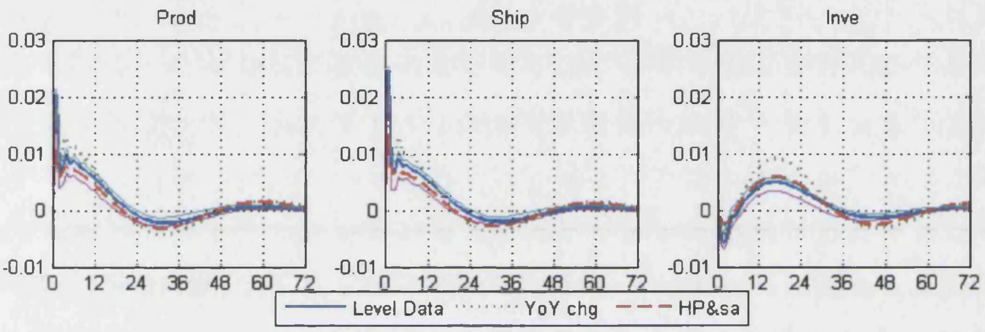

Figure 1.7: IRFs due to a positive shock in the shipment equation (three-variable VARs with Japanese data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

Technology Shock: Figure 1.6 shows impulse responses to a production shock, which can be regarded as a technology or supply shock. After a positive shock, both production and shipment increase. Inventories increase due to the law of motion of inventories (1.2). Sales do not increase as much as production does; hence, for production shocks, output is more volatile than sales. This corresponds with the theory of cost shock models.[15]

However, more importantly, production returns to zero roughly 9 months after the shock. The effects of a positive production shock disappear quickly. This is

[15]Cost shock models in the inventory literature emphasise the effect of production cost. The idea is that because the source of shock lies on the production side, production is more volatile than sales. In addition, inventory investment increases when production increases due to a low cost shock (procyclical inventory investment).

because a positive production shock induces an increase in inventories[16] – but, because having excess inventories is costly for firms, they want to reduce such excess inventories by cutting production.

Demand Shock: On the other hand, Figure 1.7 shows that after a positive sales shock, which can be regarded as a demand shock, production stays above zero for more than 20 months. Right after a positive demand shock, inventories decrease due to the law of motion of inventories (1.2).[17] However, such a level of inventories is too low, and firms want to increase their production in order to recover their inventories. Also, note that the initial impacts of a demand shock are much larger than those of a supply shock (compare the units of the y-axes).

Indeed, we can draw more implications. In the theoretical literature, the target inventory models – including the stockout avoidance model – suggest that the target level of inventories is an increasing function of sales. Thus, after a positive demand shock, firms want not only to replenish their reduced inventories, but also to raise the level of inventories so that it meets the new, higher level of sales. Actually, the subsequent increase in production is slightly larger than that of sales (otherwise, inventories would decrease). As a result, even though the source of the shock is on the demand side, output is more volatile than sales. In the sense that demand shocks are magnified by inventories, inventories are regarded as *destabilising factors in business cycle frequencies.*

In contrast, while inventories drop sharply right after a positive demand shock, more than half of the initial effect of the shock on production and shipment disappears within one period. This shows that inventories work as buffers in a very short time period. In this sense, production smoothing theory is still very much alive at very high frequencies.[18]

[16]In phase diagrams such as Figures 1.1 and 1.2, starting from the origin, a positive supply shock is plotted as a jump to the northeast of the origin.

[17]In phase diagrams such as Figures 1.1 and 1.2, starting from the origin, a positive demand shock is plotted as a jump to the northwest of the origin.

[18]Originally, inventory literature started with the production smoothing theory, which says that firms have an incentive to smooth the time-path of production due to a convex cost function;

These findings can be summarised as follows. Inventories are destabilising factors at business cycle frequencies but are stabilising factors at very high frequencies. This view is in line with Wen (2002).

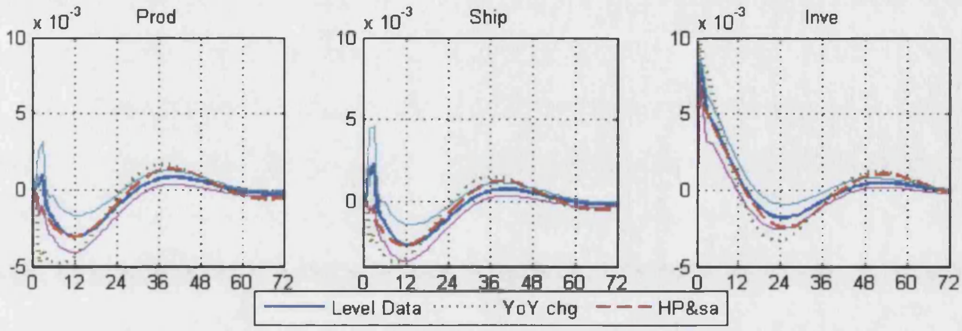

Figure 1.8: IRFs due to a positive shock in the inventory equation (three-variable VARs with Japanese data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

Inventory Shock: After a positive shock to the inventory equation, both sales and production decline (Figure 1.8). In a sense, a shortage of inventories is akin to an increase in demand, and vice versa, because firms have an incentive to replenish (or cut) them to their normal level.

1.3.4 Cross Correlations and Spectral Densities

The cross correlations and spectra are computed form the estimated coefficients in equation (1.1).[19] Note that with non-stationary processes, neither is well defined; thus, we should focus on the cross correlations and spectra in the HP-s.a. and YoY data sets. Nonetheless, the results in the benchmark data quite markedly resemble those based on the two stationary data sets. Both cross correlations and spectral densities show that (a) there is a cycle with business cycle frequency, and (b)

they hold inventories to protect themselves from unexpected demand shocks. However, though this theory is at first glance very clear-cut, it cannot explain the two famous inventory stylised facts (see Sections 2.2 and 1.5.4). This failure has been the biggest motivator for subsequent inventory research.

[19]See Chapter 10 of Hamilton (1994) for the computation of cross correlation and spectra. However, note that the phase shifts are computed in this chapter based on a different method from that shown in Hamilton (1994) (see the Appendix).

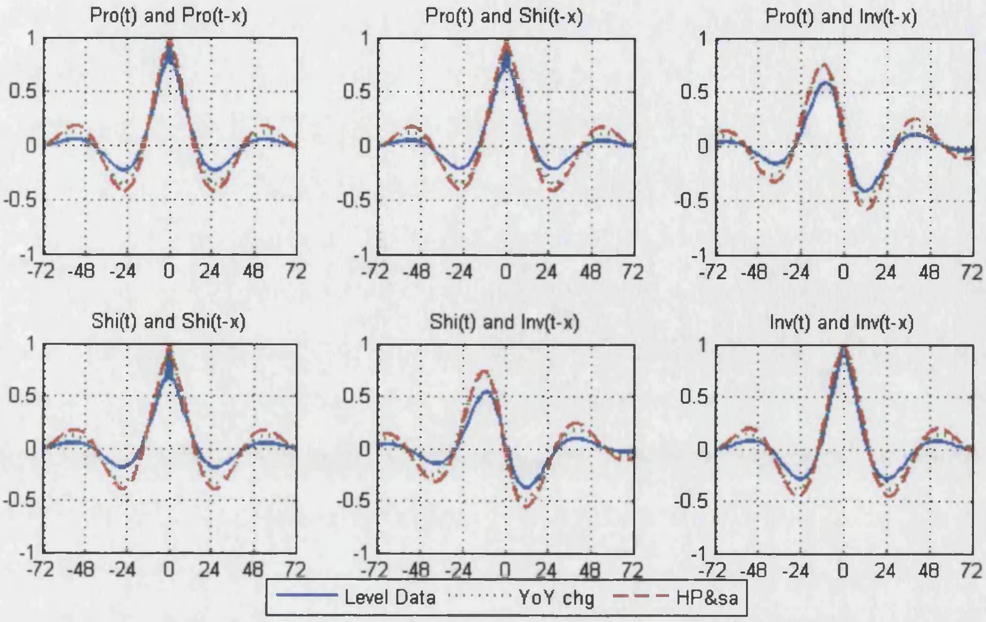

Figure 1.9: Cross correlations (three-variable VARs with Japanese data).

the contemporaneous correlation fails to capture the dynamic relationship among variables.

Cross Correlations

The cross correlations (Figure 1.9) show several observations worth mentioning. First, the cross correlation between production/shipment and inventories reaches its peak and bottom when the time lag is around ∓ 12 months, which is consistent with the estimated phase shift. Second, the contemporaneous correlation between production/shipment and inventories is close to zero; thus, the contemporaneous correlation alone cannot capture their dynamic relationship. Third, the autocorrelations reach their bottom around ± 25 months, implying that the dominant cycle is around 50 months in length ($= 25 \times 2$), which is not very different from the finding in Section 1.3.2 (see also Section 1.A.4). Fourth, the spikes in autocorrelations of production and shipment at 0 month imply a very high frequency component that affects both production and shipment. This is indirect evidence of buffer inventory models (see Figure 1.7).

Spectral Densities

The spectral densities[20] (Figure 1.10) show several observations worth mentioning. First, all the cospectra and quadrature spectra reach their peaks or bottoms at around 56 months, which again implies that the cyclical component with a period length of around 56 months is most influential. Second, the cospectrum between production/shipment and inventories is almost zero for all period lengths, which implies that the contemporaneous covariance cannot capture their dynamic relationship in any frequency. However, the existence of a dynamic relationship between production/shipment and inventories is evident in the quadrature spectra between production/shipment and inventories. Finally, the quadrature spectrum between production and shipment is almost zero for all period lengths, which means that there is almost no time lag between them.

1.3.5 Summary of Three-Variable VAR

Among others, the following findings are important.

- A lot of evidence supports the existence of the inventory cycle, and its estimated cycle length is close to the post-war average of business cycles.
- The estimated phase shift between production/shipment and inventories is

[20]It may be worth reviewing the two spectral densities for multiple variables.

First, a cospectrum has the same meaning as a spectrum with one variable. For the components of cross covariances reflected in contemporaneous covariance, a cospectrum attributes such components to each frequency. For example, if the absolute value of a cospectrum density reaches its peak at frequency f, it implies that the cycle with frequency f makes the largest contribution to the contemporaneous covariance. The integral of cospectral densities over the whole frequency domain $0 \leq f \leq 2\pi$ is equal to the contemporaneous covariance.

Second, a quadrature spectrum essentially represents anything other than the corresponding cospectrum. For the components of cross covariances *not* reflected in contemporaneous covariance, a quadrature spectrum attributes such components to each frequency. For example, if the absolute value of a quadrature spectrum density reaches its peak at frequency f, it implies that the cycle with frequency f makes the largest contribution to the cross covariance with a time lag of $\pi/2f$ periods (1/4 of the period length $2\pi/f$). Remember that if two variables follow a sine curve, and the phase shift between them is 1/4 of the period length, then the contemporaneous correlation of these two variables is zero, even though both follow essentially the same process. In other words, a quadrature spectrum represents the relationship that is *not* reflected in contemporaneous covariance due to phase shift. The integral of quadrature spectral densities over $0 \leq f \leq 2\pi$ is equal to zero.

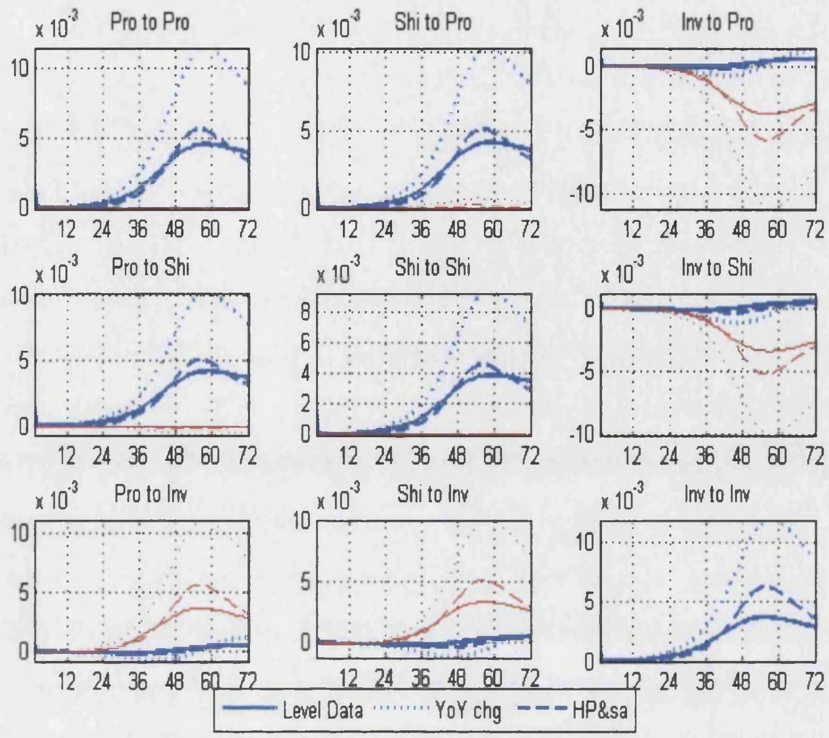

Figure 1.10: Co- and quadrature spectra (three-variable VARs with Japanese data). Bold lines show cospectra and narrow lines show quadrature spectra.

12 months (close to 1/4 of one period length). For example, if inventories are bottoming out now, then the production will peak and start declining about 16 months later.[21]

- Due to inventories, a boom lasts longer with a demand shock than a supply shock.

- For a demand shock, inventories work as destabilising factors in business cycle frequencies, but work as buffers within very short periods.

- Although the bias due to the unit root seems very minimal, the estimated cycle length is sensitive to the time trend specification.

There is a supplementary remark in terms of the theoretical research on inventories. The findings in our VAR estimations support *all* of the following three

[21] 16 months $\simeq 56.1/2 - 12.4$ (one half of the cycle length minus the phase shift between production and inventories).

leading theories: production-smoothing, target inventory and cost shock models. While the cost shock model is consistent with the IRFs to a supply shock, the target inventory and production smoothing models are each in line with the IRFs to a demand shock. However, because (i) the initial impacts of demand shocks on production and shipment are much larger than those of supply shocks, and (ii) the effects of demand shocks last longer than those of supply shocks, it seems that demand shocks are a more important source of economic fluctuation. Hence, if business cycles are of interest, the target inventory model is perhaps the most relevant.

1.4 Six-Variable VAR

This section describes the results of the six-variable VAR estimation, used to investigate the interaction between monetary policy and inventories.

1.4.1 Description of Details

Original Data Though the BoJ's direct policy instrument is the *un*collateralised O/N call rate (and excess reserves under the zero-interest rate policy), its data length is short. Hence, the collateralised O/N call rate, which exhibits movements quite similar to those of the uncollateralised O/N call rate, is adopted in this analysis.[22] For the Consumer Price Index (CPI), the general (overall) index excluding fresh foods and imputed rents is used,[23] while the material price index in the Corporate Goods Price Index (CGPI) is included as a leading inflation indicator.[24] To avoid zero-interest rate periods, the estimation period is from January 1978 to December 1998.

In the HP-s.a. data set, following convention, the O/N call rate and CGPI are

[22] See "How to Download Long-Term Time-Series Data Files" on http://www.boj.or.jp/en/theme/research/stat/market/short_mk/tanki_rate/index.htm
[23] See http://www.stat.go.jp/english/data/cpi/index.htm
[24] See "Index by Stage of Demand and Use" on http://www.boj.or.jp/en/theme/research/stat/pi/cgpi/index.htm#04

not seasonally adjusted. In the YoY data set, the YoY change in the O/N rate is used, although it is presumably stationary.

Recursiveness Assumption The six-variable VAR, following Christiano, Eichenbaum and Evans (1999),[25] assumes that the O/N call rate can respond to any of the current shocks. It also assumes that neither CPI nor material prices can respond to the current shocks to the three real variables. Because material price index is regarded as a leading indicator of CPI, it can respond to contemporaneous CPI shocks.

1.4.2 Roots of Coefficient Matrix

Each estimation finds two or three pairs of complex roots that correspond to the business cycle. Selected point estimates of the roots are shown in Table 1.4. Roots omitted from the table are complex roots with very high frequencies (i.e., shorter than 6 months).

Table 1.4: Estimated business cycle roots (six-variable VARs with Japanese data).

Panel I: Level						
Roots	0.96±0.10i	0.82±0.12i	0.9139	0.6127	0.5026	-0.3693
Norm	0.9640	0.8287	0.9139	0.6127	0.5026	0.3693
Angle	±0.0316π	±0.0461π	0	0	0	0
Cycle length	63.2	43.4	+inf	+inf	+inf	+inf
Panel II: HP-s.a.						
Roots	0.97±0.11i	0.85±0.09i	0.80±0.01i	0.40±0.25i		
Norm	0.9755	0.8545	0.8010	0.4735		
Angle	±0.0352π	±0.0339π	±0.0036π	±0.1746π		
Cycle length	56.8	59.0	552.8	11.5		
Panel III: Year-on-Year						
Roots	0.96±0.11i	0.92±0.10i	0.9741	0.7934	0.50964	-0.3746
Norm	0.9655	0.9270	0.9741	0.7934	0.50964	0.3746
Angle	±0.0376π	±0.0361π	0	0	0	0
Cycle length	53.2	55.4	+inf	+inf	+inf	+inf

The roots in the second column, at first glance, may seem to indicate one identical cycle, but the point estimates of the phase shifts differ considerably among

[25]See Sims (1986), Leeper et al. (1996), Leeper et al. (2003) and Kim (1999), among others, for the opposing view.

Table 1.5: Estimated phase shifts (six-variable VARs with Japanese data).

unit: months	(Cycle length)	Shipment	Inventories	O/N call	CPI	Com. Price
Level data	(63.2)	0.2828	11.655	3.2501	-13.619	-6.8463
	(43.4)	2.4008	6.4567	4.7944	9.3550	-3.3411
HP-s.a.	(56.8)	-0.0434	12.660	4.4041	-13.348	-2.4408
	(59.0)	7.1328	8.5986	7.2003	11.619	6.9404
YoY	(53.2)	-0.0095	12.824	1.9726	12.267	-4.1758
	(55.4)	0.0013	11.386	-8.7080	0.5821	11.306

Note: Time-lags from production.

the three data sets. On the other hand, the phase shifts of the largest norm roots are consistent among the three data sets (except for CPI in the YoY estimation), and are compatible with those in the three-variable estimations. In addition, none of the other roots is robust against a change in the VAR order. Overall, it is concluded that there exists one business cycle pair of complex roots (perhaps the same cycle as in the three-variable estimations) in the six-variable estimations. This conclusion is also supported by the cross correlation and spectrum analysis below.

Compared to the three-variable estimations, the cycle length (63.2 months) now becomes longer in the level data estimation, while it becomes shorter in the YoY data estimation.

Phase Shifts

The O/N call rate lags behind production by 3.3 months in the level data, suggesting that the BoJ reacts to real variables fairly quickly.[26] However, it is not forward-looking; perhaps good monetary policy would anticipate the cyclical patterns of economic variables, given the long time lag before the effects of monetary policy are realised (shown below). Indeed, it seems that the Fed's monetary policy anticipates such cyclical patterns (see the Appendix).

[26] It is important to note that phase shifts do not indicate the speed of responses to *shocks*. Instead, for example, we can interpret the phase shift between the O/N call rate and an endogenous variable as a speed of the BoJ's response to the *cyclical component* of that endogenous variable.

1.4.3 Impulse Response Functions

One caveat of the six-variable analysis is the price puzzle.[27] In other respects, however, the estimation results are consistent with theoretical predictions.

Supply vs. Demand Shocks: Monetary policy is tightened after a positive shipment (demand) shock (Figure 1.12), while its response to a positive production (supply) shock is ambiguous (Figure 1.11). Indeed, following a positive supply shock, although the response is not estimated tightly, the point estimates of the all three IRFs show that the BoJ loosens its monetary policy. Considering the behaviours of other IRFs, this is because (i) a boom lasts longer after a positive demand shock than after a positive supply shock, and (ii) the leading inflation indicator and CPI increase after a positive demand shock but not after a positive supply shock. Hence, it is important to discriminate between demand and supply shocks, in order to analyse monetary policy.

Inventory Shock: A positive deviation of inventories from the steady state is akin to a negative demand shock (Figure 1.13). As a result, the O/N call rate declines after a positive inventory shock.

Price Shocks: The O/N call rate increases after a positive material price shock, but decreases after a positive CPI shock. These patterns seem to reflect the features of the BoJ's monetary policy.

On one hand, after a positive CPI shock, both the O/N call rate and production decline possibly because the major CPI shocks tend to arise from increases in public prices and energy prices in Japan.[28] In other words, large CPI shocks are often regarded as exogenous negative shocks; indeed, production and shipment decline after a positive CPI shock.

[27] See Sugihara et al. (2000), Teruyama (2001) and Yoshikawa et al. (1993). Almost all versions in these studies show temporal price increases after a tight monetary policy shock in Japanese data.

[28] The effects of the changes in VAT rate on CPI are adjusted in our data.

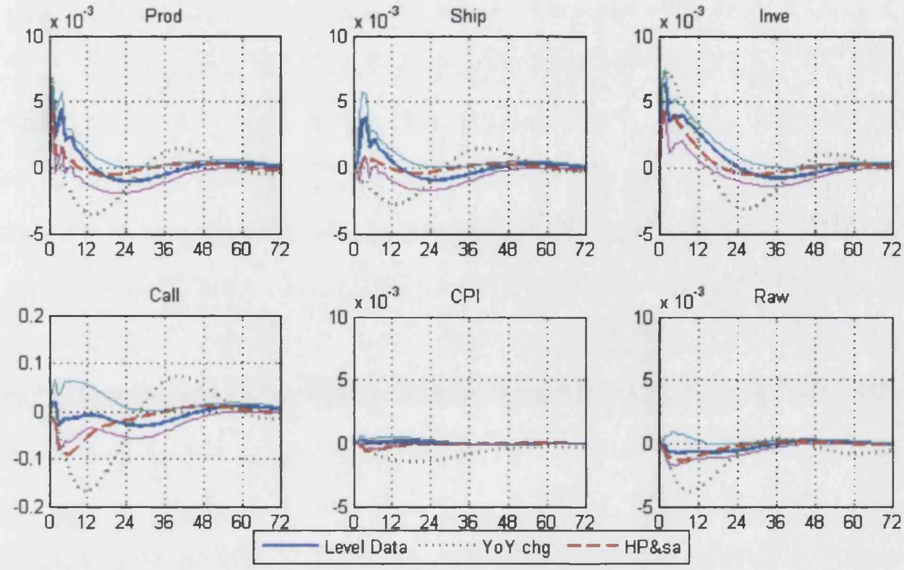

Figure 1.11: IRFs due to a positive shock in the production equation (six-variable VARs with Japanese data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

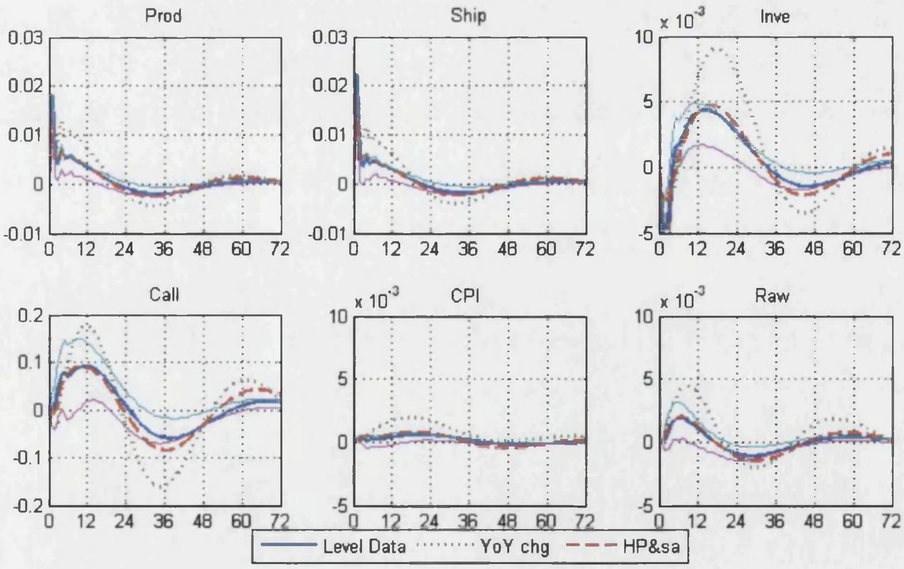

Figure 1.12: IRFs due to a positive shock in the shipment equation. Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

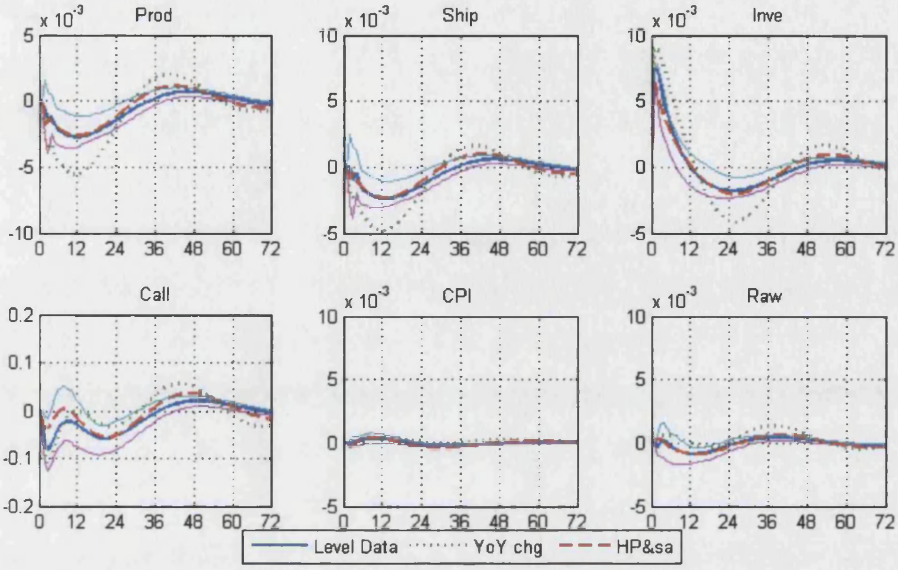

Figure 1.13: IRFs due to a positive shock in the inventory equation (six-variable VARs with Japanese data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

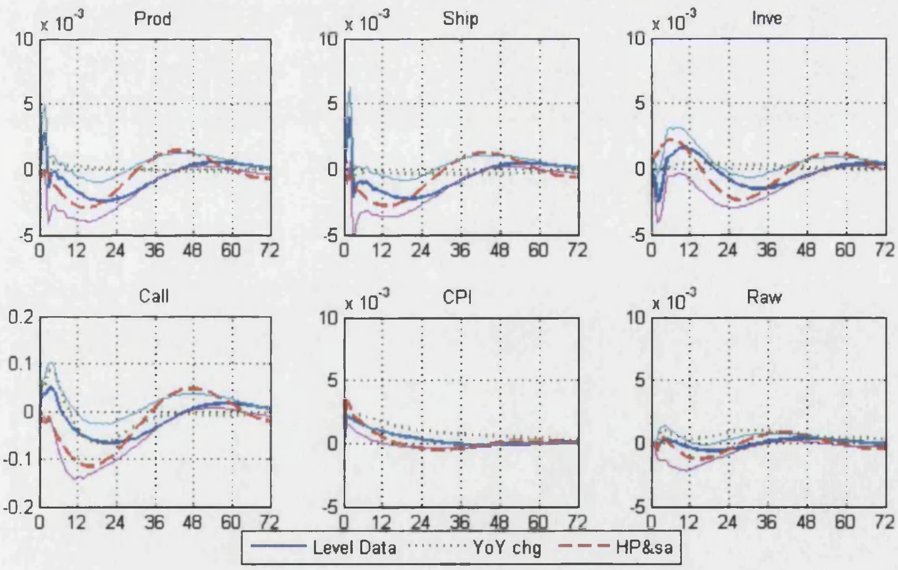

Figure 1.14: IRFs due to a positive shock in the CPI equation (six-variable VARs with Japanese data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

On the other hand, the BoJ tends to focus on leading inflation indicators, while CPI is often considered as a lagging indicator. Moreover, the BoJ traditionally has been concerned with the exchange rate. Because exports are the growth engine of the Japanese economy (though this situation is changing), a strong yen, which reduces the exporters' profit margins and competitiveness, has been considered something that the central bank has to defeat. Hence, the BoJ's reaction to the leading inflation indicator may represent its reaction to the exchange rate; a strong yen implies low import prices (especially on raw materials), and is followed by an expansionary monetary policy.

Call Rate Shock: The effects of O/N call rate shocks (monetary policy shocks) on production, shipment, and inventories are unclear and mixed. In the level data estimation, production and shipment decline several periods after a positive call rate shock, although they decline right after the shock in the HP-s.a. and YoY data. Existing studies find a long time lag before the effects of monetary policy materialise.[29]

Bils and Kahn (2000) find that the inventory investment is positively correlated to the interest rate; this is considered a puzzle because a high interest rate gives rise to a high inventory carry cost. There is one natural way to address this puzzle; if demand decreases sharply while production cannot adjust quickly, firms are "forced" to accumulate inventories due to the law of motion of inventories (1.2). However, VAR estimations show no substantial differences between the IRFs of production and shipment.

1.4.4 Spectral Analysis and Cross Correlations

Cross correlations and spectra confirm the findings discussed above. First, the cross correlation between the O/N rate and production/shipment (and material price) reaches the peak with a 2 to 4 months lag. This is consistent with the es-

[29] See Bernanke and Gertler (1995) and Christiano et al. (1999) among others.

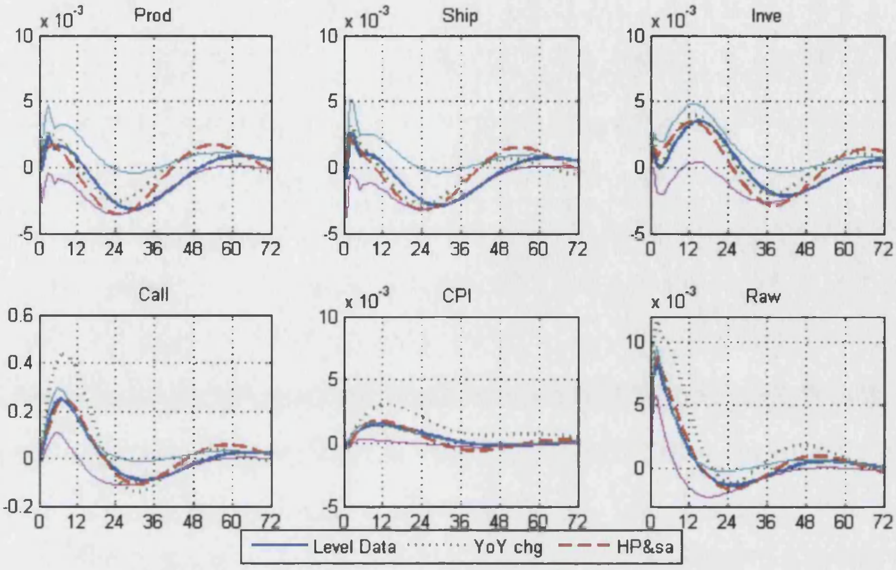

Figure 1.15: IRFs due to a positive shock in the leading inflation indicator equation. Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

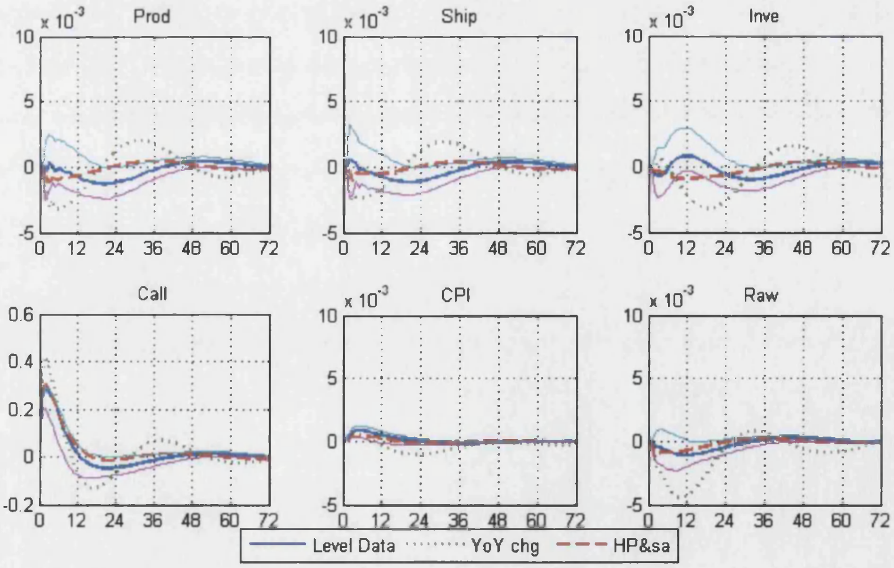

Figure 1.16: IRFs due to a positive shock in the O/N call rate equation (six-variable VARs with Japanese data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

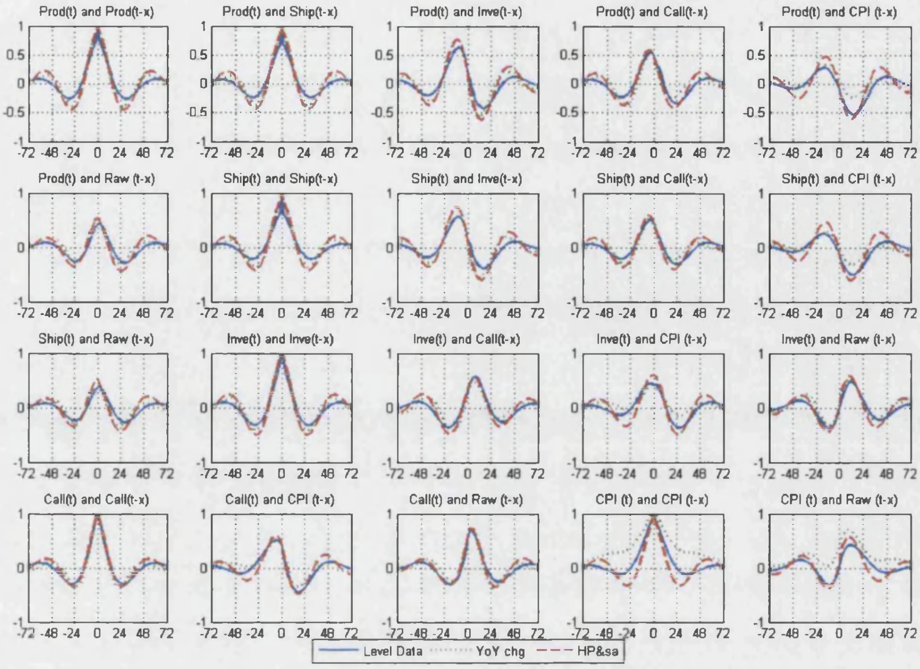

Figure 1.17: Cross correlations (six-variable VARs with Japanese data).

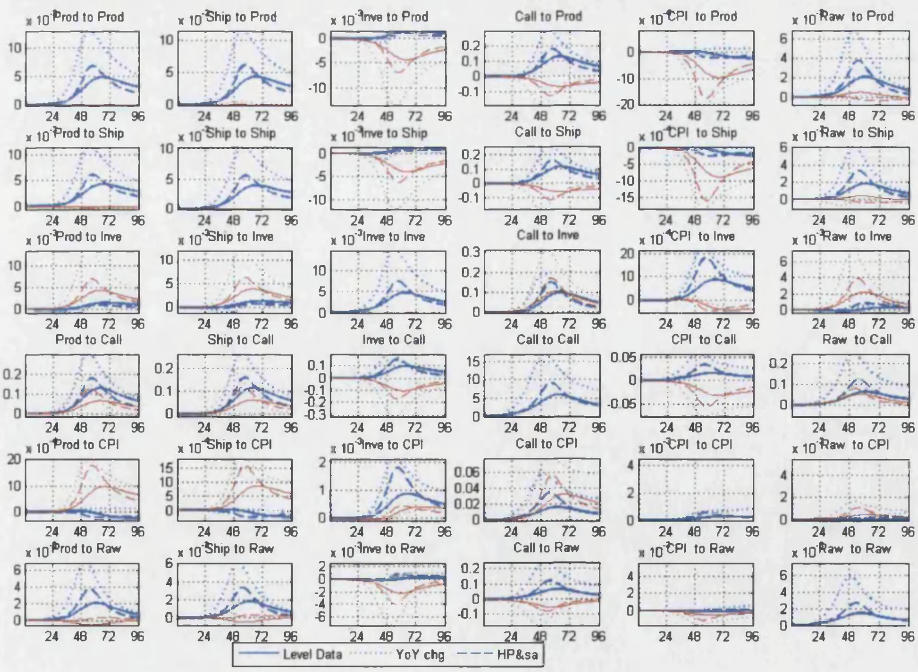

Figure 1.18: Co- and quadrature spectra (six-variable VARs with Japanese data). Bold lines show cospectra and narrow lines show quadrature spectra.

timated phase shifts between them. Second, as quadrature spectra suggest, there are dynamic relationships between the O/N rate and other variables that are not reflected in the contemporaneous correlations. Finally, most of the co- and quadrature spectra reach their peaks or bottoms at around 53 to 63 months, implying that the cycle with a period length of 53 to 63 months is the most important cyclical factor.

1.4.5 Summary of Six-Variable VAR

The most important discoveries in the six-variable estimations are that (i) the BoJ reacts to demand shocks, but its reaction to supply shocks is vague, and (ii) it reacts to shocks within a quarter, which is reasonably quick, but its monetary policy does not seem to be forward-looking.

1.5 Discussion

This section discusses some miscellaneous albeit important issues.

1.5.1 Modern vs. Old Thoughts

The VAR estimations in this chapter shed light on some old thoughts regarding business cycles. By the early 20th century, Kitchin (1923), Juglar (1860), Kuznets (1930), and Kondrachieff (1935) found cycles of roughly of 3.4, 10, 20, and 50 years, respectively.[30] Later, Schumpeter (1939) excavated and sorted out their

[30] A summary of the major old thoughts is as follows.

Name	Period (yrs)	Main Driving Force
Kitchin Cycle	3.4	Inventories
Juglar Cycle	10	Investment
Kuznets Cycle	20	Construction
Kondratieff Cycle	50	Technological Revolution

Consider the implications of these numbers. First, in terms of the cycles lengths, most of them are integer multiples of the shorter ones. This implies that observed cycles are not completely distinguishable from one another. For example, three Kitchin cycles could be misidentified as one Juglar cycle.

Second, the main driving forces in the table are provided by later analyses. For example, the data used by Kitchin are bank clearings, commodity prices, and interest rates, whereas findings (excluding Kuznets (1930)), and Burns and Mitchell (1946) conducted a comprehensive study of business cycles. Of particular importance, it seems that most of these old studies presupposed a damping oscillation, or perhaps a limit cycle, though not explicitly.

On the other hand, the old views contrast with some modern views. For example, Prescott (1986) points out that the term *business cycle* is inaccurate. He suggests instead the concept of *business cycle phenomena,* because "some systems of low-order linear stochastic difference equations with a nonoscillatory deterministic part, and therefore no cycle, display key business cycle features."[31]

Essentially, Prescott's business cycle phenomena are successive exponential decays: successive deviations of variables from their steady states and their returning processes. In contrast, this chapter shows substantial evidence of periodicity, which means that variables follow sine curves.[32]

In contrast to exponential decays, damping oscillations (stable sine curves) imply that a boom is the seed of the following recession, which, in turn, is the seed of the next boom. In this sense, the cycles in this chapter have a meaning closer to those of the old studies. Specifically, the cycles reported in this chapter seem to correspond to those found by Kitchen. Indeed, the length of the Kitchen cycle is close to our estimations, and researchers consider inventories to be the driving force behind the Kitchen cycle (see Knetsch (2004), for example).

Kondratieff uses wholesale prices, interest rates and wages, foreign trade and the production of some metals. Hence, Kitchin himself supposedly did not recognise his finding as an inventory cycle.

Third, all of these cycles are empirical findings with little theoretical background, and their empirical techniques may not be defensible by modern standards. Indeed, Harvey (1993, pp.195-196) demonstrates that the moving average that Kuznets uses generates spurious cycles. Hence, it should be understood that the existence of these cycles has not yet been confirmed econometrically.

[31] Prescott (1986), p.10.

[32] See Hassler et al. (1992) for a related discussion.

1.5.2 Level of Inventories vs. Inventory Investment

This subsection shows the risk of considering inventory investment as a proxy for the level of inventories, especially when complex roots have a dominant influence.

An Implication for Contemporaneous Correlations

Suppose that both production Y_t and the *level* of inventories U_t follow sine curves. Ignoring the effect of other terms, if the phase shift s between them is roughly $\pi/2$, then their correlation is almost zero.

$$\begin{aligned} Y_t &= A_Y r^t \sin(\theta t) + \textit{other terms} \\ U_t &= A_U r^t \sin(\theta t - s) + \textit{other terms} \simeq -A_U r^t \cos(\theta t) + \textit{other terms} \end{aligned}$$

where A_Y and A_U are coefficients and r is the norm of the relevant complex roots. Because inventory investment is a time difference of the level of inventories, by approximating such difference by time derivative,

$$U_t - U_{t-1} \simeq \frac{\partial U_t}{\partial t} = A_U r^t \sin(\theta t) - (\log r)\, A_U r^t \cos(\theta t) + \textit{other terms}$$

Note that $\log r \simeq r-1$ is a small negative number, because most economic variables are persistent in the data (i.e., the norm of roots is less than but close to 1), and hence the effect of the second term is very small. Thus, in terms of frequency θ, both production and inventory investment are governed by the same term $r^t \sin(\theta t)$. In sum, a zero correlation between production and the *level of inventories* implies a positive correlation between production and *inventory investment.*

This is one of the salient features of complex roots. Suppose that both output and the level of inventories follow exponential decays. In this case, if the correlation between production and the level of inventories is close to zero, then that between production and inventory investment is also close to zero. Indeed, we observe both a near-zero correlation between output and inventories and a positive correlation between production and inventory investment, which is (indirect) evidence of periodicity in inventories.

1.5.3 Pseudo Propagation

This chapter has shown many hump-shaped IRFs. If individual IRFs are examined separately, then hump-shaped IRFs may appear to suggest the existence of some mechanism that magnifies initial shocks. However, if variables are examined jointly, the locus monotonically converges to the steady state under a proper metric. For example, in the phase diagrams (Figures 1.1 and 1.2), which jointly examine production/shipment and inventories, the distance from the origin (steady state) is monotonically shrinking. In this regard, it is difficult to interpret the hump-shaped IRFs in this chapter as evidence of a propagation mechanism, if the word "propagation" means that the initial small shock is gradually magnified by an endogenous mechanism.

Intuition

Consider the target inventory model as an example.[33] It implies that the desired level of inventories is an increasing function of sales (or production). Thus, if the level of inventories becomes too low, firms have an incentive to increase their production to replenish inventories. In this class of models, either an increase in sales (demand) or a decrease in inventories can stimulate production. In this sense, it may be reasonable to consider sales and inventory shortages jointly as, say, an "effective demand." It is quite possible that a positive deviation of sales from the steady state individually appears to grow after a positive demand shock (hump-shaped IRF), but the effective demand is monotonically shrinking (stable spiral in the phase plane) if inventories are concurrently increasing.[34]

[33] The target inventory model is a class of models that includes the stockout avoidance model in the theoretical literature and the linear quadratic specification in empirical research.

[34] Mathematically, in linear models, it is possible for a small initial shock to grow only in explosive systems.

1.5.4 Inventory Stylised Facts

One of the important findings in the VAR analyses is that production and shipment (sales) move very closely with each other. This itself may not sound interesting, but it has a strong implication; the following two inventory stylised facts actually describe a single fact from two different perspectives.

Fact 1: Inventory investment is procyclical.

Fact 2: Output is more volatile than sales.

To see this, consider the law of motion of inventories (1.2). Then, it is quite straightforward to show

$$Var\,(S_t) = Var\,(Y_t) + Var\,(U_t - U_{t-1}) - 2Cov\,(Y_t, U_{t+1} - U_t)$$

which means that $Cov\,(Y_t, U_{t+1} - U_t) > 0$ is a *necessary* condition of $Var\,(S_t) < Var\,(Y_t)$. With a similar manipulation, we can easily show that $Cov\,(S_t, U_t - U_{t-1}) > 0$ is a *sufficient* condition of $Var\,(S_t) < Var\,(Y_t)$. In sum,

- If the word *procyclical* in Fact 1 means $Cov\,(Y_t, U_t - U_{t-1}) > 0$, then Fact 1 is a necessary condition for the Fact 2.
- If *procyclical* in Fact 1 means $Cov\,(S_t, U_t - U_{t-1}) > 0$, Fact 1 is a sufficient condition for Fact 2.
- In data, sales and production move very closely with one another. Thus, $Cov\,(Y_t, U_t - U_{t-1}) > 0$ and $Cov\,(S_t, U_t - U_{t-1}) > 0$ are nearly interchangeable.

Therefore, roughly speaking, stylised Fact 1 is a necessary and sufficient condition of stylised Fact 2.[35]

[35]Note that output Y_t here is defined as gross output, not value-added.

1.6 Conclusion

To study inventory cycles (see Figures 1.1 and 1.2), VAR estimations (equation (1.1)) are conducted in this chapter. The main findings are as follows.

1. The length of the detected cycle is relatively close to the average length of the post-war business cycles, and the existence of business cycle complex roots is statistically significant.

2. Inventories seem to work as buffers at very high frequencies, while they seem to destabilise an economy at business cycle frequencies.

3. The estimated phase shift between inventories and production/shipment is 12 months. Hence, for example, if inventories have bottomed out, then production will peak around 16 months later. Inventories are very informative for predicting near-future economic conditions.

4. Contemporaneous correlations are not enough to capture the dynamic relationship among variables. This is especially true for inventories and, to a lesser extent, the policy interest rate.

5. Due to the behaviour of inventories, a boom lasts longer after a positive demand shock than after a positive supply shock.[36]

6. As a result, the BoJ tightens its monetary policy after positive demand shocks, while it does not clearly react to supply shocks.

7. The BoJ's monetary policy is timely but not forward-looking.

Perhaps the most critical weakness of this research is that the detected cycle length is sensitive to the time trend in the level data estimation. However, the fact that the HP-s.a. and YoY data sets find the similar results suggests the robustness of the estimations.

[36]Among these seven findings, this and the following two observations do not hold for the U.S. data. See the Appendix.

For the sake of argument, let us consider the worst-case scenario. Certainly, the chosen time trend, the 5th-order polynomial, mimics the HP-filter, which, in turn, has similar effects to the YoY change. Hence, it is possible that the detected cycle is just an artefact generated by the HP-filter $\simeq$ YoY $\simeq$ 5th-order polynomial time trend. Nevertheless, even in this worst-case scenario, we can still claim that, given the popular use of the HP-filter and YoY change, the detected cycle may be something that practitioners consider to be a business cycle (even if it may not be a "true" business cycle).

In terms of estimation technique, each of the three data sets is subject to its own problem. However, the estimated results among these three data sets are similar to each other, and, in addition, most of the estimates are very precise. Perhaps, then, it is safe to claim that the estimates are not considerably distorted.

The key reason for the successful estimations is the quality of inventory level data. Most theories suggest that the *level of inventories* plays a major role, but almost all existing empirical studies are based on *inventory investment*. In general, the quality of inventory level data is poor, but Japan is one of the few exceptions to this. Considering that practitioners pay close attention to inventory behaviour, it is advisable for other governments to construct reliable inventory level data, thereby providing useful information about near-future economic conditions.

Appendices for Chapter 1

1.A Six-Variable VAR with U.S. Data

This section describes the estimation results of the six-variable VAR with the U.S. data. The main problems with the U.S. data are (a) the pool of surveyed firms and survey methods are perhaps different between production and shipment/inventories because they are provided by different institutions, (b) the quality of real inventory data is not very good, and (c) data of real inventory before seasonal adjustment is not available.

Compared to Japanese data, the estimations with the U.S. data are less precise. In addition, a couple of IRFs are not consistent among the (i) level, (ii) HP-s.a. and (iii) YoY data sets. Hence, it seems that the results based on the U.S. data are less reliable than those based on the Japanese data.

Nonetheless, we find that (1) one pair of complex roots exists, and the implied cycle length is fairly close to the post-WWII average, (2) inventories lag behind production/shipment by 1/5 to 1/6 of the business cycle length, and (3) the Fed reacts to supply shocks less sharply than to demand shocks. However, unlike the estimations for Japan, the last finding is not very clear. In addition, the lifespans of booms due to a positive demand and supply shocks are almost the same in the U.S. estimation, and the behaviours of inventories are not very different in response to those two types of shocks.

1.A.1 Description of Details

Most of the details are the same as those for the Japanese data sets. Hence, this subsection mainly explains the differences from the estimations made with Japanese data.

Original Data All data are monthly data from January 1978 to December 1998. Although more data are available for the United States, the same period used in the Japanese estimations is used here for the sake of comparison (expanding the data period makes the estimation more precise, but only slightly). Although production data are compiled by the Board of Governors of the Federal Reserve System,[37] real shipment and inventory data are estimated by the U.S. Bureau of Economic Analysis.[38] The latter are, as building blocks, compiled to estimate U.S. national income (GDP), and "their quality is significantly less than that of the higher level aggregates," according to the Bureau. Shipment and inventories are of "manufacturing" (not including trading sectors) for comparison. As a monetary policy indicator, the effective monthly Fed funds rate (FF rate) is used.[39] Inflation is measured by the Consumer Price Index for All Urban Consumers (CPI-U) excluding food and energy, while PPI (raw materials) is used as a leading inflation indicator.[40]

Data Formats Again, there are three data sets: (i) level, (ii) HP-s.a. and (iii) YoY data. All estimations are based on equation (1.1) with order 3. The estimation with the level data uses the 5th-order time trend without seasonal dummies because only seasonally adjusted real shipment and inventories are available. For simplicity, seasonally adjusted CPI-U is used for all three data sets, while not seasonally adjusted FF rate and PPI (raw materials) are used because they are not considered to have seasonality.

Unit Root For the three-variable VAR with the level data, Monte Carlo experiments again suggest that there exists one real (not complex) unit root in the U.S. data set (the results are omitted). The results based on the stationary data sets

[37] U.S. production data are available at http://www.federalreserve.gov/releases/G17/

[38] Shipment and inventory data in nominal terms are available from the U.S. Census Bureau: http://www.census.gov/indicator/www/m3/hist/naicshist.htm

For the estimations of real shipment and inventories, see Herman et al. (1976). For data, see the website of the Bureau of Economic Analysis:

http://www.bea.gov/national/nipaweb/nipa_underlying/SelectTable.asp

[39] See the Fed's website: http://www.federalreserve.gov/Releases/H15/data.htm

[40] Both are available at http://www.bls.gov/home.htm

Table 1.6: Estimated business cycle roots (six-variable VARs with U.S. data).

Panel I: Level

Roots	0.93±0.09i	0.88±0.06i	0.73±0.08i	0.43±0.37i	0.8051	-0.5162
Norm	0.9374	0.8870	0.7307	0.5622	0.8051	0.5162
Angle	±0.0291π	±0.0219π	±0.0365π	0.2261π	0	0
Cycle length	68.52	91.15	54.83	8.85	+inf	+inf

Panel II: HP-s.a.

Roots	0.95±0.11i	0.77±0.03i	0.74±0.11i	0.43±0.30i	0.9511	-0.4983
Norm	0.9443	0.7667	0.7454	0.5221	0.9511	0.4983
Angle	±0.0363π	±0.0109π	±0.0455π	±0.1970π	0	0
Cycle length	55.12	182.80	43.93	10.15	+inf	+inf

Panel III: YoY

Roots	0.93±0.13i	0.9084	0.8128	0.7677	0.7323	0.52±0.39i	0.9855	-0.5763
Norm	0.9376	0.9084	0.8128	0.7677	0.7323	0.6445	0.9855	0.5763
Angle	±0.0428π	0	0	0	0	±0.2047π	0	0
Cycle length	46.73	+inf	+inf	+inf	+inf	9.77	+inf	+inf

(HP-s.a. and YoY data) are relatively similar to those based on the level data, though such similarities are not as strong as in the Japanese estimations.

1.A.2 Roots of Coefficient Matrix

Selected point estimates of the roots are shown in Table 1.6. Roots omitted from the table are complex roots with very high frequencies (shorter than 8 months) and some short real roots.

There are many conjugate pairs of complex roots that correspond to long cycles, but only the first pair in each panel seems to be robust against a change in the VAR order. For this cycle, phase shifts are consistent among all three data sets. In addition, cross correlations and spectra also show that the dominant cycle is 47 to 69 months in length, which is close to the post-war average (67 months).[41]

Phase Shifts

The phase shift between production and inventories is 1/5 to 1/6 of the cycle length, implying that the trajectory of the inventory cycle is a (shrinking) ellipse with a major (longer) axis running from the northeast to the southwest around

[41] See NBER's "U.S. Business Cycle Expansions and Contractions" at http://nber.nber.org/cycles/cyclesmain.html

Table 1.7: Estimated phase shifts (six-variable VARs with U.S. data).

unit: months	(Cycle length)	Shipment	Inventories	FF rate	CPI-U	Com. Price
Level data	(68.5)	1.9612	12.748	-8.0767	-0.6753	1.7430
	(91.2)	2.9152	19.939	16.323	-16.551	22.389
	(54.8)	-4.5560	1.5431	0.6742	-0.3231	-3.0252
HP-s.a.	(55.1)	2.1547	10.436	-4.0955	1.3363	2.7509
	(182.8)	0.5981	5.2254	39.217	5.5583	18.872
	(43.9)	-3.3573	2.9764	-0.2556	-1.0580	-4.0754
YoY	(46.7)	1.9730	7.4541	-3.6056	-2.9659	0.7352

Note: Time-lags from production.

the origin (see Figure 1.2).

The FF rate precedes production by 4 to 8 months. It seems that the Fed's monetary policy is forward-looking/pre-emptive; it anticipates the cyclical patterns of economic variables.

1.A.3 Impulse Response Functions

As with the estimation for Japan, there exists a somewhat perverse price puzzle. In addition, the estimated IRFs have a wide confidence interval (especially for the FF rate and prices).

Supply vs. Demand Shocks: Monetary policy is tightened after both positive demand and supply shocks (Figures 1.20 and 1.19). However, the Fed raises the FF rate much more sharply in response to a demand shock than a supply shock, because the leading inflation indicator increases after a demand shock but decreases after a supply shock. In addition, the initial effect of a demand shock is stronger than that of a supply shock.

Unlike Japanese estimations, the lifespans of booms do not differ between demand and supply shocks. The author's conjecture is that this is because of differences between the surveyed firms in production and shipment/inventories statistics. For example, if a firm's figures are included in production statistics but not in shipment statistics, then the demand shock that hits that firm increases production but not shipment. In any event, the U.S. estimations are less precise, and thus it might be safer not to draw too many conclusions from them.

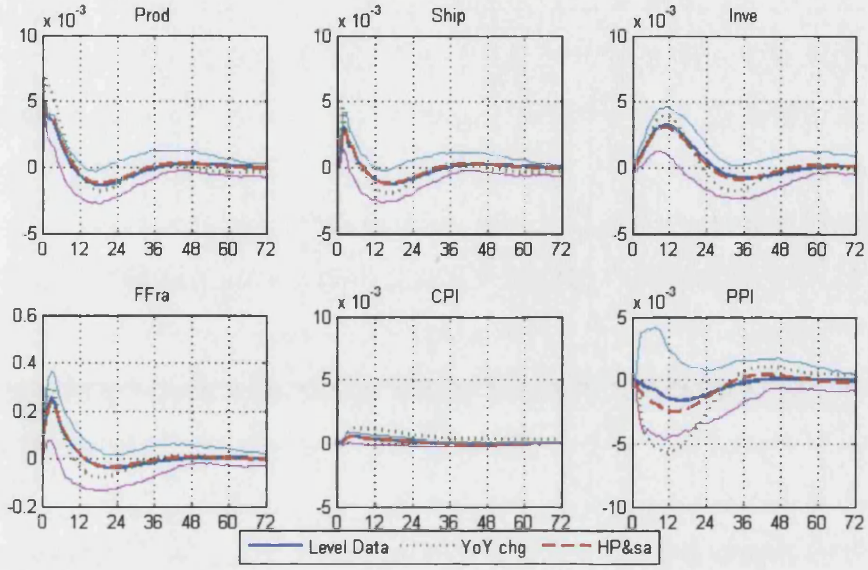

Figure 1.19: IRFs due to a positive shock in the production equation (six-variable VARs with U.S. data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

Price Shocks: The IRFs to shocks to CPI and PPI raw materials are similar to each other, but the latter, a leading inflation indicator, has stronger effects than the former. It seems that the central banks react to leading inflation indicators but not to CPI both in Japan and in the United States.

Fed Funds Rate Shock: Again, the price puzzle arises; after a positive FF rate shock, CPI rises (Figure 1.23). Though the confidence interval is very wide, inventories also increase after a positive FF rate shock. This could be because firms cannot cut their production quickly enough to counterbalance the decline in demand, but this is difficult to verify because data are collected from different pools of sampled firms.

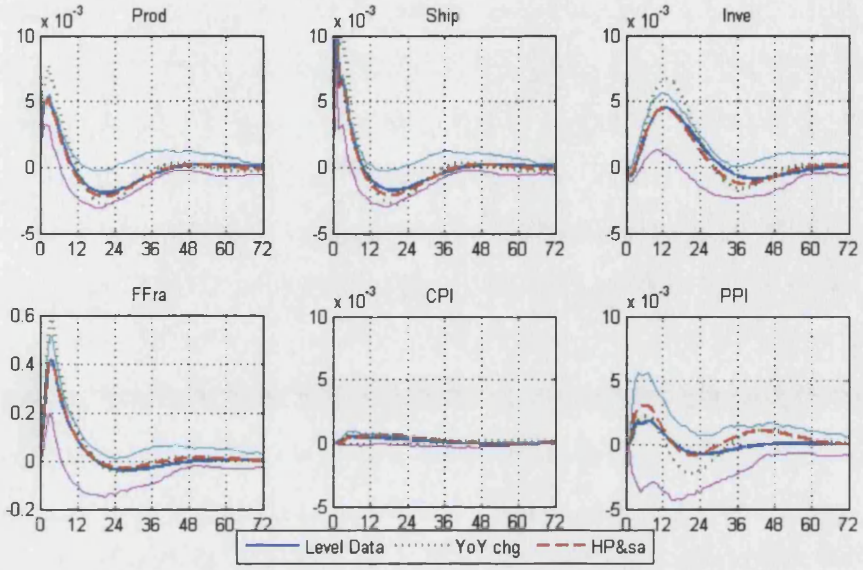

Figure 1.20: IRFs due to a positive shock in the shipment equation. Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

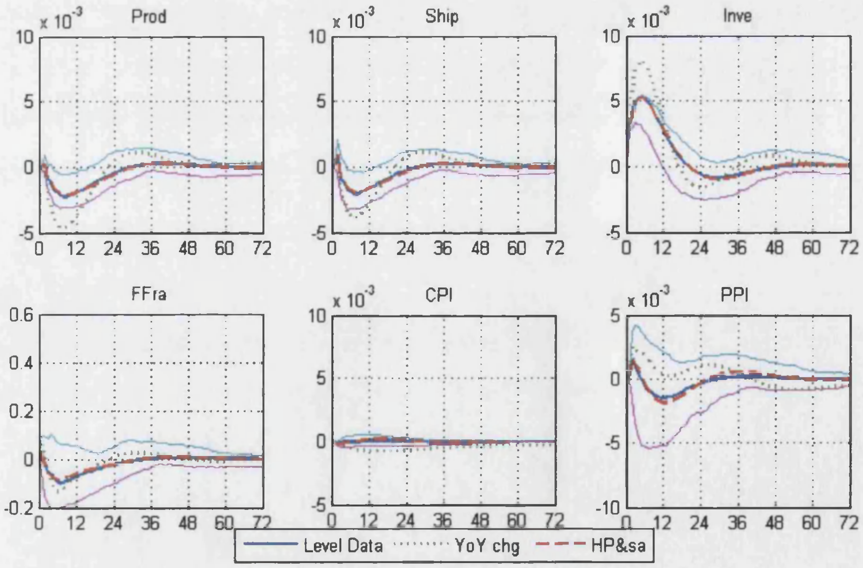

Figure 1.21: IRFs due to a positive shock in the inventory equation (six-variable VARs with U.S. data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

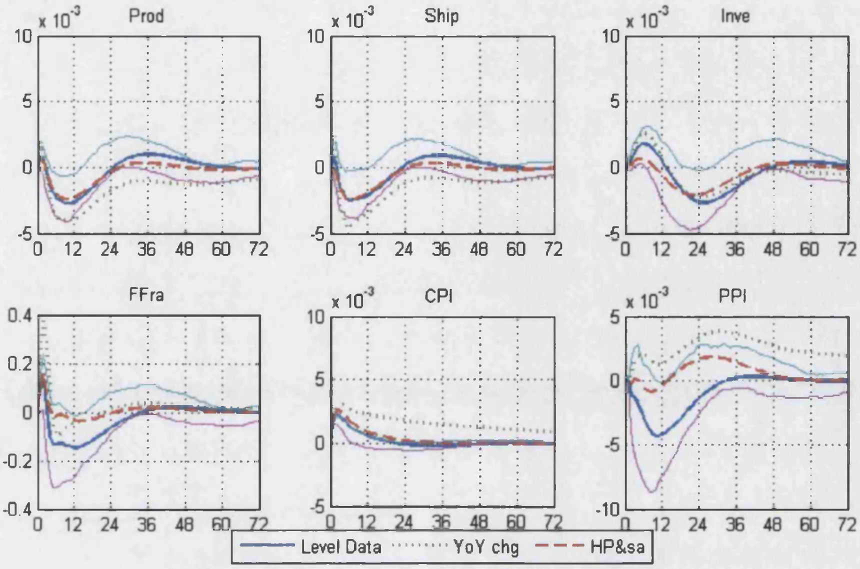

Figure 1.22: IRFs due to a positive shock in the CPI equation. Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

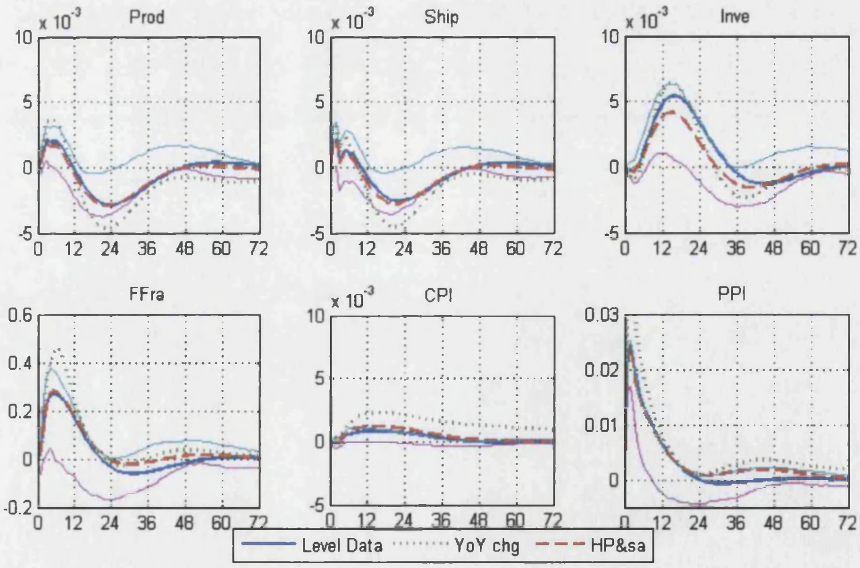

Figure 1.23: IRFs due to a positive shock in the leading inflation indicator equation. Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

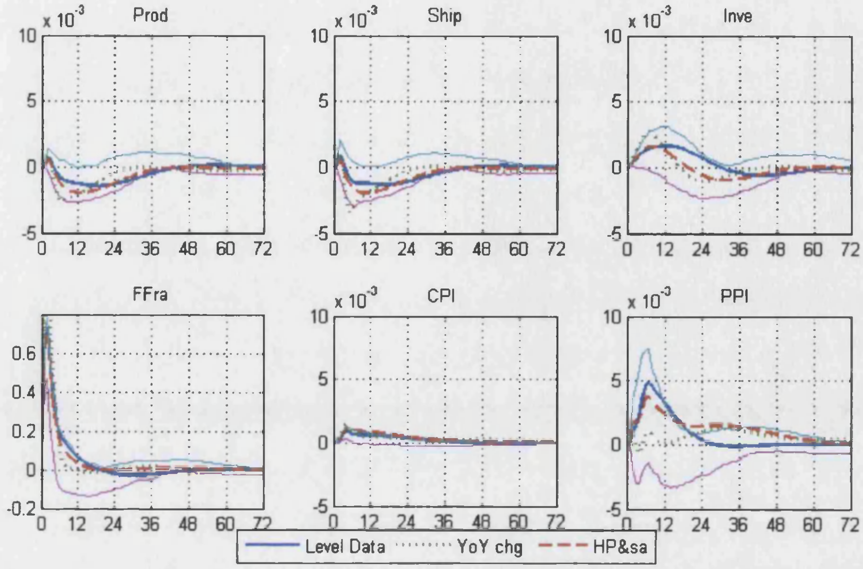

Figure 1.24: IRFs due to a positive shock in the FF rate equation (six-variable VARs with U.S. data). Narrow lines show the 95% confidence intervals of level data estimations based on the bootstrapping method.

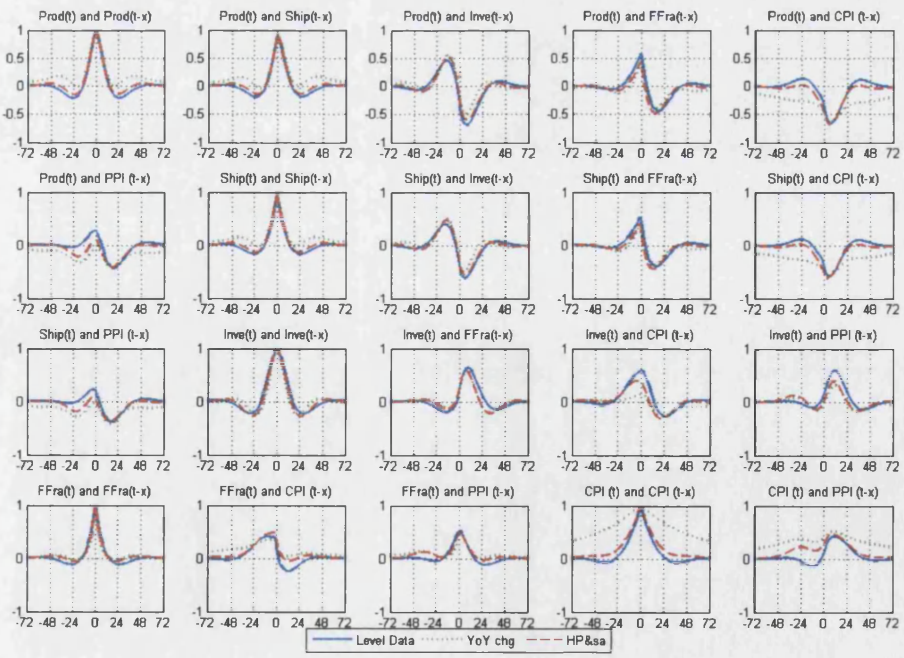

Figure 1.25: Cross correlations (six-variable VARs with U.S. data).

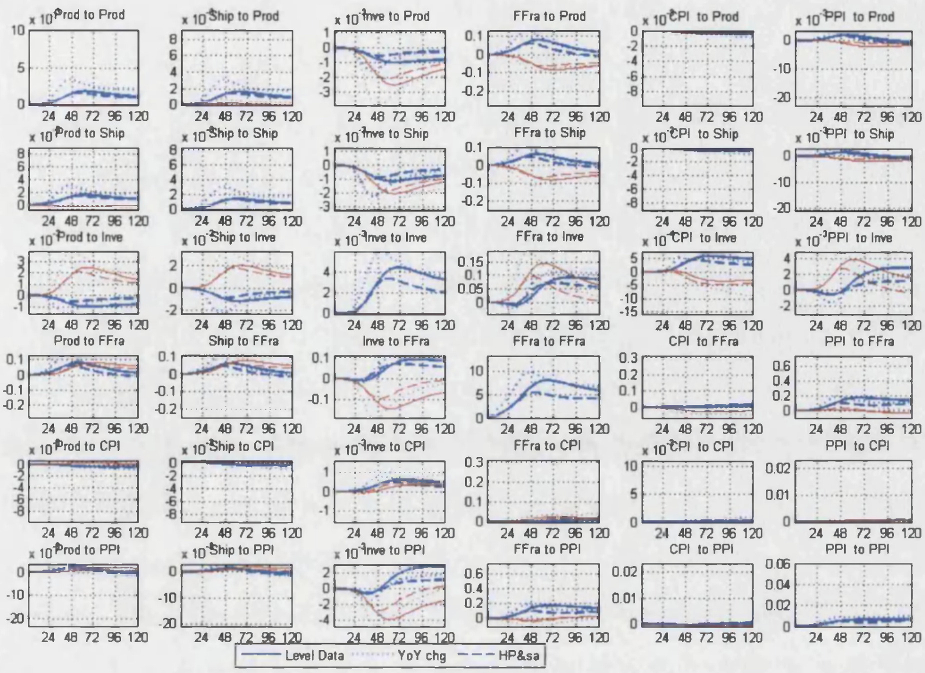

Figure 1.26: Co- and quadrature spectra (six-variable VARs with U.S. data). Bold lines show cospectra and narrow lines show quadrature spectra.

1.A.4 Spectral Analysis and Cross Correlations

Like Japanese data, U.S. data also show the S-shape cross correlations between inventories and other variables, which shows the existence of time lags between them. The correlations between production/shipment and the FF rate peak around 0 to -2 months, showing that the Fed reacts to these variables with a short time lag, which may seem to be inconsistent with the finding in the phase shift between them (see Section 1.A.2). However, this is because of very high frequency components; by definition, the Fed cannot react to *iid* shocks in advance. Remember that the phase shift between production and the FF rate shows the Fed's reaction to the cyclical component of, but not to shocks to, production, but the cross correlation between them reflects the Fed's reaction to both the cyclical component and shocks. On the other hand, the correlations between production/shipment and the FF rate reach their bottom at around 15 to 20 months, which shows that it takes more than one year for the effect of monetary policy to fully materialise.

The spectra show that the quadrature spectrum plays a major role mainly with inventories (Figure 1.26). Most of the spectra of CPI and PPI raw materials with other variables have a sharp spike at 0 month (making it difficult to distinguish them from the y-axis), which means that their behaviour is dominated by shocks, with weak cyclical linkages with other variables. Also note that most of the spectra have their peak or bottom at around 60 months, which means that the cyclical component with a 60 months long is a key driving factor in the business cycle. The quadrature spectra of the FF rate with other variables have their peak or bottom at business cycle frequencies, showing that contemporaneous covariances are not sufficient to evaluate the Fed's monetary policy.

1.B Computation of Phase Shifts

The mathematical techniques used in this chapter are found in any elementary textbook (hence, most of the derivations are omitted). However, this section briefly describes how to compute phase shifts in a given system of difference equations.[42] It may be useful to some readers since the author personally experienced some difficulty in finding references for the computation of phase shifts.

1.B.1 Computational Summary

Suppose that we have obtained a VAR(M) estimation without exogenous variables (see equation (1.1)). Then, it can be rewritten in the form of VAR(1) by redefining

[42]Note that the phase shifts in this chapter are computed by a different algorithm discussed in Chapter 10 of Hamilton (1994).

the vector of endogenous variables.

$$
\begin{aligned}
Y_t &= Y_{t-1}\mathbf{B} + \xi_t \mathbf{C} \qquad (1.3)\\
\mathbf{B} &\equiv \begin{bmatrix} B_1 & \cdots & B_{M-1} & B_M \\ I & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & I & 0 \end{bmatrix}, \qquad \begin{aligned} \mathbf{C} &\equiv \begin{bmatrix} C & 0 & \cdots & 0 \end{bmatrix}, \\ Y_t &\equiv \begin{bmatrix} y_t & \cdots & y_{t-M+1} \end{bmatrix} \end{aligned}
\end{aligned}
$$

where y_t and ξ_t are row vectors of endogenous and exogenous variables, respectively. ξ_t is assumed to be *iid* over time and equations.

Let Λ and V be the matrices of eigenvalues and eigenvectors of $\mathbf{B}$ in (1.3), respectively.

$$
\Lambda = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}, \quad V = \begin{bmatrix} V_1 & \cdots & V_n \end{bmatrix} \qquad (1.4)
$$

where n is the number of roots ($M \times$ number of endogenous variables) and V_j is the eigenvector that corresponds to the j-th eigenvalue. Then,

- Frequencies (θ_j): $\Theta = diag\begin{bmatrix} \theta_1 & \cdots & \theta_n \end{bmatrix} = \arctan(\operatorname{Im}\Lambda./\operatorname{Re}\Lambda)$
- Cycle lengths ($2\pi/\theta_j$): $2\pi./\Theta = diag\begin{bmatrix} 2\pi/\theta_1 & \cdots & 2\pi/\theta_n \end{bmatrix}$
- Phase (β_{lj}): $\Phi = \arctan(\operatorname{Im}V./\operatorname{Re}V) + \textit{nuisance term}$
- Phase shifts between k and l: $\Phi_{k.} - \Phi_{l.} = \begin{bmatrix} \Phi_{k1} - \Phi_{l1} & \cdots & \Phi_{kn} - \Phi_{ln} \end{bmatrix}$

There are a few comments. In terms of notations, "$./\Theta$" signifies the element-by-element multiplication of Θ^{-1} from right, $\Phi_{k.}$ is the k-th row of Φ and $\operatorname{Re}V$ and $\operatorname{Im}V$ mean the real and imaginary parts of V, respectively. Φ_{lj} is the phase of the l-th endogenous variable with respect to the cycle corresponding to the j-th eigenvalue.

If the r-th eigenvalue is real, then frequency θ_r is positive infinity and phase shifts between any variables are zero.

The unit of $\Phi_{k.} - \Phi_{l.}$ is radian. To convert the unit from radian to time, it should be divided by a proper frequency, as in the main text.

In actual computation, it is necessary to take care the fact that any $\hat{\beta}_{lj} \equiv \beta_{lj} \pmod{2\pi}$ are equivalent to β_{lj}. Also, with some computer software, it is difficult to distinguish β_{lj} and $-\beta_{lj}$.

1.B.2 Derivation

If λ_j and λ_i are conjugate each other (denote conjugate by upper bar: $\lambda_i = \bar{\lambda}_j$), then V_j and V_i are also conjugate each other ($V_i = \bar{V}_j$). This is evident because $\bar{\lambda}_j$ and $\bar{V}_j$ must satisfy the definition of the eigenvalue-eigenvector if λ_j and V_j satisfy it.

$$(\mathbf{B} - \lambda_j I) V_j = 0 \Leftrightarrow \overline{(\mathbf{B} - \lambda_j I) V_j} = 0 \Leftrightarrow \left(\mathbf{B} - \bar{\lambda}_j I\right) \bar{V}_j = 0$$

Note that $\bar{\mathbf{B}} = \mathbf{B}$ and $\bar{I} = I$ since the identity matrix and $\mathbf{B}$ are both real.

Denote such λ_j and V_j as follows.

$$\begin{aligned}
\lambda_j &= a_j + b_j i = \rho_j (\cos\theta_j + i \sin\theta_j) \\
\bar{\lambda}_j &= a_j - b_j i = \rho_j (\cos\theta_j - i \sin\theta_j) \\
V_j &= R_j + M_j i \\
\bar{V}_j &= R_j - M_j i \\
R_j &= \begin{bmatrix} R_{1j} \\ \vdots \\ R_{nj} \end{bmatrix}, \; M_j = \begin{bmatrix} M_{1j} \\ \vdots \\ M_{nj} \end{bmatrix}
\end{aligned}$$

where $\rho_j = \sqrt{a_j^2 + b_j^2}$ and $\theta_j = \arctan b_j / a_j$. It is obvious that both $\lambda_j^t V_j$ and $\bar{\lambda}_j^t \bar{V}_j$ are elementary solutions of the difference equations 1.3. Note that by De Moivre's formula,

$$\begin{aligned}
\lambda_j^t &= \left(\rho_j\left(\cos\theta_j + i\sin\theta_j\right)\right)^t = \rho_j^t\left(\cos\theta_j t + i\sin\theta_j t\right) \\
\bar{\lambda}_j^t &= \left(\rho_j\left(\cos\theta_j - i\sin\theta_j\right)\right)^t = \rho_j^t\left(\cos\theta_j t - i\sin\theta_j t\right)
\end{aligned}$$

However, we prefer the elementary solutions that do not have imaginary root i. Note that any linear combination of these solutions can be also elementary solutions. Thus, define

$$\begin{aligned}
\eta_i^{\mathrm{Re}} &= \frac{1}{2}\left(\lambda_i^t V_i + \bar{\lambda}_i^t \bar{V}_i\right) = \rho_i^t\left(R_i\cos\theta_i t - M_i\sin\theta_i t\right) \\
\eta_i^{\mathrm{Im}} &= \frac{1}{2i}\left(\lambda_i^t V_i - \bar{\lambda}_i^t \bar{V}_i\right) = \rho_i^t\left(M_i\cos\theta_i t + R_i\sin\theta_i t\right)
\end{aligned}$$

By the formula of linear combination of trigonometric functions (synthesis formula),

$$\begin{aligned}
R_j\cos\theta_j t - M_j\sin\theta_j t &= \psi_j\cdot\sin\left(\theta_j t + \hat{\beta}_j\right) \\
R_j\cos\theta_j t + M_j\sin\theta_j t &= \psi_j\cdot\sin\left(\theta_j t + \tilde{\beta}_j\right) = \psi_j\cdot\cos\left(\theta_j t + \hat{\beta}_j\right)
\end{aligned}$$

where $\cdot$ signifies element-by-element multiplication, and

$$\hat{\beta}_j = \begin{bmatrix} \hat{\beta}_{1j} \\ \vdots \\ \hat{\beta}_{nj} \end{bmatrix} = \arctan\left(\frac{M_j}{R_j}\right), \quad \psi_j = \begin{bmatrix} \psi_{1j} \\ \vdots \\ \psi_{nj} \end{bmatrix} = \begin{bmatrix} \sqrt{R_{1j}^2 + M_{1j}^2} \\ \vdots \\ \sqrt{R_{nj}^2 + M_{nj}^2} \end{bmatrix}$$

Interestingly, there is a kind of duality between eigenvalues and eigenvectors. Therefore, the two real elementary solutions are written as

$$\begin{aligned}
\eta_j^{\mathrm{Re}} &= \psi_j\cdot\rho_i^t\sin\left(\theta_j t + \hat{\beta}_j\right) \\
\eta_j^{\mathrm{Im}} &= \psi_j\cdot\rho_i^t\cos\left(\theta_j t + \hat{\beta}_j\right)
\end{aligned}$$

The solution of linear differential equations is a linear combination of the elementary solutions.

$$y_t = \cdots + \omega_j \psi_j \cdot \rho_i^t \sin\left(\theta_j t + \hat{\beta}_j\right) + \omega_{j'} \psi_j \cdot \rho_i^t \cos\left(\theta_j t + \hat{\beta}_j\right) + \cdots$$

Weights $\{\omega_\tau\}_{\tau=1}^n$ are typically determined by the initial condition (past and present innovations in our case) of a given problem. By using the same formula again, it is shown that the phase of the l-th variable with respect to the j-th eigenvalue β_{lj} must satisfy

$$\begin{aligned}
& \alpha_{lj} \rho_i^t \sin\left(\theta_j t + \beta_{lj}\right) \\
= \; & \omega_j \psi_{lj} \rho_i^t \sin\left(\theta_j t + \hat{\beta}_{lj}\right) + \omega_{j'} \psi_{lj} \rho_i^t \cos\left(\theta_j t + \hat{\beta}_{lj}\right) \\
= \; & \left(\psi_{lj} \sqrt{\omega_j^2 + \omega_{j'}^2}\right) \rho_i^t \sin\left(\theta_j t + \hat{\beta}_{lj} + \bar{\beta}_{lj}\right)
\end{aligned}$$

where $\bar{\beta}_{lj} = \bar{\beta}_j = \arctan\left(\omega_j / \omega_{j'}\right)$ is common to all l.

Hence,

$$\begin{aligned}
\alpha_{lj} &= \psi_{lj} \sqrt{\omega_j^2 + \omega_{j'}^2} \\
\beta_{lj} &= \hat{\beta}_{lj} + \bar{\beta}_j
\end{aligned}$$

It is clear that the phase shift between the k-th and l-th variables is independent from the initial value (past and present innovations in our case) because $\bar{\beta}_j$ is cancelled out.

$$\beta_{kj} - \beta_{lj} = \hat{\beta}_{kj} - \hat{\beta}_{lj}$$

Remember that $\bar{\beta}_j$ is dependent on ω_τ but $\hat{\beta}_{lj}$ is not.

1.C Smoothing Parameter for Monthly HP-Filter

According to the rule of thumb, the smoothing parameter of the Hodrick-Prescott filter (HP-filter) should be 100, 1600 and 14400 for annual, quarterly and monthly data, respectively. However, these values are not consistent with one another. This note, instead, numerically demonstrates that the smoothing parameter that is consistent with 1600 for quarterly data is 7-8 for annual data and slightly less than $130,000$ for monthly data. In general, if the favourite smoothing parameter for quarterly data is λ_Q, then mnemonically

$$4^4\lambda_Y \simeq \lambda_Q \simeq \lambda_M/3^4$$

where λ_Y and λ_M are smoothing parameters for annual and monthly data, respectively. These linear relationships among λ_Y, λ_Q and λ_M are stable for most economic variables.

This conclusion provides a numerical support for the analytical finding in Ravn and Uhlig (2002). It is important to note that this note does not propose any single best smoothing parameter. Instead, it simply states the consistency among smoothing parameters for different data frequencies.

The idea behind the two exercises in this note is quite simple. Suppose that the frequency of time series data is quarterly. Then, we can construct annual data from the original quarterly data by a proper method (e.g., by simply taking the average of four quarters in one year). Let Y and Q be the column vectors of annual and quarterly data, respectively.

Next, the HP-filter is applied to Y and Q to obtain smooth series. Define $HP(Q, \lambda_Q)$ as a matrix such that $Q^{HP} = HP(Q, \lambda_Q)\, Q$ and $Y^{HP} = HP(Y, \lambda_Y)\, Y$, where Q^{HP} and Y^{HP} are the vectors of HP-filtered series.

The first exercise uses the cubic spline to convert annual HP-filtered data Y^{HP} to quarterly data Y^{HP2Q}. Note that the cubic spline should perform very well because Y^{HP} is a very smooth series by construction. Given the original annual data Y, Y^{HP2Q} is a function of only λ_Y. Similarly, given Q, Q^{HP} is a function of only λ_Q. The first exercise obtains the optimal λ_Y as a function of λ_Q; i.e., λ_Y that minimises the following quadratic error function for each λ_Q.

$$\min_{\lambda_Y} \left(Y^{HP2Q}(\lambda_Y) - Q^{HP}(\lambda_Q) \right)^2 \qquad \text{for each } \lambda_Q \tag{1.5}$$

The second exercise converts quarterly HP-filtered data Q^{HP} to annual data Q^{HP2Y} by a proper method (e.g., taking the average). Again, note that Y^{HP} and Q^{HP2Y} are functions of only λ_Y and λ_Q, respectively. The second exercise obtains the optimal λ_Q as a function of λ_Y; i.e., λ_Q that minimises the following quadratic error function for each λ_Y.

$$\min_{\lambda_Q} \left(Q^{HP2Y}(\lambda_Q) - Y^{HP}(\lambda_Y) \right)^2 \qquad \text{for each } \lambda_Y \tag{1.6}$$

By similar exercises, it is possible to obtain the optimal λ_M and λ_Q as functions of each λ_Q and λ_M, respectively. Then, the same exercises are done for several data.

The results are almost identical in both types of exercises, and hence the results of only the first type of exercises (1.5) are shown. Figure 1.27 shows the optimal λ_M as a function of λ_Q (all lines are too close to distinguish), and Figure 1.28 shows the optimal λ_Q as a function of λ_Y. There are clear linear relationships regardless of data.

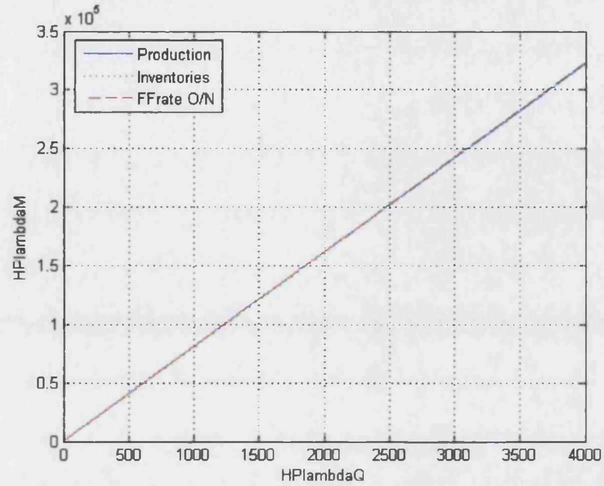

Figure 1.27: Optimal λ_M as functions of given λ_Q.

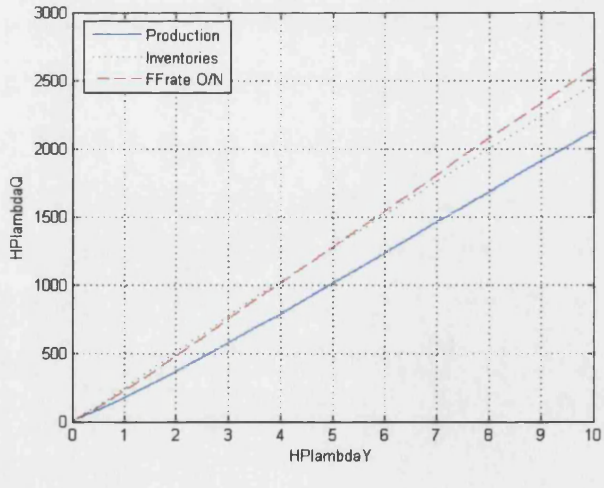

Figure 1.28: Optimal λ_Q as functions of given λ_Y.

Chapter 2

Inventory Cycles

This chapter investigates a rational dynamic stochastic general equilibrium model with a stockout constraint and a production chain.

Our model shows that the stockout avoidance and cost shock models satisfy stylised inventory facts – production is more volatile than sales and inventory investment is procyclical – for demand and supply shocks, respectively, while production smoothing works at very high frequencies. Note that the cost shock and production smoothing models are naturally embedded in our micro-founded general equilibrium framework. Moreover, as a by-product, the production chain causes the slow adjustment of inventories in aggregate. Consequently, our model generates (a) high labour volatility and (b) low correlation between labour productivity and output; the standard RBC cannot produce these two empirical findings. Finally, our model yields inventory cycles.

2.1 Introduction

Inventories are important in understanding business cycles. Inventory investment accounts for a large share of GDP fluctuations, especially during recessions.[1] Despite this importance, most existing theoretical studies of inventories focus only

[1] For example, Fitzgerald (1997) reports that "changes in inventory investment are, on average, more than one-third the size of quarterly changes in real GDP over the postwar period." See also Blinder and Maccini (1991).

on firm/industry level analyses; only a few general equilibrium analyses exist. The motivation of this chapter is to investigate a micro-founded rational dynamic stochastic general equilibrium (DSGE) model that satisfies two stylised inventory facts: (1) production is more volatile than sales and (2) inventory investment is procyclical. Specifically, we constructs a DSGE model with a stockout constraint and a production chain; the stockout constraint means that no seller can sell more products than the inventories she holds, and the production chain means that one firm's output is used as a production factor by other firms, and this repeats.

In a sense, this chapter is a general equilibrium extension of Kahn (1987, 1992), who first analysed the stockout constraint. The key trade-off under the stockout constraint is that having too much inventory is costly because unsold goods impose a carrying cost (Jorgenson's user cost), while having too little inventory is also costly because the risk of losing sales opportunity due to stockout is too high. Balancing carrying cost against stockout probability, firms choose the optimal level of inventories. As a result, the optimal level of inventories is an increasing function of demand; given the level of inventories, strong demand reduces the expected amount of unsold goods and raises the stockout probability.

Our research, however, is most closely related to Khan and Thomas' (2004b) fully rational DSGE for inventories. In comparing the (S,s) and stockout avoidance models, they conclude that the former is superior to the latter, partly because firms have almost no inventories in the stockout avoidance model.

However, we conjecture that the competitive goods market in their model is not compatible with the existence of unsold goods (inventories carried over to the next period). Consider firms' decisions at different points in one period. Certainly, *when firms decide their production*, there is an incentive to hold inventories as buffers, because some factor inputs are decided before the realisation of aggregate shocks in their model. However, *when firms decide their sales*, there is little incentive to hold inventories, because all aggregate shocks are already revealed. In their competitive goods market, the price of goods should rise if demand is strong and vice versa, until the market clears (i.e., no inventories exist). At the end of the day, no inventories are carried to the next period.

In contrast, in our non-Walrasian goods markets, price does not equate demand and supply; instead, we assume price posting. Indeed, we claim that neither instances of stockouts nor unsold goods take place under flexible price. In sum, the most important difference between Khan and Thomas' model and ours is that they assume a competitive goods market, while we assume non-Walrasian goods markets.

Simulating our model, we find several observations. First, our model quantitatively satisfies the two stylised inventory facts. The intuition is as follows. When a positive demand shock hits firms, their inventories are *initially* reduced, and thus firms want to replenish inventories. Moreover, the target level of inventories becomes higher than the normal level, because the demand is stronger than usual. Hence, *in subsequent periods*, firms have to produce more than they sell in order to accumulate inventories. Thus, inventory investment is positive when sales and production are high, while production is more volatile than sales. Although this mechanism was predicted by Kahn (1987) in his firm level analysis, one of our contributions is to quantitatively endorse his prediction in the dynamic stochastic general equilibrium framework.

However, it is important to note that not only the stockout constraint is embedded in our model. Indeed, our model includes the mechanisms predicted by the cost shock and production smoothing models. Importantly, even though we do not intend to explicitly build these mechanisms in our model, they must, naturally and inevitably, appear in our fully rational, micro-founded environment. On one hand, with a positive productivity shock (i.e., a negative cost shock), production increases but sales do not increase very much; as a result, inventories increase when production increase, while production is more volatile than sales. On the other hand, inventories certainly decrease right after a positive demand shock, and production does not react quickly because of the convex cost function. More specifically, if a band-pass filter is applied to the simulated data series, our model finds that production is *less* volatile than sales and inventory investment is *countercyclical* at very high frequencies. In sum, in our model, the following three leading inventory models are all working: cost shock, production smoothing and stockout avoidance models. Or, equivalently, our model finds that these three mechanisms predicted by firm/industry level analyses are all alive even in the DSGE framework.

Another important finding in our model simulation is the slow adjustment of inventories, which is found in several empirical studies.[2] The key mechanism behind this is the production chain[3]. When an intermediate goods producer (M-firm) wants to replenish its inventories of intermediate goods (M-goods),[4] it has to increase its own production and its use of M-goods provided by other M-firms. This is demands for other M-firms' goods and reduces their inventories. This process repeats. In other words, increasing inventories in one firm decreases inventories in other firms. Thus, *the adjustment of inventories (or intermediate goods) in aggregate is indeed slow.*

This slow adjustment of inventories also generates two by-products: higher volatility of working hours, and lower correlation between labour productivity and output, than the standard real business cycle (RBC) model. For the former, different from the standard RBC model, there is one extra production factor in our model – M-goods. However, because the adjustment of M-goods is slow, firms are forced to use more labour input to compensate for the sluggish adjustment of M-goods during booms. Indeed, our model predicts that M-goods' price increases sharply after a positive demand shock, which encourages firms to substitute M-goods with labour. As a result, labour productivity (= output/hours) does not increase when output increases, because the increases in working hours are large

[2] See Blinder and Maccini (1991), among others. Also, Ramey and West (1997) interpret the persistent inventory to sales ratio as one expression of the slow adjustment of inventories.

[3] However, the primary purpose of explicitly modelling the production chain is to generate a realistic sales volume, which is much larger than the volume of production due to the use of intermediate goods. Note that under representative firm models, production is (almost) equal to sales.

[4] Note that our model analyses the stockout constraint in M-goods markets. Thus, inventories in our model mean inventories of M-goods, unless otherwise mentioned.

enough to offset those in output; thus the correlation between labour productivity and output is low in our model. In sum, by adding stockout constraint and production chain, our model improves the standard RBC model in terms of labour.

Finally, our model can replicate so-called inventory cycles (see the Introduction of Chapter 1). However, although VAR-based analyses find sine curve impulse response functions (IRFs), our theoretical model generates only over-damped oscillations, which means that there is a mechanism that generates oscillation, but its effect is not strong enough to exhibit sine curve IRFs. Nonetheless, the model exhibits cycles in the phase diagrams.

The plan of this chapter is as follows. Section 2.2 reviews both theoretical and empirical literature, and summarises the stylised inventory facts. Our model satisfies not only the two famous stylised facts, but also additional detailed facts. Section 2.3 establishes the model environment. The key features of our model include: (i) in addition to the representative household, there are two types of firms: final goods producers (F-firms) and intermediate goods producers (M-firms), both of which use capital, labour and M-goods as inputs, while the former produce final goods (F-goods) which are used as consumption or investment goods, while the latter supply M-goods; (ii) individual M-goods are differentiated from each other, and hence an M-firm must use M-goods produced by other M-firms (production chain); and (iii) the sales of M-firms are subject to the stockout constraint. Section 2.4 presents numerical results. Section 2.5 section concludes. The technical details are relegated to the Appendix.

In terms of terminology, note that this chapter uses "she" for a seller and "he" for a buyer. Also, the concept of inventories includes "goods on shelf" GoS_t and "unsold goods" U_{t+1}. Though this may sound ambiguous, we often need a word that represents both, because they are closely related to one another; indeed, $GoS_t = U_t$ under a simplified parameter setting. Inventory investment always means $U_{t+1} - U_t$.

2.2 Literature Review and Stylised Facts

This section reviews existing researches. Despite inventory's importance in business cycle research, most existing theoretical inventory models focus only on firm/industry level analyses. There have been only a limited number of analyses of inventories in the setting of the DSGE model. In addition, key empirical research is also reviewed to reconsider stylised inventory facts.

2.2.1 Theories in Firm/Industry Level Analyses

Although we adopt more detailed facts to evaluate the model performance, the following two traditional stylised inventory facts have motivated the theoretical inventory research:

(i) Production is more volatile than sales.

(ii) Inventory investment is procyclical.

Production Smoothing

The first attempt to understand inventories was the simple **production smoothing** (or **buffer inventories**) model, in which, analogous to consumption smoothing, firms want to avoid wild fluctuations in production because of a convex cost function (which should be present even with the CRS production function in general equilibrium), and inventories are used as buffers against demand shocks. However, it is obvious that smooth production cannot explain volatile production, and it predicts that inventory investment is negative when there is a positive demand shock. Thus, its predictions contradict both of the above stylised facts.

Subsequent Models

Hence, subsequent researchers have made efforts to reconcile the production smoothing motive and the two stylised facts. In firm/industry level analyses, there are several strands of literature:[5]

- **Serially correlated demand shocks** may explain to some extent why production is not very smooth, but it alone cannot explain why production is more volatile than sales.[6]

- The **non-convex cost function** (or **bunching production**) has much empirical evidence from plant level studies, but it is uncertain as to whether the same mechanism works in aggregate.[7]

- The **cost shock model** successfully explains stylised fact (i), while its empirical evidence is mixed. However, without any additional assumptions, it predicts that sales and inventories should be uncorrelated.

- **(S,s) ordering policies** successfully explains (i) under the assumption that production takes place no sooner than the order is placed; a fixed ordering cost induces **bunching orders**, and hence orders (production by suppliers) are more volatile than sales (of retailers). However, it does not predict (ii). Moreover, it has difficulty in aggregation, and it alone cannot explain why the stylised facts also hold at individual firms.[8]

- **Inventories as production factors** can explain (ii) but not (i). In aggregate level analyses, where some simplification is inevitable, it may be difficult to discriminate inventory investment from capital investment in this model.[9]

It seems that the above lines of research have not yet reached successful results.

Target Inventory Models

However, the following two models appear to be more promising than those above.

[5] Of course, some researchers have contrived tricks to amend the problems pointed out here. The comments in the following list simply offer a glimpse of the models' basic features.

[6] See Blinder (1986).

[7] See Ramey (1991) and Ramey and Vine (2004) for this line of research.

[8] See Caplin (1985) and Caballero and Engel Caballero and Engel (1991), among others.

[9] See Ramey (1989).

- The **inventories as sales facilities** model is suggested from the standpoint of empirical studies.[10] The idea is that inventories, e.g., in showcases, are necessary to sell goods as samples or specimens. When sales are strong and serially correlated, a firm has to make up for the drop in inventories and, in addition, has to accumulate additional inventories to keep up with the new sales level, which is higher than before. Hence, in principle the model can explain both (i) and (ii).

- The **stockout avoidance motive** is probably the most natural setting, at least as a casual conjecture. Similar reasoning to that of the inventories as sales facility model shows that this can also explain (i) and (ii).[11]

Note that the inventories as sales facilities and stockout avoidance models are indeed special cases of a more general class of models. The generalised target inventory model has the following sales function:

$$S_t = \left(D_t(P_t)^{\psi} + \phi GoS_t^{\psi} \right)^{\frac{1}{\psi}} \tag{2.1}$$

where $D_t(.)$ is demand as a function of price P_t, GoS_t is goods on shelf (inventories), and ψ and ϕ are parameters. The model reduces to the inventories as sales facilities model in Bils and Kahn (2000) if $\psi = 0$, while it reduces to the stockout avoidance model when $\psi = -\infty$. It is important to note that both models imply that the (target) *level of inventories*, rather than *inventory investment*, is an increasing function of demand.

2.2.2 General Equilibrium Analyses

As mentioned above, only a few general equilibrium analyses have been done to date. We list some of the theoretical works below.

[10] See Bils and Kahn (2000) and Pindyck (1994).

[11] See Kahn (1987, 1992). Abel (1985) provides early work on the stockout constraint. Wen (2002) also gives some support for this idea.

(S,s) Models

Fisher and Hornstein (2000) and Kahn and Thomas (2004a, 2004b) focus on the (S,s) model in the settings of DSGE.

Although the (S,s) model seems unsuccessful in firm/industry level analyses, Fisher and Hornstein (2000) construct a DSGE model that satisfies the two stylised facts. In their model, general equilibrium feedback seems to be the key to understanding inventories.[12] By incorporating a matching scheme in the goods market,[13] they embed a mechanism by which a high level of inventories induces retailers to lower their sales prices so that consumers increase their search efforts (thus, sales are positively correlated with inventories).[14]

On the other hand Khan and Thomas (2004a, 2004b) also find that the (S,s) model can explain two stylised inventory facts. In Khan and Thomas (2004b), they compare (S,s) and stockout models and conclude that the former is better than the latter in terms of the two traditional stylised facts (see the next subsection).

Target Inventory Models

Kahn et al. (2002) constructed an **inventory in the utility model** as a proxy for the stockout avoidance motive with imperfect information. Their intuition is essentially the same as ours; when a positive shock hits a firm, its inventories decline, but the firm then has to replenish inventories and build up inventories to achieve the new, higher target level (because the sales shock is assumed to be persistent). They emphasise informational imperfection; firms cannot sell all of today's products in today's market due to a informational problem. However, inventory in the utility is not based on a micro-foundation, though it could be a useful short-cut.

[12]For the aggregation problem, they restrict the state space; the possible level of inventory holdings are limited to a few natural numbers.

[13]Note that in this sense their model also can be regarded as a non-Walrasian model. Their pricing mechanism is marginal (reservation) utility pricing, which is a special case of the Nash Bargain (sellers have all the bargaining power), and similar to ours.

[14]See Blinder (1982) and Bental and Eden (1993) for similar insights.

Khan and Thomas (2004b) analyse the **stockout constraint** in a non-linear DSGE framework. In comparing the (S,s) and stockout avoidance models, they conclude that the former is superior to the latter, partly because firms have almost no inventories in the stockout avoidance model.

However, we conjecture that the competitive goods market in their model is not compatible with the existence of unsold goods (inventories carried over to the next period). Consider firms' decisions at different points in one period. Certainly, *when firms decide their production*, there is an incentive to hold *inventories as buffers* against imperfect information during one period.[15] This is because some factor inputs are decided before the realisation of aggregate shocks in their model. However, *when firms decide their sales*, there is little incentive to hold inventories,[16] because all aggregate shocks are already revealed. Having inventories just leads to a carry cost, but it no longer protects firms unless the marginal cost of the next period is very high. In their competitive goods market, the price of goods should rise if demand is strong and vice versa, until the market clears (i.e., no inventories exist), although Khan and Thomas (2004b) do not report the change in the goods prices. At the end of the day, no inventories are carried to the next period. In a sense, their goods market is a Walrasian market with a vertical supply curve; unless the demand curve is unorthodox, the market finds a price to equate demand and supply.

In contrast, in our non-Walrasian goods markets, price does not adjust demand and supply; instead, we assume price posting. Indeed, we claim that neither instances of stockouts nor unsold goods take place under flexible price. In sum, our research is most closely related to Khan and Thomas' (2004b), but the most important difference between their and our models is that they assume a competitive goods market, while we assume non-Walrasian goods markets.[17] Note that, be-

[15] Note that inventories in this sentence are goods on shelf in our terminology. However, because there is no unsold goods carried from the previous period in their model, goods on shelf are equal to today's production.

[16] Note that inventories in this sentence are unsold goods in our terminology. Note also that inventory investment means the time difference of unsold goods in general.

[17] In addition, while our model is solved by linearisation, they employ a non-linear solution

cause goods prices respond to all the aggregate shocks, though not to idiosyncratic shocks, our model falls into the class of flexible price models.

Inventories with Sticky Price

Hornstein and Sarte (2001) and Boileau and Letendre (2004) incorporate inventories into a dynamic sticky price model.

The motivation to hold inventories used by Hornstein and Sarte is production smoothing. In their model, after a positive monetary shock, (i) for agents who have an opportunity to change prices, sales plummet down because their new prices become higher than other agents', but production does not move very much due to convex cost function, while (ii) for agents who do not change their price, sales and production increase. According to them, initial changes in sales are offset in aggregate, while changes in production are not. Thus, production is more volatile than sales.

Boileau and Letendre studied three types of models in the dynamic sticky price model. The most successful one is the model they call the shopping-cost model,[18] and it creates more persistence in output and inflation than the standard sticky price model. At first glance, their shopping-cost model seems to be similar to the micro-founded target inventory model such as ours, in the sense that both models share the feature that inventories help sales. However, it appears that their model should be regarded as an inventories as production factors model, at least in aggregate. This is because, while inventories reduce the retailers' shopping cost, the authors impose the zero profit condition on the retailers at the same time. This means that, if retailers and producers can be regarded as one big sector, inventories work as a production factor in this big sector. Indeed, their final algebraic results look like those of the inventories as production factors model. In this sense, it is slightly questionable whether or not their model should be classified as the same

method.

[18]The other two model investigated by Boileau and Letendre (2002) are a linear-quadratic model and inventories as factors of production.

class of the models as ours.

Other Important Research

Another important general equilibrium inventory paper is Diamond and Fudenberg (1989).[19] Although their model yields interesting results, including cyclical movements and multiple equilibria, their economy is highly stylised. They assume that each agent cannot have a (stochastic) production opportunity until she sells her products, and hence their "inventories" represent the number of people who had a production opportunity but have not yet sold their products. Thus, we think their model is qualitatively interesting from a purely theoretical viewpoint, but may not be able to allow numerical experiments.

2.2.3 Empirical Studies and Stylised Facts

This subsection briefly reviews empirical research and draws implications.

Stylised Inventory Facts

Although, as mentioned in the previous subsection, two stylised inventory facts are well known, we use more detailed facts in order to evaluate the model performance.

Most importantly, Wen (2002) reveals that the two traditional findings hold only at the business cycle frequencies (8 to 40 quarters); production is less volatile than sales and inventory investment is countercyclical at very high frequencies (2 to 3 quarters).[20] In addition, Ramey and West (1997) suggest that the I/S ratio is persistent, which is perhaps essentially equivalent to the slow adjustment of inventories estimated by Blinder and Maccini (1991).[21] Finally, Bils and Kahn (2000) show that the I/S ratio is countercyclical.

In sum,

[19] See also Diamond (1982).

[20] In this connection, Hornstein (1998) states that inventory investments are important for short-term output fluctuations (6 quarters or less), rather than business cycle fluctuations.

[21] Their model is often called an (empirical) target inventory model (though they are typically not micro-founded). See also Blanchard (1983) and West (1986).

1.a Inventory investment is strongly countercyclical at very high frequencies (2 to 3 quarters).

1.b Inventory investment is procyclical at business cycle frequencies (8 to 40 quarters).

2.a Production is less volatile than sales at high frequencies.

2.b Production is more volatile than sales at business cycle frequencies.

3.a The inventory/sales ratio is persistent and the adjustment of inventories is very slow.

3.b The inventory/sales ratio is countercyclical.

There are a couple of supplementary comments. First, facts 1b and 2b (and hence traditional facts (i) and (ii)) are essentially equivalent to one another (see Section 1.5.4). Second, while facts 1.a and 2.a support the production smoothing motive model, 1.b and 2.b are consistent with the target inventory models (see Wen (2002)).

Inventory Cycles

Inventory cycles are cyclical movements in the phase plan, wherein typical year-on-year change (YoY) in inventories is on the x-axis, and YoY changes in production/shipment are on the y-axis. This phenomenon is stable over time. The conjugate pair of complex roots in VAR coefficients is detected in Chapter 1, which is necessary for generating inventory cycles. Hence, in addition to the stylised facts listed above, the objective of this theoretical research is to construct a DSGE model that exhibits inventory cycles, as mentioned in the Introduction.

Other Empirical Issues

Negative Correlation Between I/S Ratio and Interest Rate: Bils and Kahn (2000) report that the correlation between the real interest rate and I/S is negative (see Table 2 in Bils and Kahn (2000)). They compute the correlation between expectations of real interest rate and I/S conditional on proper information sets. Then they argue that there must be some mechanism such as countercyclical markup to reconcile the FOC w.r.t. inventories to the data. Their finding is puzzling because the target inventory models suggest that the optimal inventories are decreasing in the interest rate (carrying cost). One possible way to understand this finding is that they essentially estimate the monetary policy rule, rather than the optimisation condition of inventories.[22] Nonetheless, we want to point out that a serious puzzle exists in the inventory literature.

Inventories as Collateral: Related to the financial side of the economy, Kashyap et al. (1994) and Gertler and Gilchrist (1994) empirically show that small firms, whose access to financial markets is presumably limited, reduce their inventory holdings more than large firms during recessions. Thus, they both conclude that, for small firms, there is some form of interactions between inventories and financial/liquidity constraints.

Diminishing GDP Volatility and New Inventory Management: Since mid 1980s, many industrialised countries have experienced a decline in the volatility of their GDP and prices (though some authors, such as Comin and Philippon (2005), find that the variability of output is increasing over time at the firm level). In this regard, Kahn et al. (2002) argue that improved inventory management (due to, say, new information technology) allows firms to protect themselves from shocks. They show that the decline in output volatility is salient more in the durable goods sector than in others. Their claim is also numerically evaluated by using our model.

[22]Though controlling the information set looks like using the two-stage regression, their information set is presumably not independent of disturbances (i.e., the variables in the information sets do not work as IVs). Suppose, for example, that the monetary authority has a rule that it raises its policy interest rate when sales are strong and the inventory level is low. With no remedy, if the estimation of the FOC is less stable than that of the monetary policy rule, such computation essentially detects the monetary policy rule, rather than the FOC w.r.t. inventories.

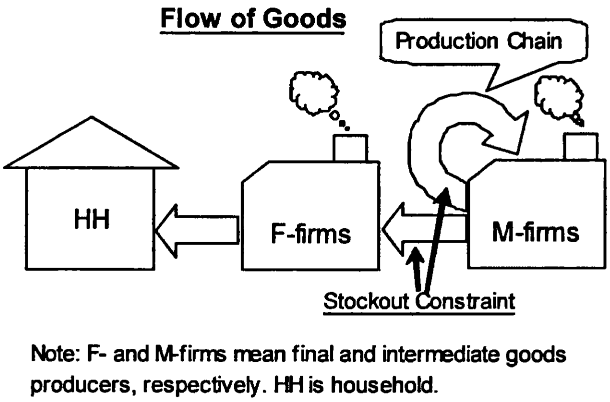

Figure 2.1: An illustration of the structure of the model economy.

2.3 Model Environment and Some Intuitions

This section illustrates the key features of the model, but the full derivation of the most general model is relegated to the Appendix. First, the general setup of the model economy is described, and then the optimisation problem of each agent is defined.

Among other assumptions, the stockout constraint and the production chain are essential – the model aims to analyse them in general equilibrium –, while idiosyncratic demand shock, price posting rule, etc. are rather technical assumptions. The latter are necessary devices for modelling the non-Walrasian goods markets; stockout implies that the goods markets do not clear.

2.3.1 Production Chain

There are three types of agents in the model: a representative household (HH), intermediate goods producers (M-firms) and final goods producers (F-firms), all of which optimise. HH works, consumes and invests. Production factors for both types of firms are labour, capital and intermediate goods (M-goods). Final goods (F-goods) are converted into consumption and investment goods (it is possible to interpret F-firms as retailers). A continuum of M-firms produce mutually differentiated M-goods (à la Dixit-Stiglitz monopolistic competition). A bundle of M-goods are necessary to produce not only F-goods, but also M-goods – production chain.

Looking at the Leontief's input-output table, any two industries demand and supply M-goods from and to one another. Because the input of M-goods is subtracted from sales to compute value-added, sales are much larger than value-added in reality. On the other hand, the stockout constraint implies that the target level of inventories (or goods on shelf) is an increasing function of sales, not value-added. Hence, without modelling the production chain, we underestimate the volume of sales – and hence the volume of the target level of inventories.[23]

Note that if the M-goods markets are frictionless, then the model reduces to a single production sector model; the stockout constraint – a friction in M-goods markets – makes the production chain worth analysing.

Implications of Production Chain

Different from the standard RBC model, however, there is one additional production factor – M-goods. When a shock hits the model economy, capital cannot adjust quickly, as in the standard RBC model, because its evolution is governed by the capital accumulation equation. The adjustment of the additional production factor – M-goods – is also sluggish. This is because of the production chain; when one M-firm wants to increase its supply, it must use other firms' M-goods, which, in turn, implies that other firms want to increase their production by using other firms' M-goods.[24] In aggregate, to produce M-goods, M-firms must consume M-goods! In sum, due to the production chain, the adjustment of inventories is very sluggish in aggregate. In addition, this slow adjustment of M-goods inventories

[23]In this connection, consider the Leontief production function where the elasticity of substitution between labour/capital and M-goods is zero $\eta_M = 0$ (see the Appendix for notations). Then the use of M-goods is proportional to the gross output: $Y_t^M = Z_t^{Mn} M_{t-1}^M/(1-\phi_M)(= Z_t^{Mn} V_t^M/\phi_M)$, where Z_t^{Mn} is the technology and ϕ_M is the share parameter of value-added component V_t^M. The **Leontief's inverse matrix** – the most important concept in the input-output table analysis – shows the increase in the output of one sector due to a unit increase in final demand. Noting that M-goods produced are used as inputs of F-firms and M-firms: $Y_t^M = M_t^F + M_t^M$, $\partial Y_{ss}^M/\partial M_{ss}^F = (1-(1-\phi_M))^{-1} = \phi_M^{-1} > 1$ in the symmetric steady state (in our model, the matrix is actually 1×1). Hence, gross output fluctuates more when the share of intermediate goods is larger. In principle, it is possible to simulate Leontief's inverse matrix analysis dynamically.

[24]To gain further intuition, see also the previous footnote.

has several important implications for labour (see below for details).

2.3.2 Stockout Constraint

Our model explicitly analyses the effect of the stockout constraint, which was first examined by Kahn (1987), and our study is a general equilibrium extension of his market equilibrium analysis.

Our model considers the stockout constraint on the M-good markets, and defines goods on shelf GoS_t as the sum of unsold goods U_t and (a portion of) today's production Y_t^M. In terms of terminology, GoS_t and U_t are both (the level of) "inventories," but the former is measured before the opening of M-goods markets, while the latter is after the markets close.

The stockout constraint, the main friction in our model, means that no seller can sell more products than the stocks on shelf GoS_t. Hence,

$$S_t = \min\{GoS_t, M_t^p\} \tag{2.2}$$

where S_t is the sales and M_t^p is the potential demand for M-goods. The potential demand is "potential" simply because it may not be realised due to stockout.[25] There is a fundamental trade-off; stockout is costly because it means the loss of a profitable sales opportunity, but unsold goods are also costly because they impose a carrying cost (or Jorgenson's user cost) of unsold goods. Note that the nature of the carrying cost is the cost of financing inventories (plus capital and income gain/loss of inventories) while the (marginal) opportunity cost of missing profitable sales opportunity is measured in terms of the forgone profit margin in our model.

Hence, the target level of inventories is an increasing function of the potential demand (which moves closely with sales), but is a decreasing function of the interest rate (financing cost). When the potential demand is strong, for example, if GoS_t were kept unchanged, the stockout probability would be too high while the

[25]It may be possible to express the stockout constraint in the form of a non-negativity constraint on GoS_t, but adding the non-negativity constraint complicates the algebra.

level of expected unsold goods would be too low; hence, firms have an incentive to accumulate inventories, and vice versa. Note that choosing optimal GoS_t is equivalent to choosing optimal stockout probability.

Implications of Stockout Constraint

The stockout constraint can (at least potentially) explain the two inventory stylised facts (see also Kahn (1987)). One of the goals of this chapter is to quantitatively evaluate the effects of the stockout constraint in the DSGE framework.

The intuition is as follows. As mentioned above, under the stockout constraint, the target level of inventory is an increasing function of the potential demand, which shows movements quite similar to sales. Hence, inventory investment is naturally procyclical (fact 1b). Furthermore, production must increase more than sales because, otherwise, inventories decrease (fact 2b).

In addition, the I/S ratio is countercyclical because, during a recession, the interest rate is low and thus the carrying cost is low as well, which stimulates inventory holdings relative to sales.

Inventories as Buffers

It is important to note that the mechanism explained in the previous subsection is expected to materialise at business cycle frequencies.

At very high frequencies, on the other hand, production smoothing can be explained by the very basic convex cost function. Inventories work as buffers against demand shocks. Even if production technology ensures constant returns to scale (CRS), as long as the labour supply is convex (due to the concave utility function), this mechanism works. Because firms do not want to adjust their production quickly, inventories will decrease right after a positive demand shock, and vice versa.

Note that both mechanisms – buffer stocks and stockout constraint – do not contradict to one another, and they indeed coexist in our model. Moreover, of course, our model also incorporates the cost shock model. Our objective is not to pick up one single "true" mechanism out of the three models, but to compare them to evaluate their relative importance.

2.3.3 Structure of M-goods Markets

This subsection provides rather technical basis of the model. We recommend that interested readers consult the Appendix. Here, only the key assumptions are listed:

- Due to **idiosyncratic shocks**, individual sellers face different levels of demand. Both stockout and unsold goods exist, implying that the M-market is **non-Walrasian.**
- Hence, we cannot use market clearing conditions as a pricing mechanism. Instead, we assume the **price posting** by sellers, wherein buyers decide on the trading quantities. Buyers' FOCs are regarded as demand curves.
- Due to the price posting and CRS production function, our model falls in the class of **representative agent models in aggregate**, despite the heterogeneity caused by stockout.
- M-goods are differentiated from each other (Dixit-Stiglitz' monopolistic competition). Two-stage budgeting is modified by the **cost effect of losing variety.**

Note that it is possible to linearise the stockout constraint (2.2), because *the numbers of sellers and buyers* with binding (2.2) are smooth functions in aggregate, even though (2.2) is not a smooth function from the individual sellers' viewpoint. Also, note that F-firms play only the role of buyers, but M-firms behave as both sellers and buyers.

2.3.4 Household

The infinitely-lived representative household (HH) maximises its expected lifetime utility.

$$\max_{\{C_s, H_s^H\}_{s=0}^{\infty}} E_0 \left[\sum_{t=0}^{\infty} \beta^t U \left[C_t, 1 - H_t^H \right] \right]$$

s.t.

$$C_t + B_{t,t+1} = R_{t-1,t} B_{t-1,t} + W_t H_t^H + D_t^{iv}$$

The period utility $U[.,.]$ is time additive, is discounted by the subjective discount factor β^t, and takes consumption C_t and leisure $1 - H_t^H$ as arguments, where the total time endowment is normalized to one and H_t^H is the labour supply.

The period budget constraint has cash outflow in the LHS and inflow in the RHS. The LHS means that HH expense their resources to consumption or one-period bonds $B_{t,t+1}$, while the RHS implies that cash inflows are the sum of bond redemption $R_{t-1,t} B_{t-1,t}$, wage income $W_t H_t^H$ and dividends D_t^{iv}.[26]

HH takes the real interest rate $R_{t-1,t}$, wage rate W_t and D_t^{iv} as givens. All the first order conditions (FOCs) are quite standard.

Functional Form

Throughout this chapter, we assume the following functional form for the period utility.

$$U\left[C_t, 1 - H_t^H \right] = (1 - \psi) \frac{\left(C_t \right)^{1-\gamma}}{1 - \gamma} + \psi \frac{\left(1 - H_t^H \right)^{1-\gamma_L}}{1 - \gamma_L}$$

where ψ is the weight for leisure, and γ and γ_L are the elasticities of intertemporal substitutions of consumption and leisure, respectively. When $\gamma_L = 0$, our utility function reduces to Hansen's indivisible labour model.

[26] Alternatively, we can assume that there are infinitely many HHs which own both F- and M-firms. In that case, dividends are assumed to be state contingent, and thus all household enjoy the same level of cash inflow; as a result, whole HHs reduce to one sector in aggregate.

2.3.5 Firms

We assume that quadratic adjustment costs apply to changing labour demand and input of M-goods, as well as investment.

M-Firms' Optimization Problem

As shown in the Appendix, we can exploit the slightly modified two-stage budgeting ($\int_0^1 Q_t^j P_t^j M_t^j dj = Q_t^{\frac{-1}{\theta-1}} P_t^M M_t^M$).

$$\max E_0 \left[\sum_{t=0}^{\infty} \beta^t \frac{\lambda_t^H}{\lambda_0^T} \left\{ \begin{array}{c} P_t^i S_t - W_t H_t^{Mp} - I_t^M - Q_t^{\frac{-1}{\theta-1}} P_t^M M_t^M \\ -\chi_{MH} \left(H_t^{Mp} - H_{t-1}^{Mp} \right)^2 / H_{t-1}^{Mp} \\ -\chi_{MM} \left(M_t^M - M_{t-1}^M \right)^2 / M_{t-1}^M \end{array} \right\} \right]$$

s.t.

$$\begin{aligned} U_{t+1} &= U_t - S_t + Y_t^M \\ S_t &= \min \left\{ U_t + \upsilon Y_t^M, M_t^p \right\} \\ Y_t^M &= Y^M \left[K_t^M, H_t^{Mp}, M_{t-1}^M; \mathbf{Z}_t^M \right] \\ K_{t+1}^M &= (1 - \delta_M) K_t^M + I_t^M - \chi_{MK} (I_t^M - \delta_M K_t^M)^2 / K_t^M \end{aligned}$$

The objective function says that M-firms maximise the present value (PV) of their net cash inflows, which are discounted by the stochastic discount factor $SDF_t = \beta^t \lambda_t^H / \lambda_0^H = \beta^t (\partial U_t / \partial C_t) / (\partial U_0 / \partial C_0)$. The cash inflow is only the sales revenue $P_t^i S_t$, where P_t^i is the sales price of producer i. While sales price is a choice variable, the purchase price P_t^M is given for all agents, though $P_t^i = P_t^M$ for $\forall i$ in equilibrium. On the other hand, cash outflow is composed of the wage payment $W_t H_t^{Mp}$, which is wage rate W_t times labour hours H_t^{Mp}, the expenditure on investment goods I_t^M (the price of F-goods is normalized to 1) and the expenditure on M-goods $Q_t^{\frac{-1}{\theta-1}} P_t^M M_t^M$, where Q_t is the number of available varieties, and P_t^M and M_t^M are the price and quantity indices of M-goods, respectively. In addition, the adjustment costs of labour and M-goods inputs $\chi_{MH}\left(H_t^{Mp}-H_{t-1}^{Mp}\right)^2/H_{t-1}^{Mp}$ and $\chi_{MM}\left(M_t^M-M_{t-1}^M\right)^2/M_{t-1}^M$ also constitute M-firms' cash outflow. χ_{MH} and χ_{MM} are both given parameters. These costs are evaluated in terms of F-goods. In sum, the net cash inflow is the sales revenue minus expenditure on labour, investment goods and M-goods, as well as the adjustment costs.

The first constraint is the evolution of unsold goods. The second represents the stockout constraint. Sales S_t is the minimum of GoS_t or potential demand M_t^p. Note that M_t^p is the sum of the baseline demand and idiosyncratic shock in our notation. The third constraint shows the production function, in which $\mathbf{Z}_t^M$ is exogenous shocks. The production function takes capital K_t^M, labour H_t^{Mp} and M-goods M_{t-1}^M as production factors. The fourth constraint is the evolution of capital, in which we assume a quadratic adjustment cost, where δ_M and χ_{MK} are given parameters.

F-firms' Optimization Problem

The optimization problem of F-firms is as follows.

$$\max E_0\left[\sum_{t=0}^{\infty}\beta^t\frac{\lambda_t^H}{\lambda_0^H}\left\{\begin{array}{c} Y^F\left[K_t^F,H_t^{Fp},M_{t-1}^F;Y_t^{Ftot},\mathbf{Z}_t^F\right]-W_tH_t^{Fp} \\ -Q_t^{F\frac{-1}{\theta-1}}P_t^MM_t^F-I_t^F \\ -\chi_{FH}\left(H_t^{Fp}-H_{t-1}^{Fp}\right)^2/H_{t-1}^{Fp} \\ -\chi_{FM}\left(M_t^F-M_{t-1}^F\right)^2/M_{t-1}^F \end{array}\right\}\right]$$

s.t.

$$K_{t+1}^F=\left(1-\delta_F\right)K_t^F+I_t^F-\chi_{FK}(I_t^F-\delta_FK_t^F)^2/K_t^F$$

The objective function again says that firms maximize the PV of their net cash inflows. The modified version of two-stage budgeting holds, as in the case of M-firms. $W_tH_t^{Fp}$ and I_t^F refer to labour costs and expense on investment, respectively. The production of final goods Y_t^F takes capital K_t^F, labour H_t^{Fp} and M-goods M_{t-1}^F as production factors, where the superscript F implies F-firms.

$\chi_{FH}\left(H_t^{Fp}-H_{t-1}^{Fp}\right)^2/H_{t-1}^{Fp}$ and $\chi_{FM}\left(M_t^F-M_{t-1}^F\right)^2/M_{t-1}^F$ denote the adjustment costs of labour and M-goods, respectively, in which χ_{FH} and χ_{FM} are given parameters.

The constraint represents the evolution of capital with the quadratic adjustment cost. Note that, in this formulation, the level of capital in the steady state is not affected by the parameter χ_{FK}, which governs the adjustment cost of investment.

Functional Form

We assume a CES production function with a Hicks-neutral technology shock $\mathbf{Z}_t^K = Z_t^{Kn}$. For $K = F, M$,

$$
\begin{aligned}
Y_t^K &= Y^K\left[K_t^K, H_t^{Kp}, M_{t-1}^K; \mathbf{Z}_t^K\right] \\
&= Z_t^{Kn}\left[\phi_K\left(\frac{V_t^K}{\phi_K}\right)^{\frac{\eta_K-1}{\eta_K}} + (1-\phi_K)\left(\frac{M_{t-1}^K}{1-\phi_K}\right)^{\frac{\eta_K-1}{\eta_K}}\right]^{\frac{\eta_K}{\eta_K-1}} \\
V_t^K &= {K_t^K}^{\alpha_K}\, {H_t^{Kp}}^{1-\alpha_K}
\end{aligned}
$$

where ϕ_K is the share parameter of the value-added component and η_K is the elasticity of substitution between the value-added component and M-goods as inputs. The value-added component V_t^K is assumed to be a Cobb-Douglas function, in which the share of capital is α_K.Parameters ϕ_K, η_K and α_K are exogenous.

2.4 Numerical Experiments

This section shows the calibration results. We implement the linearisation around the non-stochastic steady state, and simulate the model to obtain the second moments and impulse response functions (IRFs).

Table 2.1: Benchmark parameters for model simulations

Symbol	Meaning	Benchmark value
β	Subjective discount factor (4% annual interest rate)	$1.04^{-1/4}$
γ	Reciprocal of elasticity of intertemporal substitution of consumption	1.00
γ_L	Reciprocal of elasticity of intertemporal substitution of labour	0.00
ψ	Weight on leisure in period utility (Working hours = 1/3)	0.68
θ	Elasticity of substitution among M-goods	10.0
v	Range parameter of idiosyncratic shock (U_{ss}/S_{ss} = 2 months)	0.40
υ	Share of today's output that can be sold in today's market	0.50
α_M, α_F	Capital share in value added	0.35
η_M, η_F	Elasticity of substitution btw M-goods and value-added compo.	0.30
ϕ_M	Weight on value-added compo. of M-firms	0.50
ϕ_F	Weight on value-added compo. of F-firms	0.05
δ_M, δ_F	Depreciation rate of capital (Capital/GDP = 10)	0.015
χ_{MK}, χ_{FK}	Coefficient on quadratic adjustment cost of investment	0.10
χ_{MH}, χ_{FH}	Coefficient on quadratic adjustment cost of labour	1.50
χ_{MM}, χ_{FM}	Coefficient on quadratic adjustment cost of M-goods use	1.00
ρ_{Mn}	AR(1) coefficient of Hicks-neutral technology shock to M-firms	0.75
ρ_{Fn}	AR(1) coefficient of Hicks-neutral technology shock to F-firms	0.85

Table 2.2: Endogenous variables in the steady state.

Symbol	Meaning	Steady state value
SDF_t	Stochastic discount factor (= real interest rate)	0.04
W_t	Wage rate	1.76
P_t^M	M-goods price	0.9996
Q_t	Pr[cannot buy] (= number of available varieties)	0.999
Pr_t	Pr[stockout]	0.074
λ_t^M	Marginal cost of M-goods production (shadow price of M-goods)	0.89
C_t	Consumption	0.83
H_t^H	Labour supply (= $1-$ leisure = $H_t^M + H_t^F$)	0.28
S_t	Sales of M-goods	1.79
M_t^M, M_t^F	Use of M-goods as production factors	0.86, 0.93
Y_t^M	Gross output of M-goods (= $M_{t+1}^M + M_{t+1}^F$)	1.79
Y_t^F	Gross output of F-goods (= $C_t + I_t^M + I_t^F$)	0.98
V_t^M	(Notional) value-added in M-firms (= $K_t^{M\ \alpha_M} H_t^{M\ (1-\alpha_M)}$)	0.93
V_t^F	(Notional) value-added in F-firms (= $K_t^{F\ \alpha_F} H_t^{F\ (1-\alpha_F)}$)	0.053
$H_t^{M_p}$, $H_t^{F_p}$	Labour input for production	0.27, 0.015
I_t^M, I_t^F	Investment	0.14, 0.008
K_t^M, K_t^F	Capital at the beginning of period t	9.37, 0.53
U_t	Unsold M-goods at the beginning of period t	1.25
Z_t^H	Preference shock	1.00
Z_t^{Mn}, Z_t^{Fn}	Hicks-neutral technology shock in production function	1.00, 1.00

2.4.1 Parameter Selection

To select parameters, we do not employ any optimal selection criteria. Rather, for the sake of comparability, we follow the convention in the RBC literature. For the parameters that are specific to our model, we select values to match some steady state values to the data. The difficulty, however, is that we have more than one parameter that governs one steady state value: ν vs. υ for the steady state I/S ratio. In addition, there are six coefficients for the adjustment costs, which are not pinned down by the first moments. Hence, perhaps one possible criticism is that our model has too many degrees of freedom in choosing parameters.

RBC Parameters

For exact values of the RBC parameters, please see Table 2.1. For the elasticity of substitution among varieties, we borrow the number that is commonly used in the sticky price model ($\theta = 10$). We select values for AR(1) coefficients for technology shocks to match the autocorrelation function of GDP (i.e., $Corr\left\{GDP_t, GDP_{t-5}\right\} \simeq 0$). Though these values are smaller than in the standard RBC model, perhaps this is merely due to the existence of adjustment costs and does not signify endogenous persistence.

Parameters Specific to the Model

Share Parameter of Value-Added: For the share parameter of the (notional) value-added in production functions, we set $\phi_M = 0.5$ so that the share of M-goods M_t^M / Y_t^M in the M-firms is roughly 45%; the value-added is roughly 55% of sales. This number is taken from the Japanese and U.S. Leontief's input-output tables. Also, we set $\phi_F = 0.05$ so that F-firms act as if they were the retailers who simply convert M-goods into F-goods.

Note that the notional value-added V_t^M and V_t^F, which appear in the definitions of our production functions, are not consistent with the statistical concept of GDP. For example, $GDP_t^M \equiv Y_t^M - P_{ss}^M M_{t-1}^M$ for M-firms. Hence, note that the terminology "GDP" in this chapter means gross output minus the use of M-goods. Also, note that we assume the Laspeyres price index so that goods are evaluated by the price of the steady state (base year).

Elasticity of Substitution Between Value-Added and Input M-goods: For the elasticity of substitution between the notional value-added component and intermediate goods η_K $(K = F, M)$, we do not have much guidance. Rotemberg and Woodford (1996) used a value of 0.7, while Bruno (1984) suggested 0.3 to 0.4.[27] Because, presumably, the substitution should be low, we use 0.3.

Magnitude of Idiosyncratic Shock and Proportion of Output that Can be Sold in Today's Markets: There are two parameters that affect the steady state I/S ratio: the upper and lower supports of the uniform idiosyncratic shock $\nu/2$, and the portion of today's products that can be sold in today's market υ. In the data, the I/S ratio is roughly 2 months (0.67 quarter).[28]

On one hand, if we set $\upsilon = 1$, as in most firm/industry level analyses, inventories have no significant effect. This is because we assume that production is decided after observing all of the aggregate shock. Hence, if M-goods firms can sell all of their products in the current period market, they, as a collective agent, can respond to aggregate shocks almost fully. Certainly, inventories still vary over time as the interest rate changes over time, and so does the carrying cost. However, in a sense, inventories merely follow other key variables in this case; hence, the model behaves very similarly to the standard RBC model. On the other hand, if we set $\upsilon = 0$ (i.e., $GoS_t = U_t$), it must be the case that $U_{ss} > S_{ss}$, which clearly contradicts the data. If we could know how well firms responded to contemporary aggregate shocks in the real world, we could pin down the value of υ.

Our strategy is as follows. We first naively set $\upsilon = 1/2$, as simply the midpoint between the two extremes, and then choose $\nu = 0.4$ so that the I/S in the model

[27] Basu (1996) regards Bruno's survey as an upper bound.
[28] See Ramey and West (1997), for example.

economy is 2 months.

Convenience Yield on Inventories: Stockout probability, which is roughly 5% to 9% in the data according to Bils (2004), is mainly affected by the subjective discount factor β elasticity of substitution among varieties θ and convenience yield c_1. Essentially, any parameters that determine the opportunity cost of holding inventories affect the steady state stockout probability. If the opportunity cost of lost sales is high, the optimal stockout probability is lower. Given $\theta = 10$, we select $c_1 = 0.00$ (we assume no convenience yield), so that $Pr_{ss} = 7.4\%$.

Adjustment Costs: We assume quadratic adjustment costs, which are rather standard in DSGE research. Specifically, we set $\chi_{MK} = \chi_{FK} = 0.1$, $\chi_{MH} = \chi_{FH} = 1.5$ and $\chi_{MM} = \chi_{FM} = 1.0$.

2.4.2 Numerical Results

A shock to F-firms' production function (F-shock) can be regarded as a *pure* demand shock for M-firms, while a shock to M-firms' production function (M-shock) works as a demand shock and a supply shock from the viewpoint of *individual M-firms.*

In this subsection, all the simulated data are HP-filtered, unless otherwise mentioned. Also, "relative volatilities" are standard deviations relative to that of total GDP or M-firms' GDP. Similarly, "correlations" are correlations with total GDP or M-firms' GDP.

Second Moments

Table 2.3 on page 102 summarises the second moments generated by the model. The results show that, compared to the RBC model, our model considerably decreases the correlation between labour productivity and hours worked, and it satisfies the two stylised inventory facts.

Table 2.3: Simulation results (comparison to the standard RBC model).

Cited from Cooley and Prescott (1995)

	Output	Sales	Hours	Consumption	Investment	d(inventories)	Output/ Hours	Corr{Productivity, Hours}
Standard RBC Model								
relative s.d.	*1.35*	-	0.57	0.24	4.41	-	0.45	almost 1
corr	1.00	-	0.99	0.84	0.99	-	0.98	
Data								
relative s.d.	*1.72*	0.71^	0.92	0.50	4.79	0.271^	0.52	-0.26*
corr	1.00	0.94^	0.86	0.83	0.91	0.658^	0.41	

Notes: "relative s.d." means s.d. relative to s.d. of output. Italics are s.d., not relative s.d.
"corr" means correlation with GDP.
^ indicates that numbers are taken from Khan & Thomas (2004)
* indicates that numbers are taken from Gali (1999).

Stockout Model (elasticity btw Value-add & M-goods = 0.3)

	Output	Sales	Hours	Consumption	Investment	d(inventories)	Output/ Hours	Corr{Productivity, Hours}
Technology shock to M-firms: rho = 0.75, sigma = 0.7%								
relative s.d.	*2.83*	0.77	0.99	0.18	4.64	0.29	0.36	-0.13
corr	1.00	0.81	0.93	0.44	0.53	0.62	0.23	
of which M-firms								
relative s.d.	1.04	0.77	0.96		4.25	0.28	0.36	-0.06
corr	1.00	0.90	0.93		0.51	0.65	0.30	
Technology shock to F-firms: rho = 0.85, sigma = 0.7%								
relative s.d.	*1.57*	0.55	0.87	0.20	4.52	0.15	0.26	0.42
corr	1.00	0.96	0.97	0.73	0.99	0.01	0.63	
of which M-firms								
relative s.d.	0.53	0.77	1.63		8.46	0.28	0.67	-0.95
corr	0.94	0.99	0.98		0.96	0.31	-0.88	

Notes: For "of which M-fims," "relative s.d." and "corr" show s.d. relative to that of M-firms' output and correlation with M-firms' output, respectively.
Relative s.d. of M-firms' output shows s.d. of M-firms' output relative to that of total output. See also notes above.

Correlation of Inventory Investment with GDP: Inventory investment is positively correlated with M-firms' GDP for both shocks. With M-shocks, it is not surprising to observe this positive correlation (0.65); this is exactly what the cost shock model expects. However, it is more important to find a positive correlation (0.31) even with a pure demand shock (see the next subsection for intuition); the stockout model also can generate procyclical inventory investment, though the correlation is lower than data (0.66)

The near-zero correlation between inventory investment and total GDP (M-firms' GDP plus F-firms' GDP) with F-shocks is the artefact of the model assumptions because the F-shock directly increases the F-firms' value-added, but it decreases M-firms' inventories. Indeed, if we use preference shocks instead of the F-shocks, the correlation is even higher. However, preference shocks deteriorate other dimensions of the model performance, so we do not choose this option.

Relative Volatility of Sales: Sales are less volatile than output for both types of shocks. Moreover, the model performs quantitatively very well in this respect; the standard deviation of sales relative to that of M-firms' GDP is 0.77 for both F- and M- shocks in our model, while this value is 0.71 in the data. With M-shocks, this is not surprising, because the source of the shock lies on the production side, as the cost shock models predict. However, it is important to note that, even when the source of the shock lies on the demand side, production is more volatile than sales.

Intuition: For F-shocks, the target inventory models explain the mechanism behind two observations: (i) procyclical inventory investment and (ii) output more volatile than sales, as follows. When a positive demand shock hits M-firms, of course, their inventories initially decline, simply because buyers take away M-goods from the shelf of M-firms. However, keeping such a low level of inventories is costly, because it leads to a too high stockout probability (in the stockout model) and because of an inefficient sales activity without enough samples in showcases (in the inventories as sales facility model). The common prediction among the target inventory models is that the target level of inventories is an increasing function of demand/sales. Hence, with a positive demand shock, the target level of inventories is higher than usual and, as a result, M-firms have an incentive not only to replenish their declined inventories but also to accumulate more inventories to meet the higher demand. However, as the law of motion of inventories (2.7j) shows,

$$U_{t+1} - U_t = Y_t^M - S_t$$

the output of M-goods Y_t^M must increase more than the sales of M-goods S_t to build up inventories U_{t+1}, suggesting that (i) Y_t^M increase more volatile than S_t and (ii) $U_{t+1} - U_t$ is positive when Y_t^M and S_t increase. Indeed, this chapter confirms this mechanism quantitatively in the DSGE setting.

Relative Volatilities of Consumption and Investment: For both shocks, our model inherits the basic nature of the standard RBC model. That is, the relative volatility of consumption is too low, while that of investment roughly matches the data. This is not surprising since our model is an extension of the standard RBC model. The correlation of investment and value-added is too low for the M-shock (0.53), though. The reason for this is that an increase in M-shock, opposed to F-shock, raises the price of investment goods (F-goods), relative to M-goods price.

Persistence of I/S Ratio: According to Ramey and West (1997), the first and second autocorrelations of the inventory-sales relationship (akin to I/S ratio) range from 0.88 to 0.97 and 0.80 to 0.91, respectively. This persistency is regarded as another expression of the slow adjustment of inventories. In our model, the first and second autocorrelations of the I/S ratio are 0.88 and 0.61 for F-shocks and 0.71 and 0.25 for M-shocks, respectively.[29] The I/S ratios in our model are

[29] These values are defined as U_t/S_t, where S_t is the M-firms' sales. The results are almost the same if we define the I/S ratio as unsold goods divided by total sales.

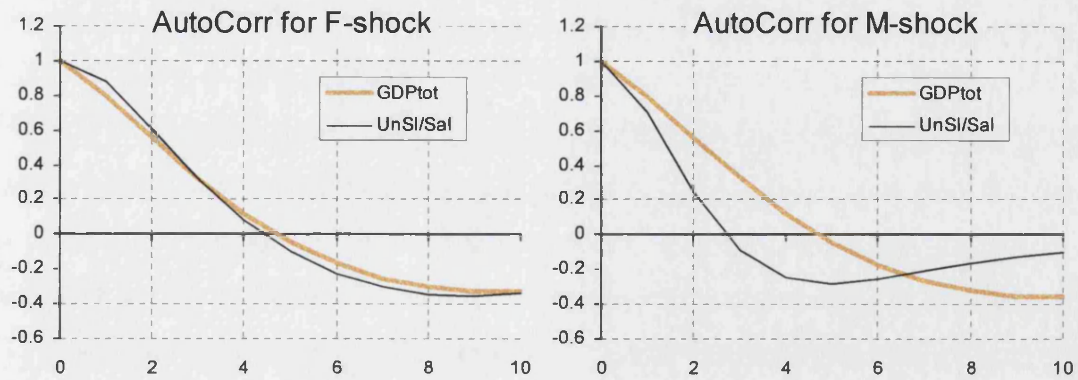

Figure 2.2: Autocorrelation functions. "GDPtot" and "Unsl/Sal" mean gross output minus the use of M-goods, and unsold goods divided by sales (I/S ratio), respectively.

considerably persistent (see Figure 2.2), though they are somewhat lower than the data. Moreover, in our model the I/S ratio is countercyclical because of the procyclical interest rate.

The key mechanism behind this is the production chain. Suppose a positive demand shock hits an M-firm. This firm faces a decrease in its inventories and expects strong future sales, so it wants to replenish its inventories; much more, it raises its inventory level to catch up with the new higher level of sales. As a consequence, it has to increase its production and, hence, the use of production factors, including M-goods. However, this, in turn, implies that the demands (and hence the sales) of other M-firms increase, and that their inventories are reduced. In other words, the production chain implies that one firm's replenishment of inventories reduces other firms' inventories. Therefore, the adjustment of inventories is slow in aggregate. It is important to note that M-goods price increase sharply after a positive F-shock, while M-goods price does not decrease very much after a positive M-shock. Note that unit labour cost (wage/labour productivity) decreases after a positive M-shock (= a negative cost shock), implying that M-goods becomes expensive in relative term.

In this regard, our model can suggest a very simple reason that reduced form target inventory models estimate an *implausibly* slow adjustment speed; it is indeed slow! Certainly, Blinder and Maccini (1991) persuasively argue that "One major difficulty with stock-adjustment models is that adjustment speeds generally turn out to be extremely low; the estimated λ is often less than 10 percent per month. This is implausible when even the widest swings in inventory stocks amount to no more than a few days of production."[30] Reiterating our finding, the inventories' adjustment is slow in aggregate due to production chain, although it seems to be implausible from the viewpoint of individual firms. Partial equilibrium analyses may miss the general equilibrium feedback through volatile M-prices; during a boom, high M-prices discourage M-firms from replenishing their inventories quickly by producing more.

Working Hours: In our model, working hours are more volatile than in the standard RBC model. As a result, the correlation between hours and labour productivity is lower than the standard RBC model. If we focus on M-firms, this correlation is -0.06 and -0.95 with M- and F-shocks, respectively.

One of the major drawbacks of the standard RBC model is that it counterfactually exhibits an almost perfect correlation between labour productivity and working hours. Although one way to overcome this caveat is to add demand shocks (see Christiano and Eichenbaum (1992) for government expenditure, and Bencivenga (1992) for preference shocks), such demand shock models are criticized by Gali (1999), in which a structural VAR shows that the correlation between labour productivity and hours is negative for technology shocks, but positive for other shocks. Gali (1999) suggested that a dynamic sticky price model with a labour effort model can, at least potentially, generate a negative correlation. However, our model improves the model performance in this respect even without price rigidity.

The mechanism that generates volatile working hours in our model is the slow adjustment of inventories; due to the production chain, one firm's replenishment

[30] See Blinder and Maccini (1991, p.81).

Table 2.4: Model behaviour at different frequency domains.

	High Frequencies (2-3quaters)		Business Cycle Frequencies (8-40quaters)	
Data	Var(sales)/Var(output)	Cor(d(inventory),	Var(sales)/Var(output)	Cor(d(inventory), sales)
	1.10	**-0.43**	**0.72**	**0.58**
Model	Var(sales)/Var(output)	Cor(d(inventory),	Var(sales)/Var(output)	Cor(d(inventory), sales)
tech shock to M-firms: rho = 0.75, sigma = 0.7%				
	0.18 (0.32)	**0.21 (0.36)**	**0.83 (0.82)**	**0.60 (0.29)**
tech shock to F-firms: rho = 0.85, sigma = 0.7%				
	0.36 (1.02)	**-0.97 (-0.92)**	**0.56 (0.76)**	**0.62 (0.63)**

Note: Data is OECD average (cited from Wen (2003)). Parentheses indicate for M-firms.

of inventories reduces other firms' inventories in aggregate. The right panels of Figures 2.3 and 2.4 show the IRFs of production factors. It is clear that, for both types of shocks, the increase in M-goods use is less volatile than M-goods production and labour input compensates such sluggish adjustment of M-goods. Note that, because an increase in technology directly contributes to the increase in output, the increase in labour is roughly 50 to 60% of that in output (see Table 2.3 on page 102) in the standard RBC model.

The overly low volatility of working hours predicted by the standard RBC model is closely related to the overly high correlation between labour productivity and output. For example, in the standard RBC model, the increase in working hours during a boom is not large relative to the increase in output, and hence output/hours increases during a boom. However, in our model, hours increase enough to decrease output/hours, and hence corr{output/labour, output} becomes negative.

Frequency Analysis

This subsection exploits the band-pass filter developed by Baxter and King (1999) to the simulated data. For the summary, see Table 2.4. At business cycle frequencies (8-40 quarters), both shocks perform quantitatively well.

At high frequencies (2-3 quarters) the results with M-shocks fail to mimic the

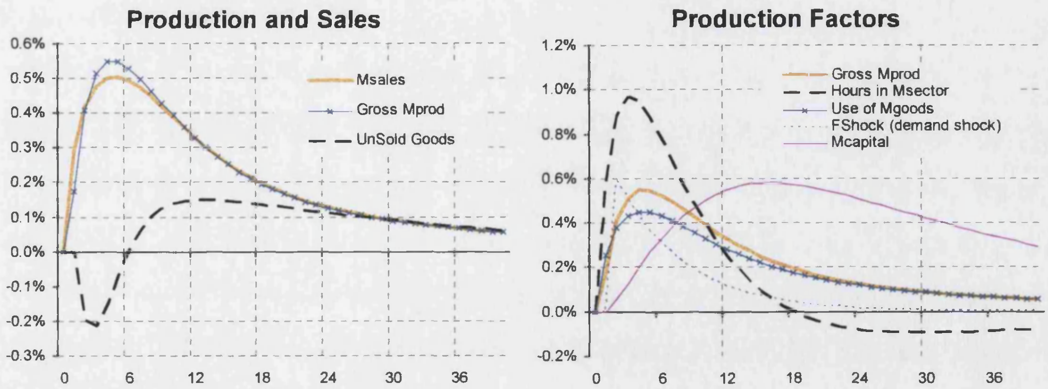

Figure 2.3: Selected impulse response functions to a positive demand shock (a shock to F-firms' production).

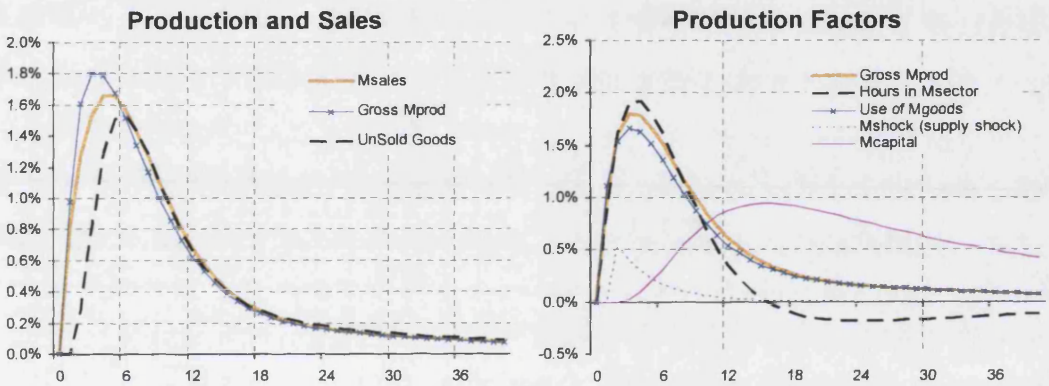

Figure 2.4: Selected impulse response functions to a positive supply shock (a shock to M-firms' production).

data. On the other hand, F-shocks generate results qualitatively similar to the data, especially for M-firms; inventory investment is negatively correlated to sales and sales is more volatile than output.

Intuitively, as the production smoothing model predicts, inventories work as buffers at high frequencies. Due to the convex cost function, it is costly to change the production level very frequently; hence firms use inventories as buffers to prevent their production from wildly varying over time.

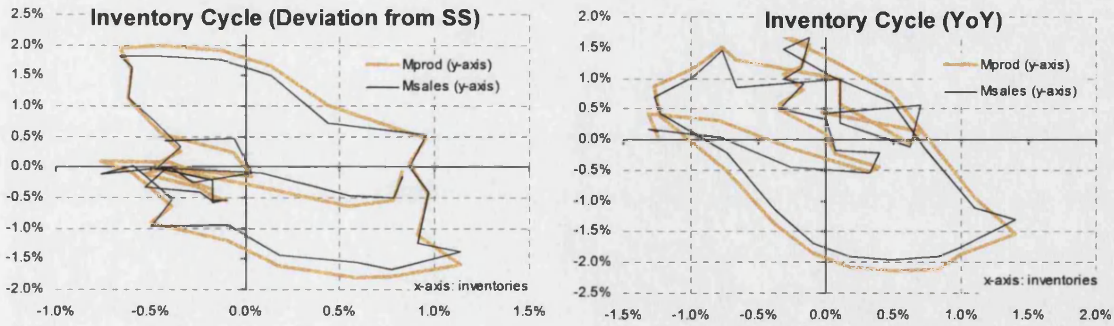

Figure 2.5: A sample path in phase diagrams generated by shocks to F-firms' production. Simulated data are converted to the year-on-year (YoY) growth rate in the right panel.

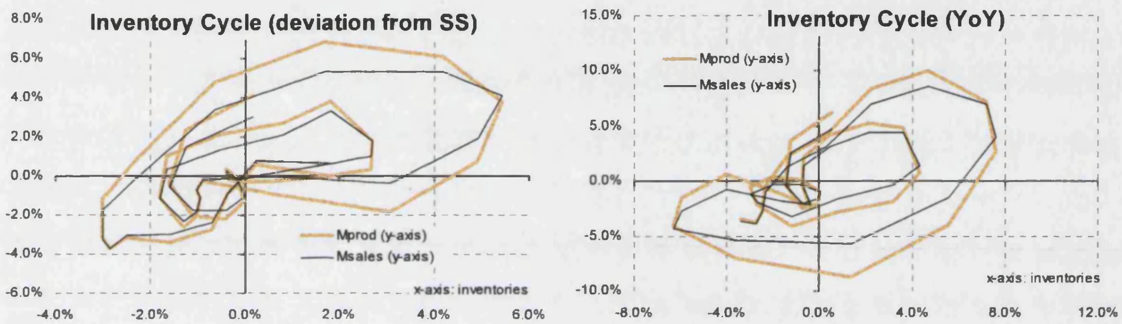

Figure 2.6: A sample path in phase diagrams generated by shocks to M-firms' production. Simulated data are converted to the year-on-year (YoY) growth rate in the right panel.

Impulse Response Functions and Inventory Cycles

Our model has two (or one, depending on parameters) pairs of conjugate complex roots whose absolute values are less than one. Because no impulse response functions exhibit clear oscillations (see Figures 2.3 and 2.4), we can say that our model shows over-damped oscillations. Roughly speaking, in our model, there exist a potential mechanism to yield cycles, but it is not strong enough to generate sine waves IRFs.

However, *in sample paths*, our model yields cycles that are quite similar to the observed inventory cycles (see Figures 2.5 and 2.6), although the shape of cycle is not clear with F-shocks. The typical length of cycles (if they exist) seems to be around 15 to 19 quarters, which is somewhat longer than Kitchin cycles (13

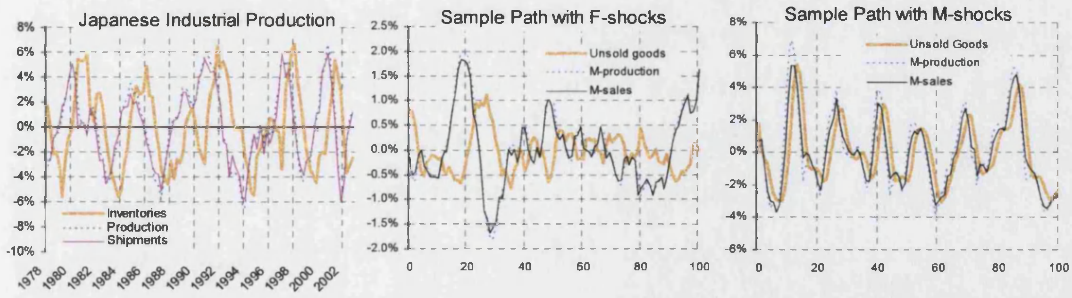

Figure 2.7: Sample paths of selected variables. Left panel shows the actual data (Japanese industrial production); middle and right panels show samples paths generated by F- and M-shocks, respectively.

quarters), is close to the Japanese post-war average (16.8 quarters), and is shorter than the U.S. post-war average (21 quarters).[31] Importantly, the sample paths with M-shocks show a time lag between peaks and bottoms of production/sales and unsold goods (inventories). Such a time lag, perhaps caused by the slow adjustment of inventories, is called a phase shift. The phase shift between production (or sales) and inventories is important to generate inventory cycles.

Changing Magnitude of Friction

Kahn et al. (2002) and McConnell and Perez-Quiros (2000) argue that the decline in GDP volatility is due to an improvement in inventory management technology. To test this idea, we simulate the model for various values of ν and υ. We interpret an improvement in inventory management as a lower value of ν (smaller magnitude of idiosyncratic shock) or a higher value of υ (a larger portion of today's output that can be sold in today's market). The results are summarised in Figures 2.8 and 2.9.

Changing the magnitude of idiosyncratic shock ν does not significantly change the volatility of GDP in either case (see the lower-right panels). Interestingly, an increase in the portion of today's products that can be sold in today's market υ increases, rather than decreases, GDP volatility for F-shocks, as opposed to

[31] For Japanese business cycles, the number is the average of all business cycles See Economic and Social Research Insutitute, Cabinet Office, Government of Japan (2004) and NBER (n.d.).

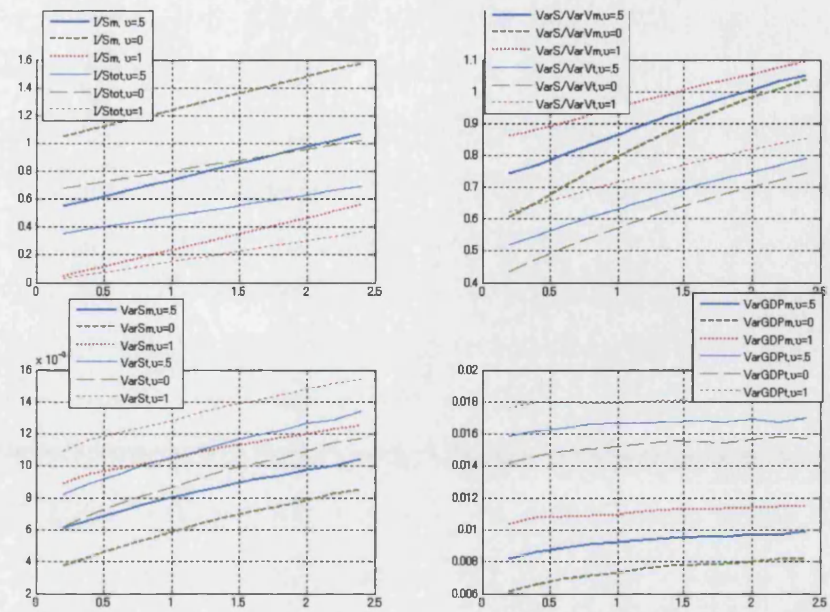

Figure 2.8: Effects of changing variation in idiosyncratic shocks (with demand shocks); lower ν (x-axis) implies lower goods market frictions. The source of aggregate shock is shocks to F-firms.

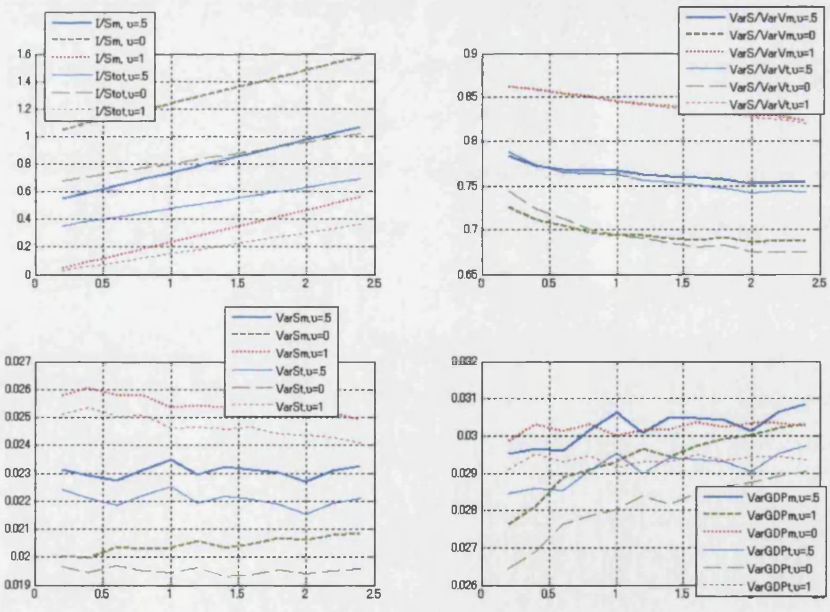

Figure 2.9: Effects of changing variation in idiosyncratic shock (with supply shocks); lower ν (x-axis) implies lower goods market frictions. The source of aggregate shock is shocks to M-firms.

their conjecture. This is perhaps because inventories are a stabilising factor at very high frequencies, as shown above. The more quickly M-firms can react to today's demand shocks, the more quickly those shocks are transmitted to M-firms' production.

The I/S ratio decreases when either ν goes down or v goes up in our experiments. This supports Kahn et al. (2002), in the sense that they regard a declining I/S ratio as evidence for their hypothesis. However, judging from the results of other experiments, it seems that the observed decline in the durable goods sector's I/S ratio is not the cause, rather than the result, of the decline in GDP volatility; the less volatile an economy is, the weaker is firms' incentive to hold inventories to hedge their loss of sales opportunities.

Overall, our model shows a negative implication for the hypothesis that an improvement in inventory management is the main reason for the decline in GDP volatility. The key intuition is that inventories are destabilising factors at business cycle frequencies but stabilising factors at very high frequencies; hence, it is uncertain whether holding lower inventories implies a more stable economy.

2.5 Conclusion

This chapter investigates a fully rational dynamic stochastic general equilibrium model with a stockout constraint and a production chain. Here, the stockout constraint simply means that no seller can sell more goods than goods than she holds on the shelf (i.e., inventories), even if she faces a strong demand. The key trade-off in this market friction is that a stockout is costly because it means the loss of a profitable sales opportunity, while having excess inventories is also costly because it imposes a too high carrying cost (financing cost). The production chain means that a firm's product is used as an input by other firms. Our model has two types of firms: final goods producers and intermediate goods producers, both of which take a basket of intermediate goods as production factors. The model constructed in this chapter is in the class of representative agent models without any price rigidity; however, the intermediate goods market is non-Walrasian.

The model quantitatively satisfies stylised inventory facts. On one hand, if the source of the shock lies on the supply side, as the cost shock model suggests, a positive technology shock pushes up production, and such an increase in production is absorbed by an increase in inventories; sales do not increase very much. One the other hand, if the source of the shock lies on the demand side, as the target inventory models predict, a positive demand shock increases sales, and inventories *initially* decrease. Hence, if we limit our focus only to very high frequency behaviours, inventories work as buffers; the production smoothing model is alive at very high frequencies. However, due to stronger demand, the target level of inventories also increases. *In subsequent periods*, production must increase more than sales, because firms must not only replenish decreased inventories but also accumulate inventories to meet the stronger demand. Because inventories increase as demand increases, inventory investment is procyclical at business cycle frequencies. In this sense, our model supports three leading inventory models at firm/industry level analyses: cost shock, production smoothing and target inventory models.

In addition, due to the production chain, adjustment of inventories is quite slow. When one firm want to replenish its inventories, it must increase its production. However, such an increase in production must use other firms' inventories as production factors. Hence, the adjustment of inventories is slow *in aggregate*; if the change in intermediate goods price is ignored (i.e., the general equilibrium feedback through price is ignored), it may seem easy to adjust inventory level quickly; but the price of intermediate goods increases wildly, which discourages firms from using them.

The most important finding in this chapter is that the stockout constraint and production chain generate a low correlation between labour productivity and output. The key intuition behind this is the slow adjustment of inventories. When a positive shock hits the model economy, firms cannot increase their use of intermediate goods because inventories of intermediate goods cannot adjust swiftly in aggregate; as a result, intermediate goods price increases. Thus, firms are encouraged to substitute their intermediate goods input with more labour input (capital cannot adjust as in the standard RBC model). Although the standard RBC model predicts the low volatility of working hours, our model yields working hours volatile enough to match the data. When output increases, because working hours increase considerably, labour productivity (i.e., output/hours) does not increase very much. Compared to the standard RBC model, the stockout constraint and production chain improve the behaviour of labour without deteriorating other properties of the model.

Appendices for Chapter 2

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The full derivation is shown in the following. The equation numbers indicated in the MATLAB codes correspond exactly to the equation numbers in the Appendix. We use the word "number" instead of "measure" unless there is a risk of confusion.

2.A Structure of M-goods Markets

This subsection provides the details of technical assumptions.

2.A.1 Agents Distribute over $[0,1] \times [0,1]$ ($\subset \mathbb{R}^2$)

Unlike the standard monopolistic competition models, we assume that agents distribute over a rectangle rather than over a line segment. Specifically, there is a continuum of markets over $[0,1]$, and there is a continuum of sellers distributed over $[0,1]$ in each market. In different markets, different varieties (types) of goods are traded; in each market, all sellers sell the same variety of goods (there are one-to-one correspondences between markets and varieties of goods).

In a discrete example, there are, say, 1,000 markets and 1,000 sellers in each market, yielding a total of 1,000,000 sellers. If all sellers behave as buyers at the same time (production chain), then there are 1,000,000 buyers as well. If each buyer visits all markets, then 1,000,000 buyers appear in every market.[32] Thus, each seller in a market meets (on average) 1,000 buyers.[33] Note that, though the discrete example is often used in the sequel, the formal derivation is based on the continuum of agents.

[32]Note that this exposition ignores F-firms. If F-firms are taken into account as buyers, then there is a total of 2,000,000 buyers in each market. In the continuous model, the measure of sellers (M-firms) is 1 (in $\mathbb{R}^2$), and the measure of buyers (M-firms plus F-firms) is 2 (in $\mathbb{R}^2$).

[33]Note that in a continuous setting, this means that each seller meets a positive measure of buyers.

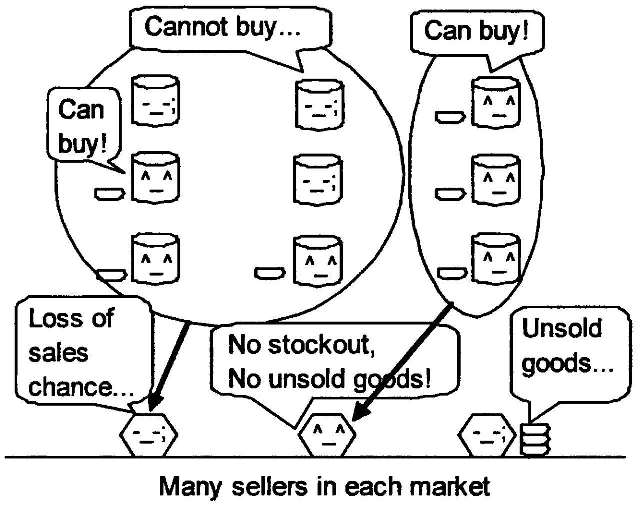

Figure 2.10: An illustration of the market structure. Buyers do not distribute evenly over sellers.

2.A.2 Idiosyncratic Shock

Next, we assume that buyers do not distribute evenly in each market. That is, some sellers meet many buyers while others meet only a few in every market. The uncertainty in the number of buyers is called an idiosyncratic shock. A simple example is illustrated in Figure 2.10. It should be clear that the idiosyncratic shock causes the mismatch between buyers and sellers in every market.

From the sellers' viewpoint, if a seller meets more buyers than GoS_t/M_t^b, where M_t^b be the baseline demand (demand per buyer), she faces a stockout; she sells all of goods on her shelf but she loses some of her customers due to the stockout. Otherwise, she has unsold goods U_{t+1} which she carries to the next period. There is a key trade-off between stockout and unsold goods. Having too low GoS_t leads to too high a stockout probability (loss of sales opportunity), but having too high GoS_t leads to too high a carrying cost of U_t.

In each market, one specific type (variety) of goods are traded. Thus, from the buyers' viewpoint, some buyers, who visit a busy seller in a market, cannot buy that specific type of goods; because we assume imperfect substitution among varieties, these buyers experience a utility cost.[34] Buyers determine M_t^p taking

[34]We assume that, once buyers visit a shop, they cannot visit other shops in the same market.

into account such losses in variety.

Uniform Distribution

We assume that the idiosyncratic demand shock follows a uniform distribution.[35] More specifically, we assume that the potential demand for a seller M_t^p is the sum of the baseline demand M_t^b (= demand per buyer) and the idiosyncratic shock e_t^p,[36] where M stands for M-goods.

$$M_t^p = M_t^b + e_t^p, \qquad e_t^p \sim U\left[-\frac{\nu}{2}, \frac{\nu}{2}\right]$$

where ν is the parameter that governs the support, and the variance $(\nu^2/12)$ of the distribution of e_t^p.

Derivation of Key Equations

The easiest way to understand the following results is by examining Figure 2.11. The two panels in the upper half show how to derive the lower right panel; the downward sloping lines in the three panels are all identical, and represent potential demand M_t^p. If the number of buyers is normalised to one, the area under this line (i.e., (A) and (B)) is equal to baseline demand M_t^b.

In the lower left panel, the downward sloping line M_t^p shows how buyers distribute over sellers. Each point on the x-axis represents a seller, and the height of the downward sloping line at each point on the x-axis shows the number of buyers who meet that seller. Note that our assumptions about CRS and price posting (see below) guarantee that all sellers hold the same level of GoS_t, which is, thus, represented by the horizontal line in the lower left panel. Hence, area

This assumption is necessary to make the idiosyncratic shock meaningful; otherwise, all buyers will buy each variety of goods in the end, reducing our M-goods markets to Walrasian markets.

[35] This assumption is only for computational simplicity. A simple urn-ball analysis concludes the degenerate distribution; if buyers visit sellers randomly, all sellers meet an equal number (measure) of buyers.

[36] It could be more natural to assume that e_t^i is the shock on the number of buyers, so that $M_t^p = M_t^i(N_b + e_t^i)$ where N_b is the average number of buyers. However, it turns out that the following computation becomes extremely messy with this specification.

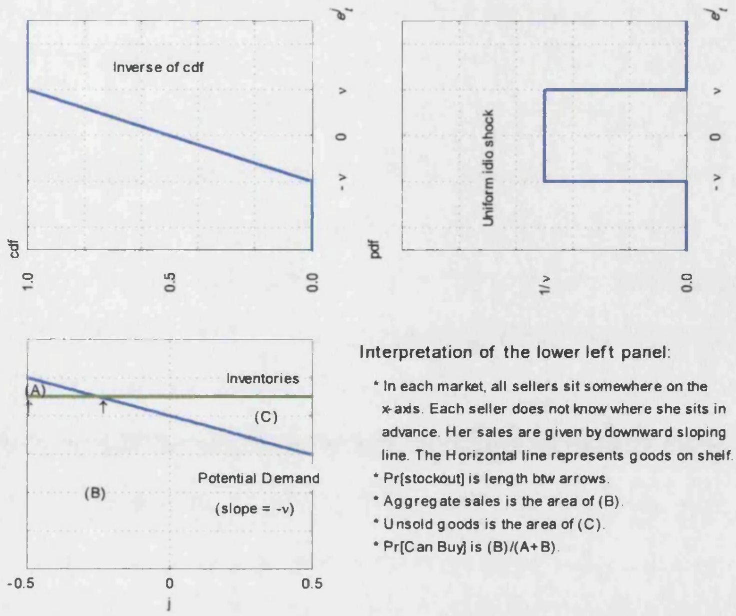

Figure 2.11: Derivation of the stockout probability and unsold goods.

(A) implies that potential demand M_t^p exceeds GoS_t, and thus the area shows unsatisfied (potential) demand. From areas (A) and (B), we can compute the probability that, in the market for a type of goods, a buyer can buy that type of good: $\Pr[\text{a buyer can buy a good}] = Q_t = (B)/((A)+(B))$.

From the viewpoint of each seller, she does not know in advance where her position is on the x-axis in the lower left panel before the realisation of the idiosyncratic shock. Hence, the probability that a seller faces a stockout is represented by the line segment between the two arrows in the lower left panel.

Area (C) implies that GoS_t exceeds M_t^p; such excess goods are carried to the next period as unsold goods U_t. However, a portion of today's production $(1-v)\,Y_t^M$ is not shown in today's market. Hence, U_t equals the area of (C) plus $(1-v)\,Y_t^M$. Also, the area of (B) shows the aggregate sales S_t, which equals $E\,[\text{sales of each seller}]$.

Key Equations

Therefore, primary school arithmetic yields the following results:

$$
\begin{aligned}
\text{Pr [a seller faces stockout]} &\equiv Pr_t = \frac{M_t^i - GoS_t}{\nu} + \frac{1}{2} && \text{(2.3a)} \\
\text{aggregate sales of market} &\equiv S_t = GoS_t - \frac{\nu}{2}\left\{\frac{M_t^i - GoS_t}{\nu} - \frac{1}{2}\right\}^2 \\
&= E\,[\text{sales of a seller}] && \text{(2.3b)} \\
\text{unsold goods} &= \frac{\nu}{2}\left\{\frac{M_t^i - GoS_t}{\nu} - \frac{1}{2}\right\}^2 + (1-v)\,Y_t^M \\
&\equiv U_{t+1} && \text{(2.3c)} \\
\text{Pr [a buyer can buy a good]} &= \frac{1}{M_t^i}\left\{GoS_t - \frac{\nu}{2}\left\{\frac{M_t^i - GoS_t}{\nu} - \frac{1}{2}\right\}^2\right\} \\
&= Q_t && \text{(2.3d)}
\end{aligned}
$$

Several comments are in order. First, neither M_t^p nor e_t^p appears in these expressions, which implies that the idiosyncratic shocks in all markets average out. Second, because there is a continuum of markets with a unit measure, Pr[a buyer can buy a good] is equal to Q_t, the measure (number) of the available varieties for each buyer. If goods are considered collectively, a low Q_t deteriorates the quality of goods due to imperfect substitution among varieties (see below for an intuitive example). Third, because there is a continuum of sellers in each market with a unit measure, and because the measure of market is unity, E [sales of a seller] is equal to the aggregate sales S_t. Fourth, regardless of the distribution assumption, the following relationship must hold:

$$
\begin{aligned}
U_{t+1} &= GoS_t + (1-v)\,Y_t^M - S_t, \qquad GoS_t = U_t + vY_T^M \\
Q_t &= S_t/M_t^i
\end{aligned}
$$

where v is the portion of today's output that can be sold in today's market. Note that we assume that only a portion of today's output can be sold in today's market. Finally, in this connection, the first term of (2.3c) represents the unsold goods that cannot be sold due to the idiosyncratic shock (i.e., the area of (C)), even though they are on the shelf, and the second term represents goods that are not on sale in today's market.

2.A.3 Miscellaneous Comments for Assumptions

The idiosyncratic shock is necessary to deal with a kinked constraint; the stockout constraint $S_t = \min\{GoS_t, M_t^p\}$ is not smooth and non-differentiable. However, $E[S_t]$ becomes smooth by adding idiosyncratic shock from the viewpoint of each agent. This technique to smooth non-smooth constraints by adding shocks is not new; it is commonly used in analyses of voting behaviour, and was first used for inventory analysis by Kahn (1987). However, this chapter shows a nice interpretation: inventories as options to sell (see the next subsection for details).

The large number of agents is necessary for aggregation. In terms of sellers, due to the law of large numbers (LLN), aggregate sales equal the expected sales ($S_t = E[S_t]$), which is a smooth function. Hence, we can linearise aggregate S_t. In terms of buyers, Q_t (the number of available varieties = probability of facing stockout) is also a smooth function, because there are infinitely many varieties (LLN).

It is also important to note that we need to confine our focus to the constant returns to scale (CRS) for aggregation. Individual M-firms (sellers) have different levels of U_t carried from the previous period, while the target level of goods on shelf $GoS_t(= U_t + vY_T^M)$ is the same for all M-firms, meaning that Y_t varies among M-firms. Hence, if production technology is not CRS, it is not possible to aggregate individual productions.

Timing Assumption

There is another assumption; firms cannot use M-goods they purchase today for today's production. This assumption is logically necessary, especially for M-firms, because M-firms must produce before M-markets open, while they can use M-goods only after M-markets close.[37]

2.A.4 Monopolistic Competition and Cost of Losing Varieties

Imperfect Competition

In addition, we assume monopolistic competition à la Dixit and Stiglitz (1977). There are two reasons not to assume perfect substitution among varieties. First, if goods were perfect substitutes for each other, buyers would not need to visit all markets. Second, because perfect substitution implies zero profit, no seller wants to hold inventories; sellers earn zero profit from their sales if they can sell their inventories, while they suffer from a carrying cost of unsold goods if they cannot.

In our environment, two-stage budgeting with quantity and price indices still holds. However, as mentioned above, because the number of available varieties fluctuates over time, we need to consider the cost effect of losing varieties.[38]

The intuition of the utility cost is as follows. Let us consider a familiar example, say, ice cream. Suppose a consumer prefers vanilla and chocolate ice creams equally, but vanilla and chocolate ice creams are not perfect substitutes for one another. Also suppose that their costs are the same. Then, one vanilla and one chocolate give higher utility than two vanillas, because they are differentiated from one another. However, the costs of vanilla + vanilla and vanilla + chocolate are the same. Thus, given the level of expenditures, having fewer varieties provides lower utility, and vice versa. Or, equivalently, with fewer varieties available, the pecuniary cost of achieving a certain level of utility is higher.

[37] Certainly, it is possible to assume that F-firms (but not M-firms) produce, say, in the second half of each period, while M-firms produce in the first half. However, it is a bit cumbersome if the timing assumptions differ between F- and M-firms.

[38] Interestingly, one of the main motivations of Dixit and Stiglitz Dixit and Stiglitz (1977) is to analyse firms' entry and exit, explicitly addressing the effect of a changing number of firms (or varieties, in our language).

Number of Available Varieties

The cost effect of losing varieties is not, in itself, of interest, and quantitatively its effect seems very weak under the plausible parameter range. However, it is a logical consequence of the combination of Dixit-Stiglitz monopolistic competition and stockout. Thus, we show only the key results without derivations. Note that they are defined and discussed *from the viewpoint of a buyer.*

First, Q_t^j is defined as an indicator function which is 1 if a buyer can buy the j-th good, and 0 otherwise. Then, the measure of the available varieties Q_t is:

$$\begin{aligned} Q_t &= \int_0^1 Q_t^j dj \\ Q_t^j &= \begin{cases} 1 & \text{if } j\text{-th variety is available} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Due to LLN, Q_t has two meanings: the number (measure) of available varieties and the probability that a *buyer* can buy a variety without encountering a stockout. Note that Q_t is a distinct concept from $1 - Pr_t$, the probability that a *seller* does not face a stockout.

Price Index

Next, we define the price index of intermediate goods as:

$$P_t^M \equiv \left[\int_0^1 \frac{Q_t^j}{Q_t} \left(P_t^{j\ 1-\theta} \right) dj \right]^{\frac{1}{1-\theta}} = \left[\int_0^1 \frac{Q_t^j}{Q_t} dj \int_0^1 \left(P_t^{j\ 1-\theta} \right) dj \right]^{\frac{1}{1-\theta}} = \left[\int_0^1 P_t^{j\ 1-\theta} dj \right]^{\frac{1}{1-\theta}}$$

where θ is the elasticity of intratemporal substitution among varieties. Several comments are in order. First, (a) multiplying by Q_t^j means that unavailable goods are not taken into account,[39] and (b) dividing by Q_t means that the index is the "average" of individual prices. Second, the integral is factorised as shown by the

[39] In general, the price index could be different among buyers, because they have different baskets of goods. However, in our model, the price index is common to all buyers because of LLN.

second equality because Q_t^j and P_t^j are, in a sense, not correlated; P_t^j is assumed to be fixed before the realisation of the idiosyncratic shock (see below), while Q_t^j is not the choice of an agent (determined exogenously by the idiosyncratic shock). Third, at optimum all sellers set the same price (i.e. $P_t^i = P_t^j$ for $\forall i, j \in [0, 1]$) due to the price posting and CRS production technology. As a result, $P_t^j = P_t^M$ for $\forall j \in [0, 1]$. Indeed, many combinations of definitions of price and quantity indices are logically consistent. We have chosen our definitions so that $P_t^j = P_t^M$ at optimum.

Quality-Adjusted Quantity Index

In this regard, the definition of the quantity index of M-goods that is consistent with our price index is:

$$M_t^K \equiv \left[\int_0^1 Q_t^j \left(M_t^j{}^{\frac{\theta-1}{\theta}} \right) dj \right]^{\frac{\theta}{\theta-1}}$$

where $K = F, M$; i.e., M_t^F is the index of M-goods purchased by F-firms, and M_t^M is that of M-firms. Again, there are several comments parallel to the price index. First, multiplying by Q_t^j means that unavailable goods are not taken into account, and (b) *not* dividing by Q_t means that the index is the "sum" of individual quantities. Second, at optimum $M_t^i = M_t^j$ for $\forall i, j \in [0, 1]$, because all prices are equal due to symmetricity. Third, it is shown that the baseline demand M_t^j in equations (2.3) is not an index, but instead is measured in terms of a physical unit. Thus,

$$M_t^j = Q_t^{\frac{-\theta}{\theta-1}} M_t^F + Q_t^{\frac{-\theta}{\theta-1}} M_t^M \tag{2.4}$$

since both F- and M-firms use M-goods for their production. Since $Q_t < 1$ and $\theta > 1$, $M_t^j > M_t^F + M_t^M$. In other words, physical demand is larger than the index. This difference becomes larger as Q_t becomes smaller. In this connection, M_t^K can be interpreted as a quality adjusted quantity index – with fewer varieties, the quality of the M-goods index becomes lower. Finally, Q_t and hence M_t^K have the same value for any buyer due to LLN.[40]

Two Stage Budgeting

From these two indices, the expenditure for M-goods of a buyer in sector K can be written as

$$\int_0^1 Q_t^j P_t^j M_t^j dj = Q_t^{\frac{-1}{\theta-1}} P_t^M M_t^K \quad \text{for} \quad K = F, M \tag{2.5}$$

where the LHS is the direct definition of expenditures on M-goods, and the RHS means that we can restate this definition with price and quantity indices. The first multiplicative term $Q_t^{\frac{-1}{\theta-1}} (= Q_t Q_t^{\frac{-\theta}{\theta-1}})$ in (2.5) represents the cost of losing varieties. This is because, under non-perfect substitution, to achieve a certain level of quantity index, an increase in quantity in each variety must compensate for a loss of varieties (see (2.4)).

2.A.5 Price Posting

An important consequence of non-Walrasian intermediate goods markets is that we cannot use the market clearing conditions as a pricing mechanism. Hence, we assume the following price posting rule as an alternative. The rule follows a simple extensive game, in which first sellers set their price, then buyers are distributed among sellers *unevenly* (idiosyncratic shock), and finally buyers choose optimal quantity if they are not subject to a stockout. This extensive game is played in each M-market in every period. We assume that (i) in each market, only one identical variety of goods are traded (varieties and markets are one-to-one correspondences to each other), (ii) in each period, each buyer visits only one seller for each variety (i.e., only one visit in each market), and (iii) even if he fails to buy a variety due to a stockout, he cannot visit other shops in that market.

[40] Although the exact components of available varieties may differ among buyers (say, some can buy vanilla+strawberry, while others mint+chocolate), the number (measure) of available varieties is the same (2 varieties in this ice cream example).

0. All the aggregate shocks are revealed.

1. Anticipating the buyers' action, sellers set their sales price before the realisation of the idiosyncratic shock. Once a seller decides her price, she cannot change it until the next period (price posting).

2. Idiosyncratic shock is revealed; buyers are distributed among sellers unevenly. As a result, some sellers meet many buyers while others meet only a few.

3. At each shop, all buyers stand in a queue, and then buyers, in order, choose an optimum amount to buy until goods on shelf run out. The order in the queue is stochastic for buyers; a buyer cannot buy the good if goods on shelf run out before his turn. In this case, he simply loses one variety.

A few remarks are in order here. First, due to the assumption that sellers set their sales price before observing the idiosyncratic shock, and the assumption of constant returns to scale, all sellers choose the same sales price. Second, the measure of available goods varies over time but, in each period, the LLN guarantees that all buyers enjoy the same measure of available varieties, although the varieties' components differ among agents.

Third, analytically this price posting rule implies that sellers take buyers' demand function as a given, while the buyers take the M-price as a given. Algebraically, we first obtain the FOC w.r.t the use of M-goods for *each* M-price, and then we obtain the FOC w.r.t. M-price subject to the demand function. Note that (i), individual sellers cannot deprive other sellers' customers in our market structure (ii) sellers exploit the slope of the demand curve as monopolists, and (iii) the quantity traded is not socially optimal.[41]

Finally, the resulting pricing is a slightly generalised version of the markup formula in the standard Dixit-Stiglitz monopolistic competition model. Namely, there exists $\tilde{\theta}$ such that $P_t^M = \tilde{\theta} / \left(\tilde{\theta} - 1 \right) \lambda_t^M$, where λ_t^M is the marginal cost of

[41]This is not only because of the price posting, but also because of externalities (see below).

producing M-goods and $\tilde{\theta} \gtrless \theta$ is the elasticity of substitution that is adjusted by Q_t and Pr_t.

2.B Analytical Results

This section summarises the analytical results.

2.B.1 Optimisations of Individual Agents

See the main text.

2.B.2 Equilibrium

There are 26 endogenous variables and 26 equations (excluding the law of motions of exogenous shocks), of which four variables and four equations (H^{Mp}_{t-1}, H^{Fp}_{t-1}, M^{M}_{t-1} and M^{F}_{t-1}) are merely lagged variables and their definitions due to the adjustment costs. With proper initial and terminal conditions, these equations define the equilibrium.

Omitting lagged variables and their definition equations, this subsection summarises the 22 equations. See Table 2.2 on page 98 for the list of variables used.

Two equations are derived from the FOCs of the representative household's optimisation.[42]

$$\beta^t \frac{\partial U_t}{\partial C_t} = \beta^t \lambda^H_t = \lambda^H_0 SDF_t \tag{2.6a}$$

$$W_t = \frac{\partial U_t / \partial L_t}{\partial U_t / \partial C_t} \tag{2.6b}$$

Nine equations come from the FOCs and constraints of M-firms' optimisation.

[42] We omit the equation for the real interest rate $R_{t-1,t}$

$$\beta E_t \left[\frac{\partial U_{t+1} / \partial C_{t+1}}{\partial U_t / \partial C_t} R_{t,t+1} \right] = 1$$

because SDF_t and $R_{t-1,t}$ move in exactly the same way in the linearised model.

$$
E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\left\{\lambda_{t+1}^M\frac{\partial Y_{t+1}^M}{\partial K_{t+1}^M}+\frac{(1-\delta_M)+\chi_{MK}\left(\left(I_{t+1}^M/K_{t+1}^M\right)^2-\delta_M^2\right)}{1-2\chi_{MK}\left(I_{t+1}^M/K_{t+1}^M-\delta_M\right)}\right\}\right]
$$

$$
=\frac{1}{1-2\chi_{MK}\left(I_t^M/K_t^M-\delta_M\right)} \tag{2.7a}
$$

$$
\chi_{MH}\left\{E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\left\{\left(\frac{H_{t+1}^{Mp}}{H_t^{Mp}}\right)^2-1\right\}\right]-2\left\{\frac{H_t^{Mp}}{H_{t-1}^{Mp}}-1\right\}\right\}+\lambda_t^M\frac{\partial Y_t^M}{\partial H_t^{Mp}}=W_t \tag{2.7b}
$$

$$
\chi_{MM}\left\{\begin{array}{c} E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\left\{\left(\frac{M_{t+1}^M}{M_t^M}\right)^2-1\right\}\right] \\ -2\left\{\frac{M_t^M}{M_{t-1}^M}-1\right\}\end{array}\right\}+E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\lambda_{t+1}^M\frac{\partial Y_{t+1}^M}{\partial M_t^M}\right]=Q_t^{M\,\frac{-1}{\theta-1}}P_t^M \tag{2.7c}
$$

$$
S_t=\theta Pr_t^-\left(1-\frac{\lambda_t^M}{P_t^M}\right)\left(Q_t^{F\,\frac{-\theta}{\theta-1}}M_t^F+Q_t^{M\,\frac{-\theta}{\theta-1}}M_t^M\right) \tag{2.7d}
$$

$$
\left(Pr_t^++Pr_t^-\right)\lambda_t^M = vP_t^iPr_t^++E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\left\{\begin{array}{c}(1-v)\,Pr_{t+1}^+P_{t+1}^i\\ +Pr_{t+1}^-\lambda_{t+1}^M+c_1\end{array}\right\}\right] \tag{2.7e}
$$

$$
\text{where } Pr_t^+ \equiv \frac{Pr_t}{1-vPr_t} \text{ and } Pr_t^- \equiv \frac{1-Pr_t}{1-vPr_t} \tag{2.7f}
$$

$$
Y_t^M = Z_t^{Mn}\left[\phi_M\left(\frac{V_t^M}{\phi_M}\right)^{\frac{\eta_M-1}{\eta_M}}+(1-\phi_M)\left(\frac{Z_t^{Mm}M_{t-1}^M}{1-\phi_M}\right)^{\frac{\eta_M-1}{\eta_M}}\right]^{\frac{\eta_M}{\eta_M-1}} \tag{2.7g}
$$

$$
V_t^M = Z_t^{Mv}\,{K_t^M}^{\alpha_M}\,{H_t^{Mp}}^{1-\alpha_M} \tag{2.7h}
$$

$$
K_{t+1}^M = (1-\delta_M)\,K_t^M+I_t^M-\chi_{MK}(I_t^M-\delta_MK_t^M)^2/K_t^M \tag{2.7i}
$$

$$
U_{t+1} = U_t-S_t+Y_t^M \tag{2.7j}
$$

Six equations come from the FOCs and the constraints of F-firms.

$$E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\left\{\frac{\partial Y_{t+1}^F}{\partial K_{t+1}^F}+\frac{(1-\delta_F)+\chi_{FK}\left(\left(I_{t+1}^F/K_{t+1}^F\right)^2-\delta_F^2\right)}{1-2\chi_{FK}\left(I_{t+1}^F/K_{t+1}^F-\delta_F\right)}\right\}\right]$$

$$=\frac{1}{1-2\chi_{FK}\left(I_t^F/K_t^F-\delta_F\right)} \tag{2.8a}$$

$$\chi_{FH}\left\{E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\left\{\left(\frac{H_{t+1}^{Fp}}{H_t^{Fp}}\right)^2-1\right\}\right]-2\left\{\frac{H_t^{Fp}}{H_{t-1}^{Fp}}-1\right\}\right\}+\frac{\partial Y_t^F}{\partial H_t^{Fp}}=W_t \tag{2.8b}$$

$$\chi_{FM}\left\{\begin{array}{c}E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\left\{\left(\frac{M_{t+1}^F}{M_t^F}\right)^2-1\right\}\right]\\ -2\left\{\frac{M_t^F}{M_{t-1}^F}-1\right\}\end{array}\right\}+E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\frac{\partial Y_{t+1}^F}{\partial M_t^F}\right]=Q_t^{F\frac{-1}{\theta-1}}P_t^M \tag{2.8c}$$

$$\begin{aligned}
Y_t^F &= Z_t^{Fn}\left[\phi_F\left(\frac{V_t^F}{\phi_F}\right)^{\frac{\eta_F-1}{\eta_F}}+(1-\phi_F)\left(\frac{Z_t^{Fm}M_{t-1}^F}{1-\phi_F}\right)^{\frac{\eta_F-1}{\eta_F}}\right]^{\frac{\eta_F}{\eta_F-1}} && (2.8d)\\
V_t^F &= Z_t^{Fv}\,{K_t^F}^{\alpha_F}\,{H_t^{Fp}}^{1-\alpha_F} && (2.8e)\\
K_{t+1}^F &= (1-\delta_F)K_t^F+I_t^F-\chi_{FK}(I_t^F-\delta_F K_t^F)^2/K_t^F && (2.8f)
\end{aligned}$$

Two equations are the market clearing conditions for labour and F-goods. Because all adjustment costs other than investments are measured in terms of F-goods, they are deducted from the market clearing condition for the final goods.

$$\begin{aligned}
H_t^H &= H_t^{Mp}+H_t^{Fp} && (2.9a)\\
Y_t^F &= C_t+I_t^M+I_t^F-AdjC_t && (2.9b)
\end{aligned}$$

$$\begin{aligned}
AdjC_t = {} & \chi_{FH}\frac{(H_t^{Fp}-H_{t-1}^{Fp})^2}{H_{t-1}^{Fp}}+\chi_{MM}\frac{\left(M_t^M-M_{t-1}^M\right)^2}{M_{t-1}^M} && (2.9c)\\
& +\chi_{MH}\frac{(H_t^{Mp}-H_{t-1}^{Mp})^2}{H_{t-1}^M}+\chi_{FM}\frac{\left(M_t^M-M_{t-1}^M\right)^2}{M_{t-1}^M}+c_1U_t
\end{aligned}$$

Three equations are derived from the specification of the idiosyncratic shock (2.3).

$$
\begin{aligned}
S_t &= \min\left\{U_t + vY_t^M, M_t^p\right\} \\
&= GoS_t - \frac{\nu}{2}\left\{\frac{Q_t^{F\frac{-\theta}{\theta-1}}M_t^F + Q_t^{M\frac{-\theta}{\theta-1}}M_t^M - GoS_t}{\nu} - \frac{1}{2}\right\}^2 \qquad (2.10a) \\
Pr_t &= \Pr\left[\text{a seller faces stockout}\right] \\
&= \frac{Q_t^{F\frac{-\theta}{\theta-1}}M_t^F + Q_t^{M\frac{-\theta}{\theta-1}}M_t^M - GoS_t}{\nu} + \frac{1}{2} \qquad (2.10b) \\
U_{t+1} &= \frac{\nu}{2}\left\{\frac{Q_t^{F\frac{-\theta}{\theta-1}}M_t^F + Q_t^{M\frac{-\theta}{\theta-1}}M_t^M - GoS_t}{\nu} - \frac{1}{2}\right\}^2 + (1-v)\,Y_t^M \qquad (2.10c)
\end{aligned}
$$

In a sense, (2.10) is the alternative to the market clearing condition of (the index of) intermediate goods. In the limit $\nu \to 0$ (i.e., no idiosyncratic shock), if $v = 1$ (all products today can be sold in today's market), (2.10a) and (2.10c) show that $U_{t+1} = 0$ (no unsold goods) and $M_t^i = GoS_t$ (M-markets clear), where $M_t^i = Q_t^{F\frac{-\theta}{\theta-1}}M_t^F + Q_t^{M\frac{-\theta}{\theta-1}}M_t^M$.

The last two equations show the law of motions of exogenous shocks. In the basic version, we use only AR(1) Hicks-neutral technology shocks in intermediate and final goods productions.

$$
\begin{aligned}
\ln Z_t^{Mn} &= \ln Z_{t-1}^{Mn} + \xi_t^{Mn} \\
\ln Z_t^{Fn} &= \ln Z_{t-1}^{Fn} + \xi_t^{Fn}
\end{aligned}
$$

where ξ_t^{Mn} and ξ_t^{Fn} are *iid* innovations that follow proper normal distributions.

2.B.3 Inventories as Options to Sell

This subsection discusses the key trade-off in the stockout model: the FOC with respect to unsold goods (2.7e). Assume, for simplicity, that $\nu = c_1 = 0$. Then,

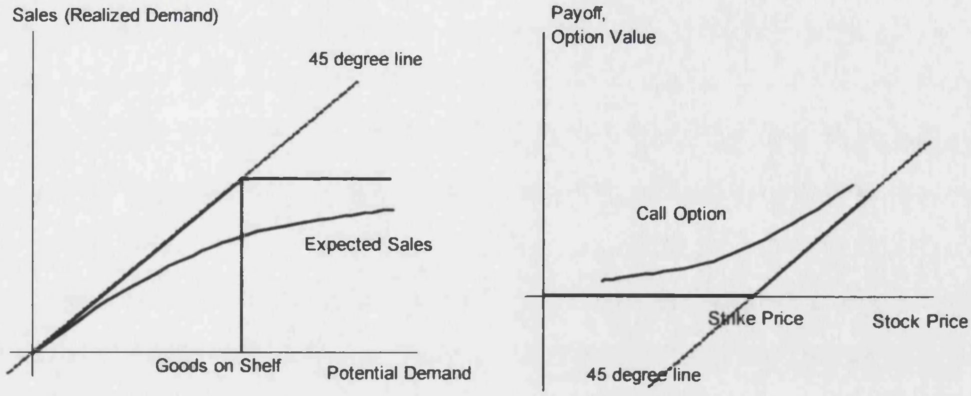

Figure 2.12: Comparison between a financial option and inventories.

(2.7e) reduces to

$$E_{t-1}\left[SDF_t\left\{\left(P_t^i-\lambda_t^M\right)\Pr\left[GoS_t<M_t^p\right]\right\}\right]=\lambda_{t-1}^M-E_{t-1}\left[SDF_t\lambda_t^M\right] \quad (2.11)$$

where λ_t^M is the marginal cost of producing M-goods (Lagrange multiplier for the law of motion of unsold goods), $SDF_t = \beta^t\lambda_t^H/\lambda_0^H$ is the stochastic discount factor, $P_t^i - \lambda_t^M$ is the marginal profit margin (P_t^i is the sales price of seller i), and $\Pr\left[GoS_t < M_t^p\right] = \partial E\left[S_t\right]/\partial U_{t-1}$ is the stockout probability from the viewpoint of individual sellers. This equation states that the carrying cost of one additional unit of inventory (RHS) is equal to the expected value of the marginal cost of the lost sales opportunity (LHS).

Equivalently, we can treat inventories as financial assets in the asset pricing equation,

$$E_{t-1}\left[SDF_t\left\{\frac{P_t^i\Pr\left[GoS_t<M_t^p\right]+\lambda_t^M\Pr\left[GoS_t>M_t^p\right]}{\lambda_{t-1}^M}\right\}\right]=1 \quad (2.12)$$

Note that the inside of the curly bracket shows the gross return on having one more unit of unsold goods.

It is important to note that the expression $\Pr[GoS_t < M_t^p]$ is essentially equivalent to an "option delta" in finance;[43] having one more unit of inventory means

[43] Remember that the delta of a call option is

having an option to sell one more unit (see Figure 2.12). In this sense, inventories have a feature similar to options on financial assets. While an option delta is defined as the sensitivity of the option price to a change in the underlying stock price in finance, $(P_t^i - \lambda_t^M)\Pr[GoS_t < D_t^p]$ is the sensitivity of profit to a change of GoS_t.[44]

$$\Delta_c = \Phi\left[\frac{(s + r\tau - k)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}\right]$$

where s is the log of the underlying stock price today, k is the strike price of the option, r is the (constant) risk-free rate, τ is the time to maturity, σ is the volatility and Φ is a (standard normal) distribution function. (One way to understand the term $\sigma\sqrt{\tau}/2$ is Jensen's inequality. The Black-Scholes model assumes a log-normal, rather than normal, distribution for stock price.)

We can see the following correspondences: value of holding inventories (value of option), derivative of the expected profit w.r.t. inventories (option delta), and demand change (price change of underlying stock) relative to the inventory holdings (strike price). The correspondence of $(P_t^M - \lambda_t^M)$ is always 1 in the case of a call option, because a 1-pound increase in stock price trivially leads to a 1-pound increase in payoff, if the stock price at the exercise date is higher than the strike price. Remember that, if the potential demand is less than goods on shelf, 1 unit of increase in the potential demand leads to an increase in profit by $(P_t^M - \lambda_t^M)$.

Related to the importance of the CRS assumption, note that, ignoring the effect of Jensen's inequality, $s + r\tau$ represents the expected stock price at the exercise date under the equivalent martingale measure (in the risk neutral world, the stock price must grow at the same rate as the risk free rate); hence, the option delta can be regarded as the probability that the stock price exceeds the strike price under the risk neutral measure. The real world probability measure should be changed to the equivalent martingale measure because investors are risk averse. However, such a change of measure is not necessary in our model, because, roughly speaking, our CRS assumption (with some other technical assumptions) implies that sellers are risk neutral. So we can use *risk neutral pricing* without changing the measure.

[44] Certainly, it is potential demand rather than inventories that is stochastic, but we can show the following result:

$$\frac{\partial}{\partial U_t} E\left[\min\{GoS_t, M_t^p\} \,|\tilde{\Omega}_t\right] = 1 - \frac{\partial}{\partial M_t^i} E\left[\min\{GoS_t, M_t^p\} \,|\tilde{\Omega}_t\right] \tag{2.13}$$

Since, as mentioned above,

$$\begin{aligned} \frac{\partial}{\partial U_t} E\left[\min\{GoS_t, M_t^p\} \,|\tilde{\Omega}_t\right] &= \Pr\left[GoS_t < M_t^p\right] \\ \frac{\partial}{\partial M_t^p} E\left[\min\{GoS_t, M_t^p\} \,|\tilde{\Omega}_t\right] &= \Pr\left[GoS_t > M_t^p\right] \end{aligned}$$

it is clear that (2.13) is equivalent to

$$\Pr\left[GoS_t < M_t^p\right] = 1 - \Pr\left[GoS_t > M_t^p\right] \tag{2.14}$$

Thus, the first derivative of the expected sales w.r.t. unsold goods means one minus a *decrease* in the expected sales due to an *increase* in the underlying demand. Here, in the equation (2.14) "one" means that, without the stockout constraint, one unit of increase in demand would trivially lead to one unit of increase in sales, but, due to the second term ($\partial E[\min\{GoS_t, M_t^p\} \,|\tilde{\Omega}_t]/\partial M_t^i$ = effect of the probability of stockout), the incremental expected sales must be smaller than they would be without the constraint. Therefore, we can restate our claim more precisely; the first derivative of sales with respect to inventories means *a reduction in the loss of sales opportunity* by holding one more unit of inventories.

2.B.4 Search Externalities

There are search externalities in M-markets.

On the buyers' side, each buyer ignores the negative effect of *congestion*. Intuitively, if buyers buy more, then available varieties ($Q_t = \Pr[\text{can buy}]$) become fewer because stockouts arise more often, but infinitesimal buyers ignore such an effect. In our model, FOC w.r.t M-goods input is

$$E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\frac{\partial Y_{t+1}^M}{\partial M_t^M}\right] = Q_t^{\frac{-1}{\theta-1}}P_t^M \tag{2.15}$$

However, if there were, say, a strong union of purchasing managers, which coordinated buyers' decisions, the FOC w.r.t M_t^M would be

$$E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\frac{\partial Y_{t+1}^M}{\partial M_t^M}\right] = Q_t^{\frac{-1}{\theta-1}}P_t^M\left(1+\frac{-1}{\theta-1}\frac{\partial Q_t/Q_t}{\partial M_t^M/M_t^M}\right) \tag{2.16}$$

which implies that the social cost (RHS of (2.16)) is larger than the private cost (RHS of (2.15)). The additional term shows the effect of congestion, which infinitesimal buyers ignore.

On the sellers' side, if there were a powerful union of sellers which coordinated sellers, the FOC w.r.t. unsold goods of intermediate goods producers would be

$$\begin{aligned}
&E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\left\{\begin{array}{c}\left(P_{t+1}^i-\lambda_{t+1}^M\right)Pr_{t+1}\\ +\frac{M_{t+1}^M}{U_{t+1}}\left(\frac{\partial M_{t+1}^M/M_{t+1}^M}{\partial Q_{t+1}/Q_{t+1}}-\frac{\theta}{\theta-1}Q^{\frac{-\theta}{\theta-1}}\right)\frac{\partial Q_{t+1}/Q_{t+1}}{\partial U_{t+1}/U_{t+1}}\left(1-Pr_{t+1}\right)\end{array}\right\}\right]\\
=\ &\lambda_t^M - E_t\left[\beta\frac{\lambda_{t+1}^H}{\lambda_t^H}\lambda_{t+1}^M\right]
\end{aligned}$$

but infinitesimal sellers ignore two effects. The first is *the cost of losing varieties* ($\frac{M_{t+1}^M}{U_{t+1}}\frac{\partial M_{t+1}^M/M_{t+1}^M}{\partial Q_{t+1}/Q_{t+1}}\frac{\partial Q_{t+1}/Q_{t+1}}{\partial U_{t+1}/U_{t+1}} \geq 0$). When inventories are higher, the measure of varieties that a buyer can enjoy is larger; hence the effective cost is lower, which, in turn, stimulates the demand for M-goods. However, such a mechanism is ignored. The second is the *squeezing effect* due to fewer varieties

$(-\frac{M^M_{t+1}}{U_{t+1}}\frac{\theta}{\theta-1}Q^{\frac{-\theta}{\theta-1}}\frac{\partial Q_{t+1}/Q_{t+1}}{\partial U_{t+1}/U_{t+1}} \leq 0)$. As mentioned in the previous subsection, when fewer varieties are available, the (physical unit of) potential demand of one buyer becomes larger to achieve a certain level of the quantity index. These two effects offset one another; the net effect may be positive or negative.

Nonetheless, some numerical experiments suggest that the overall effect of the search externality seems to be very small.

Chapter 3

A Solution Method for Linear Rational Expectation Models under Imperfect Information

This chapter has developed a solution algorithm for linear rational expectation models under imperfect information – which, in this chapter, means that some decisions are made based on smaller information sets than others.

Perhaps surprisingly, in state space representation, imperfect information does not change the coefficients on the past crawling variables. Hence, an imperfect model is saddle-path stable (sunspot, explosive), if the corresponding perfect information model is saddle path stable (sunspot, explosive, respectively). Moreover, if the minimum information set contains all the information up to time $t-S-1$, then the direct effects on the impulse response functions last for only the first S periods.

Although imperfect information does not drastically change the qualitative nature of a model, it can significantly alter its quantitative properties. This chapter demonstrates, as an example, that adding imperfect information to the RBC model remarkably improves the correlation between labour productivity and output.

3.1 Introduction

This chapter presents a solution algorithm for linear rational expectation models under imperfect information. "Imperfect information" in this chapter signifies that some decisions may be made before observing some shocks, while others may be made after observing them. For example, we can consider a variant of the RBC model, in which labour supply is decided before observing today's productivity shock. In this variant, apart from the information structure (i.e., the FOC with respect to labour supply has an expectation operator), the equations that define the equilibrium are the same as in the standard RBC model.

Imperfect information is an important consideration for several reasons. First, imperfect information plays an important role in many important classes of models, such as the sticky information model of Mankiw and Reis (2001). Second, researchers often do not know *a priori* what information is available when each decision is made; hence, they may want to estimate the information structure by parameterising it, or they may want to experiment on a model under several patterns of information structure. It is easy to implement such robustness checks with the algorithm; once structural equations are obtained, then the additional input to the algorithm is only the information structure in a model. Third, the obtained numerical result may not be robust for a small change in information structure. Indeed, imperfect information may significantly alter the second moments and the shapes of impulse response functions.

This chapter offers an easy-to-use MATLAB code to solve a general class of linear models under imperfect information.[1] The algorithm provides the solution

[1] The set of MATLAB codes is available upon request: k.shibayama@kent.ac.uk

of a model in the form of

$$\begin{aligned}\kappa_{t+1} &= H\kappa_t + J\xi^{t,S} \\ \phi_t &= F\kappa_t + G\xi^{t,S} \\ \xi^{t,S} &\equiv \begin{pmatrix} \xi_t^T & \cdots & \xi_{t-S}^T \end{pmatrix}^T\end{aligned}$$

where κ_t and ϕ_t are the vectors of crawling and jump variables, respectively, and ξ_{t-s} is the vector of innovations at time $t-s$, for $s = 0, \cdots, S$, where S is such that the minimum information set in the model includes all information up to time $t - S - 1$. The superscript T indicates transposition, and hence $\xi^{\tau,S}$ is the vertical concatenation of $\{\xi_{\tau-s}\}_{s=0}^{S}$. H, J, F and G are the solution matrices provided by the algorithm. The algorithm is an extension of the QZ method by Sims (2002).

The most important breakthrough made by this chapter is its choice of state variables. The state variables in this solution are κ_t and $\xi^{t,S}$. Imperfect information requires the expansion of the state space, but this can be done either by expanding the innovation vector or by expanding the set of crawling variables (Note that the representation of state space is not necessarily unique). Our choice of state variables works intuitively because, if past innovations are recorded, we can recover the past crawling variables and hence recover the information available in past periods.[2]

By keeping the number of crawling variables unchanged, it can be shown that the dynamic parts of the solution (i.e., H and F matrices) are the same as in the corresponding perfect information model. Thus, it is clear that if the corresponding perfect model is saddle-path stable (sunspot, explosive), then an imperfect information model is also saddle-path stable (sunspot, explosive, respectively). That is to say, the information structure does not alter the dynamic stability property.

Moreover, invariant H and F matrices imply that the direct effects of imperfect information on impulse response functions last for only S after an impulse, if

[2]Hence, even though some decisions are made without observing κ_t, for example, economic models can be formulated as in (3.2).

the minimum information set at time t in a model has all the information up to time $t - S - 1$. In subsequent periods, the impulse response functions follow essentially the same process as those in the perfect information counterpart – more specifically, if, S period after an impulse, the values of the crawling variables are κ_S, then the following impulse response functions are exactly the same as those of the perfect information counterpart that starts with κ_S and zero innovations. One such example can be found in Dupor and Tsuruga (2005), who argue that the hump-shaped impulse response functions found in Mankiw and Reis (2001) critically hinge on the assumption of the Calvo style information updating; some agents, though their population decreases over time, cannot renew their information forever. By instead constructing the Taylor style staggered information renewal, Dupor and Tsuruga (2005) show that impulse response functions jump to zero right after the last cohort renews its information set.

There are, at least allegedly, two existing treatments of imperfect information.[3] The first remedy for imperfect information is to define the *dummy* variables. For example, consider a variant of the standard RBC model, in which labour supply L_t is determined without observing today's innovations. Then, the optimal labour supply is determined by

$$0 = E_{t-1}\left[\eta L_t + \sigma C_t - W_t\right] \tag{3.1}$$

[3]There are three types of methods for perfect information models.

1. King and Watson's method (1998 and 2002) (see also Woodford (undated)) implements a two-stage substitution. First, non-dynamic jump variables are substituted out, and then dynamic jump variables are substituted out from the system of equations.
2. In the QZ method by Sims (2002) (see also Klein (2000)), the QZ decomposition is applied to matrices on endogenous variables. Recognising that (1) roots that correspond to non-dynamic jump variables are infinite, and (2) roots that correspond to dynamic jump variables are larger than one in absolute terms, the transversality conditions (TVCs) eliminate both types of jump variables at once.
3. The method of undetermined coefficients by Uhlig (1999) (see also Christiano (1998)) substitutes a guess solution into the given system of equations; the resulting matrix polynomial is solved directly. In principle, this method does not require that given equations are first-order difference equations. Higher order matrix polynomials can be numerically solved (see Appendix).

where C_t and W_t are consumption and wage at time t, η and σ are parameters provided by the theory, and $E_{t-1}[\,]$ is the expectation operator with all information up to time $t-1$. Define dummy variable L_t^* such that

$$\begin{aligned} 0 &= E_t\left[\eta L_{t+1}^* + \sigma C_{t+1} - W_{t+1}\right] \\ L_{t+1} &= L_t^* \end{aligned}$$

In this method, having additional crawling variable L_t, the set of crawling variables is expanded. The problem with this method is that it cannot solve the model if some endogenous variables are determined before observing some (not all) of today's innovations but after observing the others.

The other possibility is a modification of the method of undetermined coefficients. According to Christiano (1998), his version of method of undetermined coefficients, like ours, can deal with models in which some endogenous variables are determined before observing some (not all) of today's innovations are observed but after observing the others. The most salient difference between his method and ours is in the specification of information structure; Christiano (1998) requires a user to provide only one matrix R that specifies which innovations are to be included in the information set of each expectation operator. Roughly speaking, matrix R relates equations to observable innovations. In contrast, in the algorithm developed in this chapter, a researcher must specify two matrices: one relates innovations to equations (like Christiano (1998)), and the other relates innovations to variables. The difference is crucial. To understand this, consider the above example (3.1). It is clear that a researcher must specify the information set of the expectation operator in (3.1). However, in a given information set, there are generically three possibilities, namely that (a) the representative household fixes labour supply before observing some of today's innovations, (b) it determines wage before innovations (sticky wage), or (c) it decides consumption before innovations. Hence, one more matrix is necessary in our algorithm to specify which of C_t, W_t or

H_t is chosen while not having full information. In general, the quantitative behaviour of a model is completely different, depending on which variables are assumed to be decided before observing some information. Indeed, in the following section, it is shown that the difference between (a) and (b) is very crucial.

The plan of this chapter is as follows. In Section 3.2, we define the problem and derive the solution, and show two key observations. First, if the k-th time t variable $y_{k,t}$ is determined without observing the i-th time $t-s$ innovations $\xi_{i,t-s}$, then $y_{k.t}$ cannot respond to $\xi_{i,t-s}$, given κ_{t-S}. Second, if the expectation operator in the j-th equation has an information set that includes $\xi_{i,t-s}$, $\xi_{i,t-s}$ cannot be the source of the expectation error in the j-th equation. It turns out that these two restrictions are enough to determine the unique solution coefficients. In Section 3.3, we discuss the assumptions that are necessary for guaranteeing the existence of a solution. Each of them has some economic meaning, and the existence condition is slightly tighter under imperfect information than under perfect information. In Section 3.4, the main features of the solution of imperfect information models are briefly discussed. Most of them are direct consequences of the invariant H and F matrices. In Section 3.5, we demonstrate the effects of imperfect information on the otherwise standard RBC model as an example. Section 3.6 concludes the discussion.

3.2 Derivation of the Solution

Essentially, our algorithm is an extension of the QZ method used in Sims (2002). Our problem is to obtain the state space representation of a solution that satisfies two key zero restrictions. For the details of matrix notation, see the Appendix.

3.2.1 Definition of the Problem

The inputs and outputs of the algorithm are defined.

Given Models

Following Sims (2002), we formulate the linear rational models with expectation errors as follows.

$$0 = Ay_{t+1} + By_t + C\xi_t + D\xi_{t+1} + E\xi^{t,S} \tag{3.2}$$

where

$$\begin{aligned} E &= \begin{bmatrix} E_0 & E_1 & \cdots & E_s & \cdots & E_S \end{bmatrix} \\ &= \begin{bmatrix} E_{0,11} & \cdots & E_{0,1N} & & E_{s,11} & \cdots & E_{s,1N} & & E_{S-1,11} & \cdots & E_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ E_{0,M1} & \cdots & E_{0,MN} & & E_{s,M1} & \cdots & E_{s,MN} & & E_{S-1,M1} & \cdots & E_{S,MN} \end{bmatrix} \end{aligned}$$

$$y_t = \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}, \xi^{t,S} = \begin{pmatrix} \xi_t \\ \vdots \\ \xi_{t-S} \end{pmatrix}$$

y_t is the vector of all endogenous variables, in which κ_t is the vector of crawling variables and ϕ_t is that of jump variables. Stock variables are all recorded at the beginning of each period. M is the number of equations, which is equal to the number of endogenous variables, N is the number of innovations, and S is such that the minimum information set includes ξ_{t-S-1}.

ξ_{t-s} is a column vector of *iid* innovations at time $t-s$. Limiting ξ_t to *iid* is not restrictive since we can add the law of motions of serially correlated shocks to the system of equations and treat the shocks themselves as crawling variables.[4].

A, B and C are proper coefficient matrices, and they are provided by an economic model. D and E represent the expectation errors. D is non-zero even for perfect information models, because of dynamic jump variables (e.g., expectation error in the consumption Euler equation due to consumption at time $t+1$). An

[4]See Woodford (undated): this technique simplifies the algebra and computation significantly.

economic theory must specify the positions of zero elements in D and E,[5] while the values of non-zero elements are computed by the algorithm. ξ_{t-s} can be the source of expectation errors because some endogenous variables are decided without observing it.

Goal of the Algorithm

Our objective is to obtain the state space representation of (3.2).

$$\kappa_{t+1} = H\kappa_t + J\xi^{t,S} \tag{3.3a}$$

$$\phi_t = F\kappa_t + G\xi^{t,S} \tag{3.3b}$$

where

$$\begin{aligned}
J &\equiv \begin{bmatrix} J_0 & J_1 & \cdots & J_s & \cdots & J_S \end{bmatrix} \\
&\equiv \begin{bmatrix} J_{0,11} & \cdots & J_{0,1N} & & J_{s,11} & \cdots & J_{s,1N} & & J_{S,11} & \cdots & J_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ J_{0,M_\kappa 1} & \cdots & J_{0,M_\kappa N} & & J_{s,M_\kappa 1} & \cdots & J_{s,M_\kappa N} & & J_{S,M_\kappa 1} & \cdots & J_{S,M_\kappa N} \end{bmatrix} \\
G &\equiv \begin{bmatrix} G_0 & G_1 & \cdots & G_s & \cdots & G_S \end{bmatrix} \\
&\equiv \begin{bmatrix} G_{0,11} & \cdots & G_{0,1N} & & G_{s,11} & \cdots & G_{s,1N} & & G_{S,11} & \cdots & G_{S,1N} \\ \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ G_{0,M_\phi 1} & \cdots & G_{0,M_\phi N} & & G_{s,M_\phi 1} & \cdots & G_{s,M_\phi N} & & G_{S,M_\phi 1} & \cdots & G_{S,M_\phi N} \end{bmatrix}
\end{aligned}$$

3.2.2 Two Key Observations

This subsection shows two key zero restrictions. The algorithm seeks the solution that satisfies them.

[5]Exactly speaking, a researcher does not need to specify the zero elements in D. Instead, the number of crawling variables must be specified. Given construction of $y_t = [\kappa_t^T \ \phi_t^T]^T$, the algorithm determines the positions of zero elements in D (Only dynamic jump variables can be the sources of expectation errors D).

Repeated Substitutions

To obtain the representation of κ_{t+1} and ϕ_t as functions of κ_{t-S} and $\xi_{t-\tau}$ for $\tau = 0, \cdots, 2S-1$, repeat the substitution of the vertically concatenated guess solution (3.3) into itself. Defining $\check{H} = [H^T \ F^T]^T$,

$$\begin{aligned}
\begin{pmatrix} \kappa_{t+1} \\ \phi_t \end{pmatrix} &= \check{H}\kappa_t + \tilde{\Gamma}\xi^{t,S} = \check{H}\left(H^S\kappa_{t-S} + \textstyle\sum_{k=1}^{S} H^{k-1}J\xi^{t-k,S} \right) + \tilde{\Gamma}\xi^{t,S} \\
&= \check{H}H^S\kappa_{t-S} + \left(\Gamma_0\xi_{t-0} + \Gamma_1\xi_{t-1} + \cdots + \Gamma_S\xi_{t-S}\right) \\
&\quad +\check{H}\begin{pmatrix} H^0\left(J_0\xi_{t-1} + J_1\xi_{t-2} + \cdots + J_S\xi_{t-1-S}\right) \\ +H^1\left(J_0\xi_{t-2} + J_1\xi_{t-3} + \cdots + J_S\xi_{t-2-S}\right) + \cdots \\ +H^{S-1}\left(J_0\xi_{t-S} + J_1\xi_{t-S-1} + \cdots + J_S\xi_{t-S-S}\right) \end{pmatrix} \\
&= \check{H}H^S\kappa_{t-S} + \Pi_0\xi_t + \Pi_1\xi_{t-1} + \cdots + \Pi_s\xi_{t-s} + \cdots + \Pi_S\xi_{t-S} \\
&\quad + \text{terms with } \xi_{t-\tau} \text{ for } \tau \geq S+1 \qquad (3.4)
\end{aligned}$$

where $\tilde{\Gamma} \equiv \begin{bmatrix} \Gamma_0 & \cdots & \Gamma_s & \cdots & \Gamma_S \end{bmatrix}$ with $\Gamma_s \equiv \begin{bmatrix} J_s^T & G_s^T \end{bmatrix}^T$, and

$$\begin{aligned}
\Pi_0 &\equiv \Gamma_0 = \begin{bmatrix} J_0 \\ G_0 \end{bmatrix} \\
\Pi_1 &\equiv \Gamma_1 + \begin{bmatrix} H \\ F \end{bmatrix} J_0 = \begin{bmatrix} J_1 + HJ_0 \\ G_1 + FJ_0 \end{bmatrix} \\
\Pi_2 &\equiv \Gamma_2 + \begin{bmatrix} H \\ F \end{bmatrix} (J_1 + HJ_0) = \begin{bmatrix} J_2 + H(J_1 + HJ_0) \\ G_2 + F(J_1 + HJ_0) \end{bmatrix}, \cdots \\
\Pi_s &\equiv \Gamma_s + \begin{bmatrix} H \\ F \end{bmatrix} \left(\textstyle\sum_{k=0}^{s-1} H^{s-1-k}J_k\right) = \begin{bmatrix} J_s + H\sum_{k=0}^{s-1} H^{s-1-k}J_k \\ G_s + F\sum_{k=0}^{s-1} H^{s-1-k}J_k \end{bmatrix}, \cdots \\
\Pi_S &\equiv \Gamma_S + \begin{bmatrix} H \\ F \end{bmatrix} \left(\textstyle\sum_{k=0}^{S-1} H^{S-1-k}J_k\right) = \begin{bmatrix} J_S + H\sum_{k=0}^{S-1} H^{S-1-k}J_k \\ G_S + F\sum_{k=0}^{S-1} H^{S-1-k}J_k \end{bmatrix}
\end{aligned}$$

In the recursive representation,

$$
\begin{aligned}
\Pi_0 &= \Gamma_0 = \begin{bmatrix} J_0 \\ G_0 \end{bmatrix} \\
\Pi_s &= \Gamma_s + \tilde{H}\Pi_{s-1} \text{ for } s = 1, \cdots, S
\end{aligned}
$$

where

$$
\tilde{H} \equiv \begin{bmatrix} H & 0 \\ F & 0 \end{bmatrix} \tag{3.6}
$$

Intuitively, the j,k-th element of Π_s is the effect of $\xi_{k,t-s}$ (the k-th innovation at time $t-s$) on $y_{j,t}$ (the j-th endogenous variable at time t). Thus, given κ_{t-S}, $\Pi_{s,jk}$, which is defined as the j,k-th element of Π_s, is zero if $y_{j,t}$ is determined without observing $\xi_{k,s}$.

In the matrix representation

$$
\Gamma = M_{\Gamma\Pi}\Pi \tag{3.7}
$$

where

$$
\begin{aligned}
\Gamma &\equiv \begin{bmatrix} \Gamma_0^T & \cdots & \Gamma_s^T & \cdots & \Gamma_S^T \end{bmatrix}^T && (3.8a) \\
\Pi &\equiv \begin{bmatrix} \Pi_0^T & \cdots & \Pi_s^T & \cdots & \Pi_S^T \end{bmatrix}^T && (3.8b) \\
M_{\Gamma\Pi} &\equiv \begin{bmatrix} I & & & 0 \\ -\tilde{H} & I & & \\ & \ddots & \ddots & \\ 0 & & -\tilde{H} & I \end{bmatrix} && (3.8c)
\end{aligned}
$$

$M_{\Gamma\Pi}$ is clearly invertible, and plays a key role in the following.

Zero Restrictions

Throughout this chapter, we exploit the following two observations.

1. If the k-th set of variables $y_{k,t}$ does not observe the i-th set of time $t-s$ innovations $\xi_{i,t-s}$, given κ_{t-S} and $\xi_{t-\tau}$ for $\tau = s+1, \cdots$, $\partial y_{k,t}/\partial \xi_{t-s} = \Pi_{s,ki} = 0$. Simply put, no decision can respond to unobserved innovations.

2. If the information set of the expectation operator in the j-th equation includes the i-th time $t-s$ innovation $\xi_{i,t-s}$, then the realization of the j-th equation must hold for any realisation of the i-th innovation. The expectation error in each expectation operator occurs only due to innovations that are not included in its information sets. Thus, $E_{s,ji} = 0$.

For example, suppose that labour supply L_t (k-th variable, $y_{k,t}$) is decided on before observing today's technology shock (i-th shock, $\xi_{i,t}$), but after today's preference shock (l-th shock, $\xi_{l,t}$), both of which are *iid.* If the FOC with respect to L_t is the j-th equation,

$$\begin{aligned} \Pi_{0,ki} &= 0 \ (\xi_{i,t-0} \text{ does not affect } y_{k,t}) \\ E_{0,jl} &= 0 \ (\xi_{l,t-0} \text{ does not cause expectation error in } j\text{-th eqn}) \end{aligned}$$

Roughly speaking, $E_{0,jl} = 0$ means that if the expectation operator of the j-th equation is eliminated from the j-th equation, it still holds in terms of $\xi_{0,l}$. It is the duty of a user to specify the positions of these zero elements in Π and E.

3.2.3 Sketch of Derivation and Key Equations for Computation

The fully detailed derivation is provided in the Appendix. This subsection briefly describes the skeleton of the derivation and lists the minimum results necessary for computation.

QZ Decomposition

In order to introduce notations, this subsection briefly reviews the QZ decomposition (or generalised Schur decomposition). For matrices A and B ($\in \mathbb{C}^{n\times n}$), there exist unitary matrices Q and Z such that

$$\begin{aligned} Q^H A Z &= \Omega_A \\ Q^H B Z &= \Omega_B \end{aligned}$$

where Ω_A and Ω_B are both upper triangular matrices, and superscript H indicates a conjugate transpose. Any unitary matrix U satisfies $U^H U = UU^H = I$. Let a_{kk} and b_{kk} be the k-th diagonal elements in Ω_A and Ω_B, respectively. Assuming that a_{kk} and b_{kk} are not zero at the same time, then $\lambda_k \equiv b_{kk}/a_{kk}$ for $k = 1, \cdots, n$ are the generalised eigenvalues of the matrix pencil $B - \lambda_k A$.[6]

The basic idea is that by applying the QZ decomposition to (3.2) as in Sims (2002), the algorithm separates unstable roots from stable roots.

$$\begin{aligned} 0 &= Ay_{t+1} + By_t + C\xi_t + D\xi_{t+1} + E\xi^{t,S} \\ &= \Omega_A Z^H y_{t+1} + \Omega_B Z^H y_t + Q^H C\xi_t + Q^H D\xi_{t+1} + Q^H E\xi^{t,S} \\ &= \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix} \begin{pmatrix} s_{t+1} \\ u_{t+1} \end{pmatrix} + \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \\ &\quad + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} C\xi_t + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} D\xi_{t+1} + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} E\xi^{t,S} \end{aligned}$$

where

$$\begin{pmatrix} s_t \\ u_t \end{pmatrix} \equiv Z^H \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}$$

By using TVCs, the expected values of all unstable roots u_{t+1} are set to be equal

[6] See Appendix for a brief review of the relationship between the system of first-order difference equations and generalised eigenvalues.

to zero (Remember that all innovations are assumed to be *iid*).[7]

Notations for the Outputs of QZ Decomposition

For later use, we define submatrices as follows

$$Z^H \equiv \begin{bmatrix} Z^H_{s.} \\ Z^H_{u.} \end{bmatrix} \equiv \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix}, Z \equiv \begin{bmatrix} Z_{\kappa s} & Z_{\kappa u} \\ Z_{\phi s} & Z_{\phi u} \end{bmatrix}, Q^H \equiv \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} \tag{3.9a}$$

$$\Omega^A \equiv \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix}, \Omega^B \equiv \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \tag{3.9b}$$

where subscripts u and s imply unstable and stable roots, respectively. Note that Ω^A_{ss} and Ω^B_{uu} are both invertible by construction.

Additionally, we define the four matrices as

$$\Lambda^A_{s\kappa} \equiv \Omega^A_{ss} Z^H_{s\kappa} + \Omega^A_{su} Z^H_{u\kappa} \tag{3.10a}$$

$$\Lambda^A_{s\phi} \equiv \Omega^A_{ss} Z^H_{s\phi} + \Omega^A_{su} Z^H_{u\phi} \tag{3.10b}$$

$$\Lambda^B_{s\kappa} \equiv \Omega^B_{ss} Z^H_{s\kappa} + \Omega^B_{su} Z^H_{u\kappa} \tag{3.10c}$$

$$\Lambda^B_{s\phi} \equiv \Omega^B_{ss} Z^H_{s\phi} + \Omega^B_{su} Z^H_{u\phi} \tag{3.10d}$$

Note that all the matrices defined here are obtained from the outputs of the QZ decomposition.

Matrix Subscripts

We introduce the following notation rule for subscripts on matrices. For a matrix A,

- $A_{.x}$ is columns x of a matrix A,
- $A_{x.}$ is rows x of a matrix A,

[7] If the expectations of u_{t+1} must be zero under perfect information, they must be also zero under imperfect information. This can be shown by simply applying the iterated linear projection.

- $A_{.\neg x}$ is the columns remaining after the elimination of columns x, and
- $A_{\neg x.}$ is the rows remaining after the elimination of rows x,

where x is the name of a set of columns or rows. This notation makes certain matrix operations extremely simple. See the Appendix for further details.

Zero Restrictions

As a result of manipulating the matrix equations, it is shown that

$$0 = \Pi + M_{\Pi E}(E + \mathbf{C}) \tag{3.11}$$

$$M_{\Pi E} \equiv (M_{y\Gamma} M_{\Gamma\Pi}) \backslash \mathbf{Q} \tag{3.12}$$

where

$$\Gamma \equiv \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_{S-1} \end{pmatrix}, E \equiv \begin{pmatrix} E_0 \\ \vdots \\ E_{S-1} \end{pmatrix} \tag{3.13a}$$

$$\mathbf{C} \equiv \begin{pmatrix} C_0 \\ 0 \end{pmatrix}, \mathbf{Q} \equiv \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix} \tag{3.13b}$$

$$M_{y\Gamma} \equiv \begin{bmatrix} \Phi & \Lambda^{0A} & & 0 \\ & \ddots & & \\ & & \Phi & \Lambda^{0A} \\ 0 & & & \Phi \end{bmatrix} \tag{3.13c}$$

$$\Phi \equiv \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix}, \Lambda^{0A} \equiv \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \tag{3.13d}$$

and $X \backslash Y = X^{-1}Y$. Bear in mind that, while $M_{y\Gamma}$ is computable solely from the outputs of the QZ decomposition, $M_{\Gamma\Pi}$ and $M_{\Pi E}$ are obtained only after finding H and F.

Given $M_{\Gamma\Pi}$, E and Π are computed column by column (i.e., innovation by innovation). It is important to remember that some elements in Π and E are zero due to the two zero restrictions. Thus, for the i-th column (or equivalently for the i-th innovation),

$$0 = \begin{pmatrix} \Pi_{1,i} \\ \vdots \\ \Pi_{k,i}\,(=0) \\ \vdots \\ \Pi_{M(S+1),i} \end{pmatrix} + M_{\Pi E} \left(\begin{pmatrix} 0 \\ \vdots \\ E_{ji} \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} C_{\cdot i} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \right) \tag{3.14}$$

where M in subscripts is the number of equations and hence $M(S+1)$ is the number of rows in Π.

From the k-th set of equations in (3.14)

$$0 = \Big[M_{\Pi E} \Big]_{kj} E_{ji} + \Big[M_{\Pi E} \Big]_{kj} \mathbf{C}_{ji} + \Big[M_{\Pi E} \Big]_{k\neg j} \mathbf{C}_{\neg ji} \tag{3.15}$$

which gives the values of the non-zero elements of E. From the remaining equations in (3.14),

$$\begin{aligned} 0 \;=\; & \Pi_{\neg ki} + \Big[M_{\Pi E} \Big]_{\neg kj} \mathbf{C}_{ji} + \Big[M_{\Pi E} \Big]_{\neg k \neg j} \mathbf{C}_{\neg ji} \\ & - \Big[M_{\Pi E} \Big]_{\neg kj} \left(\Big[M_{\Pi E} \Big]_{kj} \backslash \Big[M_{\Pi E} \Big]_{k\neg j} \mathbf{C}_{\neg ji} + \mathbf{C}_{ji} \right) \end{aligned} \tag{3.16}$$

which gives the non-zero elements of Π.

It is assumed that $[M_{\Pi E}]_{kj}$ is invertible. In general, however, it is not necessarily true, and the economic meaning of its invertibility is discussed later.

Solution

The solution algorithm computes key matrices sequentially. The basic structure is as follows:

1. Obtain submatrices form the outputs of the QZ decomposition (3.9 and 3.10).

2. Obtain H and F from (3.17).

3. Obtain $M_{\Gamma\Pi}$, $M_{y\Gamma}$ and $M_{\Pi E}$ from (3.8c, 3.13c and 3.12).

4. Obtain E and Π from (3.18 and 3.19).

5. Obtain G and J from (3.20).

H **and** F**:** As in (Sims (2002)), it turns out that the H and F matrices are derived independently from the G and J matrices, based on the coefficient on κ_{t-S} in (3.4) (see the Appendix for details). Therefore, they are exactly the same as in perfect information models.

$$F = -Z_{u\phi}^{H} \backslash Z_{u\kappa}^{H} = Z_{\phi s}/Z_{\kappa s} \tag{3.17a}$$

$$H = -Z_{\kappa s}\left(\Omega_{ss}^{A} \backslash \Omega_{ss}^{B}\right)/Z_{\kappa s} \tag{3.17b}$$

E **and** Π**:** From (3.15) and (3.16), the non-zero elements of E and Π are

$$E_{ji} = -\left[\, M_{\Pi E} \,\right]_{kj} \backslash \left[\, M_{\Pi E} \,\right]_{k\neg j} \mathbf{C}_{\neg ji} - \mathbf{C}_{ji} \tag{3.18}$$

$$\Pi_{\neg ki} = -\left[\, M_{\Pi E}^{-1} \,\right]_{\neg j \neg k} \backslash \mathbf{C}_{\neg ji} \tag{3.19}$$

where $M_{\Pi E}$ can be obtained from (3.8c) and (3.13) with the solution of H and F. Note that H and F can be computed without referring E, Π or $M_{\Pi E}$. Since $[M_{\Pi E}]_{kj}$ is assumed to be invertible, $[M_{\Pi E}^{-1}]_{\neg j \neg k}$ is also invertible (see the Appendix for the proof).

J **and** G**:** From the definition of Γ (3.8a),

$$\Gamma \equiv \begin{bmatrix} J_0 \\ G_0 \\ \vdots \\ J_S \\ G_S \end{bmatrix} = M_{\Gamma\Pi}\Pi \tag{3.20}$$

Note that, with H and F matrices, $M_{\Gamma\Pi}$ are recovered from (3.8c).

D**:** From a given economic model (3.2) it is obvious that

$$D = -A \begin{bmatrix} 0 \\ G_0 \end{bmatrix}$$

3.3 Assumptions

In this section, we discuss three assumptions. Assumptions 1 and 2 in the following are the same as in the solution method for perfect information models, while assumption 3 is specific to imperfect information models. This subsection omits discussion about the Blanchard-Kahn condition, which is automatically satisfied by assumption 1.

3.3.1 Assumption 1: $Z^H_{u\phi}$ is Invertible

Klein (2000) shows that this assumption is a generalisation of the condition derived in Blanchard and Kahn (1980). Boyd and Dotsey (1990) makes it clear that the Blanchard-Kahn condition, which counts and compares the numbers of unstable roots and jump variables, is a necessary but not sufficient condition for the existence of a unique solution; they provide a counter-example that satisfies the Blanchard-Kahn counting condition but does not have a stable solution. Intuitively, an invertible $Z^H_{u\phi}$ means that we can always find the values of jump variables such that the expectation of u_{t+1} is a zero vector in any states (TVCs). Remember that $Z_{u\phi}^{H}$ maps jump variables to unstable roots, and its inverse maps unstable roots to jump variables. See King and Watson (1995) for an intuitive exposition.

The existence of the right inverse of $Z_{u\phi}^{H}$ entails the existence of jump variables, but the non-existence of its left inverse implies non-uniqueness of jump variables. See Uhlig (2000) for a treatment of non-uniqueness.

3.3.2 Assumption 2: a_{kk} and b_{kk} are Not Zero at the Same Time

If a_{kk} and b_{kk} are zero at the same time, it implies that there exist row vectors X such that $0 = X\xi$ (One such example is the k-th row of Q). The existence of such row vectors generically implies either of the following:

(a) If $X\xi$ is indeed zero, then some equations are not linearly independent of the others. Essentially, there are fewer equations than endogenous variables. At least one equation can be expressed as a linear combination of others, and such a linear combination is X.

(b) If $X\xi$ is non-zero, clearly there is an internal contradiction. One such example is a two-equation, two-variable non-dynamic model with no state variables:

$$\begin{aligned} \phi_{1,t} &= \alpha\phi_{2,t} + \xi_t \\ \phi_{1,t} &= \alpha\phi_{2,t} + \xi_t + \eta_t \end{aligned}$$

Obviously, both do not hold at the same time for non-zero η_t. Since the QZ decomposition is merely a linear transformation, this implies that there is an internal inconsistency in the original system of equations (3.2).

3.3.3 Assumption 3: $[M_{\Pi E}]_{kj}$ is Invertible

This condition is specific to imperfect information models, though it is analogous to the equation (40) in Sims (2002).[8] Intuitively, if it is not invertible, then the information structure is not consistent. Note that the inverse of $[M_{\Pi E}]_{kj}$, if it exists, maps the j-th set of expectation errors to the k-th set of innovations to which some endogenous variables cannot respond. Hence, if the inverse of $[M_{\Pi E}]_{kj}$ exists, then expectation errors can equate both sides of the equations for any realisation of innovations.

A non-invertible $[M_{\Pi E}]_{kj}$ appears in the following example. Suppose that all production factors and all demand components are decided before observing today's technology shock. In this case, output varies depending on the realisation of technology, while demand cannot respond to it. Thus, the goods market does not clear at any price. One important lesson from this is that a researcher must construct consistent models; an arbitrarily specified information structure may have internal inconsistencies.

3.4 Properties of the Solution

The solutions computed by the algorithm have the following properties. Properties 1 and 2 are simply the bases of the algorithm and properties 3 and 4 are the direct consequences of invariant H and F.

1. If a variable $y_{k,t}$ is decided without observing an innovation $\xi_{i,t-s}$, then $\xi_{i,t-s}$ does not affect $y_{k,t}$ (i.e., $\partial x_t / \partial \xi_{i,t-s} = 0$) given crawling variables κ_{t-S}.

2. If $\xi_{i,t}$ is included in the information set of expectation operators in the j-th equation, then $\xi_{i,t}$ cannot be the source of the expected error in the j-th equation.

[8]Note, however, that Sims' condition is related to time $t+1$ expectation errors, while our discussion in the following is related to time τ expectation errors ($\tau < t$).

3. The dynamic parts of the solution (H and F) are the same as in the perfect information models. Thus, imperfect information does not change the numbers of stable and unstable roots. As a consequence, if a model under imperfect information exhibits saddle-path stability, for example, then the corresponding model under perfect information must also exhibit saddle-path stability.

4. Invariant dynamic parts also imply that the direct effects of imperfect information last only S period after an impulse. The direct effects are caused by G and J matrices. In subsequent periods, they essentially follow the same dynamics as under perfect information. More specifically, let $\tilde{\kappa}_{t+S}$ be the values of crawling variables S period after an impulse. Then subsequent impulse response functions are exactly the same as those under perfect information starting with $\tilde{\kappa}_{t+S}$ with setting all innovations equal to zero.

Properties 3 and 4 show that *qualitatively* an imperfect information model inherits key properties from the corresponding perfect information model. As shown in the next section, however, imperfect information can have *quantitatively* significant effects.

There are several comments on information sets.

- In our representation of solutions, *from the viewpoint of researchers*, the set of state variables at time t is $\{\kappa_t, \xi_t, \xi_{t-1}, \cdots, \xi_{t-S}\}$ (today's crawling variables and current and past innovations). Roughly speaking, crawling variables correspond to state variables under perfect information. Past innovations are necessary for describing a model economy, because they recover past information sets.

- Similarly, *from the viewpoint of economic agents in a model*, any information set must be a subset of the state variables (it does not include jump variables). The maximum possible information set at time t is the same as the state variables.

Related to the latter, rationality imposes the following two restrictions.

- First, any information set must be internally consistent; i.e., agents in a model always infer some state variables from other state variables, if possible. Essentially, rationality requires that any information structure is consistent with the solution (3.3a).

  For example, information set $\{\kappa_{t-1}, \kappa_{t-2}, \cdots, \xi_t, \xi_{t-1}, \cdots\}$ (without κ_t) is not allowed because rational economic agents must infer κ_t from κ_{t-1} and $\xi^{t,S}$. Similarly, $\{\kappa_t, \kappa_{t-1}, \cdots, \xi_t, \xi_{t-2}, \xi_{t-3}, \cdots\}$ (without ξ_{t-1}) is not acceptable, because economic agents must know ξ_{t-1}. On the other hand, the algorithm can deal with information set $\{\kappa_{t-2}, \kappa_{t-3}, \cdots, \xi_{t-1}, \xi_{t-3}, \cdots\}$, though it is difficult to interpret economically; agents exploit the information about ξ_{t-1} temporarily, but they systematically forget it.

  In terms of our computer codes, if an information set only includes $\{\xi_{t-s}, \xi_{t-s-1}, \xi_{t-s-2}, \cdots\}$, the algorithm deems that that information set includes κ_{t-s+1} but not $\{\kappa_t, \cdots, \kappa_{t-s+2}\}$.

- Second, the algorithm does not allow for inference from jump variables; otherwise, imperfect information models reduce to the corresponding perfect information models in most cases, because rational economic agents infer most hidden information from the solution (3.3b). For example, if households observe all production factors and output, they can correctly infer today's productivity shock from production function.

Though these restrictions may seem to be exceedingly restrictive, our algorithm is still applicable to the models, in which agents make future decisions in the current period. Such a class of models includes sticky price and sticky information models, for example.

The following points are also important.

- The algorithm cannot deal with parameter uncertainty.

- The algorithm can easily deal with noisy information models. Suppose an AR(1) shock process A_t follows

$$\ln A_{t+1} = \rho \ln A_t + \sqrt{1-\eta}\xi_t^{ob} + \sqrt{\eta}\xi_t^{uo} \tag{3.21}$$

where ξ_t^{ob} and ξ_t^{uo} are the observable and unobservable components of innovation, respectively, and $(1-\eta)/\eta$ is the signal to noise ratio. This technique allows us to parameterise the extent of imperfect information.

3.5 An Example

3.5.1 Standard RBC Model

To demonstrate the quantitative effects of imperfect information, we consider the standard RBC model under imperfect information, focussing on impulse response functions and second moments.

The main economic motivation is to address an overly high $Corr\,(Y_t - H_t, Y_t)$ in the standard RBC model. Under the plausible parameter range, the standard RBC model predicts an almost perfect correlation between labour productivity $Y_t - H_t$ and output Y_t, but in the data the correlation is only slightly positive.

Hence, we modify the standard RBC model by adding imperfect information related to the labour market. The relevant equations are

$$0 = bH_t - W_t - \lambda_t \tag{3.22a}$$

$$0 = Y_t - H_t - W_t \tag{3.22b}$$

where Y_t, H_t, W_t, λ_t are output, working hours, wage and the marginal utility of consumption, respectively. All endogenous variables are measured as deviations from their steady state values in percentage terms. b is a constant, which represents (a multiple of) the elasticity of marginal disutility of labour. The first equation is of the representative household (HH) – the FOC with respect to labour supply –, while the second is of firms – it equates the marginal product of labour ($Y_t - H_t$) to wage.[9] The set of state variables under perfect information is $\{K_t, A_t, \xi_t\}$, where K_t and A_t are capital and technology at the beginning of time t, respectively, and ξ_t represents the innovation on technology. Note that A_t is regarded as an endogenous crawling variable, and there is only one *iid* exogenous variable ξ_t. That is to say, A_t is treated as a stock variable.

Assuming that today's innovation affects today's output,

$$\begin{aligned} Y_t &= A_{t+1} K_t^{\alpha} H_t^{1-\alpha} \\ \ln A_{t+1} &= \rho \ln A_t + \xi_t \end{aligned}$$

where ρ is a parameter that governs the persistence of technology shock.

Case I: HH Decides Labour Supply before Observing Innovations

In this case (3.22a) does not hold. Instead, the labour supply decision is governed by

$$0 = E\left[\ bH_t - W_t - \lambda_t \mid \{K_{t-j}, A_{t-j}, \xi_{t-j}\}_{j=S+1}^{\infty}\right]$$

Since H_t cannot react to past innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial H_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

Figure 3.1 shows the impulse response functions where $S = 5$, which means that the household decides its labour supply five quarters in advance.

There are several points worth noting here:

[9]Note that since all endogenous variables are represented as log-deviations from their steady state, $Y_t - H_t$ is the deviation of "output divided by labour hour" (i.e., labour productivity). The Cobb-Douglas production function implies that the marginal product of labour is $(1-\alpha)$ times labour productivity, which means that the percent change of labour productivity is exactly the same as that of the marginal product of labour. In other words, in the Cobb-Douglas production function, $Y_t - H_t$ represents both the percent deviations of labour productivity and marginal product of labour.

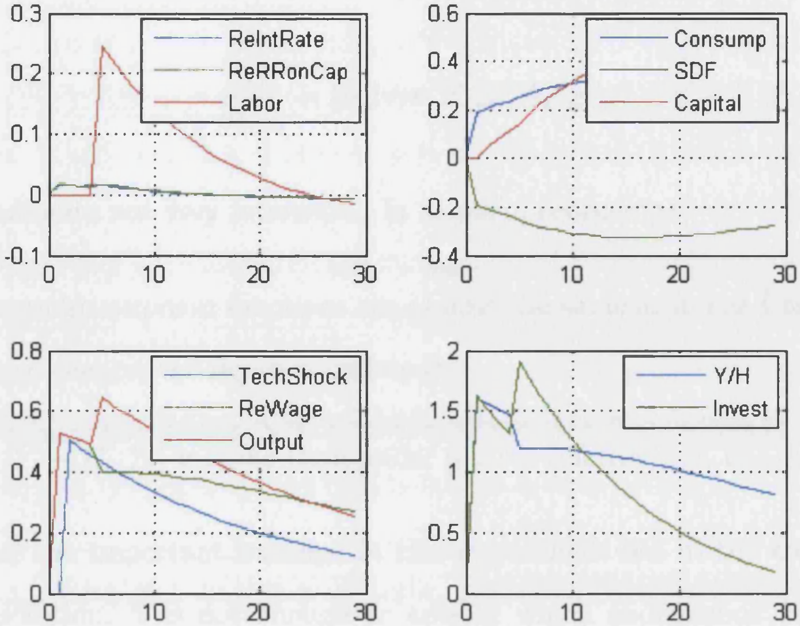

Figure 3.1: Impulse response functions to a positive technology innovation of the standard RBC model, in which labour supply is determined five periods in advance.

- Woking hours do not move for the first S periods. That is, $\partial H_t / \partial \xi_{t-s} = 0$ for $s = 0, 1, \cdots, S$.

- Labour productivity $(Y_t - H_t)$ and investment show unusual movements for the first S periods. However, after $S+1$ periods, all endogenous variables follow (linear combinations of) AR(1) processes. This is one example of the proposition that *the direct effect of imperfect information lasts only S periods after an impulse.*

- $Corr\,(Y_t - H_t, Y_t)$ is lower than under perfect information, but only slightly (exact numbers are omitted).

Case II: Firms Decide Labour Demand before Observing Innovations

In this case, (3.22b) does not hold. Instead, the labour demand decision is governed by

$$0 = E\left[\; Y_t - H_t - W_t \;\middle|\; \left\{K_{t-j}, A_{t-j}, \xi_{t-j}\right\}_{j=S+1}^{\infty} \right]$$

Since H_t cannot react to the innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial H_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

The results are not very interesting in terms of economics.

- The impulse response functions are almost the same as in the Case I, except for wage (hence, the figure is omitted).

- $Corr\,(Y_t - H_t, Y_t)$ is lower than under perfect information, but only slightly.

However, the important message in this experiment lies in the computation. To find a solution, it is not enough to specify which endogenous variables are determined with imperfect information; a researcher must also specify which information sets are imperfect. This is evident that the results of Cases I and II are not the same.

Case III: HH Decides Wage before Observing Innovations but Accommodates Labour Demand

This case can be regarded as a version of the sticky wage model. The representative household fixes wage before observing innovations, and it commits itself to supplying labour to accommodate labour demand.

In this case, (3.22a) does not hold. Instead, the labour supply decision is governed by

$$0 = E\left[\, bH_t - W_t - \lambda_t \mid \{K_{t-j}, A_{t-j}, \xi_{t-j}\}_{j=S+1}^{\infty}\right]$$

Since W_t cannot react to the innovations, for $s = 0, 1, \cdots, S$,

$$\frac{\partial W_t}{\partial \xi_{t-s}} = 0 \text{ given } K_{t-S}, A_{t-S}$$

The results are very interesting:

Table 3.1: Comparison between perfect and imperfect information RBC models.

	Output	Hours	Consumption	Investment	Corr(Output,Outpu/Hours)
			Data		
s.d.	1.72	1.59	0.86	8.24	0.41
relative	1.00	0.92	0.50	4.79	
			Standard RBC		
s.d.	1.35	0.47	0.33	5.95	0.98
relative	1.00	0.35	0.24	4.41	
			Imperfect information (RBC with Prefixed Wage)		
s.d.	2.15	2.10	0.53	7.92	0.25
relative	1.00	0.98	0.25	3.69	

Note: Figures of "Data" and "Standard RBC" are cited from Cooley and Prescott (1995).

- The volatility of labour is much higher.
- $Corr\,(Y_t - H_t, Y_t)$ is much lower than under perfect information.
- Given the standard deviation of the innovation, both output and labour are more volatile.
- The behaviours of most variables other than labour and labour productivity do not change significantly.

The intuition behind these results is quite simple. Without imperfect information, when there is a positive productivity innovation, wage increases, which discourages firms from hiring more labour. As a result, labour does not increase by very much. Indeed, another failure of the standard RBC model is that it predicts too low labour volatility relative to output volatility. During a boom both Y_t and H_t increase, while $Y_t - H_t$ increases because the increase in H_t is not large enough. Consequently, both Y_t and $Y_t - H_t$ increase in a boom, which is the (one possible) mechanism behind a high $Corr\,(Y_t - H_t, Y_t)$ in the standard RBC model.

However, if wage is determined without seeing a positive innovation, it does not change quickly; hence, firms are not discouraged from using more labour. Consequently, in a boom both Y_t and H_t increase, while $Y_t - H_t$ does not increase very much because the increase in H_t is large enough. Hence, the model predicts a low $Corr\left(Y_t - H_t, Y_t\right)$. Indeed, in the otherwise standard RBC model with one-period wage stickiness, the predicted relative volatility of labour almost matches the data. Under the standard parameter set, $Corr\left(Y_t - H_t, Y_t\right)$ is negative for $S \geq 2$.

Table 3.1 is the summary table of the selected second moments for one-period wage stickiness ($S = 1$). One-period wage stickiness improves the labour volatility and correlation between labour productivity and output, while it slightly deteriorates the model performance in terms of the relative volatility of investment.

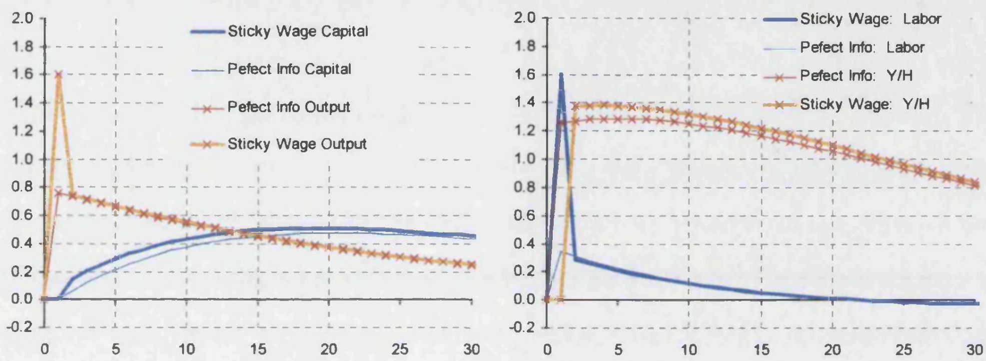

Figure 3.2: Comparison of selected impulse response functions to a positive technology innovation between standard RBC and RBC with wage stickiness.

Figure 3.2 shows the comparison of selected impulse response functions between perfect and imperfect information models. The salient differences appear only in the first period. In the sticky wage model, both labour and output jump in the first period, and the size of the jumps are the same, hence, the labour productivity does not change in the first period. Note that the Cobb-Douglas production function implies that the labour productivity is always equal to wage.

Figure 3.3 shows the relative volatilities and correlations for different degrees of imperfect information (i.e., for different values of S). As S increases, $Corr\left(Y_t - H_t, H_t\right)$ decreases.

Case III also reveals one computational requirement; simply specifying the information set in each equation is not enough to find a solution. A researcher must

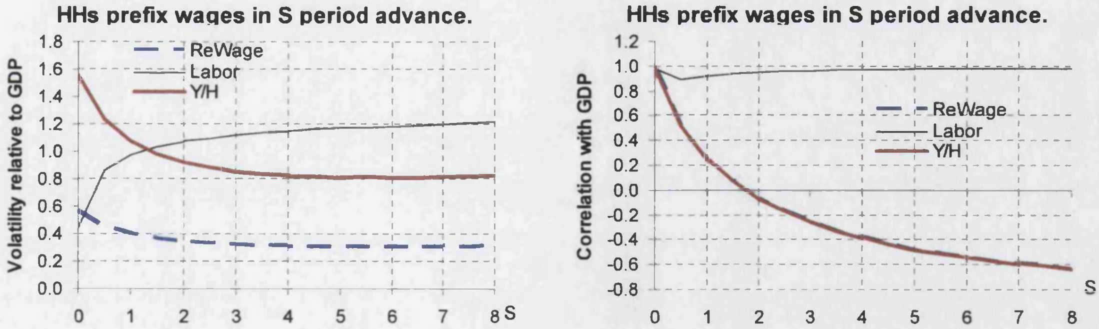

Figure 3.3: Effect of different degrees of imperfect information on selected second moments.

also specify which variables are determined without observing perfect information. This is evident in that the results of Case I and III are not the same.

Conclusion for RBC under Imperfect Information

Adding one-period wage stickiness is quantitatively enough to overcome the two drawbacks of the standard RBC model – where (a) labour volatility is too small and (b) the correlation between labour productivity and output is too high – without significantly deteriorating other dimensions of the model performance. This example shows the possibility that the information structure of a model has significant quantitative effects.

3.6 Conclusion

This chapter has developed an algorithm for linear rational models under imperfect information. Imperfect information is important because it includes many interesting classes of models such as sticky information and noisy signal models.

The algorithm exploits two observations: (1) if an endogenous variable $y_{k,t}$ is decided without observing an innovation $\xi_{i,t-s}$, then $y_{k,t}$ is not affected by $\xi_{i,t-s}$ (i.e., $\partial y_{k,t}/\partial \xi_{i,t-s} = 0$); (2) if the information set in the j-th equation includes $\xi_{i,t-s}$, then $\xi_{i,t-s}$ cannot be the source of expectation error in the j-th equation ($E_{s,ji} = 0$). The solution is defined by these two zero restrictions, and it turns out that they are enough to determine unique solutions.

The state space representation chosen in this algorithm is the set of crawling variables at the beginning of the current period and current and past innovations. This representation reveals that the dynamic parts of the solution (i.e., the H and F matrices) are the same as under the corresponding perfect information models. Invariant H and F matrices imply that (a) the dynamic property, such as sunspot or saddle-path stability is not altered by information structure, and (b) impulse response functions are not (directly) affected by the information structure after the first S periods, where S is such that the minimum information set in a model has all the information up to time S. These findings show that *qualitatively* imperfect information models inherit the properties of their perfect information counterparts.

However, as the RBC example demonstrates, *quantitatively* imperfect information may be important. Hence, it is desirable to check for robustness in terms of the information structure, and our MATLAB algorithm offers an easy way to conduct such experiments. Once structural equations are obtained, then the additional inputs to the algorithm are only two zero restrictions.

Appendices for Chapter 3

3.A Extension of Uhlig's Theorem 3

Proposition 1 *(Extension of Uhlig's Theorem 3) To find a $m \times m$ matrix X that solves the matrix polynomial*

$$\Theta_n X^n - \Theta_{n-1} X^{n-1} - \cdots - \Theta_1 X - \Theta_0 = 0 \tag{3.23}$$

Given $m \times m$ coefficient matrices $\{\Theta_{n'}\}_{n'=0}^{n}$, define the $nm \times nm$ matrices Ξ and Δ by

$$\Xi = \begin{bmatrix} \Theta_{n-1} & \cdots & \Theta_1 & \Theta_0 \\ I & & 0 & 0 \\ & \ddots & & \vdots \\ 0 & & I & 0 \end{bmatrix} \tag{3.24a}$$

$$\Delta = \begin{bmatrix} \Theta_n & 0 & \cdots & 0 \\ 0 & I & & 0 \\ \vdots & & \ddots & \\ 0 & 0 & & I \end{bmatrix} \tag{3.24b}$$

and obtain the generalized eigenvalues λ and the generalized eigenvector s such that $\lambda \Delta s = \Xi s$. Then, s can be written as

$$s = \begin{pmatrix} \lambda^{n-1} x \\ \vdots \\ \lambda x \\ x \end{pmatrix}$$

for some $x \in \mathbb{R}^m$*, and*

$$X = \Omega\Lambda\Omega^{-1}$$

where $\Omega = [x_1, \cdots, x_m]$ *and* $\Lambda = diag(\lambda_1, \cdots, \lambda_m)$.

Proof. Almost identical to Uhlig (1999). ■

3.B Matrix Operations

To pick up and drop out columns and rows from a matrix, we define

- $[A]_{.x}$ as columns x of a matrix A,
- $[A]_{x.}$ as rows x of a matrix A,
- $[A]_{.\neg x}$ as the columns remaining after the elimination of columns x, and
- $[A]_{\neg x.}$ as the rows remaining after the elimination of rows x,

where x is the name of a set of columns or rows. The brackets are used simply because they often clarify notations, and often can be omitted (i.e., $[B]_{.\neg y} = B_{.\neg y}$). The dot . implies all rows or columns (e.g., $B_{..} = B$). It is quite easy to show the following formulae:

$$\begin{aligned}
[AB] &= [A]_{.\neg x}[B]_{\neg x.} + [A]_{.x}[B]_{x.} \\
[AB]_{.\neg y} &= [A][B]_{.\neg y} \\
[AB]_{\neg x.} &= [A]_{\neg x.}[B] \\
[AB]_{\neg x \neg y} &= [A]_{\neg x.}[B]_{.\neg y}
\end{aligned}$$

An example for the first formula is

$$
\begin{aligned}
\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} &= \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \end{bmatrix} + \begin{bmatrix} a_{12} \\ a_{22} \end{bmatrix} \begin{bmatrix} b_{21} & b_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} \\ a_{21}b_{11} & a_{21}b_{12} \end{bmatrix} + \begin{bmatrix} a_{12}b_{21} & a_{12}b_{22} \\ a_{22}b_{21} & a_{22}b_{22} \end{bmatrix} \\
&= \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} & a_{11}b_{12} + a_{12}b_{22} \\ a_{21}b_{11} + a_{22}b_{21} & a_{21}b_{12} + a_{22}b_{22} \end{bmatrix}
\end{aligned}
$$

where $x = 2$.

Note that this notation is consistent with other matrix subscripts; for example, the rows of $Z_{s\kappa}$ are related to stable roots s and its columns are related to crawling variables κ.

3.C Invertible $Z_{u\phi}^{H}$ Implies Invertible $Z_{s\kappa}^{H}$

Proposition 2 *For an invertible matrix Z, which is partitioned as*

$$
Z = \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}
$$

if Z_{11} is invertible, then $[Z^{-1}]_{22}$ is also invertible.

Proof. Define

$$
\begin{aligned}
Z_L &\equiv \begin{bmatrix} I & 0 \\ -Z_{21}Z_{11}^{-1} & I \end{bmatrix} \\
Z_R &\equiv \begin{bmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{bmatrix}
\end{aligned}
$$

Note that $Z_L Z Z_R$ has full rank because all of Z_L, Z and Z_R have full rank, and note that

$$\begin{bmatrix} I & 0 \\ -Z_{21}Z_{11}^{-1} & I \end{bmatrix} \begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix} \begin{bmatrix} I & -Z_{11}^{-1}Z_{12} \\ 0 & I \end{bmatrix} = \begin{bmatrix} Z_{11} & 0 \\ 0 & Z_{22} - Z_{21}Z_{11}^{-1}Z_{12} \end{bmatrix}$$

Hence, $G \equiv Z_{22} - Z_{21}Z_{11}^{-1}Z_{12}$ must have full rank.

For a full rank matrix with an invertible upper left submatrix, the well-known formula tells us

$$\begin{bmatrix} Z_{11} & Z_{12} \\ Z_{21} & Z_{22} \end{bmatrix}^{-1} = \begin{bmatrix} Z_{11}^{-1} + Z_{11}^{-1}Z_{12}G^{-1}Z_{21}Z_{11}^{-1} & -Z_{11}^{-1}Z_{12}G^{-1} \\ -G^{-1}Z_{21}Z_{11}^{-1} & G^{-1} \end{bmatrix}$$

Note that the RHS exists since we know that both Z_{11} and G are invertible. Thus, $[Z^{-1}]_{22}$ is invertible. ■

Since Z is unitary, $Z^{-1} = Z^H$, which implies $G^{-1} = [Z^{-1}]_{22} = Z_{22}^H$. Since Z_{22}^H has full rank, its conjugate transpose $Z_{22} \left(= \left[Z_{22}^H\right]^H\right)$ also has full rank.

3.D Invertibility of Block Triangular Matrices

Due to the following introductory result, we know that Ω_{ss}^A, Ω_{uu}^B, Φ, $M_{y\Gamma}$ and $M_{\Gamma\Pi}$ are all invertible.

Consider a block triangular Δ which has invertible block diagonal submatrices Δ_{dd}

$$\Delta = \begin{bmatrix} \Delta_{11} & & & & \\ & \ddots & & & \\ & & \Delta_{dd} & & \\ & & & \ddots & \\ & & & & \Delta_{DD} \end{bmatrix}$$

Δ is either an upper or lower block diagonal. Then, Δ is invertible.

To show this, focus on the upper left part first

$$\det \begin{bmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{bmatrix} = \Big(\det \Delta_{11} \Big) \Big(\det \left(\Delta_{22} - \Delta_{21} \left(\Delta_{11} \backslash \Delta_{12}\right)\right) \Big)$$
$$= \Big(\det \Delta_{11} \Big) \Big(\det \Delta_{22} \Big) \neq 0$$

Note that $\Delta_{21} \left(\Delta_{11} \backslash \Delta_{12}\right) = 0$ since either Δ_{21} or Δ_{12} is zero. We can repeat this process until it shows $\det \Delta \neq 0$.

3.E Rank Deficient H Matrix and Expansion of Innovation Vector

The representation of a solution under imperfect information is not necessarily unique. This section shows the equivalence of two representations.

Consider the RBC model, in which labour supply is decided without observing today's innovation. The vector of crawling variables is

$$\kappa_t = \begin{bmatrix} K_t \\ H_t^S \end{bmatrix}$$

where K_t and H_t^S are capital and labour supply at time t, respectively. Then, the solution has an H matrix that is rank deficient.

We can decompose such an H matrix by using eigenvalue-eigenvector decomposition

$$
\begin{aligned}
V\mathcal{K}_{t+1} &= \lambda V\mathcal{K}_t + VJ\xi_t \\
\lambda &= \begin{bmatrix} \lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \\
V &= \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}
\end{aligned}
$$

From the first row

$$
V_{11}K_{t+1} + V_{12}H_{t+1}^S = \lambda_1 \left(V_{11}K_t + V_{12}H_t^S \right) + \left(V_{11}J_1 + V_{12}J_2 \right) \xi_t
$$

where J is 2x1. From the second row

$$
\begin{aligned}
V_{21}K_{t+1} + H_{t+1}^S &= \left(V_{21}J_1 + V_{22}J_2 \right) \xi_t \\
H_{t+1}^S &= \left(V_{22} \backslash V_{21}J_1 + J_2 \right) \xi_t - V_{22} \backslash V_{21}K_{t+1}
\end{aligned}
$$

Under our assumption 3 (invertible $[M_{\Pi E}]_{kj}$), V_{11} and V_{22} are non-zero. Hence,

$$
K_{t+1} = \lambda_1 K_t + J_1 \xi_t + \lambda_1 \left(\frac{V_{12}(V_{22} \backslash V_{21})J_1 + V_{12}J_2}{V_{11} - V_{12}(V_{22} \backslash V_{21})} \right) \xi_{t-1}
$$

Thus, it is shown that with a rank deficient H matrix we can reduce the set of crawling variables by increasing the number of innovations.

3.F Full Derivation

This section provides the full derivation. For the notation, see the main text.

3.F.1 QZ Decomposition

Applying the QZ decomposition to (3.2)

$$
\begin{aligned}
0 &= \Omega_A Z^H y_{t+1} + \Omega_B Z^H y_t + Q^H C\xi_t + Q^H D\xi_{t+1} + Q^H E\xi^{t,S} \\
&= \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ 0 & \Omega^A_{uu} \end{bmatrix} \begin{pmatrix} s_{t+1} \\ u_{t+1} \end{pmatrix} + \begin{bmatrix} \Omega^B_{ss} & \Omega^B_{su} \\ 0 & \Omega^B_{uu} \end{bmatrix} \begin{pmatrix} s_t \\ u_t \end{pmatrix} \\
&\quad + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} C\xi_t + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} D\xi_{t+1} + \begin{bmatrix} Q^H_{s.} \\ Q^H_{u.} \end{bmatrix} E\xi^{t,S} \qquad (3.25)
\end{aligned}
$$

where s_t and u_t are stable and unstable roots, respectively, such that

$$
\begin{pmatrix} s_t \\ u_t \end{pmatrix} \equiv \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \begin{pmatrix} \kappa_t \\ \phi_t \end{pmatrix}
$$

Unstable Roots and Transversality Conditions (TVCs)

Imperfect information requires a slightly careful treatment of TVCs. Focussing on the lower half of (3.25)

$$
0 = \Omega^A_{uu} u_{t+1} + \Omega^B_{uu} u_t + Q^H_{u.} C\xi_t + Q^H_{u.} D\xi_{t+1} + Q^H_{u.} E\xi^{t,S} \qquad (3.26)
$$

Iterating it forward

$$
\begin{aligned}
&\lim_{l\to\infty} \left\{ \begin{array}{c} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^l u_{t+l} \\ + \sum_{s=1}^{l-1} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^s \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) \left(C\xi_{t+s} + D\xi_{t+1+s} + E\tilde{\xi}^{t+s,S} \right) \end{array} \right\} \\
&= -u_t - \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) C\xi_t - \sum_{l=0}^{S} \left(-\Omega^B_{uu} \backslash \Omega^A_{uu} \right)^l \left(\Omega^B_{uu} \backslash Q^H_{u.} \right) E\hat{\xi}^{t+l,S} \qquad (3.27)
\end{aligned}
$$

where

$$\xi^{t+l,S} = \begin{pmatrix} \xi_{t+l} \\ \vdots \\ \xi_{t+1} \\ \xi_t \\ \vdots \\ \xi_{t+l-S} \end{pmatrix} = \hat{\xi}^{t+l,S} + \tilde{\xi}^{t+l,S} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \xi_t \\ \vdots \\ \xi_{t+l-S} \end{pmatrix} + \begin{pmatrix} \xi_{t+l} \\ \vdots \\ \xi_{t+1} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where $A\backslash B = A^{-1}B$, and $A/B = AB^{-1}$.

There are many information sets, under each of which TVCs must be satisfied. – that is, TVCs are (seemingly) tighter under imperfect information. However, if the perfect information counterpart satisfies TVCs, corresponding imperfect information models also satisfy them automatically due to the law of iterated linear projection.[10] Thus, the same logic as in the perfect information case holds; because $\left(-\Omega_{uu}^B\backslash\Omega_{uu}^A\right)^l \to 0$ as $l \to \infty$ by construction, the expected value of u_{t+l} explodes for any non-zero value of the RHS of (3.27), which contradicts the TVCs. Note that the inside the limit operator in the LHS shows the expected value of u_{t+l} (the realisation of u_{t+l} plus expectation errors) times $\left(-\Omega_{uu}^B\backslash\Omega_{uu}^A\right)^l$. Hence, the RHS of (3.27) must be zero.

Therefore,

$$\begin{aligned} -\Omega_{uu}^B u_t &= -\Omega_{uu}^B Z_{u\kappa}^H \kappa_t - \Omega_{uu}^B Z_{u\phi}^H \phi_t \\ &= Q_{u.}^H C\xi_t + \Omega_{uu}^B \sum_{l=0}^{S} \left(-\Omega_{uu}^B\backslash\Omega_{uu}^A \right)^l \left(\Omega_{uu}^B\backslash Q_{u.}^H \right) E\hat{\xi}^{t+l,S} \\ &= Q_{u.}^H C\xi_t + \sum_{l=0}^{S} \left(-\Omega_{uu}^A/\Omega_{uu}^B \right)^l Q_{u.}^H E\hat{\xi}^{t+l,S} \end{aligned} \tag{3.28}$$

[10]There are two comments. First, (3.27) must hold for *any* realisation of κ_{t-1} and ξ_{t-s} for $s = 0, 1, \cdots$. Hence, it is *not* possible that TVCs hold under imperfect information but not under perfect information. Second, if an information set does not include, for example, $\xi_{i,t-s}$ then the relevant expected value of u_{t+s} is the RHS with setting $\xi_{i,t-s} = 0$. Hence, if TVCs hold for the full information set, they hold for non-full information sets as well.

Substituting our "guess solution" (3.3) into (3.28),

$$
\begin{aligned}
0 &= \left(\Omega^B_{uu} Z^H_{u\kappa} + \Omega^B_{uu} Z^H_{u\phi} F \right) \kappa_t + \Omega^B_{uu} Z^H_{u\phi} G \xi^{t,S} + Q^H_{u.} C \xi_t \\
&\quad + \sum_{l=0}^{S} \left(-\Omega^A_{uu} / \Omega^B_{uu} \right)^l Q^H_{u.} E \hat{\xi}^{t+l,S}
\end{aligned} \tag{3.29}
$$

Stable Roots

Similarly, from the upper half,

$$
\begin{aligned}
0 &= \Omega^A_{ss} \left(Z^H_{s\kappa} \kappa_{t+1} + Z^H_{s\phi} \phi_{t+1} \right) + \Omega^A_{su} \left(Z^H_{u\kappa} \kappa_{t+1} + Z^H_{u\phi} \phi_{t+1} \right) \\
&\quad + \Omega^B_{ss} \left(Z^H_{s\kappa} \kappa_t + Z^H_{s\phi} \phi_t \right) + \Omega^B_{su} \left(Z^H_{u\kappa} \kappa_t + Z^H_{u\phi} \phi_t \right) \\
&\quad + Q^H_{s.} C \xi_t + Q^H_{s.} D \xi_{t+1} + Q^H_{s.} E \xi^{t,S}
\end{aligned} \tag{3.30}
$$

Again, by substituting (3.3) into (3.30), after some manipulation,

$$
\begin{aligned}
0 &= \left(\Lambda^A_{s\phi} F H + \Lambda^A_{s\kappa} H + \Lambda^B_{s\phi} F + \Lambda^B_{s\kappa} \right) \kappa_t \\
&\quad + \Lambda^A_{s\phi} G \xi^{t+1,S} + Q^H_{s.} D \xi_{t+1} + Q^H_{s.} C \xi_t \\
&\quad + \left(\Lambda^A_{s\phi} F J + \Lambda^A_{s\kappa} J + \Lambda^B_{s\phi} G + Q^H_{s.} E \right) \xi^{t,S}
\end{aligned} \tag{3.31}
$$

Though the definitions of $\Lambda^A_{s\kappa}$, $\Lambda^A_{s\phi}$, $\Lambda^B_{s\kappa}$ and $\Lambda^B_{s\phi}$ are (3.10) in the main text, the following result may be more useful.

$$
\begin{bmatrix} \Lambda^A_{s\kappa} & \Lambda^A_{s\phi} \\ \Lambda^B_{s\kappa} & \Lambda^B_{s\phi} \end{bmatrix} = \begin{bmatrix} \Omega^A_{ss} & \Omega^A_{su} \\ \Omega^B_{ss} & \Omega^B_{su} \end{bmatrix} \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \tag{3.32}
$$

3.F.2 Expansion of $\xi^{t+1,S}$ and $\xi^{t,S}$

Expanding $\xi^{t+1,S}$ and $\xi^{t,S}$ in (3.31) and (3.29),

$$
\begin{aligned}
0 \;=\; & \left(\Lambda^A_{s\phi} F H + \Lambda^A_{s\kappa} H + \Lambda^B_{s\phi} F + \Lambda^B_{s\kappa} \right) \kappa_t \\
& + \left(\Lambda^A_{s\phi} G_0 + Q^H_{s.} D \right) \xi_{t+1} \\
& + \left(\Lambda^A_{s\phi} G_1 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_0 + \Lambda^B_{s\phi} G_{0.} + Q^H_{s.} E_{0.} + Q^H_{s.} C \right) \xi_t \\
& + \left(\Lambda^A_{s\phi} G_2 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_1 + \Lambda^B_{s\phi} G_{1.} + Q^H_{s.} E_{1.} \right) \xi_{t-1} + \cdots \\
& + \left(\Lambda^A_{s\phi} G_S + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_{S-1} + \Lambda^B_{s\phi} G_{S-1.} + Q^H_{s.} E_{S-1.} \right) \xi_{t-(S-1)} \\
& + \left(\left(\Omega^A_{ss}/Z_{\kappa s}\right) J_S + \Lambda^B_{s\phi} G_{S.} + Q^H_{s.} E_{S.} \right) \xi_{t-S}
\end{aligned}
$$

$$
\begin{aligned}
0 \;=\; & \left(\Omega^B_{uu} Z^H_{u\kappa} + \Omega^B_{uu} Z^H_{u\phi} F \right) \kappa_t \\
& + \sum_{s=1}^{S} \left(\Omega^B_{uu} Z^H_{u\phi} G_s + \left(\textstyle\sum_{k=0}^{S-s} \left(-\Omega^A_{uu}/\Omega^B_{uu} \right)^k Q^H_{u.} E_{k+s} \right) \right) \xi_{t-s} \\
& + \left(Q^H_{u.} C + \Omega^B_{uu} Z^H_{u\phi} G_0 + \left(\textstyle\sum_{k=0}^{S} \left(-\Omega^A_{uu}/\Omega^B_{uu} \right)^k Q^H_{u.} E_k \right) \right) \xi_t
\end{aligned}
$$

Since these matrix equations must hold for any realisation of κ_t, $\xi_{t-\tau}$ for $\tau = -1, 0, 1, \cdots, S$,

$$0 = \Lambda^A_{s\phi} F H + \Lambda^A_{s\kappa} H + \Lambda^B_{s\phi} F + \Lambda^B_{s\kappa} \tag{3.33a}$$

$$0 = \Omega^B_{uu} Z^H_{u\kappa} + \Omega^B_{uu} Z^H_{u\phi} F \tag{3.33b}$$

$$0 = \Lambda^A_{s\phi} G_{0.} + Q^H_{s.} D \tag{3.34a}$$

$$0 = 0 \tag{3.34b}$$

$$
\begin{aligned}
0 &= \Lambda^A_{s\phi} G_1 + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_0 + \Lambda^B_{s\phi} G_0 + Q^H_{s.} E_{S.} + Q^H_{s.} C && (3.35a) \\
0 &= \Omega^B_{uu} Z^H_{u\phi} G_0 + \left(\sum_{s=0}^{S} \left(-\Omega^A_{uu}/\Omega^B_{uu} \right)^s Q^H_{u.} E_s \right) + Q^H_{u.} C && (3.35b)
\end{aligned}
$$

$$
\begin{aligned}
0 &= \Lambda^A_{s\phi} G_{s+1} + \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_s + \Lambda^B_{s\phi} G_s + Q^H_{s.} E_s && (3.36a) \\
& \quad \text{for } s = 1, \cdots, S-1 \\
0 &= \Omega^B_{uu} Z^H_{u\phi} G_s + \left(\sum_{k=0}^{S-s} \left(-\Omega^A_{uu}/\Omega^B_{uu} \right)^k Q^H_{u.} E_{k+s} \right) && (3.36b) \\
& \quad \text{for } s = 1, \cdots, S-1
\end{aligned}
$$

$$
\begin{aligned}
0 &= \left(\Omega^A_{ss}/Z_{\kappa s}\right) J_S + \Lambda^B_{s\phi} G_S + Q^H_{s.} E_S && (3.37a) \\
0 &= \Omega^B_{uu} Z^H_{u\phi} G_S + Q^H_{u.} E_S && (3.37b)
\end{aligned}
$$

3.F.3 Dynamic Parts (H and F)

Since (3.33a) and (3.33b) do not include G, J, D, E or Π, these two matrix equations can be solved for H and F independently. Thus, assuming $Z^H_{u\phi}$ has a (right) inverse,[11]

$$
\begin{aligned}
F &= -Z^H_{u\phi} \backslash Z^H_{u\kappa} = Z_{\phi s}/Z_{\kappa s} \\
H &= -Z_{\kappa s} \left(\Omega^A_{ss} \backslash \Omega^B_{ss} \right) / Z_{\kappa s}
\end{aligned}
$$

Note that the H and F matrices are the same as in the corresponding perfect

[11]Remember that an invertible $Z^H_{u\phi}$ implies an invertible $Z^H_{s\kappa}$.

information model.[12]

3.F.4 Zero Restrictions on E and Π

Vertically concatenating matrix equations (3.35a)-(3.37b) in pairs,

$$
\begin{aligned}
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_1 + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_0 \\
&\quad + \sum_{k=0}^{S} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_k + Q^H C && \text{(3.38a)} \\
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_{s+1} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_s \\
&\quad + \sum_{k=0}^{S-s} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s} \text{for } s = 1, \cdots, S-1 && \text{(3.38b)} \\
0 &= \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_S + Q^H E_S && \text{(3.38c)}
\end{aligned}
$$

---

[12]For the F matrix, note

$$
Z^H Z = \begin{bmatrix} Z^H_{s\kappa} & Z^H_{s\phi} \\ Z^H_{u\kappa} & Z^H_{u\phi} \end{bmatrix} \begin{bmatrix} Z_{\kappa s} & Z_{\kappa u} \\ Z_{\phi s} & Z_{\phi u} \end{bmatrix} = \begin{bmatrix} Z^H_{s\kappa} Z_{\kappa s} + Z^H_{s\phi} Z_{\phi s} & Z^H_{s\kappa} Z_{\kappa u} + Z^H_{s\phi} Z_{\phi u} \\ Z^H_{u\kappa} Z_{\kappa s} + Z^H_{u\phi} Z_{\phi s} & Z^H_{u\kappa} Z_{\kappa u} + Z^H_{u\phi} Z_{\phi u} \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}
$$

Looking at the lower left element

$$
\begin{aligned}
Z^H_{u\kappa} Z_{\kappa s} + Z^H_{u\phi} Z_{\phi s} &= 0 \\
-Z^H_{u\kappa} Z_{\kappa s} &= Z^H_{u\phi} Z_{\phi s} \\
-Z^H_{u\phi} \backslash Z^H_{u\kappa} &= Z_{\phi s}/Z_{\kappa s}
\end{aligned}
$$

Also, remember that

$$
Z^{-1}_{\kappa s} = Z^H_{s\kappa} - Z^H_{s\phi} \left(Z^H_{u\phi} \backslash Z^H_{u\kappa} \right)
$$

and that Ω^A_{ss} is invertible by the reordering of QZ decomposition.

Note that

$$
\begin{aligned}
0 &= \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix} \left(\begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & 0 \end{bmatrix} \Gamma_{s+2} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} \right. \\
&\qquad \left. + \textstyle\sum_{k=0}^{S-(s+1)} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s+1} \right) \\
&= \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \sum_{k=1}^{S-s} \begin{bmatrix} 0 & 0 \\ 0 & -\Omega^A_{uu}/\Omega^B_{uu} \end{bmatrix}^k Q^H E_{k+s} \qquad (3.39)
\end{aligned}
$$

Subtracting (3.39) from each of (3.38),[13]

$$
\begin{aligned}
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_1 + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_0 + Q^H (E_k + C) \qquad (3.40a) \\
0 &= \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_{s+1} + \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_s + Q^H E_{k+s} \qquad (3.40b) \\
&\quad \text{for } s = 1, \cdots, S-1 \\
0 &= \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix} \Gamma_S + Q^H E_S \qquad (3.40c)
\end{aligned}
$$

[13]Though this process is not necessary, it reduces the computational burden.

and again vertically concatenating these equations,

$$
\begin{aligned}
0 &= M_{y\Gamma}\Gamma + \mathbf{Q}\left(E + \mathbf{C}\right) \\
\Gamma &\equiv \begin{pmatrix} \Gamma_0 \\ \vdots \\ \Gamma_S \end{pmatrix}, E \equiv \begin{pmatrix} E_0 \\ \vdots \\ E_S \end{pmatrix}, \mathbf{C} \equiv \begin{pmatrix} C_0 \\ 0 \end{pmatrix}, \mathbf{Q} \equiv \begin{bmatrix} Q & & 0 \\ & \ddots & \\ 0 & & Q \end{bmatrix} \\
M_{y\Gamma} &\equiv \begin{bmatrix} \Phi & \Lambda^{0A} & & & \\ & \Phi & \Lambda^{0A} & 0 & \\ & & \ddots & \ddots & \\ & 0 & & \Phi & \Lambda^{0A} \\ & & & & \Phi \end{bmatrix}, \quad \begin{matrix} \Phi \equiv \begin{bmatrix} \Omega^A_{ss}/Z_{\kappa s} & \Lambda^B_{s\phi} \\ 0 & \Omega^B_{uu} Z^H_{u\phi} \end{bmatrix}, \\ \Lambda^{0A} \equiv \begin{bmatrix} 0 & \Lambda^A_{s\phi} \\ 0 & \Omega^A_{uu} Z^H_{u\phi} \end{bmatrix} \end{matrix}
\end{aligned}
$$

Note that since Φ is invertible, $M_{y\Gamma}$ is also clearly invertible. Hence,

$$
\begin{aligned}
0 &= \Gamma + M_{y\Gamma} \backslash \mathbf{Q}\left(E + \mathbf{C}\right) \\
&= M_{\Gamma\Pi}\Pi + M_{y\Gamma} \backslash \mathbf{Q}\left(E + \mathbf{C}\right)
\end{aligned}
$$

where (3.7) is used to derive the second line. Hence,

$$
\begin{aligned}
0 &= \Pi + M_{\Pi E}\left(E + \mathbf{C}\right) && \text{(3.41a)} \\
M_{\Pi E} &\equiv \left(M_{y\Gamma} M_{\Gamma\Pi}\right) \backslash \mathbf{Q} && \text{(3.41b)}
\end{aligned}
$$

In the following, we compute E and Π *column by column.*

$$
\Pi_{.i} = M_{\Pi E}\left(E_{.i} + \mathbf{C}_{.i}\right)
$$

Remember that some elements in $\Pi_{.i}$ are zero due to imperfect information, while some elements in $E_{.i}$ are non-zero. For example,

$$0 = \begin{pmatrix} \Pi_{1,i} \\ \vdots \\ \Pi_{k,i}\,(=0) \\ \vdots \\ \Pi_{M(S+1),i} \end{pmatrix} + M_{\Pi E} \left(\begin{pmatrix} 0 \\ \vdots \\ E_{ji} \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} C_{.i} \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix} \right) \tag{3.42}$$

E Matrix

From the k-th set of equations in (3.42)

$$0 = \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{kj} E_{ji} + \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{kj} \mathbf{C}_{ji} + \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{k\neg j} \mathbf{C}_{\neg ji}$$

Hence, assuming $\begin{bmatrix} M_{\Pi E} \end{bmatrix}_{kj}$ is invertible,

$$E_{ji} = -\begin{bmatrix} M_{\Pi E} \end{bmatrix}_{kj} \backslash \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{k\neg j} \mathbf{C}_{\neg ji} - \mathbf{C}_{ji}$$

Π matrix

From the other equations in (3.42), we eliminate the expectation errors E_{ji}.

$$\begin{aligned}
\Pi_{\neg ki} &= \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{\neg kj} \left(\begin{bmatrix} M_{\Pi E} \end{bmatrix}_{kj} \backslash \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{k\neg j} \mathbf{C}_{\neg ji} + \mathbf{C}_{ji} \right) \\
&\quad - \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{\neg kj} \mathbf{C}_{ji} - \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{\neg k\neg j} \mathbf{C}_{\neg ji} \\
&= \left(\begin{bmatrix} M_{\Pi E} \end{bmatrix}_{\neg kj} \left(\begin{bmatrix} M_{\Pi E} \end{bmatrix}_{kj} \backslash \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{k\neg j} \right) - \begin{bmatrix} M_{\Pi E} \end{bmatrix}_{\neg k\neg j} \right) \mathbf{C}_{\neg ji} \\
&= -\begin{bmatrix} M_{\Pi E}^{-1} \end{bmatrix}_{\neg j\neg k} \backslash \mathbf{C}_{\neg ji}
\end{aligned}$$

The vector $\Pi_{\neg ki}$ and $\Pi_{ki} = 0$ can be vertically merged to recover $\Pi_{.i}$, and the vectors $\Pi_{.i}$ are horizontally concatenated to recover full Π matrix. Note that an invertible $\left[M_{\Pi E} \right]_{kj}$ implies an invertible $\left[M_{\Pi E}^{-1} \right]_{\neg j \neg k}$. Not surprisingly, $\mathbf{C}_{ji}$ does not affect the coefficient matrix $\Pi_{.i}$, because the j-th set of equations does not hold for the i-th innovation in any case; it only affects the expectation error E_{ji}.

3.F.5 Other Matrices (J, G and D)

J and G Matrices

To obtain the J and G matrices, from (3.7),

$$\Gamma \equiv \begin{bmatrix} J_0 \\ G_0 \\ \vdots \\ J_S \\ G_S \end{bmatrix} = M_{\Gamma\Pi}\Pi$$

D Matrix

From the A matrix in a given model (3.2),

$$D = -A \begin{bmatrix} 0 \\ G_0 \end{bmatrix}$$

which always satisfies (3.34a). It can be shown that the j-th rows in D are zeros if the j-th equation does not include $t+1$ dynamic jump variable (see the next section).

3.G A Comment on the D Matrix

The derivation of the D matrix is a bit tricky, and requires careful attention concerning non-square matrices $\Lambda_{s\phi}^{A}$ and $Q_{s.}^{H}$. We do not show the straightforward derivation of D – which is perhaps not intuitive – but instead we simply show that our solution always satisfies (3.34a), which, in turn, reveals an important intuition.

First, we define dynamic and non-dynamic jump variables: $\phi_{t+1} = \begin{bmatrix} (\phi^d_{t+1})^T & (\phi^n_{t+1})^T \end{bmatrix}^T$. Note that the coefficients of the non-dynamic jump variables ϕ^n_{t+1} in A matrix must be zero by the definition of "non-dynamic".

$$Ay_{t+1} \equiv \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \begin{pmatrix} \kappa_{t+1} \\ \phi^d_{t+1} \\ \phi^n_{t+1} \end{pmatrix}$$

where ϕ^d_{t+1} is the vector of dynamic variables, such as consumption in the Euler equation. The submatrices in G_0 and Q^H are defined as

$$\tilde{G}_0 \equiv \begin{bmatrix} 0 \\ G_0 \end{bmatrix} \equiv \begin{bmatrix} 0 \\ G_{0,\phi^d \cdot} \\ G_{0,\phi^n \cdot} \end{bmatrix}$$

$$Q^H \equiv \begin{bmatrix} Q^H_{s\cdot} \\ Q^H_{\phi\cdot} \end{bmatrix}, Q^H_{s\cdot} \equiv \begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \end{bmatrix}, Q^H_{u\cdot} \equiv \begin{bmatrix} Q^H_{u^f\kappa} & Q^H_{u^f\phi^d} & Q^H_{u^f\phi^n} \\ Q^H_{u^i\kappa} & Q^H_{u^i\phi^d} & Q^H_{u^i\phi^n} \end{bmatrix}$$

where u^f and u^i imply finite and infinite unstable roots, respectively.

Focussing on the second term of (3.34a)

$$\begin{aligned} Q^H_{s\cdot}D &= Q^H_{s\cdot}A\tilde{G}_0 = \begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \end{bmatrix} \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \begin{bmatrix} 0 \\ G_{0,\phi^d \cdot} \\ G_{0,\phi^n \cdot} \end{bmatrix} \\ &= \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d} \right) G_{0,\phi^d \cdot} \end{aligned} \tag{3.43}$$

For the first term of (3.34a) note that $\Lambda^A_{s\phi}$ is the $s\phi$-th elements in $\Omega^A Z^H$, i.e.,

$$\begin{aligned}
\Lambda^A_{s\phi} &= \begin{bmatrix} \Omega^A Z^H \end{bmatrix}_{s\phi} = \begin{bmatrix} QQ^H\Omega^A Z^H \end{bmatrix}_{s\phi} = \begin{bmatrix} QA \end{bmatrix}_{s\phi} \\
&= \left[\begin{bmatrix} Q^H_{s\kappa} & Q^H_{s\phi^d} & Q^H_{s\phi^n} \\ Q^H_{u^f\kappa} & Q^H_{u^f\phi^d} & Q^H_{u^f\phi^n} \\ Q^H_{u^i\kappa} & Q^H_{u^i\phi^d} & Q^H_{u^i\phi^n} \end{bmatrix} \begin{bmatrix} A_{\kappa\kappa} & A_{\kappa\phi^d} & 0 \\ A_{\phi^d\kappa} & A_{\phi^d\phi^d} & 0 \\ A_{\phi^n\kappa} & A_{\phi^n\phi^d} & 0 \end{bmatrix} \right]_{s\phi} \\
&= \left[\begin{bmatrix} * & \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d} \right) & 0 \\ * & * & 0 \\ * & * & 0 \end{bmatrix} \right]_{s\phi} \\
&= \begin{bmatrix} \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d} \right) & 0 \end{bmatrix}
\end{aligned}$$

where $*$ elements are irrelevant for our current interest. Hence,

$$\begin{aligned}
\Lambda^A_{s\phi}G_0 &= \begin{bmatrix} \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d} \right) & 0 \end{bmatrix} \begin{bmatrix} G_{0,\phi^d.} \\ G_{0,\phi^n.} \end{bmatrix} \\
&= \left(Q^H_{s\kappa}A_{\kappa\phi^d} + Q^H_{s\phi^d}A_{\phi^d\phi^d} + Q^H_{s\phi^n}A_{\phi^n\phi^d} \right) G_{0,\phi^d.} \qquad (3.44)
\end{aligned}$$

(3.43) and (3.44) show that (3.34a) holds. The key to the solution is a sort of zero restriction; A matrix has zero columns by the definition of "non-dynamic" variables.

A further question is the consistency of D (i.e. whether the computed D always has zeros at the proper positions?). Specifically, if the j-th equation does not have ϕ^d_{t+1}, it should not have an expectation error due to ξ_{t+1}, and hence the row vector $D_{j.}$ must be zero; this zero restriction on D is analogous to that on E. This is surely satisfied because the rows corresponding to non-dynamic equations in D $(= A\tilde{G}_0)$ is always zero by the construction of A; i.e., the j-th row in A is zero if the j-th equation does not include dynamic jump variables ϕ^d_{t+1}. For example, in the standard RBC model, all but the Euler equation have zero rows in A and hence in D.

What this section discusses is the correspondence between expectation errors and the source of such errors. If, for example, expectation errors with respect to full information up to time ι_t appears in the equations without dynamic jump variables, then it is a logical contradiction (expectation errors without their causes), and hence (3.34a) is not satisfied. Conceptually, the consistency of the D matrix is parallel to the invertibility of $[M_{\Pi E}]_{kj}$. As mentioned in the main text, the non-invertibility of $[M_{\Pi E}]_{kj}$ implies an incorrect specification of the information structure with respect to $\xi_{t+\tau}$ ($\tau = 0, 1, \cdots, S$). Similarly, an inconsistent D (or the non-existence of a consistent D) implies an incorrect specification of information structure with respect to ξ_{t+1}. Such inconsistency/non-existence happens, for example, if a researcher puts an expectation operator on the evolution of capital, rather than on the consumption Euler equation.

Finally, note that a consistent D matrix exists $\Leftrightarrow$ Equation (40) in Sims (2002) holds. Thus, it is now clear that equation (40) in Sims (2002) must always be satisfied if expectation errors appear only in the equations with dynamic jump variables, *regardless of the dynamic property such as saddle-path stable, sunspot, or explosive.*

Concluding Remarks

This thesis analyses inventories empirically and theoretically. It shows a battery of evidence for periodicity; economic variables, such as production, follow sine waves. This is in stark contrast with the modern view, where business cycles are successive deviations from the steady state and their returning process. Instead, this thesis find that booms and recessions occur alternately; more precisely, a boom is the seed of the recession that follows, and vice versa. This thesis also finds that inventories play a key role in generating endogenous cycles – namely, inventory cycles.

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