Rationality, Uncertainty Aversion and Equilibrium Concepts in Normal and Extensive Form Games

by

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for my parents

"To the many genres in the art of lying developed in the past, we must now add two more recent varieties. There is, first, *the apparently innocuous one of the public-relations managers in government (...). The* second *new variety of the art of lying, [is] less frequently met in everyday life (...). [It] appeals to (...) professional 'problem solvers', (...) some of them equipped with game theories and system analyses (...).*

Hence they were not just intelligent, but prided themselves on being 'rational'. (T)hey were indeed to a rather frightening degree above 'sentimentality' and in love with 'theory', the world of sheer mental effort. They were eager to find formulas, (...) that would unify the most disparate phenomena with which reality presented them; that is they were eager to discover laws *by which to explain and predict political and historical facts as though they were as necessary, and thus as reliable, as the physicists once believed natural phenomena to be. (...)*

The problem-solvers did not judge; *they calculated. (...) An utterly irrational confidence in the calculability of reality [became] the leitmotif of the decision-making processes (...)."*

Hannah Arendt (1969)

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Abstract

This thesis contributes to a re-examination and extension of the equilibrium concept in normal and extensive form games. The equilibrium concept is a solution concept for games that is consistent with individual rationality and various assumptions about players' knowledge about the nature of their strategic interaction. The thesis argues that further consistency conditions can be imposed on a rational solution concept.

By its very nature, a rational solution concept implicitly defines which strategies are non-rational. A rational player's beliefs about play by non-rational opponents should be consistent with this implicit definition of non-rational play. The thesis shows that equilibrium concepts that satisfy additional consistency requirements can be formulated in Choquet-expected utility theory, i.e. non-expected utility theory with non-additive or set-valued beliefs, together with an empirical assumption about players' attitude toward uncertainty.

Chapter 1 introduces the background of this thesis. We present the conceptual problems in the foundations of game theory that motivate our approach. We then survey the decision-theoretic foundations of Choquet-expected utility theory and game-theoretic applications of Choquet-expected utility theory that are related to the present approach.

Chapter 2 formulates this equilibrium concept for normal form games. This concept, called Choquet-Nash Equilibrium, is shown to be a generalization of Nash Equilibrium in normal form games. We establish an existence result for finite games, derive various properties of equilibria and establish robustness results for Nash equilibria.

Chapter 3 extends the analysis to extensive games. We present the equivalent of subgame-perfect equilibrium, called perfect Choquet Equilibrium, for extensive games. Our main finding here is that perfect Choquet equilibrium does not generalize, but is qualitatively different from subgame-perfect equilibrium.

Finally, in chapter 4 we examine the centipede game. It is shown that the plausible assumption of bounded uncertainty aversion leads to an 'interior' equilibrium of the centipede game.

Contents

Chapter 1

Introduction

Game theory is closely related to the question what constitutes rationality in strategic interaction. The precise nature of this relationship, however, is subtle and controversial. From the point of view of experimental game theory, bounded rationality seems a natural explanatory hypothesis of actual behaviour. From the point of view of theories of evolution and learning in games, behaviour is the result of a dynamic process and rationality need not play an explanatory role. Yet, the question what strategic rationality means has motivated many concepts in game theory from its very beginning (von Neumann 1928, von Neumann & Morgenstern 1944) to the present (Dekel & Gul 1997).

In decision theory, following Savage (1954), rationality is defined as a consistency property. Under uncertainty, the consequence of an act depends on the state of nature. When the ranking of these acts is consistent, i.e. satisfies Savage's axioms, then it corresponds to the maximisation of the subjectively expected utility. Thus a decision-maker is rational if and only if he ranks acts as if he has both a utility function over consequences and a probability measure over states that expresses his beliefs, and an act is ranked higher than another if and only if his expected utility is higher

What distinguishes a game from a decision under uncertainty is that the players know that the uncertainty arises from strategic interaction, that is the decisions of other players. Consequently, in a game the decision-theoretic definition of rationality is not sufficient: A player's beliefs must be rational as well, i.e. consistent with what the player knows. Intuitively, then, a player will be rational if and only if he maximises his expected utility given rational beliefs.

If the rationality of the players is common knowledge, this requirement of rational beliefs leads to an infinite regress: A player's beliefs must be consistent with the rationality of the opponent, the opponent's beliefs must be consistent with the rationality of the player, and so the player's beliefs must also be consistent with these beliefs of the opponent, ... ad infinitum.

The central solution concepts in non-cooperative game theory are Nash's equilibrium concept (Nash 1950) in normal form games, and its refinements and extensions in extensive form games (Selten 1965, Harsanyi 1967–68, Kreps & Wilson 1982*b*). Their status as definitions or implications of game-theoretic rationality is under much debate. Alternative game-theoretic solution concepts have been motivated by rationality considerations, and, in turn, other justifications have been proposed for the equilibrium concept. Yet, the classical motivation for Nash equilibrium and subgame-perfect equilibrium — the lack of an incentive to deviate and the exclusion of incredible threats — refer explicitly to the rationality and the knowledge of the players. In particular, the equilibrium concept is consistent with the infinite regress that arises if rationality is common knowledge. Finally, the equilibrium concept is, so far, the unrivalled solution concept in applied game theory.

For these reasons, this thesis concentrates on the equilibrium concept as an expression of game-theoretic rationality. Our starting point is the hypothesis that Nash equilibrium can be understood as a definition of game-theoretic rationality in games in which rationality of the players is common knowledge. We argue, however, that further consistency conditions have to be imposed on a rationality concept in games without common knowledge of rationality. The objective of this thesis is to show that equilibrium concepts that satisfy these consistency requirements can be defined on the basis of non-expected utility theory.

The thesis does not claim that Nash equilibrium is indeed an adequate definition of rationality in normal form games, even if rationality is common knowledge, or that it is the only one. Rather, this view is adopted as a working hypothesis. Although a defence of this view can be given (e.g., Aumann & Brandenburger (1995)), the importance of the equilibrium concept alone justifies the study of the consequences of this view.

The acceptance of the equilibrium concept as a rationality definition for normal form games thus leads to two questions: How can this definition be extended to normal form games if rationality is not common knowledge? And how can this definition be extended to extensive form games? These questions are closely related, because a strategy in an extensive form game must specify an action after every possible history of the game, and not only after those that are consistent with rational play. Equilibrium refinements typically require that the players' rationality is common knowledge even after histories in which players deviated from rational play. It is this aspect that is responsible for the controversy whether equilibrium refinements capture rationality in extensive games. The analysis of normal form games without mutual knowledge of rationality thus serves two purposes, first to relax a restrictive assumption about normal form games, and thus to extend the scope of the rationality concept, and secondly to provide a basis for the analysis of extensive form games.

The acceptance of any game-theoretic solution concept as a definition of rationality has the consequence that it also serves as an implicit definition of non-rational play: All strategies that do not satisfy the game-theoretic solution concept have to be considered non-rational. It follows that every deviation from rational play must be considered as evidence of non-rationality of the deviating player. On the other hand, compliance with the game-theoretic solution concept cannot be so clearly interpreted as evidence of rationality. A conforming player can have followed the game-theoretic solution "for the wrong reasons", for instance by chance.

We are thus led to impose the following consistency conditions on an equilibrium concept if rationality is not common knowledge: First, players who do not play equilibrium strategies must be considered non-rational. Secondly, any non-equilibrium strategy is a possible strategy that a non-rational player might choose. When deriving equilibrium strategies for rational players, it must be taken into account that the opponent need not be rational and therefore that he may play any non-rational strategy.

In this situation, the decision-theoretic rationality concept of Savage (1954), that provided the starting point of our considerations, is no longer adequate: It forces the rational player to form a specific probabilistic belief about the play of a non-rational opponent in a situation in which, by definition, the rational player must believe that an opponent, if non-rational, chooses any other strategy.

This thesis argues that non-additive decision theory, pioneered by Schmeidler (1989), allows the extension of the equilibrium concept that satisfies the additional consistency conditions. Thus, by weakening the demands on individual rationality it becomes possible to fulfill stronger consistency requirements in strategic interaction.

In the rest of this introduction we aim to present the background of this thesis in more detail. In chapter 2 formulate the equilibrium concept for normal-form games in which rationality is not common knowledge. In chapter 3 we formulate an analogue of subgame-perfect equilibrium for extensive games. Finally, in chapter 4 we analyse the centipede game, which plays an important role in the discussion of rationality in game theory.

The rest of this chapter is organized as follows. In the first section we describe how classical game theory captures strategic rationality. Section 2 describes the subjective expected utility model in decision theory. Section 3 explains how subjective expected utility theory led to Bayesian game theory, and describes some of the difficulties that remain in defining strategic rationality. Section 4 describes the Ellsberg paradox, and how this empirical evidence led to the development of decision theory with non-additive beliefs. Section 5 further investigates the meaning of non-additive beliefs. Section 6 presents the decision-theoretic aspects of non-additive beliefs in more detail and section 7 discusses the problem how to update such beliefs. Section 8 describes various equilibrium concepts for games that allow non-additive beliefs. Section 9 lists some economic applications, and section 10 reviews the experimental evidence on uncertainty aversion. Section 11 concludes. Finally, section 12 outlines the limitations of our analysis and section 13 specifies the terminological conventions.

Before I start, I would like to make some personal remarks. The main purpose of this introduction is to try to convince the reader that Choquet expected utility theory is not just a technique looking for an application, but a solution to a problem in game theory. Thus I owe it to the reader to identify the the relevant underlying currents of game theory, and to a degree to evaluate them. When writing about such topics as foundations of rationality and probability, I felt the difficulty of avoiding formulations that sound grand, or even pretentious. Insofar as I failed in this, nothing could be further from my intention.

Also, to keep this introduction to a reasonable length while including the relevant aspects, some aspects are so densely argued that I hope that the reader will not be put off. In spite of this, I had to leave out many minor aspects and would like to refer the reader to other accounts of rationality in game theory, e.g. Dekel & Gul (1997) or Rubinstein (1998).

Finally, each chapter has been written as a self-contained paper. As a consequence, the introductions and the description of Choquet expected utility theory in the later chapters introduce a degree of redundancy. However, this also allows the presentation of the material from the slightly different perspectives taken in the analysis of normal form games, extensive form games, and backward induction. In addition, I hope that the reader finds it convenient that the chapters are self-contained.

In the following I will use the pronoun 'we', meant to include the reader.

1.1 Rationality in Classical Game Theory

'Classical game theory' (the expression used by Harsanyi & Selten (1988)) can be characterized by two aspects: First, the theory attempts to capture rationality in strategic interaction, and is not concerned with evolution or bounded rationality. Secondly, the rationality concept itself is implicit and informal. While the solution concept is explicitly motivated by rationality considerations, only the solution concept is defined formally, and it is not explicitly derived from a formal definition of rationality. Since this thesis contributes to a clarification of the rationality concept in games, we begin with a review of the relationship between rationality and game theory.

Rationality first enters game theory implicitly in the concepts of 'correct play' and 'value of a game'.[1] These are the concepts that Zermelo (1913) used in his proof that chess is strictly determined. His motivation is to find out if every position in chess is similar to an endgame, i.e. has a well-defined value for the players, assuming that they play according to backward induction, which is 'surely the only correct way' (Zermelo 1913, my translation). Thus the concept of rationality is implicitly already used, but it is not seen as problematic and not made explicit.

Von Neumann's publication of the minimax theorem of 1928,[2] usually regarded

[1] It is interesting to note that this is entirely analagous to the way both probability theory and decision theory began (Hacking 1975).

[2] Zermelo's argument contained a gap that was closed in the famous paper on graph theory of Kőnig (1927). Kőnig (1927) also reports how Zermelo's own solution, and that Zermelo's paper attracted von Neumann to game theory. This is noteworthy because Zermelo's paper is not even quoted in von Neumann & Morgenstern (1944): Backward induction gives a completely different

as the beginning of game theory proper, embodies a different rationality concept.[3] Von Neumann (1928) for the first time argues explicitly — but informally — on the basis of rationality. Playing a minimax strategy is rational because it guarantees the security level, and in zero-sum games it is not possible to gain if the opponent is guaranteed to get his security level. In other words, a player whose payoff is lower than the amount that he can guarantee himself cannot be rational, but in a zero-sum game the players cannot both get more than their security level. In particular, minimax strategies are equilibria, but that was not von Neumann's concern; it is well-known that he objected to the equilibrium concept as a general solution concept.[4]

Von Neumann & Morgenstern (1944) attempted to base the theory of all games on the minimax solution. In order to capture the cooperative element present in non-zero-sum games, the coalitional structure associated with an extensive game had to be analysed, and this association was based on the minimax principle by adding a fictitious player (whose payoff transformed the game into a zero-sum game). While they thus laid the foundation for the analysis of cooperative games (although the distinction was only made explicit by Nash (1951)), the derivation of the coalitional structure from the extensive game was unfounded.

Nash's equilibrium concept (1950, 1951) allowed a direct noncooperative analysis of finite nonzero-sum games with finitely many players. Like von Neumann, Nash argued explicitly (and informally) on the basis of rationality: Equilibrium strategies are rational because unilateral deviations are not profitable. Again, this marks a shift in the rationality concept. In fact, one of the objections against the equilibrium concept was based on the fact that equilibrium strategies are not minimax strategies (Aumann & Maschler 1972, Harsanyi 1977): There are games in which

solution to non-zerosum games than that proposed in von Neumann & Morgenstern (1944).

[3] See also the first minimax argument attributed to Waldegrave in the 2nd edition of de Montmort (1708, 1713), and the notes of Borel (1921,1924,1927)).

[4] It seems that one can only speculate on the reasons. Binmore (1996a) suggests that von Neumann objected to the possible multiplicity of Nash equilibria, since equilibria do not possess the payoff-equivalence and exchangeability properties of minimax strategies. Another reason might have been that the characteristic feature of non-zero-sum games according to von Neumann & Morgenstern (1944) is the presence of both the competitive and the cooperative element, and the latter required for von Neumann a coalitional analysis (von Neumann & Morgenstern 1944, Section 20).

the unique equilibrium payoffs give the value, without guaranteeing it. However, this argument ignores that players would then have an incentive to play a best reply to the opponent's minimax strategy instead of their own minimax strategy (Harsanyi 1982*a*).

The subsequent refinement of Nash's equilibrium concept, initiated by Selten's concept of subgame perfection (Selten 1965) is again based on a shift in the rationality concept. While Nash's reasoning is accepted for games in which all choices are simultaneous, it is regarded as too weak in extensive games: Nash's requirement that players have no incentive to deviate is necessary, but not sufficient for rationality, because it allows 'incredible threats' to sustain the equilibrium path. Subsequently, game theory attempted to capture rationality by more and more subtle refinement criteria for Nash equilibrium, for instance proper equilibrium (Myerson 1978), strictly perfect equilibrium (Okada 1981) and strategic stability (Kohlberg & Mertens 1986).

Finally, Nash's original justification of equilibria, i.e. the lack of an incentive to deviate, seems to require that players choose their equilibrium strategies because they believe that their opponents play their equilibrium strategies. Since the equilibrium need not be unique, this rationality concept cannot be considered as complete before a rational way to select among equilibria is found (Harsanyi & Selten 1988).

We have thus seen that classical game theory has attempted to capture rationality in strategic interaction, but the rationality concept itself has changed with the game-theoretic solution concept. Moreover, both the ideas of minimax play and equilibrium play have been important throughout. But while game theory has made significant progress in our understanding of rationality, classical game theory has not been able to provide a definitive characterization.

1.2 Subjective Expected Utility Theory

Building on the expected utility theory of von Neumann & Morgenstern (1944) and the probability theory of de Finetti (1931a, 1937), Savage (1954) characterizes rationality in decision-making.[5] His theory was simplified by Anscombe & Aumann

[5]See also Ramsey (1926), whose construction takes objective probability (in the form of an objective randomization device) as given.

(1963), and since the following developments take their system as a starting point we will present it in more detail.

Anscombe & Aumann (1963) consider a (finite) set of states S and a (finite) set of consequences ('prizes') Z. They then assume the existence of an objective randomization device that allows to specify all probability distributions ΔZ over Z ('roulette lotteries', or 'lotteries' for short). An (Anscombe-Aumann) act f, or 'horse lottery', is then a mapping that associates a probability distribution over Z with each state $s \in S$.

A decision maker (henceforth player) is characterised by a binary relation $\succeq$ over acts. Anscombe and Aumann consider the following axioms:[6]

(A1) $\succeq$ is a complete pre-ordering, i.e. complete and transitive.

(A2) The decision problem is non-trivial, i.e. there exist f, f' with $f \succ f'$.

(A3) The preference relation is continuous, i.e. if $f \succ g \succ h$ then there exist $\alpha, \beta \in (0,1)$ such that[7] $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.

(A4) Preferences are monotonic,[8] i.e. if $f(s) \succeq f'(s)$ for all $s \in S$ then $f \succeq f'$.

(A5) Preferences satisfy the independence axiom, i.e. if $f \succ f'$ then for all α with $0 < \alpha \leq 1$ and all acts g also $\alpha f + (1-\alpha)g \succ \alpha f' + (1-\alpha)g$.

They prove, that if, and only if, the binary relation $\succeq$ over acts satisfies the above axioms then there exist

(1) a utility function over consequences $u : Z \longrightarrow \mathbb{R}$ that is 'cardinal', i.e. unique up to affine transformations, and
(2) a unique probability measure μ over states,
(3) such that one act is preferred to another if and only if its expected utility is higher, i.e.

$$f \succeq f' \iff \sum_{s \in S} u(f(s)) \cdot \mu(s) \geq \sum_{s \in S} u(f'(s)) \cdot \mu(s).$$

[6] This version of the Anscombe-Aumann model follows Schmeidler (1989) and Fishburn (1970).

[7] Here, the 'compound act' $[\alpha f + (1-\alpha)g](s)$ is defined as $\alpha f(s) + (1-\alpha)g(s)$. This is well-defined, since ΔZ is a mixture space in the sense of Herstein & Milnor (1953), i.e. it is possible to take convex combinations of probability distributions.

[8] Here, $f(s)$ and $f'(s)$ are roulette lotteries, i.e. probability distributions p_1, p_2 over consequences, and the preference relation over roulette lotteries (as usual, also denoted $\succeq$) is derived from the preference over constant acts, i.e. $p_1 \succeq p_2 \;:\iff\; p_1^* \succeq p_2^*$, where $p^*(s) = p$ for all $s \in S$.

In Savage's (1954) framework, (Savage) acts map states directly into consequences. Since the set of consequences does not have a mixture space structure, the axioms cannot be expressed in terms of compound acts. Savage's axioms allow the derivation of a qualitative likelihood relation over events that is shown to be representable by a 'probability' measure. Given this measure, in turn, it is possible to derive the existence of a utility function and the expected utility characterization.

However, the subtleties of Savage's theorem (see, e.g., Wakker (1993)) will be important for the application of game theory. The set of states must be infinite in Savage's framework and the space of events must be a σ-algebra. The set of acts must be sufficiently rich, i.e. include all constant acts. Finally, the probability measure is only finitely additive, not countably additive,[9] and it cannot have atoms.

1.3 Bayesian Game Theory

On the basis of subjective expected utility theory, classical game theory was criticised by Kadane & Larkey (1982). They argued that for each player, the opponents' strategies correspond to the states of uncertainty, so that decision-theoretic rationality alone does not go further than specifying that each player has beliefs about his opponents' play. However, Harsanyi (1982*b*) and Aumann (1987) argued that this argument overlooks that the players are assumed to know that their opponents are rational, and indeed that this is common knowledge.[10] Consequently, these are restrictions on the beliefs of the rational players that have to be taken into account.

The argument that the game-theoretic solution concept should be formally derived from subjective expected utility theory and assumptions about the knowledge of the players was given by Spohn (1982) (see Osborne & Rubinstein (1994)). The first such derivation was first carried out by Bernheim (1984) and Pearce (1984). They showed that common knowledge of rationality alone only leads to 'rationalizable' strategy profiles, but not necessarily to equilibrium.

Their work marked the beginning of 'Bayesian game theory'.[11] Subsequent studies tried to identify the epistemological assumptions that underlie game-theoretic

[9] Recently, Stinchcombe (1997) has extended the Savage model to the countably additive case.

[10] A fact is common knowledge if everybody knows it, everybody knows that everybody knows it, everybody knows that, ad infinitum.

[11] Of course, Harsanyi (1967–68) argued explicitly from a Bayesian point of view.

solution concepts (see, e.g., Tan & Werlang (1988), Battigalli & Bonanno (1998)). Particularly influential has been the interactive belief model proposed by Aumann (1981, Appendix 4), which adopts Harsanyi's (1967–68) methodology of games with incomplete information. This model was used, in particular, by Aumann (1987) to study the epistemic foundations of correlated equilibria, by Aumann & Brandenburger (1995) for Nash equilibria, and by Aumann (1995) for backward induction.[12]

In Aumann & Brandenburger (1995), each player can be one of several 'types'. A type specifies the strategy that the player chooses, and his beliefs about the opponents' types. Thus in this model, the set of states is the set of 'type profiles'. To each state corresponds a strategy profile chosen by the players, and for each player a hierarchy of beliefs about the opponents. Aumann & Brandenburger (1995) show that in a state in which players are rational in the sense of subjective expected utility theory, this is mutually known, and the players have a common prior about states, and conjectures about the strategy choices that are commonly known, then the strategy profile in this state is a Nash equilibrium.

On the basis of Aumann's and Brandenburger's result, a Nash equilibrium can be interpreted as an equilibrium in beliefs.[13] As a consequence, neither the multiplicity of equilibria in a specific game, nor the question whether an equilibrium is pure or mixed, leads to the difficulties that arise in justifying equilibria as actual strategy choices. It is also interesting that full common knowledge of rationality is not necessary to justify equilibria. On the other hand, the assumptions of common priors and that the players' conjectures are commonly known is obviously very strong.[14]

In extensive games, Aumann (1995) distinguishes between material and substantive rationality and between ex-ante and ex-post rationality. In all cases, a player is rational if he does not know of another strategy that yields higher utility. Material rationality means that a player is rational at all decision nodes that are reached (in the play associated with the state of the world); substantive rationality means that he is rational at all decision nodes, whether play reaches them or not. If a player is ex-ante rational then he does not know at the beginning of the game that, at some

[12] For a survey see Dekel & Gul (1997). See also Binmore (1996*b*), Aumann (1996, 1998)

[13] This interpretation goes back to Harsanyi (1973).

[14] Therefore the equilibrium selection problem also arises in Bayesian game theory.

node, another strategy yields higher utility (conditional on the node being reached). If a player is ex-post rational, then he does not know at the time of his choice of action that another action yields higher utility. Since under perfect recall a player knows at the time of his choice what he knew at the beginning of the game, and since rationality is defined as the absence of knowledge (of a superior strategy), ex-post rationality implies ex-ante rationality. Aumann (1995) shows that, in perfect information games, common knowledge of ex-ante substantive rationality implies the backward induction outcome.

It is exciting to see how the development described above constitutes progress in our understanding of rationality in games. In spite of this, even the study of the epistemic foundations of game theory has so far not been able to provide a conclusive characterization of strategic rationality:

First, the results for extensive form games leave open questions. Aumann's (1995) result depends on his definition of rationality. Also, it does not apply to extensive form games in general, or the backward induction strategy profile. Also, stronger equilibrium refinements still lack an epistemological foundation.[15]

Secondly, so far Savage's theorem is not formally applied to the interactive belief model, and such a formal application still faces difficulties.[16] It is not clear whether the states in an interactive belief system can be states of the world in Savage's framework. They must 'leave no relevant aspect undescribed' (Savage 1954) and thus must themselves include a description of the players' knowledge and beliefs. It is not clear whether strategies can be viewed as Savage acts, because the state of the world alone already determines the consequence in an interactive belief system. Even if this is assumed, strategy spaces are typically not rich enough to satisfy Savage's theorem. Further, in games players choose a strategy, rather than rank their available ones. Also, the utility function is derived on the basis of beliefs, whereas in games the utility functions are assumed to be (mutually) known. And, as mentioned above, the beliefs resulting from Savage's theorem are non-atomic and only finitely additive measures.

A third objection to the view that game theory captures rationality is based on

[15] However, such studies are under way, see e.g. Battigalli & Bonanno (1998). For related difficulties in extensive games see also Basu (1988, 1990) and Reny (1993).

[16] Mariotti (1998) formulates many of these difficulties for the case in which opponents' strategies are taken as states, and they also apply essentially to interactive belief systems.

the principle of sequential rationality that underlies subgame perfection. According to this principle (see, e.g., van Damme (1992)), deviations from rationality are not taken as evidence of non-rationality. For other equilibrium refinements, e.g. sequential equilibrium (Kreps & Wilson 1982*b*) and trembling-hand perfect equilibrium (Selten 1975), this is justified by assuming that (otherwise fully rational) players 'tremble' when they choose their actions. Thus deviations are interpreted as mistakes, that allow no inference about the lack of rationality of the opponent,[17] and 'complete rationality' is regarded as a 'limiting case of incomplete rationality' (Selten 1975).

Quite apart from its empirical implausibility, no argument is given why rationality would have to be conceptualized in this way. In fact, von Neumann and Morgenstern argued that

> *... the rules of rational behavior must provide definitely for the possibility of irrational conduct on the part of others. In other words: Imagine that we have discovered a set of rules for all participants — to be termed as "optimal" or "rational" — each of which is indeed optimal provided that the other participants conform. Then the question remains as to what will happen if some of the participants do not conform. If that should turn out to be advantageous for them — and, quite particularly, disadvantageous to the conformists — then the above "solution" would seem very questionable. We are in no position to give a positive discussion of these things as yet — but we want to make it clear that under such conditions the "solution", or at least its motivation, must be considered as imperfect and incomplete. In whatever way we formulate the guiding principles and the objective justification of "rational behavior," provisos will have to be made for every possible conduct of "the others." Only in this way can a satisfactory and exhaustive theory be developed. But if the superiority of "rational behavior" over any other kind is to be established, then its description must include rules of conduct for all conceivable situations — including those where "the others" behaved irrationally, in the sense of the standards which the theory will set for them."*
>
> *von Neumann & Morgenstern (1947, p.32)*

Finally, even the assumption that rationality is mutual knowledge is empirically implausible, and theoretically raises the question what constitutes rationality if it is not. Our preceding argument implies that lack of mutual knowledge of rationality arises endogeneously in extensive form games, after a deviation from rational play.

[17] This criticism of backward induction is well-known, see, e.g., Fudenberg & Tirole (1991).

So the question arises, what strategic rationality means if already at the beginning of the game there is this lack of mutual knowledge of rationality.

This question was first addressed in a seminal series of papers by Kreps, Milgrom, Roberts and Wilson (1982, henceforth KMRW). Again on the basis of subjective expected utility theory, they considered the case in which the rational players have a specific belief about the 'type' of a non-rational opponent in an incomplete information game. For instance, in the finitely repeated prisoners' dilemma, they showed that if the non-rational types are believed to be tit-for-tat players and the game is repeated sufficiently often, then mimicking tit-for-tat is also an equilibrium strategy for the rational players. This approach sheds light on the experimental evidence (e.g. Selten & Stoecker (1986)). More generally, this approach has been extremely useful in Industrial Organization, where it made it possible to rationalize intuitively important strategic phenomena like limit pricing and predatory pricing.

From a theoretical point of view, the KMRW approach allows to circumvent the difficulties of the principle of sequential rationality. Here, a deviation from the equilibrium strategy is indeed taken as evidence that players are not rational. In this case, the rational players conclude that they face the non-rational type. However, also this approach has its difficulties:

First, the theory does not explain why the rational players should hold a specific belief about the non-rational type. Subjective expected utility theory alone does not provide any restrictions, and in fact does not even provide a reason why different players should agree on such beliefs. Expressing this ignorance simply through a uniform probability distribution is not satisfactory either, because game trees can be changed in ways that should be strategically irrelevant (Thompson 1952, Dalkey 1953). For instance, postulating that the uniform distribution should describe beliefs about non-rational players would imply that the rational strategies are not invariant with respect to additions of duplicate strategies.

A second difficulty with the KMRW approach is that it relies on Harsanyi's (1967) analysis of games with incomplete information. Thus the possibility that a player is non-rational is captured by the assumption that a corresponding 'non-rational' type exists. But in Harsanyi's analysis, a type is a convenient way to capture the infinite hierarchy of beliefs that arises in incomplete information. Moreover, assuming that such types exist corresponds to assuming that these hierarchies satisfy consistency requirements (see, e.g., Brandenburger & Dekel (1993). These assumptions are not in the spirit of interpreting players as not rational.

Finally, if the KMRW approach is interpreted as capturing rationality, then rational strategies result from applying the equilibrium concept, or its refinements, to the resulting incomplete information game. But then, any non-equilibrium strategy is characterized as non-rational. This, in turn, does not square with the starting assumption that the rational players have a specific belief about non-rational play.

To conclude, there are ample reasons to agree with Binmore's assessment about game theory that "much of what we say in defending what we do does not hang together properly." (Binmore 1992). The aim of this thesis is to address some of the issues raised above. Of course, it will at best be possible to provide a first step towards a better characterization of strategic rationality. We will argue that this requires a weakening of subjective expected utility theory. Such generalizations of SEU have been developed on the basis of empirical objections to its descriptive interpretation.

1.4 The Ellsberg Paradox and Choquet Expected Utility Theory

When Savage proposed the subjective expected utility model, he allowed both a descriptive and a normative interpretation of the theory.[18] Subsequently, its descriptive validity, i.e. the claim that people act in accordance with the theory was criticised, first on the basis of thought experiments and casual empiricism, later

[18] **Later, Savage (1961, 1971, 1977) was very clear that its main significance was normative:**

> *"A personal theory could be given a psychological and empirical interpretation as predicting the behavior of some class of "persons." As empirical theories, they are not very interesting, nor have they very wide domains of validity. Their real importance is as normative theories by which a person, like you, can police himself for coherence."*
>
> *Savage (1977, p.10)*

And about his book (1954):

> *"The author, though interested in personal probability, was not yet a personalistic Bayesian ..."*
>
> *Savage (1977, p.18)*

on the basis of experimental evidence. The criticism was first directed against expected utility theory under risk (e.g., Allais (1953)), and this has led to alternative, or more general decision theories under risk, e.g. generalized expected utility (Machina 1982), regret theory (Bell 1982, Loomes & Sugden 1982), prospect theory (Kahneman & Tversky 1979) and anticipated utility theory (Quiggin 1982).

The criticism of Ellsberg (1961) was specifically directed at subjective expected utility theory under uncertainty. Ellsberg considered the following thought experiments:

First, consider a decision-maker who is faced with the choice between bets on the colour of a ball drawn at random from an urn. Assume the decision maker knows that there are 90 balls in the urn, 30 of which are red. He also knows that the remaining 60 balls are blue or green, but he does not know the proportions. Ellsberg argued, that when faced with the alternative, whether to bet on a red ball or on a green ball, most people would prefer the first choice. On the other hand, when faced with the alternative of winning if the ball is either red or blue or winning if the ball is either green or blue, most people would prefer the second choice. However, taken together these decisions are inconsistent with the assumption that the decision-maker is 'probabilistically sophisticated', i.e. that his beliefs can be represented by a probability measure p: His first choice implies that[19] $p(\text{red ball}) > p(\text{green ball})$. His second choice implies that $p(\text{green ball}) + p(\text{blue ball}) > p(\text{red ball}) + p(\text{blue ball})$. So the probabilities cannot add up to 1.

Similarly, consider a decision-maker who is presented with the choice between bets on the colour of a ball drawn at random from two different urns. Assume the decision maker knows that there are 100 balls in each urn. He knows that in the first urn, 50 balls are red and 50 balls are blue. He also knows that the balls in the second urn are red or blue, but he does not know the proportions. Again, Ellsberg argued, that when faced with the alternative, whether to bet on a red ball drawn from the first urn or on a red ball drawn from the second urn, most people would prefer the first choice. But they would also prefer betting on a blue ball drawn from the first urn to betting on a blue ball drawn from the second urn. Again this violates probabilistic sophistication: His first choice implies that $p(\text{red ball from urn I}) >$

[19] The strict inequality means that we exclude the possibility that the decision-maker is indifferent. This is justified on the basis of the experimental evidence, see below.

p(red ball from urn II), his second choice implies that p(blue ball from urn I) $>$ p(blue ball from urn II).

The intuitive explanation of this behavioral pattern is that the decision-maker is uncertainty-averse, or 'ambiguity averse'.[20] Here, the uncertainty, or ambiguity, that he faces consists in his ignorance about the true proportions of balls. The decision-makers choices have a parsimonious explanation in terms of aversion to these unknown proportions: He prefers bets on events whose chances he knows. From a mechanical point of view, all that it takes for subjective expected utility theory to account for this behaviour is to allow for non-additivity of beliefs. From a decision-theoretic point of view, however, this requires a characterization of this behaviour in terms of the underlying preference relation over acts.

This characterization was achieved by Schmeidler (1989). Schmeidler considered decision-making in the Anscombe-Aumann model, and introduced a weakening of the independence axiom, called 'comonotonic independence'. This leads to 'Choquet expected utility theory' (CEU).

First, note that each (Anscombe-Aumann) act gives rise to a preference relation over states: If the decision-maker chooses act f that leads to roulette lotteries $f(s)$ and $f(s')$ in states s and s', then he prefers the occurrence of state s to the occurrence of state s' if $f(s) \succ f(s')$, where, again, $\succ$ is the preference relation over lotteries induced by the preference relation over acts by identifying lotteries with constant acts. Two acts f, g are commonly monotonic ('comonotonic') if there are no two states s and s' such that the player would prefer s to s' under f, but s' to s under g. Schmeidler's result is that this limited independence requirement still allows the identification of beliefs and utility and the separation from beliefs and utility.

Formally, Schmeidler (1989) considers the following axioms:

(A1) $\succeq$ is a complete pre-ordering, i.e. complete and transitive.

(A2) The decision problem is non-trivial, i.e. there exist f, f' with $f \succ f'$.

(A3) The preference relation is continuous, i.e. if $f \succ g \succ h$ then there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.

(A4) Preferences are monotonic, i.e. if $f(s) \succeq f'(s)$ for all $s \in S$ then $f \succeq f'$.

[20] We will use the terms interchangeable, without claiming that they are equivalent beyond the formal models of Schmeidler (1989), Gilboa & Schmeidler (1989), et al. .

(A5') Preferences satisfy the comonotonic independence axiom, i.e. if f, f' and g are pairwise comonotonic and if $f \succ f'$ then for all α with $0 < \alpha \leq 1$ and all acts g also $\alpha f + (1-\alpha)g \succ \alpha f' + (1-\alpha)g$.

Schmeidler's (1989) first main result is that preferences over acts satisfy the five axioms if, and only if, there exist

(1) a utility function over consequences $u : Z \longrightarrow \mathbb{R}$ that is 'cardinal', i.e. unique up to affine transformations, and

(2) a unique 'capacity' v over states,

(3) such that one act is preferred to another if and only if its 'expected utility' is higher, i.e.

$$f \succeq f' \iff \int_S u \circ f \, dv \geq \int_S u \circ f' \, dv.$$

Here, a capacity, or 'non-additive probability', is a generalisation of a probability measure.[21] As in the case of probabilities, $v(\emptyset) = 0$, $v(S) = 1$ and for any event $0 \leq v(E) \leq 1$. In contrast to probabilities, capacities need not be finitely additive but only monotonic, i.e. if $E \subseteq E'$ then $v(E) \leq v(E')$.

Consequently, it is necessary to define the expectation of a random variable with respect to a non-additive measure. This definition was first given by Choquet (1953), in the context of mathematical physics.[22] Choquet (1953, Section 48.1) defined the integral of a non-negative real-valued random variable X as the extended Riemann integral $\int X \, dv := \int_0^\infty v(X \geq \alpha) \, d\alpha$, where as usual $v(X \geq \alpha) = v(\{s \in S | X(s) \geq \alpha\})$. In particular, the Choquet integral of the indicator function 1_E for event E gives the capacity $v(E)$. The mathematical theory of Choquet integration is developed, e.g., in the book of Denneberg (1994).[23]

[21] Formally, capacities are closely related to games in coalitional form. However, the characteristic function of a cooperative game is usually assumed to be superadditive, i.e. $v(E \cup E') \geq v(E) + v(E')$ for coalitions E, E'.

[22] Choquet makes additional assumptions on the underlying set S (a locally compact topological space), the set of events (the set of all compact subsets of S), and the capacity v (right-continuity). These assumptions are not relevant for the definition of the integral, nor the interpretation of the capacity in a decision-theoretic context.

[23] As an aside, we want to mention that Choquet himself has already contributed to game theory in various ways. First, he presented proofs of von Neumann's minimax theorem (Choquet 1955, Choquet 1968). Secondly, he gave a game-theoretic characterization of complete metric spaces in topology (Choquet 1969). Also, he studied the continuity properties of correspondences between general topological spaces (Choquet 1948).

The Choquet integral generalizes the usual formula for the expectation (of a non-negative random variable X that is bounded above by b) in terms of the decumulative distribution function if v is additive, i.e. $\int_0^b x\ dF = \int_0^b [1 - F(x)]dx$ for a cumulative distribution function $F(x) = \text{Prob}(X \leq x)$. Dellacherie (1970) and Schmeidler (1986) proved that the Choquet integral is additive on comonotonic functions, and Schmeidler (1986) also proved the converse: A monotonic functional that extends the capacity and is additive on comonotonic functions is the Choquet integral.

Since the first four of Schmeidler's axioms coincide with those of Anscombe & Aumann (1963), this result is a proper generalization of subjective expected utility theory within the Anscombe-Aumann model. In particular, it is important to note that this result holds independently of the decision-maker's attitude towards uncertainty, e.g. whether he prefers or dislikes ambiguity. Formally, the result does not restrict the shape of the capacity v.

Schmeidler's second main result is a characterization of uncertainty aversion. Exploiting the mixture space structure of the Anscombe-Aumann setup, preferences can be defined as uncertainty-averse if the players have a (possibly strict) preference for hedging, i.e. if $f \succeq g$ and $f' \succeq g$ then $\alpha f + (1-\alpha)f' \succeq g$. Schmeidler shows that, given (A1) — (A5'), preferences display uncertainty aversion if and only if the capacity that represents beliefs is supermodular,[24] i.e. $v(E \cup E') + v(E \cap E') \geq v(E) + v(E')$ for events E and E'.

1.5 Non-Additive Beliefs

The assumption that beliefs are not necessarily additive seems counterintuitive at first, in particular because probability theory is now so deeply entrenched. The aim of this section is to clarify the role of probability theory in game theory, its limits, and the interpretation of non-additive probability.

The modern probability concept was formulated by Kolmogorov (1933).[25] One of his main achievements was the separation of probability as a mathematical

[24] In cooperative game theory, characteristic functions with this property are usually called 'convex' (see, e.g., Shapley (1971). However, mathematically this property is an instance of 'supermodularity' in lattice theory.

[25] For the history of modern probability see, e.g., van Plato (1994).

concept from its empirical interpretation. Conceptually, 'probability' is taken as a basic, undefined term, and its empirical interpretation is left open. Mathematically, probability is part of measure theory: A probability measure p is defined as a real-valued set function on a σ-algebra that is normalized ($p(\emptyset) = 0$, $p(S) = 1$), monotonic ($E \subseteq F \implies p(E) \leq p(F)$, additive ($E \cap F = \emptyset \implies p(E \cup F) = p(E) + p(F)$) and continuous with respect to limits of sequences of sets ($\lim_{i \to \infty} E_i = E \implies \lim_{i \to \infty} p(E_i) = p(E)$). The last two conditions are usually combined to 'σ-additivity' ('denumerable additivity').

However, this probability concept also has both mathematical and conceptual limitations. In fact, Kolmogorov himself writes in (1948):[26]

> *"... from the point of view of the concrete tasks of probability theory, the system in question also deserves a certain amount of criticism. This criticism ... points out, correctly, the existence of arbitrary and artificial elements. (...)*
> *1st, the notion of an elementary event is an artificial superstructure ... . In reality, events are not composed of elementary events, but elementary events originate in the dismemberment of composite events.*
> *2nd, somewhat more complicated problems require, if the theory is to be simple and tractable, that probability be subject to the axiom of denumerable additivity. However, the justification of that axiom remains purely empirical ...*
> *3rd, we are forced to give up the principle, formulated in numerous classical works in probability theory, according to which an event of probability zero is absolutely impossible... ."*
>
> *Kolmogorov (1948)*

There are other mathematical limitations of the probability concept: First, if the state space is arbitrary, probability cannot be defined on the power set, so some events — although well defined — simply do not have a probability. Secondly, the uniform distribution can be defined for some state spaces (finitely many states, compact intervals) but not for others (countably many states, function spaces), and this does not depend systematically on the cardinality of the state space. Of great significance, in particular for game theory, is the difficulty that probability 0 events are not impossible. So the question of updating after probability 0 events arises. Finally, de Finetti (e.g. 1931a, 1949, 1970) maintained that the assumption of countable additivity is a useful regularity condition but not intrinsic to probability.

[26] However, in order not to mislead the reader, it has to be pointed out that in this article Kolmogorov goes on to define probability as a set function on complete metric Boolean algebras and shows that in this case finite additivity implies denumerable additivity.

Also, any application of probability theory relies on an empirical interpretation of the probability concept. Kolmogorov himself supported the frequency interpretation of probability (Kolmogorov 1933, § I.2, Kolmogorov 1968), that interprets probabilities as long-run relative frequencies. However, one of the tenets of this interpretation is that it is possible to repeatedly sample the domain of uncertainty: one-shot events do not possess probability according to the frequentist interpretation.

It is important to notice that this interpretation of probability is incompatible with its use in game theory. Game theory — and in particular the Nash equilibrium concept — rely on the interpretation that every relevant aspect of the strategic situation has been included in the description of the game. In particular, games have to be analysed as one-shot strategic interactions. For instance, Kohlberg & Mertens (1986) write

> *"We adhere to the classical point of view that the game under consideration fully describes the real situation — that any (pre)commitment possibilities,* any repetitive aspect,[27] *...have already been modelled in the game tree. ...Also, no random event (not described in the extensive form) can be observed by a player, except if it is completely independent ...of the moves of nature in the tree."*
>
> *Kohlberg & Mertens (1986)*

So instead, game theory relies on the subjective interpretation of game theory, created by de Finetti (1931a,b, 1937). According to this 'Bayesian' interpretation, a probability measure (the 'prior distribution') represents coherent beliefs. Coherence is defined as consistency in qualitative likelihood or in betting rates that do not allow systematic exploitation ('Dutch Book' arguments). This is also the interpretation of probability that arises from the expected utility theories of Savage (1954) and Anscombe & Aumann (1963).

On the other hand, also the subjective interpretation of probabilities has its limitations (Savage 1954, Savage 1967). First, probability reflects the internal coherence of beliefs about an uncertain event, without relating the shape of these beliefs to the available evidence. There is no requirement regarding the external consistency between beliefs and evidence, and beliefs also need not take into account the amount

[27] Emphasis added.

of available evidence.[28] Secondly, in the application of Bayesianism to statistics the prior distribution has to be specified. This results in a concern about the robustness of conclusions that are based on a single prior. This has lead to interval-valued and set-valued probabilities in statistics (e.g. Smith (1961)), and to the independent use of Choquet capacities in 'robust statistics' (Huber & Strassen 1973, Huber 1981, Walley 1981). Finally, probability emerges as a purely personalistic concept: any belief that is internally coherent is equally justified. In particular, there is no requirement that specifies when different individuals have to have identical beliefs. In turn, this approach does not explain why different individuals sometimes agree.

As a consequence, even the subjective interpretation of probabilities faces difficulties when it is applied to game theory. For example, Harsanyi (1967–68) introduced the 'common prior assumption' (CPA) to analyse games with incomplete information, and Aumann (1987) used it to give an epistemic foundation for correlated equilibria. Yet the common prior assumption demands complete agreement of probabilities that are purely personalistic. It is indeed logically "possible that Savage may have welcomed the CPA" (Aumann 1987, p. 13), but the assumption violates the spirit of personalistic probabilities.

Non-additive beliefs therefore emerge as a generalisation of subjective probabilities. Capacities are measures that represent coherent degrees of belief. The possible non-additivity merely reflects the fact that weaker coherence requirements are imposed.[29] To conclude, non-additive beliefs do not constitute as big a deviation from probability theory as one might suppose.

It is interesting that non-additive beliefs have a long history. In fact, at first probabilities were not necessarily additive; additivity emerged as a result of Bernoulli's (1713) weak law of large numbers, which is also one of the main sources of the frequency interpretation of probability (Hacking 1975). Both Bernoulli (1713) and Lambert (1764) were using non-additive probabilities. In economics, Knight (1921) argued that the distinction between uncertainty and risk is important and that the former is not quantifiable through probabilities. Shackle's (1949a,b) argued for

[28] Under repeated sampling, Bayesianism would reflect the amount of evidence through updating of a prior probability. But the amount of evidence may vary for other reasons than repeated sampling.

[29] There are also attempts to give a frequency interpretation of non-additive probabilities, see Walley & Fine (1982) and Marinacci (1999). In this thesis we have no need for this interpretation.

non-additive beliefs in his development of a post-Keynesian theory. Non-additive probabilities are also used in physics (e.g. Feynman (1963, 1985)). 'Belief functions', i.e. capacities that satisfy additional requirements (Dempster 1966, Dempster 1968, Shafer 1976) are widely used in Artificial Intelligence (see, e.g., Shapiro (1992)).

The need for a modification of subjective expected utility theory has recently also been realized in Bayesian game theory (Battigalli 1996, Battigalli 1997). Its origin lies in the realization that a strategy combination induces a 'conditional probability system' on the actions of an extensive form game. Conditional probability systems are due to Renyi (1955), and were introduced into game theory by Myerson (1986) (see also McLennan (1989), Vieille (1996)). A conditional probability system specifies a probability conditional on each event of a σ-algebra, i.e. even those events with probability 0, and they are related by Bayes' Rule. From a game-theoretic point of view, if a strategy combination of the opponents is to be interpreted as a belief of a player, these beliefs thus correspond to a conditional probability system. From a decision-theoretic point of view, such beliefs correspond to a strengthening of the Savage and Anscombe-Aumann axioms on preferences over acts (Myerson 1991, Chapter 1).

We mention conditional probability systems in order to show that deviations from subjective expected utility theory have already been accepted in game theory. However, if a player's belief about his opponents corresponds to a conditional probability system, deviations from rational play are not taken as evidence of non-rationality. The player's updated belief corresponds to parts of the strategy combinations of the rational opponents.

1.6 Uncertainty Aversion

Schmeidler's (1989) development of Choquet expected utility theory led to a large literature that further investigated the various aspects of decision theory under uncertainty aversion. The aim of this section is to address some of the issues that are relevant for its use in game theory.

First, Gilboa & Schmeidler (1989) have introduced another model of uncertainty aversion ('maximin expected utility theory', or 'MEU' for short). They also consider the Anscombe & Aumann (1963) model and impose the following axioms:

(A1) $\succeq$ is a complete pre-ordering, i.e. complete and transitive.

(A2) The decision problem is non-trivial, i.e. there exist f, f' with $f \succ f'$.

(A3) The preference relation is continuous, i.e. if $f \succ g \succ h$ then there exist $\alpha, \beta \in (0,1)$ such that $\alpha f + (1-\alpha)h \succ g \succ \beta f + (1-\beta)h$.

(A4) Preferences are monotonic, i.e. if $f(s) \succeq f'(s)$ for all $s \in S$ then $f \succeq f'$.

(A5") Preferences satisfy the constant-independence axiom, i.e. if f and f' are arbitrary acts and p^* is a constant act and if $f \succ f'$ then for all α with $0 < \alpha \leq 1$ and all acts g also $\alpha f + (1-\alpha)p^* \succ \alpha f' + (1-\alpha)p^*$.

(A6) Preferences display uncertainty aversion, i.e. if $f \sim f'$ then for all α with $0 \leq \alpha \leq 1$ we have $\alpha f + (1-\alpha)f' \succeq f$.

Gilboa & Schmeidler (1989) prove that that preferences over acts satisfy the six axioms if, and only if, there exist

(1) a utility function over consequences $u : Z \longrightarrow \mathbb{R}$ that is 'cardinal', i.e. unique up to affine transformations, and

(2) a closed[30] and convex set Q of finitely additive probability q,

(3) such that one act is preferred to another if and only if its minimal expected utility is higher, i.e.

$$f \succeq f' \iff \min_{q \in Q} \int_S u \circ f \, dq \geq \min_{q \in Q} \int_S u \circ f' \, dq.$$

It is interesting to notice how the models of Schmeidler (1989) and Gilboa & Schmeidler (1989) are related. First, the axioms (A1) – (A4) are the same. Secondly, the constant-independence axiom is neither weaker nor stronger than the comonotonic independence axiom: While a constant act is comonotonic with any other act, $(A5'')$ must also hold for acts f and f' that are not comonotonic. Finally, the Gilboa & Schmeidler (1989) model assumes uncertainty aversion; this assumption is not necessary for Schmeidler's first result. If, however, uncertainty aversion is also assumed in Schmeidler's model, it becomes more specific than the model of Gilboa & Schmeidler (1989): Supermodular capacities have a non-empty core (Shapley 1971), but the lower envelope of a convex set of probabilities need not be a supermodular ca-

[30] This refers to the weak* topology.

pacity. It is well-known from cooperative game theory, that the core of a capacity is non-empty if and only if the capacity is 'balanced' (Bondareva 1963, Shapley 1967).

As a consequence, the literature debates which property adequately represents uncertainty aversion (Epstein 1997*b*, Ghirardato & Marinacci 1997), and, so far, no consensus has arisen. However, this question is not a conceptual problem for the use we make of uncertainty aversion in this thesis. The definition of a Choquet-Nash equilibrium will be based on 'basic capacities', i.e. characteristic functions of sets. This class of capacities falls into both decision models, and we will have no need to distinguish them conceptually.

Secondly, Schmeidler's (1989) model is based on the Anscombe & Aumann (1963) decision model, in which acts are mappings from states into probability distributions over consequences. Since the motivation for developing the Anscombe-Aumann model was a simplification of Savage's decision theory, the axioms that describe decision-makers with non-additive beliefs should not depend on the mixture space structure of the Anscombe-Aumann model. This leads to attempts to develop Choquet-expected utility theory for the Savage model. A very transparent approach is due to Sarin & Wakker (1992).[31]

Sarin & Wakker (1992) take a distinction between ambiguous and unambiguous events as given. As a consequence, an act is unambiguous if it is measurable with respect to unambiguous events, and ambiguous otherwise. This allows them to impose, and restrict, the Savage axioms to the unambiguous acts, while to impose a weaker set of axioms on ambiguous acts. Sarin & Wakker (1992) show that this results in the Choquet expected utility representation of these preferences over acts.[32]

That the distinction between the Anscombe-Aumann model and the Savage-model is important for Choquet expected utility theory has been emphasized by Eichberger & Kelsey (1996*c*). In the Anscombe-Aumann model, an uncertainty-averse decision-maker will display a 'preference for randomization'; in fact, Schmeidler (1989) takes

[31] Other studies were conducted by Gilboa (1987), Wakker (1989*b*), Wakker (1989*a*), Nakamura (1990), Chateauneuf (1991), Chew & Karni (1994), Oginuma (1994), Nehring (1994), Sarin & Wakker (1994), Grant, Kajii & Polak (1997) for CEU and by Casadesus-Masanell, Klibanoff & Ozdenoren (1998) for MEU.

[32] In addition, the resulting utility function is bounded, and the capacity is additive on unambiguous events.

this preference as the defining property. Eichberger & Kelsey (1996*c*) show, however, that the decision-maker will have no such preference in the Savage model, even if the capacity is supermodular (convex).[33] On the one hand, Eichberger & Kelsey (1996*c*) argue that assuming a preference for randomization is counter-intuitive. On the other hand, the Anscombe-Aumann setup can be reproduced in experiments, so that the 'preference for randomization' assumption does have empirical meaning. Finally, some open questions remain in the Sarin & Wakker (1992) model: There is no axiom that corresponds to uncertainty aversion, and nothing that guarantees convexity or balancedness of the belief-representing capacity. Also, taking a distinction between ambiguous events and unambiguous events as given has two unsatisfactory aspects: First, even the ambiguous events are sets in the sense of set theory, and well-defined by their extensions; they are not ambiguous formally but only in the interpretation. Secondly, it is not the preferences that make some events ambiguous. In particular, beliefs over unambiguous acts have to be additive, which is, however, an empirical question.

Here, we take the view that the important insight of Eichberger & Kelsey (1996*c*) is that CEU in the Anscombe-Aumann model and in the Savage model correspond to two different kind of preferences. Beyond this, however, we argue that there is no a priori 'correct' model of decision-making under non-additive beliefs: Since the deviation from subjective expected utility theory was motivated by the empirical evidence of the Ellsberg paradox, it is also an empirical question which model better describes decision-makers. Eichberger & Kelsey (1996*c*) note, however, that there is no strong empirical evidence for a preference for uncertainty in decision situations.

Finally, it is important to mention that other theories for decision-making under uncertainty have been developed, some of them also with the objective to explain the Ellsberg paradox. In particular, Bewley (1986, 1987a, 1987b, 1988, 1998) developed 'Knightian Decision Theory', in which preferences over acts may be incomplete. This results in a representation of preferences in which beliefs may be sets of probabilities, and acts are only chosen when they are unambiguously better than a 'status quo' alternative. Quiggin (1982) has developed 'anticipated utility theory' (later called 'rank-dependent utility theory'), which applies to risk rather than uncertainty, but which is formally closely related to Choquet expected utility

[33] Mathematically, this is due to the fact that Fubini's theorem does not hold for general non-additive measures, see, e.g., Ghirardato (1997).

theory: The exogenously given probabilities are distorted and the expectation of the distorted function is calculated as a Choquet integral.

1.7 Updating

The problem how to update non-additive beliefs arises naturally in extensive form games, in which the opponent's strategies are at least partly observable. Several approaches have been developed how to update non-additive beliefs. In this section, we survey some of these and explain our choice of the 'Dempster-Shafer rule'.

The natural starting point for the investigation of the problem how to update non-additive beliefs is conditional probability, respectively Bayes' rule, i.e. $v(A|B) := \frac{v(A\cap B)}{v(B)}$. Formally, Bayes' rule is still well-defined even if v is not additive. However, Bayes' rule leads to conceptual difficulties.

Since the capacity v is derived from the preference relation over acts, it is appropriate to consider the problem of updating this preference relation. Following Machina (1989), we can define that act f is preferred to act f' conditional on event B and 'reference act' g if the act[34] (f_B, g) is preferred to the act (f'_B, g). Gilboa & Schmeidler (1993) have shown that, in this framework, the application of Bayes' rule to non-additive beliefs corresponds to 'optimistic updating', in the sense that the decision-maker updates his beliefs as if his reference act g assigns the worst consequence to the event that B does not occur. This is optimistic, because the event B is then always regarded as good news. The assumption of optimistic updating is consistent with non-additive beliefs in general, but does not seem to agree with the assumption that players are uncertainty-averse, i.e. take a pessimistic attitude towards uncertainty.

A second problem associated with Bayes' rule is the presence of belief 0 events. In particular, consider the 'basic capacity'[35] that associates belief 1 to the event S but probability 0 to any event $B \subset S$. This capacity is supermodular, taking the

[34] Here, act (f_B, g) is the act that gives the same consequence (or lottery over consequences) as f if the state is in B, and the same as g if not. The definition is independent of the question whether acts are Savage acts or Anscombe-Aumann acts.

[35] We call these capacities basic because they form a basis for the linear space of all capacities over a finite state space (Shapley 1953). In cooperative game theory these capacities are called 'simple games', but in the literature on CEU 'simple capacities' are distorted probability distributions.

Choquet integration of a random variable X with respect to this capacity yields the minimal value $\min_{s \in S} X(s)$. Consider now the case that an event $B \subset S$ occurs. Bayes' rule is unable to update the capacity because the event B has belief 0. Yet, intuitively, it is clear that the natural updated capacity associates belief 1 to the event B and 0 to every event $A \subset B$.

Both of these difficulties can be avoided through the 'Dempster-Shafer rule'. This updating rule has been proposed in statistics (Dempster 1967, 1968) and was further developed to a theory of evidence based on 'belief functions' by Shafer (1976). It has found many applications in Artificial Intelligence (e.g. Guan & Bell (1991), Yager, Kacprzyk & Fedrizzi (1994), Smets (1994)).

In order to derive a belief function, consider a finite state space S and a 'basic probability assignment' on its power set 2^S, i.e. $m(E) \geq 0$, $m(\emptyset) = 0$ and $\sum_{E \in 2^S} m(E) = 1$. Assume that $m(E)$ is interpreted as the weight of evidence that supports the belief that the true event is E, but the evidence does not allow a more restrictive interpretation. Then a belief function is a set function on S that is defined as $v(E) := \sum_{A \subseteq E} m(A)$, i.e. the belief in the event E is given by the sum of the weights of evidence on events that imply E. Shafer (1976) shows that the resulting set function v is a capacity that is 'completely monotonic', i.e. for all $n \in \mathbb{N}$ and for all $E_1, ..., E_n \in 2^S$

$$v(\bigcup_{i=1}^{n} E_i) \geq \sum_{\substack{I \subseteq \{1,...,n\} \\ I \neq \emptyset}} (-1)^{1+card\, I}\, v(\bigcap_{i \in I} E_i).$$

In particular, belief functions are supermodular (n=2), i.e. express uncertainty aversion.

The Dempster-Shafer rule for updating v conditional on event B is given by the formula

$$v(A|B) := \frac{v(A \cup \overline{B}) - v(\overline{B})}{1 - v(\overline{B})}.$$

It is only defined for $v(\overline{B}) < 1$.

The Dempster-Shafer rule reduces to Bayes' rule if v is additive. This updating rule allows the intuitive updating of basic capacities described above. Moreover, Gilboa & Schmeidler (1993) have shown that the Dempster-Shafer rule corresponds to pessimistic updating, i.e. the reference act assigns the best outcome to the event that B does not occur, so that B is regarded as bad news. The assumption of pessimistic updating naturally complements the assumption of uncertainty aversion. Also, the Dempster-Shafer rule preserves uncertainty aversion, i.e. an updated supermodular capacity is itself supermodular.

However, there are also conceptual difficulties associated with the Dempster-Shafer rule. First, before the work of Gilboa & Schmeidler (1993) it was criticised for lacking a behavioural foundation. Secondly, recall from above that that dynamically consistent updating of preferences over acts means that act f is preferred to act f' conditional on event B and 'reference act' g if the act (f_B, g) is preferred to the act (f'_B, g), because then the decision maker will not change his behaviour if he learns that B occurred and he originally preferred (f_B, g) to (f'_B, g). Epstein & Breton (1993) and Eichberger & Kelsey (1996*b*) show that the Dempster-Shafer rule is not dynamically consistent in the following sense: If dynamic consistency holds for all acts g then beliefs must be additive, and there is no updating rule that is dynamically consistent if beliefs are not additive.

As a consequence, Eichberger & Kelsey (1996*b*) conclude that there is no 'correct' updating rule for non-additive beliefs, and argue that the updating rule should depend on the application. As argued above, the pessimistic updating captured by the Dempster-Shafer rule agrees with the assumption of uncertainty aversion. In addition, given that the deviation from subjective expected utility to decision theory with non-additive beliefs was motivated empirically by uncertainty aversion in the Ellsberg paradox, the requirement of a theoretical consistency argument like dynamic consistency is not very compelling. We take the view that it is an empirical question which updating rule better describes decision-making in experiments.

Other updating rules have been suggested for non-additive beliefs (e.g. Walley (1981) and Fagin & Halpern (1990)). However, the Walley-Fagin-Halpern rule is also not dynamically consistent, and in addition would not allow updating of basic capacities. Further, Gilboa & Schmeidler (1993) have shown that the Dempster-Shafer rule coincides with 'maximum likelihood updating' of sets Q of additive beliefs, which selects those beliefs from Q that maximise the probability of the condition B and updates those by Bayes' rule, i.e. $v(A|B) = \min\{q(A|B) = \frac{q(A \cap B)}{q(B)} \mid q(B) = \max_{q' \in Q} q'(B)\}$.

1.8 Non-Additive Beliefs in Game Theory

After the development of Choquet expected utility theory the question arises naturally what its implications for Bayesian game theory are. The aim of this section is to survey the solution concepts that have been suggested.

The analysis of the consequences of non-expected utility theories for games began early (e.g. Fishburn (1972)). For instance, Blume, Brandenburger & Dekel (1991) extend non-Archimedean expected utility theory to games. In particular, Crawford (1990) presented a game-theoretic analysis of the consequences of a general failure of the independence axiom for individual preferences. He proposed a general 'Equilibrium in Beliefs' for two-player normal form games. Since Choquet expected utility theory assumes a weak form of independence, and therefore beliefs are further restricted, the following equilibrium concepts can be seen as special cases of equilibria in beliefs. However, due to the functional form of preferences, the presence of more than two players, or an extensive form structure many additional aspects arise.

Equilibrium and rationalizability concepts based on some form of CEU have been proposed by Dow & Werlang (1994), Eichberger & Kelsey (1993), Epstein (1997*a*), Haller (1995), Hendon, Jacobsen, Sloth & Tranæs (1996), Klibanoff (1993), Lo (1995*b*), Lo (1996), Marinacci (1994), Mukerji (1994), and Ritzberger (1996) for normal form games, and by Eichberger & Kelsey (1995), Lo (1995*a*) and Ryan (1997*a*) for extensive form games.[36] The literature is surveyed in Eichberger & Kelsey (1993) and Haller (1997).

The common objective of these studies is the question, in which sense the assumption of non-additive beliefs can be consistent with game-theoretic reasoning, and what the consequences of non-additive beliefs in strategic interaction are. Thus these studies do not explicitly focus on the conceptual problems in the foundations of game theory described above.[37]

In section 8.1, we present the equilibrium concepts for normal form games that incorperate non-additive beliefs. In section 8.2 we discuss solution concepts for extensive form games. The aim of these sections is to convey which approaches

[36] See also Hart, Modica & Schmeidler (1994), who use MEU for a joint derivation of utility and value in two-player zero-sum games. However, this work is stronger linked with decision theory than with non-cooperative game theory.

[37] However, it can be argued that they do so implicitly.

have been taken in the literature, and which aspects have been identified as critical, but we will not attempt to give an exhaustive survey. Section 8.3 relates these approaches to that taken in this thesis.

1.8.1 Normal Form Concepts

In the following, consider a finite normal form game $G := (I, S, u)$, where I is the set of players, S_i is the set of pure strategies of player $i \in I$, and $S = \times_{i \in I} S_i$ is the set of pure strategy profiles. Let 2^{S_i} be the power sets of S_i. The sets of mixed strategies are given by Σ_i, and $\Sigma = \times_{i \in I} \Sigma_i$ is the set of mixed strategy profiles. As usual, S_{-i} and Σ_{-i} are the sets of i-incomplete strategy profiles, that specify a pure, respectively mixed, strategy for every player other than i. Let $u = (u_i)$ be the I-tuple of von Neumann - Morgenstern utility functions for the players.

• Dow & Werlang (1994) consider finite two-person games in normal form. They define a 'Nash Equilibrium under Uncertainty' as a pair of capacities v_1, v_2 that correspond to the players' beliefs, i.e. $v_1 : 2^{S_1} \longrightarrow [0,1]$ is player 2's belief about the strategy choice of player 1, and analogously for player 2. In order to formulate how these beliefs have to be consistent, they study the question how the support of a capacity should be defined. They propose that the support of a capacity v is a minimal event for which the complement has capacity zero, i.e. $S = supp_1\ v\ :\Longleftrightarrow v(\overline{S}) = 0$ and $v(\overline{T}) > 0$ for any event $T \subsetneq S$. A pair of capacities then forms a Nash equilibrium under uncertainty if there exist supports of v_1 and v_2 such that each pure strategy in these supports maximises the Choquet expected utility of that player given his belief, i.e.

$$\forall s_1^* \in supp\ v_1 :\ s_1^* \in \arg\max_{s_1 \in S_1} \int_{S_2} u_1(s_1, s_2)\ d\ v_2,$$

$$\forall s_2^* \in supp\ v_2 :\ s_2^* \in \arg\max_{s_2 \in S_2} \int_{S_1} u_2(s_1, s_2)\ d\ v_1.$$

First, notice that this equilibrium concept is formulated in terms of pure strategies, so it does not allow for a strict preference for mixed strategies. Secondly, notice that the support concept implicitly defines the knowledge of the players[38]. Since 'non-additive beliefs' cannot be correct if the opponent chooses a mixed strategy, the players are only assumed to correctly anticipate the support. The non-additivity

[38] See also Morris (1993).

of beliefs then allows the expression of uncertainty aversion. Dow & Werlang (1994) show that non-rationalizable outcomes may be Nash equilibria under uncertainty, and that there are equilibria that differ from the backward induction equilibrium if the once-repeated prisoners' dilemma is analysed as a normal form game. They also show that their approach is different from the 'crazy type' approach of Kreps, Milgrom, Roberts & Wilson (1982), because uncertainty averse players would ignore the possibility of benevolent crazy types, so that only harmful types would enter the players' calculations.

However, there are several difficulties with their approach: First, a natural starting point for defining the support of a capacity is to define it as the smallest set that has belief 1, i.e. $S = supp_2\ v\ :\Longleftrightarrow\ v(S) = 1$ and $v(T) < 1$ for any event $T \subsetneq S$. The problem with this definition is that under uncertainty aversion only the universal set would qualify as a support. On the other hand, Dow & Werlang (1994) show that a support in their sense need not be unique. In addition, consider again the basic capacities that give weight 1 to the universal set S and weight 0 to any other set. Under the definition of support of Dow & Werlang (1994), each individual element is a support of this capacity. Secondly, in equilibrium players do maximise utility and choose their strategies accordingly. This is not anticipated by the players. For this reason, Dow & Werlang (1994) propose their equilibrium concept as a model in which players lack logical omniscience. Thirdly, their equilibrium concept does not address the perfection aspect of extensive games, and the backward induction equilibrium is also a Nash Equilibrium under Uncertainty.[39,40]

• Klibanoff (1993) considers finite normal form games with finitely many players.

[39]Haller (1995) considers finite two-player games in normal form. He investigates the consequences of introducing non-additivity on the solvability of games. A game is solvable if for any two Nash equilibria (σ_1^*, σ_2^*) and (τ_1^*, τ_2^*) the strategy combinations (σ_1^*, τ_2^*) and (σ_1^*, τ_2^*) are also equilibria, and all Nash equilibria are payoff-equivalent for each player. It is well-known that all zero-sum games are solvable under Nash equilibrium. Haller (1995) adopts the solution concept of Dow & Werlang (1994) and shows that even zero-sum games need not be solvable under Nash Equilibrium under Uncertainty.

[40]Marinacci (1994) considers finite two-player games in normal form. He suggests a variation of the solution concept of Dow & Werlang (1994). As in Dow & Werlang (1994), players maximise Choquet expected utility given their possibly non-additive beliefs, and beliefs have to be correct in the sense that all strategies in the support of the beliefs are best replies. The difference is that he adopts as support the set of all pure strategies that have strictly positive weight, i.e. $supp_3\ v_i := \{s_i \in S_i \mid v_i(s_i) > 0\}$.

He defines an 'Equilibrium with Uncertainty Aversion' as a mixed strategy σ_i for each player, together with, for each player, a (closed and convex) set of additive beliefs Q_i over the opponents' pure strategy combinations. Each player chooses his mixed strategy in order to maximise the minimal expected utility in the sense of Gilboa & Schmeidler (1989) given his beliefs Q_i, and the beliefs have to be consistent in the sense that player i considers the i-incomplete equilibrium profile as possible. Formally, for all $i \in I$,

$$\sigma_i^* \in \arg\max_{\sigma_i \in \Sigma_i} \min_{q_i \in Q_i} u_i(\sigma_i, q_i), \text{and}$$

$$\sigma_{-i}^* \in Q_i.$$

First, notice that the equilibrium concept is formulated in mixed strategies. Here, players are allowed to have a strict preference for mixed strategies. Secondly, in games with more than two players, the players are allowed to believe that their opponents' choices are correlated, even though in equilibrium the actual choices are independent.

• Eichberger & Kelsey (1993) extend Dow's and Werlang's equilibrium concept to games with finitely many players. The players' beliefs are given by capacities v_i defined on the power set of the opponents' pure strategy combinations $2^{S_{-i}}$. The support concept is defined as in Dow & Werlang (1994). Let $S_i^*(v_i)$ be the set of pure strategies that maximise Choquet expected utility of player i. Then a profile of non-additive beliefs v_i form an Equilibrium under Uncertainty iff, for all $i \in I$,

$$supp_1 \ v_i \subseteq \times_{i' \in I, i' \neq i} S_i^*(v_i).$$

As in Dow & Werlang (1994), an 'Equilibrium under Uncertainty' is defined in terms of pure strategies, i.e. excludes a strict preference for mixtures. As in Klibanoff (1993), players are allowed to believe that their opponents' choices are correlated.

• Mukerji (1994) also studies finite normal form games with finitely many players. He defines an 'Equilibrium in ϵ-ambiguous Beliefs' as a profile of 'simple capacities', i.e. contracted probabilities ($v(E) = \alpha \cdot p(E)$ for events $E \subsetneq S$ and $v(S) = 1$ for the universal set S, $\alpha > 0$). As noted by Dow & Werlang (1994), for such capacities the support coincides with that of the underlying probability and is therefore uniquely determined. Similarly, a simple capacity models independence if the underlying probability is a product of marginals. The beliefs modelled as contracted probabilities correspond to a game in which rationality is not mutual knowledge and ϵ is the probability that the opponent is not rational. The players are completely ignorant about non-rational play and are uncertainty-averse. An equilibrium in ϵ-ambiguous beliefs is then defined as a the equilibrium of Dow & Werlang (1994). In particular, it excludes a strict preference for mixed strategies.

• Lo (1995*b*) studies finite two-player games in normal form on the basis of Gilboa's and Schmeidler's (1989) maxmin expected utility theory. He defines two support concepts for sets of additive beliefs over the opponent's strategy choice. The support of Q_i is the set of all opponent's pure strategies that have strictly positive probability under any belief in Q_i, i.e. $supp\ Q_i := \{s_j \in S_j \mid q_i(s_j) > 0,\ \forall q_i \in Q_i\}$. The extended support of Q_i is the set of all opponent's pure strategies that have strictly positive probability under some belief in Q_i, i.e. $xsupp\ Q_i := \{s_j \in S_j \mid q_i(s_j) > 0,\ \exists q_i \in Q_i\}$. Lo (1995*b*) then defines a 'Cautious Nash Equilibrium' as a pair of sets of additive beliefs Q_1^*, Q_2^* such that for both players

$$\forall s_i^* \in supp\ Q_j^* \quad \forall s_i \in S_i : \quad \min_{q_i \in Q_i^*} u_i(s_i^*, q_i) \geq \min_{q_i \in Q_i^*} u_i(s_i, q_i), \text{ and}$$

$$\forall s_i^* \in xsupp\ Q_j^* \quad \forall s_i \notin supp\ Q_j^* : \quad \min_{q_i \in Q_i^*} u_i(s_i^*, q_i) \geq \min_{q_i \in Q_i^*} u_i(s_i, q_i).$$

The interpretation of this equilibrium concept is that every strategy in the support of Q_j is infinitely more likely than any strategy that is not, because for the first strategy $q_j(s_i) > 0$ for all $q_j \in Q_j$, whereas for the second strategy $q_j(s_i) = 0$ for some $q_j \in Q_j$. So the first condition requires that such strategies maximise maxmin expected utility of Gilboa & Schmeidler (1989). On the other hand, every strategy in the extended support of Q_j is still considered possible, although infinitely less likely than those strategies in the support. The second condition requires that these possible strategies are also rational for player i in the sense that only the infinitely more likely strategies are strictly better.

Since Lo's (1995*b*) equilibrium concept is formulated in terms of pure strategies, he does not allow a strict preference for mixtures. For this reason, his solution concept is a strengthening of 'Nash Equilibrium under Uncertainty' of Dow & Werlang (1994). Under the concept of Dow & Werlang (1994), no requirement is imposed on strategies that are not in the support of the players' beliefs.

• Lo (1996) studies finite normal form games with finitely many players. Again players' beliefs are modelled as sets of additive beliefs in the sense of Gilboa & Schmeidler (1989). For the set Q_j of beliefs of player j (additive probabilities over

S_{-i}) he defines the set of marginal distributions as $marg_{S_i} Q_j := \{\sigma_i \in \Sigma_i \mid \exists q_j \in Q_j : \sigma_i = marg_{S_i} q_j \}$. In a 'Beliefs Equilibrium' $(Q_i^*)_{i \in I}$ each such marginal must be a mixed strategy that maximises the minimal expected utility of that player, i.e.

$$marg_{S_i} Q_j^* \subseteq \arg \max_{\sigma_i \in \Sigma_i} \min_{q_i \in Q_i^*} u_i(\sigma_i, q_i).$$

The interpretation of this equilibrium condition is that player j, knowing that Q_i are player i's beliefs, can anticipate that player i will choose a utility maximising strategy. In analogy with Nash equilibrium, the set of strategies which do not maximise utility should have measure 0 under the marginal $marg_{S_i} Q_j$. In a 'Beliefs Equilibrium', two players' beliefs about a third need not coincide. For this reason Lo (1996) suggests a 'Beliefs Equilibrium with Agreement' as a set of beliefs Q_i^* for each player for which there exist sets of mixed strategies Σ_i^* such that, for all players i and j, the beliefs Q_i^* are the closed convex hull of all additive beliefs q on S_{-i} whose marginal $marg_{S_j} q$ is contained in Σ_j. That is, two different players agree on the play of a third player k in the sense that their sets of marginals are Σ_k. Also, a player believes that his opponents play independently in the sense that his set of beliefs contains all product measures (this independence concept is due to Gilboa & Schmeidler (1989)). Since the solution concepts are defined in terms of mixed strategies, strict preference for randomization is allowed.

Other solution concepts have been proposed by Epstein (1997*a*),[41] Hendon, Jacobsen, Sloth & Tranæs (1995),[42] and Ritzberger (1996).[43]

[41] Epstein (1997*a*) considers finite normal form games with finitely many players. He considers a class of general preferences that do not imply specific functional forms, but both CEU and MEU fall into this class. He then extends the rationalizability concept of Bernheim (1984) and Pearce (1984) to this class of preferences. Epstein (1997*a*) excludes a strict preference for mixtures, but allows correlated beliefs about the play of the opponents.

[42] Hendon et al. (1995) study finite normal form games with finitely many players. They propose a 'Nash Equilibrium with Lower Probabilities' which is based on belief functions. In an earlier paper (Hendon, Jacobsen, Sloth & Tranæs 1994) they show that the mixture space axioms of Herstein & Milnor (1953), when applied to belief functions, yield a utility function on the power set of consequences such that $v_1 \succeq v_2 \iff \sum_{E \in 2^Z} m_1(E) u(E) \geq \sum_{E \in 2^Z} m_2(E) u(E)$, where m_1 and m_2 are the basic probability assignments of belief functions v_1 and v_2.

On the basis of this result, they define the expected utility of a profile of belief functions through the basic probability assignments as

$$u_i(v_1, ..., v_n) := \sum_{E_1 \times ... \times E_n \subseteq S} m_1(E_1) \cdot ... \cdot m_n(E_n) u_i(E_1, ..., E_n).$$

In particular, the expected utility of a pure strategies $s_i \in S_i$ is defined by choosing for v_i the basic

1.8.2 Extensive Form Concepts

Eichberger & Kelsey (1995) are the first to consider a class of extensive form games. Specifically, they consider finite two-player signalling games: Player 1 receives full information about the 'state of nature' $\theta \in \Theta$ and takes an action $s_1 \in S_1$. Player 2 is able to observe s_2 but is ignorant of θ. Player 2 takes an action $s_2 \in S_2$. The actions and the state of nature then determine the players' utilities $u_i(s_1, s_2, \theta)$ for $i = 1, 2$. Eichberger & Kelsey (1995) extend the equilibrium concept of Dow & Werlang (1994) to this class of games. The extensive structure requires a specification of the updating rule, and they adopt the Dempster-Shafer rule. A 'Dempster-Shafer Equilibrium' consists of a pair of non-additive beliefs v_1 and v_2. The belief v_2 specifies player 1's belief about player 1's strategy choice. The belief v_1 specifies player 2's belief about player 1's strategy choice and the state of nature. For a Dempster-Shafer equilibrium, there must exist a support (in the sense of Dow & Werlang (1994)) of v_2 such that each pure strategy in the support is a Choquet expected utility maximising strategy for player 1. In addition, player 2 will update his belief v_1 after observing s_1. So, for a Dempster-Shafer equilibrium, for each strategy s_1 the updated capacity v_1' must have a support of pure strategies that maximises Choquet expected utility of player 1. Again, Eichberger & Kelsey (1995) do not allow a preference for randomization.[44]

probability assignment that puts weight 1 on strategy set $\{s_i\}$. Hendon et al. (1995) then adopt the support concept $supp_1$ and define a Nash equilibrium with lower probabilities as a profile of belief functions such that each strategy in its support maximises expected utility. They show that even cooperation in the one-shot prisoners' dilemma can be an equilibrium outcome under this solution concept.

[43] Ritzberger (1996) considers finite normal form games with finitely many players. In Ritzberger (1996), the equilibrium concept is defined in terms of strategy choice rather than belief. As in a Nash equilibrium, each player can anticipate the mixed strategies of the opponent, but evaluates lotteries over outcomes according to rank-dependent expected utility theory. That is, he distorts these probabilities when he calculates his expected utility. Ritzberger (1996) does not require that this distortion is equivalent to uncertainty aversion, in fact he treats the players' decision problem as one under risk rather than uncertainty. A 'Nash Equilibrium under Expected Utility with Rank-Dependent Probabilities' is then a Nash equilibrium in which it is common knowledge how the different players distort the probabilities.

[44] Ryan (1997*a*) extends the work of Eichberger & Kelsey (1995). He notes that the support of Dow & Werlang (1994) is not invariant after updating through the Dempster-Shafer rule (see also Ryan (1997*b*)). He proposes a different support concept that gives a unique support if it exists: $S = supp_4\ v \iff v(\overline{S}) = 0$ and $v(T) > 0$ for any non-empty event $T \subsetneq S$. He defines a 'Robust

Finally, Lo (1995*a*) studies finite extensive games with perfect recall. His equilibrium concept is too intricate to be given formally here. Lo (1995*a*) adopts the maxmin expected utility theory of Gilboa & Schmeidler (1989), so that each player's beliefs (at the beginning of the game) are given by sets of additive probability measures over the opponents' pure strategy sets. He requires that in equilibrium all such measures have the same support. Further, he postulates that the beliefs are updated by the maximum likelihood rule, and that the updated beliefs (at each information set) reflect independence and agreement as in Lo (1996). In a 'Multiple Priors Nash Equilibrium', every pure strategy in the support of the beliefs must then maximise the minimum expected utility, given the updated beliefs, on the paths of play that are compatible with the beliefs.

1.8.3 Summary

The equilibrium concepts described above have achieved two objectives: First, the introduction of non-additive beliefs can account for strategic phenomena that are unexplained by Nash equilibrium analysis, e.g. that players may have a strict preference for mixed strategies over the pure strategies in its support, or that non-rationalizable strategy profiles may be equilibria. Secondly, the solution concepts have identified the difficulties that the extension of the Nash equilibrium concept faces. In particular, the definition of support of a capacity, the inclusion of a preference for mixtures, and the accommodation of correlated beliefs about the opponents' play are choices of the modeller rather than determined by the structure of the interaction. Also, some of these concepts only apply to specific classes of games, or are very difficult to apply. Finally, some requirements seem to be motivated by analogy with the Nash equilibrium concept rather than an intrinsic consideration of rationality or strategic interaction, e.g. requirements about compatibility of marginals.

More fundamentally, all solution concepts with the exception of Mukerji (1994) and Lo (1995*b*) assume that rationality in the sense of Choquet expected utility maximisation is mutual or common knowledge. There is no source of ambiguity or uncertainty beyond the individual preferences. The players are assumed to under-

Dempster-Shafer Equilibrium' as a Dempster-Shafer equilibrium of Eichberger & Kelsey (1995) in which such supports exist and are preserved on the equilibrium path.

stand the structure of the strategic interaction less well than the modeller. Their failure to anticipate the consequences of rationality are due to a lack of logical omniscience. To be sure, these assumptions are indeed logically consistent. However, the solution concepts proposed in this thesis do not rely on these assumptions.

The solution concept of Mukerji (1994) is closely related to the (independently developed) concept of 'Choquet-Nash equilibrium' proposed in this thesis.[45] As in this thesis, Mukerji (1994) considers non-additive beliefs as a concept to analyse games in which rationality is not mutual knowledge.[46] He also adopts the framework of Kreps et al. (1982) and assumes that rational players face complete uncertainty about non-rational play and are uncertainty-averse. If the players' beliefs then are simple capacities, his solution concept coincides, for two-player games, with the 'Choquet-Nash equilibrium' defined in this thesis. This construction overcomes the conceptual problems described above and allows to extend the equilibrium concept to non-additive beliefs: The possible non-rationality of the opponent forms the source of uncertainty, and the rational players can anticipate the play of their rational opponents.

The solution concepts in this thesis are also, but not only motivated by the question how the equilibrium concept can be extended to non-additive beliefs. First, a 'Choquet-Nash equilibrium' is derived as a special case of a 'Weak Choquet-Nash equilibrium'. There the rational players have general non-additive beliefs about their non-rational opponents. Thus there is no role for 'simple capacities' in the solution concept. Also, the solution concepts differ for games with more than two players. In a Choquet-Nash equilibrium the events that more opponents are non-rational are independent. However, there are two main differences between the approach of Mukerji (1994) and the approach in this thesis:

First, we argue that the assumption of complete ignorance about the play of non-rational opponents does not only overcome the problem of specifying a prior in the Kreps et al. (1982) approach, but is a necessary consequence of the fact that the solution concept itself only specifies rational strategies. The consistency requirement

[45]Mukerji (1994) considers normal form games only. He does not extend his approach to a robustness analysis for Nash equilibria.

[46]In Lo (1995*b*), rationality is not mutual knowledge either, but it is mutual knowledge that it is infinitely more likely that the opponent is rational than that he is not. In Mukerji (1994) and in this thesis, these likelihoods are of the same order.

is that all deviations then must be considered non-rational. This forces beliefs to be non-additive, respectively set-valued.

Secondly, our solution concepts are mainly motivated by extensive form considerations. There lack of mutual knowledge of rationality arises endogenously after a deviation from rational play. Thus we aim to suggest a solution concept in which this deviation is taken as evidence of non-rationality. As a consequence, the 'Perfect Choquet equilibrium' proposed in this thesis is an equilibrium concept in which equilibrium arguments only apply on the equilibrium path. Off the equilibrium path a different solution concept applies because equilibrium arguments only hold for rational players. Again on the basis that we propose a rational solution concept only, all deviations are possible, and under uncertainty aversion this corresponds to minimax reasoning off the equilibrium path.

In formulating our solution concept for extensive form games we emply the Dempster-Shafer rule. Here our solution concept (independently) adopts the choice of Eichberger & Kelsey (1995). They note that normal form concepts "fail to exploit one of the strengths of non-additive probabilities, namely that unlike additive probabilities, they can be updated after events of probability zero." Eichberger & Kelsey (1995) extend the Dow & Werlang (1994) equilibrium concept to signalling games. In this thesis we also use the Dempster-Shafer rule to update after belief-zero events, but the equilibrium concepts differ both formally and in motivation and interpretation.

Finally, it remains to mention that there are other equilibrium concepts that address the extensive form aspect of interpreting deviations as evidence of non-rationality. In particular, Battigalli (1987), Reny (1992), Fudenberg & Levine (1993) and Rubinstein & Wolinsky (1994) considered equilibrium concepts that only require players to choose best replies to conjectures that are confirmed on the equilibrium path (and various other assumptions). Thus the players' conjectures may be wrong off the equilibrium path, but they never observe this; beliefs off the equilibrium path are not constrained. Typically, this leads to a generalization of perfect equilibrium, and to a multiplicity of equilibria. In contrast, in this thesis we do make strong assumptions on behaviour off the equilibrium path. This behaviour results from the players' attitude towards uncertainty, not his beliefs about the opponents. As a consequence, perfect Choquet equilibria do not just generalize perfect equilibria.

1.9 Economic Applications

The development of decision theory with non-additive beliefs was inspired by deviations from subjective expected utility theory as in the Ellsberg paradox. The extension to game theory with non-additive beliefs was a natural next step. Ultimately, however, the success of these theories will depend on the question whether they have interesting applications. A recent review is Mukerji (1999). Here we only want to list some of them:

In finance, Choquet expected utility theory has been applied to portfolio choice (Dow & Werlang 1992*b*, Dekel 1989, Simonsen & Werlang 1990), initial public offerings (Yoo 1990) and asset pricing (Dow & Werlang 1992*a*, Kelsey & Milne 1995, Epstein & Wang 1994).

In agency and contract theory some applications are Ghirardato (1994), Kelsey & Spanjers (1997), Mukerji (1996), Rigotti (1997), Rigotti (1998*b*), Rigotti (1998*a*), and Rigotti & Ryan (1998).

In information economics CEU is used in Dow, Madrigal & Werlang (1989), Eichberger & Kelsey (1995), Persico (1995), and Tallon (1995a, b).

Chateauneuf (1994) and Ben-Porath, Gilboa & Schmeidler (1997) analyse inequality measurement in an approach similar to CEU. Eichberger & Kelsey (1996*a*) study the effect of non-additive beliefs on the provision of public goods. Lo (1998) considers auctions with uncertainty-averse bidders. Wang, Young & Panjer (1997) apply anticipated utility theory to the study of insurance prices.

1.10 Experimental Evidence on Uncertainty Aversion

The aim of this thesis is to propose a rationality concept for extensive games for uncertainty averse players. It remains to survey the experimental evidence on uncertainty aversion.[47]

First, although Ellsberg's paradox has stimulated so much research, Ellsberg (1961) did not conduct scientific experiments. His reasoning was based on thought experi-

[47] Camerer (1995) offers a general survey of experiments about decision-making under risk and uncertainty, Camerer & Weber (1992) concentrate on non-additive beliefs. This section is based on Smithson (1997).

ments and casual observation.[48] However, careful experiments later confirmed Ellsberg's intuition. In the original urn examples, experimental subjects regularly show a strict aversion to ambiguity in the sense that they are willing to pay for avoiding it (Camerer & Weber 1992). Moreover, this aversion is robust: Even written arguments cannot convince them to change their beliefs. (Slovic & Tversky 1974). In addition, there is a low correlation between the revealed attitudes towards risk and the attitudes towards uncertainty[49] (Cohen, Jaffray & Saïd 1985).

Apart from confirming this basic intuition, the experimental studies further refined the understanding of uncertainty aversion. Some studies found that there is less uncertainty aversion when uncertain acts may avoid losses than when uncertain acts may provide gains (Cohen et al. 1985). Also, uncertainty aversion effects recedes when people believe that they have sufficient expertise in case the ambiguous outcome is realized (Heath & Tversky 1991).

It is important, however, that uncertainty aversion is by no means a universal empirical phenomenon. Experiments found that subjects prefer ambiguity when there are low probabilities of gains and high probabilities of losses (Curley & Yates 1985). Subjects also prefer consensual but ambiguous information to conflicting but precise information, i.e. people prefer ambiguity to conflict. And when asked to evaluate the likelihood of causes for specific events, it is found that subjects give too little weight to (ambiguous) catch-all categories ('catch-all underestimation bias') (Fischhoff, Slovic & Lichtenstein 1978).

The only experimental study so far that has investigated how players update their beliefs (Cohen, Gilboa, Jaffray & Schmeidler 1999) has not led to a clear-cut conclusion.

Similarly, there are not many experimental studies relevant for an evaluation of uncertainty aversion in games, possibly due to the lack of theoretical frameworks until recently. One related study is Beard & Beil (1994), who report on experiments whether players rely on the rationality of others. They find strong evidence against relying on the self-interest of opponents in games, even though most players indeed

[48] In the same vein, Gilboa & Schmeidler (1993) argue that the widespread use of classical statistics can be viewed as evidence against Bayesianism.

[49] Note that in CEU the attitude towards risk (expressed as the curvature of the utility function) is indeed independent of the attitude towards uncertainty (expressed as the shape of the belief capacity).

act in their self-interest.

Finally, it is also important to mention that there is evidence against the assumption that beliefs are monotonic, which is a finding that even calls into question the modelling of beliefs as capacities. Tversky & Kahneman (1983) (see also Selten (1991)) devised the famous 'Linda problem', in which they confronted subjects with the following information:

> *Linda is 31 years old, single, outspoken, and very bright. She majored in philosophy. As a student, she was deeply concerned with issues of discrimination and social justice, and also participated in anti-nuclear demonstrations.*
>
> *Tversky & Kahneman (1983)*

When asked to rank the likelihood of different propositions about Linda, Tversky & Kahneman (1983) report that 90 % rank the statement that "Linda is a bank teller and is active in the feminist movement" before "Linda is a bank teller", even though the former obviously implies the latter ('conjunction fallacy'). Tversky & Kahneman (1983) argue that decision-making is not based on beliefs at all, but on heuristics that lead to biases, i.e. systematic deviations from rationality. In the 'Linda problem', the 'representation heuristic', i.e. the hypothesis that subjects rank statements on the basis of their representativeness of the given information, is proposed as an explanation of observed behaviour. However, the status of both the evidence and the explanation is a subject of debate (e.g. Gigerenzer (1996), Kahneman & Tversky (1996).

1.11 Conclusion

This chapter has shown how game theory approaches the question what constitutes rationality in games. Despite significant progress, however, open questions remain.

In contrast, subjective expected utility theory in decision theory has received conflicting empirical evidence. This has led to the development of decision theories based on non-additive beliefs that explain the empirical evidence as uncertainty aversion. Under the subjective interpretation of probability, the non-additivity has a natural interpretation as a weaker consistency requirement on likelihood judgements. The extension of non-additive beliefs to game theory yields new strategic insights, but also faces conceptual difficulties.

The objective of this thesis is to bring these two strands together: In the following chapters we will argue for a particular way to introduce non-additive beliefs into game theory, based on consistency requirements for a rationality concept. In turn, the resulting equilibrium concept addresses some of the conceptual difficulties of capturing rationality in games.

1.12 Limitations

Throughout this work we will focus on non-cooperative games rather than cooperative games. In particular, both communication between players and their ability to write binding contracts are excluded unless formally modelled as part of the game. For the most part, we will focus on finite games, even though many games of economic interest are infinite.

We will concentrate on the extension of Nash's (1950) equilibrium concept and not consider rationalizability as a basis for the extension. Since communication is excluded, we will also not consider extensions of correlated equilibria.

We will present the motivation for the definition of these equilibrium concepts informally, and not give a formal epistemic characterisation. We also have nothing to say about other justifications of equilibrium ideas based on learning or evolution.

Finally, we will only present theoretical concepts and will not confront them with experimental evidence.

1.13 Terminology

In agreement with the literature, we use the concept 'non-additive' in the sense of 'not necessarily additive': Non-additive set functions may be, but need not be additive.

We often speak of a 'player' even when he is the single decision-maker.

We distinguish between 'normal form games' and 'strategic form games' in the following way: In a normal form game each player has one action and the players choose their actions simultaneously or in ignorance of each other's choice. A strategic form game is associated with an extensive form game in which players may take more than one action and may move sequentially. The actions of the strategic form game are the pure strategies of the extensive form game. We make this convention in order to avoid any misunderstandings when we discuss normal form games, i.e. they are not the strategic form games of extensive form games with a sequential structure.

Following Myerson (1991) (and Halmos (1950)), we sometimes use 'iff' to abbreviate 'if and only if'.

Chapter 2

Uncertainty Aversion and Equilibrium in Normal Form Games

Abstract

This chapter presents an analysis of games in which rationality is not necessarily mutual knowledge. We argue that a player who faces a non-rational opponent faces genuine uncertainty that is best captured by non-additive beliefs. Optimal strategies can then be derived from assumptions about the rational player's attitude towards uncertainty. This paper investigates the consequences of this view of strategic interaction. We present an equilibrium concept for normal form games, called Choquet-Nash Equilibrium, that formalizes this intuition, and study existence and properties of these equilibria. Our results suggest new robustness concepts for Nash equilibria.

2.1 Introduction

From a classical point of view, game theory is about the question what constitutes rationality in a situation of strategic interaction (von Neumann & Morgenstern 1944, particularly sections 2.1 and 4.1). The players are assumed to be rational in a decision-theoretic sense, i.e. they act as if they possess a utility function over outcomes and beliefs given by a probability distribution over states, and maximise (subjective) expected utility (von Neumann & Morgenstern 1944, Savage 1954). Beliefs, in turn, have to be compatible with what the players know. In particular, players are assumed to know that their opponents are themselves rational. Under additional assumptions, the equilibrium concept (Nash 1950) can then be interpreted as a rationality concept (see, e.g., Tan & Werlang (1988), Aumann & Brandenburger (1995)).

However, the assumptions that players are rational, and that they know that their opponents are rational, are restrictive, both from an introspective and an experimental point of view. This paper addresses the question what constitutes rationality if rationality is not mutual knowledge. As in Kreps et al. (1982), we distinguish between rational and non-rational players. However, we argue that the possibility that the opponent is not rational leads to uncertainty that cannot be adequately captured by beliefs that are necessarily representable by a probability measure. Thus, the analysis of games without mutual knowledge of rationality has to be based on a weaker definition of decision-theoretic rationality. In particular, Choquet-expected utility theory allows more general beliefs. Thus, we combine the analysis of Kreps et al. (1982) with Choquet-expected utility theory.

Choquet-expected utility theory (henceforth CEU) is due to Schmeidler (1989). Under Choquet-expected utility theory players are maximising expected utility subject to their beliefs, but their beliefs do not have to be additive. CEU is closely related to, but not quite identical with maxmin expected utility theory (Gilboa & Schmeidler 1989), which allows sets of additive beliefs. Whereas Savage's subjective expected utility theory reduces uncertainty to risk, CEU and its variants gives rise to a qualitative difference between risk and uncertainty.

This difference is important in games if we distinguish between rational and non-rational players as in Kreps et al. (1982). A rational player is one who chooses his strategy as to maximise utility given his beliefs. A rational player who faces a rational opponent can anticipate her strategy if he knows her utility function and can anticipate her beliefs. Consequently, a rational player who faces a rational opponent faces risk, in the sense that his beliefs are given by objective probabilities determined by best-reply considerations. Thus his beliefs are necessarily additive.

On the other hand, a rational player who faces a non-rational opponent faces true uncertainty, if all he knows is that a non-rational player does not necessarily choose a utility-maximising strategy. Under CEU, a rational player's beliefs reflect his attitude towards uncertainty. As a result, it becomes possible to base a theory of rational decisions in games not on a player's theory about how non-rational opponents play, but on his attitude towards uncertainty. Since CEU was motivated by phenomena that can be explained as uncertainty aversion — for instance the Ellsberg paradox — we also make this assumption.

We present an equilibrium concept, called Choquet-Nash equilibrium, that formalizes this intuition and discuss existence and properties of these equilibria in normal form games. We show that

- in normal form games Choquet-Nash equilibria always exist,
- not every rationalizable strategy is a Choquet-Nash equilibrium, and, conversely, non-rationalizable strategies may be equilibria,
- strictly dominated strategies are never rational, but elimination of such strategies cannot be iterated,
- robustness with respect to doubts about the rationality of the opponents is not captured by payoff-dominance or risk-dominance,
- mixed strategies may or may not be robust, depending on the game in question.

On this basis we formulate two equilibrium refinements: A Nash equilibrium is called strictly uncertainty aversion perfect if it continues to be an equilibrium as long as the belief in the opponents' rationality is sufficiently strong. Such equilibria need not exist. A Nash equilibrium is called uncertainty aversion perfect if it can be approximated by equilibria that do not require mutual knowledge of rationality. We show that such equilibria always exist, and that these refinements differ from those that are based on 'trembles' of otherwise fully rational opponents, i.e. trembling-hand perfect, proper and strictly perfect equilibria.

This paper makes three contributions. First, we extend the analysis of Kreps et al. (1982, henceforth KMRW). In contrast to KMRW, we do not need to specify a particular belief about the 'type' of an irrational opponent. Due to the absence of a theory of non-rational decision-making, such a specification is necessarily ad hoc. Moreover, the uniform distribution does not adequately model the ignorance about an irrational opponent, because it is not invariant under irrelevant changes of the game, for instance when adding a superfluous strategy that is a mere copy of an existing one. In our approach, ignorance can naturally be expressed as a non-additive probability.

More fundamentally, two difficulties arise with interpreting equilibria as rational strategies in the KMRW framework. First, interpreting equilibrium strategies as rational implicitly defines all non-equilibrium strategies as non-rational. Thus, a rational player's beliefs about an non-rational opponent should be consistent with this definition of non-rationality. This means that his beliefs should be consistent with any non-equilibrium strategy of the opponent. Secondly, a 'type' in a game with incomplete information corresponds to a consistent infinite hierarchy of beliefs. Thus, in KMRW the rational player believes that the opponent possesses such beliefs, even if he is not rational. In contrast, in our analysis an irrational opponent is a source of genuine uncertainty, and the question what constitutes a rational strategy is determined by a rational player's attitude towards uncertainty. Consequently, our analysis applies independently of the question whether the opponent can be modelled as a type.

The second contribution of this paper consists in a robustness analysis of Nash equilibria. Applying our solution concept to normal form games allows us to formalize how robust a Nash equilibrium is with respect to doubts about the rationality of the opponent. This robustness concept differs from existing ones, and shows how robustness is not a property of an equilibrium concept in general, but rather a property of specific equilibria in specific games.

The third contribution of this paper is that it extends the equilibrium concept to games in which players have non-additive beliefs. Here we extend solution concepts proposed by Dow & Werlang (1994), Eichberger & Kelsey (1993), Epstein (1997*a*), Haller (1995), Hendon et al. (1995), Klibanoff (1993), Lo (1995*b*), Lo (1996), Marinacci (1994), Mukerji (1994), Ritzberger (1996), and Ryan (1997*a*). This literature considers games in which players maximise CEU, or some variant of CEU. These papers show that it is possible to capture strategic phenomena that cannot be explained when players maximise subjective expected utility, and have also uncovered the difficulties that an extension of the equilibrium concept has to address. In our analysis we provide an explicit reason for the existence of uncertainty, and on this basis some of these difficulties can be avoided. In particular, it is not necessary to use simple capacities in the definition of an equilibrium, or to decide between the different support concepts that have been proposed for capacities, or to formulate an independence concept for capacities.[50]

This paper is organized as follows. In section 2 we define the equilibrium concept for two-player games and prove existence of Choquet-Nash equilibria. In section 3 we derive properties of Choquet-Nash equilibria, formulate the two refinements of Nash equilibria, and compare them with standard solution concepts. In section 4 we discuss the extension to infinitely many strategies and more than two players. Section 5 compares the equilibrium concept with other equilibrium concepts that are based on Choquet expected utility and uncertainty aversion. Section 6 presents an equilibrium concept that allows players to have a strict preference for mixed strategies. Section 7 concludes.

2.2 Choquet-Nash Equilibrium

A game in normal form is defined by specifying the set of players N, for each player a set of strategies S_i and each player's von Neumann - Morgenstern utility function u_i. In particular, players are assumed to be rational: when faced with uncertainty they maximise subjective expected utility. This concept of rationality has been axiomatized by Savage (1954).

In a game, rational beliefs must not only satisfy Savage's axioms, but must in addition be consistent with what players know about the structure of the game and about each other's rationality. In particular, if a player can anticipate which strategies are rational and if he knows that his opponent is rational, then he can

[50] After a first version of this paper was completed, I learnt of the closely related approach of Sujoy Mukerji (1994). His main concern is the consistent introduction of CEU into game theory, and he argues that this requires the KMRW framework. We fully agree with this, in addition we argue in this paper that the converse also holds, i.e. non-additive beliefs overcome the limitations of the KMRW approach.

anticipate his opponent's play. Precise arguments along this line are developed, e.g., in Tan & Werlang (1988) and Aumann & Brandenburger (1995).

If rationality is not mutual knowledge the question thus arises how a rational player should act if he *knew* that the opponent is not rational. In that case Savage's axioms imply that the rational player should have a belief given by a unique probability measure over the opponent's actions. If neither a theory of bounded rationality nor a stable empirical regularity of non-rational behaviour is available, there seems to be no foundation for this belief. The idea of this paper is that a weaker rationality concept allows further assumptions about the *rational* player from which rational actions can be derived.

2.2.1 Uncertainty Aversion

A key axiom in subjective expected utility theory is the independence axiom (Anscombe & Aumann 1963, Samuelson1952). Intuitively, the independence axiom says that if a decision maker prefers one act over another then he should also prefer a probability mixture of the first and a third act over the same mixture of the second and the third act: Either this probability mixture will reduce to a choice between the first two acts, or not, in which case the decision-maker is left with the third act in either case.[51] The descriptive validity of the independence axiom is questioned by the Allais paradox, the Ellsberg paradox and similar findings. Since its consequence is that a decision maker's beliefs can be represented by a probability measure, it also places a high demand on a player's rationality.

CEU weakens the independence axiom (Schmeidler 1989). Under CEU, the independence axiom is not assumed to hold for all acts, but only for acts that are "comonotonic". Two acts[52] f, f' are comonotonic if $f(\omega) > f(\omega')$ implies $f'(\omega) \geq f'(\omega')$, i.e. both acts give rise to the same preference ordering over states. In the following figures, acts f, g and h are pairwise comonotonic, f (or g or h) and h' are not.

[51] However, this interpretation equates the probability mixture with a two-stage lottery, i.e. also assumes a version of the 'reduction of compound lotteries axiom', see Kreps (1988, p.50 – 52) for the expected utility case.

[52] Here, acts $f \in \mathcal{F}$ map states $\omega \in \Omega$ into von Neumann - Morgenstern utilities. The acts are measurable with respect to events $E \in \Sigma \subseteq 2^{\Omega}$.

	ω_1	ω_2
f	10	6
g	16	0
h	10	0
h'	0	4

FIGURE 1

	ω_1	ω_2
$\frac{1}{2}f + \frac{1}{2}h$	10	3
$\frac{1}{2}g + \frac{1}{2}h$	13	0
$\frac{1}{2}f + \frac{1}{2}h'$	5	5
$\frac{1}{2}g + \frac{1}{2}h'$	8	2

FIGURE 2

Restricting the Sure-Thing Principle to comonotonic acts means that if the player is indifferent between f and g then he must also be indifferent between $\frac{1}{2}f + \frac{1}{2}h$ and $\frac{1}{2}g + \frac{1}{2}h$, because f, g and h are comonotonic. However, he may, e.g., strictly prefer $\frac{1}{2}f + \frac{1}{2}h'$ to $\frac{1}{2}g + \frac{1}{2}h'$. The reason is that mixtures of non-comonotonic acts can be interpreted as "hedging", i.e. distributing utility across states. Uncertainty aversion means that players may rationally act as if they hedged against uncertainty. Thus, in contrast to subjective expected utility theory, CEU allows the introduction of an additional assumption about rational preferences over acts that characterizes the player's attitude towards uncertainty.[53]

Schmeidler (1989) has shown that behaviour that is rational in this weaker sense can still be described by expected-utility maximisation. Players do still act as if they possess a von Neumann - Morgenstern utility function and beliefs, and take expected values. These beliefs, however, are no longer given by a probability measure over events, but a capacity, i.e. non-additive measure over events. Formally, a capacity v maps Σ into $[0,1]$ such that (i) $v(\emptyset) = 0$, (ii) $v(\Omega) = 1$ and (iii) $E \subseteq E' \Longrightarrow v(E) \leq v(E')$. Property (iii) weakens the finite-additivity requirement for finitely-additive measures: $E \cap E' = \emptyset \Longrightarrow v(E \cup E') = v(E) + v(E')$. Note that non-additive beliefs still may, but in general need not be additive.

The expectation of a real-valued random variable X with respect to a non-additive measure v is defined in Choquet (1953). If X takes finitely many values $\alpha_1 > \ldots >$

[53]This preference for randomisation argument exploits the structure of the Anscombe-Aumann model (Eichberger & Kelsey 1996c). Also, comonotonic independence may be too strong a requirement for uncertainty aversion (Epstein 1997b, Ghirardato & Marinacci 1997). In our game-theoretic context these are side issues, however.

α_n the Choquet integral is given by[54]

$$\int X dv := \sum_{i=1}^{n} v(X \geq \alpha_i) \cdot \Delta\alpha_i,$$

where $\Delta\alpha_i := \alpha_i - \alpha_{i+1}$ and $\alpha_{n+1} := 0$.

Formally, uncertainty aversion can be characterized in terms of the capacity v. The capacity v displays uncertainty aversion iff it is supermodular, i.e. $v(E) + v(E') \leq v(E \cap E') + v(E \cup E')$. The 'probability weights' $v(E)$ of an uncertainty averse decision maker do not add up to 1. Maximisation of Choquet expected utility under uncertainty aversion corresponds to allocating probability residuals to outcomes that are worst for the player.

2.2.2 Equilibrium

Let (I, S, u) be a finite two-player game in normal form. If player i knew that his opponent was non-rational, CEU implies that his belief is given by a not necessarily additive capacity v_j over S_j. Moreover, his expected utility from his pure strategy s_i is given by the Choquet expectation $u_i(s_i, v_j) := \int_{S_j} u_i(s_i, s_j)\, dv_j$. We define his payoff from a mixed strategy $\sigma_i \in \Delta S_i$ as $u_i(\sigma_i, v_j) := \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, v_j)$.

In a game in which rationality is not mutual knowledge, player 1 will thus take both possibilities into account: that the opponent is rational and that he need not be. If he can anticipate the rational strategies, his overall expected utility will be the weighted sum of his expected utility from interacting with a rational opponent, and the Choquet expected utility from interacting with a non-rational opponent. The weight corresponds to his degree of belief in the opponent's rationality. In a weak Choquet-Nash equilibrium, these rational strategies are determined endogenously.

Definition. *Let (I, S, u) be a finite two-player game in normal form. Let $0 \leq \epsilon_1, \epsilon_2 \leq 1$. Let v_1 be a capacity on S_1 and v_2 be a capacity on S_2. Then σ^* is a* **weak Choquet-Nash equilibrium** *iff (if and only if)*

$$\sigma_1^* \in \arg\max_{\sigma_1 \in \Sigma_1} [\, (1 - \epsilon_1) \cdot u_1(\sigma_1, \sigma_2^*) + \epsilon_1 \cdot u_1(\sigma_1, v_2) \,],$$

$$\sigma_2^* \in \arg\max_{\sigma_2 \in \Sigma_2} [\, (1 - \epsilon_2) \cdot u_2(\sigma_1^*, \sigma_2) + \epsilon_2 \cdot u_2(v_1, \sigma_2) \,].$$

[54] As usual, we write $v(X \geq t)$ for $v(\{\omega \in \Omega | X(\omega) \geq t\})$. The integrals on the right hand side are extended Riemann integrals. If v is additive this is the usual expectation.

Note that if $\epsilon_1 = \epsilon_2 = 1$ then each player believes that he faces a non-rational opponent, and thus the question what constitutes a rational strategy is purely decision-theoretic. On the other hand, if $\epsilon_1 = \epsilon_2 = 0$ then rationality is mutual knowledge. Note also that this definition assumes that the rational players know each others beliefs. Finally, notice that this equilibrium concept makes no assumption about the players' attitudes towards uncertainty, in particular, they may be uncertainty loving.

In general, when players are not expected utility maximisers, an equilibrium need not exist (Crawford 1990, Dekel, Safra & Segal 1991). However, the following proposition shows that this problem does not arise under CEU.[55]

Proposition 1. *For all* $\epsilon_1, \epsilon_2, v_1$ *and* v_2 *a weak Choquet-Nash equilibrium exists.*

Proof. The proof is the standard argument due to Nash (1950). The best reply correspondence $\sigma_i^*(\sigma_j) = \arg\max_{\sigma_i \in \Sigma_i} [\ (1-\epsilon_i) \cdot u_i(\sigma_i, \sigma_j) + \epsilon_i \cdot u_i(\sigma_i, v_j)\ ]$ maps the $(n-1)$ dimensional unit simplex into itself. Since the objective function is linear in σ_i, it is continuous, therefore a maximum exists and the best reply correspondence is non-empty and convex-valued. Since u_i is continuous in σ_j, it also has a closed graph. By Kakutani's Fixed Point Theorem, $(\sigma_1^*(\sigma_2), \sigma_2^*(\sigma_1))$ has a fixed point, which is, by definition, a general Choquet-Nash equilibrium. **q.e.d.**

In this generality the equilibrium concept is difficult to apply, because the beliefs ϵ_i and v_i have to be specified. We therefore make three simplifying assumptions: First, we assume that players share a common prior about the degree of mutual knowledge of rationality. This assumption is for simplicity only, but also has two useful side effects. It avoids any ad hoc asymmetry, and it makes the assumption that players know each others beliefs less demanding. Secondly, we assume that players are totally ignorant about the behaviour of a non-rational opponent. This ignorance has two reasons: Our solution concept specifies rational strategies only, so it does not restrict at all the range of non-rational strategies. Thus, complete ignorance is a consistency requirement. Also, there is no exogenous theory of non-rational decision making. As a consequence, every assumption about the shape of a rational player's beliefs about his non-rational opponent are ad hoc. In addition, a useful

[55] Note that this existence result also holds under uncertainty love. However, this is due to the order of integration, see section 6.

side effect is that the assumption that players know the rational opponent's beliefs is less restrictive. Finally, we consider the case that the players are uncertainty averse. Uncertainty aversion is the natural explanation of behavior observed in the Ellsberg paradox.

Complete ignorance can naturally be captured by 'simple capacities':

$$v_j(E_j) = \begin{cases} 0 & \text{, if } E_j \subset S_j, \\ 1 & \text{, if } E_j = S_j. \end{cases}$$

If player i holds this belief v_j about a non-rational opponent, he is only certain that the opponent will choose one of his available actions, but is unable to assign positive probability to any particular set of actions.

The Choquet-expectation of a utility function with respect to a simple capacity reflects uncertainty aversion, since all probability is allocated to the worst realization, i.e.

$$\int_{S_j} u_i(s_i, s_j)\ dv_j = \min_{s_j \in S_j} u_i(s_i, s_j).$$

A Choquet-Nash equilibrium is a weak Choquet-Nash equilibrium with these additional assumptions.

Definition. *Let (I, S, u) be a finite two-player game in normal form. Let $0 \leq \epsilon \leq 1$. Then σ^* is a* **Choquet-Nash equilibrium** *iff*[56]

$$\sigma_1^* \in \arg\max_{\sigma_1 \in \Sigma_1} \left[\ (1-\epsilon) \cdot u_1(\sigma_1, \sigma_2^*) + \epsilon \cdot \sum_{s_1 \in S_1} \sigma_1(s_1) \cdot \min_{s_2 \in S_2} u_1(s_1, s_2)\ \right],$$

$$\sigma_2^* \in \arg\max_{\sigma_2 \in \Sigma_2} \left[\ (1-\epsilon) \cdot u_2(\sigma_1^*, \sigma_2) + \epsilon \cdot \sum_{s_2 \in S_2} \sigma_2(s_2) \cdot \min_{s_1 \in S_1} u_2(s_1, s_2)\ \right].$$

It follows from proposition 1 that in every finite two-player game in normal form a Choquet-Nash equilibrium (henceforth CNE) exists. Moreover, every symmetric game also has a symmetric Choquet-Nash equilibrium.

Definition. *Let (I, S, u) be a finite two-player game in normal form. The game is symmetric iff $S_i = S_j$ and $u_i(s_i, s_j) = u_j(s_j, s_i)$. A strategy combination is symmetric iff $s_i = s_j$.*

[56] In the remaining sections, we also use the notation $\sum_{s_i \in S_i} \sigma_i(s_i) \cdot \min_{s_j \in S_j} u_i(s_i, s_j) = \int_{S_i} \min_{s_j \in S_j} u_i(s_i, s_j)\ d\sigma_i^*$.

Remark 1. *For all ϵ, in a symmetric game a symmetric Choquet-Nash equilibrium exists.*

Proof. Again, the proof is standard. The result is proved as in proposition 1, except that the fixed point argument is applied to the best reply correspondence $\sigma_i^*(\sigma_i)$. **q.e.d.**

2.3 Properties of Choquet-Nash Equilibria

The aim of this section is to present the properties of Choquet-Nash equilibria. Section 3.1 relates them to dominance and rationalizability. In section 3.2, we relate Choquet-Nash equilibria to the robustness of Nash equilibria This will lead to the definition of two equilibrium refinements (sections 3.3 and 3.4). Section 3.5 compares them with minimax strategies in zero-sum games. Finally, section 3.6 compares them with other equilibrium refinements (trembling-hand perfect, proper and strictly perfect equilibria).

2.3.1 Dominance and Rationalizability

The following result implies that, independently of the degree of mutual knowledge of rationality, no strictly dominated strategy is rational.

Lemma 1. *Let (I, S, u) be a finite two-player game in normal form. Let $0 \leq \epsilon \leq 1$. Let σ^* be a Choquet-Nash equilibrium. Then if $\sigma_i^*(s_i) > 0$, then s_i is a best response to σ_j^* and ϵ, i.e.*

$$s_i \in \arg\max_{s_i \in S_i} \left[\ (1-\epsilon) \cdot u_i(s_i, \sigma_j^*) + \epsilon \cdot \min_{s_j \in S_j} u_i(s_i, s_j)\ \right].$$

Proof. Again, the proof is standard. If s_i is not a best response then some other strategy s_i' gives higher expected utility than s_i. Thus the player can increase his overall utility from σ_i^* by playing $\hat{\sigma}_i(s_i') = \sigma_i^*(s_i) + \sigma_i^*(s_i')$, $\hat{\sigma}_i(s_i) = 0$ and $\hat{\sigma}_i(s_i'') = \sigma_i^*(s_i'')$ for all other strategies s_i'', which contradicts the assumption that σ_i^* is a best reply. **q.e.d.**

It is important to notice, however, that strict dominance cannot be iterated, as the game in Figure 3 shows:

	L	R
T	1,1	−99,0
B	0,1	0,0

FIGURE 3

In this game playing L is a strictly dominant strategy for player 2. Consequently, iterated strict dominance yields T as the unique rational strategy for player 1, if rationality is mutual knowledge. In particular, (T, L) is the unique equilibrium and the unique rationalizable strategy profile of the game.

However, (T, L) is not a plausible profile unless player 1 is convinced that player 2 is rational. The CNE in this game depends on ϵ. In every CNE, player 2 will play L because this is his strictly dominant strategy. However, unless $\epsilon \leq \frac{1}{100}$ only strategy B is rational for player 1.

Note that this shows that non-rationalizable strategies may be CNE-strategies. The 'Matching Pennies' game in figure 4 shows that, conversely, not every rationalizable strategy is a CNE.

	L	R
T	1,−1	−1,1
B	−1,1	1,−1

FIGURE 4

Note that the best reply correspondence for a Choquet-Nash equilibrium in the 'Matching Pennies' game is given by

$$\sigma_i^*(\sigma_j) = \arg\max_{\sigma_i \in \Sigma_i} \left[\ (1-\epsilon) \cdot u_i(\sigma_i, \sigma_j) + \epsilon_i \cdot (-1)\ \right],$$

which differs from the Nash best reply correspondence only by a factor and a constant. Consequently, independently of ϵ, only the mixed strategies $\sigma_1(T) = \sigma_1(B) = \sigma_2(L) = \sigma_2(R) = \frac{1}{2}$ form a CNE. This is also the unique Nash equilibrium, but every strategy profile is rationalizable. We have thus established proposition 2:

Proposition 2. *Non-rationalizable strategy profiles may be Choquet-Nash equilibria. Conversely, not every rationalizable strategy profile is a Choquet-Nash equilibrium.*

2.3.2 The Robustness of Nash Equilibria

The definition of a Choquet-Nash equilibrium collapses to the definition of Nash equilibrium if $\epsilon = 0$. So any Nash equilibrium is a CNE for $\epsilon = 0$. We will show that a given Nash equilibrium may also be a CNE for $\epsilon > 0$, and that the highest such ϵ can be regarded as a measure of robustness of a given Nash equilibrium.[57] To establish this claim, we first need the following lemma:

Lemma 2. *Let (I, S, u) be a finite two-player game in normal form. Let $0 < \epsilon \leq 1$. Let σ^* be a Choquet-Nash equilibrium. If σ^* is a Nash equilibrium, then it is also a Choquet-Nash equilibrium for all $0 \leq \epsilon' \leq \epsilon$.*

Proof. Let $0 < \epsilon \leq 1$ and $0 \leq \epsilon' \leq \epsilon$. Since σ^* is a Nash equilibrium, $u_i(\sigma_i^*, \sigma_j^*) \geq u_i(\sigma_i, \sigma_j^*)$. Since σ^* is a CNE for ϵ,

$$\begin{array}{rl} & (1-\epsilon) \cdot u_i(\sigma_i^*, \sigma_j^*) + \epsilon \cdot \int_{S_i} \min_{s_j \in S_j} u_i(s_i, s_j)\ d\sigma_i^*\ ] \\ \geq & (1-\epsilon) \cdot u_i(\sigma_i, \sigma_j^*) + \epsilon \cdot \int_{S_i} \min_{s_j \in S_j} u_i(s_i, s_j)\ d\sigma_i\ ] \end{array}$$

for all σ_i and all i. Consequently, for any $\alpha \in [0, 1]$,

$$\begin{array}{rl} & \alpha \cdot u_i(\sigma_i^*, \sigma_j^*) + (1-\alpha) \cdot [(1-\epsilon) \cdot u_i(\sigma_i^*, \sigma_j^*) + \epsilon \cdot \int_{S_i} \min_{s_j \in S_j} u_i(s_i, s_j)\ d\sigma_i^*\ ] \\ \geq & \alpha \cdot u_i(\sigma_i, \sigma_j^*) + (1-\alpha) \cdot [(1-\epsilon) \cdot u_i(\sigma_i, \sigma_j^*) + \epsilon \cdot \int_{S_i} \min_{s_j \in S_j} u_i(s_i, s_j)\ d\sigma_i\ ] \end{array}$$

for all σ_i. So for $\alpha = 1 - \frac{\epsilon'}{\epsilon}$ we have $\alpha \in [0, 1]$ and

$$\begin{array}{rl} & (1-\epsilon') \cdot u_i(\sigma_i^*, \sigma_j^*) + \epsilon' \cdot \int_{S_i} \min_{s_j \in S_j} u_i(s_i, s_j)\ d\sigma_i^* \\ \geq & (1-\epsilon') \cdot u_i(\sigma_i, \sigma_j^*) + \epsilon' \cdot \int_{S_i} \min_{s_j \in S_j} u_i(s_i, s_j)\ d\sigma_i \end{array}$$

for all σ_i, and for all $i \in I$, i.e. σ^* is also a CNE for ϵ'. **q.e.d.**

On the basis of lemma 2, we can now define a measure of robustness of a Nash equilibrium with respect to doubts about the rationality of the opponent:

[57] See also Eichberger & Kelsey (1993).

Definition. *Let (I, S, u) be a finite two-player game in normal form. Let $0 \leq \epsilon \leq 1$. Let σ^* be a Nash equilibrium. Then the degree $\overline{\epsilon}(\sigma^*)$ of uncertainty aversion robustness of σ^* is given by the largest ϵ for which σ^* is a Choquet-Nash equilibrium.*

Note that $\overline{\epsilon}$ exists because the expected utility functions are continuous in ϵ.

As the following game shows, this measure of robustness formalizes a different intuition about robustness than payoff-dominance and risk-dominance. The game in figure 5 has two strict Nash equilibria:

	L	R
T	5,5	0,1
B	1,0	3,3

FIGURE 5

The equilibrium (T, L) dominates the equilibrium (B, R) both with respect to payoff-dominance and with respect to risk-dominance. However, $\overline{\epsilon}(T, L) = \frac{4}{5}$, since if a rational opponent plays L (respectively T) then it is only rational to play T (respectively L) as long as $\epsilon \leq \frac{4}{5}$. On the other hand, $\overline{\epsilon}(B, R) = 1$, since if a rational opponent plays R (respectively B) then it is never rational to deviate from B to T (respectively from R to L).

We next show that strict Nash equilibria are robust with respect to doubts of the rationality of the opponent:[58]

Remark 2. *Let (I, S, u) be a finite two-player game in normal form. Let s^* be a Choquet-Nash equilibrium. If s^* is a strict Nash equilibrium, then there exists an $\overline{\epsilon} > 0$ such that s^* is a Choquet-Nash equilibrium for all $0 \leq \epsilon \leq \overline{\epsilon}$.*

Proof. Define for each $i \in I$

$$\begin{aligned} \delta_i &:= \max_{s_i \in S_i} \left[\min_{s_j \in S_j} u_i(s_i, s_j) - \min_{s_j \in S_j} u_i(s_i^*, s_j)\right] \\ \alpha_i &:= \min_{s_i \in S_i} \left[u_i(s_i^*, s_j^*) - u_i(s_i, s_j^*)\right] \end{aligned}$$

Note that $\alpha_i > 0$ and $\delta_i \geq 0$. Define $\epsilon_i := \frac{\alpha_i}{\delta_i + \alpha_i}$ and $\overline{\epsilon} := \min_{i \in I} \epsilon_i$. Note that

[58] Note that strict Nash equilibria are pure.

$\bar{\epsilon} > 0$. Then for any i and any $s_i \in S_i$

$$\begin{array}{cc} & (1-\bar{\epsilon}) \cdot [u_i(s_i^*, s_j^*) - u_i(s_i, s_j^*)] \\ \geq & (1-\bar{\epsilon}) \cdot \alpha_i \\ \geq & (1-\epsilon_i) \cdot \alpha_i \\ = & \epsilon_i \cdot \delta_i \\ \geq & \bar{\epsilon} \cdot \delta_i \\ \geq & \bar{\epsilon} \cdot [\min_{s_j \in S_j} u_i(s_i, s_j) - \min_{s_j \in S_j} u_i(s_i^*, s_j)] \end{array}$$

It follows from lemma 1 that only pure strategy deviations are relevant, so s^* is a CNE for $\bar{\epsilon}$, and by lemma 2 for all $\epsilon \leq \bar{\epsilon}$. **q.e.d.**

The requirement that a Nash equilibrium is strict is sufficient for $\bar{\epsilon} > 0$, but it is not necessary, as the 'Matching Pennies' game in figure 4 shows. The non-strictness of mixed strategy equilibria is sometimes regarded as a conceptual weakness, because the players, while having no incentive to deviate, still seem to lack a positive incentive to choose their equilibrium strategies. This has led to a justification of mixed strategy equilibria by purification arguments, i.e. in terms of an embedding of the original game into a game with (slight) incomplete information.[59] However, both this criticism of mixed equilibria and their defense apply equally to all mixed strategy equilibria. Next, we show that the robustness measure $\bar{\epsilon}$ formalizes that in some games mixed strategy equilibria are more plausible than in other games.

	L	R
T	9,9	0,7
B	7,0	8,8

FIGURE 6

The game in figure 6 has a mixed strategy Nash equilibrium $\sigma_1^*(T) = \sigma_2^*(L) = \frac{4}{5}$, $\sigma_1^*(B) = \sigma_2^*(R) = \frac{1}{5}$. Given that the rational opponent plays σ_j^*, a player's expected payoff from a rational opponent is independent of his own strategy. Thus a rational player will only take into account the expected payoff from a non-rational opponent.

[59] Note, however, that the justification of Nash equilibria given in Aumann & Brandenburger (1995) is independent of the question whether the equilibrium is pure or mixed.

This payoff is 0 when he plays T (respectively L) and 7 when he plays B (respectively R). So a rational player will always deviate to B (respectively R) if he expects a rational opponent to play according to σ^* and there is doubt about the opponent's rationality, however small it is, unless $\epsilon = 0$.

The stability property of mixed strategy equilibria are given by remark 3:

Remark 3. *Let (I, S, u) be a finite two-player game in normal form. Let σ^* be a Nash equilibrium. Let $i \in I$, $s_i, s_i' \in S_i$, $\sigma_i^*(s_i) > 0$ and $\sigma_i^*(s_i') > 0$. Then if $\min_{s_j \in S_j} u_i(s_i, s_j) \neq \min_{s_j \in S_j} u_i(s_i', s_j)$ then $\bar{\epsilon}(\sigma^*) = 0$.*

Proof. If σ^* is a CNE, both s_i and s_i' must be best replies to ϵ. However, since σ^* is also a Nash equilibrium, both s_i and s_i' are also best replies to σ_{-i}^* if $\epsilon = 0$. So if $\bar{\epsilon} > 0$ we must have $\min_{s_j \in S_j} u_i(s_i, s_j) = \min_{s_j \in S_j} u_i(s_i', s_j)$, a contradiction. **q.e.d.**

The following example shows that even for a genuinely mixed Nash equilibrium[60] we may have $0 < \bar{\epsilon} < 1$, i.e. the Nash equilibrium is robust, but not trivially so:

	L	R
T	2,1	0,1
B	1,0	1,0

FIGURE 7

Consider the mixed equilibrium T and $q^* \equiv \text{Prob}(L) = \frac{3}{4}$. Then player 1 will prefer T as long as $(1-\epsilon)2 \cdot \frac{3}{4} \geq 1$, i.e. $\epsilon \leq \frac{1}{3}$. Player 2 is always indifferent between L and R, so $\bar{\epsilon} = \frac{1}{3}$. Note that for $q^* = \frac{1}{2}$ every $p^* \equiv \text{Prob}(T) \in [0, 1]$ is also a Nash equilibrium, however, for any equilibrium with $p^* > 0$ we have $\bar{\epsilon} = 0$, i.e. such equilibria are not robust. The reason is that if there is a positive probability, however small, that player 2 is not rational, player 1 will prefer to play B if a rational opponent plays $q^* = \frac{1}{2}$.

Note, however, that we cannot have $0 < \bar{\epsilon} < 1$ for Nash equilibria in 2×2 games in which *both* players use genuinely mixed strategies. The following game shows

[60] A Nash equilibrium is genuinely mixed if at least one player chooses a non-degenerate mixed strategy.

that $0 < \bar{\epsilon} < 1$ is possible even if both players use genuinely mixed strategies:

	L	C	R
T	4,4	0,0	0,1
C	0,0	4,4	0,1
B	1,0	1,0	1,1

FIGURE 8

Consider the mixed strategy Nash equilibrium $p_1^* \equiv \text{Prob}(T) = p_2^* \equiv \text{Prob}(M) = \frac{1}{2}, q_1^* \equiv \text{Prob}(L) = q_2^* \equiv \text{Prob}(C) = \frac{1}{2}$. This is also a CNE as long as $\epsilon \leq \bar{\epsilon} := \frac{1}{2}$, because a rational player will receive 2 from a rational opponent whom he meets with probability $(1 - \epsilon)$, but 0 from a non-rational opponent if he plays the equilibrium strategy. Deviating to his third pure strategy will give him 1 in either case.

So far, all robust equilibria were quasi-strict. Recall that a Nash equilibrium is quasi-strict if every pure best reply to the equilibrium strategies of the opponent is in the support of the equilibrium strategy (Harsanyi 1973). We next show that this is not true in general, i.e. that robustness in our sense neither implies nor is implied by quasi-strictness of a Nash equilibrium.

	L	R
T	2,1	1,0
B	2,1	0,0

FIGURE 9

Consider the Nash equilibrium (T, L) of the game in figure 9. It is not quasi-strict, because B is also a best reply to L. Yet it is robust, i.e. $\bar{\epsilon} = 1$, because for player 2 L is strictly dominant. Player 1 knows that a rational opponent will play L, and in case the opponent is non-rational he will strictly prefer T to B. This shows that robustness does not imply quasi-strictness. Conversely, the mixed strategy equilibrium in the game in figure 6 is quasi-strict, yet it is not robust. We have thus established proposition 3, which shows that our robustness concept differs from quasi-strictness:

Proposition 3. *Robustness and quasi-strictness are unrelated, i.e. Nash equilibria may be robust and quasi-strict, non-robust and quasi-strict, robust and non-quasi-strict, or neither.*

It remains to consider the most important special case of non-quasi-strict equilibria, namely Nash equilibria in weakly dominated strategies. We show that such equilibria may or may not be robust.

	L	R
T	2,2	0,2
B	2,0	1,1

FIGURE 10

The Nash equilibrium (T, L) is payoff-dominant, but involves weakly dominated strategies and is therefore not quasi-strict. For this equilibrium $\bar{\epsilon} = 0$, so it is not robust. However, consider the game in figure 11:

	L	C	R
T	2,2	0,0	0,1
B	2,0	0,0	1,1

FIGURE 11

Again (T, L) is a payoff-dominant Nash equilibrium in weakly dominated strategies. However, it is indeed robust. For both T and B, a rational player 1 expects 2 from a rational opponent playing his equilibrium strategy L, and 0 from a non-rational opponent. Player 2, on the other hand, strictly prefers L to R as long as $\epsilon \leq \frac{1}{2}$, so $\bar{\epsilon} = \frac{1}{2}$.

To summarize, we have shown that $\bar{\epsilon}$ can be interpreted as a measure of robustness of a Nash equilibrium with respect to doubts about the rationality of the opponent. This robustness concept differs from payoff dominance, risk dominance, strictness or quasi-strictness. This leads us to suggest two refinements of Nash equilibrium.

2.3.3 Strict Uncertainty Aversion Perfection

So far, we have shown how the concept of Choquet-Nash equilibrium sheds light on the robustness of Nash equilibria. This suggests to use this robustness analysis as a basis for equilibrium refinements. Intuitively, Nash equilibria are robust if they can be approximated by Choquet-Nash equilibria. Since this approximation can take different forms, we define two equilibrium refinements: strictly uncertainty aversion perfect equilibria (section 3.3) and uncertainty aversion perfect equilibria (section 3.4).

Definition. *Let (I, S, u) be a finite two-player game in normal form. Let σ^* be a strategy combination. Then σ^* is a* **strictly uncertainty aversion perfect** *Nash equilibrium if and only if there exists a sequence $(\epsilon_k)_{k\in\mathrm{N}}$, with $0 < \epsilon_k < 1$ and $\lim_{k\to\infty} \epsilon_k = 0$, such that σ^* is a Choquet-Nash equilibrium for every ϵ_k.*

We first note that the strictly uncertainty aversion perfect equilibria are those with a strictly positive degree of uncertainty aversion robustness:

Lemma 3. *Let (I, S, u) be a finite two-player game in normal form. Let σ^* be a Nash equilibrium. Then σ^* is a strictly uncertainty aversion perfect Nash equilibrium if and only if $\bar{\epsilon}(\sigma^*) > 0$.*

Proof. Necessity ('only if') is immediate because $\bar{\epsilon} \geq \epsilon_k > 0$. Sufficiency ('if') follows from lemma 2 by considering the sequence $(\frac{\bar{\epsilon}}{k})$. **q.e.d.**

Next, we show that a strictly uncertainty aversion perfect equilibrium is indeed a Nash equilibrium. This establishes that this concept is indeed an equilibrium refinement:

Remark 4. *Let (I, S, u) be a finite two-player game in normal form. A strictly uncertainty aversion perfect Nash equilibrium is indeed a Nash equilibrium.*

Proof. Let $\epsilon_k > 0$. Since σ^* is a CNE for ϵ_k we have

$$\begin{aligned} & (1-\epsilon_k)\cdot u_i(\sigma_i^*, \sigma_j^*) + \epsilon_k \cdot \textstyle\int_{S_i} \min_{s_j\in S_j} u_i(s_i, s_j)\ d\sigma_i^*\ ] \\ \geq\ & (1-\epsilon_k)\cdot u_i(\sigma_i, \sigma_j^*) + \epsilon_k \cdot \textstyle\int_{S_i} \min_{s_j\in S_j} u_i(s_i, s_j)\ d\sigma_i\ ] \end{aligned}$$

for all σ_i and all $i \in I$. These expected utility functions are continuous in ϵ_k, so the inequalities also hold in the limit as $\epsilon_k \to 0$. **q.e.d.**

We next study the existence question:

Proposition 4. *A strictly uncertainty aversion perfect Nash equilibrium need not exist.*

Proof. Consider the game in figure 12:

	L	C	R
T	2,2	2,0	0,1
B	2,0	1,1	1,0

FIGURE 12

Let $p_1 \equiv \text{Prob}(T), p_2 \equiv \text{Prob}(B), q_1 \equiv \text{Prob}(L), q_2 \equiv \text{Prob}(C), q_3 \equiv \text{Prob}(R)$. Any Nash equilibrium of this game takes the form $p_1^* \geq \frac{1}{3}, q_1^* = 1$. Each such (p^*, q^*) is an equilibrium, and there can be no equilibrium with $p_1 = 0$ (else $q_2^* = 1$ and $p_1 \neq 0$), so $q_3^* = 0$, and if $q_2 > 0$ then $p_1^* = 1$ and $q_2 = 0$, a contradiction.
However, none of these equilibria is strictly uncertainty aversion perfect: Player 1 knows that he can expect 2 from a rational opponent both if he plays T and B, but from a non-rational opponent he will expect 0 from T and 1 from B. As long as $\epsilon_k > 0$, he will play B. **q.e.d.**

This result suggest to look for existence in a subclass of games. Surprisingly, not even 2×2-games always possess a strictly uncertainty aversion perfect Nash equilibrium, as the game in figure 13 shows:

	L	R
T	2,0	0,2
B	1,2	2,1

FIGURE 13

This game has a unique Nash equilibria in genuinely mixed strategies $p^* \equiv \text{Prob}(T) = \frac{1}{3}, q^* \equiv \text{Prob}(L) = \frac{2}{3}$. However, if player 1 expects a rational opponent to play q^*, he will strictly prefer B to T, since he will achieve the same utility from a rational opponent, but a higher utility in case the opponent is non-rational. So $\bar{\epsilon} = 0$, and the claim follows from lemma 3.

Finally, the following remark characterises strictly uncertainty aversion perfect equilibria. It will be useful when we study zero-sum games and standard equilibrium refinements in sections 3.5 and 3.6.

Remark 5. *Let (I, S, u) be a finite two-player game in normal form. Let σ^* be a Nash equilibrium. Then σ^* is strictly uncertainty aversion perfect if and only if*

$$\nexists i \in I,\ \nexists s_i \in S_i,\ \nexists s_i' \in \text{supp}\ \sigma_i^* :$$

$$u_i(s_i, \sigma_j^*) = u_i(\sigma_i^*, \sigma_j^*)$$

$$\text{and} \min_{s_j \in S_j} u_i(s_i, s_j) > \min_{s_j \in S_j} u_i(s_i', s_j).$$

Proof. Suppose for some player $i \in I$ there exist s_i and $s_i' \in$ supp σ_i^* such that $u_i(s_i, \sigma_j^*) = u_i(\sigma_i^*, \sigma_j^*)$ and $\min_{s_j \in S_j} u_i(s_i, s_j) > \min_{s_j \in S_j} u_i(s_i', s_j)$. Then, as long as $\epsilon > 0$, for player i a deviation from σ_i^* to s_i is profitable, because he will expect the same utility as σ_i^* from a rational opponent, but a higher utility from a non-rational opponent. So $\bar{\epsilon}(\sigma^*) = 0$, i.e. σ^* is not strictly uncertainty aversion perfect. Conversely, if these conditions hold, player i does not have a profitable deviation.

q.e.d.

The following proposition summarizes the above results on the robustness of Nash equilibria, in case a strictly uncertainty aversion perfect equilibrium exists:

Proposition 5. *Let (I, S, u) be a finite two-player game in normal form. Let σ^* be a strictly uncertainty aversion perfect Nash equilibrium.*

(1) Every strict equilibrium is strictly uncertainty aversion perfect. However, strictly uncertainty aversion perfect equilibria need not be strict.

(2) Quasi-strict equilibria in general, and mixed strategy equilibria and equilibria in weakly dominated strategies in particular, may be, but need not be, strictly uncertainty aversion perfect.

2.3.4 Uncertainty Aversion Perfection

Because strictly uncertainty aversion perfect equilibria need not exist, we suggest the following weaker refinement of Nash equilibria:

Definition. *Let (I, S, u) be a finite two-player game in normal form. Let σ^* be a strategy combination. Then σ^* is an* **uncertainty aversion perfect** *Nash equilibrium if and only if there exists a sequence $(\epsilon_k)_{k\in\mathbb{N}}$, with $0 < \epsilon_k < 1$ and $\lim_{k\to\infty}\epsilon_k = 0$, and a sequence of strategy profiles $(\sigma_k^*)_{k\in\mathbb{N}}$, such that each σ_k^* is a Choquet-Nash equilibrium for ϵ_k and $\lim_{k\to\infty}\sigma_k^* = \sigma^*$.*

Since this definition allows constant sequences of strategy profiles, every strictly uncertainty aversion perfect equilibrium is indeed uncertainty aversion perfect.

Remark 6. *Let (I, S, u) be a finite two-player game in normal form. An uncertainty aversion perfect Nash equilibrium is indeed a Nash equilibrium.*

Proof. Let $\epsilon_k > 0$. Since σ_k^* is a CNE for ϵ_k we have

$$\begin{array}{rl} & (1-\epsilon_k)\cdot u_i(\sigma_{i,k}^*, \sigma_{j,k}^*) + \epsilon_k \cdot \int_{S_i} \min_{s_j\in S_j} u_i(s_i, s_j)\ d\sigma_{i,k}^*\ ] \\ \geq & (1-\epsilon_k)\cdot u_i(\sigma_i, \sigma_{j,k}^*) + \epsilon_k \cdot \int_{S_i} \min_{s_j\in S_j} u_i(s_i, s_j)\ d\sigma_i\ ] \end{array}$$

for all σ_i and all $i \in I$. These expected utility functions are continuous in ϵ_k, σ_i and σ_j, so the inequalities also hold in the limit as $\epsilon_k \to 0$ and $\sigma_k^* \to \sigma^*$. **q.e.d.**

Proposition 6. *Every finite two-player game in normal form has at least one uncertainty aversion perfect Nash equilibrium.*

Proof. Consider a sequence $\epsilon_k \to 0$. By proposition 1, there exists a CNE for every ϵ_k. Since the strategy sets are compact subsets of finite-dimensional euclidean spaces, by the Bolzano-Weierstraß Theorem, every sequence of CNEs σ_k^* has a convergent subsequence σ_l^*. Since the associated sequence ϵ_l also converges to 0, the limit of σ_l^* is an uncertainty aversion perfect Nash equilibrium. **q.e.d.**

We end this section with an example of an equilibrium in *pure* strategies that are *not* weakly dominated that is *not* uncertainty aversion perfect:

	L	C	R
T	1,1	2,0	0,0
B	1,0	1,0	1,0

FIGURE 14

Consider the equilibrium (T, L). The strategy T is undominated, and L is weakly dominant. Yet (T, L) is not uncertainty aversion perfect: As long as $\epsilon > 0$, a rational player 2 will play L because it is weakly dominant. But given L, player 1 will expect utility 1 from a rational opponent both if he plays T or B, but since $\epsilon > 0$ he will strictly prefer B.

2.3.5 Zero-Sum Games

Under complete ignorance, an uncertainty averse player will allocate probability weight 1 to the outcome that is worst for himself. Intuitively, this suggests a close relationship of Choquet-Nash equilibria with minimax strategies in zero-sum games.

We next show, however, that this is *not* the case[61]. First, consider strictly uncertainty aversion perfect equilibria:

	L	R
T	0,0	2,–2
B	2,–2	1,–1

FIGURE 15

In the game in figure 15, the Nash equilibrium is unique, and since the game is zero-sum the strategies are minimax strategies. However, remark 5 implies that this equilibrium is not strictly uncertainty aversion perfect. This example also shows that in even in zero-sum games a strictly uncertainty aversion perfect equilibrium

[61] This result is due to a lack of preference for uncertainty, see section 4.

need not exist.

However, in the previous game the minimax strategies are uncertainty aversion perfect. The following example shows that not every Nash equilibrium in a zero-sum game is uncertainty aversion perfect:

	L	R
T	1,−1	1,−1
B	0,0	1,−1

FIGURE 16

The pair of minimax strategies (T, R) is not uncertainty aversion perfect: As long as $\epsilon > 0$, player 2 prefers to play L because L is weakly dominant.

2.3.6 Equilibrium Refinements

The fact that not all Nash equilibria are robust in the sense of (strict) uncertainty aversion perfection raises the question whether *perfect* Nash equilibria are more robust with respect to doubt about the rationality of the opponent. In this section we present the relationship between uncertainty aversion perfection and other equilibrium refinements.

First, note that the equilibrium (T, L) in figure 14 is proper, because C and R are equally costly mistakes for player 2. So for $\epsilon = \frac{1}{k}$ the strategy combinations σ_k with $\text{Prob}(T) = 1 - \frac{1}{k}, \text{Prob}(B) = \frac{1}{k}$, $\text{Prob}(L) = \frac{k}{k+2}, \text{Prob}(C) = \text{Prob}(R) = \frac{1}{k+2}$ is an ϵ-proper equilibrium, and as $k \to \infty$ we have $\epsilon \to 0$ and $\sigma_k \to (T, L)$. However, as shown above, this equilibrium is not uncertainty aversion perfect. So properness does not imply uncertainty aversion perfection.

Note also, however, that this equilibrium is not strictly perfect. Our next example shows that strict perfection does not imply strict uncertainty aversion perfection:

	L	R
T	2,0	0,2
B	1,2	2,1

FIGURE 17

The mixed strategy equilibrium $p^* \equiv \text{Prob}(T) = \frac{1}{3}, q^* \equiv \text{Prob}(L) = \frac{2}{3}$ is strictly perfect, because it is completely mixed. But by remark 5, it is not strictly uncertainty aversion perfect.

The next game shows that strictly perfect equilibria even need not be uncertainty aversion perfect:

	L	R
T	2,2	0,0
M	0,0	2,2
B	1,1	1,1

FIGURE 18

Consider the mixed strategy Nash equilibrium $p_1^* \equiv \text{Prob}(T) = \frac{1}{2}, q^* \equiv \text{Prob}(L) = \frac{1}{2}$. This equilibrium is strictly perfect: Let $\mu_T, \mu_M, \mu_B, \mu_L, \mu_R$ be any strictly positive trembles (minimum probabilities). Then the strategy combination σ_μ with $p_1 = \frac{1}{2}(1-\mu_B), p_2 = \frac{1}{2}(1-\mu_B), p_3 = \mu_B$, $q_1 = \frac{1-2\mu_L}{2(1-\mu_L-\mu_R)}, q_2 = 1-q_1$ is a Nash equilibrium of the perturbed game, and as $\mu_L, \ldots \to 0$ we have $\sigma_\mu \to \sigma^*$. However, the equilibrium is not uncertainty aversion perfect: For player 1, as long as $\epsilon > 0$, strategy T gives $2(1-\epsilon)q_\epsilon$ and strategy M gives $2(1-\epsilon)(1-q_\epsilon)$, where q_ϵ is the strategy of a rational player 2. In order to find a sequence of mixed strategies for player 1 that converges to $p = \frac{1}{2}$, he must be willing to mix between T and M, which implies that in any Choquet-Nash equilibrium $q_\epsilon = \frac{1}{2}$. But then both T and M yield less than B, so for $\epsilon > 0$ no such equilibrium exists.

Conversely, we can ask whether (strictly) uncertainty averse equilibria also satisfy refinement criteria for Nash equilibria. However, the equilibrium (T, L) in figure 11 is strictly uncertainty aversion perfect, yet T is weakly dominated for player 1, and therefore (T, L) is not trembling-hand perfect.

We have thus established a lack of relationships between robustness with respect to lack of mutual knowledge of rationality and equilibrium refinements that is summarized by the following proposition:

Proposition 7. *Neither a proper equilibrium nor a strictly perfect equilibrium need be uncertainty aversion perfect. Conversely, even a strictly uncertainty aversion perfect equilibrium need not be trembling-hand perfect.*

2.4 Extensions

So far, we have defined the solution concept only for 2-player games with finitely many strategies. Typically, in economic games the strategy spaces are infinite, for instance if firms choose prices, quantities, a location, a point in time, or a certain probability.

The Choquet-integral of a general random variable X is defined as

$$\int X dv := \int_0^\infty v(X \geq t)dt + \int_{-\infty}^0 [v(X \geq t) - 1]dt.$$

As before, we define the expected utility from a non-rational opponent as $u_i(s_i, v_j) := \int_{S_j} u_i(s_i, s_j)\ dv_j$ and the payoff from a mixed strategy $\sigma_i \in \triangle S_i$ as $u_i(\sigma_i, v_j) := \int_{S_i} u_1(s_i, v_j)\ d\sigma_i$.

As before, we can thus define a weak Choquet-Nash equilibrium for 2-player games with possibly infinite strategy spaces. Under the assumptions of a common prior about rationality, complete ignorance about non-rationality and uncertainty aversion this reduces to:

Definition. *Let (I, S, u) be a two-player game in normal form. Let $0 \leq \epsilon \leq 1$. Then σ^* is a* **Choquet-Nash equilibrium** *iff*

$$\sigma_1^* \in \arg\max_{\sigma_1 \in \Sigma_1} [\ (1-\epsilon) \cdot u_1(\sigma_1, \sigma_2^*) + \epsilon \cdot \int_{S_1} \min_{s_2 \in S_2} u_1(s_1, s_2)\ d\sigma_1\ ],$$

$$\sigma_2^* \in \arg\max_{\sigma_2 \in \Sigma_2} [\ (1-\epsilon) \cdot u_2(\sigma_1^*, \sigma_2) + \epsilon \cdot \int_{S_2} \min_{s_1 \in S_1} u_2(s_1, s_2)\ d\sigma_2\ ].$$

As an example, consider a symmetric duopoly with linear cost and demand curve. Under Bertrand competition, setting price equal to marginal cost is a Choquet-Nash equilibrium independently of ϵ. Under Cournot competition, however, the firms have an incentive to offer less than the Cournot equilibrium output, and set higher prices, since for any given production there is a small chance that a non-rational opponent swamps the market and drives down profits.

The extension to n players is conceptually straightforward. However, it has to take into account that the events that different opponents are non-rational are independent. For instance, if there are three players, then player 1 should maximise[62]

$$\begin{aligned}\max_{\sigma_1\in\Sigma_1} \quad [& \quad (1-\epsilon)^2 \cdot u_1(\sigma_1, \sigma_2^*, \sigma_3^*) \\ + & \quad \epsilon(1-\epsilon) \cdot \sum_{s_1\in S_1} \sigma_1(s_1) \cdot \min_{s_2\in S_2} u_1(s_1, s_2, \sigma_3^*) \\ + & \quad (1-\epsilon)\epsilon \cdot \sum_{s_1\in S_1} \sigma_1(s_1) \cdot \min_{s_3\in S_3} u_1(s_1, \sigma_2^*, s_3) \\ + & \quad \epsilon^2 \cdot \sum_{s_1\in S_1} \sigma_1(s_1) \cdot \min_{(s_2,s_3)\in S_2\times S_3} u_1(s_1, s_2, s_3) \ ].\end{aligned}$$

In general, we can formulate the solution concept in the following way: Let I be the player set, and for $J \subseteq I$ let s_J be a strategy profile that specifies a pure strategy for each player in J. Let S_J be the set of such profiles, i.e. $S_j = \times_{i\in J} S_i$. Let s_{-J} be a strategy profile that specifies a pure strategy for all players not in J.

Definition. *Let (I, S, u) be a finite two-player game in normal form. Let $0 \leq \epsilon \leq 1$. Then σ^* is a* **Choquet-Nash equilibrium** *iff for every player $i \in I$*

$$\begin{aligned}\sigma_i^* \in \arg\max_{\sigma_1\in\Sigma_1} \quad [& \quad (1-\epsilon)^{|I|} \cdot u_i(\sigma_i, \sigma_{-i}^*) \\ + & \quad \sum_{\substack{J\subseteq I\setminus\{i\} \\ J\neq\emptyset}} [\ \epsilon^{|J|}(1-\epsilon)^{I\setminus(J\cup\{i\})}\ ] \\ & \quad \cdot [\ \sum_{s_i\in S_i} \sigma_i(s_i) \cdot \min_{s_J\in S_J} u_i(s_i, \sigma_{-(J\cup\{i\})}^*, s_J)\ ]\ ],\end{aligned}$$

where $|J|$ denotes the number of players in $J \subseteq I$.

[62] We continue to make the assumptions for Choquet-Nash equilibria: common priors ϵ, complete ignorance and uncertainty aversion.

2.5 Related Literature

The aim of this section is to argue that our equilibrium concept circumvents some of the controversial aspects of previous attempts to generalize the equilibrium concept to non-additive beliefs: the definition of support of a non-additive measure, the requirement that players' beliefs are simple capacities, and the definition of independence of several non-additive beliefs.

Previous solution concepts — with the exception of Mukerji (1994) — have not distinguished between rational and non-rational players. In those models, the rational player is allowed to have non-additive beliefs about the opponent's play. An equilibrium is then interpreted as an equilibrium in beliefs. However, since beliefs are non-additive, they cannot be correct, so the weaker consistency requirement that players are not wrong is imposed on equilibrium beliefs. Following Dow & Werlang (1994), this is formalised as the requirement that the players anticipate the support of the opponent's beliefs. This raises the question, however, how the support of a non-additive capacity should be defined, and different support concepts give rise to different equilibrium concepts. These issues are surveyed, e.g., in Eichberger & Kelsey (1993) and Haller (1997).

Since defining the support as the smallest set of strategies that has belief 1 under uncertainty aversion does not impose any restriction on the support, Dow & Werlang (1994) define the support as the smallest set of strategies whose complement has belief 0. The support, so defined, need not be unique. The approach of Dow & Werlang (1994) models a situation in which rational players lack logical omniscience, in that they do not draw the logical conclusions of their knowledge.

The question how to define the support of a non-additive capacity does not arise in our model. Here, players have additive beliefs about the rational opponents. So their expectations can be correct in the usual, literal, sense. Also, the rational players are assumed to be logically omniscient.

In the Dow & Werlang (1994) model, the support question has a natural answer in the special case, in which the non-additive beliefs are 'simple capacities', i.e. capacities that uniformly distort probabilities

$$v(E) = \begin{cases} \alpha \cdot p(E) & , \quad E \neq \Omega, \\ 1 & , \quad E = \Omega, \end{cases}$$

where uncertainty aversion corresponds to the assumption that $\alpha < 1$. For such simple capacities, the Choquet-integral of a random variable X takes the form

$$\int X \, dv = \alpha \cdot \int X \, dp + (1 - \alpha) \cdot \min_{\omega \in \Omega} X(\omega).$$

Thus, our concept of Choquet-Nash equilibrium corresponds formally to the case where $\alpha = (1 - \epsilon)$. However, this analogy is purely formal: A weak Choquet-Nash equilibrium cannot be re-interpreted as a simple capacity, and for non-simple capacities the above decomposition does not hold. We are not requiring that rational players' beliefs about the opponents' play are simple, but that beliefs about rational opponents are additive, whereas those about non-rational opponents may be non-additive, but otherwise arbitrary (i.e. non-simple).

Finally, Dow & Werlang (1994) define their equilibrium concept for 2-player games. Eichberger & Kelsey (1993) extend their solution concept to n-player games and allow for the possibility that a rational player beliefs that his opponents do not act independently. In their approach, imposing such a restriction requires an independence concept for capacities (see, e.g., Ghirardato (1997) and Hendon et al. (1996)).

In our approach, this issue also does not arise. Since rational players have additive beliefs about their rational opponents, the usual independence concept applies and the equilibrium concept for n-player games assumes that rational players believe that their rational players act independently. This is in line with the underlying assumption that the game form models a non-cooperative situation and is common knowledge among the rational players.

2.6 Preference for Uncertainty

Recall that in section 2 we have defined the expected utility from pure strategy s_i against a non-rational opponent by the Choquet expectation $u_i(s_i, v_j) := \int_{S_j} u_i(s_i, s_j) \, dv_j$. Then we defined his payoff from a mixed strategy $\sigma_i \in \Delta S_i$ as $u_i(\sigma_i, v_j) := \sum_{s_i \in S_i} \sigma_i(s_i) \cdot u_i(s_i, v_j)$. As a consequence, the overall expected utility is linear in the probabilities $\sigma_i(s_i)$. Since v_j is non-additive, the order of integration in $u_i(\sigma_i, v_j)$ is important. In this section we present and analyse an alternative equilibrium concept in which this order is reversed.

We continue to make the assumptions of a common prior about rationality, complete ignorance about non-rational play and uncertainty aversion. Note that then

$$u_i(\sigma_i, v_j) = \min_{s_j \in S_j} u_i(\sigma_i, s_j) = \min_{s_j \in S_j} \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, s_j).$$

First note the following lemma:

Lemma 4.

$$\sum_{s_i \in S_i} \sigma_i(s_i) \min_{s_j \in S_j} u_i(s_i, s_j) \leq \min_{s_j \in S_j} \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, s_j).$$

The inequality may be strict.

Proof. For all s_i and s_j

$$\min_{s_j \in S_j} u_i(s_i, s_j) \leq u_i(s_i, s_j).$$

Therefore for all s_j

$$\sum_{s_i \in S_i} \sigma_i(s_i) \min_{s_j \in S_j} u_i(s_i, s_j) \leq \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, s_j).$$

So this holds in particular for the smallest value of the right-hand side. To see that the inequality may be strict, consider the following example:

	L	R
T	2	0
B	0	2
$\frac{1}{2}T + \frac{1}{2}B$	1	1

FIGURE 19

Here

$$\sum_{s_i \in S_i} \sigma_i(s_i) \min_{s_j \in S_j} u_i(s_i, s_j) = 0 < \min_{s_j \in S_j} \sum_{s_i \in S_i} \sigma_i(s_i) u_i(s_i, s_j) = 1.$$

q.e.d.

Thus, reversing the order of integration allows players to have a *strict* preference for mixed strategies in a game. The first equilibrium concept that captures this phenomenon in strategic interaction is given by Klibanoff (1993), who based it on maxmin expected utility theory of Gilboa & Schmeidler (1989), in which players have set-valued beliefs. Allowing strict preference for mixed strategies gives rise to the following definition:

Definition. *Let (I, S, u) be a finite two-player game in normal form. Let $0 < \epsilon_1 \leq 1$. Then σ^* is a* **strong Choquet-Nash equilibrium** *iff*

$$\sigma_1^* \in \arg\max_{\sigma_1 \in \Sigma_1} [\ (1-\epsilon_1) \cdot u_1(\sigma_1, \sigma_2^*) + \epsilon_1 \cdot \min_{s_2 \in S_2} u_1(\sigma_1, s_2)\ ],$$

$$\sigma_2^* \in \arg\max_{\sigma_2 \in \Sigma_2} [\ (1-\epsilon_2) \cdot u_2(\sigma_1^*, \sigma_2) + \epsilon_2 \cdot \min_{s_1 \in S_1} u_2(s_1, \sigma_2)\ ].$$

Under uncertainty aversion, a strong Choquet-Nash equilibrium always exists. This is essentially the same argument as in proposition 1, except that objective function is now quasi-concave in the probabilities σ_i. However, the analogous solution concept for uncertainty love is now no longer guaranteed to exist, since the objective function need not be quasi-concave. As a consequence, the best-reply correspondence need not be convex-valued (see, e.g., Crawford (1990) and Dekel et al. (1991)).

The main characteristic of a strong Choquet-Nash equilibrium is that in zero-sum games, the solution concept coincides with Nash equilibrium: Since it is already rational to play maxmin strategies against rational opponents, and since this is also rational against non-rational opponents, it is overall rational. More generally:

Remark 7. *Let (I, S, u) be a finite two-player game in normal form. Let $0 \leq \epsilon \leq 1$. Then every equilibrium in maxmin-strategies is also a strong Choquet-Nash equilibrium independently of ϵ.*

Proof. If σ^* is an equilibrium then for all $i \in I$ and all $\sigma_i \in \Sigma_i$

$$u_i(\sigma_i^*, \sigma_j^*) \geq u_i(\sigma_i, \sigma_j^*).$$

Since σ_i^* are maxmin strategies, for all $\sigma_i \in \Sigma_i$

$$\min_{s_j \in S_j} u_i(\sigma_i^*, s_j) = \min_{\sigma_j \in \Sigma_j} u_i(\sigma_i^*, \sigma_j) \geq \min_{\sigma_j \in \Sigma_j} u_i(\sigma_i, \sigma_j) = \min_{s_j \in S_j} u_i(\sigma_i, s_j).$$

Combining both inequalities gives the result. **q.e.d.**

2.7 Conclusion

The paper presented equilibrium concepts that formalize the idea that lack of mutual knowledge of rationality together with a lack of a theory of non-rationality create genuine uncertainty. However, on the basis of decision theory with non-additive, or set-valued, beliefs, rational behaviour is still well-defined, if the attitude towards uncertainty is specified.

The motivation for developing our solution concepts were deviations from subjective expected utility in experiments, as in the Ellsberg paradox. This behaviour can be parsimoniously explained as uncertainty aversion. Thus we also formulated the solution concepts under this assumption. To what extent these solution concepts can model behaviour is an empirical question; this also holds for the question which solution concept is relevant in a particular situation. For instance, we see the question whether players have a strict preference for mixed strategies as an empirical one.

The assumption of extreme uncertainty aversion is rather crude; however, in the absence of a theory of bounded rationality that imposes restrictions on deviations from rational play, it seems the only assumption consistent with the fact that only rational strategies are derived.

Our results suggest robustness concepts for Nash equilibria. In so doing, we consider mutual knowledge of rationality as a limiting case of lack thereof. This is entirely analogous to Selten's (1975, p. 35) view of "complete rationality as a limiting case of incomplete rationality". However, we would argue that robustness with respect to ignorance about non-rational play is more plausible than robustness with respect to 'trembles' of otherwise fully rational players.

The following are suggestions for future research: First, the question arises whether there are epistemic foundations for Choquet-Nash equilibria in a model similar to that of Aumann & Brandenburger (1995). Secondly, it will be interesting to study the effects of communication and correlation on a Choquet-Nash equilibrium in the spirit of Aumann's (1974) correlated equilibrium. On the other hand, the equilibrium concepts could also be weakened to rationalizability concepts along the lines of Bernheim (1984) and Pearce (1984). Finally, combining our robustness concepts with equilibrium refinements for rational players will further narrow down the set of equilibria.

Chapter 3

Uncertainty Aversion and Equilibrium in Extensive Form Games

Abstract

This chapter formulates a rationality concept for extensive games in which deviations from rational play are interpreted as evidence of irrationality. Instead of confirming some prior belief about the nature of non-rational play, we assume that such a deviation leads to genuine uncertainty. Assuming complete ignorance about the nature of non-rational play and extreme uncertainty aversion of the rational players, we formulate an equilibrium concept on the basis of Choquet expected utility theory. We apply the equilibrium concept to ~~the~~ the centipede game and the finitely repeated prisoners' dilemma.

3.1 Introduction

According to the principle of sequential rationality, a rational player of an extensive form game regards his opponents as rational even after a deviation from rational play. The internal consistency of this principle is subject of much debate (see, e.g., Aumann (1995, 1996, 1998), Binmore (1996), Reny (1993)).

Attributing non-rational deviations to 'trembles' of otherwise perfectly rational players (Selten 1975) is logically consistent, but raises the second concern with the principle of sequential rationality, that is its empirical plausibility. Quite independently of the question whether there exists a rationality concept that implies, or is at least consistent with sequential rationality, the question arises if there is room for an alternative rationality concept, in which deviations from the solution concept are interpreted as evidence of non-rationality. In this paper, we attempt to formulate such a rationality concept on the basis of Choquet expected utility theory.

In a seminal series of papers, Kreps, Milgrom, Roberts, and Wilson (1982) (henceforth KMRW) developed the methodology for analysing games with possibly non-rational opponents. In their models, there is some a priori uncertainty about the rationality of the opponent. Under subjective expected utility, players act as if they possess a probability distribution over the 'type' of the opponents' non-rationality. They maximise utility given their beliefs, and in sequential equilibrium their beliefs are consistent with the play of rational opponents. KMRW have shown how even small degrees of uncertainty about rationality can have large equilibrium effects. They showed that this can explain both intuitive strategic phenomena, particularly in industrial organization, and, at least to some degree, experimental evidence.

One problem in this approach, however, is for an outside observer to specify the probability distribution over the types of the non-rational opponents before experimental or field data are available. A second problem is that analysing the strategic interaction as a game with incomplete information implies that the other players, whether rational or not, can be modelled as 'types', who possess a consistent infinite hierarchy of beliefs about the strategic interaction. Thus, the players in this methodology are not really non-rational; rather, they are rational but have preferences that differ from those that the game attributes to 'rational' players.

In this paper, we argue that a consistency argument addresses both of these problems. A game-theoretic solution concept that singles out rational strategies implicitly defines all other strategies as non-rational. Thus, consistency requires that beliefs about non-rational players should not exclude any of these non-rational strategies. In other words, if the rationality concept is point-valued, the beliefs about non-rational play should include all deviations, and thus must be set-valued. So in this sense, the rationality concept itself pins down beliefs about non-rational play, but excludes subjective expected utility theory (henceforth SEU) as the adequate model of these beliefs. Thus, SEU is not an appropriate framework for beliefs about non-rationality when rationality is endogenous.

Thus, this paper argues that, after an opponent deviates from rational play, a rational player faces genuine uncertainty. What matters, then, is the rational player's attitude towards uncertainty. This paper formulates the equilibrium concept for the case in which rational players are completely uncertainty averse. It is this case that has led to the development of decision theories with set-valued and non-additive beliefs as an explanation of the Ellsberg (1961) paradox. Consequently, we base the equilibrium concept on Choquet expected utility theory (henceforth CEU) developed by Schmeidler (1989).

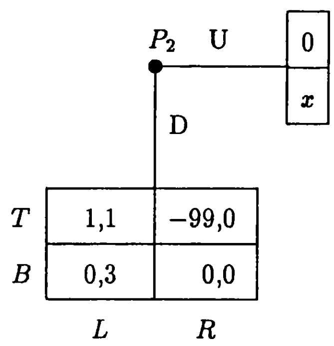
This paper joins a growing literature that applies CEU to games. The first of these were Dow & Werlang (1994) and Klibanoff (1993). Dow & Werlang (1994) consider normal form games in which players are CEU maximisers. Klibanoff (1993) similarly considers normal form games in which players follow maxmin-expected utility theory (Gilboa & Schmeidler 1989), which is closely related to CEU. In Hendon, Jacobsen, Sloth & Tranæs (1995) players have belief functions, which amounts to a special case of CEU. Extensions and refinements have been proposed by Eichberger & Kelsey (1993), Lo (1995*b*), Lo (1996), Marinacci (1994) and Ryan (1997). Epstein (1997*a*) analysed rationalizability in normal form games. These authors consider normal form games and do not distinguish between rational and non-rational players. The paper closest to ours is Mukerji (1994), who considers normal form games only but argues that the distinction between rational and non-rational players is necessary to reconcile CEU with the equilibrium concept. For normal form games our concepts differ only in motivation and technical detail. The present paper mainly concerns extensive form games. Extensive games have been studied by Lo

(1995a) and Eichberger & Kelsey (1995). Lo (1995a) extends Klibanoff's approach to extensive games, Eichberger & Kelsey (1995) are the first to use the Dempster-Shafer updating rule (see section 3) in extensive games. They do not distinguish between rational and non-rational players.

This paper is organized as follows: The next section discusses an example. Section 3 presents Choquet expected utility theory and discusses the problem of updating non-additive beliefs. In section 4 we formulate the equilibrium concept for two player games with perfect information. In section 5 we discuss the centipede game and the finitely repeated prisoners' dilemma in order to relate the equilibrium concept to the foundations of game theory. Section 6 elaborates on the extension of the solution concept to general extensive games. Section 7 concludes. There is one appendix on details of updating non-additive beliefs.

3.2 An Example

Consider the following extensive form game, in which payoffs are given in von Neumann - Morgenstern utilities:[1]



First, consider the case that $x = 4$. Then D cannot be rational for P_2, because it is strictly dominated. Therefore, P_1 knows at the beginning of the subgame that P_2 is not rational and, consequently, has no reason to assume that P_2 will play his strictly

[1]After D, both players P_1 and P_2 know that P_2 chose D and they play the normal form subgame, i.e. choose simultaneously between T and B, respectively L and R.

dominant strategy L in the subgame. P_1's best reply to L is T, but, intuitively, it is very risky.

In the absence of a theory of rationality, P_1 faces true uncertainty about P_2's play in the subgame. Therefore, if P_1 is sufficiently uncertainty averse, it becomes rational for him to play B. Thus, under these assumptions the rational strategies are U, L (because it is strictly dominant in the subgame) and B. This strategy combination is not a subgame-perfect Nash equilibrium, yet no player has an incentive to deviate unilaterally from these rational strategies. Moreover, if there is some initial doubt $\epsilon > 0$ about P_2's rationality,[2] then D is also not a probability zero event, because nothing is known about a non-rational player, who might therefore well play D.[3]

All that it takes to reach these conclusion formally is a calculus that allows non-additive, or set-valued, beliefs, and that captures P_1's uncertainty as well as his uncertainty aversion. In addition, in order to conclude that that P_2 must be non-rational after D we need an updating rule for non-additive beliefs.

Secondly, consider the case that $x = 2$. The above criticism of subgame perfection still applies: The equilibrium (T, L) in the subgame makes U rational, but once U is designated as rational, P_1 faces true uncertainty after D and, if uncertainty averse, will rationally deviate to B. So (U, L, T) is not a rational solution. However, neither is (D, L, B), because if D is rational then P_1 is justified in anticipating strategy L, and should play his best reply T. Now P_2 has an incentive to deviate.

So suppose it is rational for P_2 to play D with probability p. Suppose further that there is a probability $\epsilon > 0$ that P_2 is not rational at the beginning of the game. Then P_1's optimal strategy in the subgame will depend on his belief about the rationality of P_2, given p and ϵ. The same updating rule that for $x = 4$ allows the natural conclusion that P_2 is rational after D gives the result[4] that

$$v(P_2 \text{ rational} \mid D) = \frac{(1-\epsilon)p}{1-(1-\epsilon)(1-p)}.$$

[2] For simplicity, assume in this example that there is no doubt about the rationality of P_1.

[3] If rationality is common knowledge at the beginning of the game, then D is indeed a probability zero event. In general, we take the view that there is a difference between probability zero events in decision theory and probability zero events in games, where an event for one player is an act for another. Here, D is an act that might destroy this common knowledge. It is still intuitive that P_1 should consider P_2 as non-rational. This conclusion could be formally reached by taking limits as $\epsilon \to 0$. However, in this paper we concentrate on the case $\epsilon > 0$.

[4] See the next section and the appendix.

Note that $v(P_2 \text{ rational} \mid D) = (1-\epsilon)\frac{p}{p+\epsilon(1-p)} < 1-\epsilon$. Thus, in line with his uncertainty aversion P_1 considers the worst case when he updates his belief ϵ. This worst case is that a non-rational player will play D with probability 1, because this makes it most likely that his behavior in the subgame is unpredictable, and, again due to uncertainty aversion, should be evaluated with the worst outcome.

Since a rational P_2 will play L, P_1 knows that T will give utility 1 with probability $\frac{(1-\epsilon)p}{1-(1-\epsilon)(1-p)}$. With the complementary probability, P_2 is non-rational and the theory is silent about what this means. Again, P_1 faces true uncertainty, and if he is extremely uncertainty averse, he will allocate the complementary probability to the worst outcome -99. So P_1's expected utility from T is

$$\frac{(1-\epsilon)p}{1-(1-\epsilon)(1-p)} \cdot 1 - 99 \cdot \frac{\epsilon}{1-(1-\epsilon)(1-p)}.$$

In a mixed strategy equilibrium P_1 must be willing to randomize, so we must have

$$\frac{(1-\epsilon)p - 99\epsilon}{1-(1-\epsilon)(1-p)} = 0,$$

i.e. $p = 99\frac{\epsilon}{1-\epsilon}$. Note, first, that P_1 is willing to mix only if $\epsilon < \bar{\epsilon} = \frac{1}{100}$, i.e. if the initial doubt about rationality is small enough, otherwise T will be too risky. Secondly, as ϵ goes to zero, p goes to zero, i.e. it is less and less rational for P_2 to play D. Both aspects are quite intuitive.

Further, P_2 must be willing to randomize as well, so that we must have $2 = q \cdot 1 + (1-q) \cdot 3$, i.e. $q = \frac{1}{2}$, where q is the probability that P_1 plays T. Overall, the rational strategies for given $\epsilon > 0$ are given by $(p^* = 99\frac{\epsilon}{1-\epsilon}, L, q^* = \frac{1}{2})$ if $\epsilon < \bar{\epsilon}$ and (D, L, B) if $\epsilon \geq \bar{\epsilon}$. Again, no player has an incentive to deviate.[5]

3.3 Choquet Expected Utility and Updating

Under SEU, a player has preferences over acts that map a set of states of nature S into a set of consequences Z. Under consistency assumptions about this preference ordering, the player acts as if he possesses a utility function u over consequences (cardinal, i.e. unique up to affine transformations), and a probability distribution p

[5] For completeness, we can finally consider the case $\epsilon \longrightarrow 0$. This gives the strategy profile $(U, L, q^* = \frac{1}{2})$. Note, however, that if $\epsilon = 0$ (as opposed to $\epsilon \searrow 0$), then, intuitively, P_1 has an incentive to deviate from $q^* = \frac{1}{2}$.

over states that represents subjective beliefs, and maximises expected utility. This axiomatisation of SEU is due to Savage (1954). Anscombe & Aumann (1963) have simplified this approach by assuming that acts map states into lotteries (probability distributions) over states.

Ellsberg's paradox (1961) provides evidence, however, that players do not necessarily act as if their beliefs are probability distributions. On the contrary, these experiments provide evidence for the hypothesis that beliefs are non-additive, and that players are uncertainty averse.

CEU also considers a preference relation over acts. Under weaker consistency assumptions, a player still acts as if he possesses a cardinal utility function u and subjective beliefs v, and maximises 'expected utility'. The difference to SEU is that beliefs no longer have to be additive. Non-additive beliefs are given by a set function v that maps events (sets of states) into $\mathbb{R}$ such that

(i) $v(\emptyset) = 0$,

(ii) $v(\Omega) = 1$,

(iii) $E \subseteq F \implies v(E) \leq v(F)$.

CEU was first axiomatised by Schmeidler (1989).[6]

The expectation of a utility function with respect to non-additive beliefs v is defined by Choquet (1953). For $u \geq 0$ the Choquet integral is given by the extended Riemann integral

$$\int u \, dv := \int_0^\infty v(u \geq t)dt,$$

where $v(u \geq t)$ is short for $v(\{s \in S | u(s) \geq t\})$. For arbitrary $u = u^+ - u^-$, where $u^+ := \max\{u, 0\}$ and $u^- := \max\{-u, 0\}$ denote the positive and the negative part, the Choquet integral is defined as $\int u \, dv := \int u^+ \, dv - \int u^- \, d\tilde{v}$, where $\tilde{v}$ is the dual of v, i.e. $\tilde{v}(E) := 1 - v(\overline{E})$ and $\overline{E}$ is the complement of E.

Non-additive beliefs express uncertainty aversion[7] if v is supermodular, i.e. $v(E \cup E') + v(E \cap E') \geq v(E) + v(E')$. If v is supermodular, then its core $Core(v) := \{p | p(E) \geq v(E)\}$ of additive set functions p that eventwise dominate v is non-

[6] See also Gilboa (1987), Wakker (1989) and Sarin & Wakker (1992).

[7] Note that we only claim that supermodularity is sufficient for uncertainty aversion, not that it is also necessary. Necessity is a controversial question (see Epstein (1997*b*) and Ghirardato & Marinacci (1997)). The reason why we associate uncertainty aversion with supermodularity is that we can then think interchangeably of non-additive and set-valued beliefs.

empty (Shapley 1971). In that case, we can equivalently think of the players as possessing the set of additive beliefs $Core(v)$. The Choquet integral of u is then given by $\int u\ dv = \min_{p \in Core(v)} \int u\ dp$ (Schmeidler 1986, Schmeidler 1989, Gilboa & Schmeidler 1989).

Finally, we have to specify how players update beliefs. There is no universally agreed upon updating rule for non-additive beliefs. Instead, the consensus is that different updating rules are appropriate for different circumstances. In line with the assumption that players are uncertainty averse, we use the Dempster-Shafer rule (Dempster 1967, Shafer 1976), which is given by

$$v(A|B) := \frac{v(A \cup \overline{B}) - v(\overline{B})}{1 - v(\overline{B})}$$

The Dempster-Shafer rule reduces to Bayes' Rule if the capacity v is additive. Gilboa & Schmeidler (1993) show that the Dempster-Shafer rule corresponds to pessimistic updating. The Dempster-Shafer rule is not dynamically consistent, but there is no dynamically consistent updating rule for non-additive beliefs (see, e.g., Epstein & Breton (1993) and Eichberger & Kelsey (1996)). Thus the Ellsberg paradox implies that updating must be dynamically inconsistent. The Dempster-Shafer rule preserves supermodularity (Fagin & Halpern 1990), and is commutative (Gilboa & Schmeidler 1993). Finally, we note that our approach does not rely on the details of the Dempster-Shafer rule. Any updating rule that takes into account that there are no probability zero events when non-rational play is unrestricted is admissible. Which updating rule will eventually prove to be the correct one is an issue that will have to be settled experimentally, for a first step in this direction see Cohen, Gilboa, Jaffray & Schmeidler (1999).

3.4 Perfect Choquet Equilibria

We use the following notation for extensive form games as defined in Selten (1975) and Kreps & Wilson (1982*b*): Let Γ be an extensive game, finite and with perfect recall. Let V be the set of vertices, with decision nodes X and endnodes Z. Let $\emptyset$ be the origin (empty history). Let $\preceq$ be the precedence relation, i.e. $v \preceq v'$ means that there is a path from v to v'. The relation $\preceq$ is an arborescence, i.e. a partial ordering in which different nodes have disjoint successor sets. Let I be the player set. Let X_i be the decision nodes of player $i \in I$. Let H_i be the set of player i's information sets $h_i \in H_i$. Let $A(h_i)$ be the set of actions that are available to player i at his information set h_i, similarly let $A(x_i)$ be the set of actions that are available to player i at his decision node x_i. Let A_i be the set of actions available to player i at some information set. Let X_0 be the set of all nodes at which there is a random move, and for $x_0 \in X_0$ let $\pi(x_0)$ be the probability distribution over $A(x_0)$.

Let S_i be the set of pure strategies $s_i : H_i \longrightarrow A_i$ of player i, $s_i(h_i) \in A(h_i)$. Let Σ_i be the set of behavior strategies of player i, i.e. $\sigma_i(h_i)$ is a probability distribution over $A(h_i)$. The sets S and Σ are the sets of pure and behavior strategy profiles $s \in S = \times_{i \in I} S_i$, $\sigma \in \Sigma = \times_{i \in I} \Sigma_i$. As usual, s_{-i} and σ_{-i} denote i-incomplete strategy combinations. Similarly, $s_{i,-h_i}$ and $\sigma_{i,-h_i}$ denote h_i-incomplete strategies of player i, i.e. strategies that do not specify an action at information set h_i.

Let $u_i : Z \longrightarrow \mathbb{R}$ be the von Neumann - Morgenstern utility function of player i. For $s \in S$, let $u_i(s)$ be the expected utility of player i if the pure strategy combination s is played and random moves are drawn according to the distributions $\pi(x_0)$. For $\sigma \in \Sigma$, let $u_i(\sigma)$ be the expected utility of player i if the behavior strategy combination σ is played. For a decision node $x \in X$, let $u_i(\sigma|x)$ be the conditional expected utility of player i, if the game starts at decision node x and the behavior strategy combination σ is played.

The definition of a perfect Choquet equilibrium will become quite involved for general extensive games. For this reason, we first restrict attention to two-player games with perfect information.

Since in extensive games lack of mutual knowledge of rationality arises endogenously whenever a player deviates from his rational strategy, we consider a situation in which rationality is in general not mutual knowledge. We aim to define what rational strategies are. We assume that rational players maximise Choquet - expected utility, i.e. possess a utility function u and maximise utility given their beliefs. Since the opponent may be rational or not, their beliefs can be expressed as two capacities v_R and $v_{\overline{R}}$, where v_R is the belief about the play of rational opponents and $v_{\overline{R}}$ the belief about the play of non-rational opponents. Let $\epsilon_{ij}(x_i)$ be player i's belief that player j is not rational at decision node x_i.

So for given beliefs the rational player chooses his action at decision node x_i by maximising

$$\max_{a\in A(x_i)} \quad [1-\epsilon_{ij}(x_i)] \int u_i(a,\sigma^*_{i,-x_i},s_j \mid x_i)\; d\, v_R$$
$$+\epsilon_{ij}(x_i) \int u_i(a,\sigma^*_{i,-x_i},s_j \mid x_i)\; d\, v_{\overline{R}},$$

where $\sigma^*_{i,-x_i}$ is player i's plan how to continue playing.

The strategy of a rational opponent has to be determined endogenously. So, in equilibrium beliefs v_R gave to coincide with the opponent's rational strategy σ^*_j. In particular, v_R is an additive belief and the Choquet integral reduces to the usual integral, i.e.

$$\int u_i(a,\sigma^*_{i,-x_i},s_j \mid x_i)\; d\, v_R = \int u_i(a,\sigma^*_{i,-x_i},s_j \mid x_i)\; d\, \sigma^*_j = u_i(a,\sigma^*_{i,-x_i},\sigma^*_j \mid x_i).$$

It remains to specify the beliefs $v_{\overline{R}}$ about play of non-rational opponents. Since the solution concept specifies rational strategies only, every deviation has to be considered non-rational. Thus $v_{\overline{R}}$ should not impose any restriction on the play of a non-rational player, so that the rational player faces non-additive uncertainty. What matters then is the rational players attitude towards uncertainty. We define the solution concept for the case in which rational players are uncertainty-averse.[8] Consequently, we assume that $v_{\overline{R}}$ is the basic capacity that assigns belief

$$v_{\overline{R}}(S'_j) = \begin{cases} 1, & S'_j = S_j \\ 0, & else \end{cases}$$

to the event that a non-rational player's strategy is in the set S'_j.

Modelling complete uncertainty as a basic capacity as opposed to a uniform probability distribution also has the practical advantage that the expected value of the utility function does not depend on the description of the state space. For instance, if a superfluous move, i.e. a copy one of the opponent's strategies, is added to the opponent's strategy set, Choquet expected utility under a basic capacity is the same, whereas the expected utility under a uniform probability distribution would, in general, change.

[8]The Ellsberg paradox seems to point towards uncertainty aversion, and this has been the main motivation for developing CEU. Smithson (1997) reports that uncertainty aversion is a robust phenomenon in the Ellsberg experiment. However, Smithson (1997) also draws attention to the fact that uncertainty aversion is not a universal empirical fact.

For this capacity, the Choquet integral reduces to[9]

$$\int u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i)\; d\, v_{\overline{R}} = \min_{s_j \in S_j} u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i).$$

Overall, a rational player thus maximises his expected utility, given his beliefs ϵ_{ij}, v_R and $v_{\overline{R}}$. The perfection requirement now means that a player maximises his utility at each decision node in the game, conditional on that node being reached. Moreover, as the game progresses he updates his beliefs, and since his beliefs about non-rational opponents are non-additive he does so on the basis of the Dempster-Shafer rule. In a perfect Choquet equilibrium, a rational player correctly anticipates the play of a rational opponent and has no incentive to deviate. Formally:

Definition. *Let Γ be a finite extensive two-player game with perfect information. Then σ^* is a perfect Choquet equilibrium iff (if and only if) for each players i, each of his decision nodes x_i, and each pure action $a^*(x_i)$ in the support of $\sigma^*_i(x_i)$*

$$\begin{aligned} a^*(x_i) \quad &\in \quad \arg\max_{a \in A(x_i)} \; [1-\epsilon_{ij}(x_i)]\; u_i(a, \sigma^*_{i,-x_i}, \sigma^*_j \mid x_i) \\ &\qquad\qquad + \epsilon_{ij}(x_i)\; [\min_{s_j \in S_j} \; u_i(a, \sigma^*_{i,-x_i}, s_j \mid x_i)], \\ \epsilon_{ij}(x_i) \quad &:= \quad \frac{\epsilon_{ij}(x'_i)}{1 - [1-\epsilon_{ij}(x'_i)][1 - \prod_{x'_i \prec x_j \prec x_i} \sigma^*_j(x_j)]}, \end{aligned}$$

where x'_i is player i's decision node that precedes x_i, the product is taken over all decision nodes of player j that lie between x'_i and x_i, and $\sigma^*_j(x_j)$ is the probability that player j takes the action that leads from x'_i to x_i.[10]

Thus, a perfect Choquet equilibrium resolves the infinite regress that arises in a situation in which rationality is not mutual knowledge: Rationality means to maximise utility given beliefs; these beliefs take into account that a rational opponent will do the same, and that the rationality concept does not restrict the play of a non-rational opponent.

[9] Note that in contrast to some of the literature on CEU in games we do not restrict rational players' beliefs to 'simple capacities', i.e. distorted probability distributions. In principle, the players may have arbitrary beliefs about non-rational play. Here, we assume instead that a rational player distinguishes between rational and non-rational opponents.

[10] The updating rule takes into account that the opponent may move more than once between x'_i and x_i. See remark (6) in the appendix.

Note that in a perfect Choquet-equilibrium the equilibrium path is supported by a different solution concept, i.e. minimax play, off the equilibrium path. Consequently, the solution concept does not suffer from the logical deficiency of subgame perfection, where the equilibrium path is supported by equilibrium reasoning off the equilibrium path.

Note also the important difference between subjective expected utility theory and Choquet expected utility in justifying the minimax-strategy against non-rational opponents. Under subjective expected utility the maximin-strategy is rational only if the rational player believes that the non-rational opponent minimaxes him. This belief seems difficult to justify. Under CEU the maximin-strategy is rational because the rational player cannot exclude the possibility that the non-rational opponent plays, perhaps by chance, a minimax-strategy and because he reacts aversely towards the uncertainty created by the lack of possibility to forecast a non-rational opponent's play.

This solution concept generalizes immediately to repeated normal form games, i.e. multi-stage games with observed actions (Fudenberg & Tirole 1991), in which the players move simultaneously in each stage, and learn the (pure) actions after each stage.

In section 6 we discuss the extension of this equilibrium concept to general extensive games and to more than two players. In the next section we relate the solution concept to the foundations of game theory.

3.5 Subgame Perfection

The aim of this section is to relate our solution concept to the discussion on the foundations of game theory. The example in section 2 already shows how a perfect Choquet equilibrium differs from subgame perfection. The following discussion of the centipede game shows that backward induction need not be based on common knowledge of rationality. Thus, we argue that our solution concept provides a robustness criterion for subgame-perfect equilibria.

3.5.1 The Centipede Game

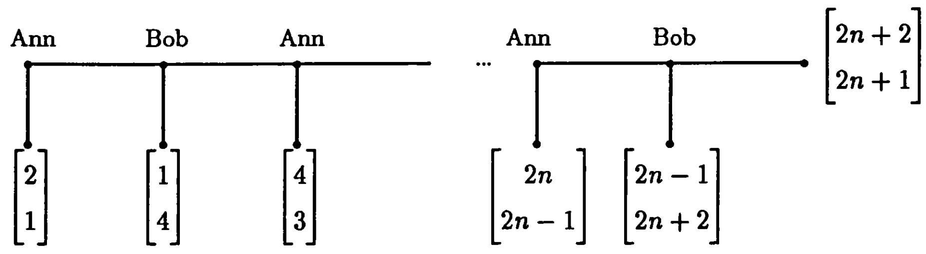
The logical consistency of subgame perfection has been controversial for a long time (Binmore 1987–88, Reny 1993). Selten's (1975) concept of trembling-hand perfection circumvents these difficulties by explaining deviations from rationality as unsystematic trembles of otherwise rational players, so that deviations are not evidence of non-rationality. Rationality is then defined as a limiting case of non-rationality where the probability of mistakes approaches zero. Though this approach is empirically implausible, it is logically consistent.

The logical status of subgame perfection was further clarified by Aumann (1995, 1998) (see also Binmore (1996), Aumann (1996)). Aumann (1995) shows that common knowledge of 'ex ante substantive rationality' implies the backward induction outcome in perfect information games. Here, a player is 'rational' if there is no other strategy that the player knows to give him higher expected utility than the one he chooses.

The distinction between 'ex ante' and 'ex post' rationality refers to the point in the game when his knowledge matters. 'Ex ante rationality' at some decision node v means that at the beginning of the game he knows of no better action at v, 'ex post' rationality means that when v is reached he knows of no better strategy. Consequently, 'ex ante' rationality is weaker than 'ex post' rationality.

The distinction between 'substantive' and 'material' rationality refers to the decision nodes where the player is assumed to be rational. Thus 'substantive' rationality means that a player is rational at all decision nodes, whether they are reached by rational play or not. 'Material' rationality, on the other hand, means that players are only assumed to be rational at reached decision nodes. 'Material' rationality is weaker than substantive rationality, and Aumann shows that 'material' rationality does not imply the backward induction outcome.

Aumann (1998) notices that his result can be sharpened for the centipede game. The centipede game (Rosenthal 1981) has become a cornerstone for the discussion of the foundations of game theory. Aumann shows that, due to its specific payoff structure, common knowledge of 'ex post material rationality' implies the backward induction outcome in the centipede game. The rationality concept cannot be weakened to 'ex ante material rationality'. Note that common knowledge of rationality implies the backward induction outcome, not the backward induction strategy profile.



The Centipede Game

It is immediate that the perfect Choquet equilibrium in the centipede game is to play down everywhere: At the last node 'down' is strictly dominant, at the penultimate node the player knows that a rational opponent will go down at the last node, or will consider the non-rational opponent unpredictable and assume the worst. In either case the continuation payoff is less than that from going 'down', so again 'down' is optimal. The same reasoning applies at every earlier node.

This conclusion is interesting for the following reasons: First, not only do we get the backward induction outcome, but also the backward induction strategy profile. Moreover, this profile is achieved using the same logic as subgame perfection. Secondly, the backward induction solution arises without mutual knowledge of rationality. Finally, the original objection to backward induction no longer applies: players are not assumed to be rational off the equilibrium path.

3.5.2 The Finitely Repeated Prisoners' Dilemma

One of the first papers to apply CEU to normal form games was Dow & Werlang (1994). In particular, Dow and Werlang develop an equilibrium concept for players who have non-additive beliefs and analyse the once-repeated prisoners' dilemma.[11] They show that players with non-additive beliefs may not backward induct, and contrast their result with that of Kreps, Milgrom, Roberts & Wilson (1982).

[11] We have so far only considered extensive form games with perfect information; this is extended in the next section. However, as the following remarks make clear, we do not need the the general equilibrium concept to deal with the once-repeated prisoners' dilemma.

One of the differences between a perfect Choquet equilibrium and the Nash equilibrium under uncertainty of Dow & Werlang (1994) is the way in which uncertainty arises in the game. Dow & Werlang (1994) do not distinguish between rational and non-rational players, in their equilibrium concept players are CEU maximisers who lack, to some degree, logical omniscience. They anticipate that their opponents maximise CEU, but do not draw the conclusion about the strategy choice. In other words, theirs is an equilibrium in beliefs. In our model, it is the lack of mutual knowledge of rationality that gives rise to uncertainty. The rational players can anticipate how rational opponents will act, but not how non-rational opponents will

The main difference, however, that Nash equilibrium under uncertainty is a normal form concept. Thus when players may have non-additive beliefs, cooperation in the first period and defection in the second can be an equilibrium: If the players believe that the opponent cooperates in the second period if, and only if they do so at the first stage, they have an incentive to cooperate early, and for lack of logical omniscience both players' beliefs may be non-additive.[12,13]

Under complete uncertainty aversion, in a perfect Choquet equilibrium both players defect at all stages: In the second stage defection is strictly dominant, in the first stage a player can anticipate that a rational opponent will defect in the next stage, and a non-rational opponent causes uncertainty that is evaluated by its worst outcome, i.e. defection. So the second stage action is independent of the first stage action, and again defection is strictly dominant in the first stage.

Note that Aumann's (1995) justification of the backward induction outcome does

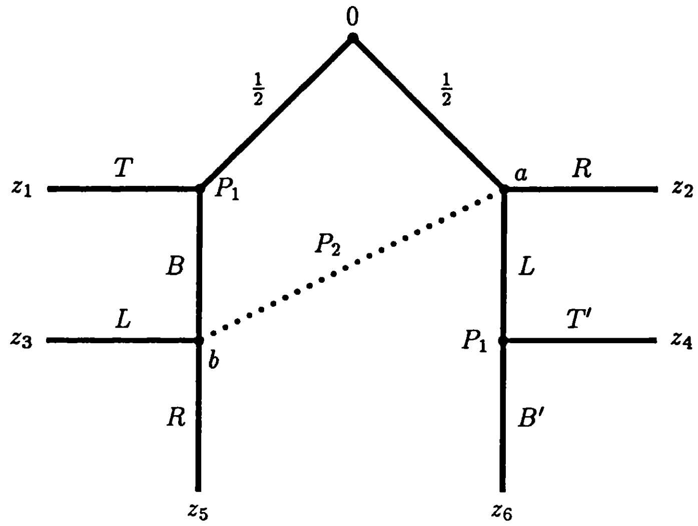
[12] In fact this phenomenon is also related to another aspect of the Nash equilibrium under uncertainty in Dow & Werlang (1994), i.e. their definition of support of a non-additive measure. The support implicitly defines the knowledge of the players. What the correct support notion is for non-additive beliefs is controversial. This issue does not arise in a perfect Choquet equilibrium, in which players know the rational strategies in the usual sense.

[13] It is also easy to construct such an equilibrium in the KMRW framework: Again, assume that the rational players believe that the non-rational opponent cooperates in the first period and cooperates in the second period if and only if both players cooperated in the first period. If the probability of having such a 'crazy' opponent is sufficiently high, it is indeed rational to cooperate in the first period. This result is interesting because it shows that backward induction may break down even if the strategies of the rational opponent can be correctly anticipated and subgame perfection is required. Yet, a basic shortcoming of this result is that it rests on this specific belief about non-rational play.

not apply to the repeated prisoners' dilemma, since it is not a perfect information game. Thus, the perfect Choquet equilibrium concept sheds some light on the robustness of subgame-perfect equilibria. It is perhaps surprising that backward induction is robust in games like the centipede game or the finitely repeated prisoners' dilemma, in which it is most counterintuitive.

3.6 Extension

The aim of this section is to discuss the extension of the solution concept to general extensive games. This extension is not straightforward, as the following example shows:



Note, first, that in a general extensive game a player should hold different beliefs about the degree of his opponent's rationality at different decision nodes in the same information sets. In the above example, at note a player P_2 should have belief $\epsilon(\emptyset)$, i.e. the prior with which he started the game. However, if T strictly dominates B, then at decision node b P_2 should hold the updated belief that the opponent is non-rational, i.e. $\epsilon(b) = 1$.

The second problem, also illustrated in the above example, is that not all decision nodes of an information set matter equally for the player who moves there. Above, P_2's belief $\epsilon(b)$ does not matter at all, because P_1 does not move after b. Only at decision node a is P_2's belief about P_1's rationality relevant for his decision.

Overall, in the example the intuitively correct belief for P_2 to have at his information set is therefore his prior $\epsilon(\emptyset)$. We suggest to generalize this observation in the following way:

Let h_i be an information set of player i. Call a decision node $x_i \in h_i$ relevant for player i if his opponent has a decision node in the subtree that starts at x_i. For each relevant decision node x_i in h_i, calculate the (in general non-additive) belief $\mu'(x_i)$ that the node is reached given the optimal strategies and beliefs $\epsilon(h_i')$ at preceding information sets h_i'. Form an updated belief $\epsilon(x_i)$ for each relevant decision node, where $\epsilon(x_i)$ is the Dempster-Shafer update from the preceding information sets and the equilibrium strategies. Finally, define $\epsilon(h_i)$ as the expected value of $\epsilon(x_i)$ given $\mu(x_i) := \frac{\mu'(x_i)}{\sum_{x_i \in h_i} \mu'(x_i)}$. Formally:

Definition. *Let Γ be a finite two-player game in extensive form.*

Then σ^* is a perfect Choquet equilibrium iff (if and only if) for each players i, each of his information sets h_i, and each pure action $a^*(h_i)$ in the support of $\sigma_i^*(h_i)$

$$a^*(h_i) \in \arg\max_{a \in A(h_i)} [1 - \epsilon_{ij}(h_i)]\, u_i(a, \sigma^*_{i,-h_i}, \sigma^*_j \mid h_i) + \epsilon_{ij}(h_i)\, [\min_{s_j \in S_j} u_i(a, \sigma^*_{i,-x_i}, s_j \mid h_i)],$$

$$\mu'(x_i) := \prod_{x_i' \prec x_i} \sigma^*(x_i') \prod_{x_j' \prec x_i} (\, [1 - \epsilon_{ij}(x_j')]\sigma_j^*(x_j') + \epsilon_{ij}(x_i') \,)$$

$$\mu'(h_i) := \textstyle\sum_{x_i \in h_i} \mu'(x_i)$$

$$\mu(x_i) := \frac{\mu'(x_i)}{\mu'(h_i)}$$

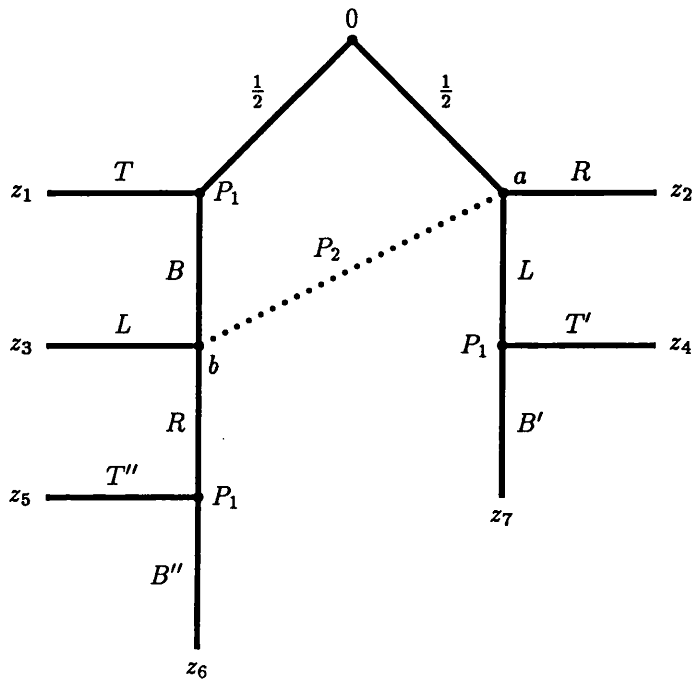
$$u_i(a, \sigma^*_{i,-h_i}, \sigma^*_j \mid h_i) = \sum_{x_i \in h_i} \mu(x_i) u_i(a, \sigma^*_{i,-h_i}, \sigma^*_j \mid x_i),$$

$$\epsilon_{ij}(x_i) := \frac{\epsilon_{ij}(x_i')}{1 - [1 - \epsilon_{ij}(x_i')][1 - \prod_{x_i' \prec x_j \prec x_i} \sigma_j^*(x_j)]},$$

$$\epsilon_{ij}(h_i) := \sum_{x_i \in h_i} \mu(x_i)\epsilon_{ij}(x_i)$$

where x_i' (x_j') are player i's (j's) decision nodes that precedes x_i, $\sigma_i^*(x_i')$ ($\sigma_j^*(x_j')$) is the probability that player i (j) takes the action that leads from x_i' (x_j') toward x_i.

Consider the following example:



Consider P_2 at his information set h: Let $\epsilon \equiv \epsilon(\emptyset)$, then we have $\epsilon(a) = \epsilon$. Again, assume that T is strictly dominant, so that $\epsilon(b) = 1$. Also, $\mu'(a) = \frac{1}{2}$ and $\mu'(b) = \frac{1}{2}\epsilon$. Note that the calculation of $\mu'(b)$ again reflects uncertainty aversion, because the worst case is that non-rational players would play B, since this would give most weight to the worst outcome as P_2 weighs his decision at his information set. Finally, $\mu(a) = \frac{\frac{1}{2}}{\frac{1}{2}+\frac{1}{2}\epsilon} = \frac{1}{1+\epsilon}$ and $\mu(b) = \frac{\epsilon}{1+\epsilon}$. So $\epsilon(h) = \mu(a)\epsilon(a) + \mu(b)\epsilon(b) = \frac{2\epsilon}{1+\epsilon}$.

The extension of the equilibrium concept to more than two players is conceptually straightforward, but computationally demanding. For two opponents P_j and P_k of player i, and beliefs ϵ_j and ϵ_k about their lack of rationality, the rational player has to take all four cases into account: that either of the players is non-rational, that both are rational and that neither is. So at decision node x_i he should maximise

$$\begin{aligned}\max_{a\in A(x_i)} \quad & [(1-\epsilon_j)(1-\epsilon_k)]\ u_i(a,\sigma^*_{i,-x_i},\sigma^*_j,\sigma^*_k) \\ & +[(1-\epsilon_j)\epsilon_k]\ \min_{s_k\in S_k}\ u_i(a,\sigma^*_{i,-x_i},\sigma^*_j,s_k) \\ & +[\epsilon_j(1-\epsilon_k)]\ \min_{s_j\in S_j}\ u_i(a,\sigma^*_{i,-x_i},s_j,\sigma^*_k) \\ & +[\epsilon_j\epsilon_k]\ \min_{(s_j,s_k)\in S_j\times S_k}\ u_i(a,\sigma^*_{i,-x_i},s_j,s_k).\end{aligned}$$

In particular, taking into account that both players may be non-rational means that in the worst case their actions may be correlated. Beliefs ϵ_j and ϵ_k are then updated separately on the basis of the Dempster-Shafer rule.

3.7 Conclusion

The paper has suggested a solution concept that combines equilibrium logic with maxmin play off the equilibrium path. The solution concept is natural if rationality is not mutual knowledge, no restriction is imposed on deviations from rationality, and if players are uncertainty-averse.

Nevertheless, the solution concept also has some limitations. First, the computational effort of calculating equilibria may be quite high. Secondly, experiments show that players sometimes systematically deviate from rational play, so that it may be possible to formulate more restrictive assumptions on non-rational play after all. Note that this would give rise to a difference between a descriptive solution concept with such restrictions, and a normative concept like ours where we based the lack of restrictions on the consistency argument that a rationality concept alone does not restrict non-rational play.

Thirdly, the assumption that players are completely uncertainty-averse is extreme. Yet, at the current stage of the development of Choquet expected utility theory there is no ideal alternative.[14]

Finally, it seems a drastic consequence that strategic interaction comes to a complete halt after a deviation from rational play. Note, however, that the solution concept

[14] It would be possible to assume that players take a weighted average of the best and the worst outcome if they are certain to face a non-rational opponent. This is Hurwicz's optimism-pessimism index (see Arrow & Hurwicz (1972). However, by introducing another free parameter the model would lose predictive power.

applies to one-shot games, and therefore leaves no room for real-world strategies to deal with doubts about rationality, e.g. experimentation and communication. Note also that trembling-hand perfection makes an equally extreme assumption to ensure logical consistency by postulating that otherwise fully rational players 'tremble', and that deviations therefore provide no evidence against rationality.

Needless to say, our solution concept provides nothing but a first step that may be a basis for a more refined analysis.

Appendix

Let v be a capacity and consider the events $E, F \in \Sigma$. The Dempster-Shafer rule specifies that the posterior capacity of event E is given by (if $v(\overline{F}) < 1$)

$$v(E|F) := \frac{v(E \cup \overline{F}) - v(\overline{F})}{1 - v(\overline{F})}.$$

Let ϵ be the prior probability that a player is not rational with $0 < \epsilon < 1$. Assume that a rational player chooses an action A with probability p. Then the posterior belief about the opponent's rationality after action A is given by

$$v(R|A) := \frac{(1-\epsilon)p}{1-(1-\epsilon)(1-p)}.$$

This is derived as follows:

Let R be the event that the player is rational, let $\overline{R}$ be the event that he is non-rational.

We want to calculate

(3.1) $$v(R|A) := \frac{v(R \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})}.$$

Since a player is either rational or not, we have

(3.2) $$v(R|A) + v(\overline{R}|A) = 1.$$

Consequently,

(3) $v(R|A) = \frac{v(R \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})}$,

(4) $v(\overline{R}|A) = \frac{v(\overline{R} \cup \overline{A}) - v(\overline{A})}{1 - v(\overline{A})}$

imply

$$v(\overline{A}) = v(R \cup \overline{A}) + v(\overline{R} \cup \overline{A}) - 1.$$

Further,

(5) $v(\overline{A}|R) = \frac{v(\overline{A} \cup \overline{R}) - v(\overline{R})}{1 - v(\overline{R})}$, and

(6) $v(\overline{A}|\overline{R}) = \frac{v(\overline{A} \cup R) - v(R)}{1 - v(R)}$.

We know that

(7) $v(R) = 1 - \epsilon$,

(8) $v(\overline{R}) = \epsilon$,

(9) $v(\overline{A}|R) = 1 - p$, and

(10) $v(\overline{A}|\overline{R}) = 0$.

so that

(11) $v(\overline{A} \cup \overline{R}) = (1 - \epsilon)(1 - p) + \epsilon$, and

(12) $v(\overline{A} \cup R) = 1 - \epsilon$.

Thus

(3.13) $$v(\overline{A}) = (1 - \epsilon) + (1 - \epsilon)(1 - p) + \epsilon - 1 = (1 - \epsilon)(1 - p).$$

Consequently,

$$\begin{aligned} v(R|A) &:= \frac{(1-\epsilon) - (1-\epsilon)(1-p)}{1 - (1-\epsilon)(1-p)} \\ &= \frac{(1-\epsilon)p}{1 - (1-\epsilon)(1-p)}, \\ v(\overline{R}|A) &= \frac{\epsilon}{1 - (1-\epsilon)(1-p)} \end{aligned}$$

Remarks:

(1) The derivation is only valid under lack of mutual knowledge of rationality, i.e. for $\epsilon > 0$ and $\epsilon < 1$, otherwise $v(\overline{A}|R)$ or $v(\overline{A}|\overline{R})$ are not well-defined.

(2) With $0 < \epsilon < 1$ there are no probability zero events, since

$$v(\overline{A}) = (1 - \epsilon)(1 - p) < 1.$$

This holds for any $p \in [0, 1]$, including the boundaries.

(3) In particular, if $\epsilon > 0$ then $\epsilon' > 0$, independently of p. However, if $p = 0$, then $\epsilon' = 1$. Thus we also need to be able to update the belief $\epsilon = 1$. Intuitively, if the prior belief about the opponent is that he is non-rational and beliefs about

his behavior are uncertainty averse, then there are no probability zero events, and the posterior belief should also be that the opponent is non-rational.This can be justified directly from the Dempster-Shafer rule (1): From monotonicity, $v(\overline{R}) \leq v(\overline{R} \cup olA)$, therefore $v(\overline{R} \cup \overline{A}) = 1$. Also, (6) implies $v(\overline{A}|\overline{R}) = v(\overline{A} \cup R)$, so again by monotonicity, $v(\overline{A}) \leq v(\overline{A} \cup R) = 0$. Since this result also follows if we substitute $\epsilon = 1$ into (13), we do not have to explicitly track this special case.

(4) The reason why $\epsilon = 0$ has to be excluded is that there is no parallel argument that $\epsilon = 0$ and $p = 0$ should give $\epsilon' = 1$. (3) implies $v(\overline{A} \cup \overline{R}) = v(\overline{A}|R) = 1$ and (1) gives $\epsilon' = \frac{1-v(\overline{A})}{1-v(A)}$, but $v(\overline{A}) \not< 1$.

(5) Note that action A is always interpreted as evidence of non-rationality: $v(\overline{R}|A) = \epsilon \frac{1}{1-(1-\epsilon)(1-p)} > \epsilon$. Thus updating is in line with uncertainty aversion, which gives rise to non-additive beliefs in the first place. For the player, the worst case is that the non-rational opponent chose action A with probability 1, because this makes it more likely, under uncertainty aversion, to receive the worst outcome in the next stage.

(6) Note that if $\epsilon' = \frac{\epsilon}{1-(1-\epsilon)(1-p)}$ and $\epsilon'' = \frac{\epsilon'}{1-(1-\epsilon')(1-p')}$ then $\epsilon'' = \frac{\epsilon}{1-(1-\epsilon)(1-p \cdot p')}$.

(7) Finally, note that the argument rests heavily on (2), i.e. the the requirement about beliefs that an opponent is either rational or non-rational, so that these beliefs have to be additive.

Chapter 4

Uncertainty Aversion and Backward Induction in the Centipede Game

Abstract

In the context of the centipede game, this chapter discusses a solution concept for extensive games that is based on subgame perfection and uncertainty aversion. Players who deviate from the equilibrium path are considered non-rational. Rational players who face non-rational opponents face genuine uncertainty and may have non-additive beliefs about their future play. Rational players are boundedly uncertainty averse and maximise Choquet expected utility. It is shown that if the centipede game is sufficiently long, then the equilibrium strategy is to play 'Across' early in the game and to play 'Down' late in the game.

4.1 Introduction

The centipede game has become a benchmark both for the empirical adequacy and the theoretical consistency of game theoretic concepts. In any Nash equilibrium — and thus in every equilibrium refinement — the first player chooses 'Down' immediately; in the unique subgame-perfect equilibrium the players choose 'Down' everywhere.

Empirically, experimental evidence suggests that players do not act in this way (see, e.g. , McKelvey & Palfrey (1992)). Theoretically, subgame perfection applies equilibrium arguments, that hold for rational players, off the equilibrium path. This is consistent only under the assumption that deviations from rational play are not evidence of non-rationality, e.g. because rational players might tremble (Selten 1975). This aspect has led to a controversial debate about backward induction (see, e.g. , Basu (1988), Reny (1993), Aumann (1995), Binmore (1996), Aumann (1996)).

McKelvey & Palfrey (1992) are able to interpret experimental evidence in the sense of Kreps et al. (1982, henceforth KMRW). In their model, the structure of the game is not mutual knowledge. Instead there is a small probability of being matched with an 'altruistic' opponent who always plays 'Across'. McKelvey & Palfrey (1992) show that, as a consequence, it is indeed rational to play 'Across' early in the game.

There are two arguments against this way of interpreting the experimental evidence. First, if taken as an explanation of evidence rather than an equilibrium effect, it relies on the actual existence of such altruists in the subject pool. The second, formulated by Selten (1991) in the context of the KMRW approach to the finitely repeated prisoner's dilemma, is that the analysis proceeds by changing the game, and not by analysing the same game in which the paradox arises. However, both criticisms do not apply if the players are assumed to know the game, but lack mutual knowledge of rationality, as suggested by Milgrom & Roberts (1982*b*, p.303). If the rational players believe that non-rational opponents always play 'Across', the analysis of McKelvey & Palfrey (1992) is an explanation of the actual evidence in the original game.

Still, this approach to modelling lack of mutual knowledge of rationality leads to conceptual difficulties:
First, there is no reason why rational players should hold this specific belief about

opponents that they do not consider to be rational. Therefore, not only is the specification of the belief that non-rational players always play 'Across' ad hoc, in the absence of a theory of non-rationality there is no basis for specifying any particular belief.

Secondly, this also holds in particular for the uniform distribution as a model of complete ignorance. There is no reason why a non-rational player should be assumed to choose all his strategies with equal probability. In addition, there is the well-known problem that a uniform probability depends on the description of the space of uncertainty: For instance, if a state is split into two sub-states, the combined probability of the two sub-states under the uniform distribution is higher than the probability of the original state.

Thirdly, and more fundamentally, if the Bayesian-Nash equilibrium is identified with rational play, then any deviation must be considered non-rational. This problem is related to, but different from the first: Not only need the players not have a particular belief about non-rational opponents, according to the rationality concept they must not have any particular belief. This consistency requirement follows from an identification of Bayesian-Nash equilibrium with rational play, because this implicitly defines all other strategies as non-rational.

Finally, the analysis of games under incomplete information on the basis of the Bayesian-Nash equilibrium assumes that the types of a player correspond to a consistent hierarchy of beliefs about the underlying uncertainty (Harsanyi 1967–68). This leads to the usual infinite regress. Thus in this analysis the rational player not only believes that a 'non-rational' opponent always plays 'Across', but also believes that the non-rational opponent believes a rational player to believe this, ... ad infinitum. But this means that a rational player must believe that his non-rational opponent has an infinite and consistent hierarchy of beliefs. This, of course, is at odds with the interpretation of this opponent as non-rational. It is for this reason that McKelvey & Palfrey (1992) refer to structural uncertainty and 'altruistic' types.

Nevertheless, the KMRW approach has been extremely useful in helping to understand strategic interaction, particularly in industrial organization (Kreps & Wilson 1982*a*, Milgrom & Roberts 1982*b*) and, as in McKelvey & Palfrey (1992), in experimental game theory.

Our model is in the same spirit as KMRW (1982) and McKelvey & Palfrey (1992).

We postulate that rationality is not mutual knowledge, i.e. an opponent may or may not be rational. We replace the assumption that players have a specific belief about non-rational play with the assumption that players are genuinely uncertain about the way non-rational opponents play. When facing uncertainty, players maximise Choquet expected utility (Schmeidler 1989, henceforth CEU). According to CEU, players act in face of uncertainty as if they maximise subjective expected utility. However, in contrast to a situation in which players face risk, players' beliefs do not have to be additive, i.e. the 'probabilities' that the players use to weigh consequences do not have to add to 1.

Our contribution in this paper is to define an equilibrium concept that extends subgame perfection to a game with genuine uncertainty due to lack of mutual knowledge of rationality. Thus we do not need to make any assumption about the behavior of non-rational players, and we can avoid modelling them as types. Instead, we can make an assumption about the rational players' attitude towards uncertainty. We assume that they are uncertainty averse, but only boundedly so. We show that this results in an equilibrium in the centipede game in which rational players play 'Across' early in the game and 'Down' late in the game. Moreover, it is subgame-perfect in the sense that decisions are optimal at every node in the game.

Our result is due to an interaction between the game-theoretic definition of strategy as a contingent plan and the players' attitude towards uncertainty. In calculating expected utilities, a player who is uncertainty averse will use 'probability weights' that do not add up to 1, and a 'probability residual' (the difference between the sum of the weights and 1) that he will allocate to the worst outcome. As long as the degree of uncertainty aversion is bounded, however, every strategy of the non-rational opponent will enter the calculation with some positive weight, however small. Since a strategy is a contingent plan, it specifies an action — 'Across' or 'Down' — after every history of the game, even those that are excluded by the strategy itself (because it specifies 'Down' very early). Consequently, the number of strategies increases exponentially in the length of the centipede game. This means that early in the game the 'probability residual' that is allocated to the worst outcome is small. Thus even uncertainty-averse players will find it profitable to go 'Across'. Late in the game, however, the number of remaining strategies is small, and uncertainty averse players will prefer 'Down'. We show that this phenomenon is an equilibrium, i.e. it is stable even if other rational players act in a similar way.

CEU has been introduced into game theory by Dow & Werlang (1994) and Klibanoff (1993). Dow & Werlang (1994) show that in the presence of uncertainty the backward induction outcome may break down if the finitely repeated prisoner's dilemma is analysed as a normal form game. Our model extends this result in two directions: First, we give an explicit reason for non-additive uncertainty, the lack of mutual knowledge of rationality. Secondly, we formulate a solution concept in the spirit of subgame perfection and show that the backward induction outcome breaks down in the subgame-perfect equilibrium of the centipede game, analysed in its extensive form. This allows the conclusion that these two concepts — backward induction and subgame perfection — differ fundamentally in the presence of uncertainty.

Other papers that combine the analysis of extensive form games with CEU are Eichberger & Kelsey (1995) and Lo (1995a). In these papers there is no explicit distinction between rational and non-rational players. Eichberger & Kelsey (1995) use the Dempster-Shafer rule to update non-additive beliefs. Closest to the spirit of our analysis is Mukerji (1994), however he considers normal form games only.

The paper is organized as follows: Section 2 contains the model, section 3 an example, section 4 the result, and section 5 concludes. There is one appendix.

4.2 The Model

4.2.1 The Centipede Game

Consider the following version of the centipede game:

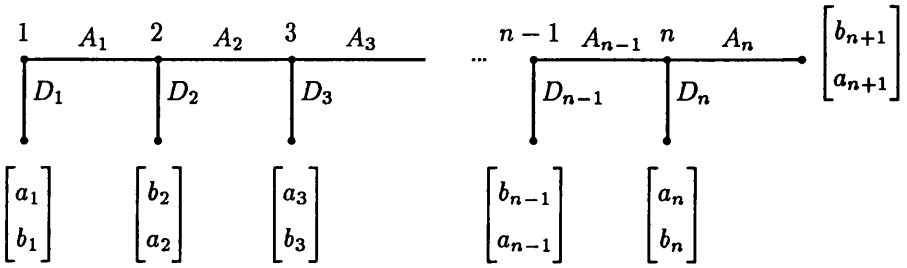


Figure 1

The decision nodes are numbered from 1 to n. For definiteness we assume that n is odd. Player P_1 moves at odd nodes, player P_2 at even nodes. At node i, a player chooses between 'Across' A_i and 'Down' D_i. The leader payoff is a_i, i.e. a_i is the payoff to the player who plays D_i. The follower payoff is b_i.

The payoffs are such that the game is a centipede game, i.e.

(1) a_i and b_i are strictly increasing in i,

(2) $a_i > b_{i+1}$,

(3) $\eta_i := \frac{a_i - b_{i+1}}{a_{i+2} - b_{i+1}}$ is weakly increasing in i,

(4) $\eta_i \leq \frac{1}{8}$ for all $i \in N$.

Thus the game corresponds to a situation in which two players can share a certain profit, but only in unequal terms. Overall profit $a_i + b_i$ is increasing, but every player prefers to be the leader now than to be the follower in the next stage. If the opponent could be relied upon to play 'Across', however, each player would play 'Across' earlier[15]. The centipede game is due to Rosenthal (1981), its name is due to Binmore (1987–88).

A pure strategy of player j is a mapping that associates with each of his decision nodes i an action A_i or D_i. Thus, if a player has m decision nodes he has 2^m many pure strategies, i.e. the number of strategies grows exponentially in the length of the game.

The players are assumed to have a prior probability that specifies the probability that the opponent is non-rational. For simplicity we assume that this prior is common to both players[16]. We denote this prior probability by ϵ, and assume $0 < \epsilon < 1$.

Our equilibrium concept aims to capture the optimal strategies of rational players. Thus a rational player must not have an incentive to deviate from his equilibrium strategy, as long as a rational opponent does not deviate either. However, a ra-

[15] Conditions (3) and (4) are conditions on the payoff increases. It means that the sure gains from playing 'Down' in relation to the possible gains from playing 'Across' increase, i.e. that playing 'Down' does not become less attractive in relative terms (3). Condition (4) says that these gains must not be too high; this is sufficient, but not necessary, to ensure that playing across does not result from uncertainty love alone. In their experiments, McKelvey & Palfrey (1992) assume that η_i is constant with $\eta_i = \frac{1}{7}$ and $n = 4$, resp. $n = 6$. (See also footnote 9.)

[16] Allowing different priors only introduces one more degree of freedom. This would not make the analysis conceptually deeper, and would make it easier to generate different equilibria.

tional player does not know what a non-rational opponent will do, and so faces genuine uncertainty. We assume that, when facing this uncertainty, rational players maximise Choquet expected utility in the sense of Schmeidler (1989).

4.2.2 Choquet Expected Utility Theory

According to CEU, players act in the face of uncertainty as if they possess a utility function over consequences and subjective beliefs over the domain of uncertainty, and maximise subjective expected utility. However, in contrast to a situation in which players face risk, players' beliefs do not have to be additive, i.e. representable by a probability measure. Instead, players' beliefs are represented by a capacity, i.e. a not necessarily additive 'probability' measure.

This model thus corresponds to a situation in which uncertainty cannot be reduced to probability. This model allows a parsimonious explanation of the Ellsberg paradox that people do not act as if their beliefs can be represented by probability measures. CEU retains the useful notion of belief and explains lack of probabilistic sophistication as a result of the players' attitude towards uncertainty.

Formally, let S be a set of states of nature. Let $s \in S$ and let $\Sigma \subseteq 2^S$ be a σ-algebra of events $E \in \Sigma$. A capacity associates with each event a real number such that[17]

(1) $v(\emptyset) = 0$,

(2) $v(S) = 1$, and

(3) $E \subseteq E' \implies v(E) \leq v(E')$.

The expected utility with respect to a capacity is defined as the Choquet (1953) integral: Let X be a simple positive random variable, i.e. X takes the positive values $x_1, x_2, ..., x_k$ on the events $E_1, E_2, ..., E_n$. The sets are measurable, pairwise disjoint, and their union is S. Without loss of generality, assume $x_1 > x_2 > ... > x_k$ and set $x_{k+1} := 0$. As usual, let $v(X \geq t) := v(\{\omega \in \Omega | X(\omega) \geq t\})$.

Then the Choquet integral is defined as[18]

$$\int X dv \quad := \quad \int_0^\infty v(X \geq t) dt$$

[17]The monotonicity property (*iii*) weakens the finite-additivity axiom $E \cap E' = \emptyset \implies v(E \cup E') = v(E) + v(E')$ for finitely-additive measures.

[18]The integral on the right hand side is the extended Riemann integral.

$$= \sum_{i=1}^{k}(x_i - x_{i+1})v(\cup_{j=1}^{i} E_j).$$

If v is additive this is the usual expectation. Thus the Choquet integral generalizes the usual formula for the expectation in terms of the decumulative distribution function E $X = \int_0^\infty F(X \geq t)dt$. It is a natural definition for an integral because it assigns to a characteristic function 1_E of an event E the capacity $v(E)$ of this event, and preserves monotonicity, i.e. if $X(s) \leq X'(s)$ for all $s \in S$ then $\int_S X dv \leq \int_S X' dv$.

4.2.3 Uncertainty Aversion

The non-additivity of v allows the formalisation of the player's attitude towards uncertainty. According to the definition of the integral, if probability weights are not additive then the probability residual is allocated to the worse outcome: Consider two events E and E' Let $E \cap E' = \emptyset$ and $E \cup E' = S$. Assume that the random variable X takes value x_1 on E and x_2 on E', and that $x_1 > x_2$. Let $v(E) + v(E') < 1$. Then by the definition of the integral $\int_S X dv = x_1 \cdot v(E) + x_2 \cdot (1 - v(E))$. This means that the probability residual $1 - v(E) - v(E')$ is allocated to the worse outcome. Thus subadditivity of a players' beliefs corresponds to his uncertainty aversion when facing genuine uncertainty. A decision-theoretic axiomatisation of uncertainty aversion in terms of preferences over acts is due to Schmeidler (1989)[19].

When a player faces a non-rational opponent his relevant space of uncertainty is the opponent's pure strategy set.Therefore, we assume that a rational player assigns to each of his opponent's pure strategies $s_j \in S_j$ some "probability weight" $\theta_{s_j} \geq 0$. Since any deviation from rationality is as non-rational as another, the player has no reason to regard any of a non-rational opponent's strategies more likely than another. For this reason we assume $\theta_{s_j} = \theta$, for all $s_j \in S_j$. This also simplifies the analysis. For simplicity we also assume that the players are identical, i.e. that the θ is the same for both of them[20].

[19] For related axiomatisations see, e.g. Gilboa & Schmeidler (1989) and Sarin & Wakker (1994)

[20] As before, introducing a different degree of uncertainty aversion for the second player corresponds to an additional degree of freedom. We think it is desirable not to introduce any ad hoc asymmetry.

If a rational player is completely uncertainty averse, we have $\theta = 0$, and in evaluating one of his pure strategies the player will assign probability 1 to the opponent's strategy that minimizes his utility. As long as $\theta > 0$, the player is only boundedly uncertainty averse, in that he gives some weight, however small, to other strategies of his opponent. Formally, this means that a rational player's beliefs about the strategy choice of a non-rational opponent is given by the capacity[21]

$$v(E) = \begin{cases} 1 & , \quad E = S_j \\ \theta|E| & , \quad E \subset S_j. \end{cases}$$

The assumption that the rational player is uncertainty averse thus translates into $\theta < \frac{1}{|S_j|}$. The main point of this paper is that there is an interaction between uncertainty aversion and the game-theoretic definition of strategy, as long as the uncertainty aversion is bounded.

4.2.4 Expected Payoffs

The specification of this capacity now allows us to define the payoff, that a rational opponent expects if he plays his pure strategy $s_i \in S_i$ and believes that his opponent is non-rational, as the CEU of his utility:

$$u(s_i, v) := \int_{S_j} u(s_i, s_j) dv.$$

Since a player does not know, however, if his opponent is rational or not, but has a prior belief ϵ that the opponent is non-rational, his expected payoff from his strategy s_i given that a rational opponent uses strategy s_j^* is given by

$$(1 - \epsilon) u(s_i, s_j^*) + \epsilon u(s_i, v).$$

A rational player will choose a strategy that maximises his payoff not only at the beginning of the game, but also in each subgame. It thus remains to specify how a rational player's beliefs change during the course of the game.

4.2.5 Updating and the Dempster-Shafer Rule

An updating rule has to generalize Bayes' Rule to non-additive probability measures. We assume that non-additive beliefs are updated through the Dempster-Shafer rule.

[21] Here $|E|$ denotes the cardinality of the set E.

Formally, let v be a capacity and consider the events $E, F \in \Sigma$. The Dempster-Shafer rule specifies that the posterior capacity of event E is given by

$$v(E|F) := \frac{v(E \cup \overline{F}) - v(\overline{F})}{1 - v(\overline{F})}.$$

The Dempster-Shafer rule (Dempster 1968, Shafer 1976) corresponds to Bayes' Rule if the capacity is additive. When it is not, it reflects the uncertainty aversion, or pessimism, of the player (Gilboa & Schmeidler 1993).

The main use we make of the Dempster-Shafer rule is that it allows the formalization of the updating process after an action that is only taken by a non-rational player: Let ϵ be the prior probability that the opponent is not rational. Assume that the opponent has two actions A and D, and that a rational opponent chooses action A with probability p. Then the posterior belief ϵ' about the opponents' rationality is given by

$$\epsilon' := \frac{\epsilon \cdot (1 - |S_j|\theta)}{1 - \epsilon|S_j|\theta - (1-\epsilon)(1-p)},$$

where $|S_j|$ is the number of the opponents' strategies, in the subgame starting at the given node, that specify D. This is formally derived in the appendix.

Note that, first, if $p = 0$ and only a non-rational player chooses A the Dempster-Shafer rule gives the result that $\epsilon' = 1$. Secondly, if $p = 1$ then $\epsilon' < \epsilon$, i.e. a rational action is interpreted as evidence of rationality. Finally, as long as there is some doubt about the rationality of the opponent at the beginning of the game, there are no probability zero events.

We can now define the solution concept.

4.2.6 The Equilibrium Concept

An equilibrium is a strategy combination from which no rational player has an incentive to deviate unilaterally. We are considering an extensive game in which rationality is not mutual knowledge, so we have to extend this definition in two ways: First, we incorporate the assumption that rational players face genuine uncertainty, maximise Choquet expected utility, are boundedly uncertainty averse and update their beliefs according to the Dempster-Shafer rule. Secondly, in the spirit of subgame perfection we require optimality at each decision node.

A *θ-perfect Choquet-Nash equilibrium* is a pair of behavior strategies (σ_1^*, σ_2^*) such that

(1) at each node, each pure strategy of a rational player in their support maximises his expected utility given his beliefs about the opponent's rationality, the rational opponent's strategy, and the degree of uncertainty aversion,

(2) the beliefs about rational opponents are correct, and

(3) the beliefs about the opponent's rationality are updated according to the Dempster-Shafer rule.

We now have the following results:

Result 1:

Every centipede game has at least on θ-perfect Choquet-Nash equilibrium, for every common degree of uncertainty aversion θ and every degree ϵ of mutual knowledge of rationality.

Result 2:

However small the degree ϵ of lack of mutual knowledge of rationality, and however small the degree of uncertainty aversion, as long as they are positive, in the θ-perfect Choquet-Nash equilibrium the first player will not play 'Down' with probability 1.

The results are formally stated and in section 4. In the next section we illustrate them by an example.

4.3 An Example

Consider the following centipede game:

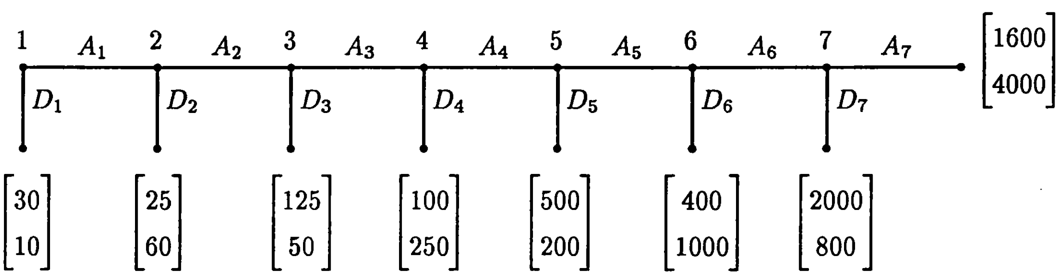


Figure 2

We assume that players have a common prior $\epsilon = \frac{1}{3}$ that the opponent is non-rational. We assume that players are boundedly uncertainty averse with degree of uncertainty aversion $\theta = \frac{1}{20}$ for both players.

By backward induction, we analyse this game starting from node 7.

At **node 7**, player P_1 will achieve 2000 if he plays D_7 as opposed to 1600 if he plays A_7. He is no longer in a situation of strategic interaction but in a pure decision situation. Therefore D_7 is his optimal choice.

At **node 6**, player P_2 faces both risk and uncertainty. He faces the risk that the opponent is non-rational, which is given by player P_2's belief ϵ_6 at node 6. Moreover, he faces the uncertainty what a non-rational opponent might play. The opponent has two strategies at node 7. Since P_2 is uncertainty averse, each of these strategies receives probability weight θ. The residual $1 - 2\theta$ is allocated to the strategy that is worst for P_2. Thus his Choquet expected utility from a non-rational opponent is given by

$$\begin{aligned} u_2(v_6, A_6) &= (1-2\theta)800 + \theta 800 + \theta 4000 \\ &= 800 + \theta(4000 - 800) \\ &= 960. \end{aligned}$$

In calculating his overall payoff from A_6, P_2 knows, by backward induction, that a rational player P_1 will play D_7, which results for P_2 in a payoff of 800. Thus his overall payoff is given by

$$(1 - \epsilon_6)800 + \epsilon_6 960.$$

P_2 can ensure 1000 by playing D_6, so D_6 is optimal.

At **node 5**, it follows by the same reasoning that D_5 is optimal.

At **node 4**, player P_2 knows that a non-rational opponent has four strategies in the continuation game, and that it is optimal to play D_6 at node 6. Thus P_2's Choquet expected utility from a non-rational opponent is given by

$$\begin{aligned} u_2(v_4, A_4) &= (1-4\theta)200 + 2\theta 200 + 2\theta 1000 \\ &= 200 + 2\theta(1000 - 200) \\ &= 280. \end{aligned}$$

Thus his overall payoff is given by

$$(1-\epsilon_4)200+\epsilon_4 280. \tag{4.14}$$

P_2 can only ensure 250 by playing D_4, so the optimal strategy depends on his beliefs ϵ_4.

By the Dempster-Shafer rule, ϵ_4 and ϵ_2 are related as follows:

$$\epsilon_4 := \frac{\epsilon_2 \cdot (1-\frac{8}{20})}{1-\frac{8}{20}\epsilon_2-(1-\epsilon_2)(1-p_3^*)}. \tag{4.15}$$

At **node 3**, player P_1 knows that a non-rational opponent has four strategies in the continuation game, and that it is optimal to play D_5 at node 5. Thus P_1's Choquet expected utility from a non-rational opponent is given by

$$\begin{aligned} u_1(A_3, v_3) &= (1-4\theta)100+2\theta 100+2\theta 500 \\ &= 100+2\theta(500-100) \\ &\doteq 140. \end{aligned}$$

Thus his overall payoff is given by

$$(1-\epsilon_3)[p_4^* 500+(1-p_4^*)100]+\epsilon_3 140, \tag{4.16}$$

where p_4^* is the probability with which a rational player P_2 plays A_4. P_1 can only ensure 125 by playing D_3, so the optimal strategy depends on his beliefs ϵ_3 and on P_2's optimal strategy p_4^*.

By the Dempster-Shafer rule, ϵ_3 and ϵ are related as follows:

$$\epsilon_3 := \frac{\epsilon \cdot (1-\frac{8}{20})}{1-\frac{8}{20}\epsilon-(1-\epsilon)(1-p_2^*)}. \tag{4.17}$$

At **node 2**, player P_2 knows that a non-rational opponent has eight strategies in the continuation game, and that it is optimal to play A_4 with probability p_4^* at node 4. However, he also knows that at node 4 he can ensure 250, so that p_4^*, due to its optimality, ensures at least as much. Thus P_2's Choquet expected utility from a non-rational opponent is bounded below:

$$\begin{aligned} u_2(v_2, A_2) &\geq (1-8\theta)50+4\theta 50+4\theta 250 \\ &= 50+4\theta(250-50) \\ &= 90. \end{aligned}$$

Thus his overall payoff is bounded below by

$$(4.18) \qquad (1-\epsilon_2)[p_3^*250 + (1-p_3^*)50] + \epsilon_2 90,$$

where p_3^* is the probability with which a rational player P_1 plays A_3. This payoff is bounded below by $50 + 40\epsilon_2$ for $p_3^* = 0$.

By the Dempster-Shafer rule, ϵ_2 and ϵ are related as follows:

$$(4.19) \qquad \epsilon_2 := \frac{\epsilon \cdot (1 - \frac{16}{20})}{1 - \frac{16}{20}\epsilon - (1-\epsilon)(1-p_1^*)}.$$

At **node 1**, player P_1 knows that a non-rational opponent has eight strategies in the continuation game, and that it is optimal to play A_3 with probability p_3^* at node 3, which gives at least 125. Thus P_1's Choquet expected utility from a non-rational opponent is bounded below:

$$\begin{aligned}
(4.20) \qquad u_1(A_1, v_1) &\geq (1-8\theta)25 + 4\theta 25 + 4\theta 125 \\
(4.21) \qquad &= 25 + 4\theta(125 - 25) \\
(4.22) \qquad &= 45.
\end{aligned}$$

Thus his overall payoff is bounded below by

$$(1-\epsilon_1)[p_2^*125 + (1-p_2^*)25] + \epsilon_1 45,$$

where p_2^* is the probability with which a rational player P_2 plays A_2.

Since $\epsilon_1 := \epsilon = \frac{1}{3}$, it follows that

$$(1-\epsilon_1)[p_2^*125 + (1-p_2^*)25] + \epsilon_1 45 = \frac{95}{3} + \frac{200}{3}p_2^*.$$

Since D_1 gives 30, P_1 will prefer A_1.

From the Dempster-Shafer rule, this implies

$$\begin{aligned}
(4.23) \qquad \epsilon_2 &:= \frac{\epsilon \cdot (1 - \frac{16}{20})}{1 - \frac{16}{20}\epsilon - (1-\epsilon)(1-p_1^*)} \\
(4.24) \qquad &= \frac{1}{11}.
\end{aligned}$$

This, in turn, implies that at node 2 the continuation payoff is bounded below by $50 + 40\epsilon_2 = \frac{590}{11} < 60$. This shows that despite the boundedness of uncertainty aversion the increasing payoffs alone do not lead player P_2 to choose 'Across' at node 2. If he does so, then because he expects a rational opponent also to be willing to go 'Across'. In equilibrium, these beliefs are self-fulfilling.

We now show that an equilibrium is given by

$$p_2^* = 1,$$

$$p_3^* = \frac{36}{1000},$$

and

$$p_4^* = \frac{41}{800}.$$

First, $p_2^* = 1$ is optimal because, from (5) with $\epsilon_2 = \frac{1}{11}$ and $p_3^* = \frac{36}{1000}$,

$$\begin{aligned} & (1-\epsilon_2)[p_3^* 250 + (1-p_3^*)50] + \epsilon_2 90 \\ = & \frac{590}{11} + 200\frac{10}{11}\frac{36}{1000} \\ = & \frac{662}{11} \\ > & 60. \end{aligned}$$

From (4) with $p_2^* = 1$ we have

$$\begin{aligned} \epsilon_3 &= \frac{6\epsilon}{10-4\epsilon} \\ &= \frac{3}{13}. \end{aligned}$$

Secondly, given p_4^* and ϵ_3, player P_1 is indifferent between A_3 and D_3 at node 3, and so is willing to mix. From (3) his continuation payoff is given by

$$\begin{aligned} & (1-\epsilon_3)[p_4^* 500 + (1-p_4^*)100] + \epsilon_3 140 \\ = & \frac{1}{13}[(1000 + 205 + 420] \\ = & 125. \end{aligned}$$

From (2), with $p_3^* = \frac{36}{1000}$ and $\epsilon_2 = \frac{1}{11}$ we have

$$\begin{aligned} \epsilon_4 &= \frac{6\epsilon_2}{6\epsilon_2 + 10p_3^*(1-\epsilon_2)} \\ &= \frac{6}{6+\frac{36}{10}} \\ &= \frac{5}{8}. \end{aligned}$$

Finally, given $\epsilon_4 = \frac{5}{8}$, player P_2 is indifferent between A_4 and D_4, because his continuation payoff is, from (1),

$$\begin{aligned} & (1-\epsilon_4)200 + \epsilon_4 280 \\ = & \frac{2000}{8} = 250. \end{aligned}$$

To summarize, the equilibrium is given by:

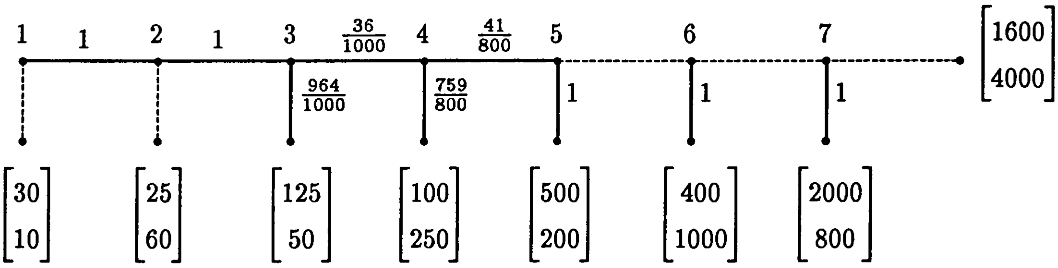


Figure 3

We end this section with some remarks:

(1) In this example, no pure strategy equilibrium exists. This can be seen from equations (1) and (2): If $p_3^* = 0$ then $\epsilon_4 = 1$, thus A_4 is optimal, which leads to $p_3^* = 1$, a contradiction. Conversely, if $p_3^* = 1$ then (5) implies $p_2^* = 1$. But then $\epsilon_3 = \frac{3}{13}$ and the continuation payoff (3) at node 3 is $(1-\epsilon_3)100 + \epsilon_3 140 < 125$, which leads to $p_3^* = 0$, another contradiction. In general, however, a pure strategy equilibrium may exist.

(2) In our example, $\epsilon = \frac{1}{3}$ is larger than in Kreps et al. (1982) and McKelvey & Palfrey (1992). It can be shown, however, that for no $\epsilon > 0$ will D_1 be chosen with probability one. More generally, here ϵ refers to a player's belief that the opponent is rational, and reasons in the same way as the player himself. This makes a high ϵ a plausible parameter value.

(3) Players adjust the belief ϵ about the opponent's rationality both upward and downward, and not just in one direction. An action that is taken by a rational player with high probability is taken as evidence of rationality and ϵ is adjusted downward. Conversely, an action that a rational player only chooses with low probability is considered as evidence of non-rationality and ϵ is adjusted upward.

(4) It is interesting to note that the taking probability does not increase monotonically. Also, in contrast to the sequential equilibrium in McKelvey & Palfrey (1992) the taking probability may be 1 not only at the last two nodes of the game.

(5) The analysis does not give a bell-shaped distribution over the terminal nodes. In McKelvey & Palfrey (1992), the sequential equilibrium alone does not either, however, they are able to show that the incorporation of learning can explain the empirical data.

4.4 Results

We now state and prove the results formally.

Definition.

A *centipede game* $\Gamma = (n, (\{D_i, A_i\})_{i=1,\dots,n}, (a_i, b_i)_{i=1,\dots,n+1})$ is given by a set N of n nodes $i \in N$, for each node two actions D_i and A_i, and for each action D_i and for A_n two payoffs a_i and b_i such that

(1) a_i and b_i are strictly increasing in i,

(2) $a_i > b_{i+1}$,

(3) $\eta_i := \frac{a_i - b_{i+1}}{a_{i+2} - b_{i+1}}$ is weakly increasing in i,

(4) $\eta_i \leq \frac{1}{8}$ for all $i \in N$.

For pure strategies s_1 and s_2 let $u_j(s_1, s_2)$ be a_k or b_k, where $k := \min\{k' | s_j(k') = D_{k'}$ for some player j $\}$, depending on whether $k \in N_j$ or not. Let σ_j be a behavior strategy of player j, where $\sigma_j(i)$ specifies the probability of 'Across' at node i under σ_j. Let $u_j(\sigma_1, \sigma_2)$ be the expected utility[22] of player j under the behavior strategies σ_1, σ_2.

Let θ be the common degree of uncertainty aversion of the two players, where[23] $0 \leq \theta < \overline{\theta} := \frac{1}{2^{\frac{n+1}{2}}}$. For given n, $\overline{\theta}$ is the upper bound on θ to ensure uncertainty aversion.

[22] Behavior strategies define additive probabilities over the pure strategy sets, so this is the usual expectation.

[23] The upper bound on θ preserves uncertainty aversion. If it is violated, both propositions still hold, but proposition 2 is due to uncertainty love alone.

Note that $\theta < \frac{1}{2^{\frac{n+1}{2}}}$ is equivalent[24] to $n \leq \overline{n} := [-(2\ (\text{ld}\ \theta) + 1)]$, where ld denotes the logarithm to the base 2. For given θ, $\overline{n}$ is the upper bound on n that ensures that players are uncertainty averse even at the beginning of the game.

Definition.

Let Γ be a centipede game. Let N_1 and N_2 be the set of player 1's and 2's decision nodes i. Let θ be the degree of the players' attitude towards uncertainty aversion, and let ϵ be the common prior about rationality. let $\epsilon_0 = \epsilon_1 := \epsilon$ Then a *θ-perfect Choquet-Nash equilibrium* is a pair of behavior strategies (σ_1^*, σ_2^*) such that if s_1^* and s_2^* are in the support of σ_1^* and σ_2^*

(1) $s_1^* \in \arg\max_{s_1}(1-\epsilon_i)u_1(s_1, \sigma_2^*) + \epsilon_i u_1(s_1, v_i), \quad \forall i \in N_1,$

$s_2^* \in \arg\max_{s_2}(1-\epsilon_i)u_2(\sigma_1^*, s_2) + \epsilon_i u_2(v_i, s_2), \quad \forall i \in N_2,$

(2) $\epsilon_{i+2} = \frac{\epsilon_i \cdot (1-|S_{j,i+1}|\theta)}{1-\epsilon_i|S_{j,i+1}|\theta-(1-\epsilon_i)(1-\sigma_j^*(i+1))},$

(3) $u_1(s_1, v_i) := \int_{S_{2,i+1}} u_1(s_1, s_2) dv_i,$

$u_2(v_i, s_2) := \int_{S_{1,i+1}} u_2(s_1, s_2) dv_i,$

(4) $v_i(E) = \begin{cases} 1 & , \quad E = S_{j,i} \\ \theta|E| & , \quad E \subset S_{j,i}, \end{cases}$

where $S_{j,i}$ is the strategy set of the opponent in the subgame beginning at node i.

Proposition 1.

For all ϵ and all θ, there exists a θ-perfect Choquet-Nash equilibrium.

Proof.[25]

Let $u_1(s_1, v_i)$ and $u_2(v_i, s_2)$ be defined as in (3) and (4), and let $u_1(\sigma_1, v_i)$ and $u_2(v_i, \sigma_2)$ be the (additive) expectations of $u_1(s_1, v_i)$ and $u_2(v_i, s_2)$ under the behavior strategies σ_1 and σ_2. Consider the correspondence[26] $\varphi : [0,1]^n \times [0,1]^{n-1} \longrightarrow [0,1]^n \times [0,1]^{n-1}$, $(\sigma_1(i), \sigma_2(i), \epsilon_i) \mapsto (\sigma_1'(i), \sigma_2'(i), \epsilon_i')$:

$$(4.5) \qquad \sigma_1'(i) := \arg\max_p (1-\epsilon) u_1(p, \sigma_2) + \epsilon_i u_1(p, v_i) \ \ \forall i \in N_1,$$

[24]Following Kolmogorov & Fomin (1954), we denote for $a \in \mathbb{R}$ the integral part by $[a]$ (the largest integer smaller than a), and the fractional part by $< a >$ ($< a >= a - [a]$).

[25]The only difference to the standard existence proof is that we apply fixed point arguments directly to the extensive form. The reason for this is that there is no agent normal form, since non-rational players cannot be modelled as players, who would choose additive behavior strategies. On the other hand, applying non-additive equilibrium concepts to the normal form game between rational agents only would require a model of independent choices by more than two players with heterogeneous priors about the rationality of the opponents.

[26]Note that equation (7) is well-defined for $\epsilon_i = 0$. Given our assumption that $\epsilon > 0$, ϵ_i will not assume this value, yet it must be included in order to have a compact domain.

$$\sigma_2'(i) := \arg\max_p (1-\epsilon)u_2(\sigma_1,p) + \epsilon_i u_2(v_i,p) \quad \forall i \in N_2, \tag{4.6}$$

$$\epsilon_{i+2}' := \frac{\epsilon_i \cdot (1-|S_{j,i+1}|\theta)}{1-\epsilon_i|S_{j,i+1}|\theta - (1-\epsilon_i)(1-\sigma_j(i+1))}. \tag{4.7}$$

We first show that a fixed point of this correspondence is a θ-perfect Choquet-Nash equilibrium:

Let $(\hat{\sigma}_1(i), \hat{\sigma}_2(i), \hat{\epsilon}_i) \in \varphi(\hat{\sigma}_1(i), \hat{\sigma}_2(i), \hat{\epsilon}_i)$. This means

$$\hat{\sigma}_1(i) \in \arg\max_p (1-\hat{\epsilon})u_1(p,\hat{\sigma}_2) + \hat{\epsilon}_i u_1(p,v_i) \quad \forall i \in N_1, \tag{4.8}$$

$$\hat{\sigma}_2(i) \in \arg\max_p (1-\hat{\epsilon})u_2(\hat{\sigma}_1,p) + \hat{\epsilon}_i u_2(v_i,p) \tag{4.9}$$

$$\hat{\epsilon}_{i+2} := \frac{\hat{\epsilon}_i \cdot (1-|S_{j,i+1}|\theta)}{1-\hat{\epsilon}_i|S_{j,i+1}|\theta - (1-\hat{\epsilon}_i)(1-\hat{\sigma}_j(i+1))}. \tag{4.10}$$

By their definitions, $u_1(p,\sigma_2)$, $u_2(\sigma_1,p)$, $u_1(p,v_i)$ and $u_2(v_i,p)$ are linear in p, so if $\hat{\sigma}_1(i)$ and $\hat{\sigma}_2(i)$ are maximisers then so are the pure strategies $\hat{s}_1(i)$ and $\hat{s}_2(i)$ in their support.[27] Consequently $(\hat{\sigma}_1(i), \hat{\sigma}_2(i), \hat{\epsilon}_i)$ satisfy (1) — (4) for any given $\epsilon \equiv \epsilon_0 \equiv \epsilon_1$.

It remains to be shown that such a fixed point exists. Since φ maps a closed, bounded and convex subset of a finite-dimensional Euclidean space into itself, Kakutani's Theorem (1941) implies that a fixed point exists if φ is non-empty, convex-valued and has a closed graph. Since the maximands are linear in p, they are continuous over a compact domain and, by Weierstraß' Theorem, the maxima in (5) and (6) exist. Moreover, (7) uniquely determines ϵ_{i+2}'. So φ is non-empty. Also, from the linearity of (5) and (6) and the uniqueness of (7), φ is convex-valued. Finally, by Berge's Maximum Theorem (1959), φ is closed-valued and upper hemi-continuous. This implies that φ has a closed graph (Border 1985, p.56, Theorem 11.9 (a)). This completes the proof.

Note that the equilibrium is not unique. Intuitively, a rational player will go across if his expected utility from a non-rational opponent — determined by his uncertainty aversion — and the expected utility from a rational opponent — weighted by his belief about the likelihood of non-rationality — is higher or equal than his payoff from going down. His belief at this node is his update given his initial beliefs and the rational strategies. It may be that the initial belief is exactly such to make him indifferent. Generically, however, this will not be the case.

[27]Note that for $u_1(p,v_i)$ and $u_2(v_i,p)$ this is due to the order of integration.

We refer to player P_i as the player who moves at node i, and denote by S_i the set of pure strategies of player P_i in the subgame starting at node i. We denote by $\sigma^*(i)$ the equilibrium probability with which player P_i plays A_i at node i.

Proposition 2.

$\forall\theta > 0\ \forall\epsilon > 0\ \exists N\ \forall n:$ *If* $N \leq n \leq \overline{n}$ *then* $\sigma^*(1) \neq 0$.

Proof. Indirect. Suppose $\sigma^*(1) = 0$. Then $\epsilon_2 = 1$ and P_2 will choose A_2 if

$$\begin{aligned} a_2 &\leq (1-\epsilon_2)\,[\sigma^*(3)a_4 + (1-\sigma^*(3))b_3] \\ &\quad +\epsilon_2\,[(1-\theta|S_3|)b_3 + \theta\frac{|S_3|}{2}b_3 + \theta\frac{|S_3|}{2}a_4], \end{aligned}$$

which is equivalent to

$$\eta_2 \leq \theta\frac{|S_3|}{2}.$$

Now define N as the smallest integer bigger than $4 + 2(\mathrm{ld}\ \eta_2) - 2(\mathrm{ld}\ \theta)$. Note that

$$\begin{aligned} & 4 + 2(\mathrm{ld}\ \eta_2) - 2(\mathrm{ld}\ \theta) && \leq -2(\mathrm{ld}\ \theta) - 2 \\ \Longleftrightarrow\ & \mathrm{ld}\ \eta_2 && \leq -3 \\ \Longleftrightarrow\ & \eta_2 && \leq \frac{1}{8} \end{aligned}$$

so that $N \leq \overline{n}$. Finally consider n with $N \leq n \leq \overline{n}$: Note that

$$\begin{aligned} & 4 + 2(\mathrm{ld}\ \eta_2) - 2(\mathrm{ld}\ \theta) && \leq n \\ \Longleftrightarrow\ & (\mathrm{ld}\ \tfrac{\eta_2}{\theta}) && \leq \frac{n-2}{2} - 1 \\ \Longleftrightarrow\ & \eta_2 && \leq \theta\frac{2^{\frac{n-2}{2}}}{2}. \end{aligned}$$

But $|S_3| \geq 2^{\frac{n-2}{2}}$, and thus $\sigma^*(2) = 1$. But since

$$1 > \eta_1,$$

$$|S_2| \geq |S_3|,$$

$$\theta\frac{|S_3|}{2} \geq \eta_2$$

and

$$\eta_2 \geq \eta_1,$$

we have independently of ϵ

$$\eta_1 < (1-\epsilon) + \epsilon\theta\frac{|S_2|}{2}.$$

This would imply $\sigma^*(1) \neq 0$, a contradiction. So indeed $\sigma^*(1) > 0$. This completes the proof.

4.5 Conclusion

A θ-perfect Choquet-Nash equilibrium is a solution concept for the centipede game that combines subgame-perfection with uncertainty aversion. We suggest as a reason why players choose 'Across' early in the game the boundedness of uncertainty aversion. Even though players are uncertainty averse, if there is enough uncertainty from which players can profit and if they expect their rational opponents also to play 'Across' then it is indeed rational to play 'Across'.

On a conceptual level, the equilibrium concept allows the analysis of the centipede game without the assumption that rationality is mutual knowledge. It avoids several difficulties that arise in the Kreps et al. (1982) approach: First, non-rational players are not necessarily 'altruistic' and always play 'Across'. Secondly, we do not need to specify any particular belief about non-rational opponents, which in the absence of a theory of non-rational play would necessarily be ad hoc. In particular, we can avoid the difficulties associated with the uniform distribution as a model of ignorance. Thirdly, we do not need to refer to non-rational players as types, which would ascribe to them a consistent hierarchy of beliefs. Finally, the solution concept is consistent with the interpretation of equilibrium strategies as rational strategies, which implicitly defines all other strategies as non-rational. As a result, the structure of the game may be assumed to be mutual knowledge.

At the same time, our solution concept builds on existing game-theoretic concepts. First, the analysis is in the same spirit as Kreps et al. (1982), which has proved to be so useful in industrial organization. Secondly, the solution concept is an equilibrium concept, and avoids the indeterminateness associated with weaker solution concepts. Similarly, the solution concept is static, and does not rest on the specification of a dynamic learning or evolutionary process. Finally, we preserve the spirit of subgame perfection in requiring optimality at all decision nodes. Thus we extend the approach of Dow & Werlang (1994) to extensive games.

The limitations of our approach are the following: First, the actual computation of an equilibrium may be complicated, it corresponds to the computation of a fixed point, as does the sequential equilibrium in McKelvey & Palfrey (1992). Secondly, the degree of uncertainty aversion is not directly observable. How to elicit this degree from a purely decision-theoretic environment is an issue for future research.

That the degree of uncertainty aversion is bounded, however, seems a plausible hypothesis whose usefulness can only be established empirically. Finally, while our solution concept gives an 'inner' equilibrium for the centipede game, it does not replicate the distribution of actual choices. While the sequential equilibrium with 'altruistic' types in McKelvey & Palfrey (1992) alone does not give this distribution either, McKelvey & Palfrey (1992) show that additional hypotheses, both about how players make mistakes and how they learn during the game, do. To introduce such hypotheses in a consistent way is another topic for future research.

Appendix

Let v be a capacity and consider the events $E, F \in \Sigma$. The Dempster-Shafer rule specifies that the posterior capacity of event E is given by

$$v(E|F) := \frac{v(E \cup \overline{F}) - v(\overline{F})}{1 - v(\overline{F})}.$$

Let ϵ be the prior probability that the opponent is not rational. Assume that the opponent has two actions A and D at the given node n. Assume that a rational opponent chooses action A with probability p. Finally, assume S is the set of the opponent's pure strategies that specify the action D at the given node.

Then the posterior belief ϵ' about the opponent's rationality after action A is given by

$$\epsilon' := \frac{\epsilon \cdot (1 - |S|\theta)}{1 - (1-\epsilon)(1-p) - |S|\theta\epsilon},$$

where $|S|$ is the number of the opponents' strategies S, and ϵ the prior belief about the opponent's rationality, with $0 < \epsilon < 1$.

This is derived as follows:

Let R be the event that the opponent is rational, let $\overline{R}$ be the event that he is non-rational.

We want to calculate

$$\epsilon' \equiv v(\overline{R}|A) := \frac{v(\overline{R} \cup D) - v(D)}{1 - v(D)}. \tag{4.11}$$

First,

$$v(R|A) + v(\overline{R}|A) = 1, \tag{4.12}$$

and

(3) $v(R|A) = \frac{v(R \cup D) - v(D)}{1 - v(D)}$,

(4) $v(\overline{R}|A) = \frac{v(\overline{R} \cup D) - v(D)}{1 - v(D)}$

imply

$$v(D) = v(R \cup D) + v(\overline{R} \cup D) - 1.$$

Secondly,

(5) $v(D|R) = \frac{v(D \cup \overline{R}) - v(\overline{R})}{1 - v(\overline{R})}$, and

(6) $v(D|\overline{R}) = \frac{v(D \cup R) - v(R)}{1 - v(R)}$.

We know that

(7) $v(R) = 1 - \epsilon$,

(8) $v(\overline{R}) = \epsilon$,

(9) $v(D|R) = 1 - p$, and

(10) $v(D|\overline{R}) = |S|\theta$,

so that

(11) $v(D \cup \overline{R}) = (1 - \epsilon)(1 - p) + \epsilon$, and

(12) $v(D \cup R) = |S|\theta\epsilon + (1 - \epsilon)$.

Thus

$$v(D) = (1 - \epsilon)(1 - p) + |S|\theta\epsilon.$$

Consequently,

(4.13) $$\epsilon' := \frac{\epsilon \cdot (1 - |S|\theta)}{1 - (1 - \epsilon)(1 - p) - |S|\theta\epsilon}.$$

Note:

- The derivation is only valid under lack of mutual knowledge of rationality, i.e. for $\epsilon > 0$ and $\epsilon < 1$, otherwise $v(D|R)$ or $v(D|\overline{R})$ are not well-defined.
- With $0 < \epsilon < 1$ there are no probability zero events. Since $|S|$ strategies specify action D and there are two actions at this node, the number of strategies is $2|S|$. So uncertainty aversion means $\theta < \frac{1}{2|S|}$. It follows that

$$v(D) = (1 - \epsilon)(1 - p) + |S|\theta\epsilon < (1 - \epsilon)(1 - p) + \frac{1}{2}\epsilon < 1.$$

  This holds for any $p \in [0, 1]$, including the boundaries.
- In particular, if $\epsilon > 0$ then $\epsilon' > 0$, independently of p. However, if $p = 0$, then $\epsilon' = 1$. Thus we also need to be able to update the belief $\epsilon = 1$. Intuitively, if

the prior belief about the opponent is that he is non-rational and beliefs about his behavior are boundedly uncertainty averse, then there are no probability zero events, and the posterior belief should also be that the opponent is non-rational.This can be justified directly from the Dempster-Shafer rule (1): From monotonicity, $v(\overline{R}) \leq v(\overline{R} \cup D)$, therefore $v(\overline{R} \cup D) = 1$. Also, (6) implies $v(D|\overline{R}) = v(D \cup R)$, so again by monotonicity, $v(D) \leq v(D \cup R) = |S|\theta < 1$. Since this result also follows if we substitute $\epsilon = 1$ into (13), we do not have to explicitly track this special case.

- The reason why $\epsilon = 0$ has to be excluded is that there is no parallel argument that $\epsilon = 0$ and $p = 0$ should give $\epsilon' = 1$. (3) implies $v(D \cup \overline{R}) = v(D|R) = 1$ and (1) gives $\epsilon' = \frac{1-v(D)}{1-v(D)}$, but $v(D) \not< 1$.
- Whether action A is interpreted as evidence of rationality or evidence of non-rationality depends on p, $|S|$ and θ:

$$
\begin{array}{rcl}
 & \epsilon' & \leq \epsilon \\
\Longleftrightarrow & \frac{\epsilon \cdot (1-|S|\theta)}{1-(1-\epsilon)(1-p)-|S|\theta\epsilon} & \leq \epsilon \\
\Longleftrightarrow & (1-\epsilon)(1-p) & \leq (1-\epsilon)|S|\theta \\
\Longleftrightarrow & p & \geq 1 - |S|\theta .
\end{array}
$$

Other things equal, the higher the probability of A, the more likely it is that A is evidence of rationality, because A is taken with high probability by rational players. The lower θ and $|S|$, the less likely it is that A is interpreted as evidence of rationality, because the greater is the uncertainty that A is taken by a non-rational player.

- Finally, note that the argument rests heavily on (2), i.e. the the requirement about beliefs that an opponent is either rational or non-rational, so that these beliefs have to be additive.

Chapter 5

Conclusion

The thesis has attempted to contribute to a clarification of the concept of strategic rationality.

We assumed that the rationality concept is an equilibrium concept. On the basis that a rationality concept does not restrict non-rational play, and that deviations from rational play should be taken as evidence of non-rationality, we argued for non-additive beliefs about non-rational play. For the case of uncertainty-averse players we developed equilibrium concepts that capture this intuition. Their main feature in extensive games is that equilibrium reasoning is only applied to the equilibrium path; due to complete ignorance and uncertainty aversion players apply maximin reasoning off the equilibrium path. The solution concept than endogenously determines the equilibrium path.

There are, of course, several limitations of the analysis. We only consider uncertainty aversion, not uncertainty love, even though uncertainty aversion is not a universal empirical phenomenon. An extreme consequence of our equilibrium concept is that the strategic interaction comes to an end after a move that reveals the non-rationality of the opponent. On the other hand, the view that such deviations say nothing at all about the rationality of the opponent, and that is underlying standard solution concepts, is equally extreme. Still, we view the equilibrium concept proposed here at best as a basis for further refinements.

On a theoretical level, we feel that further developments will largely depend on progress in the decision-theoretic aspects of Choquet expected utility theory and its variants. In particular, a joint axiomatisation of preferences and their updating seems an important area of research. Within game theory proper, we feel that there is a need to give formal epistemic characterisations for equilibrium concepts with non-additive beliefs, without assuming that players already start with non-additive beliefs.

Ultimately, however, we think that the success of any theoretical construct not only depends on its intrinsic plausibility, but also on its usefulness in applications. In this respect, the following may be areas where approaches like ours may be useful:

First, game theory with non-additive beliefs may offer new insights into institutions. That one reason for the existence of institutions may be the objective to overcome non-rational individual decision making is a widespread intuition. In addition, they may exist in order to overcome non-additive beliefs in situations in which relative frequencies decide on the long-run success of the institution. Or alternatively, they may reflect non-additive beliefs by institutionalising the implementation of mixed strategies if there is a strict preference for mixtures. Also, these considerations are then not only relevant for the existence, but also the internal organisation of institutions.

Secondly, allowing for non-additive beliefs may help to explain evidence in experimental game theory. Since uncertainty aversion effects are well-documented in decision situations, a theoretical framework may help to identify them in games and to distinguish them from other dimensions of boundedly rational decision-making. Also, experimental games may further refine our understanding of uncertainty aversion just as experiments in decision theory do.

Finally, an important area of economics for applications of game-theoretic concepts that use weaker rationality requirements is mechanism design. Optimal mechanism should be robust with respect to doubts about the rationality of the players, and should still lead to satisfying outcomes if non-rational players deviate. Minimax considerations off the equilibrium path may be an attractive starting point for such research.

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