Applied Financial Econometric Analysis: The Dynamics of Swap Spreads and the Estimation of Volatility

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Abstract

This Thesis contains an examination of the time-series properties of swap spreads, their relation with credit spreads and an estimation of the risk premium embedded in the swap spread curve. Chapter 2 introduces the main institutional aspects of swap markets, and studies the time-series properties of swap spreads. These are shown to be non-stationary and display a time-varying conditional volatility. Chapter 3 provides evidence of cointegration between corporate bond spreads and swap spreads. We estimate an error-correction model, including additional variables such as the level and slope of the yield curve, taking into account the exogenous structural break due to the crisis of August 1998. We find evidence that the relation between swap and credit spreads arises from the swap cash flows being indexed to Libor rates. Chapter 4 studies the risk premium in the term structure of the swap spreads, obtaining evidence that it is time-varying. The slope of the swap spread curve is shown to predict the changes in swap spreads. These results are relevant for the study of the risk premium in credit markets, and extend the existing literature on riskless Treasury securities. Chapter 6 develops the asymptotic properties of the quadratic variation estimator of the volatility of a continuous time diffusion process. We explore the case in which the number of observations tends to infinity, while the time between them remains fixed. For the case of a geometric Brownian motion, we show that the estimator is asymptotically biased, but the bias is a random variable that converges. We study the behaviour of this random variable via a simulation study, that shows that it typically has a "small" effect. We conclude by exploring some practical applications related the specification of the volatility for financial time series.

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Chapter 1

Introduction to the Empirical Analysis of Swap Spreads and the Estimation of Volatility

The present Thesis is a step in the investigation of the empirical properties and determinants of swap spreads and the estimation of volatility. Swap spreads are a very important variable in financial markets, being defined as the difference between the fixed rate in an interest rate swap and the yield of a government bond with comparable maturity. Therefore, swap spreads represent the compensation for entering into a swap, instead of holding a riskless bond. Interest rate swaps and their markets are discussed extensively in the next Chapter, and we develop their empirical study in the subsequent chapters. The estimation of volatility for a class of continuous-time processes using its quadratic variation, is studied in the last substantive Chapter of this Thesis. In the rest of the current Chapter, we will introduce the reasons to be of this Thesis, its objectives, and its main conclusions.

Almost every work related to interest rate swaps begins with the same motivation: swaps have become by far the most important interest rate derivative[1] and represent one of the finest examples of the success of a financial innovation. We need to bear in mind that swaps as we understand them were first designed and transacted in the early

[1] See the periodic surveys by the Bank for International Settlements.

80's[2], and didn't really mature until the second half of the 90's.

As swap markets developed, more and more academic interest has been devoted to their study. This Thesis represents a contribution to the empirical research on swaps, and more particularly, to the research on swap spreads. Throughout our work, we emphasise two methodological points: the use of a meaningful and reliable set of data (especially relevant when treating the topic of the relation of swap spreads with credit risk) and the adoption of an econometric model –error-correction– that is consistent with the time series properties of the variables involved.

This Thesis contains three Chapters on swap spreads. The motivation of the Chapter on the "Introduction to Swap Markets and the Time-Series Properties of Swap Spreads" is to provide the background for the subsequent work. However, it turns out to be the case that we already obtain some substantive results on the time-series properties of spreads.

First, we describe the institutional features of the swap contract and some of the conventions of swap markets. We will see that interest rate swaps are periodical exchanges of two cashflows indexed to different rates, one variable (e.g. 6-month Libor) and one fixed, determined in the marketplace. The fixed rate can be quoted as a spread over a government bond of comparable maturity, where the spread is positive because the floating leg is indexed to a risky rate.

We next explain how to take a trading stance on swap spreads, by taking opposite positions in the swap and the government bond markets, and provide with an approximate expression for the profit and loss (P&L) of such a strategy. The profitability of a trade will depend on whether the swap spread widens or tightens (the "capital gain" component) vs. the cashflow component, related to the swap spread level (minus the financing cost of the position).

A trade where we receive fixed in a swap and short a government bond (via the repo market), results in a position that makes a profit when spreads tighten and has a cashflow equal to the swap spread at the time of inception of the trade minus the Libor minus repo differential. This risky position should produce a positive expected return due to the presence of a risk premium. The risk of such strategy is that spreads can widen so much that this outweighs the generally positive income.[3]

Given that the evolution of swap spreads is key in the profitability and risk of swap

[2]See Baz (1997).

[3]Note that we can make an analogy with corporate bonds.

positions relative to government bonds, we next begin the study of their time series properties. We show how for USD and EUR, the two most important currencies in swap markets, swap spreads are positive, display large cyclical variations and have experienced a major regime-change, in August 1998, with a persistent increase in both levels and volatilities due to the financial crisis unleashed by the default of Russia on her sovereign debt and the consequences on the financial system. The term structure of swap spreads is generally increasing, although the 30-year spread is frequently low and has recently been even lower than the 2-year spread.

The subsequent analysis of the time-series properties of swap spreads is centered around two topics. First, we find that the levels of swap spreads are very highly autocorrelated. This can have strong implications for an econometric analysis, if, as we show, there are strong signs on non-stationarity in the data. The usual tests for unit roots –Augmented Dickey-Fuller, Perron test allowing for structural breaks– cannot reject the null hypothesis of swap spreads being I(1) processes. We do note, however, that it is difficult to justify swap spreads being I(1) on theoretical grounds, more so if we notice that these are the difference between two rates (swap rate vs. a government bond yield) which are related by no-arbitrage conditions. We tend to see our results as indication of non-stationarity over our particular sample, and that we shouldn't rely on the typical asymptotic results. We do note that the power of the unit root tests we have used is considered to be small[4] although all our results point to swap spreads being integrated or near-integrated.

The second time-series feature of swap spreads that we study in that Chapter is their apparent time-varying and persistent volatility. This is a well-documented feature of many financial variables -stock prices, exchange rates, etc- where we observe clusters of volatility, due to possible factors such as differences in the rate of news arrival or changes in investors risk aversion. We will approach this topic within the GARCH framework[5]. Our results show that GARCH is an appropiate choice to model the first differences in swap spreads. We extend the GARCH model to take into account that volatility may be asymmetric (it increases when "bad news" hit the market by more than it decreases when "good news" arrive), and that there is a change in regime for volatility after August 1998, to a higher level.

After setting the stage in Chapter 2, we continue with a Chapter on "The Relation

[4]See Banarjee et al (1993).

[5]See Engle (1982).

Between Swap and Credit Spreads." This contains a strong message, as we argue that swap spreads and credit spreads are related in the long term by a cointegrating relation. We proceed to estimate that relation using a very specific econometric technique, an error-correction model ("ECM").

The study of the relation between credit risk and swap spreads is probably the first topic treated in the empirical literature on swaps (see Sun et al. (1993), Minton (1994), etc.). At that time, this relation was explained by swaps being seen as defaultable instruments, and hence researchers focused on counterparty default risk. This idea was examined in the most direct way by Cossin and Pirotte (1997), who for the swap book of a particular bank, compared the market quotes with the actual transaction rates and related the difference to the credit rating of the counterparty. As expected, better rated counterparties obtained better terms in their swaps.

As swap markets have developed, and especially since the crisis of August 1998, market participants have designed an array of measures (collateralization, netting, material adverse event clauses, etc.) to reduce the counterparty risk of swaps to almost negligible levels. We, together with other researchers (see for instance He (2000), Collin-Dufresne and Solnik (2001)) argue that the relation between swap and credit spreads is due to the fact that the swap cash-flows, being default-free, are still indexed to a risky rate, Libor. This is a structural feature of swaps, by their own design, that links swap and credit spread in the long run. Together with the evidence that both series are I(1) or near-integrated in small samples, this justifies our econometric approach to model them as cointegrated processes.

In the literature, there are a number of references with econometric approaches to the relation between swap and credit spreads, e.g. Minton (1994) and Lang et al. (1998). We believe that one of our main advantages over them is a very carefully constructed, exhaustive and reliable corporate bond spread data. In this Chapter we use the credit spread curves constructed from the Lehman Brothers Index database, a bank that at least for USD is considered the leading provider of corporate bond data. The index contains all the bonds in the market that satisfy a set of minimum conditions and have an acceptable liquidity. The bonds utilised are only those which have been priced by a trader, and the estimation methodology used to obtain the spread curves is robust to outliers.

The other advance in that Chapter is the careful discussion of the time-series properties of the processes involved and the choice of a methodology that addresses the non-stationarity of the data (an error-correction model) in a way that maximises the amount of information extracted from the data. The ECM consists of two relations, one long-term (in levels) and one short-term (in changes). The short-term relation includes the lagged error from the long-term relation to account for the mean reversion. We augment the long-term relation with the slope of the yield curve, to take into account the effect of the business cycle in the relation between swaps and credit (see Lang et al. (1998), and Harvey (1988)).

The results from the estimation of the ECM with USD data between 1994 and 2001, substantiate most of our claims. In particular, we obtain a strong relation between swaps and credit spreads. This relation depends on the position in the business cycle, i.e. swap spreads are relatively tight when the yield curve is steep, which corresponds to periods with a high expected inflation and growth. The mean reversion variable is strongly significant and negative, as expected. Interestingly, variables related to differential counterparty risk are not significant or have the "wrong" sign. Finally, there's evidence of a regime-shift in August 1998, with the relation between swaps and credit spreads becoming more positive and the impact of the slope being more important.

After the empirical investigation on the relation between swap and credit spreads, in Chapter 4 we explore the properties of the risk premium embedded in swaps. As we discuss in Chapter 2, a useful analogy to understand swap positions is to compare them with holding a corporate bond. That is, when we are short the spread (receiving in a swap and shorting a government bond), we make a capital gain/loss when swap spreads tighten/widen, and we receive a sure income of the spread minus the running financing cost. This income is generally positive and can be interpreted as the coupon of a corporate bond over a government bond.

We just showed in Chapter 3 a positive relation between swap and credit spreads, hence holding swap risk is akin to holding credit risk, and so we would expect a risk premium to be present in swaps. Actually, some researchers, when studying the risk premium for corporate bonds, deal with the lack of data on those instruments by taking swap spreads as proxies of credit spreads (Liu et al. (2000)). Chapter 4 is a direct empirical study of the properties of the risk premium in swap spreads, whether there is one, is it constant or time-varying and whether we can use it to predict swap spread movements.

In our opinion, this is an important previous step before a formal model of the credit risk premium.

Our empirical methodology is based on the classical approach to the risk premium in riskless government bonds (see Campbell and Shiller, (1991)). Taking the Expectations Hypothesis "EH" [6], as a starting point or null hypothesis, Campbell and Shiller construct a regression where changes in bond yields are related to the slope of the curve. It turns out that the data does not support the EH: there is evidence of the risk premium being time-varying and that a steep yield curve predicts long-dated bonds to outperform.

In our study, we construct a version of the EH that holds for the spreads between zero-coupon swap rates and zero coupon Government bonds. It is worth noting that the zero-coupon swap rates or "Libor zeros" are theoretical constructions. One way to see this is recalling that swap rates can be interpreted as par rates. From these, we can obtain the zero curve swap and the discount factors, by analogy with government bonds or within a term structure model. However, these zeros cannot be synthetized from elemental par swaps and hence are not tradeable.

The empirical test is a regression of the change in the swap spread vs. the slope of the spread curve, computed as the difference between that swap spread and the 1-year spread. According to the EH, the coefficient of the slope of the swap spread curve should be 1. It turns out that in most of the cases that we examine, for USD and EUR, the slope coefficient is significantly different from 1, implying a time-varying risk premium. Furthermore, the coefficient tends to be negative and significantly different from 0, hence swap spreads turn out to be predictable to a certain extent.

The results above are to our knowledge, new in the literature and have important implications for modeling the price of credit risk. In the rest of the chapter, we present additional evidence in support of our findings. Essentially, we have re-done our regressions with a longer data sample (since 1987 for USD), with weekly data and for swap spreads vs. benchmark bonds. The results are comparable to those in the base case.

In summary, our empirical study of swap spreads has followed three steps. First, after discussing the main institutional features of swap markets, we have characterized the time-series properties of swap spreads. This has been relevant in the adoption of the error-correction model to analyze the relation between swap and credit spreads. Given

[6] See Campbell, Lo and MacKinlay (1997).

the strong, positive relation between the two, we have investigated the properties of the compensation for holding swap spread risk, which turns out to be time-varying, and to a certain extent, predictable.

The final chapter of this Thesis is devoted to the study of the properties of an estimator of the volatility parameter of a continuous-time process, when we have discrete-time observations. This estimator is based in the corresponding quadratic variation process (see Karatzas and Shreve (1991)). This chapter is more technical in nature than the rest, but we still understand it as a work of applied financial econometrics. Once we have dealt with the technicalities, the main result tells us something about the behaviour of an estimator in a realistic situation in terms of what we generally face when doing empirical research. This is because we focus on what are the properties of the estimator when the sample size is longer, rather than when the data frequency is higher. In the process of pursuing that, we develop a general idea for the proof of our main theorem that can be extended to the more "relevant" cases. Hence, it will turn out that the most clear way to present and extend our results is in terms of "model misspecification", in this case, the exploration the properties of the estimator when the parametric process we use is not the actual data generating process.

The estimator whose properties we study is based on the quadratic variation of the given continous time diffusion process, and has been presented and used for a long time in the mathematical statistics literature (see for instance Fournié (1994)). When the time between observations tends to zero, the quadratic variation estimator can be shown to be consistent. However, the more realistic situation is one in which the time between observations is fixed (at one day, one week, or even longer) but we can increase the number of observations. Our work shows that even though the quadratic variation estimator will be biased, it is the case that the bias is finite. Of course, this does not guarantee that the quadratic variation estimator is "usable" in normal situations, e.g. weekly data for 5 years, or daily data for 1 year. An exhaustive simulation study shows that the bias is typically "small", and that the frequency of the obervations does not need to be high or the number of observations large, for the bias to so. It is when we move to data frequencies lower than one week that the estimator deteriorates quite strongly.

Regarding the underlying parametric model, we have specialized this Chapter to the case of a geometric Brownian motion. This is a case for which the maximum likelihood estimator can be found in closed form and is proved to be "optimal" or most efficient. Hence it would seem that the quadratic variation estimator is not necessary. On the other hand, the geometric Brownian motion is one of the simplest cases, and allows us to develop a method of proof that can be extended to more complicated cases. As it turns out, the proof only relies crucially on the paths of the process being bounded. For the case of the geometric Brownian motion, such bounds are given by the law of the iterated logarithm. The key point is that bounds do exist for more complicated processes, hence an extension the proof should be possible. Indeed, we have been working on a similar proof for the family of Lévy processes, which allow for the possibility of jumps. When moving to these more general processes, the issue of the behaviour of the quadratic variation estimator when the data generating process is badly specified gains more relevance.

We finally explore a number of possible practical applications for the results we have obtained. For instance, we could assess the possibility of a time-varying volatility by looking at the evolution of the quadratic variation estimator over time, or whether a geometric Brownian motion is a reasonable representation for actual stock prices by comparing the maximum likelihood estimates with those given by the quadratic variation. The conclusions of our work will be summarized in detail in the final Chapter of this Thesis.

Chapter 2

Introduction to Swap Markets and the Time-Series Properties of Swap Spreads

2.1 Introduction to Swap Markets

A swap is a derivative contract in which two parties agree to periodically exchange a stream of cashflows. These cashflows are determined according to a certain formula, and are referenced to some market price(s). It is how the cashflows are calculated that determines the particular nature of a swap. As an illustration, we may have a debt-to-equity swap, where one party receives the return on a certain stock index and pays the yield on some debt instrument, or a basis swap, where the parties exchange payments based on two different short–term interest rates. Swaps are one of the finest examples of the success of the financial innovation process in the last 30 years. Nowadays, the dealer community is able to tailor an "exotic" swap to meet almost any requirement. The present work will focus on interest rate swaps.

In an interest rate swap (swap hereafter), one party –the "payer"– periodically pays a fixed rate –the swap rate– to the other party –the "receiver"– and, in exchange, receives a floating coupon, generally equal to the Libor rate (Euribor in the case of EUR swaps).[1] The Libor rates are fixed every day and represent an average lending rate

[1]The typical market convention for USD is that the fixed payments are semiannual, and the floating

between a panel of major international banks in London. The Euribor rate is also an interbank lending rate, in this case the average for a broad panel of banks in the Euro-area. The payoffs of the swap are calculated on a notional principal, which is not exchanged.

The swap transaction between two counterparties is generally governed by the so-called International Swaps and Derivatives Association (ISDA) Master Agreement. This standard contract establishes the major features of the swap, i.e. how are the payments going to be computed, which legal jurisdiction can be used in case of disagreement, or the events that determine the premature termination of the swap due to the deterioration in the credit quality of a counterparty. The high level of legal security that the ISDA Master Agreement provides has been a major factor behind the success of swap markets. Dealers and clients typically agree on additional contract clauses with the main purpose of reducing the default risk on the swap payments.

Interest rate swap markets have experienced enormous growth in the last few years. According to the Bank for International Settlements (BIS), the notional on interest rate swaps at the end of year 2000 stood at \$48.768trn, being by far the most important interest rate contract (the notional of all "over-the-counter" interest rate contracts was \$64.668trn). In contrast, the outstanding notional in swaps in June 1998 was only \$29.363trn. As a recognition of the importance of swaps, the Federal Reserve began including the swap rates in its selected interest rate releases in July 2000.

Interest rate swaps are "over-the-counter" derivative instruments. In other words, there is no centralized exchange or clearing house, but they are traded by a network of broker-dealers. Representative swap rates are available in real time from providers of financial information such as Reuters or Bloomberg. The swap rate is typically quoted as the spread in basis points over the benchmark Treasury bond with the closest maturity.[2] That is, for maturity m, the swap spread is calculated as

$$Swap\,Spread_t^m = Swap\,Rate_t^m - Treasury\,Bond\,Yield_t^m. \tag{2.1}$$

An example of a swap transaction can be seen in Figure 2.1. Since Libor rates apply to unsecured lending operations, they include a default risk premium that varies

payments are equal to the 3-month Libor rate, paid every three months. For EUR, the convention is annual fixed payments and floating payments equal to the 6-month Euribor, paid twice a year.

[2]A benchmark or on-the-run government bond is, for a particular maturity, the most recently issued (and generally the most liquid) bond.

over time. Moreover, Libor rates are sensitive to short term liquidity conditions, which translate into short–lived rate spikes.[3] The default and liquidity premia are reflected in the swap rates, which are generally higher than the risk-free Treasury rate. The fact that the swap payoffs are referenced to risky rates, together with the evidence that all sectors of the credit market are tightly correlated, translates into a significant correlation between swap spreads and corporate bond spreads (see Chapter 3). Given the liquidity and transparency of swaps, some market participants interpret swap spreads as a good measure of the evolution of global credit risk.

Regarding the pricing of swaps, the usual market practice is to set the swap rate so that the net present value of the swap at inception is nil. We will not go into swap pricing issues here, just mention that a swap can be replicated as a portfolio of forward rate agreements (FRA) or short term interest rate futures (see Minton (1994)). A good reference that discusses swap pricing from a practitioner point of view is Campbell and Temel (1999).

2.1.1 The evolution of swap markets

The evolution of the swap market has been extremely fast.[4] Primitive forms of swap contract ("back-to-back loans") appeared at the end of the 70's in order to sidestep legal and regulatory restrictions. For instance, limitations to international flows of capital created incentives for multinational companies to issue debt in their own country and then swap the proceeds to fund their subsidiaries abroad. These restrictions gave strong comparative advantages to different agents that swaps could help to monetize. The gradual elimination of the legal and regulatory motivations for swaps did not stop the growth of swap markets. The reason is because swaps are natural instruments to hedge interest rate risk and to arbitrage away comparative advantages in different segments of the debt markets. In the latter case, swaps could be used to exploit the fact that lower-rated firms tend to have a comparative advantage in issuing floating-rate debt, while higher quality firms (or entities such as supranational organizations -e.g. the World Bank) have an advantage in issuing fixed-rate debt. Swaps allow each firm to issue where it has the advantage and then transform the terms of financing as desired. In the

[3]For instance, in the period around Y2K, Libor rates experienced a large temporary increase as the value of liquidity increased, due to concerns about the possible failure of the technology supporting the payments system.

[4]The following discussion is largely based on Baz (1997).

process, there is an arbitrage gain that can be shared by the two swap counterparties and possibly a broker–dealer.

As Finance theory prescribes, outright "issuer" arbitrage opportunities in swap markets did not persist for long. However, swaps have found extensive use in risk management as they give firms the flexibility to transform the nature of their debt in a simple and inexpensive way. For many corporates, the most important financial decision they face is whether to switch between fixed and floating financing by entering into swaps. Also, swaps are particularly suited to the management of balance sheet duration. This is particularly relevant for commercial banks and other financial institutions which tend to have a gap in the duration of their assets (long term loans at fixed rate) and liabilities (short term deposits). Also, given their sensitivity to interest rate changes, financial institutions tend to swap their issues of fixed-rate debt into floating to reduce their overall risk.

In recent years, and especially since the financial crisis of 1998, the relation between swap and credit spreads has been the main focus of swap markets. Empirically, we observe a strong correlation between swap and credit spreads.[5] Moreover, swap markets tend to be more liquid than corporate bond markets, making swaps a natural instrument with which to hedge and speculate credit risk. As an example, many dealers use swaps to hedge their inventories of corporate bonds.

It is increasingly common to take the swap curve (instead of the Treasuries curve) as the benchmark curve to reference instruments with credit risk. This is a more important issue in the Euro-area, where it does not exist a single Government bond yield curve, but a number of country curves that depend on the credit standing and liquidity of each country's debt. In contrast, the EUR swap curve is the same for all the countries in the Euro-area.[6]

2.1.2 Trading swap spreads

Engaging in swap spread trades is a common strategy for hedge funds, investment banks and other significant participants in the fixed income markets. Swap spreads are traded by taking simultaneous positions in the swap and the Treasuries market. A position that results in obtaining the spread as income consists in receiving fixed in

[5] This is studied in Chapter 3.

[6] As an illustration of this point, the usual practice for EUR corporate debt issues is to price them as a spread over the swap curve, while USD corporate bonds are still generally priced over US Treasuries.

a swap and short-selling a Treasury bond in the repo market. The cash flows of this position are as follows: from the swap we receive the swap rate and pay Libor. In the repo, we pay the Treasury coupon[7] and receive the repo rate on the cash that we lend. All together, we get the swap spread minus the difference between the Libor rate and the repo "GC" rate.[8] Since repo rates are the rates for collateralized lending, while Libor is a rate for unsecured lending, the Libor-GC spread will typically be positive, and can be interpreted as the cost of financing the position. The symmetric position (paying the swap spread) can be obtained by paying in a swap and buying the Treasury, financing it at the repo rate. In the example in Figure 2.2, the dealer is receiving the spread as income, at the cost of paying the Libor-GC spread.

The discussion above dealt with the cashflow component of the swap spread trade, but obviously there is also a capital gain (or loss) component in it, as the swap and the Treasury involved in the position change in value as time goes by. We say that a long (short) position in spreads produces a capital gain when the spread widens (tightens). By convention, a party is long the spread when is long the Treasury and pays in the swap. Conversely, a party will be short the spread when is short the Treasury bond and receives in the swap. The rule of thumb is that the position in the spread is equal to the position in the Treasury. In a long spread position, if swap spreads widen, we have a capital gain: either the yield on the Treasury went down (price went up), or the swap rate went up (the swap becomes more valuable, as we pay less than we would if we entered into a new contract), or both.

The nuances of trading swap spreads can be seen more clearly by writing down the profit and loss (P&L) formula from a position in swap spreads. For a long position, this is given approximately by

$$P\&L_{t,t+\delta} = D_m \Delta Spr_{t,t+\delta} - \frac{\delta}{365}[Spr_t - (Libor_t - GC_t)] \tag{2.2}$$

and the same with an opposite sign for a short spread position. By D_m we denote the (modified) duration of the swap[9], while δ is the investment horizon in days. The repo

[7] Assume it is a par bond.

[8] GC stands for "general collateral", meaning that the security that is pledged as collateral is readily available in the market and hence does not command a "specialness" premium because of an excess demand for it.

[9] The duration of the swap is in slightly lower than the duration of a fixed-rate par bond where the coupon is equal to the swap rate. This is because we have to subtract the duration of the floating leg of the swap, which is generally small but not insignificant. In practice, we will use an approximation and assume that the duration of the swap is equal to that of the fixed bond.

rate is taken to be the rate for bonds which are "general collateral," denoted by GC. In the formula for the P&L, the first element is the capital gain resulting from the swap spread change. The second element is the income (positive or negative) from the position or "carry". For a spread position, the carry is equal to the swap spread minus the financing cost, given by the Libor-GC spread. Notice that in a "short" spread position we have a positive income (the swap spread minus the Libor-GC) and the the capital gain is positive when swap spreads tighten. In other words, we hold spread risk, and we can think of this as buying a corporate bond, where there is a loss if the bond spread (relative to Treasuries) widens, but there is positive income in form of a coupon. In this way we can see how taking a short position in swap spreads represents taking a view on tighter spreads, or at least that the carry is large enough to compensate for the risk assumed.

2.1.3 The dynamics of swap spreads

In the following we describe and comment on the evolution of swap spreads for USD and EUR (DEM before 1999), which are the two largest swap markets.[10] Other important markets are those for GBP, JPY and CHF. We will focus in the maturities of 2, 5, 10 and 30 years, which are the focal points in the swap and Treasuries markets. As such, these are the most liquid points in the curve and also those that receive more attention by market participants. Our data consists of daily mid-market swap spreads relative to benchmark Treasuries, spanning the period between January 1995 and April 2001.[11]

In Figure 2.3, we can see that USD swap spreads display a remarkable degree of variability, having moved in a range between 20bp and 160bp. In general, spreads have been increasing with maturity, or in other words, the spread curve is upward sloping. However, the 10-year to 30-year sector has often been inverted, sometimes for a sustained period of time, e.g. between December 2000 and May 2001. Clearly, the volatility of spreads has been much higher after the global financial crisis of the second half of 1998, a fact that will be discussed below. We also observe two additional peri-

[10]The European Monetary Union in January 1999 represented the consolidation of the Euro-area swap markets. In particular, the floating rate references to the different national currencies were substituted by a unique Libor or Euribor rate. For existing swaps, the ISDA protocol gave the quidelines for the conversion, with the result that EMU did not represent a particular break in the market.

[11]Although it is possible to obtain longer samples, we have decided to restrict the data to this period because the more recent data tends to be more accurate and representative, given the rapid development of the swap markets.

ods where the level of spreads increased remarkably, namely during Summer 1999 and the first half of 2000. The major reasons behind these spread widenings were the complicated macroeconomic situation of the US, with the Fed increasing rates to restrain an overheated economy, the gyrations in the equity markets driven by the technology bubble, and the impact of the US Treasury debt buyback program.

The picture for EUR (see Figure 2.4) is slightly different than for USD, especially in the period between 1995 and August 1998. The EUR swap spreads fluctuated at lower levels than USD, between 0bp and 70bp. The volatility pre-crisis was substantial, and we also can observe how the 5y spread was typically higher than the 10y spread, a situation that reversed only around the end of 1997. Also, the evolution of EUR spreads displays a strong degree of mean reversion inside regimes. After the crisis of 1998, EUR spreads have been quite unstable, similar to the USD, experiencing the same trends, i.e. widening during summer 1999 and again at the beginning of 2000, then tightening after the second half of 2000 and into 2001. This is another manifestation of the fact that although the business cycles for Europe and US tend to be decoupled, there is a important degree of integration in their fixed income markets (see Baz et al. (1999)).

Next, in Table 2.1 we present descriptive statistics for swap spreads. For USD, the average quoted spreads between 1995 and 2001 have ranged from 40bp in the 2-year sector, to 63.5bp in the 30-year sector. The long end of the spread curve has been flat on average with a 10-year spread average of 64.3bp. These average levels hide a large degree of variability: for instance, the 10-year spread moved between 31bp and 137bp. The standard deviation of 1-day changes has been of around 1.5bp, with the largest changes being around 10bp per day. For EUR, the spread curve has a different shape, with the 5-year to 10-year sector flat on average at around 35bp. The long end of the curve has been downward sloping on average, with the 30-year spread at 31.7bp. As in the US, swap spreads have shown a large degree of variability around their mean, with the 10-year spread moving between 14bp and 74bp. Also, EUR swap spreads have been remarkably tight on a number of ocasions, for instance the minimum level for the 5-year spread was 5bp. The volatility of daily changes in spreads has been significantly higher than for USD, driven by a few large daily changes that EUR spreads have experienced. For instance, the standard deviation of daily changes in 5-year spreads has been 2.4bp per day, with daily changes as large as 29bp or -24bp. In Chapter 4 it is shown that the standard deviation of *weekly* changes in EUR spreads is significantly lower than that of USD spreads.

The empirical observation that USD swap spreads are significantly higher than EUR spreads has not been studied formally to our knowledge. There are a number of possible explanations for this difference in levels. For instance, He (2000) shows how the swap spread is the present value of the forward Libor-GC spread. This spread is the difference between rates for unsecured and collateralized lending, a fairly stable variable. The Libor-GC spread in the USD is currently around 20bp per year, while for EUR it is around 5bp. In He's model, this would generally translate into higher USD swap spreads.[12]

2.1.4 The effects of the financial crisis of August 1998

The most important recent event regarding the evolution of swap markets –and fixed income markets in general– was probably the financial crisis of the second half of 1998. This crisis was initiated in August 1998 by the default of Russia on its sovereign debt, and was compounded by the delicate economic situation in Asia and the possibility of default in emerging countries like Brazil. The crisis had an immediate impact on financial markets, with a flight-to-quality from credit markets into safe assets, mainly Treasury bonds. The delicate situation in terms of low liquidity and high volatility was complicated by the collapse of LTCM, a large hedge fund with open positions with a large number of dealers. The rescue of LTCM had to be coordinated by the US Federal Reserve in order to avoid a major liquidity crisis for the global financial system.

The effects of the crisis of 1998 and the subsequent instability in swap and credit markets can be clearly seen when we compute the statistics of spreads by subsamples. For the USD spreads (see Table 2.2), after the crisis of 1998, there is a very clear increase in both the spread levels and the volatility of spread changes. For instance, the 10-year average spread after the crisis was 94bp, against 41bp before. Correspondingly, the volatility of daily spread changes went from 0.6bp to 2.2bp. It is also important to note that the spread widening was accompanied by a large increase in the slope of the swap spread curve. The difference between the 10-year and the 2-year average swap spread

[12] Baz et al. (1999) present a number of conjectures on why Libor-GC spreads are higher for USD than for EUR. It is the case that European banks tend to have a higher credit quality than US banks. Actually, some European banks are guaranteed by the State, e.g. the German Landesbanks. Second, the money markets in the US are more developed, providing investors with more short–term types of investments. In Europe, investors are mainly restricted to bank deposits, which as a consequence, may command a premium.

went from an average of 15bp before the crisis to 34bp after.

For EUR (see Table 2.3) the implications of the crisis of 1998 are less clear. We do observe an important increase in average spread levels for the 2-year and 10-year spreads, that went from 16bp to 25 bp and from 27bp to 48.5bp respectively. However, the 5-year and 30-year average spreads remained virtually unchanged. The slope of the spread curve, measured by the 10-year minus the 2-year spread, did increase substantially from 9bp before the crisis to 24bp after. The volatility of daily changes after the crisis is generally smaller than before the crisis, but we find that to be mostly driven by the presence of a small number of large daily changes in the first period. When the changes are computed for weekly intervals, we find the opposite result (see additional evidence in Chapter 4).

In Figures 2.3 and 2.4, we observe that after the financial crisis subsided at the beginning of 1999, the instability in swap markets persisted. One of the reasons for this is the impact of the crisis on how the most sophisticated players in fixed income markets –hedge funds and investment banks– conduct their business. The lessons from the meltdown of LTCM[13] made them reduce their leverage and adopt trading strategies with lower risk. In particular, hedge funds and investment banks have become less active in pursuing relative value strategies, as these involve mantaining open positions for a medium-term horizon and the use of substantial leverage. This results in an environment in which mean-reversion is slower, relative value strategies are more risky and, possibly, misalignments across instruments and markets become more persistent.

On the other hand, we can argue that a positive consequence of the crisis of 1998 has been the increase in the awareness of swaps by market participants. The correlation of swaps with credit and their high liquidity makes them natural instruments to hedge and speculate on credit markets. Also, since the evolution of corporate bond markets is difficult to track through looking at a myriad of corporate bonds, swap spreads are becoming popular as a measure of global credit risk. The effect of a structural break in August 1998 on the time series properties of spreads will be studied in the next section. The nonstationarity that this induces is going to be a major factor in any subsequent econometric analysis.

[13]See Perold (1999) for a full exposition of the LTCM investment strategies and their pitfalls.

2.2 The Time Series Properties of Swap Spreads

In this section we explore and model the time-series properties of swap spreads. In contrast with the previous section, in which we worked with quoted swap spreads, now the data will consist of swap spreads relative to constant maturity par fitted (CMPF) Treasury yields. The CMPF curve is a spline that fits the yields of the universe of Treasury bonds excluding benchmark issues. It is the case that benchmark yields tend to be lower than the yields of the rest of the bonds with similar maturities, due to the presence of a liquidity or "benchmark" premium. The properties of the benchmark premium have not been been characterized, and they could obscure the conclusions of our study of swap spreads. The second main reason for using fitted spreads is to avoid distortions due to differences in maturity between the swap and the bond. While the maturity of the swap is constant, the maturity of the particular benchmark bond decreases over time. In environments where yield curves are very steep, or when bonds are issued less frequently, the difference in maturities can have an important impact.

For the sake of concreteness, we will analyse the time-series properties of the USD and EUR 10-year fitted swap spreads. The 10-year swap spread is the probably the most liquid point in the Treasury curve and in the swap curve and hence has the most accurate and representative data. The evolution of the 10-year fitted swap spread for USD and EUR can be found in Figures 2.5 and 2.6, and the main descriptive statistics are presented in Table 2.4.

Comparing the average spreads in Tables 2.1 and 2.4, we can see how fitted spreads are on average lower than quoted spreads, due to the presence of the benchmark premium in the latter. For USD, the benchmark premium on the 10-year T-Note was on average 15.2bp, the difference between the 64.3bp average quoted swap spread and the 49.1bp average fitted spread. Additionally, the benchmark premium has changed over the sample period, from an average of 8.4bp before the 1998 crisis, to 23.8bp after. This is due to the fact that the 10-year US Treasury Note is the preferred instrument in situations of flight-to-quality. The effect of the benchmark premium in EUR swap spreads is important but does not display the dramatic change in 1998 that the USD premium displays. The average benchmark premium for the full sample was 12bp, almost identical for both subperiods.

2.2.1 Persistence and stationarity

We now begin the analysis of the persistence of the swap spread series by looking at their autocorrelation and partial autocorrelation functions. The discussion will then turn to testing for the presence of a unit root in the series. This is key for the econometric analysis of spreads that we perform in Chapter 3. One important point to mention is that the presence of a possible exogenous structural change in 1998 needs to be taken into account, as it can induce artificially high autocorrelations and hence bias typical unit root tests.

The autocorrelation functions for both USD and EUR spreads (see Tables 2.5 and 2.6) show high persistence in daily spread levels, with autocorrelations as high as 0.75 at the 100th lag. In order to interpret these numbers, we need to discuss the statistical significance of the observed sample autocorrelations.

Let us denote the kth sample autocorrelation as r_k. Given a set of independent observations drawn from a fixed distribution, it can be shown that $Var(r_k) \simeq T^{-1}$, where T is the number of observations. For T large, it is the case that $\sqrt{T}r_k \sim N(0,1)$ asymptotically. Hence, a simple way to test whether a sample autocorrelation is significantly different from zero is to see whether

$$|r_k| > \frac{2}{\sqrt{T}} \tag{2.3}$$

In general, however, we have observations that are not independent. In that case, the estimate of the standard error of the sample autocorrelation r_k will be given by

$$Var(r_k) \simeq \frac{1}{T}(1 + 2\sum_{j<k}^{k} r_j^2) \tag{2.4}$$

Instead of looking at the sample autocorrelations one by one, we may want to have a single number that incorporates the information in all the autocorrelations (up to a certain lag). A Portmanteau test statistic for the case of a white noise process is given by

$$Q^*(k) = T\sum_{i=1}^{k} r_i^2 \tag{2.5}$$

which asymptotically follows a χ_k^2 distribution.

The statistic above has been shown to have poor small sample properties. Worse, even for relatively large samples it may tend to be biased towards not rejecting the null hypothesis of no serial correlation. An approximation with better properties is the Ljung-Box statistic, which is given by

$$Q(k) = T(T+2)\sum_{j\leq k}\frac{r_j^2}{T-j} \tag{2.6}$$

The Ljung-Box Q-statistic is asymptotically distributed as a χ_k^2.
In Tables 2.5 and 2.6 we report the sample autocorrelations for USD and EUR respectively, together with the standard errors and Q-statistics. In all cases, the autocorrelations of the series in levels are highly significant.
Given that the first order autocorrelations of the spread levels are close to one, it is clear that the issue of stationarity needs to be carefully addressed. We have first computed augmented Dickey-Fuller tests (ADF) for each spread series, where the null hypothesis is that the process contains a unit root. The ADF test generalizes the Dickey-Fuller test to the case in which innovations are serially correlated. Assume that the process y_t is a AR(p) process, of the form

$$y_t = \theta_0 + \sum_{i=1}^{p}\phi_i y_{t-i} + \epsilon_t \tag{2.7}$$

It is straightforward to show that this can be rewritten as

$$\Delta y_t = \theta_0 + \phi y_{t-1} + \sum_{i=1}^{k}\delta_i \Delta y_{t-i} + \epsilon_t \tag{2.8}$$

where $k = p-1$ and

$$\phi = \sum_{i=1}^{p}\phi_i \tag{2.9}$$

$$\delta_i = -\sum_{j=i+1}^{p-1}\phi_j,\ j = 1,2,\ldots,p-1 \tag{2.10}$$

The null hypothesis of a unit root is given by $\phi = 1$. This can be tested via OLS estimation, where the test statistic is constructed as

$$t_\mu = \frac{\hat{\phi}_T - 1}{se(\hat{\phi}_T)} \tag{2.11}$$

where $se(\hat{\phi}_T)$ is the standard error of the OLS estimate. The number of lags in the autorregression is chosen such that the residual is white noise. The results of the ADF tests are presented in Table 2.7. The overall conclusion is very clear in that we cannot reject the null hypothesis of a unit root in the series in levels. Additionally, in order to address the impact of the structural change in August 1998, we have computed the ADF test values for the full sample and for the subsamples before and after crisis. As we can see in Table 2.7, the results are unambiguous: the ADF tests cannot reject the null hypothesis of nonstationarity, for each subsample defined by the crisis of August 1998, as well as for the full sample.

As we mentioned in the previous section, the financial crisis of 1998 was an exogenous event that had a strong impact on both the level and the volatility of spreads. It is well-known that the presence of a structural break in a sample tends to inflate autocorrelation coefficients. In order to deal explicitly with the impact of this fact on unit root tests, we have performed the Perron (1989) test, in which the null hypothesis allows for an exogenous change both in the level and the rate of growth of the series. If we assume that the break happens at time T_B, the null hypothesis is that

$$H_0: \; y_t = \mu_1 + y_{t-1} + \eta DTB_t + (\mu_2 - \mu_1)DU_t + \epsilon_t \qquad (2.12)$$

where

$$DTB_t = 1 \text{ if } t = T_B + 1, \text{ and 0 otherwise} \qquad (2.13)$$

$$DU_t = 1 \text{ if } t > T_B + 1, \text{ and 0 otherwise} \qquad (2.14)$$

and where ϵ_t is an error term that satisfies a number of regularity conditions.[14] The alternative hypothesis is

$$H_a: \; y_t = \mu_1 + \beta_1 t + (\mu_2 - \mu_1)DU_t + (\beta_2 - \beta_1)DT_t^* + \epsilon_t \qquad (2.15)$$

where

$$DT_t^* = t - T_B \text{ if } t > T_B, \text{ and 0 otherwise} \qquad (2.16)$$

The results in Table 2.8 indicate that there is evidence of nonstationarity which persists after taking into account the presence of an exogenous structural break in August 1998.

[14] See Mills (1999) pg.75.

The issue of testing for the presence of a unit root vs. stationarity is going to appear prominently in this Thesis. It is important to note that such a test suffers from important problems. First, the I(0) alternatives which are close to to being I(1), are very plausible in economic terms. In other words, the usual tests (Dickey-Fuller, Augmented Dickey-Fuller) have a low power, in the sense that it is too difficult to reject the null hypothesis of a unit root.[15] This is compounded by the fact that the true data generating process is not known, while the critical values of the test statistics are sensitive to the structure of the process.[16] Having this in mind, we find that the evidence of non-stationarity for the series we deal with, is generally very high.

The overall conclusion from this section is that we there is evidence that the swap spread levels contain a unit root. Hence, the econometric analysis of swap spreads should be performed on the differenced series.[17] In checking for nonstationarity, we have been taken into account the presence of a structural break in swap markets, due to the financial crisis of August 1998. A final point is to mention that, apart from the statistical evidence, there is an additional reason for conducting the analysis of the series in differences: the returns from swap spread positions are driven by the changes in the spreads, hence the interest in modelling them directly.

2.2.2 Properties of the swap spreads in first differences

We first look at the statistics of the daily spread changes in Table 2.9. The overall conclusions for fitted spreads are similar to those for benchmark spreads. For USD, the standard deviation of the daily spread changes was around 1.6bp. However, if we look at this by subsamples, we can see how the volatility went up from 0.7bp per day before August 1998 to 2.3bp after. For EUR, the standard deviation of spread changes is less different across samples, being 1.4bp per day before August 1998 and 1.7bp after that date. The extreme values for USD, were 7.7bp and -8.2bp. Although the average level of spreads in EUR was much lower than that for USD, the magnitude of spread changes in EUR has been quite large, with extremal movements of 11.2bp and -8.6bp in one day.

Next, we look at the autocorrelation functions of the daily differences in USD and EUR

[15] See Hamilton (1994), chapter 17.

[16] See Banarjee et al. (1993) and Mills (1999), Section 3.1.7 for a more extensive discussion.

[17] This is true as long as the differenced series is I(0). We will show in the next section that this is indeed the case.

spreads. From the results in Tables 2.10 and 2.11, it is clear that the persistence of the series in differences is much lower than that of the series in levels. For USD, the first autocorrelation is 0.119. The autocorrelations at higher lags are much lower, between -0.05 and 0.05. The ACF for EUR daily differences in spreads is slightly different, in the sense that we find that the first autocorrelation is significantly higher than the rest, but *negative*. For the full sample, the first autocorrelation is equal to -0.140. As for USD, the autocorrelations at higher lags lie in a small range, between -0.1 and 0.1 approximately.

The fact that the first-order autocorrelation of daily changes in swap spreads is significantly different from zero for both USD and EUR may indicate the possibility that swap spreads are predictable. This predictability could be monetized using simple trading strategies, based on either momentum (USD) or mean-reversion (EUR). We find very unlikely that this simple strategies will produce significant economic profits. First, the first order autocorrelation being significantly different from zero, is a result that is not robust when we look at the data in subsamples. If we look at Table 2.10 for USD, we see that the first order autocorrelation being significantly positive happens only for the period post-98. For EUR, the result is the opposite: in Table 2.11 we can see that the negative autocorrelation disappeared after August 1998. A second factor would be the impact of transaction costs: since the correlation fades away for changes beyond 1 day, the trading would be fairly frequent, resulting in large transaction costs. In our judgement, in order to exploit swap spreads being possibly weakly-predictable, we need a more sophisticated approach. In Chapter 4, we will study the profitability of a trading strategy based on a predictive relation between changes in swap spreads and the slope of the swap spread curve. In efficient markets, we would not expect such trading opportunities to last for long. As we argue in previous sections, it is possible that the scaling down of trading operations in swap spreads after August 1998, has resulted in the persistence of such opportunities.

The evidence from the autocorrelation functions for the daily differences in spreads indicates that it is unlikely that they contain a unit root. To confirm it, we have run a set of augmented Dickey-Fuller tests, the results of which are presented in Table 2.12. For both USD and EUR 10-year spread changes, the ADF uniformly reject the null hypothesis of a unit root.

2.2.3 Modelling the conditional volatility of swap spreads

In this section we specify and estimate the conditional volatility for the first (daily) differences in swap spreads. It is clear from the statistics that there are periods in which the volatility is apparently higher than in other periods, and that these regimes in volatility are persistent. We will study these issues in the framework of generalized autoregressive conditional heteroskedasticity models for the conditional volatility of a series.

The first step in such analysis is to model the conditional mean of the process. In other words, we first need to remove the linear effects, until we obtain a series which is white noise. To this effect, we will work in the framework of the autoregressive moving average ARMA(p,q) family of processes. The general form of an ARMA(p,q) process is given by

$$y_t = \kappa + \alpha_1 y_{t-1} + \cdots + \alpha_p y_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \cdots + \theta_q \epsilon_{t-q} \tag{2.17}$$

where p and q denote the number of autoregressive and moving average terms respectively. After some experimentation, we have chosen an ARMA(1,0) (or AR(1)) as a reasonable approximation to the dynamics of the conditional mean of the differences in swap spreads. The specification should capture the properties of the conditional mean in a parsimonious way (i.e. avoiding the overparametrization of the model). There is a number of criteria that balance the two objectives, for instance the Akaike Information Criteria (AIC) or the Schwartz Information Criteria (BIC).[18] We have found that these two indicators can give very divergent results: for instance, for EUR, the AIC indicates an ARMA(4,4), while the BIC chooses an ARMA(1,0). Also, when we try to estimate the higher-order models with the spread data, we find that the coefficients tend to be not significant and that, if the estimation is via maximum likelihood, the results very much depend on the starting values. All in all, we consider that the best practical choice is to work with AR(1) processes that result in residuals with very slight linear effects, if any (see Tables 2.13 and 2.14).

When we look at the evolution of the swap spread series over time, we can observe that there are extended periods for which the volatility is higher or lower than "usual."

[18] These consist in minimizing either $AIC(p,q) = \log \hat{\sigma}^2 + 2(p+q)T^{-1}$ or $BIC(p,q) = \log \hat{\sigma}^2 + (p+q)T^{-1} \log T$, where $\hat{\sigma}^2$ is the estimated error variance. The AIC penalizes heavily parametrized models less than the BIC, as we can see by comparing the second term in the previous equations.

In other words, we observe clusters of volatility. After characterizing the conditional mean of the spread changes as an ARMA(p,q) process, the error will be denoted by u_t. This is white noise, but we want to allow for the conditional variance of this disturbance to change over time. Assume that the form of the error is given by

$$u_t = \sqrt{h_t} v_t \tag{2.18}$$

where v_t is an i.i.d process with zero mean and unit variance and h_t is the conditional variance of u_t.

There are a number of ways of modelling the conditional variance. The most popular framework for such analysis is that of autoregressive, conditional heteroskedasticity models, referred to as ARCH(m) (see Engle (1982)). In ARCH models, the conditional variance of the error terms depends on their past realizations. We say that u_t follows an ARCH(m) process if

$$h_t = \kappa + \alpha_1 u_{t-1}^2 + \cdots + \alpha_m u_{t-m}^2 \tag{2.19}$$

One simple way to test for persistence in volatility is to look at the ACF of the squared residuals after fitting an ARMA(p,q) model for the conditional mean of the series. This is presented in Tables 2.15 and 2.16. For EUR, the autocorrelations are as large as $\rho(1) = .250$, $\rho(2) = .286$ and even $\rho(20) = 0.116$. For USD the results are similar, with $\rho(1) = 0.229$, $\rho(2) = 0.248$, and $\rho(20) = 0.211$.

A more formal way to test for persistence in the conditional volatility of the series is to regress the values of u_t^2 on its lags. If there are no ARCH effects, the R^2 of this regression should be zero, fact that can be used as a basis for a formal test. Engle (1982) proved that, under the null of no ARCH effects, the statistic TR^2 follows a χ_m^2 distribution, where m is the order of the ARCH tested (and so equal to the number of lags in the regression). In our case, the test is implemented on the residuals from the ARMA filter on the differenced spread series. The results in Table 2.17 indicate quite clearly the presence of ARCH effects in volatility.

It has been observed that a simple ARCH may not be flexible enough to reflect the statistical properties of the typical financial time series. A more general approach for u_t is that of a generalized autoregressive conditionally heteroskedastic process of order (r, m), denoted by GARCH(r,m) (see Bollerslev (1986)). In a GARCH(r,m) model, we assume that h_t evolves according to

$$\begin{aligned} h_t =& \kappa + \delta_1 h_{t-1} + \cdots + \delta_r h_{t-r} \\ &+ \alpha_1 u_{t-1}^2 + \cdots + \alpha_m u_{t-m}^2 \end{aligned} \quad (2.20)$$

As an initial step, we have estimated GARCH(1,1) processes for the conditional volatilities of the first differences in swap spreads for both USD and EUR.

We have also extended the model to take into account two important remarks on the behaviour of spreads. First, we may want to extend the GARCH model by allowing for an "asymmetry" in the behaviour of the condional volatility of the process. This arises from the observation that the volatility of financial time series is perceived to be higher when there are "bad news", rather than when the market gyrations are caused by the adjustment to "good" news. One model that tries to capture this and quantify this asymmetry is proposed in Glosten, Jagannathan and Runkle (1993), who assume that h_t evolves according to

$$h_t = \kappa + \delta_1 h_{t-1} + \alpha_1 u_{t-1}^2 + \gamma u_{t-1}^2 I_{t-1} \quad (2.21)$$

where $I(t) = 1$ if $u(t) < 0$, and 0 otherwise, to allow for asymmetry or "leverage" effect. For the case of spreads, "bad news" is when spreads widen, that is, the innovation u_t is large. Hence, if there is a leverage effect, in this specification, we would expect that $\gamma < 0$. We denote this specification as asymmetric-GARCH, or A-GARCH.

Second, we look for evidence on the impact of the crisis of August 1998 in the volatility of the spreads. We can model this in the GARCH framework by introducing a dummy variable in the conditional volatility equation. This dummy is defined as $D_t = 1$ if t is after August 1998, and 0 otherwise. Then, the conditional variance is given by

$$h_t = \kappa + \alpha D_t + \delta_1 h_{t-1} + \alpha_1 u_{t-1}^2 + \gamma u_{t-1}^2 I_{t-1} \quad (2.22)$$

where we would expect α to be strictly positive.

Using the (filtered) spread changes data, we have estimated three specifications (GARCH, A-GARCH, and A-GARCH with a time dummy) for USD and EUR spreads. The conclusions are generally the same for both currencies. In the estimation of the GARCH(1,1), we find that all the coefficients are statistically significant, and the study of the normalized residuals shows that the model is statistically adequate (see Tables 2.18 and 2.21). The usual Q-tests cannot reject the null that the normalized residuals are white noise, although they are clearly non-normal. Second, an F-test of no-ARCH vs. ARCH on the normalized residuals, cannot reject the null hypothesis. In the second model, the A-GARCH(1,1) (see Tables 2.19 and 2.22), we find that the leverage term is significant and negative, indicating that volatility tends to be higher when spreads widen, or in "bad news" periods. Finally, in the third specification, the dummy variable in the A-GARCH(1,1) especification is significant, showing that the average volatility after August 1998 has been higher. However, in the USD case (Table 2.20), the leverage term is not significant, its effect being captured by the dummy variable. For the EUR series, both the leverage and the time dummy are significant, with the expected signs (negative and positive respectively).

2.3 Conclusions

In this chapter we have presented the main institutional features of swap markets and the most important time-series properties of swap spreads. Our intention is to facilitate the empirical analysis of swap spreads, a series that has become one of the most important variables in fixed income markets.

The main conclusions in this chapter come probably from the time–series analysis of spreads. First, we have shown that swap spread levels are highly persistent at a daily frequency. More concretely, we find that swap spread series contain a unit root. This implies that, in an econometric analysis, we should work with the series in first differences. This is compounded by the fact that the returns on swap spread positions are given by the changes in spreads, so it is intuitive to model those directly.

Next, we have modelled the conditional mean of the first differences in swap spreads as an ARMA process, and the conditional volatility as a GARCH process. We have studied extensions of the GARCH model, that show that higher volatility tends to occur in periods of "bad" news (spread widening) and that the volatility after August 1998 is higher on average.

The evidence we have shown in this chapter should be useful in a further econometric analysis of swap spreads, modelling the determinants of swap spreads and the dynamic interactions with other variables. These topics will be addressed in the rest of this dissertation.

Levels	USD 2y	USD 5y	USD 10y	USD 30y
Mean	40.8	53.1	64.3	63.5
Std. Deviation	19.8	26.4	29.6	32.4
Maximum	88.8	107	137	161.5
Minimum	12.5	22	31	31
Changes	USD 2y	USD 5y	USD 10y	USD 30y
Mean	0.021	0.027	0.028	0.010
Std. Deviation	1.6	1.4	1.5	1.6
Maximum	8.0	9.0	11.8	12.0
Minimum	-7.9	-6.0	-7.0	-9.7
Levels	EUR 2y	EUR 5y	EUR 10y	EUR 30y
Mean	20.0	35.3	36.1	31.7
Std. Deviation	8.0	11.9	13.4	15.2
Maximum	46.5	59.1	70.1	75.4
Minimum	1.1	4.7	13.8	7.9
Changes	EUR 2y	EUR 5y	EUR 10y	EUR 30y
Mean	0.002	0.004	0.009	0.004
Std. Deviation	2.4	2.4	1.7	2.6
Maximum	13.4	29.0	9.5	19.0
Minimum	-15.6	-24.1	-11.6	-25.0

Table 2.1: Descriptive statistics of levels and daily changes in USD and EUR swap spreads (vs. benchmark Treasury). All figures in bp. The full sample is from January 1995 to April 2001, T=1596 for USD and T=1661 for EUR.

Levels	Pre-crisis USD 2y	USD 5y	USD 10y	USD 30y	Post-crisis USD 2y	USD 5y	USD 10y	USD 30y
Mean	25.2	31.3	40.7	39.7	60.4	80.4	94.2	93.8
Std. Deviation	8.0	6.3	5.9	4.5	10.8	13.5	18.6	27.0
Maximum	45.5	47.3	56	52	88.8	107	137	161.5
Minimum	12.5	22	31	31	38.8	45.5	55.6	44.5
Changes	Pre-crisis USD 2y	USD 5y	USD 10y	USD 30y	Post-crisis USD 2y	USD 5y	USD 10y	USD 30y
Mean	0.019	0.012	0.021	-0.014	0.023	0.046	0.038	0.041
Std. Deviation	1.2	0.6	0.6	0.6	1.9	2.0	2.2	2.3
Maximum	7.1	3.8	4.0	7.1	8.0	9.0	11.8	12.0
Minimum	-5.0	-2.0	-4.5	-6.1	-7.9	-6.0	-7.0	-9.7

Table 2.2: Descriptive statistics of levels and daily changes in USD swap spreads (vs. benchmark Treasury). All figures in bp. The pre-crisis sample is from January 1995 to July 1998 (897 obs.), the post-crisis sample is between August 1998 and April 2001 (699 obs.).

Levels	Pre-crisis EUR 2y	EUR 5y	EUR 10y	EUR 30y	Post-crisis EUR 2y	EUR 5y	EUR 10y	EUR 30y
Mean	16.3	34.8	26.7	30.5	24.8	35.9	48.5	32.9
Std. Deviation	6.6	12.9	5.6	13.7	6.9	10.3	10.1	16.4
Maximum	46.5	56.8	43.4	57.5	40.1	59.1	70.1	75.4
Minimum	1.1	4.7	13.8	7.9	6.7	14.7	30.6	9.4
Changes	**Pre-crisis EUR 2y**	**EUR 5y**	**EUR 10y**	**EUR 30y**	**Post-crisis EUR 2y**	**EUR 5y**	**EUR 10y**	**EUR 30y**
Mean	0.004	-0.007	0.008	-0.017	0.000	0.019	0.011	0.022
Std. Deviation	2.8	2.8	1.7	3.1	1.8	1.7	1.7	2.1
Maximum	13.4	29.0	8.3	19.0	10.5	7.6	9.5	13.2
Minimum	-15.6	-24.1	-7.6	-25.0	-9.7	-10.8	-11.6	-9.7

Table 2.3: Descriptive statistics of levels and daily changes in EUR swap spreads (vs. benchmark Treasury). All figures in bp. The pre-crisis sample is from January 1995 to July 1998 (936 obs.), the post-crisis sample is between August 1998 and April 2001 (725 obs.).

USD	Full sample	Pre-crisis	Post-crisis
Mean	49.1	32.3	70.6
Std. Deviation	23.8	3.9	21.1
Maximum	117.5	42.8	117.5
Minimum	24.6	24.6	38.5
EUR	**Full sample**	**Pre-crisis**	**Post-crisis**
Mean	24.5	14.7	36.5
Std. Deviation	13.0	4.1	9.8
Maximum	57.6	25.9	57.6
Minimum	2.4	2.4	19.2

Table 2.4: Descriptive statistics of USD and EUR 10-year fitted swap spreads. All figures in bp. The full sample is from January 1995 to April 2001, and the pre(post)-crisis is before(after) August 1998. For USD, the total number of observations is T=1596, with 897 pre-crisis and 699 post-crisis observations. For EUR, the number of observation is T=1661, 936 pre-crisis and 725 post-crisis observations.

Lag	Full sample r_k	$s.e.(r_k)$	LB Q-Stat	sign.	Pre-crisis r_k	$s.e.(r_k)$	LB Q-Stat	sign.	Post-crisis r_k	$s.e.(r_k)$	LB Q-Stat	sign.
1	0.998	0.025	1591.4	0	0.984	0.033	872.1	0	0.994	0.038	693.3	0
2	0.995	0.043	3174.5	0	0.970	0.057	1720.7	0	0.986	0.065	1376.7	0
3	0.992	0.056	4749.5	0	0.957	0.073	2546.7	0	0.978	0.084	2050.4	0
4	0.989	0.066	6317.1	0	0.945	0.086	3353.4	0	0.971	0.099	2715.3	0
5	0.987	0.075	7878.0	0	0.934	0.097	4141.6	0	0.965	0.112	3372.2	0
6	0.984	0.082	9432.2	0	0.922	0.107	4910.8	0	0.958	0.123	4021.2	0
7	0.982	0.089	10979.6	0	0.910	0.115	5660.8	0	0.951	0.133	4662.2	0
8	0.980	0.096	12520.6	0	0.897	0.123	6391.5	0	0.945	0.143	5295.8	0
9	0.977	0.102	14054.9	0	0.886	0.130	7104.8	0	0.939	0.151	5921.6	0
10	0.975	0.108	15582.9	0	0.875	0.137	7801.3	0	0.933	0.159	6540.2	0
20	0.953	0.153	30547.9	0	0.776	0.184	13938.6	0	0.874	0.221	12368.6	0
30	0.930	0.186	44895.9	0	0.712	0.215	18955.3	0	0.814	0.264	17513.3	0
40	0.904	0.212	58630.7	0	0.663	0.239	23358.5	0	0.745	0.295	21989.6	0
50	0.878	0.234	71635.4	0	0.637	0.257	27304.5	0	0.674	0.319	25708.9	0
100	0.739	0.310	126230.4	0	0.454	0.311	40464.3	0	0.329	0.373	35596.5	0

Table 2.5: Autocorrelation function for USD 10-year fitted swap spreads. The full sample is from January 1995 to April 2001, T=1596. Pre (post)-crisis is before (after) August 1998, with 897 and 699 observations respectively.

Lag	Full sample r_k	$s.e.(r_k)$	LB Q-Stat	sign.	Pre-crisis r_k	$s.e.(r_k)$	LB Q-Stat	sign.	Post-crisis r_k	$s.e.(r_k)$	LB Q-Stat	sign.
1	0.993	0.025	1641.8	0	0.948	0.033	843.2	0	0.985	0.037	706.4	0
2	0.988	0.042	3268.7	0	0.920	0.055	1639.0	0	0.972	0.064	1395.2	0
3	0.983	0.054	4879.6	0	0.900	0.069	2401.0	0	0.956	0.082	2062.2	0
4	0.979	0.064	6477.1	0	0.883	0.081	3135.4	0	0.942	0.096	2711.0	0
5	0.974	0.073	8060.9	0	0.866	0.091	3842.0	0	0.928	0.108	3341.3	0
6	0.971	0.080	9633.1	0	0.851	0.099	4526.2	0	0.916	0.118	3955.9	0
7	0.967	0.087	11194.7	0	0.837	0.106	5187.9	0	0.905	0.128	4556.7	0
8	0.964	0.093	12746.1	0	0.823	0.113	5828.2	0	0.894	0.136	5144.4	0
9	0.961	0.099	14289.2	0	0.806	0.120	6443.9	0	0.886	0.144	5722.6	0
10	0.958	0.104	15825.2	0	0.800	0.125	7050.1	0	0.878	0.151	6290.9	0
20	0.924	0.147	30667.1	0	0.668	0.165	12041.4	0	0.770	0.205	11346.7	0
30	0.890	0.178	44460.3	0	0.561	0.188	15570.1	0	0.674	0.238	15153.6	0
40	0.860	0.202	57395.9	0	0.493	0.203	18230.5	0	0.599	0.260	18161.0	0
50	0.847	0.223	69814.9	0	0.462	0.215	20433.1	0	0.586	0.278	20854.0	0
100	0.725	0.296	124923.4	0	0.014	0.235	24571.9	0	0.295	0.336	31059.8	0

Table 2.6: Autocorrelation function for EUR 10-year fitted swap spreads. The full sample is from January 1995 to April 2001, T=1661. Pre (post)-crisis is before (after) August 1998, with 936 and 725 observations respectively.

USD				
Full sample	BIC	(lags)	AIC	(lags)
ADF t-test	-1.56	1	-1.56	1
ADF Z-test	-4.85		-4.85	
Pre-crisis				
ADF t-test	-1.81	1	-1.81	1
ADF Z-test	-9.23		-9.23	
Post-crisis				
ADF t-test	-1.92	1	-1.92	1
ADF Z-test	-6.53		-6.53	
EUR				
Full sample	BIC	(lags)	AIC	(lags)
ADF t-test	-2.09	1	-2.10	20
ADF Z-test	-8.61		-8.44	
Pre-crisis				
ADF t-test	-3.69	1	-2.15	20
ADF Z-test	-29.50		-12.98	
Post-crisis				
ADF t-test	-2.20	1	-2.45	20
ADF Z-test	-9.67		-14.95	

Table 2.7: Augmented Dickey-Fuller tests for USD and EUR 10-year fitted swap spreads. The number of lags is determined using the BIC and the AIC criteria. The null hypothesis is that there is a unit root. The rejection levels for the ADF t-test are -2.57 (10%), -2.86 (5%) and -3.43 (1%). The rejection levels for the Z-test are -11.3 (10%), -14.1 (5%) and -20.7 (1%). For USD, the full sample is from January 1995 to April 2001, T=1596. Pre (post)-crisis is before (after) August 1998, with 897 and 699 observations respectively. For EUR, the full sample is from January 1995 to April 2001, T=1661. Pre (post)-crisis is before (after) August 1998, with 936 and 725 observations respectively.

USD		
Perron test	-2.451	
Critical value (5%)	-4.24	
Lags	147	
Lambda	0.6	
	Coefficient	t-stat
constant	0.1707	0.25
DMU	-2.39176	-1.146
Trend	0.00019	0.686
DT	0.00095	1.261
DTB	0.37618	0.226
EUR		
Perron test	-3.57	
Critical value (5%)	-4.24	
Lags	20	
Lambda	0.6	
	Coefficient	t-stat
constant	0.1327	0.533
DMU	-0.1035	-0.156
Trend	0.0002	1.049
DT	0.0002	0.609
DTB	-1.7634	-1.196

Table 2.8: Perron test for USD and EUR 10-year fitted swap spreads. The null hypothesis is that there is a unit root, with an exogenous break in August 1998. The number of lags is determined by adding lags until the Ljung-Box test rejects residual correlation at level 0.05. For USD the full sample is from January 1995 to April 2001, T=1596. Pre (post)-crisis is before (after) August 1998, with 897 and 699 observations respectively. For EUR the full sample is from January 1995 to April 2001, T=1661. Pre (post)-crisis is before (after) August 1998, with 936 and 725 observations respectively.

USD	Full sample	Pre-crisis	Post-crisis
Mean	0.017	0.013	0.023
Std. Deviation	1.6	0.7	2.3
Maximum	7.7	2.9	7.7
Minimum	-8.2	-2.7	-8.2
EUR	Full sample	Pre-crisis	Post-crisis
Mean	0.010	0.019	-0.001
Std. Deviation	1.5	1.4	1.7
Maximum	11.2	6.3	11.2
Minimum	-8.6	-7.3	-8.6

Table 2.9: Descriptive statistics of daily changes in USD and EUR 10-year fitted swap spreads. All figures in bp. For USD the full sample is from January 1995 to April 2001, T=1596. Pre (post)-crisis is before (after) August 1998, with 897 and 699 observations respectively. For EUR, the full sample is from January 1995 to April 2001, T=1661. Pre (post)-crisis is before (after) August 1998, with 936 and 725 observations respectively.

Lag	Full sample r_k	$s.e.(r_k)$	LB Q-Stat	sign.	Pre-crisis r_k	$s.e.(r_k)$	LB Q-Stat	sign.	Post-crisis r_k	$s.e.(r_k)$	LB Q-Stat	sign.
1	0.119	0.025	22.5	0.000	-0.054	0.033	2.6	0.107	0.138	0.038	13.4	0.000
2	-0.021	0.025	23.2	0.000	-0.012	0.033	2.7	0.255	-0.022	0.039	13.7	0.001
3	-0.038	0.025	25.5	0.000	-0.056	0.033	5.6	0.135	-0.037	0.039	14.7	0.002
4	-0.047	0.025	29.0	0.000	-0.012	0.034	5.7	0.223	-0.051	0.039	16.5	0.002
5	-0.003	0.025	29.0	0.000	0.013	0.034	5.8	0.321	-0.004	0.039	16.5	0.006
6	0.007	0.025	29.1	0.000	0.008	0.034	5.9	0.434	0.006	0.039	16.5	0.011
7	-0.023	0.025	29.9	0.000	0.008	0.034	6.0	0.544	-0.027	0.039	17.1	0.017
8	0.024	0.025	30.9	0.000	-0.036	0.034	7.1	0.524	0.030	0.039	17.7	0.024
9	-0.041	0.026	33.6	0.000	-0.005	0.034	7.1	0.622	-0.046	0.039	19.2	0.023
10	-0.045	0.026	36.8	0.000	0.000	0.034	7.1	0.712	-0.049	0.039	20.9	0.022
20	-0.001	0.026	42.6	0.002	-0.052	0.034	21.3	0.379	0.006	0.039	24.0	0.243
30	-0.020	0.026	49.4	0.014	0.040	0.035	32.4	0.348	-0.029	0.039	29.1	0.511
40	-0.013	0.026	59.8	0.023	0.013	0.035	41.9	0.389	-0.018	0.040	35.7	0.664
50	0.052	0.026	88.8	0.001	0.032	0.035	52.2	0.389	0.059	0.040	52.5	0.378
100	-0.012	0.028	197.3	0.000	0.014	0.037	117.7	0.109	-0.021	0.043	126.5	0.038

Table 2.10: Autocorrelation function for daily changes in USD 10-year fitted swap spreads. The full sample is from January 1995 to April 2001, T=1596. Pre (post)-crisis is before (after) August 1998, with 897 and 699 observations respectively.

Lag	Full sample r_k	$s.e.(r_k)$	LB Q-Stat	sign.	Pre-crisis r_k	$s.e.(r_k)$	LB Q-Stat	sign.	Post-crisis r_k	$s.e.(r_k)$	LB Q-Stat	sign.
1	-0.140	0.025	32.5	0.000	-0.235	0.033	51.8	0.000	-0.064	0.037	2.9	0.086
2	0.029	0.025	33.9	0.000	-0.066	0.034	55.9	0.000	0.104	0.037	10.9	0.004
3	-0.058	0.025	39.5	0.000	-0.031	0.035	56.8	0.000	-0.080	0.038	15.5	0.001
4	0.005	0.025	39.5	0.000	-0.008	0.035	56.8	0.000	0.014	0.038	15.7	0.003
5	-0.044	0.025	42.7	0.000	-0.016	0.035	57.1	0.000	-0.064	0.038	18.7	0.002
6	-0.020	0.025	43.4	0.000	0.003	0.035	57.1	0.000	-0.039	0.038	19.8	0.003
7	-0.013	0.025	43.6	0.000	-0.008	0.035	57.1	0.000	-0.017	0.038	20.0	0.005
8	-0.041	0.025	46.4	0.000	0.019	0.035	57.5	0.000	-0.089	0.038	25.8	0.001
9	-0.030	0.025	47.9	0.000	-0.086	0.035	64.5	0.000	0.014	0.038	26.0	0.002
10	0.006	0.025	48.0	0.000	0.070	0.035	69.1	0.000	-0.045	0.038	27.5	0.002
20	0.049	0.026	77.7	0.000	0.017	0.036	101.4	0.000	0.063	0.040	57.1	0.000
30	0.026	0.026	92.6	0.000	-0.008	0.036	116.4	0.000	0.029	0.041	75.1	0.000
40	-0.031	0.026	104.3	0.000	-0.058	0.037	130.2	0.000	-0.003	0.041	84.7	0.000
50	0.064	0.026	118.4	0.000	0.060	0.037	138.7	0.000	0.036	0.041	93.2	0.000
100	-0.037	0.027	180.2	0.000	-0.047	0.039	220.5	0.000	-0.063	0.044	170.3	0.000

Table 2.11: Autocorrelation function for daily changes in EUR 10-year fitted swap spreads. The full sample is from January 1995 to April 2001, T=1661. Pre (post)-crisis is before (after) August 1998, with 936 and 725 observations respectively.

USD				
Full sample	BIC	(lags)	AIC	(lags)
ADF t-test	-35.42	0	-20.91	3
ADF Z-test	-1406.74		-1966.53	
Pre-crisis				
ADF t-test	-31.50	0	-16.12	3
ADF Z-test	-945.33		-1276.59	
Post-crisis				
ADF t-test	-22.97	0	-13.80	3
ADF Z-test	-602.55		-856.67	

EUR				
Full sample	BIC	(lags)	AIC	(lags)
ADF t-test	-47.05	0	-8.30	19
ADF Z-test	-1895.87		-624.70	
Pre-crisis				
ADF t-test	-39.11	0	-7.83	19
ADF Z-test	-1158.20		-599.76	
Post-crisis				
ADF t-test	-28.65	0	-4.91	19
ADF Z-test	-771.02		-131.87	

Table 2.12: Augmented Dickey-Fuller tests for daily changes in USD and EUR 10-year fitted swap spreads. The number of lags is determined using the BIC and the AIC criteria. The null hypothesis is that there is a unit root. The rejection levels for the ADF t-test are -2.57 (10%), -2.86 (5%) and -3.43 (1%). The rejection levels for the Z-test are -11.3 (10%), -14.1 (5%) and -20.7 (1%). For USD the full sample is from January 1995 to April 2001, T=1596. Pre (post)-crisis is before (after) August 1998, with 897 and 699 observations respectively. For EUR, the full sample is from January 1995 to April 2001, T=1661. Pre (post)-crisis is before (after) August 1998, with 936 and 725 observations respectively.

Lag	r_k	$s.e.(r_k)$	LB Q-Stat	sign.
1	0.004	0.025	0.0	0.871
2	-0.031	0.025	1.6	0.456
3	-0.031	0.025	3.1	0.372
4	-0.043	0.025	6.1	0.190
5	0.002	0.025	6.1	0.293
6	0.010	0.025	6.3	0.392
7	-0.027	0.025	7.5	0.381
8	0.033	0.025	9.2	0.326
9	-0.040	0.025	11.7	0.229
10	-0.039	0.025	14.1	0.167
20	-0.005	0.025	19.6	0.485
30	-0.022	0.025	27.6	0.590
40	-0.015	0.026	38.4	0.541
50	0.049	0.026	65.4	0.070

Table 2.13: Autocorrelation function of the residuals of an AR(1) model for the daily changes in USD 10-year fitted swap spreads. The full sample is from January 1995 to April 2001, T=1596.

Lag	r_k	$s.e.(r_k)$	LB Q-Stat	sign.
1	0.003	0.025	0.0	0.916
2	0.001	0.025	0.0	0.993
3	-0.055	0.025	5.1	0.162
4	-0.009	0.025	5.3	0.261
5	-0.049	0.025	9.2	0.100
6	-0.029	0.025	10.6	0.100
7	-0.022	0.025	11.4	0.121
8	-0.048	0.025	15.3	0.053
9	-0.037	0.025	17.6	0.040
10	0.005	0.025	17.6	0.062
20	0.049	0.025	46.0	0.001
30	0.032	0.025	65.6	0.000
40	-0.034	0.026	78.0	0.000
50	0.065	0.026	95.0	0.000

Table 2.14: Autocorrelation function of the residuals of an AR(1) model for the daily changes in EUR 10-year fitted swap spreads. The full sample is from January 1995 to April 2001, T=1661.

Lag	r_k	$s.e.(r_k)$	LB Q-Stat	sign.
1	0.229	0.025	84.1	0.000
2	0.248	0.026	182.1	0.000
3	0.261	0.028	291.4	0.000
4	0.229	0.029	375.1	0.000
5	0.127	0.030	401.0	0.000
6	0.172	0.031	448.4	0.000
7	0.199	0.031	511.9	0.000
8	0.190	0.032	570.0	0.000
9	0.163	0.033	612.7	0.000
10	0.211	0.033	684.5	0.000
20	0.182	0.039	1172.3	0.000
30	0.134	0.042	1498.7	0.000
40	0.113	0.046	1929.4	0.000
50	0.066	0.048	2158.5	0.000

Table 2.15: Autocorrelation function of the squared residuals of an AR(1) model for the daily changes in USD 10-year fitted swap spreads. The full sample is from January 1995 to April 2001, T=1596.

Lag	r_k	$s.e.(r_k)$	LB Q-Stat	sign.
1	0.250	0.025	104.3	0.000
2	0.286	0.026	240.7	0.000
3	0.170	0.028	288.6	0.000
4	0.085	0.028	300.6	0.000
5	0.135	0.029	331.0	0.000
6	0.163	0.029	375.3	0.000
7	0.185	0.030	432.3	0.000
8	0.198	0.030	497.9	0.000
9	0.061	0.031	504.1	0.000
10	0.132	0.031	533.0	0.000
20	0.116	0.034	764.3	0.000
30	0.050	0.036	926.8	0.000
40	0.041	0.037	1053.0	0.000
50	0.012	0.037	1057.1	0.000

Table 2.16: Autocorrelation function of the squared residuals of an AR(1) model for the daily changes in EUR 10-year fitted swap spreads. The full sample is from January 1995 to April 2001, T=1661.

USD		
ARCH order	χ^2-stat	Sign. Level
1	83.9	0
2	147.7	0
3	197.6	0
EUR		
ARCH order	χ^2-stat	Sign. Level
1	104.1	0
2	192.4	0
3	198.0	0

Table 2.17: Testing for ARCH in the filtered squared residuals of the daily changes in USD and EUR 10-year fitted swap spreads. The null hypothesis is no ARCH. The full sample is from January 1995 to April 2001.

Variable	Coeff	Std Error	T-Stat	Signif
constant	0.004	0.001	3.0	0.002
Lag sqd resid	0.069	0.007	9.8	0.000
Lag cond var	0.932	0.006	150.1	0.000
Norm. resids				
LB Q(4)	Test Stat.	9.2	Sign. Level	0.057
LB Q(12)	Test Stat.	10.8	Sign. Level	0.549
LB Q(24)	Test Stat.	17.3	Sign. Level	0.835
ARCH in norm. resids				
F-test(4)	Test Stat.	0.7	Sign. Level	0.603
F-test(12)	Test Stat.	1.2	Sign. Level	0.284
F-test(24)	Test Stat.	1.1	Sign. Level	0.354
Squared norm. resids				
LB Q(4)	Test Stat.	2.9	Sign. Level	0.574
LB Q(12)	Test Stat.	15.4	Sign. Level	0.222
LB Q(24)	Test Stat.	29.8	Sign. Level	0.190

Table 2.18: GARCH(1,1) for daily changes in USD 10-year fitted swap spreads. The conditional mean has been filtered using an AR(1) process. We test for serial correlation in the normalized residuals and their squares. The full sample is from January 1995 to April 2001. The value of the likelihood function at the optimum is -2525.

Variable	Coeff	Std Error	T-Stat	Signif
constant	0.003	0.001	2.8	0.006
Lag sqd resid	0.081	0.010	8.4	0.000
Lag cond var	0.941	0.006	148.6	0.000
Leverage	-0.043	0.013	-3.3	0.001
Norm. resids				
LB Q(4)	Test Stat.	10.5	Sign. Level	0.032
LB Q(12)	Test Stat.	12.0	Sign. Level	0.445
LB Q(24)	Test Stat.	18.9	Sign. Level	0.756
ARCH in norm. resids				
F-test(4)	Test Stat.	0.8	Sign. Level	0.508
F-test(12)	Test Stat.	1.3	Sign. Level	0.240
F-test(24)	Test Stat.	1.0	Sign. Level	0.397
Squared norm. resids				
LB Q(4)	Test Stat.	3.5	Sign. Level	0.472
LB Q(12)	Test Stat.	16.2	Sign. Level	0.180
LB Q(24)	Test Stat.	28.8	Sign. Level	0.229

Table 2.19: Asymmetric-GARCH(1,1) for daily changes in USD 10-year fitted swap spreads. The conditional mean has been filtered using an AR(1) process. We test for serial correlation in the normalized residuals and their squares. The full sample is from January 1995 to April 2001. The value of the likelihood function at the optimum is -2521.

Variable	Coeff	Std Error	T-Stat	Signif
constant	0.028	0.007	3.8	0.000
Lag sqd resid	0.079	0.019	4.2	0.000
Lag cond var	0.871	0.023	37.6	0.000
Leverage	-0.012	0.023	-0.5	0.596
Dummy (1 after Aug98)	0.295	0.075	3.9	0.000
Norm. resids				
LB Q(4)	Test Stat.	10.8	Sign. Level	0.028
LB Q(12)	Test Stat.	11.7	Sign. Level	0.467
LB Q(24)	Test Stat.	19.9	Sign. Level	0.704
ARCH in norm. resids				
F-test(4)	Test Stat.	0.3	Sign. Level	0.908
F-test(12)	Test Stat.	1.1	Sign. Level	0.345
F-test(24)	Test Stat.	1.3	Sign. Level	0.175
Squared norm. resids				
LB Q(4)	Test Stat.	1.0	Sign. Level	0.902
LB Q(12)	Test Stat.	13.1	Sign. Level	0.361
LB Q(24)	Test Stat.	30.5	Sign. Level	0.170

Table 2.20: Asymmetric GARCH(1,1) with a dummy variable in the conditional variance equation ($D_i = 1$ for the observations after August 1998, and zero otherwise) for daily changes in USD 10-year fitted swap spreads. The conditional mean has been filtered using an AR(1) process. We test for serial correlation in the normalized residuals and their squares. The full sample is from January 1995 to April 2001. The value of the likelihood function at the optimum is -2506.

Variable	Coeff	Std Error	T-Stat	Signif
constant	0.271	0.037	7.4	0.000
Lag sqd resid	0.198	0.015	13.0	0.000
Lag cond var	0.677	0.026	25.6	0.000
Norm. resids				
LB Q(4)	Test Stat.	6.6	Sign. Level	0.162
LB Q(12)	Test Stat.	15.2	Sign. Level	0.230
LB Q(24)	Test Stat.	41.0	Sign. Level	0.017
ARCH in norm. resids				
F-test(4)	Test Stat.	1.4	Sign. Level	0.223
F-test(12)	Test Stat.	1.4	Sign. Level	0.180
F-test(24)	Test Stat.	0.9	Sign. Level	0.636
Squared norm. resids				
LB Q(4)	Test Stat.	5.5	Sign. Level	0.242
LB Q(12)	Test Stat.	17.3	Sign. Level	0.139
LB Q(24)	Test Stat.	22.8	Sign. Level	0.533

Table 2.21: GARCH(1,1) for daily changes in EUR 10-year fitted swap spreads. The conditional mean has been filtered using an AR(1) process. We test for serial correlation in the normalized residuals and their squares. The full sample is from January 1995 to April 2001. The value of the likelihood function at the optimum is -2836.

Variable	Coeff	Std Error	T-Stat	Signif
constant	0.206	0.031	6.6	0.000
Lag sqd resid	0.214	0.018	11.8	0.000
Lag cond var	0.741	0.025	29.6	0.000
Leverage	-0.109	0.020	-5.4	0.000
Norm. resids				
LB Q(4)	Test Stat.	6.7	Sign. Level	0.150
LB Q(12)	Test Stat.	14.9	Sign. Level	0.245
LB Q(24)	Test Stat.	40.5	Sign. Level	0.019
ARCH in norm. resids				
F-test(4)	Test Stat.	2.0	Sign. Level	0.092
F-test(12)	Test Stat.	1.5	Sign. Level	0.113
F-test(24)	Test Stat.	1.0	Sign. Level	0.506
Squared norm. resids				
LB Q(4)	Test Stat.	7.6	Sign. Level	0.108
LB Q(12)	Test Stat.	19.2	Sign. Level	0.084
LB Q(24)	Test Stat.	25.1	Sign. Level	0.399

Table 2.22: Asymmetric-GARCH(1,1) for daily changes in EUR 10-year fitted swap spreads. The conditional mean has been filtered using an AR(1) process. We test for serial correlation in the normalized residuals and their squares. The full sample is from January 1995 to April 2001. The value of the likelihood function at the optimum is -2831.

Variable	Coeff	Std Error	T-Stat	Signif
constant	0.235	0.035	6.7	0.000
Lag sqd resid	0.238	0.021	11.4	0.000
Lag cond var	0.680	0.030	22.9	0.000
Leverage	-0.104	0.025	-4.2	0.000
Dummy (1 after Aug98)	0.108	0.031	3.5	0.000
Norm. resids				
LB Q(4)	Test Stat.	6.6	Sign. Level	0.161
LB Q(12)	Test Stat.	19.5	Sign. Level	0.242
LB Q(24)	Test Stat.	42.6	Sign. Level	0.011
ARCH in norm. resids				
F-test(4)	Test Stat.	2.3	Sign. Level	0.058
F-test(12)	Test Stat.	1.7	Sign. Level	0.062
F-test(24)	Test Stat.	1.0	Sign. Level	0.413
Squared norm. resids				
LB Q(4)	Test Stat.	8.6	Sign. Level	0.072
LB Q(12)	Test Stat.	21.2	Sign. Level	0.047
LB Q(24)	Test Stat.	26.3	Sign. Level	0.340

Table 2.23: Asymmetric GARCH(1,1) with a dummy variable in the conditional variance equation ($D_i = 1$ for the observations after August 1998, and zero otherwise) for daily changes in EUR 10-year fitted swap spreads. The conditional mean has been filtered using an AR(1) process. We test for serial correlation in the normalized residuals and their squares. The full sample is from January 1995 to April 2001. The value of the likelihood function at the optimum is -2827.

FIGURES

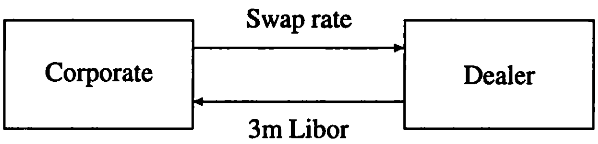

Figure 2.1: Example of an interest rate swap transaction in USD, where the dealer receives fixed.

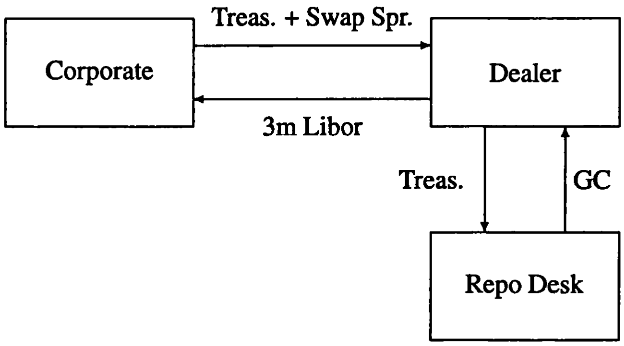

Figure 2.2: Cash flows from a swap spread trade, where the dealer receives the swap spread.

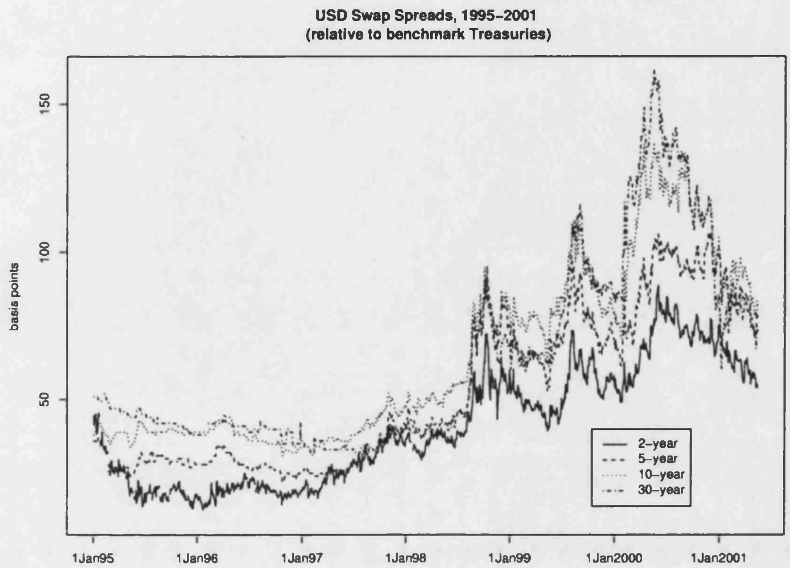

Figure 2.3: Evolution of the USD swap spreads (vs. benchmark Treasuries). Data between January 1995 and April 2001, T=1596.

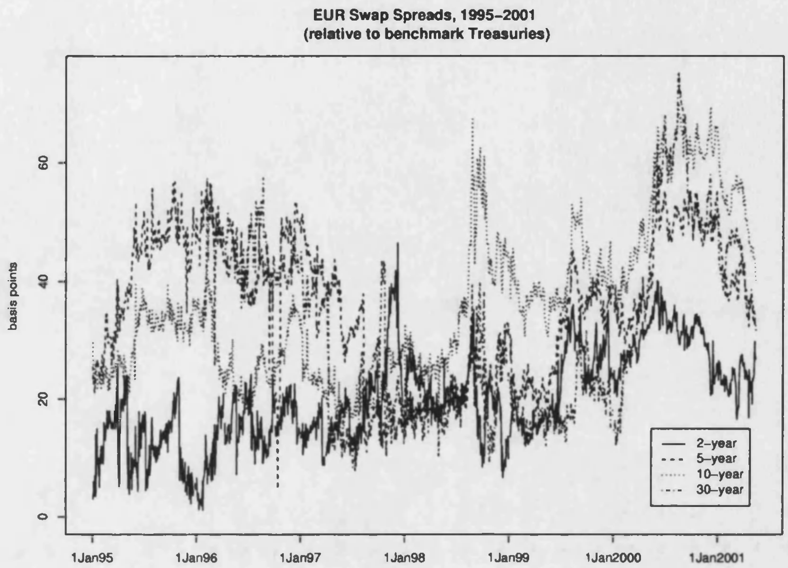

Figure 2.4: Evolution of the EUR swap spreads (vs. benchmark Treasuries). Data between January 1995 and April 2001, T=1661.

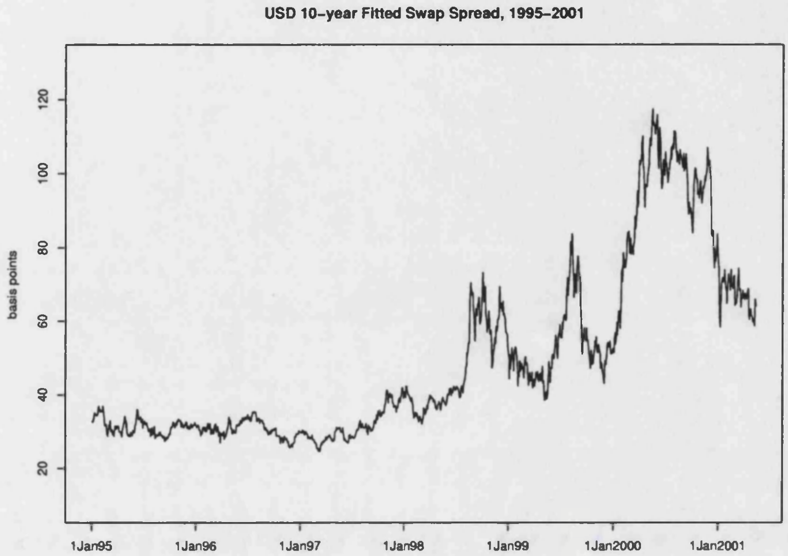

Figure 2.5: Evolution of the USD 10-year fitted swap spread (vs. constant maturity par fitted Treasuries). Data between January 1995 and April 2001, T=1596.

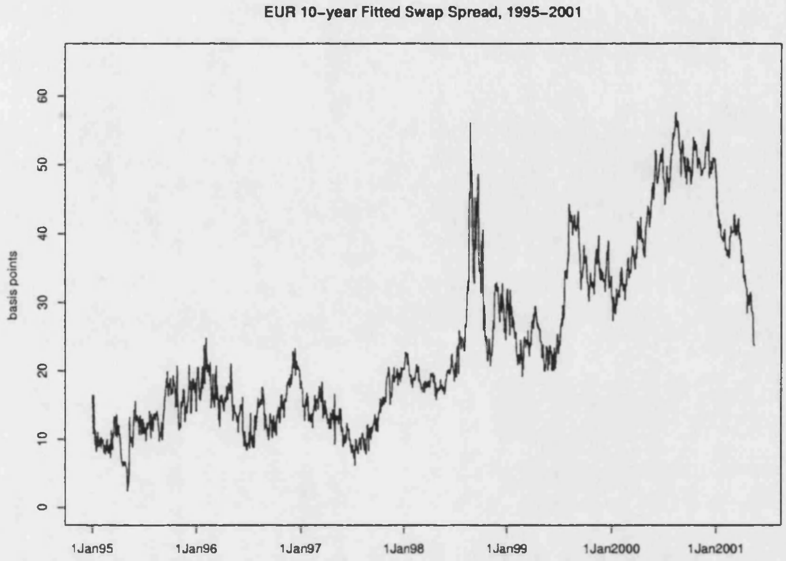

Figure 2.6: Evolution of the EUR 10-year fitted swap spread (vs. constant maturity par fitted Treasuries). Data between January 1995 and April 2001, T=1661.

Chapter 3

The Relation Between Swap and Credit Spreads: An Empirical Analysis

3.1 Introduction

It is well-known in the fixed income community that the corporate bond and the swap markets[1] are related. A high contemporaneous correlation between credit spreads and swap spreads has been empirically documented by, among others, Minton (1994), Liu, Lang and Litzenberger (1998) and Baz et al. (1999). As an illustration, we present in Figure 3.1 the evolution of 10-year USD swap spreads and credit spreads for different bond ratings.

The present chapter is an ecometric study of the relation between swap and credit spreads, conditioning for the impact of a set of variables such as the slope of the yield curve or the spread differential between high-quality and low-quality corporate bonds. The econometric framework we use is that of co-integration and error-correction models.[2] We will show evidence of spreads being integrated (or near-integrated). In this context, an error-correction model is interesting because it can shed light both on the long-term and the short-term dynamics of the processes.

[1] See Chapter 2 for an introduction to the institutional aspects of swap markets.

[2] See Banarjee et al. (1993) for a comprehensive introduction.

The final objective of this chapter is to provide empirical evidence for the discussion on the reasons and mechanisms that determine the relation between the swap and the credit spreads. An important line of research on this topic has focused on the counterparty default risk embedded in swaps (see Duffie and Huang (1996), Sun et al. (1993) and Cossin and Pirotte (1997) among others). Indeed, swaps are long term contracts between agents (typically a corporate and a financial dealer) that can default in their obligations. If the cashflows of a swap were risky, the fixed swap rate should be above the yield on Treasury securities. In practice, however, the credit risk in swaps is small. First, the notionals are not exchanged, therefore, compared to a corporate bond, the amount of money at risk at each point in time is small. Additionally, the industry has designed a whole array of credit protection mechanisms. In summary, the gross exposure to a counterparty is determined by the size of its credit line, and the net exposure is required to be collateralized regularly.

An alternative explanation is based on the fact that swap cashflows are indexed to Libor rates.[3] The Libor rate is an indicator of the cost of unsecured lending operations between major international banks in London. The swap rate is determined so that the present value of the swap at inception is nil, i.e. the value of the fixed leg is equal to the value of the floating leg, which is a stream of Libor-based payments. Hence, variations in the credit standing of major banks ("Libor sector") will have an impact on the pricing of swaps, i.e. the swap rate. Given the high correlation between all corporate sectors, this would result in a strong correlation between swap spreads and corporate spreads.

An additional factor that may contribute to increase the correlation between swap and credit spreads is the increasing role of the swap curve as the benchmark yield curve in fixed income markets. In particular, many agents hedge their credit risk exposure taking positions in swaps, or trade corporate bonds by looking at their relative value versus swaps.[4] It is also becoming common to quote spreads for corporate bonds as the difference between the corporate bond rate and the swap rate. This difference is known as "Libor" spread, and can be interpreted as the remuneration for holding the fixed-rate corporate bond after swapping it into floating.[5]

[3] Recall that a swap is an exchange of a fixed rate against a floating rate, typically Libor (see Chapter 2).

[4] The process of the swap curve replacing the traditional benchmark, the Treasury yield curve, is mainly due to the decreasing importance of US Treasuries in the debt markets. In the Lehman Brothers US Aggregate Debt index (see the Appendix for a description), at the end of November 2000 the market value of US Treasuries was 27.3% of the total of the index, while in 1990 that figure was as high as 45.5%.

[5] A paper that models the Libor spreads of corporate bonds is Collin-Dufresne and Solnik (2001), where it is argued that the Libor spread compensates for the downgrade risk of an issuer, vs. the swap rate, which

This work is organized as follows. In section 2, we describe the data and discuss the recent market evolution of swap, corporate and Libor spreads. In section 3 we make a first approximation to the study of the relation between swap and credit spreads. In particular, we look at the implications of the behaviour of two spread differentials, between financial and industrial corporations, and between high quality and low quality debt. In section 4, we estimate an error-correction model of the relation between swap and credit spreads, conditioning for a number of relevant variables. This is based on the evidence of cointegration between swap and credit spreads (after conditioning for effect of the business cycle). Section 5 concludes by summarizing the main findings in this chapter and suggesting possible topics for future research.

3.2 The Dynamics of the US Credit Markets

In this section we describe the recent evolution of US credit markets, for the period from May 1994 to May 2001. We focus on the 2, 5 and 10-year sectors, the focal points in the swap, Treasury and credit markets.

The spreads on corporate bonds ("credit spreads") have been computed from the data in the Lehman Brothers' US Corporate Investment Grade Index[6], by creating a credit yield curve.[7] The spreads of the individual corporate bonds are not weighted by market value, since that might generate large distortions in the credit curves. Instead, a spline is fitted to the cloud of points, using a robust method to deal with outliers.[8] We have two sets of credit spreads, one versus the off-the-run fitted US Treasury curve, and the other versus the swap curve (or "Libor" spreads).[9] For each maturity, we also have have the spreads by rating quality (AA, A, BBB) and by sector (Industrials and Financials). The swap spreads are measured as the difference between the mid-market quoted swap yields and the corresponding constant maturity fitted off-the-run Treasury yield.[10]

is referenced on a panel of Libor banks with "refreshed" or constant credit quality. That is, the banks in the Libor panel can change in order to keep a constant, high-quality rating. The credit spread is represented by a jump-diffusion process, where the jump component accounts for the downgrade of the issuer.

[6] See the Appendix a description of this index.

[7] For a detailed description of the methodology, refer to Monkkonen (1999).

[8] We do not consider spreads for maturities longer than 10 years. This is because the very end of the credit curve tends to have few bonds, which can display large spread differences. This makes the interpolation schemes become delicate and unstable.

[9] When the contrary is not indicated, by credit spreads we refer to spreads vs. the off-the-run fitted Treasuries including bonds from all corporate sectors.

[10] See Chapter 2 for a discussion.

3.2.1 The evolution of the spreads

In Figures 3.2 and 3.3 we can see that spreads were relatively stable until they began to widen gradually towards the end of 1997 in response to the effects of the Asian crisis. The situation deteriorated drastically with Russia's default in August 1998, as fixed income markets plunged into a global crisis. The financial crisis of Fall 1998 was a truly dramatic event in fixed income markets. As an illustration, the difference between 10-year BBB and AA spreads, which was around 45bp in the months before the crisis, went to 64bp at the end of August and to 110bp around the end of 1998. The crisis had its most important impact on the risk of the global financial system. As an indicator, the spread diferential between the 10-year single-A financial sector debt and that of the industrial sector went from 7bp right before the crisis (14-Aug-98) to 20bp at the end of August and to 54bp in the middle of September 1998. As explained in Chapter 2, the Federal Reserve had to intervene to preserve the stability and liquidity of the financial system, by among other measures, organizing the bail out of LTCM (a prominent hedge fund) and cutting short-term interest rates.

The spreads widened again during the Summer of 1999 and again, in the Spring of 2000. The persistence of high and volatile spreads can be attributed to the uncertainty regarding the US economy. Between June 1999 and June 2000, the Federal Reserve raised short term interest rates by 175bp, from 4.75% to 6.50%, in order to moderate the growth rate of the US economy. During this period, the markets had to assess whether the Fed's Chairman Greenspan would be able to engineer a "soft-landing" for the US economy, or whether it would end up in recession (at it was the case ex post). Second, the Treasury curve outperformed on the back of supply concerns and the initiation of a debt buyback program. After the rate of growth of the US economy decreased significantly in the last quarter of 2000 and the Fed began cutting interest rates aggressively at the beginning of 2001, spreads have tightened sharply, anticipating a recovery in 2002.

3.2.2 Statistics of the spread curves

Examining the statistics of swap spreads in Table 3.1, we first note that on average, the term structure of swap spreads is upward sloping, with the mean 10-year swap spread at 47.5bp. Spreads are highly variable: the 10-year has moved between 25.5bp and

117.5bp during the sample period, with a standard deviation of changes of 28bp per year. From Figure 3.1, it is clear that the properties of the series are very different before and after August 1998. In the period after that crisis, average spreads were about twice as large as averages for the period before. The impact on volatility was large too: the volatility of annual changes in the 10-year spread went from 10bp per year to 43bp.[11]

Next, we present the statistics for corporate spreads by maturity and rating (see Table 3.2). The term structure of corporate spreads is on average, upward sloping with maturity. Also, the average slope of the spread curve is generally higher for poor ratings. For instance, for AA-rated bonds, the difference between 10-year and 2-year spreads is about 22bp. The same figure, for BBB spreads, is 35bp. Regarding the crisis of 1998, we observe that average corporate spreads effectively doubled, and volatilities increased by a factor of 3 or 4, a pattern similar to that for swap spreads. In general, we see that the volatility of swap spreads is slightly lower than the volatility of credit spreads. For the 10-year maturity, the swap spread volatility was 28bp, the same as the volatility of AA spreads and lower than the 31bp and 35bp volatilities of single-A and BBB spreads.

We also report spread statistics by sector, where the division is into Industrial and Financial corporates (see Tables 3.3 and 3.4). The average level of the spread for Financials is higher than for Industrials: for single-A, 10-year corporate bonds, the average spread for Financials is of 106bp, while for Industrials it's 92bp. Also, the volatility of the spreads of Financials is much higher than that of Industrials. Again for single-A 10-year spreads, the volatility of spread changes for Financials is 44.8bp, against 28.7bp for industrials. As a consequence, the volatility of Industrial spreads is close to that of swap spreads, which is much lower than that of Financials. The reasons for the worse performance of Financials may be due to the fact that their cashflows tend to be more volatile than those of industrial companies, and that they are subject to a much higher degree of interest rate risk.

3.2.3 Properties of Libor spreads

We now look at the evolution of Libor spreads, the difference between corporate and swap spreads. Figure 3.4 presents the Libor spreads for the 10-year maturity. Their

[11] See Chapter 2 for further reference.

evolution is qualitatively similar to that of credit and swap spreads, being relatively stable until the end of 1997, deteriorating strongly in the second half of 1998, and behaving erratically afterwards, with a peak at the end of year 2000.

Looking at the descriptive statistics in Table 3.5, we see that it is the case that, on average, Libor spreads are positive, implying that corporates generally trade above the swap curve. The only exceptions are found in the 2-year maturity, for AA and A rated bonds. Also, Libor spreads can be quite large, for instance the average Libor spread for 10-year BBB was 83bp, with a maximum of 174.7bp. The volatility of Libor spreads turns out to be of a similar magnitude to that of swap spreads, for instance it is of 24.5bp for the 10-year single-A Libor spread.

3.3 Economics of the Relation Between Swap and Credit Markets

3.3.1 Swaps and credit risk

In this section we discuss the motivation for the relation between swap and credit spreads. A major line of research is based in modelling the default risk embedded in the swap contract due to the default risk of the swap counterparties (see Duffie and Huang (1996), Sun et al. (1993) and Cossin and Pirotte (1997) among others). In the following, we will argue that counterparty risk is not very relevant in practice. The first point to note is that the swap notional is not exchanged, hence the amount of money at risk is relatively small.[12] In general, the swap spreads quoted by dealers are the same for all counterparties.[13] Instead, discrimination tends to happen via quantities: the magnitude of the gross exposure to a counterparty is determined by the size of its credit lines. Additionally, the financial industry, which is to a large extent self-regulated, has developed a number of mechanisms to reduce the counterparty credit risk of swaps, which we present below.

First, the parties in a swap typically agree to collateralize the net swap exposures by regularly posting cash or Treasury securities. It is fair to note, though, that these bilat-

[12]When Duffie and Huang (1996) calibrate their model of differential counterparty default risk, they find that, in the absence of protection mechanisms, a 100bp credit spread differential between the swap counterparties in a 5-year swap, would result in a 1bp increase in swap spreads. In a currency swap, in which the notionals are indeed exchanged, the impact on the swap spread would be much higher, at 8.7bp.

[13]As long as this have been approved by the risk-control and legal departments.

eral agreements do not make swaps risk free as a clearing house would do, because the mark-to-market is less frequent and the margining system is less transparent. Second, in many instances the counterparties also agree on netting the swap cashflows, a practice by which all the swap cashflows between two counterparties are aggregated into a unique payment.[14] The netting increases the security of the counterparties because at each payment date, the amount at stake is the net payment, instead of the gross. One situation that netting helps to avoid is that in a case of liquidity crisis, the party which is a net creditor may delay its payments to the other counterparty for fear of not being paid back at all, putting the net debtor closer to insolvency. Finally, in order to reduce the risk from the dealer side, swaps are generally contracted with subsidiaries of the dealers that are structured to be AAA-rated.[15]

Due to the credit-protection mechanisms described above, swap cashflows are generally regarded as of a high credit quality.[16] This leads us to the other main motivation for the relation between swaps and credit spreads, based on the fact that the swap cashflows on the floating side are indexed to Libor rates. Libor rates apply to short–term unsecured borrowing operations (1 day to 1 year) and are determined every day by the British Bankers Association, through a poll of banks in London. The Libor panel includes 8 to 16 international banks in London, depending on the currency. Each bank communicates, by 11:00 am London time the rate at which it could borrow funds from other prime banks. The top and bottom quartiles of quotes are eliminated, and the Libor fixings are computed as the average of the rest. Since the Libor rates apply to unsecured lending operations, they incorporate a credit risk premium (in addition to a possible liquidity premium). Hence, the fixed rate will be naturally above the Treasury rate. A model of swap spreads based on this intuition is He (2000), where swap spreads are computed as the present value of the forward Libor vs. Repo spreads.

In recent years, especially since the crisis of 1998, there has been an ongoing debate

[14] In practice there can be legal impediments to consolidating payment obligations.

[15] These "special purpose vehicles" or SPVs are overcapitalized so that they have a better rating than the dealer that owns them.

[16] The development of futures contracts referenced on swaps like the LIFFE's "Swapnote" contract may be the final step in the elimination of counterparty risk in swaps. Essentially, the Swapnote contract is a future on a notional bond which is referenced on the EUR swap curve. The rationale for this contract lies in the increasing importance of the swap curve as the benchmark curve in the Euro-area. This contract tries to take advantage of two facts. First, many agents -typically portfolio managers- have mandates that do not allow them to enter into swap transactions, while they are free to enter into futures. These agents can use the new contract to take exposures to the swap curve, for instance to extend or decrease the duration of their bond portfolios. Second, and more related to our discussion, the futures contract is an exchange-traded contract. Therefore, the counterparty to all trades is a clearing house and the margining system is totally clear and transparent.

about whether the swap curve is going to become the benchmark curve in fixed income markets. In any case, the swap curve has gained a large degree of visibility. The factors behind this phenomenon have been the large distortions in the US Treasury yield curve due to concerns over supply,[17] and the very same effects of the crisis of August 1998. Before the crisis, portfolios of corporate bonds were typically hedged with short positions in Treasuries. This practice resulted in large losses in 1998, when in a period of flight to quality, spreads widened at the same time that Treasuries appreciated sharply. On the other hand, a hedge consisting of a long position in swap spreads would have worked properly. The high volatility of spread markets in the post-98 period has made the issue of how to hedge credit risk more important.

The way credit risk is traded has also had an impact on the relation between swap and credit spreads, via the financial innovation process. It has become increasingly common to trade corporate bonds on an "asset-swapped" basis. In an asset swap, the party buying a fixed-rate bond transforms it into a Libor floater by entering into a "tailored" swap paying fixed (where the fixed rate is equal to the coupon of the corporate bond), and receiving Libor plus a spread (the asset swap spread). In this way, the owner of the bond eliminates the interest rate risk of the position, and keeps an exposure to credit risk only. Notice that in the case that the corporate defaulted, he is still obliged to pay fixed in the swap.[18] Economically, buying a bond on an asset-swapped basis is similar to buying a bond and paying fixed in an interest rate swap. The asset swap arises because investors may want swap payoffs that match the date and the amount of the fixed bond payoffs. Nevertheless, we can think of the asset swap and buying a bond plus paying fixed in a swap as essentially equivalent operations, and hence the asset swap spread will be roughly equivalent to the Libor spread. The evolution of the value of the bond on an asset swapped basis will be then determined by the joint evolution of swap and credit spreads.

[17]The relative importance of US Treasury securities in fixed income markets has decreased substantially, due to the US Federal budget surpluses in the late 90's. The uncertainty generated by the future evolution of Treasury issuance and the size of the buy-back program, has made the US Treasury yield curve experience major swings. One example is the inversion of the US Treasury curve between the 2-year and the 30-year sector, for most of year 2000, after it was announced that the US Treasury would buy back $30bn of long-dated bonds during that year.

[18]The mechanics of the asset swap are explained in detail in O'Kane (1999).

3.3.2 Some empirical implications

In this section we study the relation between swap spreads and two aggregate measures of credit risk, the "quality spread" (differential between "low quality" BBB debt and "high quality" AA debt, and the financial vs. industrial spread (for single-A issues, the difference between Financial and Industrial spreads). We will refer to them as QUAL and FININD spreads respectively. We would expect the QUAL spread differential to be correlated with swap spreads if it is the case that it is the default risk of swaps what drives the relation with credit. On the other hand, we would expect the FININD spread to be more closely related to swap spreads if what matters is the specific risk of the financial sector. We advance that the evidence tends to support the second hypothesis: the correlation between fortnightly changes in 10-year swap spreads and changes in 10-year FININD spreads is 0.259, against a correlation of 0.004 with the 10-year QUAL spread.

The QUAL spread (see Figure 3.5) is generally interpreted as a proxy of the effect of the business cycle on credit markets.[19] The statistics for the QUAL spread are presented in Table 3.6. The average QUAL spread is increasing with maturity, although for the period after August 1998, we have that the average QUAL spreads are very similar for the three maturities considered, at between 73.4bp and 78bp. The evolution of the QUAL spread is interesting, in the sense that it deteriorated after the crisis of August 1998, but then tightened back to more normal levels in 1999, even though the swap spreads and corporate spreads experienced a large widening. It is in the second half of year 2000 when the QUAL spread strongly deteriorated again, as the US economy approached recession.

The FININD spread measures the differential compensation for credit risk of the financial sector with respect to the industrial sector. Since swaps spreads can be seen as the cost of transforming fixed rate debt into floating, Libor-indexed debt, they should be sensitive to the factors that determine the Libor rates. Also, on one side of the swap we will find a financial dealer, hence the relevance of trying to capture the risk of the dealer community as a whole. In Figure 3.6, we can see how the spread between Financials and Industrials increased markedly with the crisis of 1998. This can be interpreted as a reflection of the higher sensitivity to systemic risk of the financial sector relative to the

[19]The quality spread was already included in the empirical study of swap spreads by Minton (1994), as the difference between AAA and BBB corporates. Minton (1994) did not find the QUAL spread to be a significant driver of swap spreads.

industrial sector. The FININD spread came back to more normal levels at the beginning of 1999, but subsequently increased steadily, as the Federal Reserve kept raising short-term interest rates. This tends to hurt the profitability of the financial sector. From the statistics in Table 3.7, we can see that the average level of the FININD spread is relative low (e.g. 14.6bp for the 10-year), and even negative for the 2-year maturity. However, the volatility of the FININD is substantial, at 25.7bp for the 10-year. This is even more extreme when we only consider the subsample after August 1998: the average FININD for the 10-year is 20.3bp but the volatility is as high as 38.2bp.

When we put the three spreads together in Figure 3.7, it is clear that the swap spreads move more closely with the FININD spread. On the other hand, we can have significant movements in the QUAL spread which are not mirrored by movements of swap spreads. At this simple level, it seems that the sensitivity of swap spreads to credit risk is mostly driven by their sensitivity to the performance of the financial sector.

3.4 A Multivariate Analysis of Swap and Credit Spreads

3.4.1 Economic drivers of swap spreads

In this section we formulate and estimate an econometric model of swap spreads. We will first comment on a number of variables that are a priori relevant in such a model.

First, it is intuitive that swap spreads are likely to be related to the dynamics of the Treasury yield curve. The dynamics of the yield curve can be explained mostly by two factors, generally identified as the level and slope of the curve (see Mendez-Vives (2000)). In general, we would expect that when the bond markets sell off (yields increase), spreads will tend to widen reflecting the negative news. However, in periods of extreme market instability or "flight-to-quality," we observe that Treasury yields fall while spreads widen, i.e. the correlation becomes negative. In practice, it is likely to be difficult to disentangle which of these effects prevails.[20]

The slope of the yield curve is a crucial variable, since it not only reflects the way the yield curve evolves over time, but also because there is evidence that it encapsulates the market expectations regarding the business cycle. This point is argued extensively in

[20]Another common argument is that the correlation between the levels of rates and swap spreads is negative because swap markets adjust to new information more slowly than bond markets, due to their lower liquidity. For instance, lower Treasury yields would make spreads to widen temporally, in a mechanical fashion. This argument implies a degree of forecastibility for swap spreads, topic that has not been investigated to our knowledge.

Harvey (1988), where it is shown evidence that a flat or inverted yield curve predicts a recession. In empirical work on spreads, it is usual to condition them on some measure of the business cycle.[21] We will we use the slope of the Treasury yield curve for that purpose, measured as the differential between 2-year and 10-year rates. In Figure 3.9, we can see the evolution of the slope of the yield curve together with the evolution of credit spreads. Notice the strong negative correlation since January 1999, where spreads widen when the slope turns negative and tighten when the curve steepens.

Another reason to introduce the slope of the yield curve is that it has a strong impact on the financing choices of firms and their corresponding positions in swaps. When the yield curve is very steep, corporates may prefer to obtain finance on a floating basis and swap it into fixed-rate, long-term debt. In other words, when the curve is steep there is a receiver bias in swap markets, which tends to decrease swap rates and hence swap spreads.

As additional conditioning variables, we will also introduce the QUAL and the FININD spread differentials, that have an interpretation in terms of the hypotheses we have formulated for the relation between swap and credit spreads. The choice of variables may become more clear after we discuss the previous literature and their main conclusions.

3.4.2 Previous literature

In this section we comment on the work by Minton (1994), Lang et al. (1998), and Baz et al. (1999) and their results regarding the contemporaneous relation between credit and swap markets.

Minton (1994) examines the empirical implications of a valuation model for swaps based on their replication with corporate bonds. Swap rates are found to move together with corporate spreads (Baa vs. Treasuries) and to be positively (but not significantly) related to proxies for differential counterparty risk (Baa vs. Aaa spread differential). Minton also documents a positive relation between swap rates and the slope and level of the US Treasury curve. These results are obtained from a multivariate regression of first differences of all the variables, as there is evidence that swap rates are near-integrated. The fact that the credit spread (Baa vs. Treasuries) appears to be significant in the regression for swap rates is taken as an indication that differential counterparty

[21] For instance, Lang et al. (1998) condition swap spreads on detrended unemployment, and Bevan and Garzelli (2000) condition credit spreads on real GDP growth and the financing gap (difference between capital spending and internally generated funds) of US non-farm and non-financial corporations.

default risk matters. However, we would expect that to be reflected in a significant impact of the Baa vs. Aaa spread differential, which is not the case. Also, there is a number of caveats due to the nature of the data, especially that on credit.[22]

The work of Lang, Liu and Litzenberger (1998) is focused on the analyis of the determinants of the allocation of the swap surplus among the two counterparties over the business cycle. The main empirical results are obtained from a regression of swap spreads on single A spreads, either agency spreads or AAA spreads, and a proxy for the business cycle, either detrended unemployment or the percentual change in unemployment. The regressions are run in levels, and are corrected for the presence of autocorrelation at one lag. For the 10-year swap spread, the single-A spread and the agency spread are significant with a positive sign, of around 0.25 and 1, respectively. The business cycle variables tend to be significant, with a negative sign, meaning that swap spreads are pro-cyclical. [23]

We finally comment on Baz et al. (1999). This work discusses a number of possible economic drivers for swap spreads for USD, EUR and GBP. The study is motivated by the increase in the market awareness of swaps, caused by the crisis of August 1998, and the perception that swap spreads are becoming more closely related to credit markets.[24] The drivers of swap spreads that are discussed in this paper are the level and slope of the Treasury yield curve, the Libor vs Repo (for general collateral) spread, the credit spread for single A paper (AAA for EUR), and a measure of equity market volatility.

[22]The data is measured at a monthly frequency (89 observations) between August 1985 and December 1992. The corporate bond data is for newly issued bonds, fact that limits considerably the universe of bonds considered at each date. In particular, these are the bonds which are likely to be the most liquid, commanding a stochastic liquidity premium. As a measure of corporate quality spread, Minton takes the difference between Baa and Aaa bonds. However, for USD, the Aaa sector of the corporate bond market is relatively small, and it may be contaminated by the presence of issues from US Agencies. We are also surprised by the long average maturities of Aaa and Baa-rated bonds that are reported, of around 20-years in 1985, when the average maturity of our index data for that date is only about 10.5 years. Finally, as a measure of the business cycle, Minton uses monthly industrial production growth, which is a notably volatile series (about twice as volatile as GDP growth).

[23]The swap spread data in Lang et al. (1998) use is USD swap spreads for 5 and 10-years, between March 1986 and June 1992, at a monthly frequency (73 observations). The agency spreads are computed by taking the agency rates and subtracting the yields of maturity-matched Treasuries. The corporate bond spread data is obtained by taking the rates from Moody's, for A, AA and AAA bonds and subtracting a "basket" of 10-year and 30-year Treasuries that matches the average maturity of the corporate bond basket. We have two caveats. First, regarding the original data on rates, at the end of 1992, the average time to maturity of a single A bond was 27.2 years, 28.8 years for AA and 22.2 years for AAA. This seems extremely long to us (see the average maturity of the Lehman Brothers US Corporate Index in the appendix). The second caveat applies to how the authors compute the underlying goverment yield that they subtract from the corporate bond rate to obtain the spread, by taking a weighted average of the 10-year and 30-year Treasury rates. Another problem with the data, that the authors acknowledge, is that their AAA universe includes the US Agencies. It is the case that the spreads of Agencies tend to be tighter than the spreads of AAA corporates, due to their implicit or explicit government guaranty.

[24]The data on swap spreads comprises the period 1994 to 1999, on a weekly frequency, for 2, 5 and 10-year spreads, and the spreads are computed versus a constant maturity par fitted curve.

The econometric analysis consists on a regression of weekly changes of swap spreads on changes in the explanatory variables mentioned above, plus the lagged level of the swap spread itself. This is introduced in order to account for the possible presence of mean reversion in swap spreads.[25] Generally, slope, credit and lagged level of spread appear to be highly significant and with the expected sign (negative, positive, negative respectively). The Libor vs. repo is only significant for the front end of the USD swap spread curve. The equity market volatility is also mostly significant for the US. The autocorrelation coefficient is typically significant and around -0.3, consistent with the presence of mean reversion.

3.4.3 An Error Correction model for swap spreads

In this section we will argue that credit and swap spreads are cointegrated and that an appropiate methodology to capture that is an error correction model (ECM) (see Banarjee et al. (1993)). We will estimate such a model following the Engle-Granger (1987) two-step procedure. The first step consists in estimating the cointegrating vector via and OLS regression of the variables in levels. The regression residual (which should be $I(0)$), is interpreted as the deviation of the swap spread from its long-term equilibrium level. In the second step, we estimate a regression with the variables in first differences and in which we also include the lagged residual from the levels regression. This equation tries to capture the short-term adjustment dynamics of the swap spreads towards its equilibrium level. The adjustment will depend on the movements of a number of economic drivers *and* how far the swap spread is from its long-term level. If the model is well-specified, swap spreads should tend to revert to their equilibrium level, hence the coefficient of the lagged error would measure the speed of mean reversion and would be expected to be negative.

Although it is common to capture mean reversion in the cointegrating relation by introducing the lagged error term into the short-term adjustment equation, other specifications are possible. In particular, one could introduce an additional, non-linear term, for instance the lagged error to the cube. This would capture the possibility that mean reversion accelerates whenever the error is relatively large (in absolute value).[26] In

[25] The regressions are corrected for one-lag autocorrelation, by using the Maximum Likelihood method of Beach and McKinnon (see Davidson and MacKinnon (1993)). The regression for each swap spread is performed for the full sample and for the periods before and after August 1998.

[26] For a discussion, a good reference is Mills (1999), chapter 8.

our work, we have chosen to use the traditional specification, leaving changes in it for posterior research.

For the sake of concreteness, we will present the results of the analysis for the 10-year maturity, which is generally the most representative sector of the swap market. Regarding the credit spreads, we will focus on the single-A rated corporates, which is the average credit quality of the USD credit market.[27]

In order to argue that swaps and credit spreads are cointegrated, we first need to show that the series are I(1). For that purpose, we have performed a series of Augmented Dickey-Fuller tests, both for the full sample and for the two subsamples before and after the crisis of August 1998.[28] The results of the ADF tests in Table 3.8 are conclusive, as we cannot reject the null hypothesis of a unit root at the 5% level for any maturity or subsample. We have also performed ADF tests for the credit and swap spreads in first differences, for which the hypothesis of nonstationarity is strongly rejected.[29]

We have augmented the cointegrating relation with the slope of the Treasury curve (10-year yield minus 2-year yield, in basis points). This is because as Lang et. al (1994) argue, the equilibrium relation between swap and credit spreads changes along the business cycle, and the evidence in Harvey (1988) supports the idea of using the slope of the yield curve as a proxy for it. More intuitively, we tend to observe that swap spreads are lower than one would expect whenever the yield curve is very steep. Possible explanations for that have been presented in the previous section, for instance that a step yield curve results in a receiver bias in the swap market that makes swap spreads to compress sharply. The ADF test results in Table 3.8 indicate that the slope is $I(1)$, whereas the first differences are found to be stationary.

The cointegrating relation we estimate is then

$$SWAP\,SPREAD_t = \beta_0 + \beta_1 CREDIT\,SPREAD_t + \beta_2 SLOPE_t + \epsilon_t \quad (3.1)$$

The results of this regression can be seen in Table 3.9, where the credit spread has a coefficient of 0.462 and the slope has a coefficient of -0.158. In other words, in the long run swap spreads are roughly half the size of single-A credit spreads, but the slope of

[27]The conclusions for the other maturities and for the other corporate bond ratings are similar to those that we present.

[28]The reason for the tests by subsamples is because non-stationarity might be caused by the presence of a structural break of exogenous nature (see Chapter 2).

[29]Not reported, available upon request. Alternatively see the evidence in Chapter 2

the yield curve has a negative impact on it. A yield curve that is 10bp steeper will result on swap spreads 1.5bp tighter. The residual from the levels regression is shown to be $I(0)$, where we correct the critical values of the ADF test to account for the fact that the cointegrating coefficients are estimated, not known.

The results of the above regression are consistent with the observed international differences on the average level of swap spreads. It is the case that credit spreads in USD are higher than in EUR (156bp for USD single-A debt and 91bp for EUR[30]) and that the average slope of the yield curve in the Euro-area are higher than in the US. The spread between 2-year and 10-year bonds is 78bp and 18bp respectively (between January 1999 and July 2001). According to the model, this would result in higher swap spreads for USD than for EUR, as it is the case.

Next, we estimate the short-term adjustment equation, where the lagged residual from the levels regression (3.1) captures the mean reversion in the process. In addition to the the changes in credit spreads and the slope, this second equation also includes a number of variables that may influence the short term dynamics of the swap spreads: the level of interest rates and the quality and financial vs. industrial spreads.[31][32]

$$\begin{aligned} \Delta SWAP\,SPREAD_t &= \rho\epsilon_{t-1} + \gamma_0 + \gamma_1 \Delta CREDIT\,SPREAD_t \\ &+ \gamma_2 \Delta BOND_t + \gamma_3 \Delta SLOPE_t + \gamma_4 \Delta FININD_t + \gamma_5 \Delta QUAL_t + u_t \end{aligned} \quad (3.2)$$

The results from the ECM (full sample in Table 3.9, pre and post-98 crisis in Tables 3.11 and 3.13) show that the most important influence in the swap spreads comes from credit spreads, with a positive sign as expected. The coefficient on the credit spread, for the 10-year spread levels equation is 0.462, and 0.729 in the changes regression (both numbers are for the full sample). We also observe that the short term relation between swap spreads and credit spreads is stronger after the crisis of 1998. The credit spread

[30] As of July 2001, average spreads from the single-A corporate sector of the Lehman Brohters US Aggregate and Euro-Aggregate indices.

[31] All variables in equation (3.2) are introduced in first differences, as they are generally shown to be $I(1)$ or near-$I(1)$. Essentially, the bond yields and the slope of the yield curve appear to be highly nonstationary. For the spread differentials, FININD and QUAL, the results of the ADF tests are generally close to the rejection of $I(1)$ at the 5% level. For all the variables in first differences, we obtain a strong rejection of the null of stationary. We do not report the results from the Augmented Dickey-Fuller for lack of space.

[32] In the "changes" regression, all variables are as of time t, except for the lagged error from the levels regression. The intention in this chapter is to describe the contemporaneous relation between variables, and not to look for forecasting relations. In order to study the later, we would need a rigurous analysis of the exogeneity of the variables, Granger causality, etc. which is beyond the scope of the present work. Instead, we focus on the relative value relations between variables.

coefficient in the changes regression for the sample after the crisis is 0.733, against 0.468 for the sample before the crisis.

The lagged residual is found to be strongly significant and enters into the changes regression with a negative sign. Hence, deviations from the "fair" value given by the fitted level of the swap spread relative to the credit spreads, tend to be corrected during subsequent periods. The magnitude of the coefficient differs over subsamples, and in particular, the speed of the reversion to fair levels is faster in the pre-crisis period. In other words, distortions between the swap and credit markets tend to persist longer after the crisis of 1998, which is consistent with general market opinion (see Baz et al. (1999)).

The yield curve variables (level and slope) tend to be significant mostly for the period after August 1998. Also, the slope seems to have a more important effect. The changes in the 10-year bond yield have a positive coefficient, but this is significant for the full sample and the post-crisis subsample at the 10% level only. The slope has a negative sign and it's significant again for the full sample and for the post-crisis period, at the 1% level. These results confirm our conjecture that it may be difficult to obtain a clear relation between changes in yields and in swap spreads, but the slope should have a clear effect, in the form of swap spreads tightening when the curve steepens.

The impact of changes in the FININD and the QUAL spreads on the swap spreads is either not significant (FININD) or when it is, it has a puzzling negative sign (QUAL). We think that this can be due to the high correlation of these variables with the changes in credit spreads[33] that obscures their relation with swap spreads. In other words, the information in the spreads differentials seems to be already contained in the credit spread. In any case, the QUAL spread has a significantly negative sign both for the full sample and the post-crisis period, which is not consistent with the idea that differential counterparty default risk affects swap spreads.

3.5 Conclusions

This chapter addresses the relation between credit and swap spreads from an empirical point of view. The explanation of the high contemporaneous correlation between the swap and the corporate bond markets is a topic that has been not resolved theoretically.

[33]The correlation of fortnightly changes of 10-year single-A spreads with the 10-year QUAL spread is 0.30, and 0.48 for the FININD spread.

One of the reasons may be that there is scarce empirical evidence on which to base a theoretical investigation, mainly due to the difficulties in obtaining a comprehensive and accurate dataset on credit spreads.

Using a rich dataset on credit spreads constructed from bond index data, we first document the main features of swap and credit spreads. Among them, we observe that the slope of the spread curves tends to be upward sloping with maturity, that the spreads on financials tend to be higher than the spreads on industrials and that the volatility of swap spreads is generally lower than the volatility of credit spreads. Also, we note the importance of taking into account the presence of a structural break in the sample, given by the global financial crisis of August 1998. We show that spreads were higher and more volatile after the crisis.

We next discuss two approaches to explaining the relation between swap and credit spreads. We tend to find more evidence supporting the explanation that relies in the fact that swap cashflows are indexed to Libor and that the agents increasingly focus on the swap curve as the benchmark for trading and hedging credit risk. An alternative explanation, based on the default risk of the swap cashflows is hindered by the fact that the industry has developed extensive credit protection mechanisms. Swap spreads do not seem to be related to measures of the global credit risk, like the spread differential between BBB and AA debt.

Given the statistical evidence and the economics of the problem, we have investigated the possibility of swap and credit spreads being cointegrated. We estimate an error correction model where we also introduce a number of a possibly relevant economic variables. We find that the relation between credit and swap spreads is strongly positive and stable. The relation between swap spreads and the slope of the yield curve is typically negative and stronger after the crisis of 1998. The credit spread differentials (FININD and QUAL) do not appear to be significant, and we conjecture that the reason for that is because the information they contain is already incorporated in the credit spreads.

The econometric evidence tends to support the hypothesis that the relation between the swap and the corporate bond market arises from the fact that swaps are indexed to the Libor sector. The evidence in this chapter should be relevant for the construction of a theoretical model dealing with the integration between the different sectors of the fixed income markets.

Description of the Lehman Brothers Indices

The Lehman Brothers US Aggregate Index represents securities that are U.S. domestic, taxable, and dollar denominated. The index covers the U.S. investment grade fixed rate bond market, with index components for government and corporate securities, mortgage pass-through securities, and asset-backed securities. These major sectors are subdivided into more specific indices that are calculated and reported on a regular basis. The index comprises all securities that satisfy the following conditions:

- Must have at least one year to final maturity regardless of call features.
- Must have at least $150 million par amount outstanding.[34]
- Must be rated investment grade (Baa3 or better) by Moody's Investors Service. If a Moody's rating is not available, the S&P or Fitch rating is used.
- Must be fixed rate, although it can carry a coupon that steps up or changes according to a predetermined schedule.
- Must be dollar-denominated and non-convertible.
- Must be publicly issued.

The US Aggregate index can be decomposed in a number of subindexes, e.g. the US Treasury or the US Credit Investment Grade Index. The US Credit index includes both corporate and non-corporate sectors. The corporate sectors are Industrial, Utility, and Finance, which include both U.S. and non-U.S. corporations. The non-corporate sectors are Sovereign, Supranational, Foreign Agency, and Foreign Local Government. The spreads we have used in this chapter are derived from the Corporate (Investment Grade) section of US Credit. The US Corporate index includes publicly issued U.S. corporate and specified foreign debentures and secured notes that meet the specified maturity, liquidity, and quality requirements. To qualify, bonds must be SEC-registered and include:

- Subordinated issues, provided that other specified criteria are met.
- Securities with normal call and put provisions and sinking funds.

[34]The minimum amount outstanding has been increasing over the life of the index, according to the development of the corporate bond market.

- Medium-term notes (if they are publicly underwritten).
- 144A securities (if they have registration rights).
- Global issues that are SEC-registered.

The bonds which are specifically excluded are:

- Structured notes with embedded swaps or other special features.
- Private placements, floating rate securities, and Eurobonds.

We have obtained the main statistics for the US Corp index as shown in the Table 3.15. As of December 2000, the number of corporate bonds in the index was around 3500, with an average maturity of about 7 years, down from the 9 years in the beginning of the 90's. The market value of the corporate bond market stood around the $1.2 trn mark.

		SWSPR2	SWSPR5	SWSPR10
Full sample	Mean	37.7	45.8	47.5
	Std Dev Chg.	21.5	22.0	28.3
	Min	8.6	21.2	25.5
	Max	81.1	102.5	117.5
Before Aug98	Mean	25.8	28.8	32.1
	Std Dev Chg.	18.1	9.7	10.4
	Min	8.6	21.2	25.5
	Max	53.2	40.6	42.9
After Aug98	Mean	55.6	71.2	70.6
	Std Dev Chg.	25.9	32.8	43.0
	Min	35.2	46.5	38.9
	Max	81.1	102.5	117.5

Table 3.1: Descriptive statistics of USD swap spreads (vs. fitted off-the-run Treasuries). All figures in bp. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs). The std. deviation of changes is in basis points per year.

		AA2	A2	BBB2	AA5	A5	BBB5	AA10	A10	BBB10
Full sample	Mean	51.6	63.8	95.8	64.4	81.0	114.8	73.5	94.5	130.8
	Std Dev Chg.	20.4	25.7	25.2	25.7	30.1	30.4	28.2	31.5	35.0
	Min	25.5	32.2	44.6	33.1	40.6	57.6	39.8	53.1	74.4
	Max	95.7	127.2	201.4	141.8	174.3	235.3	164.0	200.3	255.2
Before Aug98	Mean	34.9	42.6	59.6	40.8	52.9	73.9	48.6	65.8	92.1
	Std Dev Chg.	13.2	14.6	15.8	11.0	12.7	12.8	12.2	14.2	14.3
	Min	25.5	32.2	44.6	33.1	40.6	57.6	39.8	53.1	74.4
	Max	57.0	71.2	87.2	55.7	68.3	93.2	70.1	87.5	114.0
After Aug98	Mean	76.7	95.7	150.1	99.9	123.1	176.2	110.9	137.4	188.9
	Std Dev Chg.	27.9	36.4	34.1	38.3	44.9	45.1	42.0	46.8	52.3
	Min	45.3	52.6	75.2	62.4	74.0	102.1	67.2	93.7	130.0
	Max	95.7	127.2	201.4	141.8	174.3	235.3	164.0	200.3	255.2

Table 3.2: Descriptive statistics of credit spreads (vs. fitted off-the-run Treasuries) for all corporate sectors. All figures in bp. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs). The std. deviation of changes is in basis points per year.

		AA2 I	A2 I	BBB2 I	AA5 I	A5 I	BBB5 I	AA10 I	A10 I	BBB10 I
Full sample	Mean	46.9	64.1	96.0	59.2	79.9	113.8	68.9	92.0	129.5
	Std Dev Chg.	18.0	21.4	26.3	21.0	25.1	30.7	26.1	28.7	35.2
	Min	22.2	30.2	44.9	27.8	40.2	57.2	35.9	52.2	75.1
	Max	89.0	135.9	213.3	127.4	179.4	247.4	152.3	199.5	265.0
Before Aug98	Mean	31.1	41.5	59.4	36.9	51.2	72.7	45.0	63.3	91.7
	Std Dev Chg.	15.3	15.7	15.4	9.8	11.6	12.6	12.3	13.6	15.6
	Min	22.2	30.2	44.9	27.8	40.2	57.2	35.9	52.2	75.1
	Max	59.1	75.2	91.7	50.3	66.4	97.4	64.4	85.2	112.1
After Aug98	Mean	70.7	98.0	150.8	92.7	123.0	175.4	104.8	135.0	186.2
	Std Dev Chg.	21.4	27.6	36.3	30.9	37.0	45.5	38.5	42.2	52.1
	Min	38.0	51.7	76.8	54.3	72.6	102.2	59.7	94.0	128.9
	Max	89.0	135.9	213.3	127.4	179.4	247.4	152.3	199.5	265.0

Table 3.3: Descriptive statistics of credit spreads (vs. fitted off-the-run Treasuries) for the Industrial corporate sector. All figures in bp. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs). The std. deviation of changes is in basis points per year.

		AA2 F	A2 F	BBB2 F	AA5 F	A5 F	BBB5 F	AA10 F	A10 F	BBB10 F
Full sample	Mean	54.2	61.8	107.0	71.3	85.2	131.3	84.8	106.6	155.1
	Std Dev Chg.	26.6	28.7	58.0	33.1	38.1	51.9	37.9	44.8	47.6
	Min	26.1	30.1	42.3	35.7	41.8	59.7	46.5	58.4	81.3
	Max	107.5	122.2	248.2	157.4	189.4	273.4	183.7	235.3	290.1
Before Aug98	Mean	35.8	41.6	61.9	45.4	56.0	82.9	57.6	74.1	108.4
	Std Dev Chg.	13.4	14.1	22.3	12.9	14.7	20.4	16.2	19.1	23.9
	Min	26.1	30.1	42.3	35.7	41.8	59.7	46.5	58.4	81.3
	Max	56.3	67.9	85.7	61.2	73.9	113.4	76.4	94.1	143.3
After Aug98	Mean	81.8	92.3	174.6	110.1	128.9	203.9	125.7	155.3	225.2
	Std Dev Chg.	38.7	41.8	87.2	49.9	57.5	78.1	56.6	66.9	69.5
	Min	46.8	52.4	74.2	65.9	74.1	114.1	82.7	97.1	156.1
	Max	107.5	122.2	248.2	157.4	189.4	273.4	183.7	235.3	290.1

Table 3.4: Descriptive statistics of credit spreads (vs. fitted off-the-run Treasuries) for the Finance corporate sector. All figures in bp. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs). The std. deviation of changes is in basis points per year.

		AA2 L	A2 L	BBB2 L	AA5 L	A5 L	BBB5 L	AA10 L	A10 L	BBB10 L
Full sample	Mean	14.4	26.7	58.7	18.9	35.5	69.3	26.3	47.2	83.6
	Std Dev Chg.	23.2	26.6	26.2	20.8	23.8	26.5	22.2	24.5	30.3
	Min	-4.5	0.0	10.9	2.5	11.1	24.8	10.7	23.8	41.9
	Max	55.2	75.8	149.3	57.3	86.6	155.0	77.3	116.6	174.7
Before Aug98	Mean	9.3	17.1	34.1	12.1	24.2	45.2	16.7	33.9	60.2
	Std Dev Chg.	14.4	15.7	16.4	12.1	13.5	14.0	12.0	13.9	14.4
	Min	-4.5	0.0	10.9	2.5	11.1	24.8	10.7	23.8	41.9
	Max	19.6	30.5	55.7	19.5	33.5	63.7	27.8	45.2	76.2
After Aug98	Mean	22.1	41.1	95.5	29.2	52.4	105.5	40.8	67.3	118.8
	Std Dev Chg.	32.1	37.5	35.6	29.5	33.8	37.8	31.9	34.7	44.5
	Min	8.3	15.6	38.2	14.3	28.0	56.1	22.2	40.1	78.9
	Max	55.2	75.8	149.3	57.3	86.6	155.0	77.3	116.6	174.7

Table 3.5: Descriptive statistics of Libor spreads (corporates vs. the swap curve) for all corporate sectors. All figures in bp. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs). The std. deviation of changes is in basis points per year.

		QUAL2	QUAL5	QUAL10
Full sample	Mean	44.2	50.4	57.3
	Std Dev Chg.	14.9	14.9	18.6
	Min	15.0	22.3	30.2
	Max	116.9	109.6	109.7
Before Aug98	Mean	24.8	33.1	43.5
	Std Dev Chg.	7.1	6.1	7.1
	Min	15.0	22.3	30.2
	Max	37.9	48.4	61.3
After Aug98	Mean	73.4	76.3	78.0
	Std Dev Chg.	21.4	21.9	28.0
	Min	29.9	39.7	50.7
	Max	116.9	109.6	109.7

Table 3.6: Descriptive statistics of the quality spread (BBB vs. AA). All figures in bp. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs). The std. deviation of changes is in basis points per year.

		FININD2	FININD5	FININD10
Full sample	Mean	-2.3	5.3	14.6
	Std Dev Chg.	17.9	19.7	25.7
	Min	-23.8	-9.6	0.9
	Max	24.6	38.8	54.0
Before Aug98	Mean	0.0	4.9	10.8
	Std Dev Chg.	8.8	7.8	11.4
	Min	-7.4	-1.2	0.9
	Max	11.9	14.0	22.4
After Aug98	Mean	-5.8	5.9	20.3
	Std Dev Chg.	26.1	29.7	38.2
	Min	-23.8	-9.6	3.0
	Max	24.6	38.8	54.0

Table 3.7: Descriptive statistics of the Financials vs. Industrial spread (single-A). All figures in bp. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs). The std. deviation of changes is in basis points per year.

Levels regression	Statistic	p-value
F-statistic (2,167) dof	806.7	0
Durbin-Watson	0.346	0
Breusch-Pagan	54.583	0
Goldfeld-Quandt	14.513	0
Residual std. error	7.211	–
Changes regression	Statistic	p-value
F-statistic (6,162) dof	36.07	0
Durbin-Watson	2.045	0.578
Breusch-Pagan	8.862	0.1815
Goldfeld-Quandt	6.638	0
Residual std. error	3.704	–

Table 3.10: Regression diagnostics for the error correction model for 10-year swap spreads, for the full sample. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs). The Durbin-Watson test has a null hypothesis of no autocorrelation in the errors, against an alternative of first order autocorrelation. The Breusch-Pagan is a Lagrange multiplier test for heteroskedasticity. Under the null hypothesis of homoskedasticity, it is distributed as a chi-square variable. The Goldfeld-Quandt is another test for heteroskedasticity. Under the null of homoskedasticity, the statistic follows an F-distribution.

Variable	Coeff	Std Error	T-Stat	Signif
Constant	8.560	3.186	2.687	0.007
A10	0.385	0.042	9.228	0.000
SLOPE	-0.037	0.011	-3.377	0.001
R_c^2	0.714	ADF t-test	-5.130	
Variable	Coeff	Std Error	T-Stat	Signif
Constant	-0.044	0.158	-0.276	0.782
ρ	-0.378	0.084	-4.486	0.000
ΔA10	0.468	0.059	7.866	0.000
ΔBOND10	0.019	0.015	1.240	0.215
ΔSLOPE	-0.037	0.022	-1.720	0.085
ΔFININD10	-0.149	0.086	-1.727	0.084
ΔQUAL10	-0.169	0.119	-1.412	0.158
R_c^2	0.354			

Table 3.11: Results of the Error Correction model for 10-year swap spreads, for the sample before the crisis of August 1998. Data between 15 May 1994 and 31 July 1998, with semi-monthly frequency (T=102 obs). The std. errors are Newey-West (with 6 lags). The critical value for the ADF t-test on the residuals of the regression in levels is -3.37 (5%) or -3.96 (1%).

Levels regression	Statistic	p-value
F-statistic (2,99) dof	123.4	0
Durbin-Watson	0.822	0
Breusch-Pagan	0.651	0.722
Goldfeld-Quandt	1.392	0.127
Residual std. error	2.125	–
Changes regression	Statistic	p-value
F-statistic (6,94) dof	8.569	0
Durbin-Watson	2.103	0.677
Breusch-Pagan	10.802	0.094
Goldfeld-Quandt	0.933	0.590
Residual std. error	1.695	–

Table 3.12: Regression diagnostics for the error correction model for 10-year swap spreads, for the sample before the crisis of August 1998. Data between 15 May 1994 and 31 July 1998, with semi-monthly frequency (T=102 obs). The Durbin-Watson test has a null hypothesis of no autocorrelation in the errors, against an alternative of first order autocorrelation. The Breusch-Pagan is a Lagrange multiplier test for heteroskedasticity. Under the null hypothesis of homoskedasticity, it is distributed as a chi-square variable. The Goldfeld-Quandt is another test for heteroskedasticity. Under the null of homoskedasticity, the statistic follows an F-distribution.

Variable	Coeff	Std Error	T-Stat	Signif
Constant	12.067	9.877	1.222	0.222
A10	0.455	0.076	6.023	0.000
SLOPE	-0.289	0.063	-4.602	0.000
R_c^2	0.986	ADF t-test	-2.870	
Variable	Coeff	Std Error	T-Stat	Signif
Constant	0.109	0.598	0.182	0.855
ρ	-0.217	0.077	-2.824	0.005
ΔA10	0.733	0.092	7.979	0.000
ΔBOND10	0.072	0.044	1.644	0.100
ΔSLOPE	-0.183	0.062	-2.971	0.003
ΔFININD10	-0.026	0.100	-0.264	0.792
ΔQUAL10	-0.281	0.111	-2.534	0.011
R_c^2	0.620			

Table 3.13: Results of the Error Correction model for 10-year swap spreads, for the sample after the crisis of August 1998. Data between 14 August 1998 and 31 May 2001, with semi-monthly frequency (T=68 obs). The std. errors are Newey-West (with 6 lags). The critical value for the ADF t-test on the residuals of the regression in levels is -3.37 (5%) or -3.96 (1%).

Levels regression	Statistic	p-value
F-statistic (2,65) dof	159.6	0
Durbin-Watson	0.519	0
Breusch-Pagan	8.385	0.015
Goldfeld-Quandt	1.715	0.069
Residual std. error	8.909	–
Changes regression	Statistic	p-value
F-statistic (6,60) dof	16.45	0
Durbin-Watson	1.945	0.349
Breusch-Pagan	7.221	0.301
Goldfeld-Quandt	1.282	0.264
Residual std. error	5.441	–

Table 3.14: Regression diagnostics for the error correction model for 10-year swap spreads, , for the sample after the crisis of August 1998. Data between 14 August 1998 and 31 May 2001, with semi-monthly frequency (T=68 obs). The Durbin-Watson test has a null hypothesis of no autocorrelation in the errors, against an alternative of first order autocorrelation. The Breusch-Pagan is a Lagrange multiplier test for heteroskedasticity. Under the null hypothesis of homoskedasticity, it is distributed as a chi-square variable. The Goldfeld-Quandt is another test for heteroskedasticity. Under the null of homoskedasticity, the statistic follows an F-distribution.

Date	Number issues	Mod. Duration	Avg. Coupon	Avg. Maturity	Yield to worst	Market Value ($MM)
12/29/00	3,527	5.52	10.89	7.2	7.36	1,235,055
12/31/99	3,639	5.66	11.56	7.04	7.7	1,165,609
12/31/98	4,613	6.03	12.85	7.18	6.15	1,196,643
12/31/97	4,017	6	13.26	7.47	6.55	962,413
12/31/96	3,651	5.9	12.44	7.57	6.98	837,783
12/29/95	3,387	5.9	12.38	7.77	6.34	793,223
12/30/94	3,109	5.57	12.16	8.03	8.67	633,999
12/31/91	3,752	4.76	12.87	9.08	7.49	600,914
12/30/88	3,836	na	13.37	9.18	10.16	410,227
12/31/85	4,643	na	14.95	9.66	10.55	316,500
12/31/82	4,467	na	16.3	9.23	12.16	233,556
12/31/79	4,177	na	17.89	7.85	11.37	178,596
12/31/76	4,053	na	18.88	7.44	7.87	190,691
12/31/73	3,460	na	19.88	6.46	8.08	118,532

Table 3.15: Statistics of the Lehman Brothers US Corporate (Investment Grade) bond index.

FIGURES

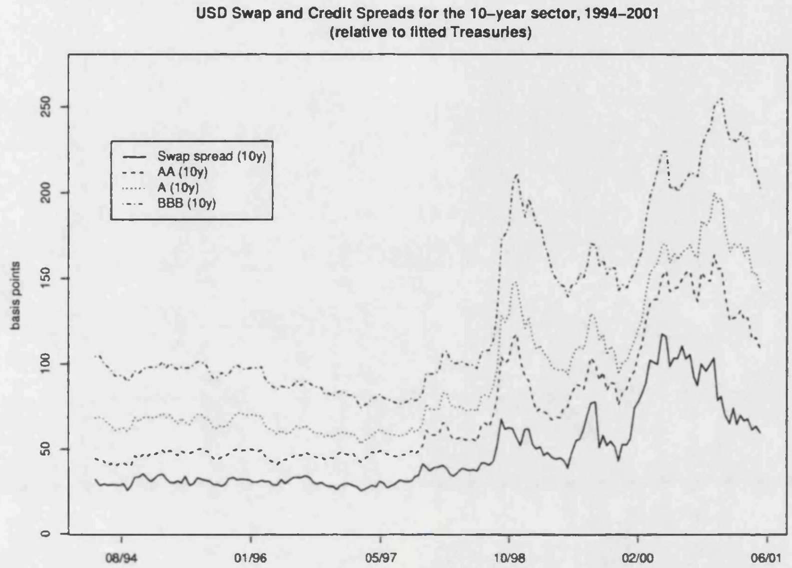

Figure 3.1: Evolution of the 10-year USD swap and credit spreads, relative to fitted off-the-run Treasuries. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs.).

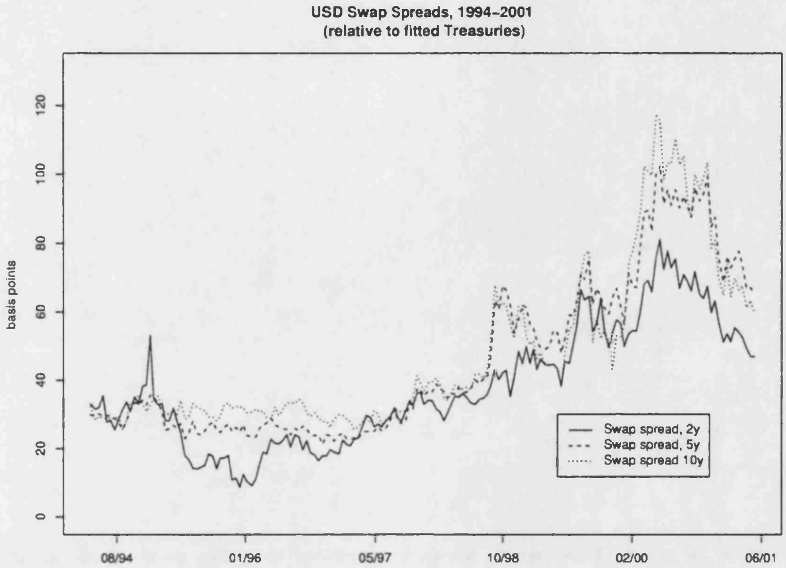

Figure 3.2: Evolution of the USD swap spreads, relative to fitted off-the-run Treasuries. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs.).

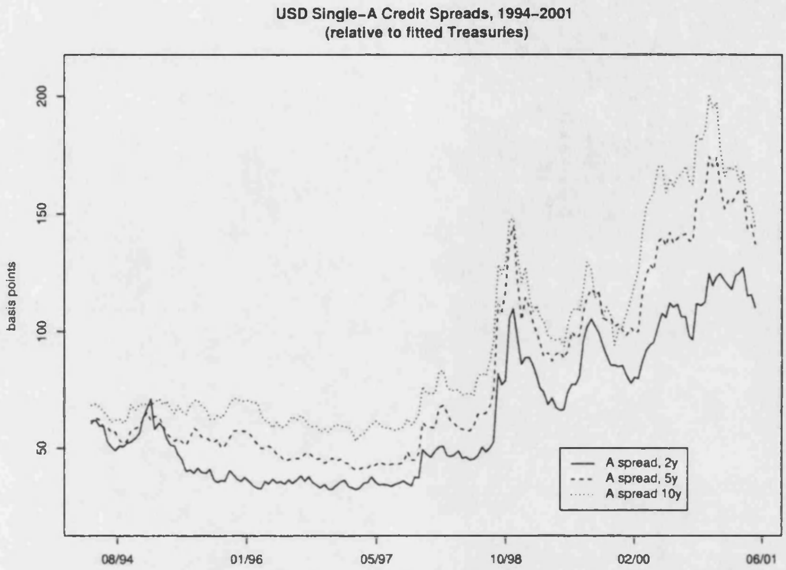

Figure 3.3: Evolution of the USD single-A credit spreads, relative to fitted off-the-run Treasuries. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs.).

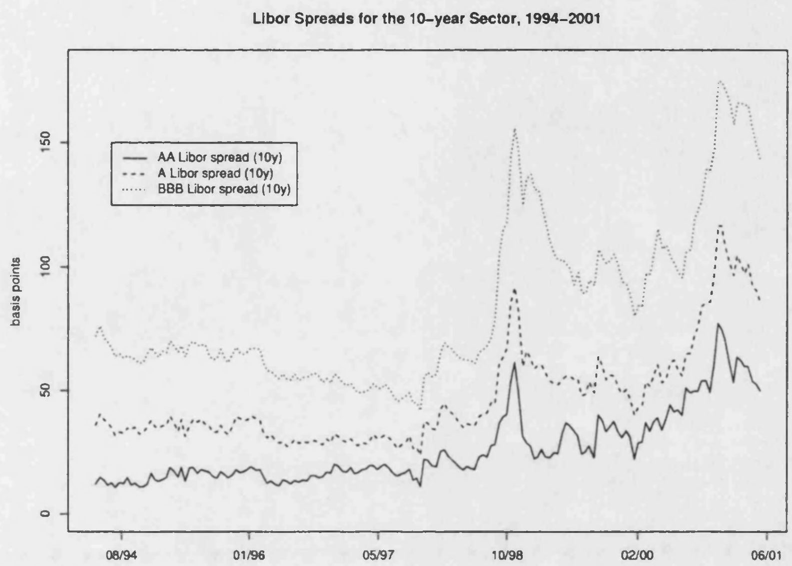

Figure 3.4: Evolution of the USD 10-year Libor spreads. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs.).

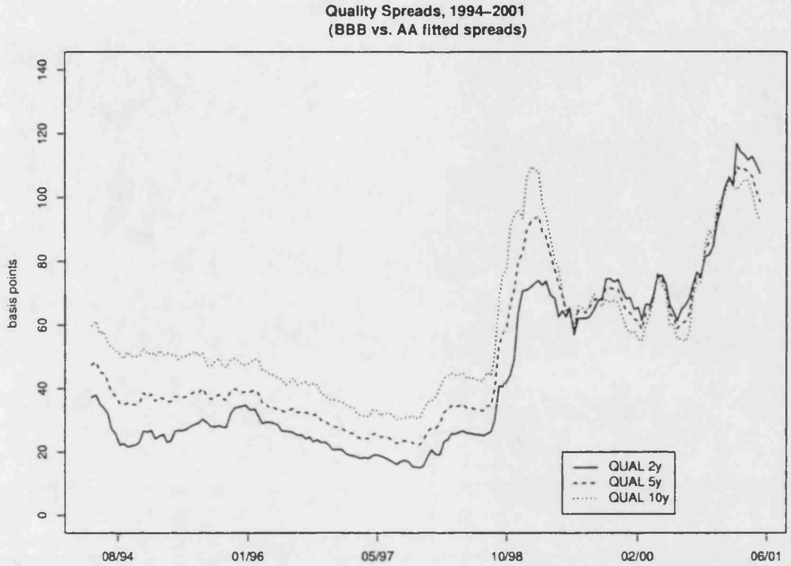

Figure 3.5: Evolution of the USD quality spreads, computed as BBB minus AA credit spreads. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs.).

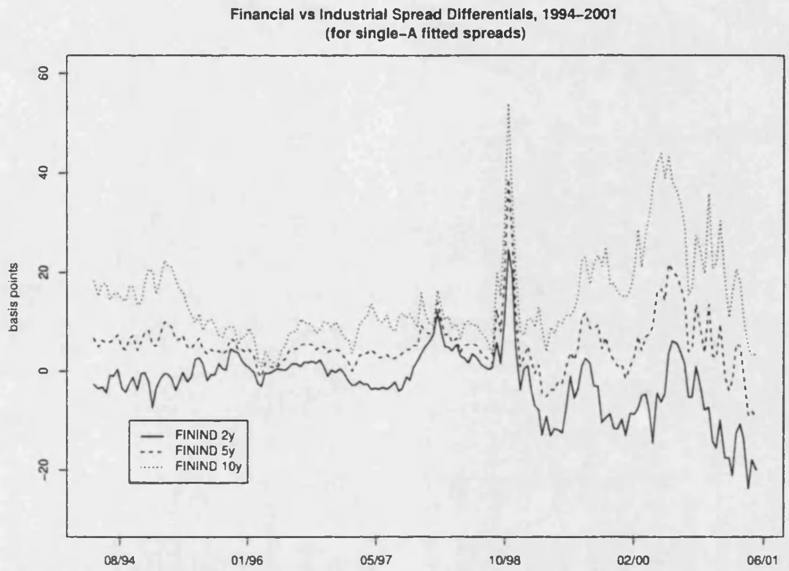

Figure 3.6: Evolution of the USD financial vs. industrial spread differential, computed for single-A corporates. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs.).

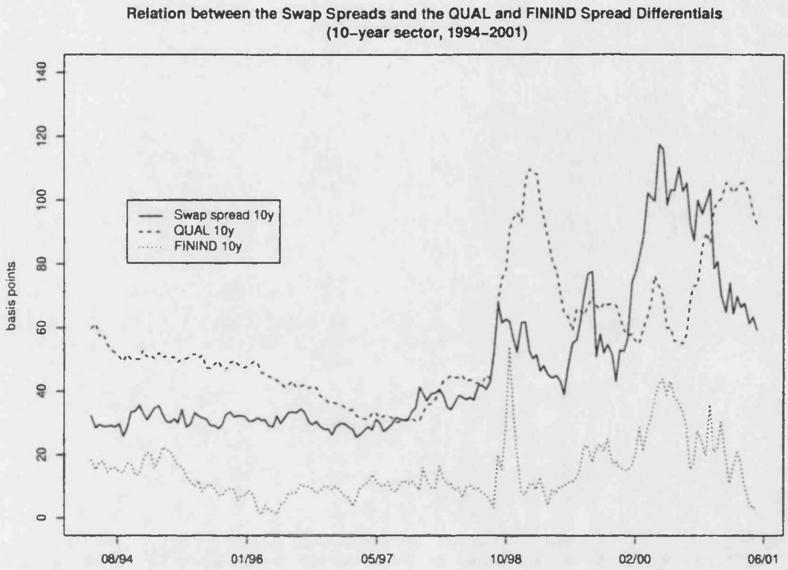

Figure 3.7: Relation between the USD 10-year swap spread and the QUAL and FININD spreads. Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs.).

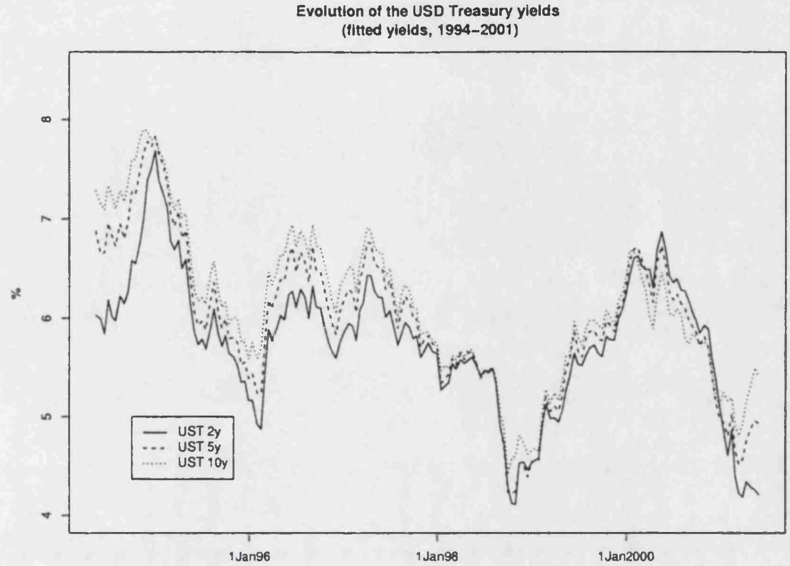

Figure 3.8: Evolution of the USD Treasury yields (from the off-the-run fitted curve). Data between 15 May 1994 and 31 May 2000, with semi-monthly frequency (T=170 obs.).

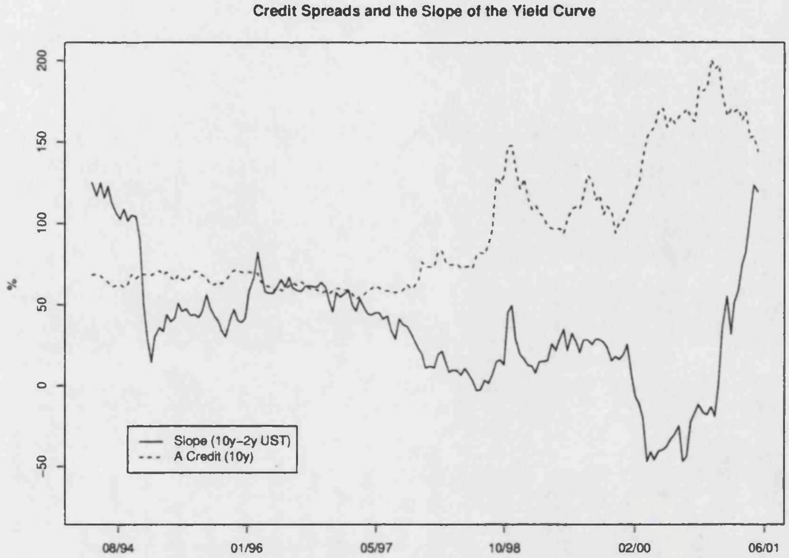

Figure 3.9: Evolution of the USD single-A 10-year credit spreads, and of the slope of the Treasury yield curve (computed as 10-year minus 2-year fitted yields). Data between 15 May 1994 and 31 May 2001, with semi-monthly frequency (T=170 obs.).

Chapter 4

The Informational Content of the Term Structure of Swap Spreads

4.1 Introduction

The present chapter is an empirical investigation of the properties of the risk premium in swap spreads.[1] Specifically, we examine whether the compensation for holding swap spread risk is time-varying. We find this to be the case: our main result is that swap spreads are to some extent predictable using the information in the slope of the term structure of spreads, in a way which is not compatible with a constant risk premium. The empirical relation is that a high current slope predicts tighter spreads in the future, and this contradicts the Expectation Hypothesis (EH hereafter), according to which the risk premium should be constant over time.

The fact that the term structure of swap spreads embeds important information regarding the evolution of swap spreads is relevant for a large number of participants in fixed income markets. This set of agents encompasses the direct users of swaps e.g. corporate treasurers managing the structure of their assets and liabilities, but also any agent sensitive to the general evolution of credit markets, given the high correlation between

[1] In an interest rate swap, the swap rate is usually quoted as a spread over the yield on a Treasury bond. See Chapter 2 for a detailed discussion of the institutional aspects of swaps.

swap and credit spreads.[2] For instance, portfolio managers increasingly use swaps to hedge the credit risk of their portfolios of corporate bonds or US Agencies. Our results can be also be useful for agents trading swap spreads, mainly hedge funds and investment banks. The academic relevance of this chapter lies in the fact that it is a first step in the study of the market price of credit risk. In particular, it complements the work by Liu, Longstaff and Mandell (2000), who use data on swap spreads for that purpose. Ideally, one would like to consider the risk premium on corporate bond spreads directly, but this is difficult due to the lack of appropiate data. Also, the corporate bond market is significantly less liquid than the swap market, making it more difficult to disentangle the different components of the compensation for risk. As we show in Chapter 3, the relation between credit spreads and swap spreads appears to be stable enough to use swap spreads as a proxy for credit spreads.

The results in this chapter have important implications on how to model swap spreads, in particular for the specification of the market price of credit risk. Some of the existing models of swap spreads are Grinblatt (1995), Duffie and Singleton (1997), He (2000) and Liu, Longstaff and Schwartz (2000). In Grinblatt (1995), swap spreads represent the compensation for the liquidity advantage of Treasuries over swaps. The swap spreads are computed as the present value of this liquidity advantage. Duffie and Singleton (1997), formulate a reduced-form model of swap yields, with 2 factors following CIR processes. In this model, swap cashflows are discounted at a "risky" short rate, the sum of a riskfree rate plus a factor that takes into account that swaps are defaultable instruments. He (2000) formalizes the relation between swap spreads and the Libor vs. repo rates for general collateral ("GC"), in the context of multifactor Gaussian models. Essentially swap spreads equal the present value of the forward Libor-GC spreads. A common feature of Grinblatt (1995), Duffie and Singleton (1997), and He (2000) is that they assume constant market prices of risk.

In the literature on the risk premium for fixed income markets, our empirical rejection of the EH for swap spreads is parallel to the results for riskless interest rates from US Treasuries. In their influential paper, Campbell and Shiller (1991) observed that the yields on long term bonds tend to decrease when the yield curve is steep, fact that implies that long bonds tend to outperform when their yields are high. This is not compatible with the expectations hypothesis for the term structure of interest rates,

[2]See Chapter 3.

according to which the excess return on a long bond is equal to that of rolling over a position in a short maturity bond (plus possibly a constant term premium). Dai and Singleton (2001) show that a large class of models of the term structure of interest rates can generate this result if the market price of risk is affine instead of constant. In other words, the compensation for interest rate risk needs to be dependent on the level of rates. Regarding the research on the market price of credit risk, given the evidence we find on the time-varying risk premium, we conjecture that a realistic model of swap spreads should account for a time varying market price of risk.

The present work can be considered, in methodological terms, as an empirical investigation of the assumptions made by Longstaff and Mandell (2000) about the market price of spread risk. The paper by Liu, Longstaff and Mandell (2000), studies the market price of credit risk using swap spread data. The model is an extension of Duffie and Singleton (1997) with four Gaussian factors, and the parameters are estimated with the yields of Treasury bonds and swap rates. The authors allow the credit-spread process to depend on the level of interest rates. They also explicitly model the time-varying risk premium in interest rates by allowing the state variables to have mean reversion and long-term mean parameters under the risk-neutral measure that are different to those under the objective measure. The results they obtain indicate that most of the variation on swap spreads is due to variations of the liquidity of Treasury bonds, rather than changes in default risk.[3] Second, they find evidence of credit premia in swap spreads, that vary over time and that were negative for most of the 90's.

The chapter is organized as follows. In Section 2 we present the concept of the risk premium for spreads and the relation with the Expectation Hypothesis. Section 3 contains the empirical exercise. In it, we estimate the predictive relation between spread changes and the slope of the spread curve, and discuss the implications of the results. We perform an additional set of regressions to check for the robustness of the results, changing the length, frequency and the type of data. Section 4 concludes by summarizing the main implications of this work and presenting topics for future research.

[3]In our chapter, we will look at spreads vs. off-the-run bonds, so that they do not incorporate an on-the-run Treasury liquidity premium. This can help to disentangle the effect of changes in liquidity in the swap market from changes in the liquidity of particular Treasury securities.

4.2 The Risk Premium in Swap Spreads

In this section we develop an expression for the expected excess return ("risk premium") for swap spreads, by noting that the swap spread is the difference between a risky swap rate[4], and a riskless Treasury rate.

The risk premium for a Treasury bond is defined as its expected return over the riskfree rate for a certain holding period. Formally, suppose that the price at time t of a Treasury zero coupon bond with maturity τ is given by $p_t^\tau = \exp\{-\tau y_t^\tau\}$ where y_t^τ is the yield-to-maturity. The realized excess return over one period is

$$e_{t,t+1}^\tau = \ln(\frac{p_{t+1}^{\tau-1}}{p_t^\tau}) - r_t \tag{4.1}$$

where r_t is the one-period riskless rate. This can be re-written in terms of yields as

$$e_{t,t+1}^\tau = -(\tau-1)(y_{t+1}^{\tau-1} - y_t^\tau) + (y_t^\tau - r_t) \tag{4.2}$$

From the equation above, we can see that in order to model the risk premium we need a model of yield changes. A simple model relates these to the slope of the yield curve;

$$y_{t+1}^{\tau-1} - y_t^\tau = \alpha_\tau + \beta_\tau \frac{(y_t^\tau - r_t)}{\tau - 1} + \tilde{\epsilon}_t \tag{4.3}$$

where ϵ is a $NID(0, \sigma_\epsilon)$ error term. Taking conditional expectations and denoting the slope by s_t^τ;

$$E_t[e_{t,t+1}^\tau] = -(\tau-1)\alpha_\tau + (1-\beta_\tau)s_t^\tau \tag{4.4}$$

According to the EH, the continuously compounded τ-period zero-coupon yield equals the average of the current and expected future short interest rates, plus a maturity-specific constant:

$$y_t^\tau = \frac{1}{\tau}\sum_{h=0}^{\tau-1} E_t[r_{t+h}] + \kappa_\tau \tag{4.5}$$

This implies that the expected returns on bonds of different maturities can differ by constants which depend on maturity but not on time. In terms of 4.3, if the EH held it would be the case that $\beta_\tau = 1$. A β coefficient significantly different than 1 implies a time-varying risk premium, as the slope is time-varying itself. The *Pure* EH, according

[4]In market terminology, swap rates are also referred to as "Libor" rates.

to which the expected excess returns on long bonds over short bond are zero, would imply that $\alpha_\tau = 0$. The empirical evidence for US Treasuries (see Campbell and Shiller (1991)) is that the EH does not hold: the slope coefficient in the regression (4.3) is significantly smaller than 1, and is negative for long maturities. This implies that long-term bonds tend to outperform when the yield curve is very steep.

We now show how we can obtain an expression for the risk premium on a swap in a similar way than we did above for a Treasury. The idea is to interpret swap rates as yields on risky par bonds, and from those obtain the implied Libor zero-coupon rates. The price b_t^τ of a zero-coupon Libor bond will be given by

$$b_t^\tau = \exp\{-\tau z_t^\tau\} \tag{4.6}$$

where z_t^τ denotes its -risky- yield-to-maturity.

The realized excess return on a Libor zero-coupon bond over one period is defined as

$$h_{t,t+1}^\tau = -(\tau - 1)[z_{t+1}^{\tau-1} - z_t^\tau] + (z_t^\tau - l_t) \tag{4.7}$$

where l_t is the Libor rate for one period of time.

After having found expressions for the excess returns on Libor and Treasury zero coupon bonds, we can obtain the excess return on a spread position as their difference;

$$h_{t,t+1}^\tau - e_{t,t+1}^\tau = -(\tau - 1)[\delta_{t+1}^{\tau-1} - \delta_t^\tau] - (l_t - r_t) \tag{4.8}$$

where δ_t^τ is equal to the spread $z_t^\tau - y_t^\tau$.

As before, in order to model the risk premium on spreads, we need a model of spread changes. We can adapt the usual simple model so that the spread changes are related to the slope of the spread curve:

$$\delta_{t+1}^{\tau-1} - \delta_t^\tau = \alpha_\tau + \beta_\tau \frac{\delta_t^\tau - (l_t - r_t)}{\tau - 1} + \tilde{u}_t \tag{4.9}$$

where u_t is a $NID(0, \sigma_u)$ error term.

Taking conditional expectations and substituting into the spread excess return formula, we have that

$$E_t[h_{t,t+1}^\tau - e_{t,t+1}^\tau] = -(\tau - 1)\alpha_\tau + (1 - \beta_\tau)[\delta_t^\tau - (l_t - r_t)] \tag{4.10}$$

where the term $\delta_t^T - (l_t - r_t)$ is the slope of the spread curve. Since this is a time-varying element, as long as β is different from 1, the spread risk premium will be time-varying too.

In the following section we will test this hypothesis by proxying the zero Libor spreads by the par spreads, and the short-term spread by the one-year swap spread, which is the shortest maturity available in the spread curve. An advantage of this over using fitted zeros from the Libor curve is the fact that the par swap spreads are tradable, while the zeros are artificial, non-tradable entities.

4.3 Estimation and Empirical Results

4.3.1 Description of the data

The dataset we will analyze consists of a panel of swap spreads for USD and EUR (DEM before January 1, 1999). We measure swap spreads as the difference between the mid-market swap rates and the constant maturity fitted par Treasury yields for the corresponding maturity.[5] The main datasets we have used consist of weekly (Friday close) observations of the spreads for USD and EUR, ending on April 20, 2001. The sample for USD begins in January 7, 1994, and has 381 observations. The series for EUR begins on February 9, 1996, and has 272 observations.[6]

In Figures 4.1 and 4.2 we plot the evolution of EUR and USD swap spreads for the most liquid maturities (2, 5, 10 and 30 years). In terms of the dynamics of the spreads, the most important feature is the structural change around August 1998. This break is due to effects of the global financial crisis originated by the default of Russia on its sovereign debt. The period after August 1998 has been characterized by high levels of spreads, together with high volatilities. Given the differences in the properties of the data for the samples before and after August 1998 (see Tables 4.1 and 4.2, in the rest of the chapter we will present the empirical results for these two subsamples, as well

[5]We use "fitted" swap spreads for two reasons. First, quoted swap spreads are spreads relative to the yields of benchmark bonds, which incorporate a stochastic liquidity premium that we would like to avoid. Second, the maturity of the benchmark bond decreases over time, while the maturity of the swap is constant, creating a mismatch that can be significant. These distortions can be addressed by computing the swap spread relative to a constant maturity par fitted yield, computed from a yield curve fitted after removing the benchmark bonds.

[6]Although there is data available for previous dates, we take these data samples as appropiate for two reasons. First, we will compute the slope of the spread curve relative to the 1-year swap spread, the point of the spread curve with the shortest maturity, and we also want to have the data for the 30-year spread, which is the longest liquid point in the curve. The data for these maturities is more recent than for the rest. Also, the accuracy of the data is likely to decrease as we move back in time, given the relative youth of swap markets.

as for the full sample.

The main properties of the swap spread data can be summarized as follows.[7] First, EUR swap spread levels are generally lower than USD spreads. Second, the term structure of spreads is on average upward sloping. However, there are flat segments, like the 5-year to 10-year segment in the USD curve. Third, the term structure of volatilities is upward sloping, and the volatilities are generally higher for the US. Finally, we observe large differences for the period before and after August 1998, with much higher levels and volatilities in the sample after the crisis.

4.3.2 Properties of the slope of the swap spread curve

For each maturity, we compute the difference between the swap spread and the 1-year swap spread, the shortest maturity point in the swap spread curve.[8]

We observe in Figures 4.3 and 4.4 that the slope of the swap spread curve tends to be positive and increasing with maturity. However, the volatility of the changes in slope is substantial, and there are periods in which some of the slopes have become negative. Especially for USD, the slopes can experience large swings. By subsamples, the average slopes were significantly higher in the period post-August 1998.

From the previous evidence, it seems to be the case that the slopes increase in periods of spread widening (and tighten in periods of spread tightening, like the first half of 2001). We have performed, for each maturity, a regression of the (weekly) changes of the swap spreads on the contemporaneous (weekly) changes of the slope of the spread curve -as the difference with the 1-year swap spread. The results in Tables 4.5 and 4.6 indicate that there is a strong positive contemporaneous correlation between changes in spreads and changes in slopes.

4.3.3 Results of the predictive regression

In this section we perform the regression (4.9) for each spread. The regressions are performed for monthly changes in spreads, which is a reasonable trading horizon.[9] As in Campbell and Shiller (1991), we increase the number of observed changes by

[7] See Chapter 2 for a more detailed discussion.

[8] The 1-year swap involves the payment of a fixed rate against the 3-month Libor (3-month Euribor for EUR). Note that for EUR, the 1-year swap contract is *not* referenced on the 6-month Euribor, contrary to the other swap maturities.

[9] In the next subsection, we present the results for the regressions over weekly, non-overlapping periods.

taking 4-week overlapping changes. This increases the precision of the estimates, at the cost of inducing autocorrelation in the regression residuals. We account for that by computing Newey-West standard errors (see Campbell, Lo, and MacKinlay (1997)).

In Tables 4.7 and 4.9 we can see that in the predictive regressions, β_1 always appears significantly different from 1, typically at the 99% level of confidence. As we showed in the previous section, this is evidence of the presence of a time-varying risk premium in swap spreads. For EUR, we find that the slope coefficients are significantly different from 0, hence the slope of the spread curve helps in predicting future changes in spreads. The sign of the slope coefficient is negative, meaning a large current slope of the spread curve indicates a future tightening in the long spread. Regarding the constant β_0, we find them to be mostly significant, both for EUR and USD, rejecting the "pure" EH.

Next, we have performed the regressions for subsamples to deal with the impact of a structural break in August 1998. For EUR (see Table 4.11), the slope coefficients before and after 1998 are in the same ballpark, and significantly different from 0 (except the 2-year spread before 1998).[10] The β_0 are significant for the subsample after August 1998. Some of the R^2 of the predictive regressions are remarkably high, like the 28.5% of the regression for the 5-year before August 1998, or the 37.5% for the 15-year regression after August 1998. For USD, when we do the regressions by subsamples (see Table 4.12), we find that the constants and the slope coefficients become significantly different from 0 for the period before August 1998. After August 1998, the predictive power of the USD regressions decreases markedly, and the coefficients become insignificantly different from zero. On the other hand, the coefficients are still significantly different from 1.

The robustness of the forecasting regressions will be studied extensively in the next section. There are, at least, two sources of problems that have been presented in the literature. First, the regression is particularly sensitive to measurement errors in the long term swap spread. Since this appears both in the RHS with a positive sign and in the LHS with a negative sign, a measurement error will tend to produce the negative sign of the β_1 estimates we obtain.[11]

Second, there is the issue of "peso" problems, that can skew the estimates from fore-

[10] The 15-year slope seems to have an abnormal behaviour, with a much more negative slope coefficient and a larger estimated β_0, both things especially after 1998. This could be due to the lower liquidity of this swap contract.

[11] For a discussion of this problem, see Campbell and Shiller (1991).

casting regressions. By "peso" problems, we mean the impact of small probability, large events. This term comes from the foreign exchange literature, when in the 70's it was observed that even though the Mexican peso was pegged to the US Dollar, and that this had been the case for a decade, the Mexican interest rates were persistently higher than the US ones. Of course, this was as an insurance for a possible collapse of the peg, which occurred in due course.

The implication is then that a small probability event, not present in a long sample, may be biasing our results. We feel quite confident that our results are robust to such problems because they survive the presence of one of the most traumatic events in financial markets in recent history, the crisis of August 1998, when swap spreads widened dramatically after Russia defaulted on its debt and the world financial system was put in doubt. Our results, tend to be valid for the whole sample (including August 1998), and for subsamples pre and post-crisis.

4.3.4 Further evidence on the predictability of swap spreads

We have performed a number of additional regressions to show that our results are robust. First, we have re-done the regressions for non-overlapping weekly changes. Second, we have extended our data sample as much as feasible, by reducing the set of maturities studied and taking proxies where necessary. We have done this only for USD, as the available data for EUR is not long enough to include a whole business cycle. Finally, we have performed the regression for quoted (instead of fitted) swap spreads, where the spread is computed against the corresponding benchmark bond. In the previous chapters, we have argued in favour of the use of fitted swap spreads, because these display less distortions than benchmark spreads. In this chapter, however, we are more interested in the implications for trading spreads, which in practice are benchmark spreads.[12]

The regression results for weekly changes in spreads in Tables 4.13 and 4.14 confirm the results obtained with 4-week overlapping changes. In the case of EUR, the slope coefficients are significantly different from zero and negative, both for the full sample and for the two subsamples. Regarding β_0, this is significant for the full sample and for the subsample after August 1998. For the USD, the regression on weekly data brings

[12]However, the fitted swap spreads can be approximately traded by using off-the-run, less liquid Treasury securities, which can be close to the fitted levels. For instance, in EUR that could be achieved by trading French government bonds, instead of the -typically- German benchmarks.

a new result, that the slope coefficients are significantly different from 0 for the full sample, and not only for the period before August 1998, as we found for the monthly changes regression. The constant coefficients are typically significantly different from 0 for the period before 98.

Next, we have taken the longest available series for USD swap spreads (since 1987) and have estimated the regression with monthly (4-week overlapping) changes. The slope is relative to the 2-year swap spreads, for lack of available data on the 1-year. As we can see in Table 4.15, we find that, except for the 3-year spread, the slope coefficients are negative and significantly different from 0 and that the β_0 are significant. These results are important because the data sample extends over a full business cycle in the US.

Finally, we run the regression for swap spreads relative to benchmark yields. In this case, the slope is computed against the 2-year benchmark swap spread and the changes are monthly (4-week overlapping). The results for EUR in Table 4.16 confirm the original results for fitted spreads, that is the slope coefficient is significantly different from 0 and from 1 and that the sign is negative. For USD (see Table 4.17), we find the slope coefficients to be negative and significantly different from 1, but not significantly different from 0.

4.4 Some Results on Trading Swap Spreads

One of the main results of this Chapter is the empirical finding that swap spread changes are predictable, to a certain extent. One may ask whether the level of predictability is high enough for a trading strategy based on the predictive regressions to be profitable. In this section, we provide some evidence that this could be the case.

The summary statistic we will use to evaluate the different trading strategies is the Sharpe ratio, which we present in annual terms. In order to judge whether an active trading strategy is desirable, we provide the Sharpe ratios of "passive" strategies, i.e. to short the swap spread week after week. That strategy is structured by receiving in swaps and shorting a government bond of comparable maturity, and balances the risk of spreads widening (capital loss) with a sure income (swap spread minus Libor vs. GC)[13]. The statistics for the passive strategy can be found in Table 4.18 for EUR and

[13]We set the Libor-GC spread for EUR to 5bp, and that for USD at 20bp. These are average levels for the periods considered, according to market practitioners.

Table 4.19 for USD. For EUR, we can see how the Sharpe ratios of the passive strategy are fairly poor, generally being below 0.3, and better for shorter-maturity spreads. Many of the Sharpe ratios are not significantly different from 0, especially after the 5-year sector. For USD, the picture is a bit bleaker, with generally lower Sharpe ratios across the curve, and most of them indistinguishable from 0. This results give an indication of the realized excess return on swap positions over our sample period, showing that most of the risk premium in the long end of the spread curve has not been large enough to prevent losses for holders of long-dated swap spread tightening positions.

The active trading strategy is based in the prediction that the regressions in Tables 4.13 and 4.14 make for the sign of the change in swap spreads in EUR and USD for the coming week. The position is rebalanced every week according to the prediction, and the P&L computed. The first hint that this strategy may be profitable comes from the "hit ratios", the percentage of times our regression predicts correctly the change in the sign of the spread over the following week. These are presented in Table 4.20 and are generally between 50% and 60%.

The results from the active trading strategy, based on our predictive regression, are presented in Tables 4.21 and 4.22, for EUR and USD respectively. The Sharpe ratios are very large for all maturities, and especially in the front end. For EUR, the Sharpe ratio for the 5-year would be 1.003, and of 0.905 for the 30-year. The only differing maturity is the 10-year, with a low but acceptable 0.346. The period before August 1998 shows higher profits for the active strategy than after the crisis. For the USD, the results are also very good, although the Sharpe ratios are not as high as for EUR. The Sharpe ratios for the full USD sample are above 0.4, with the 5-year at 0.459 and the 10-year at 0.645. The worse maturity is the 30-year, with a Sharpe ratio of 0.265. As for EUR, the period before August 1998 shows a stronger outperformance of the active strategy.

Although this results are promising, there is a caveat: these results are "in-sample" results. A full out-of-sample analysis is left for future research, together with an additional study of possible predictability in swap spreads and, if that is the case, the reasons why such predictability persists over time.

4.5 Conclusions and Further Research

In this chapter we have found that there is important information embedded in the slope of the swap spread curve regarding the future evolution of spreads. The fact that spreads tend to tighten when the slope of the spread curve is high is not compatible with a version of the expectation hypothesis for spreads. A model of swap spreads should be able to generate a time-varying risk premium. These results complement the well-known results on a time-varying risk premium in goverment bonds.

The swap market has become one of the most important elements of the fixed income markets, linking the Treasury, the derivatives and the corporate bond markets. The high correlation between swap and corporate bond spreads, but the higher liquidity and transparency of swaps, make these a crucial element in the study of the market compensation for holding credit risk. The present chapter is a step in that direction. The empirical findings of this chapter should be incorporated into a formal model of swap spreads, and possibly of credit spreads.

A topic for further research would be to proceed along the lines of Dai and Singleton (2001) but for models of swap spreads instead of models of interest rates. The essential idea is to check whether models of the term structure of swap spreads are able to generate the predictive relation between the slope and the changes in spreads. This requires the estimation (or calibration) of the model and the generation of a large number of simulated paths for the spreads. Our conjecture is that the model will needs to incorporate a linear, nonconstant market price of spread risk in order to generate the result we have presented.

Full sample	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
EURSPR1	13.5	16.8	4.2	24.4	-0.4	2.3
EURSPR2	15.9	18.0	3.1	29.3	-0.7	2.7
EURSPR3	17.8	18.4	5.2	37.0	-0.1	2.0
EURSPR5	20.9	20.0	5.6	47.0	0.1	2.2
EURSPR7	24.4	20.4	6.9	53.2	0.1	3.5
EURSPR10	26.6	22.5	7.0	56.9	-0.1	4.6
EURSPR15	28.2	25.1	8.8	57.6	0.0	2.3
EURSPR30	31.0	31.4	3.8	73.6	-1.4	10.6
Feb96 to Jul98	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
EURSPR1	12.8	15.0	4.4	23.3	-1.1	3.5
EURSPR2	13.3	14.9	3.1	28.5	-1.5	8.6
EURSPR3	14.3	15.6	5.2	25.6	-0.4	1.2
EURSPR5	14.1	15.5	5.6	21.5	-0.2	1.7
EURSPR7	14.6	15.1	6.9	20.8	-0.3	2.6
EURSPR10	15.6	16.5	7.0	25.8	-0.5	2.8
EURSPR15	17.4	19.1	8.8	29.4	-0.1	2.4
EURSPR30	27.7	34.1	3.8	52.8	-2.1	15.1
Aug98 to Apr01	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
EURSPR1	14.1	18.4	4.2	24.4	0.0	1.7
EURSPR2	18.2	20.5	6.3	29.3	-0.3	0.6
EURSPR3	21.1	20.8	5.2	37.0	-0.1	1.9
EURSPR5	27.1	23.3	10.9	47.0	0.1	1.6
EURSPR7	33.3	24.3	16.5	53.2	0.2	2.5
EURSPR10	36.8	26.9	19.9	56.9	0.0	3.5
EURSPR15	38.1	29.7	20.8	57.6	0.0	1.4
EURSPR30	34.1	28.8	6.2	73.6	-0.1	1.5

Table 4.1: Statistics of EUR swap spreads (vs. constant maturity par fitted government yields). The std. deviation of changes is in bp per year. Sample: Feb-96 to Apr-01, weekly observations, T=272 obs. First subsample: 130 obs., second subsample: 142 obs.

Full sample	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
USDSPR1	32.6	24.0	6.9	66.6	0.1	1.6
USDSPR2	36.6	24.0	7.7	77.8	0.1	5.5
USDSPR3	39.3	24.1	9.9	87.5	0.0	5.6
USDSPR5	44.4	25.9	7.5	99.5	0.1	13.7
USDSPR7	46.5	26.4	5.7	107.3	0.1	14.2
USDSPR10	46.2	27.5	11.6	114.1	-0.2	7.9
USDSPR15	42.6	30.6	15.6	120.3	-0.4	5.1
USDSPR30	52.9	28.1	25.3	138.7	-0.7	6.0
Jan94 to Jul98	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
USDSPR1	24.0	19.7	6.9	43.5	-0.4	2.4
USDSPR2	25.2	22.8	7.7	53.2	0.1	11.2
USDSPR3	25.4	20.4	9.9	43.3	0.2	16.5
USDSPR5	28.6	19.2	7.5	40.6	2.0	58.6
USDSPR7	30.9	19.3	5.7	43.4	2.5	60.9
USDSPR10	31.6	17.2	11.6	42.9	2.3	40.3
USDSPR15	28.9	21.1	15.6	39.2	0.9	11.2
USDSPR30	37.9	15.3	25.3	55.9	-1.4	22.9
Aug98 to Dec00	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
USDSPR1	47.0	30.0	36.0	66.6	0.3	0.6
USDSPR2	55.7	26.0	34.7	77.8	0.0	-0.3
USDSPR3	62.7	29.2	38.5	87.5	-0.2	0.0
USDSPR5	70.9	34.4	45.1	99.5	-0.5	1.7
USDSPR7	72.9	35.4	46.9	107.3	-0.6	2.1
USDSPR10	70.7	39.2	38.8	114.1	-0.5	1.5
USDSPR15	65.7	42.2	31.2	120.3	-0.6	1.7
USDSPR30	78.1	41.7	39.0	138.7	-0.5	1.4

Table 4.2: Statistics of USD swap spreads (vs. constant maturity par fitted government yields). The std. deviation of changes is in bp per year. Sample: Jan-94 to Apr-01, weekly observations, T=381 obs. First subsample: 239 obs., second subsample: 142 obs.

Full sample	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
EURSLP2	2.4	16.3	-7.5	17.6	0.6	1.5
EURSLP3	4.4	20.8	-10.1	20.8	0.2	1.0
EURSLP5	7.4	23.7	-9.8	33.1	0.3	0.8
EURSLP7	10.9	24.3	-9.8	38.5	0.4	1.1
EURSLP10	13.2	25.9	-10.3	37.7	0.1	1.9
EURSLP15	14.7	29.5	-9.6	36.1	0.0	1.0
EURSLP30	17.5	34.6	-8.9	49.8	-1.0	8.2
Feb96 to Jul98	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
EURSLP2	0.6	14.6	-7.5	13.0	0.2	0.2
EURSLP3	1.5	19.7	-10.1	16.7	-0.1	1.2
EURSLP5	1.3	21.4	-9.8	15.2	0.3	1.6
EURSLP7	1.8	20.9	-9.8	12.7	0.2	1.6
EURSLP10	2.8	20.6	-10.3	13.2	-0.1	1.9
EURSLP15	4.6	24.3	-9.6	15.1	0.3	1.4
EURSLP30	14.9	36.6	-8.9	45.0	-1.8	13.2
Aug98 to Dec00	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
EURSLP2	4.0	17.7	-2.8	17.6	0.7	1.9
EURSLP3	6.9	21.8	-3.7	20.8	0.4	0.9
EURSLP5	12.9	25.7	-1.9	33.1	0.3	0.4
EURSLP7	19.1	27.1	2.0	38.5	0.4	0.6
EURSLP10	22.7	30.1	11.1	37.7	0.1	1.3
EURSLP15	24.0	33.7	10.7	36.1	-0.1	0.5
EURSLP30	19.9	32.8	-4.1	49.8	0.0	0.7

Table 4.3: Statistics of the slope of the EUR swap spread curve (spread vs. constant maturity par fitted government yields), relative to the 1-year swap spread. The std. deviation of changes is in bp per year. Sample: Feb-96 to Apr-01, weekly observations, T=272 obs. First subsample: 130 obs., second subsample: 142 obs.

Full sample	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
USDSLP2	4.0	22.7	-13.6	21.0	0.1	14.5
USDSLP3	6.8	28.4	-14.5	36.8	-0.3	6.9
USDSLP5	11.8	31.8	-16.9	51.1	-0.1	10.1
USDSLP7	14.0	33.1	-18.6	57.3	-0.1	9.4
USDSLP10	13.6	35.3	-12.8	62.1	-0.2	4.4
USDSLP15	10.0	38.4	-15.7	68.3	-0.1	3.3
USDSLP30	20.3	35.3	-5.7	84.8	-0.6	4.1
Jan94 to Jul98	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
USDSLP2	1.2	22.1	-13.6	11.2	0.2	25.8
USDSLP3	1.5	25.0	-14.5	15.5	-0.3	17.2
USDSLP5	4.6	27.5	-16.9	20.3	0.5	25.4
USDSLP7	6.9	28.1	-18.6	19.8	0.7	24.1
USDSLP10	7.6	27.2	-12.8	23.0	0.6	12.8
USDSLP15	4.9	29.5	-12.4	26.0	0.4	6.1
USDSLP30	14.0	26.5	-4.7	40.4	-0.8	8.3
Aug98 to Dec00	$\bar{x}$	$\sigma(\Delta x)$	$min(x)$	$max(x)$	$skew(\Delta x)$	$exc.kurt(\Delta x)$
USDSLP2	8.7	23.8	-9.7	21.0	-0.1	0.3
USDSLP3	15.7	33.5	-2.8	36.8	-0.4	0.4
USDSLP5	23.9	38.2	2.2	51.1	-0.5	1.5
USDSLP7	25.9	40.3	-3.6	57.3	-0.6	1.7
USDSLP10	23.7	46.1	-10.4	62.1	-0.5	0.8
USDSLP15	18.7	50.1	-15.7	68.3	-0.3	1.1
USDSLP30	31.1	46.7	-5.7	84.8	-0.6	1.4

Table 4.4: Statistics of the slope of the USD swap spread curve (spread vs. constant maturity par fitted government yields), relative to the 1-year swap spread. The std. deviation of changes is in bp per year. Sample: Jan-94 to Apr-01, weekly observations, T=381 obs. First subsample: 239 obs., second subsample: 142 obs.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	0.001	0.129	0.576	0.057	0.272
EURSLP3	-0.002	0.120	0.567	0.042	0.407
EURSLP5	0.018	0.118	0.603	0.036	0.513
EURSLP7	0.019	0.118	0.613	0.035	0.532
EURSLP10	0.011	0.122	0.667	0.034	0.589
EURSLP15	0.010	0.121	0.700	0.030	0.676
EURSLP30	-0.013	0.128	0.793	0.027	0.765
Feb96-Jul98	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	0.035	0.160	0.496	0.079	0.236
EURSLP3	0.018	0.144	0.525	0.053	0.437
EURSLP5	0.041	0.133	0.516	0.045	0.509
EURSLP7	0.039	0.133	0.504	0.046	0.484
EURSLP10	0.053	0.146	0.558	0.052	0.480
EURSLP15	0.074	0.145	0.619	0.043	0.618
EURSLP30	0.018	0.172	0.850	0.034	0.832
Aug98-Apr01	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	-0.038	0.200	0.626	0.082	0.295
EURSLP3	-0.027	0.189	0.598	0.063	0.393
EURSLP5	-0.015	0.188	0.658	0.053	0.524
EURSLP7	-0.015	0.188	0.673	0.050	0.561
EURSLP10	-0.036	0.189	0.713	0.046	0.636
EURSLP15	-0.050	0.188	0.739	0.041	0.704
EURSLP30	-0.009	0.188	0.728	0.041	0.688

Table 4.5: Results of the contemporaneous regression of weekly changes of swap spreads on weekly changes in the slope of the spread curve, for EUR. Sample: Feb-96 to Apr-01. All β_1 coefficients are significantly different from 0 at the 1% level. For all the β_0 coefficients we cannot reject the null hypothesis $\beta_0 = 0$.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.084	0.151	0.501	0.048	0.223
USDSLP3	0.092	0.138	0.501	0.035	0.348
USDSLP5	0.101	0.137	0.545	0.031	0.449
USDSLP7	0.102	0.135	0.556	0.030	0.483
USDSLP10	0.091	0.133	0.571	0.027	0.539
USDSLP15	0.086	0.137	0.622	0.026	0.608
USDSLP30	0.061	0.136	0.586	0.028	0.541
Jan94-Jul98	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.097	0.162	0.635	0.053	0.378
USDSLP3	0.089	0.141	0.522	0.041	0.409
USDSLP5	0.097	0.124	0.486	0.033	0.485
USDSLP7	0.107	0.122	0.490	0.031	0.508
USDSLP10	0.102	0.112	0.438	0.030	0.479
USDSLP15	0.102	0.127	0.532	0.031	0.554
USDSLP30	0.012	0.102	0.390	0.028	0.457
Aug98-Apr01	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.080	0.293	0.305	0.090	0.077
USDSLP3	0.102	0.286	0.481	0.062	0.300
USDSLP5	0.097	0.301	0.598	0.057	0.437
USDSLP7	0.086	0.299	0.609	0.054	0.479
USDSLP10	0.064	0.296	0.650	0.047	0.581
USDSLP15	0.053	0.295	0.675	0.043	0.641
USDSLP30	0.075	0.308	0.692	0.048	0.599

Table 4.6: Results of the contemporaneous regression of weekly changes of swap spreads on weekly changes in the slope of the spread curve, for USD. Sample: Feb-96 to Apr-01. All β_1 coefficients are significantly different from 0 at the 1% level. For all the β_0 coefficients we cannot reject the null hypothesis $\beta_0 = 0$.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	0.509*	0.301	-0.435***	0.151	0.063
EURSLP3	0.932**	0.394	-0.648***	0.144	0.112
EURSLP5	1.193***	0.402	-0.621***	0.168	0.078
EURSLP7	1.305***	0.450	-0.571***	0.197	0.054
EURSLP10	1.829***	0.692	-0.965***	0.335	0.070
EURSLP15	2.623***	0.693	-1.667***	0.434	0.104
EURSLP30	2.434***	0.831	-2.093***	0.545	0.093

Table 4.7: Results of the slope regression for EUR swap spreads. Sample: Feb-96 to Apr-01. Regression on monthly changes, computed as 4-week overlapping changes. Standard errors are Newey-West with 6 lags. All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

2-year	Statistic	p-value
F-statistic (1,266) dof	18	0
Durbin-Watson	0.858	0
Breusch-Pagan	10.313	0.001
Goldfeld-Quandt	0.853	0.817
Residual std. error	3.723	–
5-year	Statistic	p-value
F-statistic (1,266) dof	22.5	0
Durbin-Watson	0.811	0
Breusch-Pagan	3.609	0.057
Goldfeld-Quandt	1.854	0.0002
Residual std. error	3.999	–
10-year	Statistic	p-value
F-statistic (1,266) dof	20.11	0
Durbin-Watson	0.687	0
Breusch-Pagan	3.928	0.0474
Goldfeld-Quandt	1.293	0.0705
Residual std. error	5.17	–
30-year	Statistic	p-value
F-statistic (1,266) dof	27.13	0
Durbin-Watson	0.684	0
Breusch-Pagan	2.279	0.131
Goldfeld-Quandt	0.863	0.799
Residual std. error	7.234	–

Table 4.8: Regression diagnostics for the slope regression for EUR swap spreads. Sample: Feb-96 to Apr-01. Regression on monthly changes, computed as 4-week overlapping changes. The Durbin-Watson test has a null hypothesis of no autocorrelation in the errors, against an alternative of first order autocorrelation. The Breusch-Pagan is a Lagrange multiplier test for heteroskedasticity. Under the null hypothesis of homoskedasticity, it is distributed as a chi-square variable. The Goldfeld-Quandt is another test for heteroskedasticity. Under the null of homoskedasticity, the statistic follows an F-distribution.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.585	0.466	-0.097	0.133	0.004
USDSLP3	0.825**	0.401	-0.144	0.129	0.012
USDSLP5	1.197**	0.522	-0.230	0.189	0.016
USDSLP7	1.550**	0.632	-0.384	0.286	0.024
USDSLP10	1.754**	0.713	-0.682	0.459	0.037
USDSLP15	1.292**	0.629	-0.839	0.588	0.036
USDSLP30	1.584*	0.884	-0.912	0.742	0.029

Table 4.9: Results of the slope regression for USD swap spreads. Sample: Jan-94 to Apr-01. Regression on monthly changes, computed as 4-week overlapping changes. Standard errors are Newey-West with 6 lags. All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

2-year	Statistic	p-value
F-statistic (1,375) dof	1.686	0.195
Durbin-Watson	0.863	0
Breusch-Pagan	0.510	0.475
Goldfeld-Quandt	1.352	0.019
Residual std. error	5.155	–
5-year	Statistic	p-value
F-statistic (1,375) dof	5.952	0.0151
Durbin-Watson	0.823	0
Breusch-Pagan	9.001	0.0027
Goldfeld-Quandt	3.934	0
Residual std. error	5.45	–
10-year	Statistic	p-value
F-statistic (1,375) dof	14.43	0.000169
Durbin-Watson	0.567	0
Breusch-Pagan	42.46	0
Goldfeld-Quandt	8.484	0
Residual std. error	6.887	–
30-year	Statistic	p-value
F-statistic (1,375) dof	11.23	0
Durbin-Watson	0.534	0
Breusch-Pagan	51.6	0
Goldfeld-Quandt	9.411	0
Residual std. error	7.553	–

Table 4.10: Regression diagnostics for the slope regression for USD swap spreads. Sample: Jan-94 to Apr-01. Regression on monthly changes, computed as 4-week overlapping changes. The Durbin-Watson test has a null hypothesis of no autocorrelation in the errors, against an alternative of first order autocorrelation. The Breusch-Pagan is a Lagrange multiplier test for heteroskedasticity. Under the null hypothesis of homoskedasticity, it is distributed as a chi-square variable. The Goldfeld-Quandt is another test for heteroskedasticity. Under the null of homoskedasticity, the statistic follows an F-distribution.

Feb96-Jul98	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	0.072	0.411	-0.323	0.330	0.036
EURSLP3	0.115	0.330	-0.580**	0.251	0.115
EURSLP5	0.451	0.252	-1.540***	0.292	0.284
EURSLP7	0.600**	0.268	-1.743***	0.453	0.228
EURSLP10	0.842	0.610	-1.841**	0.842	0.114
EURSLP15	2.044***	0.682	-3.840***	1.193	0.236
EURSLP30	1.201	1.008	-2.098***	0.756	0.096
Aug98-Apr01	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	1.250**	0.632	-0.672***	0.199	0.118
EURSLP3	2.075***	0.689	-0.929***	0.201	0.164
EURSLP5	2.864***	1.016	-0.960***	0.258	0.120
EURSLP7	4.380***	1.415	-1.329***	0.359	0.132
EURSLP10	9.496***	2.020	-3.368***	0.672	0.246
EURSLP15	16.271***	1.976	-7.058***	0.825	0.375
EURSLP30	3.391***	1.316	-2.122***	0.812	0.093

Table 4.11: Results of the slope regression for EUR swap spreads, performed by subsamples. Regression on monthly changes, computed as 4-week overlapping changes. Standard errors are Newey-West with 6 lags. All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

Jan94-Jul98	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.342	0.502	-0.199	0.253	0.011
USDSLP3	0.499	0.374	-0.530**	0.226	0.063
USDSLP5	1.062*	0.560	-0.709**	0.358	0.075
USDSLP7	1.617**	0.805	-1.004**	0.514	0.087
USDSLP10	1.681***	0.540	-1.282***	0.364	0.114
USDSLP15	1.003**	0.489	-1.348***	0.374	0.068
USDSLP30	1.317***	0.429	-1.454***	0.485	0.124
Aug98-Apr01	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.750	1.123	-0.085	0.187	0.003
USDSLP3	1.817	1.405	-0.217	0.197	0.016
USDSLP5	3.062	1.998	-0.438	0.316	0.029
USDSLP7	3.152	2.034	-0.574	0.415	0.032
USDSLP10	2.742	2.044	-0.776	0.577	0.035
USDSLP15	1.746	1.708	-0.824	0.662	0.032
USDSLP30	2.892	2.083	-1.049	0.926	0.028

Table 4.12: Results of the slope regression for USD swap spreads, performed by subsamples. Regression on monthly changes, computed as 4-week overlapping changes. Standard errors are Newey-West with 6 lags. All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	0.234	0.169	-0.189***	0.066	0.030
EURSLP3	0.404**	0.189	-0.269***	0.074	0.047
EURSLP5	0.472**	0.217	-0.260**	0.086	0.033
EURSLP7	0.477**	0.238	-0.235**	0.092	0.024
EURSLP10	0.602**	0.284	-0.337***	0.127	0.025
EURSLP15	0.969***	0.346	-0.649***	0.193	0.040
EURSLP30	0.789**	0.391	-0.679***	0.236	0.030
Feb96-Jul98	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	0.031	0.184	-0.176**	0.086	0.032
EURSLP3	0.113	0.191	-0.374***	0.111	0.084
EURSLP5	0.194	0.187	-0.773***	0.196	0.112
EURSLP7	0.308	0.194	-0.975***	0.275	0.092
EURSLP10	0.336	0.230	-0.813**	0.340	0.044
EURSLP15	0.904**	0.310	-1.747***	0.478	0.097
EURSLP30	0.490	0.568	-0.709*	0.376	0.028
Aug98-Apr01	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP2	0.592*	0.330	-0.288**	0.114	0.044
EURSLP3	0.914**	0.380	-0.372***	0.124	0.061
EURSLP5	1.392***	0.498	-0.452***	0.149	0.062
EURSLP7	2.033***	0.654	-0.603***	0.186	0.070
EURSLP10	3.953***	0.997	-1.364***	0.330	0.109
EURSLP15	7.458***	1.346	-3.198***	0.560	0.190
EURSLP30	1.164**	0.555	-0.731**	0.318	0.037

Table 4.13: Results of the slope regression for EUR swap spreads, performed for weekly changes. Sample: Feb-96 to Apr-01. All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.260	0.198	-0.082*	0.048	0.008
USDSLP3	0.359	0.205	-0.102**	0.046	0.013
USDSLP5	0.473*	0.249	-0.126**	0.061	0.011
USDSLP7	0.525*	0.270	-0.159**	0.078	0.011
USDSLP10	0.515*	0.267	-0.220**	0.099	0.013
USDSLP15	0.365	0.253	-0.254**	0.123	0.011
USDSLP30	0.400	0.295	-0.236*	0.140	0.007
Jan94-Jul98	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.190	0.212	-0.217**	0.089	0.025
USDSLP3	0.279	0.188	-0.442***	0.109	0.066
USDSLP5	0.664***	0.219	-0.556***	0.131	0.072
USDSLP7	0.929***	0.267	-0.675***	0.167	0.066
USDSLP10	0.793***	0.237	-0.679***	0.174	0.062
USDSLP15	0.433*	0.222	-0.691***	0.224	0.039
USDSLP30	0.527*	0.226	-0.558***	0.174	0.042
Aug98-Apr01	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP2	0.294	0.471	-0.050	0.079	0.003
USDSLP3	0.890	0.619	-0.130	0.089	0.015
USDSLP5	1.267	0.870	-0.197	0.138	0.014
USDSLP7	1.064	0.853	-0.199	0.162	0.011
USDSLP10	0.825	0.753	-0.226	0.185	0.011
USDSLP15	0.498	0.635	-0.215	0.203	0.008
USDSLP30	0.743	0.801	-0.254	0.269	0.006

Table 4.14: Results of the slope regression for USD swap spreads, performed for weekly changes. Sample: Jan-94 to Apr-01. All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP3	0.383	0.385	-0.314*	0.170	0.011
USDSLP5	0.908**	0.389	-0.546***	0.156	0.036
USDSLP7	0.821*	0.440	-0.540***	0.196	0.026
USDSLP10	0.756*	0.462	-0.743**	0.300	0.028

Table 4.15: Results of the slope regression for USD swap spreads, for data between Jul-87 and Apr-01 (718 weekly obs.). The slope of the spread curve is relative to the 2-year swap spread. The regression is performed on monthly changes, computed as 4-week overlapping changes. Standard errors are Newey-West with 6 lags. All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
EURSLP5	2.170**	0.993	-0.954***	0.250	0.138
EURSLP10	2.652***	0.789	-1.231***	0.297	0.117
EURSLP30	1.639**	0.719	-2.531***	0.624	0.130

Table 4.16: Results of the slope regression for EUR with benchmark swap spreads, where the slope of the spread curve is relative to the 2-year swap spread. Sample: Feb-96 to Apr-01. Regression on monthly changes, computed as 4-week overlapping changes. Standard errors are Newey-West with 6 lags. All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

Full sample	β_0	$std.Error$	β_1	$std.Error$	R_c^2
USDSLP5	1.386**	0.612	-0.300	0.267	0.010
USDSLP10	1.447	1.004	-0.283	0.454	0.005
USDSLP30	1.555	1.260	-0.783	0.871	0.015

Table 4.17: Results of the slope regression for USD with benchmark swap spreads, where the slope of the spread curve is relative to the 2-year swap spread. Sample: Jan-94 to Apr-01. Regression on monthly changes, computed as 4-week overlapping changes. Standard errors are Newey-West with 6 lags.All β_1 coefficients are significantly different from 1 at the 5% level. The significance of the coefficients at 10%, 5% and 1% levels is denoted by $*$, $**$, and $***$ respectively.

		EURSPR1	EURSPR2	EURSPR3	EURSPR5	EURSPR7	EURSPR10	EURSPR15	EURSPR30
Full sample	Mean	0.166	0.195	0.248	0.075	0.053	0.110	0.034	1.213
	Std. Dev.	2.265	4.905	7.363	12.571	16.990	24.577	35.804	61.590
	Ann. Sharpe ratio	0.529	0.286	0.243	0.043	0.023	0.032	0.007	0.142
Before Aug98	Mean	0.082	0.159	0.287	0.125	0.132	-0.059	-0.596	4.250
	Std. Dev.	2.021	4.080	6.247	9.756	12.584	18.064	27.226	66.937
	Ann. Sharpe ratio	0.294	0.281	0.331	0.092	0.075	-0.023	-0.158	0.458
After Aug98	Mean	0.242	0.227	0.213	0.029	-0.018	0.264	0.608	-1.546
	Std. Dev.	2.471	5.565	8.269	14.704	20.224	29.333	42.204	56.394
	Ann. Sharpe ratio	0.708	0.294	0.186	0.014	-0.006	0.065	0.104	-0.198

Table 4.18: Statistics of the profit and loss from holding EUR swap spread risk.Holding period: 1 week. Sample: Feb-96 to Apr-01. Mean and std. deviation are in basis points per week. The Sharpe ratios are annualized.

		USDSPR1	USDSPR2	USDSPR3	USDSPR5	USDSPR7	USDSPR10	USDSPR15	USDSPR30
Full sample	Mean	0.165	0.149	0.087	-0.075	-0.227	-0.290	-0.491	0.179
	Std. Dev.	3.244	6.435	9.093	15.330	20.657	28.024	40.425	51.399
	Ann. Sharpe ratio	0.367	0.167	0.069	-0.035	-0.079	-0.075	-0.088	0.025
Before Aug98	Mean	-0.028	-0.055	-0.081	-0.205	-0.392	-0.498	-0.747	0.989
	Std. Dev.	2.652	6.090	7.702	11.344	15.080	17.573	27.814	27.948
	Ann. Sharpe ratio	-0.075	-0.065	-0.076	-0.130	-0.187	-0.204	-0.194	0.255
After Aug98	Mean	0.488	0.490	0.368	0.143	0.050	0.058	-0.062	-1.179
	Std. Dev.	4.037	6.983	11.064	20.378	27.652	39.896	55.605	76.062
	Ann. Sharpe ratio	0.872	0.506	0.240	0.051	0.013	0.010	-0.008	-0.112

Table 4.19: Statistics of the profit and loss from holding USD swap spread risk.Holding period: 1 week. Sample: Jan-94 to Apr-01. Mean and std. deviation are in basis points per week. The Sharpe ratios are annualized.

	2-year	3-year	5-year	7-year	10-year	15-year	30-year
EUR	0.546	0.524	0.594	0.557	0.542	0.542	0.539
USD	0.494	0.554	0.517	0.520	0.568	0.557	0.509

Table 4.20: Hit ratios for trading strategy based on the predictive regressions, for EUR and USD swap spreads on a 1-week horizon. This is the percentage of times that we predict correctly the sign of the change in swap spreads.

		2-year	3-year	5-year	7-year	10-year	15-year	30-year
Full sample	Mean	0.769	1.095	1.731	1.872	1.177	3.371	7.670
	Std. Dev.	4.848	7.285	12.451	16.886	24.549	35.645	61.121
	Ann. Sharpe ratio	1.144	1.084	1.003	0.800	0.346	0.682	0.905
Before Aug98	Mean	0.720	1.256	2.447	3.094	2.182	6.412	13.325
	Std. Dev.	4.018	6.125	9.442	12.195	17.930	26.461	65.726
	Ann. Sharpe ratio	1.291	1.479	1.869	1.830	0.878	1.747	1.462
After Aug98	Mean	0.814	0.949	1.081	0.762	0.264	0.608	2.532
	Std. Dev.	5.509	8.217	14.664	20.209	29.333	42.204	56.358
	Ann. Sharpe ratio	1.066	0.833	0.531	0.272	0.065	0.104	0.324

Table 4.21: Statistics of the profit and loss from trading EUR swap spreads according to the predictive regression. Holding period: 1 week. Sample: Feb-96 to Apr-01. Mean and std. deviation are in basis points per week. The Sharpe ratios are annualized.

		2-year	3-year	5-year	7-year	10-year	15-year	30-year
Full sample	Mean	0.425	1.270	0.973	1.110	2.490	4.140	1.886
	Std. Dev.	6.422	9.004	15.299	20.628	27.914	40.215	51.365
	Ann. Sharpe ratio	0.477	1.017	0.459	0.388	0.643	0.742	0.265
Before Aug98	Mean	0.318	1.321	1.370	1.911	2.834	4.428	2.279
	Std. Dev.	6.082	7.588	11.263	14.963	17.349	27.468	27.872
	Ann. Sharpe ratio	0.377	1.255	0.877	0.921	1.178	1.162	0.590
After Aug98	Mean	0.605	1.186	0.306	-0.232	1.914	3.657	1.227
	Std. Dev.	6.974	11.006	20.376	27.651	39.850	55.484	76.061
	Ann. Sharpe ratio	0.626	0.777	0.108	-0.060	0.346	0.475	0.116

Table 4.22: Statistics of the profit and loss from trading USD swap spreads according to the predictive regression. Holding period: 1 week. Sample: Jan-94 to Apr-01. Mean and std. deviation are in basis points per week. The Sharpe ratios are annualized.

FIGURES

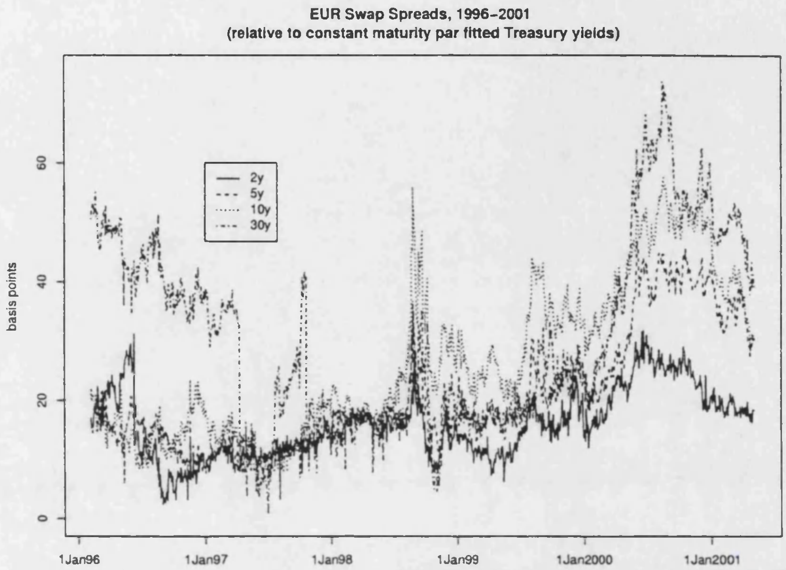

Figure 4.1: Evolution of the EUR fitted swap spreads (vs. constant maturity par fitted Treasury curve). Sample: Feb-96 to Apr-01.

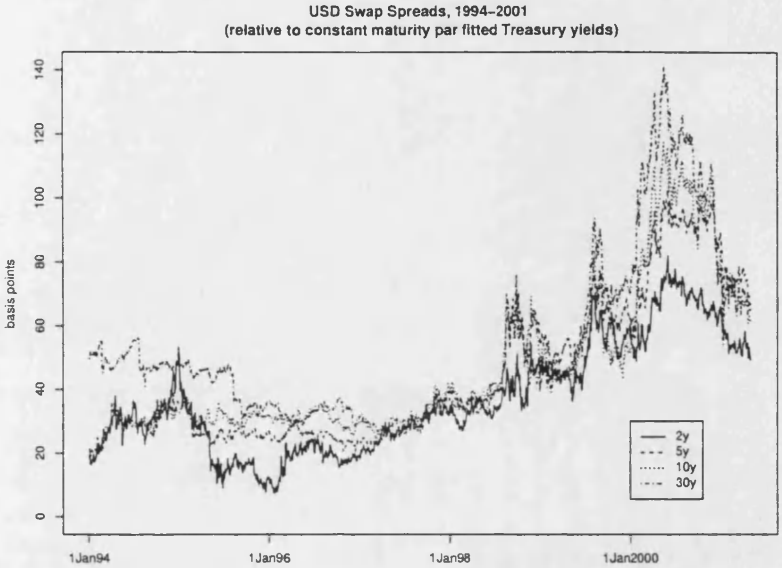

Figure 4.2: Evolution of the USD fitted swap spreads (vs. constant maturity par fitted Treasury curve). Sample: Jan-94 to Apr-01.

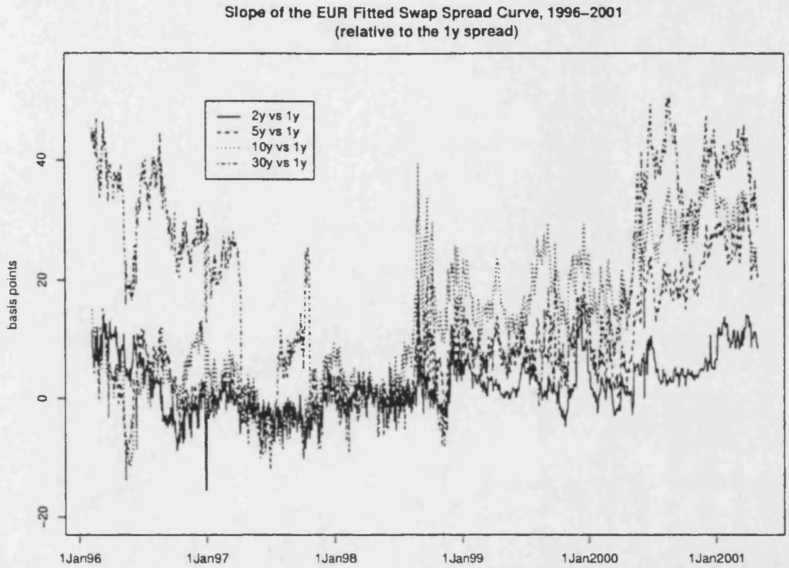

Figure 4.3: Evolution of the slope of the EUR fitted swap spread curve, relative to the 1-year swap spread. Sample: Feb-96 to Apr-01.

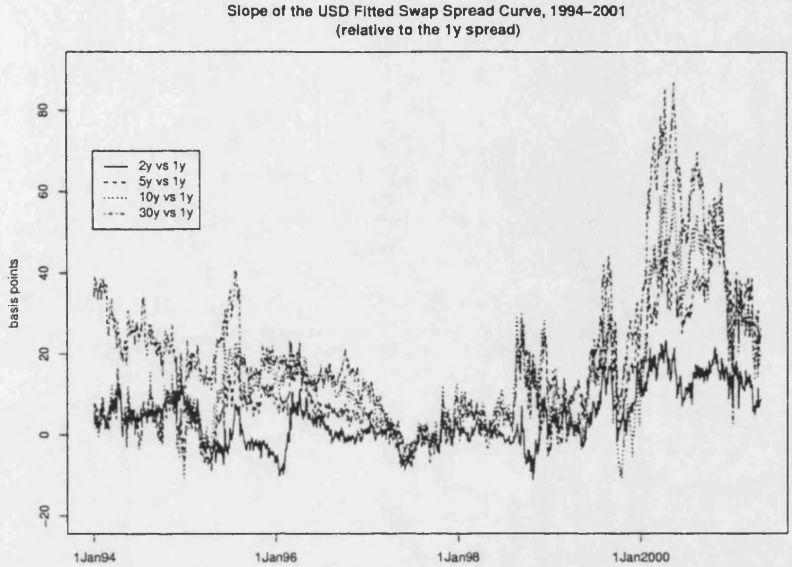

Figure 4.4: Evolution of the slope of the USD fitted swap spread curve, relative to the 1-year swap spread. Sample: Jan-94 to Apr-01.

Chapter 5

Asymptotics of the Quadratic Variation Estimator When the Sample Size Tends to Infinity

5.1 Introduction

In this Chapter we study the asymptotic properties of an estimator of the diffusion coefficient of a geometric Brownian motion based on its quadratic variation process (QV hereafter). The geometric Brownian motion is a continuous time process that is widely used to model the dynamics of financial asset prices, especially stocks. In reality, however, we are only able to observe realizations from a geometric Brownian motion in discrete time. The problem of estimating a continuous time process with discrete time observations has been studied in a number of references (see for instance Dacunha-Castelle and Florens-Zmirou (1986), Florens-Zmirou (1989), Kloeden et al. (1996), Pedersen (1995) or Yoshida (1992)). The most appropiate statistical technique under these conditions is that of maximum likelihood estimation (MLE). Dacunha-Castelle and Florens-Zmirou (1986) prove that for the case in which the observations are equidistant, the maximum likelihood estimator is consistent and asymptotically normal when the number of observations tends to infinity. Unfortunately, the joint density function can only be obtained for a limited number of diffusion processes.

The alternative methods to maximum likelihood estimation are generally based on a discretization, either of the likelihood function (see Bibby and Sorensen (1995) for a discussion) or of the process itself (see Chan et al. (1992)). The estimators obtained via discretization are typically shown to be consistent when $\Delta \to 0$, where Δ is the time between observations. However, very little is known regarding the properties of these estimators for the case in which the number of observations n tends to infinity, for a given Δ.

In this chapter, we will focus on the estimation of the *diffusion coefficient* with a set of discrete data. This particular problem has been addressed, among others, in Florens-Zmirou (1993) and in Genon-Catalot and Jacod (1993, 1994). A natural estimator of the diffusion parameter is that based on the quadratic variation of the process. More concretely, this estimator is constructed by taking a discrete approximation to the quadratic variation process corresponding to the continuous time process.

The QV estimator can be seen a general alternative to the maximum likelihood estimator for the diffusion coefficient whenever maximum likelihood is not feasible or practical. It is well-known that the QV estimator converges to the true diffusion value when the time between observations tends to 0, and it is the case that very few continuous time processes give place to known probability densities or transition functions: essentially, we can apply maximum likelihood only to the geometric Brownian motion, the Ornstein-Uhlenbeck process, and the Cox-Ingersol-Ross process (CIR). Outside this restricted set of processes, we need to resort to alternative estimation methods. Actually, even within the CIR framework, we may use the QV estimator, as a quick way to obtain an estimate for the diffusion coefficient, as we discuss below.

The QV estimator has been used, among others, by Delebecque and Quadrat (1975), Le Breton and Musiela (1984), and Fournié (1993). For instance, Fournié (1993, Ch.1) studies the estimation of a CIR process by maximum likelihood. Fournié (1993) shows that the limiting law of the process depends only on the long term mean and the ratio between the speed of mean reversion and the squared volatility. Therefore, it is not possible to estimate these two last parameters separately, hence the need for a "separate" estimator of the volatility. Fournié fixes the volatility parameter at the value given by the QV estimator. This is justified from the fact that it is well-known that the QV estimator converges almost surely to the true value of the diffusion as $\Delta \to 0$.

We will focus on the asymptotic behavior of the QV estimator for the case in which

$n \to \infty$, for a fixed Δ. The main result we obtain is that although the QV estimator is biased, it is a random variable that converges. Moreover, it is possible to isolate the bias and study its properties. The only crucial element in our proofs is a bound on the paths of the process. It is important to remark that, contrary to the proofs of classical results, we do not need any assumption on the stationarity or ergodicity of the process. For the sake of simplicity, we have specialized the discussion to the case of a geometric Brownian motion, a case for which we can use the Law of the iterated logarithm (LIL hereafter). The proof is divided in two cases depending on the relation between the drift and the volatility parameters. In the first case ($\mu < \sigma^2/2$), the LIL can be applied quite directly. For the second case, ($\mu > \sigma^2/2$), the proof is more elaborate.

Obviously, for a geometric Brownian motion, the maximum likelihood estimators can be obtained rather easily. Therefore, the interest of this chapter is more on the methodological side. In particular, we believe that the general idea and the method of proof can be extended to more complicated processes. Possibly, the most interesting extension is to Lévy processes, for which bounds on the paths of the process are available (see Wee and Kim (1995)). The proof of the geometric Brownian motion case is interesting in its own right and is a useful building block for other more difficult cases.

It could also be possible to study the properties of a QV estimator applied on the logarithm of the process. We have two possible reasons for not using the logarithmic transformation, at least in the first treatment of the topic. First, we want to make the proof as "general" as possible in order to see possible extensions to processes which are not part of the exponential family. In other words, applying a non-linear transformation to processes other than exponential can alter the time-series properties of the process in a way that makes difficult to adapt our proofs. Second, a practical reason to avoid the logarithmic transformation is when one believes that there is an additive effect in the errors of the data that can be eliminated by taking differences. Also, when the value of the variable is very small, the log transformation can magnify the impact of small errors. We leave the study of this topic for further research.

The results of this chapter are relevant in two ways. First, we think it is important to focus on the properties of the estimators of the parameters of continuous time processes when the number of observations increases, rather than for higher data frequencies. Typically, it is easier to obtain a longer data set than to obtain reliable high frequency data.[1] Also, in financial applications, high frequency data (e.g. intraday or even transaction data) is fragile, due to the presence of noise and because of market microstructure issues (e.g. bid-ask spreads).

Second, the QV estimator can be applied to a number of practical issues. For instance, there is evidence in financial markets of changes in regime in the volatility of prices and returns (see for instance Aggarwal et al. (1999) on emerging countries stock markets returns). The evolution of the QV estimator can provide with an indicator of when this changes in regime in volatility happen. Furthermore, since we have characterized the behaviour of the (random) bias, we can test whether a change in the estimates of volatility is due to estimation error or to the fact that the volatility has indeed changed. This can be done by using the empirical density of the estimation bias. Other possible applications are discussed in the concluding section.

The organization of the chapter is as follows. In section 2, we establish the basic setup and define the QV estimator. We show that this estimator can be decomposed in an element that converges to the true value when $\Delta \rightarrow 0$ plus a random variable that depends on n. Next, we study the properties of this variable, to which we will typically refer to as "bias," for the different cases given by the true values of the parameters, when the number of observations tends to infinity, for a fixed observation time step. The main result of this chapter, that the bias is a random variable that converges to a finite quantity, is proved in section 3. We then perform a simulation study to confirm the theoretical results and explore the finite sample properties of our estimator. In section 5, we present an application to the study of an actual data series, the value of the S&P 500 stock index. We conclude by summarizing the main results and discussing a number of possible extensions and applications.

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[1] It is true, however, that for samples spanning longer periods of time the issue of stationary becomes more delicate.

5.2 The Basic Setup

5.2.1 The QV estimator for a geometric Brownian motion

Assume that a certain variable -e.g. a stock price- follows a geometric Brownian motion given by the solution to the following stochastic differential equation (SDE):

$$\begin{cases} dX_t = & \mu X_t\, dt + \sigma X_t\, dW_t, \\ X_0 = & x_0 > 0,\ 0 \le t \le \infty. \end{cases}$$

where $\{W_t\}$ is a Wiener process defined in the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. The solution to the SDE is given by

$$X_t = x_0 \exp\{(\mu - \frac{\sigma^2}{2})t + \sigma W_t\},$$

where W_t is a $N(0,t)$ random variable.
Suppose that we have a sample of discrete time observations $X_{t_0}, X_{t_1}, \ldots, X_{t_n}$, where $0 = t_0 < t_1 < \cdots < t_n = t$. Let $\Delta_i \triangleq t_i - t_{i-1}$. We can specialize the solution to this case as:

$$X_{t_i} = X_{t_{i-1}} \exp\{(\mu - \frac{\sigma^2}{2})\Delta_i + \sigma(W_{t_i} - W_{t_{i-1}})\}, \tag{5.1}$$

where $W_{t_i} - W_{t_{i-1}}$ follows a $N(0, \Delta_i)$.
The *quadratic variation* estimator of σ^2 is motivated as follows. Rewrite the geometric Brownian motion as a stochastic integral equation:

$$X_t = X_0 + \int_0^t \mu X_s ds + \int_0^t \sigma X_s dW_s.$$

In the following section, we will show that the corresponding quadratic variation process is

$$\langle X \rangle_t = \sigma^2 \int_0^t X_s^2 ds,$$

hence

$$\sigma^2 = \frac{\langle X \rangle_t}{\int_0^t X_s^2 ds}.$$

In the equation above, we can approximate the numerator by a finite sum and the denominator by a Riemann sum. The QV estimator for the diffusion of a geometric Brownian motion based on its discrete-time observations is then:

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^{n}(X_{t_i} - X_{t_{i-1}})^2}{\sum_{i=1}^{n} X_{t_{i-1}}^2 \Delta_i}. \tag{5.2}$$

Notice that the quadratic variation is another way to connect the volatility with a sum of squared differences. We will motivate rigurously the concept of QV in the following section.

5.2.2 The concept of quadratic variation

The concept of QV stems from the Doob-Meyer decomposition of a submartingale.

Theorem 1 *Let the filtration* $\mathbf{F}$ *satisfy the usual conditions.*[2] *If the right–continuous submartingale* $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ *is of class DL*[3] *, then it admits the unique decomposition*

$$X_t = M_t + A_t, \quad 0 \le t < \infty,$$

where $M = \{M_t, \mathcal{F}_t; 0 \le t < \infty\}$ *is a right–continuous martingale and* $A = \{A_t, \mathcal{F}_t; 0 \le t < \infty\}$ *is an increasing process.*

We will concentrate in the class of continuous, square integrable martingales in a fixed filtered probability space $(\Omega, \mathcal{F}, \mathbf{F}, \mathbf{P})$.

Definition 1 *Let* $X = \{X_t, \mathcal{F}_t; 0 \le t < \infty\}$ *be a right-continuous martingale. We say that* X *is* square-integrable *if* $EX_t^2 < \infty$, $0 \le t < \infty$. *If in addition* $X_0 = 0$ *a.s., we say that* $X \in \mathcal{M}_2$*, and if* X *is also continuous,* $X \in \mathcal{M}_2^c$.

We can now state the definition of the quadratic variation of X:

Definition 2 *The* quadratic variation *of* $X \in \mathcal{M}_2$ *is the process* $\langle X \rangle_t \triangleq A_t$*, where* A_t *is the natural increasing process in the Doob-Meyer decomposition of* X^2*. In other*

[2]That is, $\mathcal{F}_t = \bigcap_{s \ge t} \mathcal{F}_s$ and $\mathcal{F}_0$ contains all the $\mathbf{P}$-negligible events in $\mathcal{F}$.

[3]Consider the class $\mathcal{S}_a$ of all stopping times τ of the filtration $\mathbf{F}$ which satisfy that $P(\tau \le a) = 1$ for a given finite number a. The right continuous process $X = \{X_t, \mathcal{F}_t; 0 \le t \le T\}$ is of class DL if the family $\{X_\tau\}_{\tau \in \mathcal{S}_a}$ is uniformly integrable, for any $0 < a < \infty$. The family $(X_i,\ i \in I)$ of integrable functions on a finite measure space $(\Omega, \mathcal{F}, \mathbf{P})$ is uniformly integrable if the following condition is true uniformly for i in I:

$$\lim_{c \to \infty} \int_{\{|X_i| \ge c\}} |X_i| d\mathbf{P} = 0.$$

words, $\langle X \rangle$ *is the* unique *(up to indistinguishability) adapted, natural increasing process for which* $\langle X \rangle_0 = 0$ *a.s. and* $X^2 - \langle X \rangle$ *is a martingale.*

Notice that for any $X \in \mathcal{M}_2$, the process $X = \{X_t^2, \mathcal{F}_t; 0 \leq t < \infty\}$ is a nonnegative submartingale (by Jensen's inequality), therefore it is of class DL and so it has a unique Doob-Meyer decomposition $X_t^2 = M_t + A_t;\ 0 \leq t < \infty$. If $X \in \mathcal{M}_2^c$, then $A = \langle X \rangle$ and M are continuous.

At this point, the concept of QV is not very operational. In the following, we will arrive to the QV from another direction which clarifies the idea behind it.

Definition 3 *Let* $X \in \mathcal{M}_2^c$, $X = \{X_t, \mathcal{F}_t; 0 \leq t < \infty\}$ *and fix a finite t. Let* $\Pi = \{t_0, t_1, \dots, t_n\}$ *be a partition of* $[0, t]$, *such that* $0 = t_0 \leq t_1 \leq \cdots \leq t_n = t$. *The* p-th variation *of* X *over the partition* Π *is defined as*

$$V_t^{(p)} = \sum_{i=1}^{n} |X_{t_i} - X_{t_{i-1}}|^p \tag{5.3}$$

Define the *mesh* of Π as $||\Pi|| = \max_{1 \leq i \leq n} |t_i - t_{i-1}|$. If $V_t^{(2)}(\Pi)$ converges (in some sense) as $||\Pi|| \to 0$, the limit is the quadratic variation of X on $[0, t]$, as previously defined.

Theorem 2 *Let* $X \in \mathcal{M}_2^c$. *For partitions* Π *of* $[0, t]$, *we have that*

$$lim_{||\Pi|| \to 0} V_t^{(2)} = \langle X \rangle_t \tag{5.4}$$

in probability. *That is,* $\forall \varepsilon > 0$ *and* $\forall \eta > 0$, *there exists a* $\delta > 0$ *such that* $||\Pi|| < \delta$ *implies that* $\mathrm{P}[|V_t^{(2)}(\Pi) - \langle X \rangle_t| > \varepsilon] < \eta$. *(see proof in Karatzas and Shreve (1991)).*

From the point of view of financial econometrics, we are interested in the QV of diffusions, which belong to the class of continuous square-integrable martingales. For the case of diffusions, it can be shown that all variations of order higher than 2 are zero, and that all variations of order lower than 2 are infinite. For instance, the QV of the Wiener process can be shown to be equal to t, while the *total variation* ($p = 1$) is infinite. In particular, this means that this kind of processes are nowhere differentiable. On the other hand, the quadratic variation is bounded, result that is used for instance to derive Ito's rule.

5.2.3 Decomposing the quadratic variation estimator

As we explained in the introduction, the objective of this chapter is to establish the asymptotic properties of the QV estimator in equation (5.2) when the number of observations n grows without bound, while the time between observations, Δ_i, remains fixed. We will assume throughout that $\Delta_i = \Delta$ (so that $t_i = i\Delta$).

Let

$$M_{t_i} = (X_{t_i} - X_{t_{i-1}})^2 - E[(X_{t_i} - X_{t_{i-1}})^2|\mathcal{F}_{t_{i-1}}].$$

It is immediate that

$$M_{t_i} = (X_{t_i} - X_{t_{i-1}})^2 - AX_{t_{i-1}}^2 \tag{5.5}$$

where

$$A = 1 - 2e^{\mu\Delta} + e^{(2\mu+\sigma^2)\Delta} \tag{5.6}$$

Then, the QV estimator can be decomposed in two terms:

$$\hat{\sigma}^2 = \frac{\sum_{i=1}^n M_{t_i} + A\sum_{i=1}^n X_{t_{i-1}}^2}{\Delta\sum_{i=1}^n X_{t_{i-1}}^2} = \frac{\sum_{i=1}^n M_{t_i}}{\Delta\sum_{i=1}^n X_{t_{i-1}}^2} + \frac{A}{\Delta} \tag{5.7}$$

The second term A/Δ is independent of n and converges to σ when $\Delta \to 0$. However, we are interested in what happens when $n \to \infty$, which is given by the first term. This term is the ratio of a sum of martingale differences and the process squared.

Proposition 1 *The process $\{M_{t_i}, \mathcal{F}_i\}$ is a martingale difference.*

Proof. This is proved directly using the definition of martingale difference.

Definition 4 *The process $\{M_j, \mathcal{F}\}$ is a discrete time* martingale difference *if for all j,*

(i) $E[|M_j|] < +\infty$,

(ii) $E[M_j|\mathcal{F}_{j-1}] = 0,\ a.s.$

Condition (ii) is trivially satisfied, given the definition of M_{t_i}. Condition (i) is also satisfied:

$$\begin{aligned} E[|M_{t_i}|] &\leq E[|(X_{t_i} - X_{t_{i-1}})^2| + |X_{t_{i-1}}^2||-A|] \\ &= E[(X_{t_i} - X_{t_{i-1}})^2] + |A|E[X_{t_{i-1}}^2] \\ &\leq x_0^2 \exp\{(2\mu + \sigma^2)\Delta(i-1)\}(2|A|) \end{aligned}$$

which is finite for all finite i. Therefore, $\{M_{t_i}\}$ is a martingale difference. $\square$

5.3 Asymptotics when $n \to \infty$

5.3.1 The Main Theorem

In order to study the limiting behaviour of the QV estimator when $n \to +\infty$, we will analyze the behavior of the term that depends on n, which we rewrite as

$$Z_n \equiv \frac{\sum_{i=1}^n M_{t_i}}{\sum_{i=1}^n X_{t_{i-1}}^2} = \frac{\sum_{i=1}^n X_{t_{i-1}}^2 Y_{t_i}}{\sum_{i=1}^n X_{t_{i-1}}^2} \tag{5.8}$$

where

$$Y_{t_i} = (e^{(\mu - \frac{\sigma^2}{2})\Delta + \sigma(W_{t_i} - W_{t_{i-1}})} - 1)^2 - A \tag{5.9}$$

Notice that since the $(W_{t_i} - W_{t_{i-1}})$ are distributed according to a $N(0, \Delta)$, the $\{Y_{t_i}\}$ are iid.

We now state the theorem that is the main result of this chapter:

Theorem 3 *Let $\{X_{t_i}\}_{i=1}^n$ be a set of discrete time observations of a geometric Brownian motion, where the time interval between them $\Delta = t_i - t_{i-1}$ is constant. Let the quadratic variation estimator of the diffusion coefficient be*

$$\hat{\sigma}_n^2 = \frac{\sum_{i=1}^n (X_{t_i} - X_{t_{i-1}})^2}{\Delta \sum_{i=1}^n X_{t_{i-1}}^2}.$$

Then, as the number of observations n grows without bound, it is the case that

$$\lim_{n \to +\infty} \hat{\sigma}_n^2 = (Z_\infty + A)/\Delta, \tag{5.10}$$

where $A = 1 - 2e^{\mu\Delta} + e^{(2\mu+\sigma^2)\Delta}$ and, crucially,

$$Z_\infty < \infty \; a.s. \tag{5.11}$$

This is the case both when $\mu < \sigma^2/2$ and when $\mu > \sigma^2/2$.

The most important element in the proof of the theorem is the Law of the Iterated Logarithm. In fact, the general result will then apply to other processes and families of processes as long as they satisfy some property similar to the Law of the iterated logarithm that we can use to establish bounds for the process paths. We finally note that we have not been able to prove the theorem for the case in which $\mu = \sigma^2/2$. We think this is just a technical problem, as the simulation evidence shows that Z_∞ for that case behaves very much as for the other cases.

5.3.2 Outline of the Proof

In this section we describe the general lines of the proof of the main theorem. The complete proof can be found in the Appendix. Essentially, we need to show that

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n} X_{t_{i-1}}^2 Y_{t_i}}{\sum_{i=1}^{n} X_{t_{i-1}}^2} < \infty \ a.s. \tag{5.12}$$

If we write down the denominator of equation (5.12) explicitely, we have

$$\lim_{n\to+\infty} \sum_{i=0}^{n} X_{t_i}^2 = x_0^2 \lim_{n\to+\infty} \sum_{i=0}^{n} \exp\{2(\mu - \frac{\sigma^2}{2})\Delta i + 2\sigma W_{t_i}\}.$$

When $\mu < \sigma^2/2$, it is intuitive that the first part of the exponent (negative) will dominate the random part. This is because the paths of the Wiener process are bounded. Such bounds are provided by the Law of the iterated logarithm (e.g. Karatzas and Shreve (1992), Theorem 9.23). The proof for the numerator, still assuming that $\mu < \sigma^2/2$ holds, goes along the same lines. If both numerator and denominator converge (and given that the denominator is strictly positive) it is the case that Z_∞ converges. The line of reasoning above does not help us in proving the theorem when $\mu > \sigma^2/2$, because the exponential term could grow faster than the random term. We need to take a different approach, beginning by writing the quotient

$$Z_n = \frac{\sum_{i=1}^{n} X_{t_{i-1}}^2 Y_{t_i}}{\sum_{i=1}^{n} X_{t_{i-1}}^2}$$

as a stochastic difference equation:

$$Z_n = A_n Z_{n-1} + B_n. \tag{5.13}$$

where

$$A_n = \frac{1}{1 + \frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2}}, \tag{5.14}$$

$$B_n = \frac{\frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2} Y_{t_n}}{1 + \frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2}}. \tag{5.15}$$

The objective is then to obtain a limiting relation of the type $Z_\infty = A_\infty Z_\infty + B_\infty$. In order to do that, we need to show that A_n and B_n converge and that there exists a limit for Z_n. Proving that both A_n and B_n converge when $n \to \infty$ is not very involved. Regarding Z_n we first prove that if it does converge, that will not depend on the starting point of the recursive process. This allows us to choose the most convenient starting value, which turns out to be 0. We then proceed to complete the proof, using standard arguments (definition of limit, results on Cesaro means and convergence in L^p spaces).

5.4 Simulation Study

The objective of this section is to confirm the previous theoretical results and ultimately, improve our understanding of the behaviour of the random variable Z_∞. Recall that $\hat{\sigma}^2_{QV,n} = \frac{1}{\Delta}(Z_n + A)$ and that

$$Z_n = \frac{\sum_{i=1}^n X_{t_{i-1}}^2 Y_{t_i}}{\sum_{i=1}^n X_{t_{i-1}}^2}.$$

The base case for the simulations[4] is given by the following parameters:

- Length of sample path: $n = 1{,}000$
- Number of simulated sample paths: $M = 100{,}000$
- Observation step: $\Delta = 1/252$

The process is assumed to start at $x_0 = 1$. Once we have the simulated path, we compute the value of Z_n. For each choice of parameters we will have M simulated paths of the process and values of Z_n.

[4]The simulations have been performed using the package GAUSS.

The first question that we investigate are the differences in the behavior of Z_n when we change the parameters for μ and σ. We start with a case for which the parameters satisfy $\mu < \sigma^2/2$ (Case (i)) and we alter them until we have $\mu > \sigma^2/2$ (Case(v)). In particular, we want to see what happens when. $\mu = \sigma^2/2$, a case that we have not been able to characterize theoretically. The parameter choices are given in Table 5.1, and the corresponding statistics for the simulated Z_n can be found in Table 5.2. We can see that as we move from $\mu < \sigma^2/2$ (Case (i)) to $\mu > \sigma^2/2$ (Case(v)), the typical value of Z_n remains very small and we find a lower mean and standard deviation, although the differences are very small. Note that the equality case (iii) behaves very similarly to the other cases.

We have obtained the –standardized– empirical densities for each case,[5] which we present in figure 5.1. In this figure, we have also plotted a N(0,1) density, for comparison purposes. We find clear evidence of the empirical density being positively skewed and leptokurtotic (i.e. it has "fat tails"). These two characteristics become less important as σ increases relative to μ.

Next, we check the impact of the number of simulated paths M on the simulation results. The results –for μ and σ from case (v)–, can be seen in Table 5.3. It can be seen that going from M = 5,000 to M = 100,000 does not produce important changes in the estimated behavior of Z_∞.

The sample size n is the key variable in our asymptotic study, so it is important to know how its choice has an impact on the results. We have performed a simulation study for the parameters of Case (v), ($\mu = 0.1$ and $\sigma = 0.25$), for $n = \{100, 500, 1000, 2000\}$. The results are presented in Table 5.4 and Figure 5.3. As we can see, once we have a certain number of observations n, the behaviour of Z_n becomes very similar, with a very small dispersion around zero. Even for $n = 100$, the results are very acceptable, with a very small and concentrated Z_n.

Finally, we have studied the impact on the results of the time between observations Δ. We have simulated the process with $\Delta = \{1/252, 1/52, 1/12\}$, for the parameters of Case (v), n = 1,000 and M = 100,000. The results of this exercise can be seen in Table 5.5 and Figure 5.4. Clearly, the use of monthly data produces a possibly unacceptable deterioration of the results. In particular, notice the large kurtosis (5.7610), which implies that the dispersion of the estimates is large.

[5]The empirical densities have been obtained using the package *R*, using a gaussian smoothing kernel.

The simulation results confirm the insights from the theoretical study in Section 3, namely that the QV estimator "bias" is small when n is large. Moreover, in practice, n does not need to be extremely large, nor Δ needs to be very small. For sample sizes of 1,000 daily or weekly datapoints, the QV estimator bias is negligible. This fact makes the QV estimator quite interesting for practical applications, which we study in the next section.

5.5 A Simple Application

The main result that we have derived, that the bias of the quadratic variation estimation is "small," can be used in different applications. In the following, we introduce a simple test for process misspecification based on the simulation of data.

For the case of a geometric Brownian motion, we know the maximum likelihood estimators for the drift and volatility parameters are given by

$$\hat{\alpha}_{ML} = \frac{1}{n\Delta}\sum_{i=1}^{n} x_i \tag{5.16}$$

$$\hat{\sigma}^2_{ML} = \frac{1}{n\Delta}\sum_{i=1}^{n}(x_i - \hat{\alpha}\Delta)^2 \tag{5.17}$$

where $x_i = \ln(X_{t_i}) - \ln(X_{t_{i-1}})$ and $\alpha \equiv \mu - \frac{\sigma^2}{2}$

The intution of the test is the following. Given a data sample, we first compute the maximum likelihood estimates of μ and σ. Then, we use these –consistent– estimates to obtain M simulated data samples of size n, with n = 1,000 and M = 100,000. Next, we compute the QV estimator on the simulated data and we compare it with the QV estimates obtained from the actual data. If the true process is a geometric Brownian motion, the two numbers should be close to each other. The QV estimator is not consistent as $n \to \infty$, but it is very close to being so, as the bias is small and can be estimated via simulations.

We will perform the test on the values of the S&P500 stock index, for which we have data between January 1990 and January 2001 at daily, weekly and monthly frequencies. The evolution of this index is depicted in Figure 5.5, and the results of our exercise can be seen in Table 5.6. For instance, if we take the daily frequency, and generate data assuming that the ML estimates are "true," and then we obtain the QV estimates on the simulated data, the results are that the mean estimate is $\bar{\sigma}_{QV}^{Sim} = 0.151422$, very close to the $\hat{\sigma}_{ML} = 0.151274$. However, if we then compute what is the QV estimate in the *actual* data, we find that $\hat{\sigma}_{QV} = 0.184199$, which is quite higher. The small standard deviation of the simulated QV estimates makes very implausible that the discrepancy is due to sampling error. The only exception is that of the monthly data case, where the ML and QV volatility estimates are relatively close to each other. However, this is the case in which the QV estimator is less likely to converge, due to the large discretization error.

In order to put this discrepancy in perspective, we can compare the ML and QV estimates we have with those obtained by discretizing the process, where

$$dX_t = \mu X_t \, dt + \sigma X_t \, dW_t$$

becomes

$$X_{t_i} - X_{t_{i-1}} = \mu X_{t_{i-1}} \Delta + \sigma X_{t_{i-1}} \tilde{v}_i, \tag{5.18}$$

with $\tilde{v}_i = W_{t_i} - W_{t_{i-1}} \sim N(0, \Delta)$, which we transform into

$$\frac{X_{t_i} - X_{t_{i-1}}}{X_{t_{i-1}}} = \mu\Delta + \sigma\tilde{v}$$

Then, we can obtain the "discretization" estimators as

$$\hat{\mu}_d = mean(\frac{X_{t_i} - X_{t_{i-1}}}{\Delta X_{t_{i-1}}})$$
$$\hat{\sigma}_d^2 = \frac{stdev(\hat{e})}{\sqrt{\Delta}}$$

where $\hat{e}$ are the fitted values of the quotient above.

The estimates we have obtained by discretizing the process are in Table 5.7. Notice that the estimates from discretizing the process are very close to the ML estimates in Table 5.6. This implies that the discretization bias is not very important, and hence, that the problems come from the model misspecification side.

The implication of our results is that the geometric Brownian motion is not a good statistical model for the S&P500. Estimating the volatility parameter by MLE seriously underestimates its magnitude. One indication that the geometric Brownian motion may not be adequate is the fact that the estimated volatility tends to decrease as the frequency of the data decreases. This could indicate the presence of mean reversion (see Lo and MacKinlay (1988) and (1989)). Another possibility could be that volatility changes over time. This issue, which has been studied by Aggarwal el al. (1999) among others, can be investigated in the QV framework as shown below.

Suppose now that the volatility is time-dependent, so that the true stock price process is given by

$$X_t = x_0 + \int_0^t \mu X_s \, ds + \int_0^t \sigma_s X_s \, dW_s$$

In this case, the quadratic variation process would be

$$\langle X \rangle_t = \int_0^t \sigma_s^2 X_s^2 \, ds$$

or equivalently

$$d\langle X \rangle_t = \sigma_t^2 X_t^2 \, dt$$

Therefore, $\sigma_t^2 \, dt = \frac{d\langle X \rangle_t}{X_t^2}$, which finally can be written as

$$\int_0^t \sigma_s^2 \, ds = \frac{d\langle X \rangle_s}{X_s^2} \tag{5.19}$$

The RHS of the expression above can be approximated by the Riemann sum

$$\sum_{i=1}^{n-1} \frac{(X_{t_{i+1}} - X_{t_i})^2}{X_{t_i}^2} \tag{5.20}$$

If the volatility of the series was indeed constant, the equation (5.19) would be linear in time, i.e equal to $\sigma^2 t$. To check whether that is the case, we can plot the approximation (5.20) against time. We have performed this exercise for the S&P500 data, which can be seen in Figure 5.6. It seems clear that the volatility of the S&P500 index has indeed changed over time. Possibly, there are two main regimes of volatility, one between 1990 and 1997, with a volatility lower than average (i.e. the function is concave) and then a high volatility regime (after 1997), with the function growing much faster. Inside the second regime, there are certain dates in which there is a short-lived acceleration of the volatility growth. This may correspond to price jumps, which are not compatible with the geometric Brownian motion as a statistical model. We leave for future research the extension of the results in this chapter to the case of a Lévy process, which can accommodate such behaviour.

5.6 Conclusions and Further Research

In this chapter we have shown that the quadratic variation estimator of the diffusion coefficient of a geometric Brownian motion converges asymptotically when the number of observations tends to infinity, for a fixed time between observations. The bias of the estimator is shown to be a random variable, which is finite a.s. and which we can characterize by simulations. The theoretical result, together with the insights from the simulation study, motivate the use of the quadratic variation in the statistics of diffusion processes.

The results in this chapter can be used and extended into a number of areas. One application is the study of changes of regime in volatility, by looking at the evolution of the "cumulative" quadratic variation. We can also extend the analysis to the case of the estimators for Lévy processes. The Lévy or stable family of processes seems to be better at representing actual time series, as it can accommodate the presence of jumps and generate the pervasive excess kurtosis ("fat tails") that is observed in many financial time series (e.g. stock returns, interest rate changes, etc). This extension seems feasible because the main element of the proofs for the geometric Brownian motion is the Law of the iterated logarithm, which provides bounds for the growth of the process. Equivalent bounds also exist for Lévy processes.

Another application can be derived from the fact that, for a given estimate of the volatility, we can invert the QV estimator to obtain an estimator for the drift parameter. This is interesting because it is a way to circumvent the typical obstacle of the estimation of the mean of a continuous time process. If we take the maximum likelihood estimator of the drift of a geometric Brownian motion, we see that it only depends on the first and the last observation in the sample. This makes all that happens in between irrelevant, which is intuitively unsatisfactory. Instead, the QV estimator of the volatility depends on the mean and the whole path of the process. For this idea to be feasible, we need an estimate of the volatility that is well-behaved when $\Delta > 0$, for instance the one presented in Pedersen (1995). We will investigate these issues in future research.

TABLES

Case	μ	σ	$\mu - \sigma^2/2$
(i)	0.05	0.50	-0.075
(ii)	0.06	0.45	-0.04125
(iii)	0.075	0.39	0
(iv)	0.09	0.30	0.045
(v)	0.10	0.25	0.0685

Table 5.1: Parameters used on the simulations; results in table 2.

	Mean	Std. Dev.	Minimum	Maximum	Skewness	Kurtosis
(i)	-1.67E-06	5.68E-05	-2.49E-04	3.99E-04	0.2425	3.5359
(ii)	-1.15E-06	4.43E-05	-2.04E-04	3.07E-04	0.1991	3.3634
(iii)	-7.98E-07	3.14E-05	-1.33E-04	2.08E-04	0.1873	3.2800
(iv)	-4.99E-07	1.77E-05	-7.20E-05	1.01E-04	0.1548	3.1484
(v)	-3.61E-07	1.21E-05	-5.25E-05	6.41E-05	0.1260	3.1083

Table 5.2: Simulation results for Z_n. Parameters as given by table 1, from $\mu < \sigma^2/2$, to $\mu > \sigma^2/2$. The sample paths are of length $n = 1{,}000$, with $\Delta = 1/252$. We have simulated $M = 100{,}000$ sample paths for each case.

	Mean	Std. Dev.	Minimum	Maximum	Skewness	Kurtosis
M = 5,000	-4.75E-07	1.22E-05	-3.77E-05	5.20E-05	0.1178	3.0829
M = 10,000	-4.76E-07	1.22E-05	-4.49E-05	4.62E-05	0.1463	3.0973
M = 20,000	-2.88E-07	1.20E-05	-4.88E-05	6.51E-05	0.1271	3.0665
M = 40,000	-2.71E-07	1.22E-05	-4.59E-05	5.41E-05	0.1198	3.0677

Table 5.3: Simulation results for Z_n. Parameters: $\mu = 0.1$, $\sigma = 0.25$ (Case (v)). The sample paths are of length $n = 1{,}000$, with $\Delta = 1/252$. We have simulated $M = \{5000, 10000, 20000, 40000\}$ sample paths.

	Mean	Std. Dev.	Minimum	Maximum	Skewness	Kurtosis
$n = 100$	-1.76E-07	3.55E-05	-1.29E-04	1.88E-04	0.3031	3.1683
$n = 500$	-4.03E-07	1.64E-05	-6.78E-05	9.70E-05	0.1572	3.0859
$n = 1{,}000$	-3.03E-07	1.21E-05	-5.74E-05	6.07E-05	0.1352	3.1179
$n = 2{,}000$	-2.81E-07	9.32E-06	-4.32E-05	5.61E-05	0.1302	3.2254

Table 5.4: Simulation results for Z_n. Parameters: $\mu = 0.1$ and $\sigma = 0.25$ (Case (v)). The sample paths are of length $n = \{100, 500, 1000, 2000\}$, with $\Delta = 1/252$. We have simulated $M = 100{,}000$ sample paths for each case.

	Mean	Std. Dev.	Minimum	Maximum	Skewness	Kurtosis
$\Delta = 1/252$	-2.81E-07	1.21E-05	-4.51E-05	6.15E-05	0.1480	3.0701
$\Delta = 1/52$	-6.39E-06	7.89E-05	-3.46E-04	6.05E-04	0.3314	3.8893
$\Delta = 1/12$	-1.22E-04	6.53E-04	-2.78E-03	6.56E-03	0.8396	5.7610

Table 5.5: Simulation results for Z_n. Parameters: $\mu = 0.1$ and $\sigma = 0.25$ (Case (v)). The sample paths are of length $n = 1{,}000$ with $\Delta = \{1/252, 1/52, 1/12\}$. We have simulated $M = 100{,}000$ sample paths for each case.

$\Delta = 1/252$	$\Delta = 1/52$	$\Delta = 1/12$
$\hat{\mu}_{ML} = 0.128817$ $\hat{\sigma}_{ML} = 0.151274$	$\hat{\mu}_{ML} = 0.129935$ $\hat{\sigma}_{ML} = 0.145278$	$\hat{\mu}_{ML} = 0.136603$ $\hat{\sigma}_{ML} = 0.136666$
QV: simulated data $\bar{\sigma}_{QV}^{Sim} = 0.151422$ stdev = 0.0035988	$\bar{\sigma}_{QV}^{Sim} = 0.146276$ stdev = 0.0051930	$\bar{\sigma}_{QV}^{Sim} = 0.141848$ stdev = 0.0106639
QV: actual data $\hat{\sigma}_{QV} = 0.184199$	$\hat{\sigma}_{QV} = 0.179981$	$\hat{\sigma}_{QV} = 0.158598$

Table 5.6: Hypothesis test for S&P500 stock index, 1990-2001.

$\Delta = 1/252$	$\Delta = 1/52$	$\Delta = 1/12$
$\hat{\mu}_d = 0.128887$	$\hat{\mu}_d = 0.130081$	$\hat{\mu}_d = 0.137302$
$\hat{\sigma}_d = 0.151245$	$\hat{\sigma}_d = 0.145472$	$\hat{\sigma}_d = 0.137080$

Table 5.7: Estimates for S&P500 stock index, 1990-2001, resulting from discretizing the geometric Brownian process.

Appendix: Proof of the Main Theorem

Case 1: $\mu < \frac{\sigma^2}{2}$

The result to be proved is that if $\mu < \frac{\sigma^2}{2}$, then

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n} X_{t_{i-1}}^2 Y_{t_i}}{\sum_{i=1}^{n} X_{t_{i-1}}^2} < \infty \; a.s. \tag{5.21}$$

The strategy will be to show that both numerator and denominator are finite almost surely. Then, using the property

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n} X_{t_{i-1}}^2 Y_{t_i}}{\sum_{i=1}^{n} X_{t_{i-1}}^2} = \frac{\lim_{n\to+\infty} \sum_{i=1}^{n} X_{t_{i-1}}^2 Y_{t_i}}{\lim_{n+\to\infty} \sum_{i=1}^{n} X_{t_{i-1}}^2}, \tag{5.22}$$

which is true if the denominator is nonzero a.s., we will have proved that the limit of the quotient is finite a.s. It is immediate to show that we can indeed take the quotient of limits: all the terms in the sum $\sum_{i=1}^{+\infty} X_{t_{i-1}}^2$ are positive, so we only need one term to be strictly positive for the infinite sum to be strictly positive as well. The typical term in the sum is $x_0^2 \exp\{2(\mu - \frac{\sigma^2}{2})\Delta(i-1) + 2\sigma W_{t_{i-1}}\}$. For $i = 1$, this is $x_0^2 > 0$, therefore the infinite sum is strictly positive.

The denominator

We will begin with the denominator of (5.22), proving that $\lim_{n\to+\infty} \sum_{i=1}^{n} X_{t_{i-1}}^2 < +\infty$ a.s. For notational simplicity we study the equivalent expression

$$\lim_{n\to+\infty} \sum_{i=0}^{n} X_{t_i}^2 = x_0^2 \lim_{n\to+\infty} \sum_{i=0}^{n} \exp\{2(\mu - \frac{\sigma^2}{2})\Delta i + 2\sigma W_{t_i}\}.$$

The basis of the proof is the intuition that the first part of the exponent (which is negative), dominates the random part. To prove this, we will use the Law of the iterated logarithm (e.g. Karatzas and Shreve (1992), Theorem 9.23).

Theorem 4 *Let W_t be a Wiener process on $(\Omega, \mathcal{F}, \mathbf{P})$. For almost all $\omega \in \Omega$,*

$$\limsup_{t\downarrow 0} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = 1, \qquad \liminf_{t\downarrow 0} \frac{W_t(\omega)}{\sqrt{2t \log\log(1/t)}} = -1,$$
$$\limsup_{t\to+\infty} \frac{W_t(\omega)}{\sqrt{2t \log\log t}} = 1, \qquad \liminf_{t\to+\infty} \frac{W_t(\omega)}{\sqrt{2t \log\log t}} = -1.$$

This is known as the Law of the iterated logarithm (LIL).

Notice that

$$-1 \leq \liminf_{t\to+\infty} \frac{W_t}{\sqrt{2t\log\log t}} \leq \limsup_{t\to+\infty} \frac{W_t}{\sqrt{2t\log\log t}} \leq 1, \ a.s.$$

is, by definition, equivalent to

$$-1 \leq \lim_{T_0\to+\infty} \inf_{t\geq T_0} \frac{W_t}{\sqrt{2t\log\log t}} \leq \lim_{T_0\to+\infty} \sup_{t\geq T_0} \frac{W_t}{\sqrt{2t\log\log t}} \leq 1, \ a.s.$$

By the definition of the limit we know that for all $\varepsilon > 0$, there exists a T_0^* large enough such that for all $T_0 \geq T_0^*$, the following is true:

$$1-\varepsilon \leq \inf_{t\geq T_0} \frac{W_t}{\sqrt{2t\log\log t}} \leq \sup_{t\geq T_0} \frac{W_t}{\sqrt{2t\log\log t}} \leq 1+\varepsilon, \ a.s.$$

Then, whenever $t \geq T_0^*$, we have that

$$1-\varepsilon \leq \frac{W_t}{\sqrt{2t\log\log t}} \leq 1+\varepsilon, \ a.s.$$

or, finally,

$$-(1-\varepsilon)\sqrt{2t\log\log t} \leq W_t \leq (1+\varepsilon)\sqrt{2t\log\log t}, \quad \forall t \geq T_0^*.$$

Let $a = 2(\mu - \frac{\sigma^2}{2})\Delta$ (note that a is negative by assumption). For i_0 such that $i_0\Delta \geq T_0^*$,

$$\lim_{n\to+\infty} \sum_{i\geq i_0}^{n} \exp\{ai + 2\sigma W_{t_i}\} \leq \lim_{n\to+\infty} \sum_{i\geq i_0}^{n} \exp\{ai + 2\sigma(1+\varepsilon)\sqrt{2t_i\log\log t_i}\}, \ a.s.$$

The next step is to prove that there exists a T_1^* (with $i_1\Delta = T_1^*$) such that $\forall t \geq T_1^*$, it is the case that

$$\lim_{n\to+\infty} \sum_{i\geq i_1}^{n} \exp\{ai + 2\sigma(1+\varepsilon)\sqrt{2t_i\log\log t_i}\}$$

$$\leq \lim_{n+\to\infty} \sum_{i\geq i_1}^{n} \exp\{ai + 2\sigma(1+\varepsilon)\sqrt{2}(i\Delta)^{5/8}\}, \ a.s. \quad (5.23)$$

that is, there exists T such that $\forall t \geq T$, $\sqrt{t \log\log t} \leq t^{5/8}$. Proving that $\sqrt{t \log\log t} \leq t^{5/8}$ is equivalent to proving that $\log t \leq \exp\{t^{1/4}\}$. We know that $1 + t^{1/4} \leq \exp\{t^{1/4}\}$, by the Taylor expansion of e^x, i.e. $e^x = 1 + x + x/2 + x/3! + \ldots$ Therefore, we only need to prove that $\log t \leq 1 + t^{1/4}$. We can split the domain of the functions in two regions:

- $1 \leq t \leq e$: $\log t \leq 1$, and since $t^{1/4} > 0$, then $\log t \leq 1 + t^{1/4}$.
- $t \geq e$: At $t = e$, the inequality clearly holds. Now look at the growth of the functions, i.e. the first derivative: $(\log t)' = t^{-1}$ and $(t^{1/4})' = t^{-3/4}$. Notice that $t^{-1} < t^{-3/4}$, so that $t^{1/4}$ is not only larger at $t = e$, but it grows faster.

Now, we can show that there exists a T_2^* (with $i_2\Delta = T_2^*$) such that for all i with $i \geq i_2$, it is the case that $2\sigma(1+\varepsilon)\sqrt{2}\Delta^{5/8}i^{5/8} \leq -\frac{a}{2}i$, so that

$$\lim_{n\to+\infty}\sum_{i\geq i_2}^{n} \exp\{(2\sigma(1+\varepsilon)\sqrt{2}\Delta^{5/8})i^{5/8}\} \leq \lim_{n+\to\infty}\sum_{i\geq i_2}^{n} \exp\{-\frac{a}{2}i\}.$$

To show this, we need to prove that $ci^{5/8} \leq di$ for all $i \geq i_2$, for constants $c, d > 0$, $c = 2\sigma(1+\varepsilon)\sqrt{2}\Delta^{5/8}$ and $d = -a/2$. This is equivalent to $i^{5/8} \leq \frac{d}{c}i$. Both the LHS and the RHS are 0 at $i = 0$. The first derivative of the LHS is $(5/8)i^{-3/8}$ and decreases with i, while the derivative of the RHS is $d/c > 0$, a constant. This proves our conjecture. Furthermore, we can get i_2 in closed form:

$$i_2 = (\frac{2\sigma(1+\varepsilon)\sqrt{2}\Delta^{5/8}}{-a/2})^{8/3}.$$

Finally, let $i_3 = max\{i_0, i_1, i_2\}$. Since we will take limits for n, it is important to note that the i_0, i_1, i_2 and so i_3, do not depend on n. Then,

$$\lim_{n\to+\infty}\sum_{i\geq i_3}^{n} \exp\{ai + 2\sigma W_{t_i}\} \leq \lim_{n\to+\infty}\sum_{i\geq i_3}^{n} \exp\{ai - \frac{ai}{2}\} = \lim_{n\to+\infty}\sum_{i\geq i_3}^{n} \exp\{\frac{ai}{2}\},$$

Since $a/2 < 0$, this implies that $\exp\{a/2\} < 1$, and since the exponential in the denominator grows faster than the numerator,

$$\lim_{n\to+\infty}\sum_{i\geq i_3}^{n} (\exp\{a/2\})^i < +\infty, \ a.s.$$

This is only part of the original sum:

$$\lim_{n\to+\infty}\sum_{i\geq i_0}^{n}\exp\{ai+2\sigma W_{t_i}\}=\sum_{i=0}^{i_3-1}\exp\{ai+2\sigma W_{t_i}\}+\lim_{n\to+\infty}\sum_{i\geq i_3}^{n}\exp\{ai+2\sigma W_{t_i}\}$$

The first sum is a finite sum of finite quantities, therefore it is finite. Since we have just proved that the second sum is finite a.s., we have that the denominator converges.

The numerator

Next, we turn to the numerator of (5.22). We will show that $\lim_{n\to\infty}\sum_{i=1}^{n}X_{t_{i-1}}^2 Y_{t_i} < +\infty$ a.s. We can rewrite this sum as:

$$\begin{aligned}\sum_{i=1}^{n}X_{t_{i-1}}^2 Y_{t_i} &= \sum_{i=1}^{n}M_{t_i}=\sum_{i=1}^{n}[(X_{t_i}-X_{t_{i-1}})^2-AX_{t_{i-1}}^2]\\ &=\sum_{i=1}^{n}[X_{t_i}^2-2X_{t_{i-1}}X_{t_i}+(1-A)X_{t_{i-1}}^2]\end{aligned}$$

hence,

$$\begin{aligned}\lim_{n\to\infty}\sum_{i=1}^{n}X_{t_{i-1}}^2 Y_{t_i} &= x_0^2[\lim_{n\to+\infty}\sum_{i=1}^{n}\exp\{2(\mu-\frac{\sigma^2}{2})i\Delta+2\sigma W_{t_i})\}\\ &-2\lim_{n\to+\infty}\sum_{i=1}^{n}\exp\{(\mu-\frac{\sigma^2}{2})\Delta(2i-1)+\sigma(W_{t_i}+W_{t_{i-1}})\}\\ &+(1-A)\lim_{n\to+\infty}\sum_{i=1}^{n}\exp\{2(\mu-\frac{\sigma^2}{2})\Delta(i-1)+2\sigma W_{t_{i-1}}\}]\end{aligned}\tag{5.24}$$

We have proved in the previous section that the first and the last term are finite a.s., so we only need to prove that for the second term. The proof will be done along the same lines as for conjecture 1. That is, by the law of the iterated logarithm and the definition of limit, for all $\varepsilon > 0$, there exists a T_0^* such that for all $t \geq T_0^*$, we have that

$$(1-\varepsilon)\sqrt{2t\log\log t}\leq W_t\leq(1+\varepsilon)\sqrt{2t\log\log t}\ a.s.$$

Let i_0 be such that $(i_0-1)\Delta\geq T_0^*$, then for all $i\geq i_0$,

$$\begin{aligned}(1-\varepsilon)\sqrt{2t_i\log\log t_i}&\leq W_{t_i}\leq(1+\varepsilon)\sqrt{2t_i\log\log t_i}\ a.s.\\ (1-\varepsilon)\sqrt{2t_{i-1}\log\log t_{i-1}}&\leq W_t\leq(1+\varepsilon)\sqrt{2t_{i-1}\log\log t_{i-1}}\ a.s.\end{aligned}$$

Adding the two relations, we have that

$$W_{t_i} + W_{t_{i-1}} \le (\sqrt{2t_i \log\log t_i} + \sqrt{2t_{i-1} \log\log t_{i-1}}) \ a.s. \qquad (5.25)$$

Let $a \equiv \Delta(\mu - \sigma/2) < 0$. Then

$$\begin{aligned}&\lim_{n\to+\infty} \sum_{i\ge i_0}^{n} exp\{a(2i-1) + \sigma(W_{t_i} + W_{t_{i-1}})\} \\ &\le \lim_{n\to+\infty} \sum_{i\ge i_0}^{n} exp\{a(2i-1)+\sigma(1+\varepsilon)(\sqrt{2t_i \log\log t_i}+\sqrt{2t_{i-1} \log\log t_{i-1}})\} \ a.s.\end{aligned} \qquad (5.26)$$

Next, we need to prove that there exists a T_1^* such that $\forall t_{i-1} > T_1^*$ we have that

$$\begin{aligned}&\lim_{n\to+\infty} \sum_{i\ge i_1}^{n} exp\{a(2i-1) + \sigma(1+\varepsilon)(\sqrt{2t_i \log\log t_i} + \sqrt{2t_{i-1} \log\log t_{i-1}})\} \\ &\qquad \le \lim_{n\to+\infty} \sum_{i\ge i_1}^{n} exp\{a(2i-1) + \sigma(1+\varepsilon)\sqrt{2}(t_i^{5/8} + t_{i-1}^{5/8})\} \ a.s. \qquad (5.27)\end{aligned}$$

This relation has been already proved in the previous case. Now, rewrite $t_i^{5/8} + t_{i-1}^{5/8}$ as $\Delta^{5/8}(i^{5/8} + (i-1)^{5/8})$. The next step is to prove that there exists a T_2^* such that for all $t_{i-1} \le T_2^*$ we have that

$$(\sigma(1+\varepsilon)\sqrt{2}\Delta^{5/8})(i^{5/8} + (i-1)^{5/8}) \le -a(i-1)$$

which again is proved in the same way as in the first case. Finally, letting $i_3 = \max\{i_0, i_1, i_2\}$ it is the case that

$$\begin{aligned}&\lim_{n\to+\infty} \sum_{i\ge i_3}^{n} \exp\{a(2i-1) + \sigma(W_{t_i} + W_{t_{i-1}})\} \\ &\qquad \le \lim_{n\to+\infty} \sum_{i\ge i_3}^{n} \exp\{a(2i-1) - a(i-1)\} \\ &\qquad \le \lim_{n\to+\infty} \sum_{i\ge i_3}^{n} \exp\{ai\}\end{aligned}$$

Since $a < 0$ and $e^a < 1$, we have

$$\lim_{n\to+\infty} \sum_{i\geq i_3}^{n} e^{ai} < +\infty, \; a.s.$$

As before, the sum up to i_3 is a finite sum of finite terms, so the whole sum is finite and our conjecture is proved. Since both the numerator and the denominator in (5.22) converge, we have proved that the bias converges for $\mu < \frac{\sigma^2}{2}$.

Case 2: $\mu > \frac{\sigma^2}{2}$

We now want to prove that for $\mu > \frac{\sigma^2}{2}$ it is the case that

$$\lim_{n\to+\infty} \frac{\sum_{i=1}^{n} X_{t_{i-1}}^2 Y_{t_i}}{\sum_{i=1}^{n} X_{t_{i-1}}^2} < \infty \; a.s.$$

that is, the limit of the quotient when $n \to \infty$ converges almost surely. Since $\mu - \frac{\sigma^2}{2}$, is strictly positive by assumption, we cannot use directly the Law of the iterated logarithm as we did for the Case 1.

Expressing Z_n as a stochastic difference equation

We will study the limiting behavior of our quotient, which we will refer to as Z_n, by rewriting it as a stochastic difference equation:

$$Z_n = \frac{\sum_{i=1}^{n} X_{t_{i-1}}^2 Y_{t_i}}{\sum_{i=1}^{n} X_{t_{i-1}}^2} = \frac{X_{t_{n-1}}^2 Y_{t_n} + \sum_{i=1}^{n-1} X_{t_{i-1}}^2 Y_{t_i}}{X_{t_{n-1}}^2 + \sum_{i=1}^{n-1} X_{t_{i-1}}^2} = \frac{\frac{X_{t_{n-1}}^2 Y_{t_n}}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2} + Z_{n-1}}{\frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2} + 1},$$

therefore

$$Z_n = \frac{1}{1 + \frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2}} Z_{n-1} + \frac{\frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2} Y_{t_n}}{1 + \frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2}} \tag{5.28}$$

Letting

$$A_n = \frac{1}{1 + \frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2}}, \tag{5.29}$$

$$B_n = \frac{\frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2} Y_{t_n}}{1 + \frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2}}. \tag{5.30}$$

we obtain the expression

$$Z_n = A_n Z_{n-1} + B_n. \tag{5.31}$$

Our objective is to obtain a limiting relation of the type $Z_\infty = A_\infty Z_\infty + B_\infty$. In order to do that, we need to show that A_n and B_n converge and that there exists a limit for Z_n. We start by looking at the term which is common to both A_n and B_n:

$$\begin{aligned}
\frac{X_{t_{n-1}}^2}{\sum_{i=1}^{n-1} X_{t_{i-1}}^2} &= \frac{x_0^2 \exp\{2(\mu - \frac{\sigma^2}{2})\Delta(n-1) + 2\sigma W_{t_{n-1}}\}}{x_0^2 \sum_{i=1}^{n-1} \exp\{2(\mu - \frac{\sigma^2}{2})\Delta(i-1) + 2\sigma W_{t_{i-1}}\}} \\
&= (\sum_{i=1}^{n-1} \exp\{2(\mu - \frac{\sigma^2}{2})\Delta(i-1) + 2\sigma(W_{t_{i-1}} \\
&- 2(\mu - \frac{\sigma^2}{2})\Delta(n-1) - 2\sigma W_{t_{n-1}})\})^{-1}, \\
&= (\sum_{i=1}^{n-1} \exp\{2(\mu - \frac{\sigma^2}{2})\Delta(i-n) + 2\sigma(W_{t_{i-1}} - W_{t_{n-1}})\})^{-1}, \\
&= (\sum_{i=1}^{n-1} \exp\{2(\sigma^2/2 - \mu)\Delta(n-i) + 2\sigma(V_{t_{n-1}} - V_{t_{i-1}})\})^{-1}
\end{aligned}$$

Notice that $\frac{\sigma^2}{2} - \mu < 0$, by assumption, and $n - i > 0$, so the deterministic part in the exponential becomes more negative as n grows.

Regarding the random element, we have defined a new process $V_t = -W_t$. It is immediate to prove that the process V_t is a Wiener process, by looking at the three properties that define a Wiener process. Clearly, V_t is a Gaussian process. Second, V_t has independent increments, that is $V_{t_1}, V_{t_2} - V_{t_1}, \ldots, V_{t_k} - V_{t_{k-1}}$ are independent for all $0 \leq t_1 < t_2 < \cdots < t_k$. To see this, we use the property that normal random variables are independent if they are uncorrelated, therefore we just need to prove that

$$E[(V_{t_i} - V_{t_{i-1}})(V_{t_j} - V_{t_{j-1}})] = 0, \text{when } t_i < t_j.$$

This is shown below:

$$\begin{aligned} E[(V_{t_i} - V_{t_{i-1}})(V_{t_j} - V_{t_{j-1}})] &= E[(-W_{t_i} + W_{t_{i-1}})(-W_{t_j} + W_{t_{j-1}})] \\ &= E[-(W_{t_i} - W_{t_{i-1}})(-(W_{t_j} - W_{t_{j-1}}))] \\ &= E[(W_{t_i} - W_{t_{i-1}})(W_{t_j} - W_{t_{j-1}})] \\ &= 0 \end{aligned}$$

Third, V_t has a continuous version. It is immediate that V_t satisfies the Kolmogorov Continuity Theorem:

$$\begin{aligned} E[|V_t - V_s|^4 &= E[| - W_t - (-W_s)|^4] \\ &= E[| - (W_t - W_s)|^4] \\ &= E[|W_t - W_s|^4] \\ &= n(n+2)|t-s|^2 \end{aligned}$$

where, since we have a one-dimensional Wiener process, $n = 1$.
Because the deterministic part in the exponential is negative, and the random part is a Wiener process, we can use the same arguments as in Case 1 (i.e. the Law of the iterated logarithm) to prove that that the quotient converges almost surely to a –strictly positive– random variable L:

$$\lim_{n \to +\infty} \frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1} X^2_{t_{i-1}}} = L < +\infty, \ a.s., \tag{5.32}$$

The limit L is strictly positive because we are adding strictly positive terms.
Recalling the expression for A_n and using the fact that $1 + L$ is strictly positive, we can apply the rule of quotient of limits:

$$\begin{aligned} \lim_{n \to +\infty} A_n &= \lim_{n \to +\infty} \frac{1}{1 + \frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1} X^2_{t_{i-1}}}} \\ &= \frac{1}{1 + \lim_{n \to +\infty} \frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1} X^2_{t_{i-1}}}} \\ &= \frac{1}{1+L} \ a.s. \end{aligned}$$

Regarding B_∞, the crucial point to note is that the Y_{t_i} are *iid*, so that $Y_{t_i} \xrightarrow{d} Y$. Then, we have that

$$B_n = \alpha_n \beta_n \tag{5.33}$$

where

$$\alpha_n \to \frac{L}{1+L} \; a.s. \text{ and } \beta_n \xrightarrow{d} Y,$$

therefore for $n \to +\infty$,

$$B_n \xrightarrow{d} \frac{L}{1+L} Y. \tag{5.34}$$

Proving that Z_n converges

The next and most important step is to prove that Z_n actually converges. Let us write the recursion explicitly:

$$\begin{aligned}
&Z_1 = Y_{t_1},\\
&Z_2 = A_2 Y_{t_1} + B_2,\\
&Z_3 = A_3(A_2 Y_{t_1} + B_2) + B_3 = A_3 A_2 Y_{t_1} + A_3 B_2 + B_3,\\
&Z_4 = A_4 A_3 A_2 Y_{t_1} + A_4 A_3 B_2 + A_4 B_3 + B_4,\\
&\ldots
\end{aligned}$$

We will write $Z_n(Y_{t_1})$ when we want to be explicit about the initial point for the recursion. For instance, $Z_4(Y_{t_1}^{(0)}) - Z_4(Y_{t_1}^{(1)}) = A_4 A_3 A_2 (Y_{t_1}^{(0)} - Y_{t_1}^{(1)})$.

Proposition 2 *For the general case,*

$$Z_n(Y_{t_1}^{(0)}) - Z_n(Y_{t_1}^{(1)}) = A_n \ldots A_2 (Y_{t_1}^{(0)} - Y_{t_1}^{(1)}). \tag{5.35}$$

Proof. This is proved by induction: for $n = 1$,

$$Z_1(Y_{t_1}^{(0)}) - Z_1(Y_{t_1}^{(1)}) = (Y_{t_1}^{(0)} - Y_{t_1}^{(1)}).$$

Suppose that for a general n,

$$Z_n(Y_{t_1}^{(0)}) - Z_n(Y_{t_1}^{(1)}) = \prod_{j=2}^{n} A_j (Y_{t_1}^{(0)} - Y_{t_1}^{(1)})$$

is true. For $n+1$, by the stochastic difference equation for Z_n is

$$\begin{aligned}
Z_{n+1}(Y_{t_1}^{(0)}) - Z_{n+1}(Y_{t_1}^{(1)}) &= A_{n+1}Z_n(Y_{t_1}^{(0)}) + B_n - (A_{n+1}Z_n(Y_{t_1}^{(1)}) + B_n) \\
&= A_{n+1}[Z_n(Y_{t_1}^{(0)}) - Z_n(Y_{t_1}^{(1)})] \\
&= A_{n+1}\prod_{j=2}^{n} A_j(Y_{t_1}^{(0)} - Y_{t_1}^{(1)}) \\
&= \prod_{j=2}^{n+1} A_j(Y_{t_1}^{(0)} - Y_{t_1}^{(1)}).
\end{aligned}$$

hence the statement is true. □

Proposition 3 *When $n \to +\infty$, we have that $\prod_{j=2}^{n} A_j \to 0$ a.s. Therefore, it is the case that*

$$\lim_{n\to+\infty}[Z_n(Y_{t_1}^{(0)}) - Z_n(Y_{t_1}^{(1)})] = 0, \ a.s. \tag{5.36}$$

because $Y_{t_1}^{(0)}$ and $Y_{t_1}^{(1)}$ are two constants that do not depend on n. In other words, if $Z_n(Y_{t_1})$ converges for some Y_{t_1}, it must converge for all Y_{t_1}.

Proof. In order to prove this proposition, the main point is to prove that when $n \to +\infty$, we have that $\prod_{j=2}^{n} A_j \to 0$ *a.s.* We start by rewriting this expression as

$$\begin{aligned}
\lim_{n\to+\infty}\prod_{j=2}^{n} A_j &= \lim_{n\to\infty}\prod_{j=2}^{n}\frac{1}{1+\frac{X_{t_{j-1}}^2}{\sum_{i=1}^{j-1}X_{t_{i-1}}^2}} \\
&= \lim_{n\to+\infty}\prod_{j=2}^{n}\exp\{\log(\frac{1}{1+\frac{X_{t_{j-1}}^2}{\sum_{i=1}^{j-1}X_{t_{i-1}}^2}})\} \\
&= \lim_{n\to+\infty}\exp\{\sum_{j=2}^{n}\log(\frac{1}{1+\frac{X_{t_{j-1}}^2}{\sum_{i=1}^{j-1}X_{t_{i-1}}^2}})\} \\
&= \lim_{n\to+\infty}\exp\{-\sum_{j=2}^{n}\log(1+\frac{X_{t_{j-1}}^2}{\sum_{i=1}^{j-1}X_{t_{i-1}}^2})\} \\
&= \exp\{-\lim_{n\to+\infty}(\sum_{j=2}^{n}\log(1+\frac{X_{t_{j-1}}^2}{\sum_{i=1}^{j-1}X_{t_{i-1}}^2})\}
\end{aligned}$$

By definition of limit, the fact that $\lim_{n\to+\infty}\frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1}X^2_{t_{i-1}}}=L$ a.s. (which we already have proved) means that $\forall\varepsilon>0$, $\exists N_0(\varepsilon)$ such that $\forall n\geq N_0(\varepsilon)$

$$|\frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1}X^2_{t_{i-1}}}-L|<\varepsilon,\ a.s.$$

or, equivalently,

$$L-\varepsilon<\frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1}X^2_{t_{i-1}}}<L+\varepsilon,\ a.s.\quad \forall n\geq N_0(\varepsilon)$$

We can add 1 and take logarithms, obtaining

$$\log(1+L-\varepsilon)<\log(1+\frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1}X^2_{t_{i-1}}})<\log(1+L+\varepsilon),\ a.s.\quad \forall n\geq N_0(\varepsilon).$$

Take $n^*>N_0(\varepsilon)$. Then, summing over the previous expression,

$$\begin{aligned}\sum_{n\geq N_0(\varepsilon)}^{n^*}\log(1+L-\varepsilon)&<\sum_{n\geq N_0(\varepsilon)}^{n^*}\log(1+\frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1}X^2_{t_{i-1}}})\\&<\sum_{n\geq N_0(\varepsilon)}^{n^*}\log(1+L+\varepsilon),\ a.s.\end{aligned}$$

Since this relation is true for all $\varepsilon>0$, we can pick, for instance, $\varepsilon=L/2$. In the following, let $N_0\equiv N_0(L/2)$, for notational simplicity. Then the previous equation becomes

$$\begin{aligned}(n^*-N_0)\log(1+\frac{L}{2})&<\sum_{n\geq N_0}^{n^*}\log(1+\frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1}X^2_{t_{i-1}}})\\&<(n^*-N_0)\log(1+\frac{3L}{2}),\ a.s.\end{aligned}$$

For $n^*\to+\infty$,

$$+\infty\leq\sum_{n\geq N_0}^{+\infty}\log(1+\frac{X^2_{t_{n-1}}}{\sum_{i=1}^{n-1}X^2_{t_{i-1}}})\leq+\infty,\ a.s.,\tag{5.37}$$

therefore the element in the middle must be $+\infty$ as well. When we apply the limit, the inequalities become equalities.

We just have found the limit of the sums with $n \geq N_0$. Since the sums up to n for $n < N_0$ are sums of positive elements, they are positive as well, and we finally obtain the result we were looking for:

$$\lim_{n \to +\infty} \sum_{j=2}^{n} \log(1 + \frac{X_{t_{j-1}}^2}{\sum_{i=1}^{j-1} X_{t_{i-1}}^2}) = +\infty, \ a.s. \tag{5.38}$$

Back to our original problem. Because the exponential function e^x is a continuous function and has a limit when $x \to +\infty$, by definition of continuity we finally obtain that

$$\lim_{n \to +\infty} \prod_{j=2}^{n} A_j = 0, \ a.s. \tag{5.39}$$

Therefore, it must be the case that

$$\lim_{n \to +\infty} [Z_n(Y_{t_1}^{(0)}) - Z_n(Y_{t_1}^{(1)})] = 0, \ a.s. \tag{5.40}$$

because $Y_{t_1}^{(0)}$ and $Y_{t_1}^{(1)}$ are constants that do not depend on n. □

What we have just proved is that if $Z_n(Y_{t_1})$ converges for some Y_{t_1}, it must converge for all Y_{t_1}. Up to this point, however, we have not proved yet whether Z_n converges at all. This is the objective of the next step. However, the result we have just proved allows us to choose the initial values most convenient for this purpose. Since the general expression is

$$Z_n(Y_{t_1}) = \prod_{j=2}^{n} A_j Y_{t_1} + \sum_{i=2}^{n-1} B_i \prod_{j=i+1}^{n} A_j + B_n,$$

clearly that the most convenient choice for the initial value is $Y_{t_1} = 0$, so that

$$Z_n(0) = \sum_{i=2}^{n-1} B_i \prod_{j=i+1}^{n} A_j + B_n.$$

Since we have already proved that B_∞ exists, we only need to consider the limiting behavior of the term

$$\sum_{i=2}^{n-1} B_i \prod_{j=i+1}^{n} A_j, \tag{5.41}$$

The first step is to notice the following inequalities:

$$\sum_{i=2}^{n-1} B_i \prod_{j=i+1}^{n} A_j \leq \sum_{i=2}^{n-1} |B_i \prod_{j=i+1}^{n} A_j| \leq \sum_{i=2}^{n-1} |Y_{t_i}| \prod_{j=i+1}^{n} A_j,$$

since $A_j > 0\ \forall j$ and $B_i = (c/1+c)Y_{t_i}$ where $0 < c < 1$. Our objective then is to prove that

$$\lim_{n \to +\infty} \sum_{i=2}^{n-1} |Y_{t_i}| \prod_{j=i+1}^{n} A_j < +\infty. \tag{5.42}$$

We can re-write the previous equation as

$$\begin{aligned}
\sum_{i=2}^{n-1} |Y_{t_i}| \prod_{j=i+1}^{n} A_j &= \sum_{i=2}^{n-1} \exp\{\log(|Y_{t_i}| \prod_{j=i+1}^{n} A_j)\} \\
&= \sum_{i=2}^{n-1} \exp\{\log(|Y_{t_i}| + \sum_{j=i+1}^{n} \log(A_j)\} \\
&= \sum_{i=2}^{n-1} \left(\exp\{\log(|Y_{t_i}|)\} \exp\{(n-i)\frac{1}{n-i} \sum_{j=i+1}^{n} \log(A_j)\} \right).
\end{aligned}$$

We will prove now that

$$\lim_{n \to +\infty} \sum_{i=2}^{n-1} \left(\exp\{\log(|Y_{t_i}|)\} \exp\{(n-i)\frac{1}{n-i} \sum_{j=i+1}^{n} \log(A_j)\} \right) < +\infty. \tag{5.43}$$

In order to clarify our line of reasoning, we write down explicitly the elements of the limit we are taking:

$$\begin{aligned}
&\exp\{\log(|Y_{t_2}|)\} \exp\{(n-2)\frac{1}{n-2} \sum_{j=3}^{n} \log(A_j)\} \\
&\qquad + \exp\{\log(|Y_{t_3}|)\} \exp\{(n-3)\frac{1}{n-3} \sum_{j=4}^{n} \log(A_j)\} \\
&\qquad + \cdots + \exp\{\log(|Y_{t_{n-1}}|)\} A_n
\end{aligned}$$

We first deal with the limit

$$\lim_{n \to +\infty} \frac{1}{n-i} \sum_{j=i+1}^{n} \log(A_j),$$

where we need to be careful because the limit depends on i itself. We have already proved that $\lim_{n\to+\infty}\log(A_n) = \alpha < 0$, where $\alpha = \log(1/1+L)$ and $\log(A_n) < 0$, $\forall n$. What we want to prove now is the following:

$$\lim_{n\to+\infty}\sup_{i\leq n-1}|\frac{1}{n-i}\sum_{j=i+1}^{n}\log(A_j)-\alpha| = 0. \tag{5.44}$$

First, we know that $\log(A_n)\to\alpha$, therefore, for all $\varepsilon > 0$ there exists a n_0 such that $\forall n\geq n_0$, it is the case that

$$\alpha-\varepsilon < \log(A_n) < \alpha+\varepsilon,$$

and for $i\geq n_0$ we can add the previous relation $n-i$ times so that

$$\alpha-\varepsilon < \frac{1}{n-i}\sum_{j=i+1}^{n}\log(A_n) < \alpha+\varepsilon,\ i\geq n_0.$$

This implies that

$$\sup_{n_0\leq i\leq n-1}|\frac{1}{n-i}\sum_{j=i+1}^{n}\log(A_j)-\alpha| < \varepsilon,\ \forall n\geq n_0.$$

For a fixed i such that $i\leq n_0$, it is the case that

$$\lim_{n\to+\infty}|\frac{1}{n-i}\sum_{j=i+1}^{n}\log(A_j)-\alpha| = 0,$$

This result is the direct application of the classical results on Cesaro means, according to which if $m_n\to k$ for $n\to\infty$, then

$$\frac{m_1+m_2+\cdots+m_n}{n}\to k$$

as $n\to\infty$.

Since the result is true for any fixed $i\leq n_0$ and there is obviously a finite number of i's, the result must be true for the maximum as well:

$$\lim_{n\to+\infty}\max_{i\leq n_0}|\frac{1}{n-i}\sum_{j=i+1}^{n}\log(A_j)-\alpha| = 0.$$

We now have two results, one for the a supremum and one for the maximum. Putting them together, we have that

$$\lim_{n\to+\infty} \sup_{n-1\geq i\geq n_0} \left|\frac{1}{n-i}\sum_{j=i+1}^{n} \log(A_j) - \alpha\right| \vee \max_{i\leq n_0}\left|\frac{1}{n-i}\sum_{j=i+1}^{n} \log(A_j) - \alpha\right| \leq \varepsilon$$

Finally, taking $\varepsilon \downarrow 0$, we obtain the desired result:

$$\lim_{n\to+\infty} \sup_{i\leq n-1} \left|\frac{1}{n-i}\sum_{j=i+1}^{n} \log(A_j) - \alpha\right| = 0. \tag{5.45}$$

Now let us go back to (5.42). By the definition of limit, $\exists i_0$ such that $\forall i \geq i_0$ we have that

$$\frac{1}{n-i}\sum_{j=i+1}^{n} \log(A_j) = \alpha + o(\alpha) < 0, \tag{5.46}$$

since $\alpha < 0$. Then

$$\begin{aligned}\lim_{n\to+\infty} \sum_{i=i_0}^{n} \exp\{(n-i)\frac{1}{n-i}\sum_{j=i+1}^{n} \log(A_j) + \log Y_{t_i}\} \\ \leq \lim_{n\to+\infty} \sum_{i=i_0}^{n} |Y_{t_i}| \exp\{(n-i)(\alpha+\eta)\} < +\infty \ a.s.\end{aligned} \tag{5.47}$$

This last sum is finite a.s. because it converges in L^p. That is,

$$E(\sum_{i=0}^{\infty} |Y_{t_i}| \exp\{(n-i)(\alpha+\eta)\})^p < \infty \tag{5.48}$$

For $p = 2$ the proof is as follows. We can decompose the expectation

$$E[\sum_{i=0}^{\infty} |Y_{t_i}| \exp\{(n-i)(\alpha+\eta)\}]^2 = E[\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} |Y_{t_i}||Y_{t_j}| \exp\{(2n-i-j)(\alpha+\eta)\}]^2 \tag{5.49}$$

as

$$\begin{aligned}\sum_{\substack{i=0 \\ i\neq j}}^{\infty}\sum_{j=0}^{\infty} E[|Y_{t_i}|]E[|Y_{t_j}|] \exp\{(2n-i-j)(\alpha+\eta)\} \\ + \sum_{i=0}^{\infty} E[Y_{t_i}^2] \exp\{2(n-i)(\alpha+\eta)\}\end{aligned} \tag{5.50}$$

Regarding the second component, it is not difficult to show that

$$\begin{aligned} E[Y_{t_i}^2] &= \exp\{(4\mu + 6\sigma^2)\Delta\} - 4\exp\{(3\mu + 3\sigma^2)\Delta\} + 4\exp\{(2\mu + \sigma^2)\Delta\} \\ &\quad + 4\exp\{(3\mu + \sigma^2)\Delta\} - 4\exp\{2\mu\Delta\} - \exp\{(4\mu + \sigma^2)\Delta\} \end{aligned} \tag{5.51}$$

which we denote as $E[Y_{t_i}^2] \equiv B$. Regarding the cross-products, note that to obtain an upper bound for $E[|Y_{t_i}|]$ we can use the fact that $|x - y| \le |x| + |y|$. That is

$$\begin{aligned} E[|Y_{t_i}|] &= E[|(\exp\{(\mu - \frac{\sigma^2}{2})\Delta - \sigma(W_{t_i} - W_{t_{i-1}})\} - 1)^2 - A|] \\ &\le E[(\exp\{(\mu - \frac{\sigma^2}{2})\Delta - \sigma(W_{t_i} - W_{t_{i-1}})\} - 1)^2] + |A| \end{aligned}$$

We can then show that

$$E[(\exp\{(\mu - \frac{\sigma^2}{2})\Delta - \sigma(W_{t_i} - W_{t_{i-1}})\} - 1)^2] = A \tag{5.52}$$

Hence,

$$E[|Y_{t_i}|] \le A + |A| \le 2|A| \tag{5.53}$$

Putting it all together, we obtain that

$$\begin{aligned} &E[\sum_{i=0}^{\infty} |Y_{t_i}| \exp\{(n-i)(\alpha + \eta)\}]^2 \\ &\qquad \le 4A^2 \sum_{\substack{i=0 \\ i \ne j}}^{\infty} \sum_{j=0}^{\infty} \exp\{(2n - i - j)(\alpha + \eta)\} \\ &\qquad\qquad + \sum_{i=0}^{\infty} B \exp\{2(n-i)(\alpha + \eta)\} < \infty \ a.s. \end{aligned} \tag{5.54}$$

because $\alpha < 0$ and η is of smaller order than α.

Going back to the equation (5.47), note that the summation goes from i_0 to infinity, and we have just proved that it converges. The sum for $i \in [2, i_0)$ is a finite sum of finite elements, each of which converges as $n \to +\infty$, therefore it is finite. This completes the proof:

$$\lim_{n \to +\infty} \sum_{i=2}^{n-1} B_i \prod_{j=i+1}^{n} A_j < +\infty \ a.s. \tag{5.55}$$

and hence, $Z_\infty < \infty$ a.s.

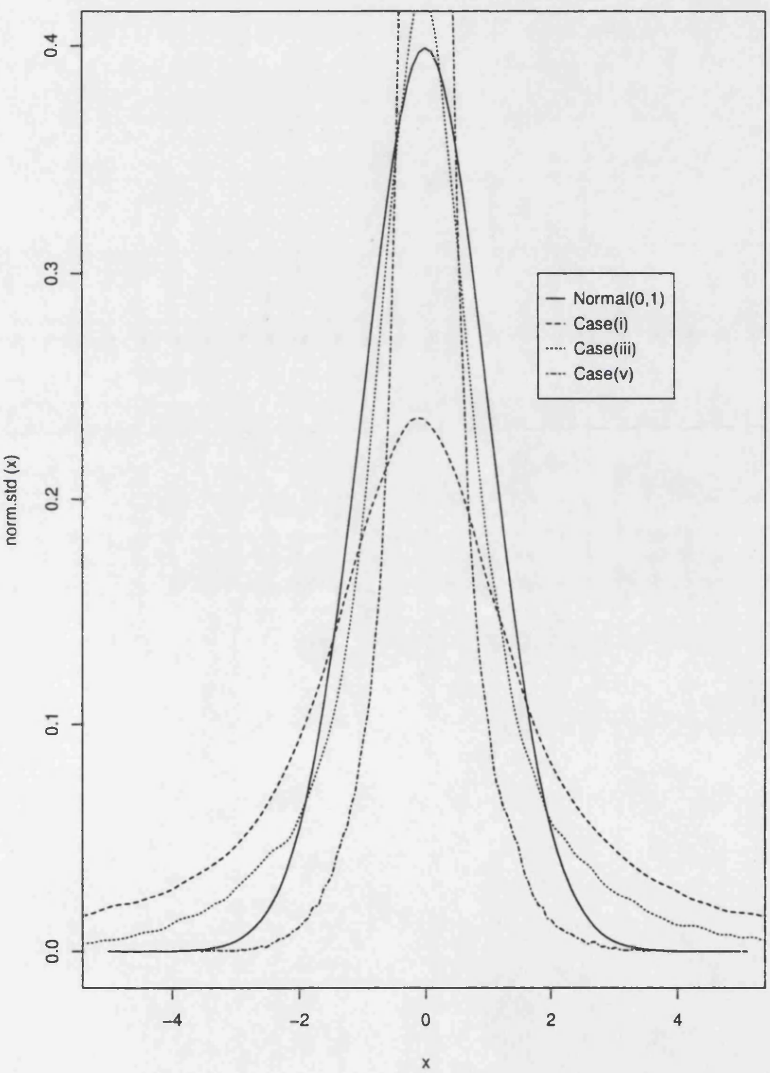

Figure 5.1: Empirical density of Z_n, $n = 1{,}000$, for the different cases in Table (2). We have simulated $M = 100{,}000$ sample paths of each case, with $\Delta = 1/252$.

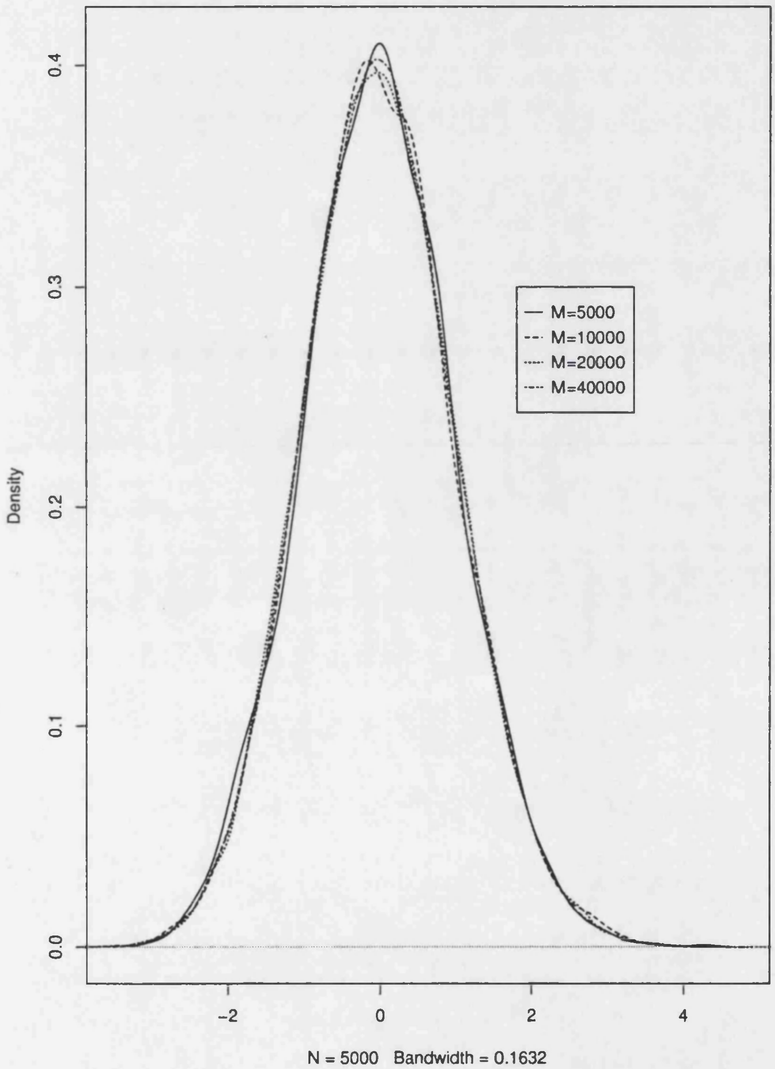

Figure 5.2: Empirical density of Z_n. Parameters: $\mu = 0.1$, $\sigma = 0.25$ (Case (v)). The sample paths are of length $n = 1{,}000$, with $\Delta = 1/252$. We have simulated $M = \{5000, 10000, 20000, 40000\}$ sample paths.

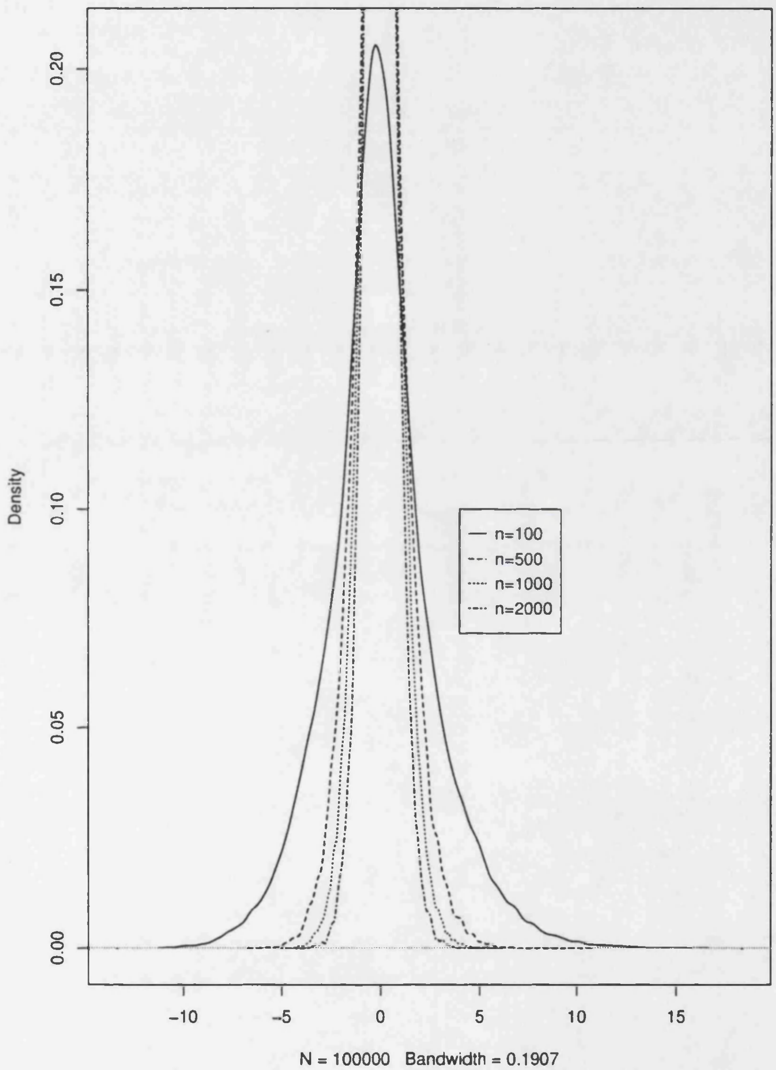

Figure 5.3: Empirical density of Z_n. Parameters: $\mu = 0.1$, $\sigma = 0.25$ (Case (v)). The sample paths are of length $n = \{100, 500, 1000, 2000\}$, with $\Delta = 1/252$. We have simulated $M = 100{,}000$ sample paths for each case.

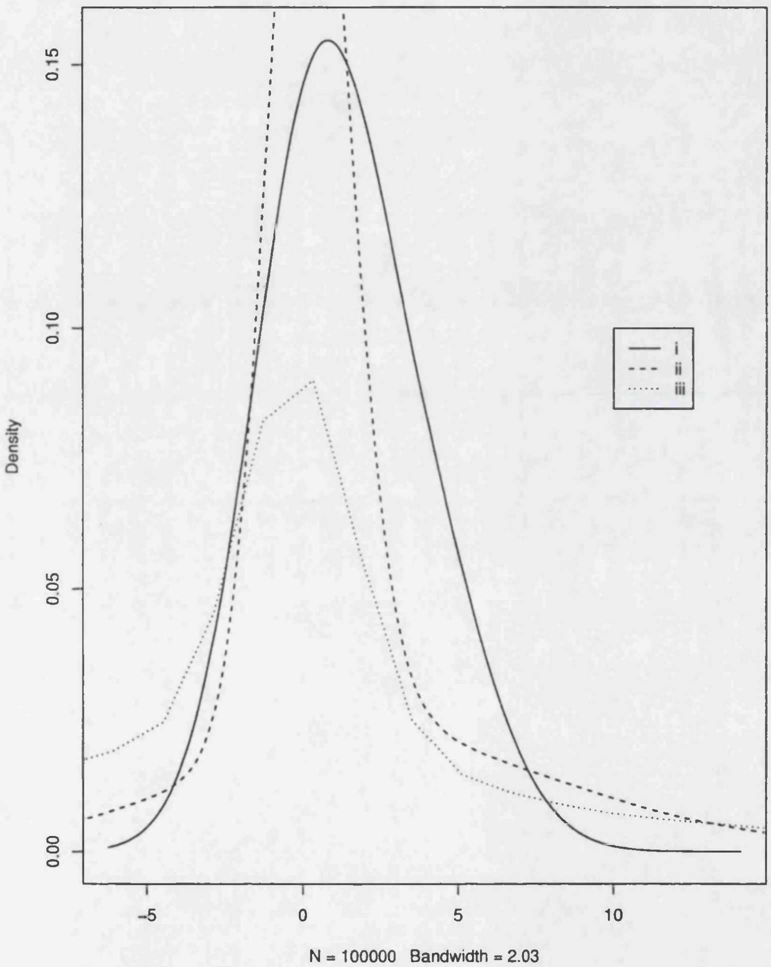

Figure 5.4: Empirical density of Z_n. Parameters: $\mu = 0.1$ and $\sigma = 0.25$ (Case (v)). The sample paths are of length $n = 1{,}000$ with $\Delta = \{1/252, 1/52, 1/12\}$ (cases i, ii and iii respectively). We have simulated $M = 100{,}000$ sample paths for each case.

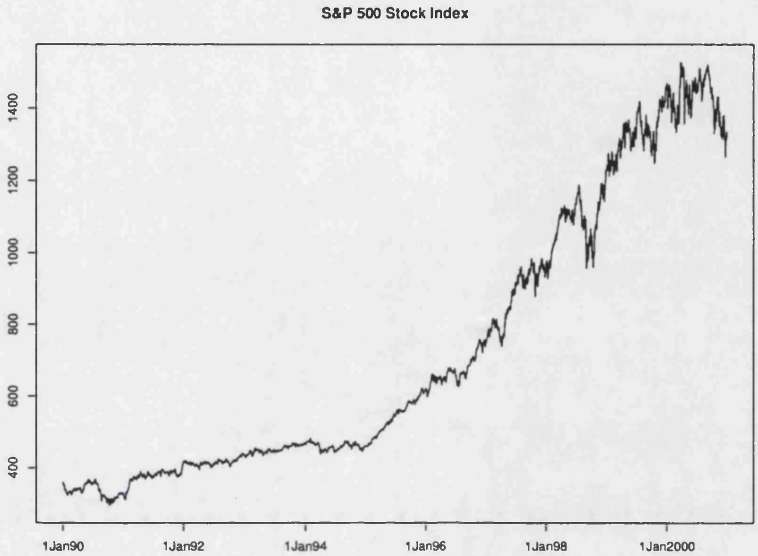

Figure 5.5: The S&P500 stock index from January 1990 to January 2001.

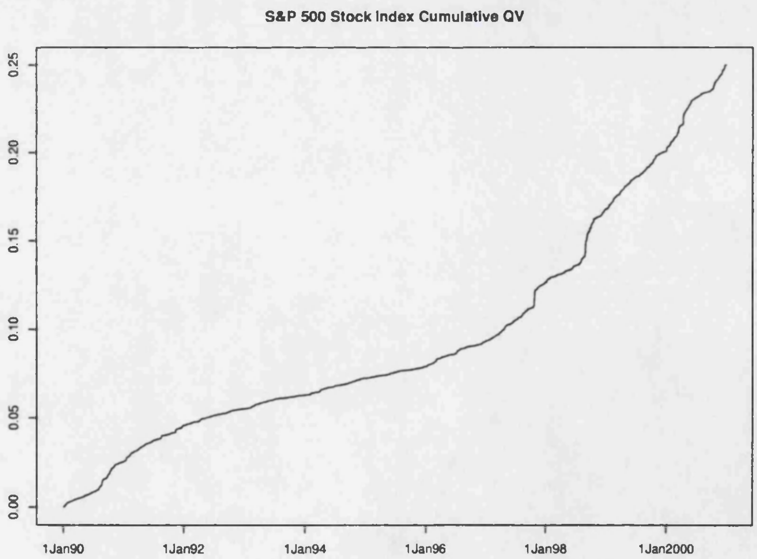

Figure 5.6: Cumulative value of $\sum_{i=1}^{n-1} \frac{(X_{t_{i+1}} - X_{t_i})^2}{X_{t_i}^2}$ for the S&P500 stock index, from January 1990 to January 2001. If the process was a geometric Brownian motion over the sample period (so that volatility was constant), the plot should be a straight line.

Chapter 6

Conclusions to the Empirical Analysis of Swap Spreads and the Estimation of Volatility

The main results in this Thesis regarding the empirical analysis of swap spreads can be classified in three different areas: the time-series properties, the relation with credit spreads and the behavior of the risk premium for holding spread risk.

Regarding the time-series properties, we document that swap spreads are positive, display large cyclical variations, and experienced a regime-shift in August 1998. We then show swap spreads to be well represented as a I(1) process, and that their demeaned first differences display a time-varying and persistent volatility. A good model for the conditional volatility of swap spreads in first differences is an Asymmetric GARCH(1,1) (see Glosten et al. (1993)), with a dummy variable for a higher volatility after August 1998. A possible line for future research is to explore fractionally integrated processes[1] as better representations for swap spreads, instead of I(1) processes.

The second step is to look at USD swap spreads and their relation with credit spreads. We find that the volatility of swap spreads is about the same as that of single-A credit spreads. Swap spreads do not seem to depend on the "corporate bond quality spread", the difference between BBB and AA spreads. On the other hand, swap spreads do move together with the differential of Financial vs. Industrial credit spreads. We inter-

[1]See Hamilton (1994).

pret this as evidence that swaps spreads are not very sensitive to counterparty default risk, but are affected by the overall risk of the financial system. This insight could be modeled more explicitly, for instance building a model of contagion risk between a number of broker/dealers. That model should recognize that the swap books of the main broker/dealers are huge and highly interlinked, and that netting and collateralizing may fail if the whole system collapses at the same time, probably due to some exogenous trigger. For instance, we mention a situation in which collateral flows are put on hold because of concerns on the financial health of a counterparty that is marginally in net losses, with the consequence that it has to make the gross payments and its liquidity deteriorates dramatically.

The main result of the Chapter on credit, and possibly of the Thesis, are the estimates of the error-correction model for the relation between credit and swap spreads. First, we show evidence that swap and credit spreads are cointegrated, that is, they are related in the long-term by an equilibrium relation. As Lang et al. (1998) point out, this relation changes over the business cycle. In order to take this into account, we add the slope of the yield curve into the cointegrating relation. Second, the "changes" regression in the ECM provides a framework to analyze the short-term adjustment of swap spreads, with a mean reversion term (the lagged error from the "levels" regression), and the changes of a number of economic drivers: level and slope of the yield curve, credit spreads, and credit spread differentials between low and high quality debt and financials vs. industrial issues. All together, this model provides with an assessment of the fundamental value of swap spreads and guide to its evolution. In terms of prediction, at time t we can use the mean reversion estimate to predict the spread at $t + 1$. The changes in the drivers are not known, and there's no a priori reason to believe they are less difficult to predict than swap spreads. This is an interesting area for future research. For instance, one could test for Granger causality and model this system in a Vector Autorregression (VAR) framework.

There are two more lines of research that come to mind. One is to refine that ECM specification, by adding or removing some economic drivers, and improving the economic justification for them. Some prime candidates for inclusion would be the VIX index of equity market implied volatility (from options on the S&P 100), swap spreads in other currencies, credit spreads of particular sectors (e.g. banks), etc. Another line of research would be to use the model to generate trading signals and compute its performance. One such model could be based on the strength of the mean reversion only. We would be particularly interested on the hit ratios (percentage of the time the sign change is predicted correctly), and the behavior of the profit and losses of such strategy.

The final chapter on swap spreads deals with the properties of the risk premium embedded in them. This is relevant on its own, given the widespread use and trading of swaps, but also because swap spreads are frequently taken to be proxies of credit spreads (Liu et al. (2000)). This can be motivated via our previous Chapter, where we establish the presence of a long-term relation between swap and credit spreads. Taking swap spreads as proxies of credit spreads also helps researchers in terms of data availability and especially, accuracy.

Our main result is that the risk premium in swap spreads is time-varying, and it changes in a way that imparts some predictability in spreads: when the slope of the swap spread curve is "high", the spreads tend to subsequently tighten. The implications of our main result are both practical and theoretical. First, we can use the predictive relation to design trading strategies. We show some evidence of excess returns resulting from this type of strategies, indicating that this may be an interesting topic for further research. Second, a realistic model of swap spreads should be able to replicate this result. In the framework of a formal model, our conjecture is that we will need a time-varying risk premium, possible affine on the state variables (see Dai and Singleton (2001)), but for spreads instead of rates. Developing a good model for swap spreads, with a time-varying risk premium, would be very valuable in order to study the market price of credit risk and its determinants.

Summarizing, the empirical analysis contained in this Thesis, represents a step in the understanding of swap spreads, prior to a more theoretical, model-based approach. A model of swap spreads should be able to generate a highly persistent process, with time-varying and persistent volatility. Also, it should relate swap to credit spreads, and introduce the effect of the business cycle. Finally, the risk premium should be time-varying. We hope that our empirical results will be useful in narrowing down the questions that the theory should try to answer, and help in the validation process of a number of models that have been proposed in the literature, as well as in the development of new ones.

The final chapter on the asymptotic properties of the quadratic variation estimator when the sample size tends to infinity deals with the estimation of the volatility parameter of a diffusion when the time between observations is fixed. This means that the usual asymptotic result (that this estimator is consistent) does not hold. For a few processes, this would not be a problem, because we could use the maximum likelihood estimator, which it is actually the one with the best properties (consistency and efficiency). However, there are very few cases in which we are able to find the likelihood function, hence the need for alternative estimators with "good" properties.

The quadratic variation estimator is a natural estimator of the volatility parameter, and it turns out to be quite robust to the problem of discrete observations. Essentially, we show that for a fixed, positive time between observations, this estimator is biased. However, the bias converges to a finite random variable when the number of observations tends to infinity. A simulation study shows that for realistic cases, the bias is small enough for the estimator to be "useful". The performance only deteriorates when we go into frequencies lower that a week or more.

As mentioned above, then, this chapter's results are twofold. First, there is an important theoretical result: the quadratic variation estimator for a geometric Brownian motion is asymptotically biased for $n \to \infty$, with $\Delta > 0$, but the bias converges to a finite random variable. This result is not obvious nor trivial. When we look at the bias, we see that it consists of a quotient of two sums of random variables, that could very well diverge. The proof is divided in two cases, depending on the relation between the drift parameter μ and the volatility parameter σ. When $\mu < \sigma^2/2$, the proof is relatively simple, with the main idea behind of the property of Wiener process that its paths are bounded. In fact, this proof can be adapted to more general families of processes with bounded paths, e.g. Lévy processes. This is left for future research. For the case $\mu > \sigma^2/2$, the proof is relatively more involved, and we needed some trials before finding a way to re-write the problem in terms of stochastic difference equations and solving the proof.

The second important set of results in the Chapter are those from the simulation study for the bias. As a basecase, we take the sample path to be 1,000 observations long, with a daily frequency. The number of simulated paths is 100,000 and troughout this section, we produce implied density plots of the normalized bias, which we compare with a N(0,1) density. It turns out that, for different parameter values and cases of (μ, σ), the bias is very small and the volatility is low. We do find evidence of a positive skew and "fat-tails" in the distribution of the bias. An obvious simulation to be done is one in which we change the number of observations. It turns out that we do not need an extremely large number of daily observations (e.g. 500 or more) to obtain a small and concentrated bias. We next perform a series of robustness checks of our simulations, changing the number of paths (this has a negligible impact on the results) and the time between observations. This last factor has an important impact on the estimates: using monthly data is not acceptable, as the kurtosis of the estimates is too large.

To close the Chapter, some applications of the quadratic variation estimator are explored and preliminary results obtained. First, we look at a test for model misspecification, that relies on the result of the bias being "small". The idea is that the quadratic variation estimator computed with the actual data, and that calculated from simulated data, should be close to each other if the process we have assumed is the right data-generating process. For the S&P 500 stock index, the simulated data gives a QV-volatility of 0.151, while the QV-volatility from the actual data is 0.181. We interpret this result as evidence that the geometric Brownian motion is not the best model to represent stock index dynamics.

The second application we present, can give us a hint why the result above is the case. It turns out that we can write the QV estimator for a process with a time-varying volatility, and plot that estimate vs. time. If the volatility was indeed constant, this number would grow linearly with time. For the S&P 500, it turns out that before 1997, volatility grew slower than linearly, and after that, it grew much faster. Also, after 1997, we observe the estimates to often jump vertically, reacting to large changes in the stock index. All in all, the QV gives us a quick way to check for the presence of regimes in volatility and where are the breakpoints. We think that these and other applications offer interesting avenues for future research.

In terms of future research, the way we would like to extend this results is actually in terms of model misspecification. That is, to investigate what is the behavior of the QV estimator when the parametric model we have assumed is not the true one. One interesting case is that of Lévy processes. These allow for the presence of jumps in the underlying price. When we increase the number of observations, we introduce more and more jumps in the sample. Hence, the question is whether there any guarantee that the QV estimator will converge in these conditions. Our preliminary results indicate that the effect of jumps will eventually die out, because the underlying Lévy noise is still producing bounded sample paths.

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