# Problems in Combinatorics: Paths in Graphs, Partial Orders of Fixed Width

Sarah Jane Goodall

London School of Economics

A Dissertation for the Degree of

Doctor of Philosophy

of

The University of London

January 1995

## Abstract

This thesis contains results in two areas, that is, graph theory and partial orders.

(1) We consider graphs $G$ with a specified subset $W$ of vertices of large degree. We look for paths in $G$ containing many vertices of $W$. The main results of the thesis are as follows. For $G$ a graph on $n$ vertices, and $W$ of size $w$ and minimum degree $d$, we show that there is always a path through at least $\left\lceil \frac{w}{\lfloor n/(d+1) \rfloor} \right\rceil$ vertices of $W$. For a connected graph $G$ on $n$ vertices, and $W$ of size $w$ and minimum degree $d$, we show that if $n \geq 2d+1$ there is always a path through at least $\min\left\{ d+1, 2\left\lfloor \frac{w-1}{\lceil n/d \rceil - 1} \right\rfloor + 1 \right\}$ vertices of $W$.

If $w \geq n-d+1$, then there is a path through at least $w+2d+1-n$ vertices of $W$. We also prove some results for graphs in which only the degree sums of sets of independent vertices in $W$ are known.

(2) Let $P = (X, <)$ be a poset on a set $\{1, 2, \ldots, N\}$. Suppose $X_1$ and $X_2$ are a pair of disjoint chains in $P$ whose union is $X$. Then $P$ is a partial order of width two. A labelled poset is a partial order on a set $\{1, 2, \ldots, N\}$. Suppose we have two labelled posets, $P_1$ and $P_2$, that are isomorphic. That is, there is a bijection between $P_1$ and $P_2$ which preserves all the order relations. Each isomorphism class of labelled posets corresponds to an unlabelled poset.

We prove that the number of (labelled) width two posets with vertex set $\{1, 2, \ldots, N\}$ is $\Omega_2(N) = \frac{(2N+1)!}{(N+2)!}\left(\frac{4}{25} + O\left(N^{-\frac{3}{8}}\right)\right)$, whereas the number of unlabelled posets on $N$ vertices is asymptotically $\frac{1.116455\ldots}{N!}\Omega_2(N)$.

# Contents

# List of Figures

## Acknowledgements

I would like to thank all those who have supported and encouraged me thoroughout my work on this thesis.

In particular, my supervisor, Dr. Graham Brightwell, has been a source of great encouragement and inspiration. For this, I am very grateful.

The Mathematics Department at the London School of Economics has provided me with a supportive and friendly atmosphere in which to carry out my research. Drs Martin Anthony, Michele Harvey and Joan Davies deserve special mention for their general concern and encouragement, as does Nicole Buckland.

I should also like to thank George Russell for his ideas and thoughts on the partially ordered sets problem.

I am grateful to Drs Christine Farmer and Brian Wilson of the Mathematics Department at Royal Holloway and Bedford New College, for their advice, and also their continued interest in my progress.

My thanks also go to Rob Hochberg for his encouragement, support and prayers, and to Emmanuel Protte.

I would like to thank my parents, my brother Matthew, and Richard Maple for their support.

Finally I would like to thank Alison and Billy.

## Declaration of Originality

I declare that this is my own unaided work.

# 1 Introduction

This thesis contains results in two areas, that is, graph theory and partial orders. In the graph theory area we deal with paths and cycles. In partially ordered sets we count the number of posets of width two.

The problem of finding paths and cycles through many of the vertices in a graph has been studied by a large number of people. Results have been obtained which involve various different conditions on the graph. These conditions include minimum degree, connnectivity, toughness, number of edges, sum of degrees of non-adjacent vertices and the size of neighbourhood unions of non-adjacent vertices. All these conditions relate to the whole graph. For example, *all* the vertices have degree at least $d$. Suppose we do not have such a condition that relates to all the vertices. We could, for example, know only the degree of the vertices of maximum degree. This gives an upper bound on the degree of the other vertices, but no more information about their individual degrees. Results have been obtained about cycles through many of the vertices of maximum degree. Now suppose we have even less information about the degree of the vertices. Let the minimum degree of some specified subset of the vertices be known, but nothing about the other vertices. This is the area we investigate. (Chapters 2–9.)

The results we prove in the second part (Chapters 10–12) concern posets. Let $P = (X, <)$ be a poset on a set $\{1, 2, \ldots, N\}$. Suppose $X_1$ and $X_2$ are a pair of disjoint chains in $P$ whose union is $X$. Then $P$ is a partial order of width two. A natural, but apparently unstudied, question is: how many width two posets are there? We answer this question asymptotically for both labelled and unlabelled posets. A labelled poset is a partial order on a set $\{1, 2, \ldots, N\}$. Suppose we have two labelled posets, $P_1$ and $P_2$ that are isomorphic. That is, there is a bijection between $P_1$ and $P_2$ which preserves all the order relations. We partition the set of labelled posets into isomorphism classes. Each isomorphism class corresponds to an unlabelled poset. If, as for random graphs, it were the case that almost every width two poset has trivial automorphism group, the two asymptotic counts would differ by a factor of $N!$. However, this does not turn out to be the case.

# 2 An Introduction to Paths in Graphs

Before we begin our discussion on graphs we need to establish the basic notation.

## 2.1 Graph Notation

We use standard notation as in Bondy and Murty [6]. Let us introduce, in particular, some standard and slightly less standard notation.

- $G = (V(G), E(G))$ is a graph on the set $V(G)$ of $n$ vertices with the edge set $E(G)$ containing no multiple edges or loops.
- $|A| = |V(A)|$, where $A$ is typically a path $P$, cycle $C$ or component $Q$ of $G \setminus C$.
- $\Gamma_A(v)$, for $A \subseteq G$, is the set of neighbours in $A$ of a vertex $v$.
- $\Gamma(v)$ is the set of neighbours in $G$ of $v$.
- $\Gamma(B) = \bigcup_{b \in B} \Gamma(b)$, the union of the neighbours of the vertices of the set $B$.
- $d_A(v) = |\Gamma_A(v)|$, the degree in $A$ of $v$.
- $d(v) = |\Gamma(v)|$, the degree in $G$ of $v$.
- $\delta = \min_{v \in G} d(v)$, the minimum degree of $G$.

We will consider graphs $G$ with a specified subset $W$ of vertices. In these circumstances, we shall normally use the following notation.

- $W$ is a specified subset of the vertices of $G$.

- $w = |W|$, the order of $W$.

- $d = \min_{v \in W} d(v)$, the minimum degree of $W$ in $G$.

Further notation will be defined as it is needed. We also say that a set of vertices $A$ is *adjacent* to a set of vertices $B$ if there is an edge $(a, b)$ such that $a \in A$ and $b \in B$.

## 2.2 Motivation and Main Graph Results

In this thesis we will be dealing with long paths and cycles through specified vertices. To put these results into context we will first talk about graphs where the specified set $W$ contains all the vertices of $G$. In particular, a path (cycle) through all the vertices of a graph is called a *Hamiltonian path (cycle)*. The first main result in the area of Hamiltonian cycles is the following well known theorem of Dirac [8].

**Theorem 2.1 (Dirac)** *Let $G$ be a graph on $n$ vertices. Let $\delta$ be the minimum degree of $G$. Then if $\delta \geq n/2$, $G$ is Hamiltonian.*

This result says that we can find a cycle through all the vertices of a graph if we know something about the minimum degree of the graph. Let $W$ be a specified subset of the vertices of $G$. Suppose we know $|W| = w$, $|G| = n$ and the minimum degree $d$ of the vertices in $W$. How well can we do then? What conditions do we need to guarantee a cycle through all of the vertices of this specified set? The answer to this is a straightforward generalisation of Theorem 2.1, which says that there is a cycle through all the vertices of degree at least $n/2$. Now this result applies to the case where the minimum degree $d$ of $W$ is large compared with $n$.

So an obvious question is to ask what happens for smaller values of $d$. Through how many vertices of $W$ can we find a cycle? The result in the case where the specified set is the whole of the vertex set of the graph was obtained by Bollobás and Häggkvist [4] following earlier work by Alon [1] and Egawa and Miyamoto [9]. A simpler proof of the Bollobás and Häggkvist result [4] was obtained by Bollobás and Brightwell [3]. Further, the result proved in [3] is more general, dealing with a specified subset $W$ of vertices of minimum degree $d$.

**Theorem 2.2 (Bollobás, Brightwell)** *Let $n$, $w$ and $d$ be positive integers, with $w \leq n$. Let $G$ be a graph on $n$ vertices and let $W$ be a subset of the vertices of $G$ of size $w$ such that the degree of each vertex in $W$ is at least $d$. Set*

$$s = \left\lceil \frac{w}{\lceil n/d \rceil - 1} \right\rceil.$$

*Then for $s \geq 3$ there is a cycle in $G$ through at least $s$ vertices of $W$.*

A special case of this result is that when $d \geq n/2$, there is a cycle through all the vertices of $W$. If we also have $W = V(G)$, then $G$ is Hamiltonian. Therefore, we recover Dirac's Theorem (Theorem 2.1) when $d \geq n/2$ and $W = V(G)$.

The results we have discussed so far relate to cycles. Instead we could, and indeed do, ask similar questions about paths. We start by looking at an immediate corollary to Dirac's Theorem. We include a proof for completeness.

**Corollary 2.3 (Corollary to Dirac's Theorem)** *Let $G$ be a graph on $n$ vertices with minimum degree $\delta \geq (n-1)/2$. Then $G$ contains a Hamiltonian path.*

**Proof.** Let $G$ be a graph on $n$ vertices such that the minimum degree is $(n-1)/2$. Let $H$ be a graph on the vertices $V(G) \cup \{v\}$ formed by adding a vertex $v$ to $G$ and all the edges between $v$ and $V(G)$. Then the minimum degree of $H$ is $(n+1)/2$. So, by Theorem 2.1, there is a Hamiltonian cycle $C$ in $H$. Now, removing $v$ from $C$ we are left with a path through all the vertices of $V(G)$. Hence there is a Hamiltonian path in $G$. □

Following Theorem 2.2, we are led to ask whether we can generalise the above result, Corollary 2.3, to graphs for which only the minimum degree of a specified set of vertices is known. Indeed, we answer this question fully as follows.

**Theorem 2.4** *Let $G$ be a graph on $n$ vertices and let $W$ be a subset of $w$ vertices of degree at least $d$. We set*

$$x = \left\lceil \frac{w}{\lfloor n/(d+1) \rfloor} \right\rceil .$$

*Then there is a path in $G$ through at least $x$ vertices of $W$.*

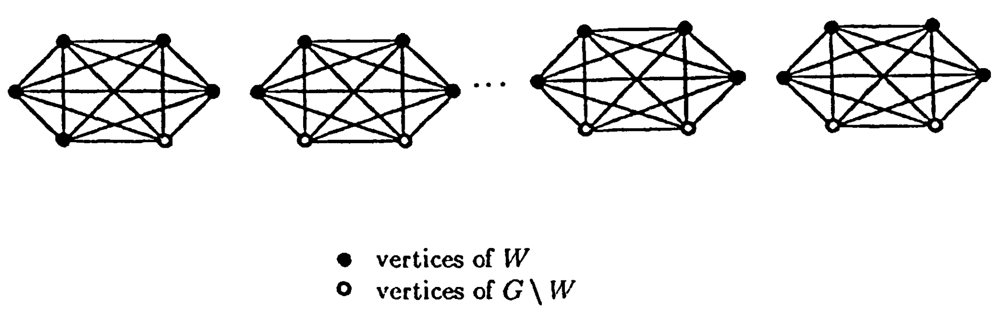


Figure 1: An extremal graph for Theorem 2.4

Theorem 2.4 is best possible, as we will see from the following family of graphs for which equality holds. This family consists of graphs which are made up of as many complete components as possible with the specified vertices evenly distributed between them. A graph of this type is illustrated in Figure 1. Obviously there is no path containing vertices from more than one component. Hence a path contains at most the number of specified vertices in a component. So what is the maximum number of specified vertices in a component? This is exactly the quantity $x$ given in Theorem 2.4. The proof of this theorem and other related results appear in Chapter 3.

As the extremal graphs are disconnected, it is natural to ask what happens if we add the constraint that the graph be connected. Now we can ask what conditions are needed to guarantee a path through *all* the specified vertices in a connected graph. In part the answer is given by Theorem 2.4, which tells us that if the minimum degree $d$ of the specified vertices is at least $(n-1)/2$ then there is a path through all the specified vertices. For values of $d$ less than this, the imposition of connectedness implies the existence of paths through more specified vertices than given by Theorem 2.4. When the minimum degree $d$ of the specified set $W$ is at least $n/3$, there is a path through all the vertices of $W$ if $W$ isn't too big, namely $|W| \leq d+1$. This result is given by the following theorem.

**Theorem 2.5** *Let $G$ be a connected graph on $n$ vertices where $n/3 \leq d$ and $W$ is a subset of $w \leq d+1$ vertices of minimum degree $d$. Then there is a path through all the vertices in $W$.*

This result is best possible in two ways. Suppose we increase the number of vertices to $3d+1$ and look at the graph consisting of three complete graphs on $d$ vertices plus one central vertex adjacent to all vertices. Let the specified vertices be evenly distributed between the three complete subgraphs. Then there is no path through all three complete subgraphs. (See Figure 2.) The necessity of the condition that $w \leq d+1$ is due to the following bipartite graph. Suppose $G = (V_1, V_2, E)$ is a bipartite graph where all the vertices of $W$ are in one part $V_1$ and the other part $V_2$

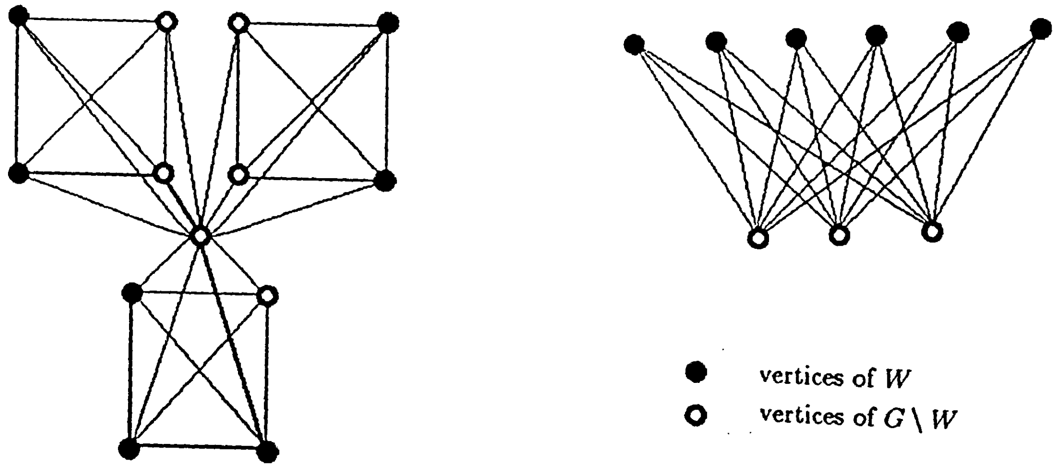


Figure 2: Two extremal graphs for Theorem 2.5

has exactly $d$ vertices. This has a path through at most $2d+1$ vertices of which at most $d+1$ can be in $W$. (See Figure 2.) We continue our discussion of this result, and other more general results, in Chapter 6.

If we remove the condition $n \leq 3d$, we obtain the main result of the thesis, Theorem 2.6, which gives the number of specified vertices in a connected graph through which we can guarantee a path.

**Theorem 2.6** *Let $n$, $w$ and $d$ be integers. Let $G$ be a connected graph on $n \geq 2d+1$ vertices and let $W$ be a subset of the vertices of $G$ of order $w$, such that the minimum degree of the vertices in $W$ is $d$. Then there is a path through*

$$\min\left\{d+1, 2\left\lfloor \frac{w-1}{\lfloor (n-1)/d \rfloor} \right\rfloor + 1\right\}$$

*vertices of $W$.*

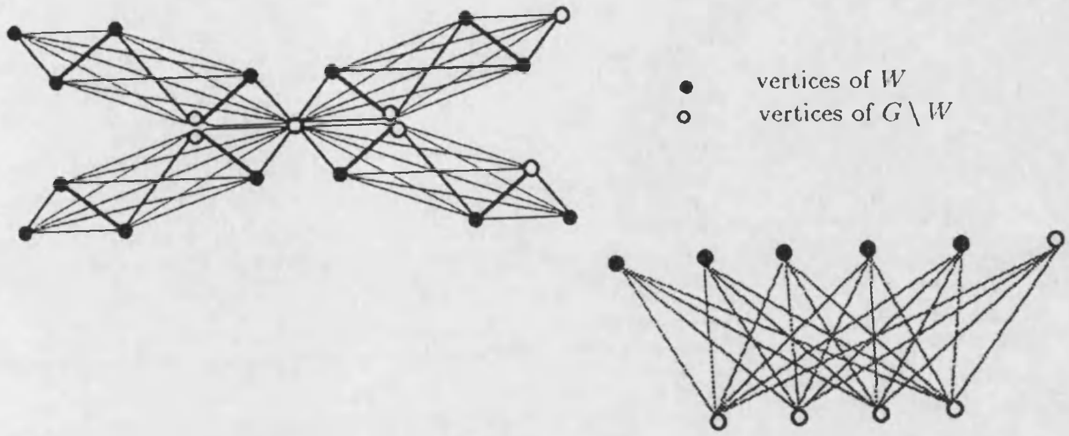


Figure 3: A flower graph and a bipartite extremal graph

If $n \leq 3d$, we recover Theorem 2.5 when $w$ is odd. When $w$ is even we have a slightly weaker result, namely a path through $w - 1$ vertices of $W$. In general, for $w \leq n - d$, we suspect that this result is not quite best possible and can often be improved by one. However, for $w \geq n - d + 1$, we prove a best possible result, which is that there is a path through at least $\min\{w, w + 2d + 1 - n\}$ specified vertices. (See Theorem 4.1.)

Next we describe the graphs which satisfy the two lower bounds given in Theorem 2.6. When $d$ is small, the theorem says that there is a path through at least $d + 1$ specified vertices. The following family of graphs shows that we can do no better than this. As in Theorem 2.5, we consider a family of bipartite graphs.

- For $1 \leq w \leq n$ and $1 \leq d \leq n-1$, we define $B(n,w,d)$ to be a complete bipartite graph on $n$ vertices, with vertex set $V_1 \cup V_2$ and specified subset $W$ of minimum degree $d$, where $|V_2| = d$ and (1) if $w \leq n-d$, then $W \subseteq V_1$, and (2) if $w \geq n-d+1$, then $V_1 \subset W$.

So for $w \leq n-d$, $B(n,w,d)$ clearly contains no path through more than $d+1$ specified vertices. (See Figure 3.)

When $d$ is large, an extremal graph is given by the following family of graphs.

- Let $G$ consist of $k$ disjoint connected subgraphs $H_i$ of at least $d$ vertices and one vertex $v$ such that for each $i$, at least one vertex $v_i \in H_i$ is adjacent to $v$. Suppose $k$ is maximum, given $n = |G|$ and $d$, that is, $k = \lfloor (n-1)/d \rfloor$, and the specified vertices are evenly distributed among the $H_i$. We call $G$ a *flower graph.* A subgraph $H_i$ is a *petal* and $v$ is the *central vertex.*

- We define $F(n,w,d)$ to be a flower graph with $n$ vertices and $w$ specified vertices of degree at least $d$, such that the petals are complete graphs, and the central vertex, which is not a specified vertex, is adjacent to all the vertices of $G$. (See Figure 3.)

The graph $F(n,w,d)$, for $w \leq n-d$, gives the bound of Theorem 2.6 in many cases. This will be discussed further in Chapter 4. The proof of Theorem 2.6 is based on Theorem 2.5 which is proved in Chapter 6, using results from Chapter 5. The remainder of the proof of Theorem 2.6 appears in Chapters 7–8.

# 3 General Graphs and Paths through Specified Vertices

In this chapter we discuss and prove some generalisations of Theorem 2.4. As already mentioned, we are interested in the maximum number of specified vertices that we can guarantee to be contained in a path. For this chapter only, we will look at the problem in a different, more general way. Suppose we label the vertices $V(G)$ of a graph $G$ by a weight function $\phi : V \to \{0, 1\}$ where $\phi(v) = 1$ if $v \in W$ and $\phi(v) = 0$ otherwise. We define the *weight* of a set $A$ by $w_A = \sum_{v \in A} \phi(v)$. Since a vertex has positive weight only when it is a specified vertex, the number of specified vertices in $A$ is $w_A$. Now we can consider our problem of finding a path through as many vertices of $W$ as possible in terms of finding a path of high weight. We extend this concept to a general weight function relating to the degree of the vertices. To do this we label the vertices of $G$ by a weight function $\psi : V \to \mathbf{N}_0 \equiv \mathbf{N} \cup \{0\}$ satisfying $\psi(v) \leq \lfloor d(v)/d \rfloor$, where $d$ is a fixed integer. We call a graph with such a weight function a graph *labelled by* $\psi$. Let $W$ be the set of vertices of non-zero weight, that is, $W = \{v : \psi(v) > 0\}$. We write $\psi(W) = \sum_{v \in G} \psi(v)$, and set $w = \psi(W)$, which we call the *weight* of $W$.

Before we state our first theorem generalising Theorem 2.4, we need some notation. We define $\eta(n, w, d)$ to be the minimum, over all graphs $G$ on $n$ vertices and of weight $w$, of the maximum weight of a path in $G$. That is,

$$\eta(n, w, d) = \min_G \max_{P \subseteq G} w_P,$$

where $P$ is a path in $G$.

**Theorem 3.1** *Let $G$ be a graph on $n$ vertices labelled by $\psi : V \to \mathrm{N}_0$. Suppose that $w \leq n$. Then*

$$\eta(n, w, d) = \left\lceil \frac{w}{\lfloor n/(d+1) \rfloor} \right\rceil.$$

To obtain Theorem 2.4 from Theorem 3.1, we simply consider functions $\psi$ with $\psi : V \to \{0, 1\}$ and $W = \{v : \psi(v) = 1\}$.

To prove Theorem 3.1, we first show that

$$\eta(n, w, d) \leq \left\lceil \frac{w}{\lfloor n/(d+1) \rfloor} \right\rceil.$$

To do this, we consider a graph with $\kappa = \lfloor n/(d+1) \rfloor$ components. Let the weighted vertices be evenly distributed between the $\kappa$ components, so that each component has either weight $\lceil w/\kappa \rceil$ or $\lfloor w/\kappa \rfloor$. Then clearly there is no path in $G$ of weight greater than $\lceil w/\kappa \rceil$, as desired. (See Figure 4.)

To complete the proof of Theorem 3.1, we need to prove that

$$\eta(n, w, d) \geq \left\lceil \frac{w}{\lfloor n/(d+1) \rfloor} \right\rceil.$$

We define $\kappa$ and $x$ as $\kappa = \lfloor n/(d+1) \rfloor$ and $x = \lceil w/\kappa \rceil$. Thus we want to prove $\eta(n, w, d) \geq x$. Therefore $n \leq (\kappa + 1)(d + 1) - 1$ and $n \geq w \geq \kappa(x - 1) + 1$. So Theorem 3.2, below, implies our lower bound for $\eta(n, w, d)$.

**Theorem 3.2** *Let $G$ be a graph of order $n$ and weight $w \leq n$ labelled by $\psi : V \to \mathrm{N}_0$. Suppose that $\kappa, x \in \mathrm{N}_0$ are such that $n \leq (\kappa+1)(d+1)-1$ and $w \geq \kappa(x-1)+1$. Then there is a path in $G$ of weight at least $x$.*

Before we start the proof, we need to introduce some terminology. A *maximum path* in $G$ is a path $P$ of maximum weight $w_P$ and of minimum length given such weight, $w_P$.

$\kappa = \lfloor n/(d+1) \rfloor$ components

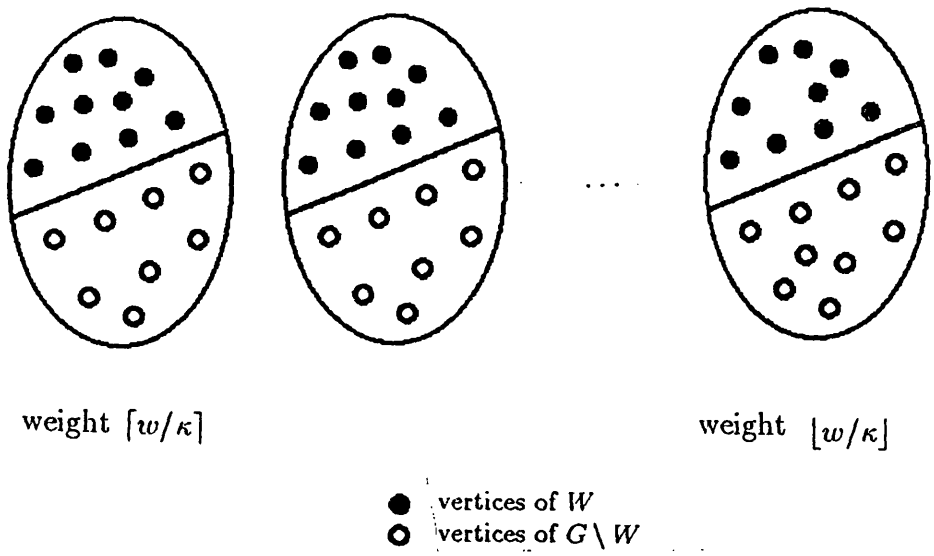


Figure 4: A construction for extremal graphs of Theorem 3.1

**Proof.** We may assume $n = (\kappa + 1)(d + 1) - 1$, since isolated vertices may be added without affecting the maximum weight of a path. Similarly, we claim that it is enough to prove the result in the case $w = \kappa(x - 1) + 1$. To see this, note that we can always relabel $G$ by a function $\psi'$, with $\psi'(v) \leq \psi(v)$ for all $v \in V(G)$ and $\sum_v \psi'(v) = \kappa(x - 1) + 1$. Therefore a path of weight $x$ in $G$ labelled by $\psi'$ has at least weight $x$ in $G$ labelled by $\psi$. We may also assume that all edges are incident with a weighted vertex, as any edge between non-weighted vertices will only serve to increase the path length. The proof is by induction on $\kappa$.

First we consider the case $\kappa = 1$ where $n = 2d + 1$ and $w = x$. The argument in this case is fairly standard but we proceed quite slowly, as the simple ideas here underlie most of the subsequent proofs. Let $P$ be a maximum path. If all the weighted vertices lie on $P$ the result holds, so suppose some vertex $w_0 \in W$ does not lie on $P$.

**Claim 1** *All the vertices of $W$ are in one component of $G$.*

If not, then $G$ consists of at least two components each containing a vertex of degree at least $d$, and hence $d+1$ vertices. Counting the vertices gives $2d+1 = n \geq 2(d+1) = 2d+2$, which is a contradiction. This establishes the claim. $\square$

In particular, we see that at least two weighted vertices lie on $P$ (for $w \geq 2$), by the connectedness of the component. Let $w_1 w_2 \ldots w_t$ be the weighted vertices on $P$ in natural order.

**Claim 2** *$P$ ends in the two weighted vertices, $w_1$ and $w_t$.*

Suppose not. Let $P$ end in some vertex $v$ where $v$ is not a weighted vertex. Then the path $P \setminus \{v\}$ has the same weight as $P$, but fewer vertices, which contradicts the definition of a maximal path. Hence the claim is established. $\square$

**Claim 3** *The vertices $w_1$ and $w_t$ are not adjacent to any weighted vertex $w_0$ in $G \setminus P$ or to any neighbours of $w_0$ outside $P$.*

Without loss of generality, suppose $w_0$ is adjacent to $w_1$. Then the path $w_0 w_1 \ldots w_t$ has weight at least $w_P + 1$ which contradicts the maximality of $P$. $\square$

**Claim 4** *There is no cycle that contains all the vertices of $P$.*

If there is a cycle containing all the vertices of $P$, then a weighted vertex $w_0 \in W \setminus P$ can be added to $P$, as it is in the same component as $P$. This contradicts the maximality of $P$. $\square$

A consequence of Claim 4 is that $w_1$ and $w_t$ have no common neighbours in $G \setminus P$.

**Claim 5** *There is no vertex $v$ on $P$ adjacent to $w_1$ whose predecessor $v^-$ on $P$ is adjacent to $w_t$.*

Suppose not. Then there is a cycle, $w_1 \ldots w_2 \ldots v^- w_t \ldots w_{t-1} \ldots v w_1$ which contains all the vertices of $P$, contradicting Claim 4. $\square$

We introduce some notation. For a subset $A$ of $V(G)$ and a vertex $v$ of $G$, let $\bar{d}_A(v) = |A \setminus \Gamma_A(v)|$. Let $|\Gamma_{G \setminus P}(w_1)| = a$ and $|\Gamma_{G \setminus P}(w_t)| = b$, and let $\Gamma_P^+(w_t)$ be the set of successors of neighbours of $w_t$ and $\Gamma_P^-(w_1)$ the set of predecessors of neighbours of $w_1$ on $P$.

Now, we count the number of vertices of $G$. On $P$, there are at least $d - b$ vertices which are not adjacent to $w_1$, as they are successors of neighbours of $w_t$. There are $d - a$ neighbours of $w_1$ on $P$, and also $w_1$ itself. Therefore on $P$ we have

$$\begin{aligned} |P| &\geq d_P(w_1) + \bar{d}_P(w_1) \geq (d-a) + (d-b) + 1 \\ &= 2d + 1 - a - b, \end{aligned}$$

and in $G \setminus P$, we have

$$\begin{aligned} |G \setminus P| &\geq d_{G \setminus P}(w_1) + d_{G \setminus P}(w_t) + |w_0| \\ &\geq a + b + 1. \end{aligned}$$

Therefore,

$$\begin{aligned} 2d + 1 &= n = |P| + |G \setminus P| \\ &\geq (2d + 1 - a - b) + (a + b + 1) \\ &\geq 2d + 2, \end{aligned}$$

which is a contradiction. Hence $P$ contains all the weighted vertices as required. This completes the proof in the case $\kappa = 1$.

Now we move to the induction step. We suppose that the result holds for values of $\kappa$ less than $k$, and let $G$ be a labelled graph for which $n = (k+1)(d+1) - 1$ and $w = k(x-1) + 1$. The aim is to reduce $w$ by $s \leq x$ and $n$ by at least $d + 1$ without changing the value of $x$, and then to use the induction hypothesis. Again let $P$ be a maximum path with initial vertex $w_1$ and final vertex $w_t$. As before, we may assume that $\psi(P) = s < x$.

Let

$$\begin{aligned} D_1 &= \Gamma_{G\setminus P}(w_1), \\ D_2 &= \Gamma_P^-(w_1), \\ D &= D_1 \cup D_2 \cup \{w_t\}, \\ W' &= W - \{w_1, w_2, \ldots, w_t\} = W \setminus P. \end{aligned}$$

So $|D| \geq d+1$ and $\psi(W') \geq \psi(W) - s = w - s$. These definitions are illustrated in Figure 5.

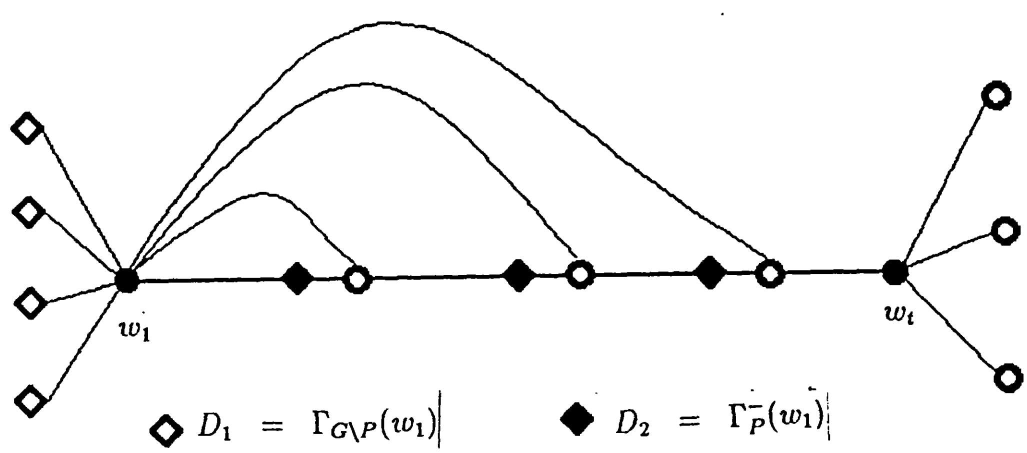


Figure 5: The path $P$ and the set of vertices $D = D_1 \cup D_2 \cup \{w_t\}$

**Claim 6** $W' \cap D = \emptyset$.

Let $v \in W'$ and suppose $v \in D_1$. This gives a path $vw_1w_2 \ldots w_t$ of weight at least $s+1$, contradicting the maximality of $P$. So $v \notin D_1$. Also $v \notin (D_2 \cup \{w_t\})$ as $W' \cap P = \emptyset$. Hence $W' \cap D = \emptyset$, as required. □

**Claim 7** $d_{G\setminus D}(v) = d_G(v)$ for all $v \in W'$.

Again, let $v \in W'$ and let $z \in \Gamma(v)$. If $z \in D_1$ then the path $vzw_1w_2\dots w_t$, contradicts the maximality of $P$. Also if $z \in D_2$, there is a path

$$vzw_jw_{j-1}\dots w_2w_1w_{j+1}w_{j+2}\dots w_t,$$

contradicting the maximality of $P$. Lastly, if $z = w_t$ the path $w_1w_2\dots w_tv$ again contradicts the maximality of $P$. Therefore $z \notin D$ and hence $d_{G\setminus D}(v) = d_G(v)$ for all $v \in W'$. □

So, by Claims 6 and 7, if we remove the set $D$ from $G$ we neither remove any vertex from $W'$ nor affect the degree of any vertex in $W'$. We consider the graph $G' = G \setminus D$ and the subset $W'$ of $V(G)$. The function $\psi' : V(G') \to \mathbb{N}_0$ defined by

$$\psi'(v) = \begin{cases} \psi(v) & v \in W', \\ 0 & \text{otherwise,} \end{cases}$$

is a weight function on $G'$. Now, the number of vertices in the new graph $G'$ is

$$\begin{aligned} |G'| &= |G| - |D| \\ &\le n - (d+1) \\ &= (k+1)(d+1) - 1 - (d+1) \\ &= k(d+1) - 1, \end{aligned}$$

and the weight of $W'$ is

$$\begin{aligned} \psi(W') &= w - s \\ &= k(x-1) + 1 - s \\ &\ge k(x-1) + 1 - (x-1) \\ &= (k-1)(x-1) + 1. \end{aligned}$$

By the induction hypothesis, there is a path of weight at least $x$ in $G'$ labelled by $\psi'$. This gives a path of weight $x$ in $G$ labelled by $\psi$ since $\psi(v) \ge \psi'(v)$ for all vertices in $G$. This completes the induction step and so there is a path of weight at least $x$ in $G$, as required. $\square$

We now generalise Theorem 2.4 for the case where just the degree sum, $d(u)+d(v)$, of two non-adjacent vertices $u$ and $v$ of $W$ is known, instead of the minimum degree of $W$. The proof is very similar to that of Theorem 3.1.

**Theorem 3.3** *Let $G$ be a graph on $n$ vertices, containing a set $W$ of $w$ vertices of $G$ such that each pair $w_1$ and $w_2$ of non-adjacent vertices of $W$ satisfies $d(w_1) + d(w_2) \geq c$. If $c = 2d$ is even, set $k = \lfloor n/(d+1) \rfloor$ : if $c = 2d+1$ is odd, set $k = \lfloor (n+1)/(d+2) \rfloor$. Then there is a path through at least $\lceil w/k \rceil$ vertices of $W$.*

**Proof.** The proof follows from the proof of Theorem 2.4 as all the counting arguments concern the endpoints of a path, which are not adjacent, and so the degree condition is sufficient for the proof. $\square$

We conclude this section by considering possible extensions of Theorem 3.1 to the case where $w > n$. In this case we have not succeeded in finding the correct value of $\eta(n, w, d)$ but we do have the following upper bounds.

**Theorem 3.4** $\eta(n, w, d) \leq \left\lceil \dfrac{w}{\lfloor n/(\lceil w/n \rceil d + 1) \rfloor} \right\rceil.$

**Proof.** A graph constructed in the following way has components with maximum weight less than or equal to the bound given and hence cannot contain a path of greater weight than this.

We define the non-negative integers $\lambda$ and $\zeta$ by $w = (\lambda - 1)n + \zeta$, and $\zeta \leq n-1$. So $\lambda = \lceil w/n \rceil$. The graph is made up of components which are all complete graphs of order at least $\lambda d+1$. The vertices and weight are divided evenly amongst these components. This gives $\kappa = \lfloor n/(\lambda d+1)\rfloor$ components of weight at most $\lceil w/\kappa\rceil$. Hence there is no path of weight greater than $\lceil w/\kappa\rceil$ in $G$, as required. $\square$

In fact, if we put a restriction on the value of $w$, we can do a little better.

**Theorem 3.5** *Let* $\lambda = \lceil w/n\rceil$ *and* $\kappa = \lfloor n/(\lambda d+1)\rfloor$. *Suppose that*

$$w \le (n-\kappa(\lambda d+1))\left\lfloor \frac{n-\kappa(\lambda d+1)-1}{d} \right\rfloor + \kappa\lambda(\lambda d+1),$$

*then*

$$\eta(n,w,d) \le \left\lceil \frac{w-(n-\kappa(\lambda d+1))\lfloor (n-\kappa(\lambda d+1)-1)/d\rfloor}{\lfloor n/(\lambda d+1)\rfloor} \right\rceil.$$

**Proof.** Again we construct a graph $G$ with a path of weight no more than the above bound. The graph consists of $\kappa$ components $Q_i$ which are complete graphs on $\lambda d+1$ vertices, and one other component $Q$. The component $Q$ consists of a complete graph on the remaining $n-\kappa(\lambda d+1)$ vertices. Now $Q$ is given its maximum possible weight, namely $(n-\kappa(\lambda d+1))\lfloor (n-\kappa(\lambda d+1)-1)/d\rfloor$. The remaining weight is then distributed amongst the $\kappa$ components, giving the required upper bound for $\eta(n,w,d)$. $\square$

We think it likely that in fact, Theorem 3.4 and Theorem 3.5 give the best possible bounds.

# 4 Connected Graphs: Preliminaries

From now on, we assume that $G$ is a connected graph on $n$ vertices. Again, $W$ is a specified subset of $w$ vertices of $G$ of minimum degree $d$. We define the *weight* $w_A$ of a subset $A$ of the vertices of $G$ to be $|A \cap W|$. We recall that a maximum path is a path of maximum weight and minimum length subject to that weight. We define $\mu(n, w, d)$ to be the minimum weight of a maximum path over all connected graphs $G$ on $n$ vertices with $w$ specified vertices of minimum degree $d$, that is,

$$\mu(n, w, d) = \min_{G} \max_{P \subseteq G} w_P$$

where $P$ is a path in a connected graph $G$.

In this chapter we continue the discussion, begun in Chapter 2, of paths through specified vertices in connected graphs. We consider extremal graphs and obtain an exact value of $\mu(n, w, d)$ for large $w$. Then we give an indication of how Theorem 2.6 will be proved.

**Theorem 2.6** *Let $n$, $w$ and $d$ be integers. Let $G$ be a connected graph on $n \geq 2d+1$ vertices and let $W$ be a subset of the vertices of $G$ of order $w$, such that the minimum degree of the vertices in $W$ is $d$. Then there is a path through*

$$\min \left\{ d+1, 2 \left\lfloor \frac{w-1}{\lfloor (n-1)/d \rfloor} \right\rfloor + 1 \right\}$$

*vertices of $W$.*

We begin our discussion of the extremal graphs of Theorem 2.6, by introducing some important notation.

- Let $k = \left\lfloor \dfrac{n-1}{d} \right\rfloor$ and and $l = \left\lfloor \dfrac{w-1}{k} \right\rfloor$.

So Theorem 2.6 asserts that $\mu(n, w, d) \geq \min\{d+1, 2l+1\}$. We shall retain this notation throughout the next four chapters.

Now we give families of examples showing that Theorem 2.6 is best possible for many values of the parameters. When $d+1 \leq 2l+1$, we want to find a graph in which there is no path of weight greater than $d+1$. For $w \leq n-d$, we consider a bipartite graph $B(n, w, d)$ on vertex set $V_1 \cup V_2$, as defined in Chapter 2. The maximum number of vertices a path can contain is $2d+1$ of which at most $d+1$ belong to $V_1$ since the path alternates between $V_1$ and $V_2$. Therefore there is no path through more than $d+1$ specified vertices, that is, $\mu(n, w, d) \leq d+1$.

For $w \geq n-d+1$, the vertices of $W$ can no longer all be contained in $V_1$. We consider $B(n, w, d)$ as defined for $w \geq n-d+1$. As $|V_2| = d$ and $V_1 \subset W$, there are $n-d$ specified vertices in $V_1$, and thus $w-n+d$ specified vertices in $V_2$. Again, the maximum length of a path $P$ is $2d+1$, of which at most $d+1$ vertices belong to $V_1$, and $d$ to $V_2$. As $V_1 \subset W$, the path contains $d+1$ weighted vertices from $V_1$. Of the vertices of $V_2$ on the path, at most $w-n+d$ are weighted, giving a path of weight at most $w-n+2d+1$, so $\mu(n, w, d) \leq w-n+2d+1$. (See Figure 6.) We will next show that this bound is tight, that is, there is a path of weight $w+2d+1-n$ in any graph with $w \geq n-d+1$, and $n \geq 2d+1$.

**Theorem 4.1** *If* $w \geq n-d+1$ *then* $\mu(n, w, d) = \min\{w, w+2d+1-n\}$.

**Proof.** Let $P$ be a maximum path. If $w_P \geq w+2d+1-n$ we are done so we assume that $w_P \leq w+2d-n$ and $w_P < w$. Let $x_1$ and $x_t$ be the endpoints of $P$, which are weighted, by the definition of a maximum path. Suppose there is a cycle containing all the weighted vertices of $P$. Then, by connectedness, there is a path of greater weight which contradicts the definition of $P$. Therefore $x_1$ and $x_t$ do not share any

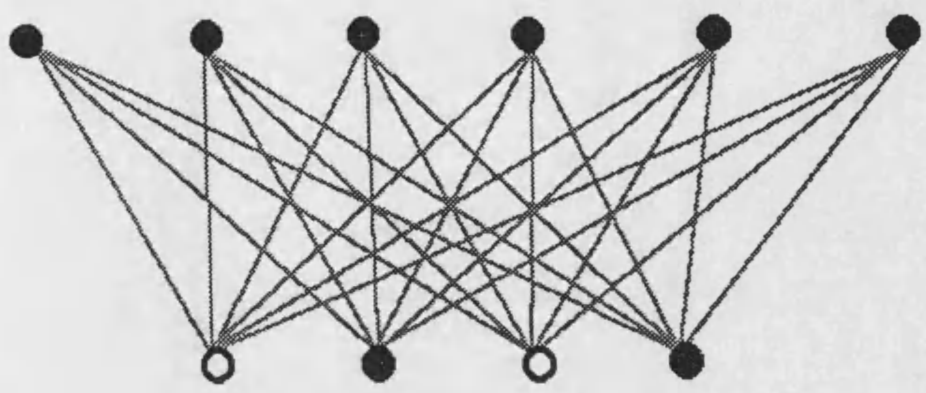


Figure 6: A graph illustrating Theorem 4.1

neighbours in $G \setminus P$ and there is no vertex $v$ on $P$ adjacent to $x_1$ whose predecessor $v^-$ is adjacent to $x_t$. Let $D = \Gamma_P^-(x_1) \cup \Gamma_{G\setminus P}(x_1) \cup \Gamma_P(x_t) \cup \Gamma_{G\setminus P}(x_t) \cup \{x_t\}$. Then $D$ contains at least $2d+1$ vertices. There are no vertices of $W$ in $\Gamma_{G\setminus P}(x_1) \cup \Gamma_{G\setminus P}(x_t)$, or there would be a path of greater weight than $P$. So there are at least $w - w_P \geq w - (w + 2d - n) = n - 2d$ vertices of $W$ in $G \setminus D$. Counting the vertices in $D$ and the weighted vertices in the rest of the graph we have

$$\begin{aligned} n &\geq |D| + |W \setminus D| \\ &\geq (2d+1) + (n - 2d) \\ &= n + 1, \end{aligned}$$

which is a contradiction. Therefore $w_P \geq w + 2d + 1 - n$, as required. $\square$

Note that, if $w = n$ and $d \geq \lceil (n-1)/2 \rceil$, we recover Theorem 2.3.

Now, returning to our discussion of extremal graphs, we consider the special intermediate case where $d = 2l + 1$ and $n \geq 2w + d - 1$. Again we consider a bipartite graph $G$. Let $G = (V_s \cup V_t, E)$ be a bipartite graph on $n$ vertices with $W \subseteq V_t$, and $V_s = V_a \cup V_b$, where $|V_a| = d - 1$ and $|V_b| = w$. We construct $G$ such that there is a complete bipartite graph on $V_a \cup V_t$ and a matching between $W$ and $V_b$. (See Figure 7.) A path in $G$ contains at most $2d - 1$ vertices, of which no more than

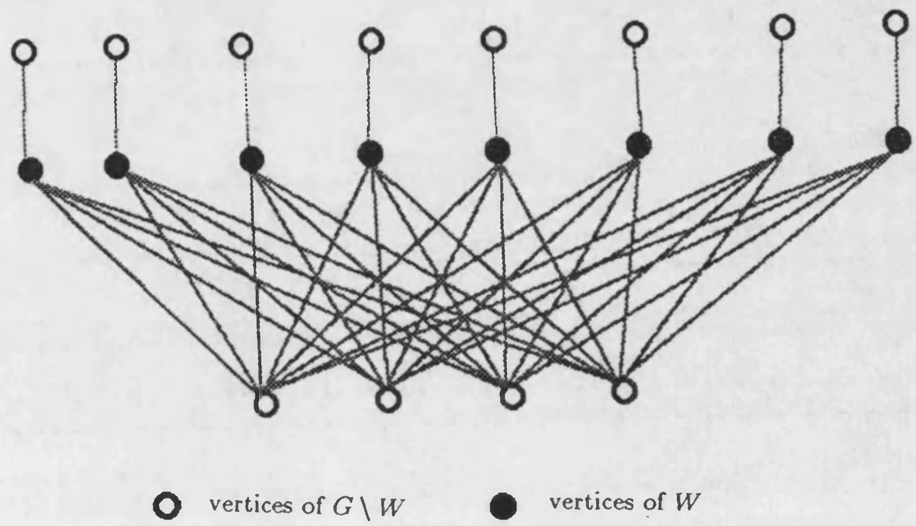


Figure 7: A bipartite extremal graph for $n = 20$, $w = 8$ and $d = 2l + 1 = 5$

$d = 2l + 1$ are specified vertices. Therefore Theorem 2.6 is tight when $d = 2l + 1$ and $n \geq 2w + d - 1$, that is, $\mu(n, w, 2l + 1) = 2l + 1$.

Now we come to the more general case, which includes the previous case, where $2l \leq d + 1$. In this case, we have only been partially successful. We show that $\mu(n, w, d) \leq 2l + 2$. We consider a flower graph $F(n, w, d)$ which consists of $k$ petals $S_i$ each containing either $l$ or $l + 1$ vertices of $W$. There can be no path through more than two of the petals of $F(n, w, d)$. Suppose $w = kl + 1$. Then there is exactly one petal of weight $l + 1$. So the maximum weight of a path is $2l + 1$, that is, $\mu(n, kl + 1, d) \leq 2l + 1$. (See Figure 8.) This gives equality in Theorem 2.6. However, if $w \geq kl + 2$, there are two or more petals of weight

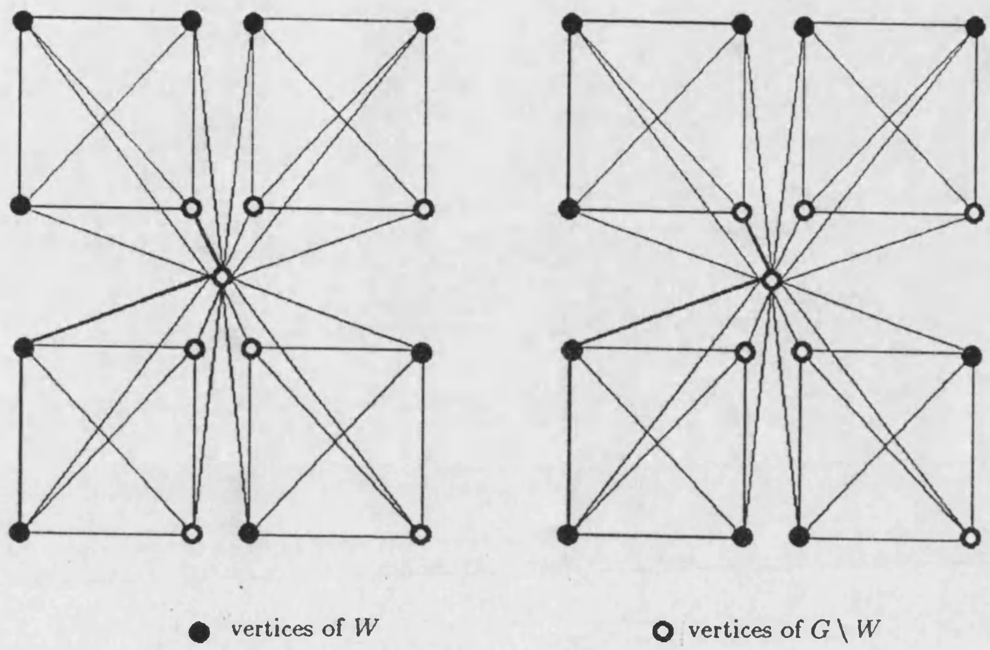


Figure 8: Two flower graphs, $F(17,9,4)$ and $F(17,10,4)$

$l+1$ in $F(n,w,d)$. So the maximum weight of a path in $F(n,w,d)$ is $2l+2$. This implies that $\mu(n,w,d) \leq 2l+2$, whereas Theorem 2.6 just tells us that $\mu \geq 2l+1$. We suspect that, for these values of the parameters, the upper bound of $2l+2$ is correct. (See Figure 8.)

So considering the flower graphs and extremal bipartite graphs, we have an upper bound for $\mu(n,w,d)$. This gives us the following theorem.

**Theorem 4.2** *Suppose* $w \leq n-d$. *If* $d = 2l+1$ *and* $n \geq 2w+d-1$, *then* $\mu(n,w,d) \leq d$. *Otherwise,*

$$\mu(n,w,d) \leq \min\left\{\left\lceil \frac{w}{\lceil n/d \rceil - 1} \right\rceil + \left\lceil \frac{w-1}{\lceil n/d \rceil - 1} \right\rceil, d+1\right\}.$$

We make the following conjecture.

**Conjecture 4.3** *The bounds in Theorem 4.2 are best possible.*

The upper bound in Theorem 4.2 differs from the lower bound in Theorem 2.6 by at most one, and then only when $w \geq kl+2$ and $d \geq 2l+2$, or $n < 2w+d-1$.

So to summarise, if $w = kl+1$ or $d \leq 2l+1$, or both $d = 2l+1$ and $n \geq 2w+d-1$, then we have equality in Theorem 2.6. If not then we suspect that the theorem can be improved by one, giving a path through $2l+2$ vertices of $W$.

As is the case for Theorems 2.2 and 2.4, the striking feature of this result is the significant difference in the values of $\mu(n,w,d)$ and $\mu(n+1,w,d)$ when $n = (k+1)d$ and $w < n-d$. This difference increases as $k$ decreases, so when $w$ is large compared with $k$, we have our most significant difference. This can easily be seen by considering the two flower graphs $F((k+1)d,w,d)$ and $F((k+1)d+1,w,d)$. The additional vertex enables us to form one more petal in $F((k+1)d+1,w,d)$ than in $F((k+1)d,w,d)$, which reduces the length of the path from $2\left\lfloor \frac{w-1}{k} \right\rfloor + 1$ to $2\left\lfloor \frac{w-1}{k+1} \right\rfloor + 1$. (See Figure 9.)

We now consider how we will prove Theorem 2.6. First we restate it as Theorem 4.4, in order to facilitate the proof.

**Theorem 4.4** *If $n$, $k$, $l$, $w$, $d$ are natural numbers such that $n \leq (k+1)d$, $w \geq kl+1$ and $k \geq 2$, then $\mu(n,w,d) \geq \min\{2l+1, d+1\}$.*

The proof is by induction on $k$. It is a slightly unusual induction as we require two base cases. This is because, in the induction step, we remove a set of vertices from

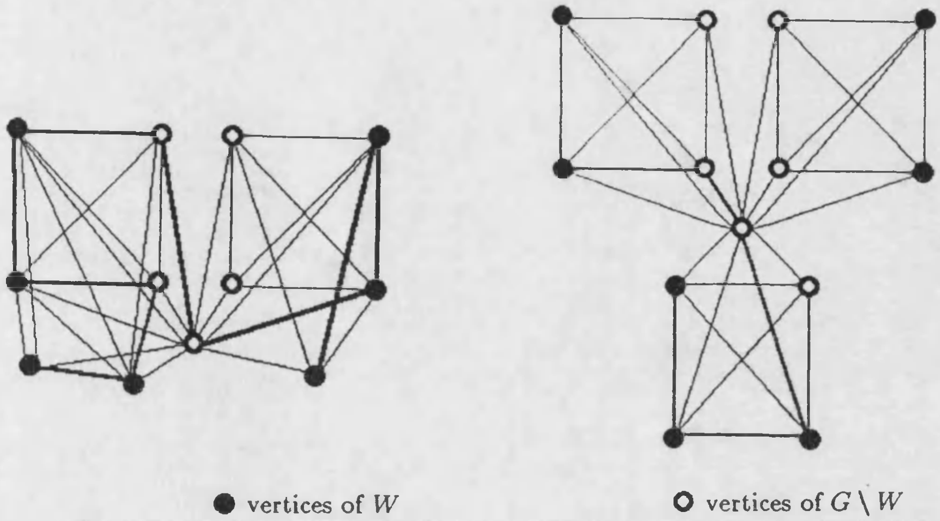


Figure 9: The flower graphs $F(12,7,4)$ and $F(13,7,4)$

$G$ of size at least $2d$ and weight at most $2l$, which decreases $k$ by 2. The two base cases we consider are the cases where $k = 2$ and $k = 3$. The case $k = 2$ follows from Theorem 4.5, below, and 4.1.

**Theorem 4.5** *If $n \leq 3d$ and $w \leq n - d$, then $\mu(n, w, d) = \min\{w, d+1\}$.*

For the next four chapters on graphs, we will consider connected graphs $G$ on $n \leq (k+1)d$ vertices containing a subset $W$ of vertices of degree at least $d$ such that the size of $W$ is $w \geq kl + 1$, where $k \geq 2$. In order to prove the theorems we may and shall assume that all edges are incident with weighted vertices. Indeed, suppose there is an edge $e$ between two non-weighted vertices $v_1$ and $v_2$. Then if $e$ is a bridge we may identify $v_1$ and $v_2$ which preserves the conditions on the graph. If $e$ is not a bridge then we may remove it without disconnecting the graph or changing any of the conditions. We may also assume that $w = kl + 1$ and that $n = (k+1)d$.

In the next chapter will introduce the main tool, a variant of the Hopping Lemma, used in the proof of Theorems 4.5 and 4.4.

# 5 The Hopping Lemma and Variations

The original Hopping Lemma, obtained by Woodall [20] is a very useful tool for problems dealing with long paths and cycles. Jackson [16] and Bondy and Kouider [5] use the Hopping Lemma and variations to prove results about long cycles in regular graphs. Jackson [17] again uses a variation to find cycles through many vertices of maximum degree.

To motivate what follows, we begin by giving the original Hopping Lemma. The Hopping Lemma involves a recursive process that determines which vertices can be swapped on and off a cycle. The starting point is a cycle with some maximal property. The original Hopping Lemma is based on a cycle $C_0$ of maximum length $c$ where $G \backslash C_0$ contains as few components as possible. Let $v_1 v_2 \dots v_c v_1$ be the vertices around a cycle $C_0$ in a graph $G$ where the suffices of the $v_i$ are reduced modulo $c$. Let $v_i^+$ be the vertex on $C_0$ after $v_i$, and for a set $A$, let $A^+ = \{v_i^+ : v_i \in A\}$. Let $v_0$ be an isolated vertex in $G \setminus C_0$. In the Hopping Lemma, we consider a set of the vertices $v_i$ that can be swapped off the cycle to form a new cycle $C_0'$ with vertex set $V(C_0) \setminus \{v_i\} \cup \{v_0\}$. This set $Y$, the union of the sets $Y_i$ given in Lemma 5.1, forms an independent set of vertices.

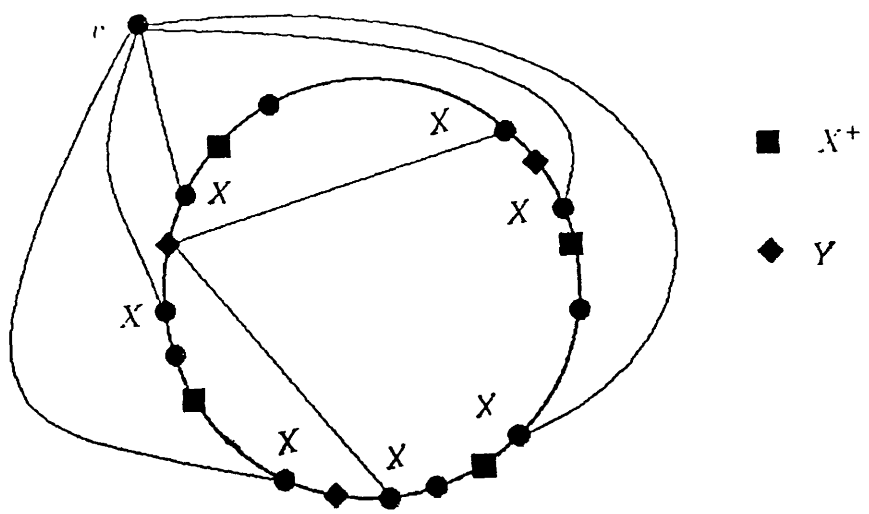


Figure 10: The sets $X$ and $Y$ in Woodall's Hopping Lemma

**Lemma 5.1 (Woodall)** *Suppose that $G$ contains no cycle of length $c+1$, and no cycle $C_0'$ of length $c$ such that $G \setminus C_0'$ contains fewer components than $G \setminus C_0$. Suppose that $v$ is an isolated vertex of $G \setminus C_0$. Let $Y_0 = \emptyset$ and, for $j \geq 1$, define*

$$X_j \quad = \quad \Gamma(Y_{j-1} \cup \{v\})$$

*and*

$$Y_j \quad = \quad \{v_i \in C_0 : v_{i-1}, v_{i+1} \in X_j\}.$$

*So $\Gamma(v) = X_1 \subseteq X_2 \subseteq \ldots$ and $\emptyset = Y_0 \subseteq Y_1 \subseteq \ldots$. Then, for all $j \geq 1$,*

1. $X_j \subseteq C_0$,
2. $X_j \cap X_j^+ = \emptyset$,
3. $X_j \cap Y_j = \emptyset$. □

Let $X = \bigcup_i X_i$, $Y = \bigcup_i Y_i$, and so $X^+ = \bigcup_i X_i^+$. Then we can deduce $X \subseteq C_0$,

$X \cap X^+ = \emptyset = X \cap Y$. The Hopping Lemma tells us that there are no edges between the vertices of $X^+$ and hence between the vertices of $Y$. The sets $X$, $Y$ and $X^+$ are illustrated in Figure 10.

As we are interested only in specified vertices, we use a variation of the Hopping Lemma which concentrates on them. We again define sets $X$ and $Y$ which have similar properties to the sets $X$ and $Y$ defined in Woodall's Hopping Lemma. We start with a cycle $C$ of maximum weight, $w_C$, and use any weighted vertex $v$ not in $C$ to generate $X$ and $Y$. Our set $Y$ is again a set of vertices that can be swapped off the cycle. By the maximality of $C$, a vertex $u$ of $Y$ must be weighted, that is $Y \subseteq W$, as we can swap the weighted vertex $v$ onto the cycle to replace it. In Woodall's Hopping Lemma, the neighbours of vertices of $Y$ lie on the cycle $C_0$, as $C_0$ is chosen such that it has maximum length and, given such length, there is minimum number of components in $G \setminus C_0$, [20]. However in our case, we cannot guarantee that the neighbours of $Y$ lie on the cycle. Therefore we have to define our set $X_j$ to be the neighbours on $C$ of vertices of $Y_{j-1} \cup \{v\}$. Another difference between Woodall's and our Hopping Lemmas is our consideration of vertices following $X$ on the cycle. Woodall's Hopping Lemma considers the vertex $v_{i+1}$ which follows $v_i \in X$, whereas we are more interested in the next *weighted* vertex after $v_i$ on $C$.

Now we will consider this variation of the Hopping Lemma more formally. Let $C$ be a cycle of maximum weight $w_C$ and of minimum length given $w_C$. We call such a cycle a *maximum cycle.* Let $C$ be the cycle $v_1v_2 \dots v_cv_1$ where the suffices of the $v_i$ are reduced modulo $c$. For a set $D$ of vertices, we define $v_i^{+D}$ to be the next vertex of $D$ on $C$ after $v_i$ and $v_i^{-D}$ to be the last vertex of $D$ on $C$ before $v_i$. Typically $D$ will be either $V$, $W$ or $X$. For simplicity we write $v^+$ and $v^-$ in place of $v^{+V}$ and $v^{-V}$.

In Woodall's Hopping Lemma, $X_1$ is defined as the set of neighbours of an isolated vertex in $G\backslash C$. Instead of using such an isolated vertex, we use one of the following sets, (1) a weighted vertex $v_0$ in $G \backslash C$, (2) all the weighted vertices in a component of $G \setminus C$, (3) the weighted vertices in all the components of $G \setminus C$. Suppose $X_1$ is given by the set of neighbours on $C$ of a set $A$. Then we say that $X$ and $Y$ are *generated* by $A$, and write $X = X(A)$ and $Y = Y(A)$. We will prove our version of the Hopping Lemma using a weighted vertex $v_0$ to generate $X = X(v_0)$ and $Y = Y(v_0)$ and show how the other two versions can be obtained from this. We define $Y_j$ and $X_j$ as follows. (See Figure 11.)

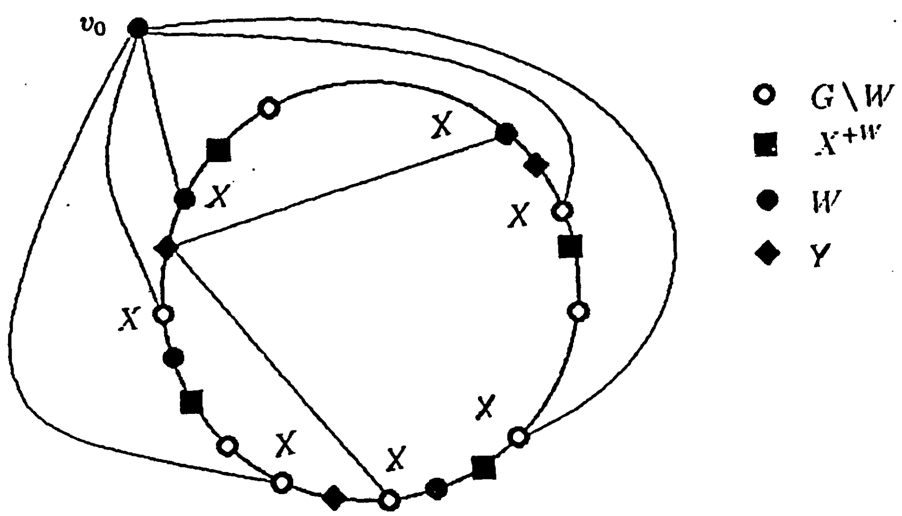


Figure 11: The sets $X$ and $Y$ in the weighted version of the Hopping Lemma

Let $Y_0 = \emptyset$ and, for $j \geq 1$, define

$$X_j = \Gamma_C(Y_{j-1} \cup \{v_0\})$$

and

$$Y_j = \{v_i \in C : v_{i-1}, v_{i+1} \in X_j\}.$$

We let $X = \bigcup_j X_j$ and $Y = \bigcup_j Y_j$. Therefore

$$X_1 \subseteq X_2 \subseteq X_3 \ldots \subseteq X$$

and

$$Y_0 \subseteq Y_1 \subseteq Y_2 \ldots \subseteq Y.$$

We define some more notation. Let

$$\begin{aligned}
X^+ &= \{x_{i+1} : x_i \in X\}, \\
X^{+W} &= \{x_i^{+W} : x_i \in X\}, \\
X^{-W} &= \{x_i^{-W} : x_i \in X\}, \\
Z^+ &= X^{+W} \setminus Y, \\
Z^- &= X^{-W} \setminus Y.
\end{aligned}$$

Let $x = |X|$ and $y = |Y|$. Note that the definitions of $X$ and $Y$ depend on the choice of $v_0$.

The version of the Hopping Lemma that we use is given next.

**Lemma 5.2** *Let $C$ be a maximum cycle and let $Y$ and $X$ be defined as above. Then,*

1. $X \cap X^+ = \emptyset$,
2. $X \cap Z^+ = \emptyset$,
3. $X \cap Y = \emptyset$,
4. $Y \subseteq W$.

We prove this in a different form for which we need the following definitions. Let $P = p_1p_2\dots p_t$ be a path. Then we let $(p_i, p_j)$ be the section of the path $p_{i+1}p_{i+2}\dots p_{j-1}$ and $[p_i, p_j]$ the section $p_ip_{i+1}\dots p_j$, and so on. Sections of cycles are defined similarly. Let $U = \Gamma_{G\setminus C}(v_0)$.

For $h \in \mathbf{N}$, we call a path $P_h = p_1p_2\dots p_z$ in $G \setminus \{v_0\}$ an *h-extendable* path if it satisfies the following three conditions.

1. $C \cap W \subseteq P_h$,
2. $p_1, p_z \in X_h \cup U$,
3. if $p_i \in Y_l$ where $l \leq h-1$, then $p_{i-1} \in X_l$ and $p_{i+1} \in X_l$.

Let $z_1, z_2, \dots, z_x, z_1$ be the vertices of $X_h$ on $C$ read cyclically. Suppose there is no $h$-extendable path in $G$ and there are two vertices $z_i, z_{i+1} \in X_h$ such that $z_{i+1} \in (z_i, z_i^{+W}]$. Then the path between $z_{i+1}$ and $z_i$ is an $h$-extendable path in $G$, which is a contradiction. Therefore the non-existence of an $h$-extendable path implies that $X_h \cap X_h^+ = \emptyset$, and $X_h \cap X_h^{+W} = \emptyset$. Suppose $v \in Y \setminus W$. As $v$ is not weighted and there are no edges between non-weighted vertices (by assumption), both $v^-, v^+ \in X \cap W$, and so $v^+ \in X^{+W}$. Therefore $v^+ \in X \cap X^{+W}$ which is a contradiction. So to prove Lemma 5.2, we prove the next lemma. The proof of this is almost the same as the proof of a variation of Woodall's Hopping Lemma by Jackson [16].

**Lemma 5.3** *Let $C$ be a maximum cycle. Then, for each $h$, there is no $h$-extendable path.*

**Proof.** The proof is by induction on $h$. Suppose there is a 1-extendable path $P_1 = p_1p_2\dots p_z$. Since $p_1, p_z \in X_1 \cup U$, we can join the path $p_1v_0p_z$ to $P_1$ to create a cycle containing more weighted vertices than $C$, which contradicts the choice of $C$. Therefore there is no 1-extendable path, completing the $h = 1$ case. We now let $h = j \geq 2$. We suppose we have a $j$-extendable path $P_j = p_1p_2 \dots p_z$ and show that this gives us a $(j-1)$-extendable path, which contradicts the induction hypothesis. We need to consider three cases.

1. $p_1, p_z \in X_{j-1} \cup U$. If $P_j$ is a $j$-extendable path then it is also a $(j-1)$-extendable path.

2. $p_1 \in X_{j-1} \cup U$, and $p_z \in X_j \setminus X_{j-1}$ (or vice versa). By the definition of $X_j$, $p_z \in \Gamma(p_m)$ for some $p_m \in Y_{j-1} \setminus Y_{j-2}$, and $p_m \neq p_1$ since $Y_{j-1} \cap X_{j-1} = \emptyset$ and $Y \cap U = \emptyset$, by definition. By assumption $P_j$ is a $j$-extendable path which implies that there is a vertex $x_i \in X_{j-1}$ such that $x_i = p_{m+1}$. Therefore the path
$$P_j' = p_1p_2 \dots p_m p_z p_{z-1} \dots x_i$$
is a $(j-1)$-extendable path since the only vertices which could fail to satisfy condition 3 are $p_m$ and $p_z$, neither of which is contained in $Y_{j-2}$.

3. $p_1, p_z \in X_j \setminus X_{j-1}$. Then $p_1 \in \Gamma(p_l)$ and $p_z \in \Gamma(p_m)$ for some $p_l, p_m \in Y_{j-1} \setminus Y_{j-2}$ and again $p_l \neq p_z$ and $p_m \neq p_1$. Hence both the predecessors and successors on $P_j$ of $p_l$ and $p_m$ are in $X_{j-1}$
If $l \leq m$, then
$$P_j' = p_{l-1} \dots p_2 p_1 p_l p_{l+1} \dots p_m p_z p_{z-1} \dots p_{m+1}$$
is a $(j-1)$-extendable path, while if $l > m$, then
$$P_j'' = p_{m-1} \dots p_2 p_1 p_l p_{l+1} \dots p_z p_m p_{m+1} \dots p_{l-1}$$
is a $(j-1)$-extendable path. This completes the proof.

The proof, in fact, shows slightly more, namely that, if we have an $h$-extendable path we have a 1-extendable path, and so the vertex $v_0$ can be included in a cycle of greater weight in $G$. Also for any vertex $v \in X^{+W}(v_0)$, there is a path $P$ through all the weighted vertices of $C$ with endpoints $v_0$ and $v$.

We now prove a corollary that provides us with information about independent sets of vertices and their neighbours.

**Corollary 5.4** *Let $C = v_1v_2 \dots v_cv_1$ be a maximum cycle. Then the following properties hold.*

1. *$X^{+W}$ and $X^{-W}$ are independent sets of vertices. If $v_{i-1} \in X$ but $v_i \notin X^{+W}$ then $v_i$ is not adjacent to any $v_j \in X^{+W}$. If $v_{i+1} \in X$ but $v_i \notin X^{-W}$ then $v_i$ is not adjacent to any $v_j \in X^{-W}$.*

2. *Given $v_i, v_j \in Z^+$ or $v_i, v_j \in Z^-$ there does not exist $v_m \in \{v_{i+2}v_{i+3} \dots v_{j-1}\}$ such that $v_i$ is adjacent to $v_m$ and $v_j$ to $v_{m-1}$.*

3. *No component of $G \setminus (C \cup \{v_0\})$ contains vertices adjacent to two vertices of $X^{+W}$ or two vertices of $X^{-W}$.*

4. *In the case where $C$ is of weight one less than the path $P$ of greatest weight, that is $w_C = w_P - 1$, no vertex of $X^{+W}$ or $X^{-W}$ is adjacent to a weighted component or a weighted vertex in $G \setminus C$.*

Proof.

1. First we consider two vertices $y_i, y_j \in Y$. Suppose there is an edge between $y_i$ and $y_j$. Then $y_j \in X$, which contradicts Lemma 5.2(3) as $Y \cap X = \emptyset$. If $y_j \in Y$ and $p_i \in Z^+$ or $p_i \in Z^-$, and there is an edge between $p_i$ and $y_j$, then $p_i \in X$, contradicting Lemma 5.2(2).

   Now we consider two vertices $v_i, v_j \in Z^+$. Suppose $v_i$ is adjacent to $v_j$. Then $v_i, v_j \in X_h^{+W} \setminus Y_h$ for some $h \geq 1$, and the path

   $$P_h = v_i^{-X} \ldots v_{j+1} v_j v_i v_{i+1} \ldots v_j^{-X}$$

   is $h$-extendable and so contradicts Lemma 5.3, since the only vertices that could fail to satisfy condition 3 are $v_i$ and $v_j$, neither of which is in $Y$. A similar proof holds if $v_i, v_j \in Z^-$.

   For the second part we suppose that there is a vertex $v_{i-1} \in X_h$ such that $v_i \notin W$ and for some $v_j \in X_h^{+W}$ there is an edge $(v_i, v_j)$. Then the path

   $$v_j^{-X} \ldots v_i^{+W} v_i v_j v_{j+1} \ldots v_{i-1}$$

   is an $h$-extendable path contradicting Lemma 5.3. Similarly if $v_{i+1} \in X$ but $v_i \notin W$.

2. Suppose $v_i, v_j \in Z^+$. Then $v_i, v_j \in X_h^{+W} \setminus Y_h$ for some $h \geq 1$. Suppose $v_j^{-X_h} = v_{j-2}$. Then $v_j$ is not adjacent to $v_{j-2}$ and so $v_m \neq v_{j-1}$. If there is a vertex $v_m$ with the given properties then the path

   $$P_h = v_i^{-X} \ldots v_j v_{m-1} \ldots v_{i+1} v_i v_m v_{m+1} \ldots v_j^{-X}$$

   contradicts Lemma 5.3. A similar proof goes through if $v_i, v_j \in Z^-$.

3. As there are no edges between unweighted vertices, components in $G \setminus C$ which have weight zero consist of an isolated vertex. Therefore one of the two cases we consider is the case where two vertices of $X^{+W}(v_0)$ are adjacent to a single vertex in $G \setminus C$. This vertex may be weighted or not. The second case is when two vertices $v_1$ and $v_2$ in a weighted component $Q$ are adjacent to two vertices of $X^{+W}$.

   (i) $u \in G \setminus (C \cup \{v_0\})$. First we suppose that $u$ is adjacent to two vertices $y_i$ and $y_j$ in $Y$. We form a new graph $G'$ by removing $u$ from $G$ and adding the edge $(y_i, y_j)$. Now in $G'$, $C$ is still a maximum cycle. Also $y_i, y_j \in Y(v_0)$, so as in part 1, we have a contradiction to Lemma 5.2(3). Similarly if $u$ is adjacent to a vertex of $Z^+$ or $Z^-$, and a vertex of $Y$, we have a contradiction to Lemma 5.2(2), and if $u$ is adjacent to two vertices of $Z^+$ or to two vertices of $Z^-$ then we have a contradiction to Corollary 5.4(1). A similar proof holds if $v_i, v_j \in Z^-$.

   (ii) Let $v_1, v_2 \in Q$ where $Q$ is a weighted component in $G \setminus C$. Suppose there are edges $(v_1, y_i)$ and $(v_2, y_j)$. Then we contract $Q$ to a single vertex, and continue as in (i). This completes the proof of part 3.

4. Let $u$ be any vertex in the same component as $v_0$. Then there ia a path from $v_0$ to $u$. Suppose that $u$ is adjacent to $v_i \in X^{+W}$. Now we identify $u$ and $v_0$, so there is a pair of vertices, namely $v_i, v_i^{-X} \in X$ which contradicts Lemma 5.2(2) or (3). A similar proof works if $v_i \in X^{-W}$.

   Now suppose $u$ ia a vertex in the same component as $u_t$ where $u_t$ is a weighted vertex in $G \setminus C$ and $u$ is adjacent to $v_i \in X^{+W}$ ($u$ and $u_t$ may be the same vertex — if not, identify them). Suppose we identify $v_0$ and $u_t$. Then we are in the case above and have a cycle of greater weight. So, if we add an edge $e$ between $v_0$ and $u_t$ in our original graph we have a cycle through $C$, $v_0$ and

$u_t$ which gives us a cycle of weight $w_C + 2$. Therefore, by removing $e$ we have a path of weight $w_C + 2$ which is a contradiction. A similar proof works if $v_i \in X^{-W}$. □

We also prove that Lemma 5.2 and Corollary 5.4 can be used in the case where $X = X(Q \cap W)$ and $Y(Q \cap W)$ are generated by all the weighted vertices in a component.

**Corollary 5.5** *Suppose that $X = X(Q \cap W)$ and $Y = Y(Q \cap W)$ are generated by all the weighted vertices in a component $Q$. Then*

1. $X(Q \cap W) \cap X^{+}(Q \cap W) = \emptyset$,
2. $X(Q \cap W) \cap Z^{+}(Q \cap W) = \emptyset$,
3. $X(Q \cap W) \cap Y(Q \cap W) = \emptyset$,
4. *$X^{+W}(Q \cap W)$ and $X^{-W}(Q \cap W)$ are independent sets of vertices, and if $v_{i-1} \in X(Q \cap W)$ but $v_i \notin W$ then $v_i$ is not adjacent to any $v_j \in X^{+W}(Q \cap W)$. Similarly for $v_{i-2}$ and vertices of $X^{-W}(Q \cap W)$.*
5. *Given $v_i, v_j \in Z^{+}(Q \cap W)$ or $v_i, v_j \in Z^{-}(Q \cap W)$ there does not exist $v_m \in \{v_{i+2}, v_{i+3}, \ldots, v_{j-1}\}$ such that $v_i$ is adjacent to $v_m$ and $v_j$ to $v_{m-1}$.*
6. *There is at most one neighbour of vertices of $G \setminus (C \cup \{Q \cap W\})$ in the set $X^{+W}(Q \cap W)$. Similarly, $|\Gamma(G \setminus (C \cup \{Q \cap W\})) \cup X^{-W}(Q \cap W)| \leq 1$.*
7. *In the case where $C$ is of weight one less than the path $P$ of greatest weight, that is $w_C = w_P - 1$, no vertex of $X^{+W}(Q \cap W)$ or $X^{-W}(Q \cap W)$ is adjacent to a weighted vertex in $G \setminus C$ or a neighbour in $G \setminus C$ of a weighted vertex in $G \setminus C$.*

**Proof.** Let $C$ be a maximum cycle in $G$. Suppose we identify all the weighted vertices in a component to a single weighted vertex $v_0$, forming a graph $G'$. Suppose there is a cycle of weight at least $w_C + 1$ in $G'$. Then this cycle must include $v_0$. By the connectedness of $Q$ the vertex $v_0$ in $G'$ can be replaced by either a weighted vertex or a path of weight at least two, so there is a cycle in $G$ of weight at least $w_C + 1$ which contradicts the maximality of $C$. Therefore $C$ is a maximum cycle in $G'$. Then we generate $X(v_0)$ and $Y(v_0)$ in $G'$. These are exactly the sets $X(Q \cap W)$ and $Y(Q \cap W)$ in $G$. Now Lemma 5.2 and Corollary 5.4 apply to the graph $G'$ and the sets $X(v_0)$ and $Y(v_0)$. Since all the vertices contracted to $v_0$ are weighted, $v_0$ can be replaced by either a weighted vertex of $Q$, or by a path with weighted endpoints, giving the required results. $\square$

The other set we use to generate the sets $X$ and $Y$ is the set of all the weighted vertices in $G \setminus C$. We only do this in the case where $w_C = w_P - 1$, that is, a cycle of greatest weight has weight one less than a path of greatest weight.

**Corollary 5.6** *Let $C$ be a maximum cycle and $P$ a maximum path, where $w_C = w_P - 1$. Then, the previous corollary applies for $X(W \setminus C)$ and $Y(W \setminus C)$.*

**Proof.** Suppose we identify all the weighted vertices in $G \setminus C$ to form a single weighted vertex $v_0$, and a graph $G'$. Then if there is a cycle in $G'$ of weight $w_C + 1$, there is either a cycle of weight $w_C + 1$ in $G$, or a path of weight $w_P + 1$, which are both contradictions. Therefore the conditions applying to $X(v_0)$ and $Y(v_0)$ in $G'$ also apply to $X(W \cap G \setminus C)$ and $Y(W \cap G \setminus C)$. $\square$

For the remaining chapters on graphs, we generate $X$ and $Y$ using a fixed set of vertices of one of the three types given above. Given a maximum cycle $C$, a *valid set* is either

1. $v$, a weighted vertex in $G \setminus C$, or

2. $Q \cap W$, all the weighted vertices in a component $Q$ of $G \setminus C$, or

3. if $w_C = w_P - 1$, $W \cap G \setminus C$, that is, all the weighted vertices not on the cycle.

Now we will introduce some lemmas which are based on the variations of the Hopping Lemma that we have just proved. Most of the proofs in the following chapters on graphs rely on counting the vertices in various disjoint sets. The most common form of the inequalities used is given by the following lemma.

**Lemma 5.7** *Let $C$ be a maximum cycle and $A$ any valid set in $G \setminus C$. Then*

$$\begin{aligned} n \;\geq\; & \textit{size of union of some subset } S \textit{ of components of } G \setminus C \\ & + i(\textit{number of neighbours of } X^{+W}(A) \textit{ in } G \setminus C) \\ & + c, \end{aligned}$$

*where $i \in \{0,1\}$ and $c = |C|$. If $i = 0$, all the components are included in $S$. If $i = 1$, the subset $S$ of components is made up of those components which contain no neighbours of $X^{+W}(A)$.* □

In the above lemma if $i = 1$, the last two terms often combine conveniently, as will be seen in Corollary 5.10. But first we will look at the neighbours of $X^{+W}(A)$.

**Lemma 5.8** *Let $C$ be a maximum cycle with $|C| = c$ and let $A$ be a valid set. Suppose $x \geq 2$, where $x = |X(A)|$. Let $x_a$ and $x_b$ be two vertices of $X^{+W}(A)$ such that $d(x_i) \geq \max\{d(x_a), d(x_b)\}$ for all $x_i \in X^{+W}(A) \setminus \{x_a, x_b\}$. Then*

$$\begin{aligned} & \left|\Gamma\left(X^{+W}(A)\right) \cup C\right| \\ & \geq \sum_{x_i \in X^{+W}(A) \setminus \{x_a, x_b\}} (d(x_i) - \lfloor c/2 \rfloor) + d(x_a) + d(x_b) + \sum_{y_j \in Y(A)} (\lfloor c/2 \rfloor - x). \end{aligned}$$

**Proof.** First we consider the neighbours of vertices of $Y$. By definition, vertices of $Y$ are only adjacent to vertices of $X$ on $C$. So the degree of a vertex $y_i$ in $G \setminus C$ is at least $d(y_i) - x$. No two vertices of $Y$ contain neighbours in the same component, by Corollary 5.4(3), Corollary 5.5(6) or Corollary 5.6, depending upon $A$. So the total number of neighbours in $G \setminus C$ of vertices of $Y$ is at least $\sum_{y_j \in Y} (d(y_i) - x)$.

Now consider neighbours of vertices of $Z^+(A)$. For any $z_i, z_j \in Z^+(A)$, let $[z_i, z_j]$ be the section of the cycle $z_i z_{i+1} \dots z_j$ and let $[z_j, z_i]$ be the section $z_j z_{j+1} \dots z_{i-1} z_i$. Then by Corollary 5.4(2),

$$\Gamma_{[z_i,z_j]}(z_j) \cap \Gamma^-_{[z_i,z_j]}(z_i) = \emptyset = \Gamma_{[z_j,z_i]}(z_i) \cap \Gamma^-_{[z_j,z_i]}(z_j).$$

So these four sets are pairwise disjoint and their union, which is the same size as the set of edges that $z_i$ and $z_j$ send to $C$, has size at most $c = |C|$. This gives

$$d_C(z_i) + d_C(z_j) \leq c.$$

Let $m$ be the maximum degree of a vertex of $Z^+(A)$ on $C$. Then for $z \in Z^+(A)$, say, $d_C(z) = m$. This means that the other vertices of $Z^+(A)$ have degree at most $\min\{m, c - m\} \leq \lfloor c/2 \rfloor$ on $C$. Therefore each vertex $z_s$ of $Z^+(A)$ has at least degree $d(z_s) - (c - m)$ in $G \setminus C$. So the number of neighbours of vertices of $Z^+(A)$ in $G \setminus C$ is at least $\sum_{z_i \in Z^+ \setminus \{z\}} (d(z_i) - c + m) + d(z) - m$. There are two cases.

1. $m \geq \lceil c/2 \rceil$. The minimum number of neighbours in $G \setminus C$ of a vertex of $Z^+(A)$ occurs when $m$ is at its minimum, $\lceil c/2 \rceil$. So the number of neighbours is at least

$$\sum_{z_i \in Z^+(A) \setminus \{z\}} (d(z_i) - c + \lceil c/2 \rceil) + d(z) - \lceil c/2 \rceil$$
$$= \sum_{z_i \in Z^+(A)} (d(z_i) - \lfloor c/2 \rfloor) + \lfloor c/2 \rfloor - \lceil c/2 \rceil .$$

2. $m < \lceil c/2 \rceil$. Then each vertex $z_i$ of $Z^+(A)$ has degree in $G \setminus C$ of at least $d(z_i) - \lceil c/2 \rceil + 1$ so there are at least

$$\sum_{z_i \in Z^+(A)} (d(z_i) - \lceil c/2 \rceil + 1) \geq \sum_{z_i \in Z^+(A)} (d(z_i) - \lfloor c/2 \rfloor) + \lfloor c/2 \rfloor - \lceil c/2 \rceil$$

neighbours of vertices of $Z^+(A)$ in $G \setminus C$.

Hence the number of vertices on $C$ and neighbours of $X^{+W}(A)$ in $G \setminus C$ is

$$\begin{aligned}
&|\Gamma_{G \setminus C}(X^{+W}(A)) \cup C| \\
&\geq \sum_{z_i \in Z^+} (d(z_i) - \lfloor c/2 \rfloor) + \sum_{y_j \in Y} (d(y_j) - x) + c \\
&\geq \sum_{x_i \in X^{+W}} (d(x_i) - \lfloor c/2 \rfloor) + \sum_{y_j \in Y} (\lfloor c/2 \rfloor - x) + c \\
&\geq \sum_{z_i \in X^{+W}(A) \setminus \{x_a, x_b\}} (d(z_i) - \lfloor c/2 \rfloor) + \sum_{y_j \in Y} (\lfloor c/2 \rfloor - x) + d(x_a) + d(x_b),
\end{aligned}$$

as required. □

We also use this result when the set $W$ has minimum degree $d$. The following two corollaries give it in a form that is easy to use.

**Corollary 5.9** *Let $C$ be a maximum cycle, $A$ a valid set, $x = |X(A)|$ and $y = |Y(A)|$. Suppose the minimum degree of the vertices of $W$ is $d$, and $x \geq 2$. Then the number of vertices on $C$ and neighbours of vertices of $X^{+W}$ is at least*

$$y(d - x) + (x - y - 2)(d - \lfloor c/2 \rfloor) + 2d.$$

**Proof.** By putting $d(x_i) \geq d$, $y = |Y|$ and $x = |X|$ into Lemma 5.8 we obtain

$$\begin{aligned}
&\left|\Gamma\left(X^{+W}(A)\right) \cup C\right| \\
&\geq \sum_{x_i \in X^{+W}(A) \setminus \{x_a, x_b\}} \left(d - \lfloor c/2 \rfloor\right) + d + d + \sum_{y_j \in Y(A)} \left(\lfloor c/2 \rfloor - x\right) \\
&\geq (x-2)\left(d - \lfloor c/2 \rfloor\right) + 2d + y\left(\lfloor c/2 \rfloor - x\right) \\
&\geq (x-y-2)\left(d - \lfloor c/2 \rfloor\right) + 2d + y\,(d-x),
\end{aligned}$$

as required. □

**Corollary 5.10** *Let $C$ be a maximum cycle, $A$ be a valid set and $x = |X(A)|$. Suppose the minimum degree of the vertices of $W$ is $d$. Suppose also that $x \geq 2$ and $w_C \leq \min\{2l - j, d - j\}$ for some $j \geq 0$. Then the size $u$ of the union of the set of neighbours of $X^{+W}(A)$ in $G \setminus C$ and the set of vertices in $C$ satisfies the inequality $u \geq 2d + j(x-2)$.*

**Proof.** By Corollary 5.9, the size of the union of the neighbours of $X^{+W}(A)$ in $G \setminus C$ and the vertices on $C$ is at least $y(\lfloor c/2 \rfloor - x) + (x-2)(d - \lfloor c/2 \rfloor) + 2d \geq (x-2)(d - \lfloor c/2 \rfloor) + 2d$, since $\lfloor c/2 \rfloor - x \geq 0$. As there are no edges between unweighted vertices, we have $d - j \geq w_C \geq \lfloor c/2 \rfloor$. So $d - \lfloor c/2 \rfloor \geq j$, and we obtain the result. □

# 6 Degree Sum Conditions on Independent Sets of Three Vertices

In this chapter we will discuss and prove a result (Theorem 6.1) to which Theorem 4.5 is a corollary. We define the *degree sum* of a set of vertices $S_i$ to be the sum of the degrees of the vertices of $S_i$, that is, $\sum_{v \in S_i} d(v)$. Instead of dealing with a set $W$ that has minimum degree $d$, we consider the degree sums of triples $S_i$ of independent vertices from $W$. That is, we consider an Ore-type problem, (see Chapter 9 for more Ore-like results). The condition we use for the vertices of $W$ is that the degree sum of three independent vertices from $W$ is at least $n$. Note that this condition is exactly what is needed to prevent there being three petals of a flower graph. What we prove, essentially, is that, under this degree sum condition, there is a path through at least as many specified vertices as in a bipartite graph $B(n, w, d)$.

Let $\xi(n,w) = \min_{G} \max_{P \subseteq G} w_P$ where $G$ is a connected graph for which the degree sum of each triple $S_i$ of vertices of $W$ is at least $n$, and $P$ is a path in $G$. That is, $\xi(n,w)$ is the maximum weight of a path that can be guaranteed given these conditions. Then we have the following theorem.

**Theorem 6.1** *If $n \leq 4$, then $\xi(n,w) = w$.*

1. *If $w \leq \left\lceil \frac{n}{3} \right\rceil + 1$, then $\xi(n,w) = w$.*
2. *If $\left\lceil \frac{n}{3} \right\rceil + 2 \leq w \leq \left\lfloor \frac{2n}{3} \right\rfloor$ then $\xi(n,w) = \left\lceil \frac{n}{3} \right\rceil + 1$.*
3. *If $w \geq \left\lfloor \frac{2n}{3} \right\rfloor + 1$, and $n \geq 5$, then $\xi(n,w) = w - n + 2\left\lceil \frac{n}{3} \right\rceil + 1$.*

Before we prove this theorem, we describe the structure of graphs satisfying the degree sum conditions, with no path through all the weighted vertices (Theorem 6.2), and discuss other related results. Theorem 6.2 is illustrated in Figure 12.

**Theorem 6.2** *Let $G$ be a graph on $n$ vertices. Suppose that $d(s)+d(u)+d(v) \geq n$ for all independent sets $\{s,u,v\}$ of $W$. Let $C$ be a maximum cycle. Then either there is a path through all the vertices of $W$, or*

1. *the weight of a maximum path $P$ is $w_C + 1$, and*
2. *every component in $G \setminus C$ has weight at most one.*

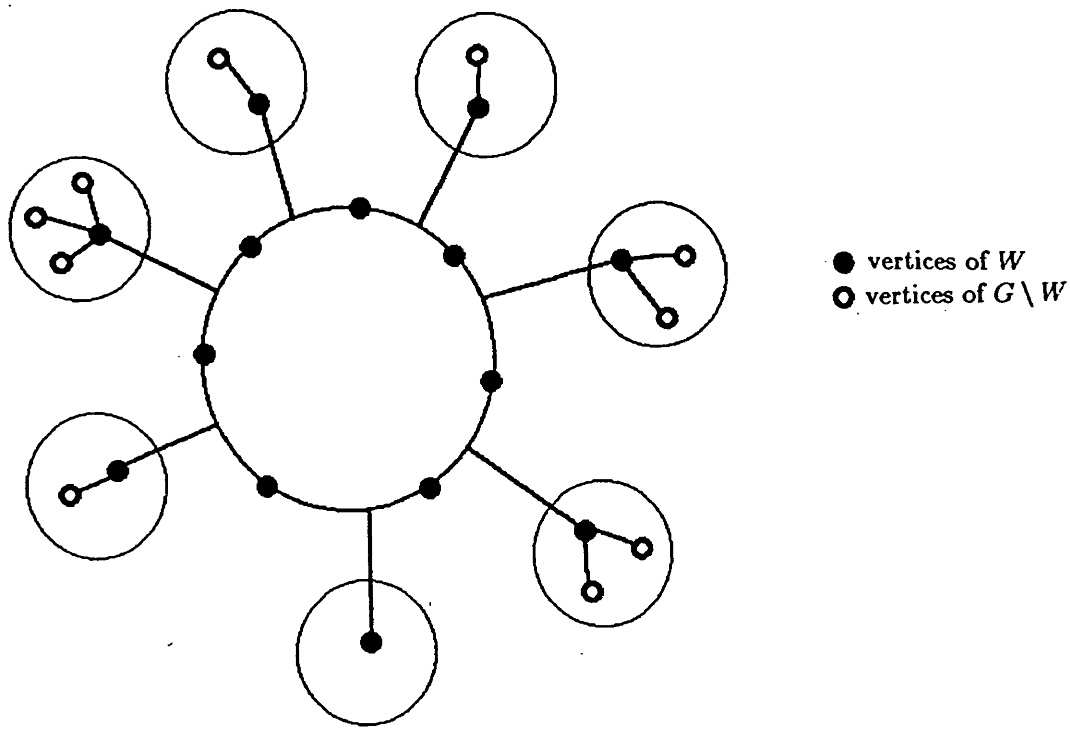


Figure 12: A graph illustrating Theorem 6.2

This theorem is a generalisation of two earlier results. Part (1) is a generalisation of a result of Enomoto, van den Heuvel, Kaneko, and Saito [19], which is recovered when $W = V(G)$. Part (2) is a generalisation of a result of Enomoto, Kaneko and Tuza [10] which was also independently proved by Bauer, (see [10]). This is again recovered from Theorem 6.1 when $W = V(G)$. The result of Enomoto, van den Heuvel, Kaneko and Saito [19] follows fairly readily from the result by Enomoto, Kaneko and Tuza [10] and Bauer (see [10]).

**Theorem 6.3 (Enomoto, Kaneko and Tuza)** *Let $G$ be a connected graph on $n$ vertices. Suppose that $d(s) + d(u) + d(v) \geq n$ for all sets of three independent vertices $s$, $u$, $v$. Let $C$ be a longest cycle. Then $G$ has a Hamiltonian path or $G \setminus C$ consists of isolated vertices.*

**Theorem 6.4 (Enomoto, van den Heuvel, Kaneko and Saito)** *Let $G$ be a connected graph on $n$ vertices. Suppose that $d(s) + d(u) + d(v) \geq n$ for all sets of three independent vertices $s$, $u$, $v$. Let $C$ be a longest cycle, and $P$ a longest path. Then $G$ has a Hamiltonian path or $|P| - |C| = 1$.*

It is easy to see that Theorem 6.3 follows from Theorem 6.4. Indeed, if $G \setminus C$ has a component consisting of more than one vertex, there is clearly a path in $G$ containing at least $|C| + 2$ vertices. However, in the weighted version, the generalisation of Theorem 6.3 does not follow as simply from the generalisation of Theorem 6.4. Let $C = v_1 v_2 \ldots v_c v_1$ be a maximum cycle in $G$. We consider a component $Q$ of $G \setminus C$. The difficulty arises if there is a cut vertex $v \in Q$, such that each component $Q_i$ of $Q \setminus \{v\}$ contains at most one weighted vertex $q_i$. Suppose also that the $Q_i$ are components of $G \setminus \{v\}$. Let $v_i$ be a vertex of $C$ which is adjacent to $v$. Such a vertex $v_i$ exists as $G$ is connected and $v$ is a cut vertex. Then we have a path of maximum weight $w_C + 1$, for example $v_{i+1}v_{i+2}\dots v_iv\dots q_i$ for any $q_i$. (See Figure 13.) However this case does not actually arise, as we will show in the proof of Theorem 6.2.

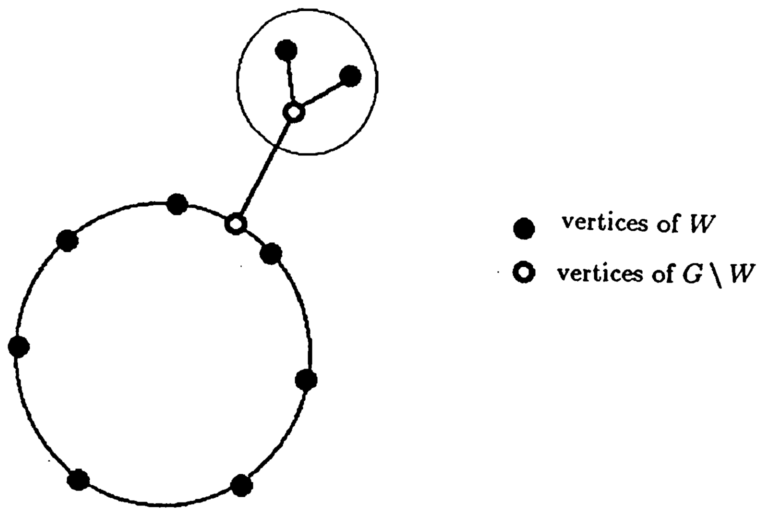


Figure 13: Theorem 6.2

**Proof of Theorem 6.2.** We will first prove part (1), which we use in the proof of part (2).

Let $P = p_1p_2\dots p_t$ be a maximum path in $G$. Suppose that $P$ does not contain all the vertices of $W$. Since $P$ is a maximum path and there is a vertex $v$, say, of $W$ in $G \setminus P$, there is no cycle containing all the vertices of $P$, as by connectedness $v$ could be added to the cycle to produce a path of weight $w_P + 1$. Let $v$ be a weighted vertex in $G \setminus P$. Then $v$, $p_1$ and $p_t$ form an independent set of vertices of $W$, since, if not, then either we have a cycle containing $P$, and so a path of greater weight by connectedness, or there is a path $vP$ again contradicting the maximality of $P$.

Let $A = \Gamma_P^-(p_1)$, $B = \Gamma_P^+(p_t)$, $D = \Gamma_{G\setminus P}(p_1)$ and $F = \Gamma_{G\setminus P}(p_t)$. Then $D \cap F = \emptyset$ and $\Gamma(v) \cap (A \cup B \cup D \cup F) = \emptyset$ or there would be a path of greater weight. Suppose

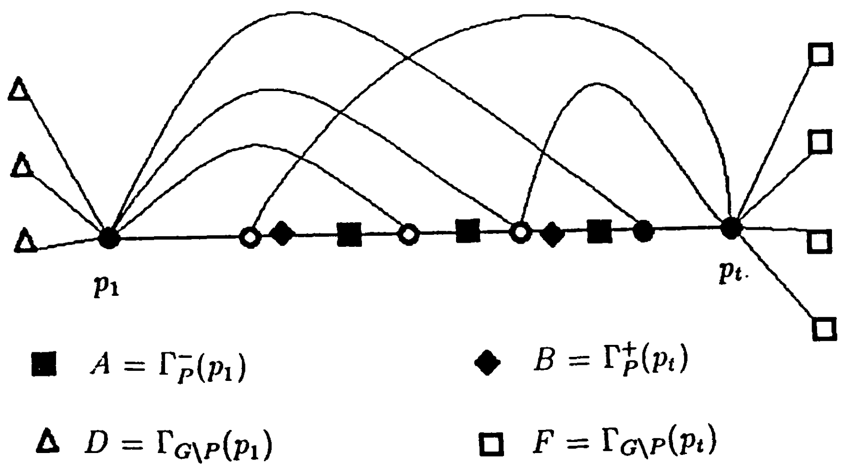


Figure 14: The sets $A$, $B$, $D$ and $F$

$A \cap B = \emptyset$. Then the sets $A$, $B$, $D$ and $F$ are disjoint and $|A| + |D| = d(p_1)$ and $|B| + |F| = d(p_t)$. So, counting the vertices, we have

$$\begin{aligned} n &\geq |A| + |D| + |B| + |F| + |\Gamma(v) \cup \{v\}| \\ &= d(p_1) + d(p_t) + d(v) + 1 \\ &\geq n + 1, \end{aligned}$$

since $p_1$, $p_t$ and $v$ form an independent set of $W$. Therefore $A \cap B \neq \emptyset$, so there is a cycle $C$ through all the weighted vertices of $P$ except one. That is $w_C = w_P - 1$, proving part (1).

Now we prove part (2). Let $C$ be a cycle of maximum weight in $G$, so $C$ has weight $w_C = w_P - 1$, by part (1). Suppose there is a component $Q$ in $G \setminus C$ of weight at least two. There is no path of weight in $Q$ at least two with an endpoint adjacent to $C$, as this contradicts part (1). Therefore there is no weighted vertex of $Q$ adjacent to $C$. Also there is no cycle of weight at least two in $Q$, as again there would be a path of weight $w_C + 2$, contradicting part (1). But by connectedness, a maximum path $P'$ in $Q$ has weight at least two. Let the weighted endpoints of $P'$ be $p'_1$ and $p'_t$. Suppose $p'_1$ and $p'_t$ are adjacent. Then by connectedness there is a path from $p'_1$ to $C$ and therefore a path $P_1$ through all the vertices of $C$ and both $p'_1$ and $p'_t$. Thus $w_{P'_1} \geq w_C + 2$, contradicting part (1). Now, since $p'_1$ and $p'_t$ are not adjacent to $C$, all their neighbours are in $Q$. As there is no cycle containing the weighted vertices of $P'$, the following sets are pairwise disjoint: $\Gamma_{P'}(p'_1)$, $\Gamma^+_{P'}(p'_t)$, $\Gamma_{Q\setminus P'}(p'_1)$ and $\Gamma_{Q\setminus P'}(p'_t)$. Then $|Q| \geq d(p'_1) + d(p'_t) + 1$. Let $v_i$ and $v_i^{+W}$ be two weighted vertices on $C$. Then either $v_i$ or $v_i^{+W}$ has at most one neighbour in $Q$. Indeed, if both $v_i$ and $v_i^{+W}$ have at least two neighbours in $Q$, there is a path $P_0$ in $Q$ of weight at least one, and endpoints adjacent to $v_i$ and $v_i^{+W}$. Thus we obtain a cycle $v_1 \dots v_{i-1} v_i P v_i^{+W} \dots v_c v_1$ of weight greater than that of $C$, contradicting the maximality of $C$. Take $v^* \in \{v_i, v_i^{+W}\}$, such that $d_Q(v^*) \leq 1$. Thus $d_{G\setminus Q}(v^*) \geq d(v^*) - 1$. As $p'_1$ and $p'_t$ are not adjacent to each other or to any vertices on $C$, we have an independent set of vertices $\{p'_1, p'_t, v^*\}$. Now counting the vertices in $Q$ and the neighbours of $v^*$, we obtain

$$\begin{aligned}
n &\geq |Q| + |\Gamma_{G\setminus Q}(v^*) \cup \{v^*\}| \\
&\geq (d(p'_1) + d(p'_t) + 1) + (d(v^*) - 1 + 1) \\
&\geq n + 1
\end{aligned}$$

which is a contradiction. Therefore there are no components of $G \setminus C$ of weight at least two. So, either there is a path through all the vertices of $W$, or each component in $G \setminus C$ is at most weight one, as required. □

Now we come to the proof of Theorem 6.1. We prove this in two parts. First we show that there are graphs with the given parameters in which there are no paths of greater weight. Then we show that a path of the required weight is always possible.

**Theorem 6.5** *1. If* $\left\lceil \frac{n}{3} \right\rceil + 2 \leq w \leq \left\lfloor \frac{2n}{3} \right\rfloor$ *then* $\xi(n, w) \leq \left\lceil \frac{n}{3} \right\rceil + 1$.

*2. If* $w \geq \left\lfloor \frac{2n}{3} \right\rfloor + 2$ *then* $\xi(n, w) \leq w + 1 - n + 2\left\lceil \frac{n}{3} \right\rceil$.

**Proof.** We consider graphs on $n$ vertices where the degree of each of the vertices of $W$ is at least $d = \lceil n/3 \rceil$. For part (1), we consider the bipartite graphs $B(n, w, d)$. For $w \leq \lfloor 2n/3 \rfloor$, we have $W \subseteq V_1$. So there is no path through more than $d + 1 = \lceil n/3 \rceil + 1$ vertices of $V_1$. Hence there is no path of weight greater than $\lceil n/3 \rceil + 1$ as required.

Now, for part (2), we again consider $B(n, w, d)$. This time, $V_1 \subseteq W$ as $w \geq n - d$, and the remainder of the vertices of $W$ are in $V_2$. Hence there is no path through more than $d + 1 = \lceil n/3 \rceil + 1$ weighted vertices of $V_1$ and $w - (n - d) = w - \lfloor 2n/3 \rfloor$ weighted vertices of $V_2$. Hence there is no path of weight more than $w - n + 2\lceil n/3 \rceil + 1$ in $G$, which completes the proof. □

Now we come to the rest of the proof of Theorem 6.1.

**Proof of Theorem 6.1.** First we consider the case where $n \leq 4$. The only connected graph on four vertices which does not have a path through all its vertices is $K_{1,3}$. In $K_{1,3}$ not all the vertices of degree one can be in $W$, as they can not satisfy the degree condition, so we are done.

Let $C$ be a maximum cycle and $P$ a maximum path.
(1) Suppose $P$ does not contain all the weighted vertices (that is, $w_P \leq w - 1$), and $w \leq \lceil n/3 \rceil + 1$. By Theorem 6.2(1), there is a cycle $C$ of weight $w_P - 1 \leq w - 2$. By

Theorem 6.2(2), all the components of $G \setminus C$ have weight at most one. Therefore there are at least two weighted components in $G \setminus C$.

As $C$ is a cycle of weight $w_P - 1$, $W \setminus C$ is a valid set. Let $x = |X(W \setminus C)|$. Suppose $x \geq 2$. Now $X^{+W}(W \setminus C)$ is an independent set, by Corollary 5.6, so any three vertices in $X^{+W}(W \setminus C)$ have degree sum at least $n$. We count the vertices in the components of $G \setminus C$, the vertices on $C$, and the neighbours of vertices of $X^{+W}(W \setminus C)$. The number of vertices in each weighted component $Q_i$ is at least $d(w_i) - x + 1$, where $w_i$ is the weighted vertex in $Q_i$. The number of vertices in $C$ and the set of neighbours of $X^{+W}(W \setminus C)$ is given by Lemma 5.8 with $d(x_a)$ and $d(x_b)$ taken to be minimum over all the vertices of $X^{+W}(W \setminus C)$. Putting $y \geq 0$ in Lemma 5.8 and applying Lemma 5.7 with $i = 1$, we obtain

$$n \geq \sum_{w_f \in W \setminus C} (d(w_f) - x + 1) + \sum_{x_j \in X^{+W} \setminus \{x_a, x_b\}} (d(x_j) - \lfloor c/2 \rfloor) + d(x_a) + d(x_b).$$

As $d(x_a)$ and $d(x_b)$ are minimum, $d(x_j) \geq \lceil n/3 \rceil$ for all other vertices $x_j$ of $X^{+W}(W \setminus C)$ as $X^{+W}(W \setminus C)$ is an independent set. So we have

$$n \geq \sum_{w_f \in W \setminus C} (d(w_f) - x + 1) + d(x_a) + d(x_b) + (x - 2)(\lceil n/3 \rceil - \lfloor c/2 \rfloor).$$

Let $w_g$ and $w_h$ be two vertices in $W \setminus C$ of highest degree. Then we have
$$n \geq (d(w_g) - x + 1) + (d(w_h) - x + 1) + d(x_a) + d(x_b) + (x - 2)(\lceil n/3 \rceil - \lfloor c/2 \rfloor).$$

Let $d(x_m) = \max\{d(w_g), d(w_h), d(x_a), d(x_b)\} \geq \lceil n/3 \rceil$, as the vertices in $W \setminus C$ and in $X^{+W}(W \setminus C)$ form an independent set by Theorem 6.2 and Corollary 5.6. The sum of the degrees of the remaining three independent vertices is at least $n$ so we obtain

$$
\begin{aligned}
n &\geq (d(x_m) - x) + n + (x-2)(\lceil n/3 \rceil - \lfloor c/2 \rfloor - 1) \\
0 &\geq (\lceil n/3 \rceil - x) + (x-2)(\lceil n/3 \rceil - \lfloor c/2 \rfloor - 1).
\end{aligned}
$$

Now since $C$ has weight at most $\lceil n/3 \rceil - 1$, it has length $c$ at most twice this because there are no edges between unweighted vertices. Also $x \leq \lfloor c/2 \rfloor$. Therefore $\lceil n/3 \rceil - \lfloor c/2 \rfloor - 1 \geq 0$ and $\lceil n/3 \rceil - x \geq 1$. Also $x \geq 2$ so we have a contradiction.

So now we have $x \leq 1$. If $x = 1$, let $x_0$ be the vertex in $X(W \setminus C)$, and if $x = 0$ let $x_0$ be a vertex of $C$ with a neighbour $u$, say, in a weighted component. Now no vertex of $W \setminus C$ is adjacent to $x_0^{+W}$, except possibly the unweighted vertex $u$. Therefore $(W \setminus C) \cup \{x_0^{+W}\}$ is an independent set of vertices. Let $w_1$ and $w_2$ be two vertices in $W \setminus C$. Any two of the vertices $w_1$, $w_2$ and $x_0^{+W}$ share at most one neighbour, namely $x_0$ when $x = 1$ and $u$ when $x = 0$, as $w_1$ and $w_2$ are in different components of $G \setminus C$ by Theorem 6.2(2). Therefore

$$
\begin{aligned}
n &\geq |\Gamma(w_1) \cup \Gamma(w_2) \cup \Gamma(x_0^{+W}) \cup \{w_1, w_2, x_0^{+W}\}| \\
&\geq d(w_1) + d(w_2) + d(x_0^{+W}) - 2 + 3 \\
&\geq n + 1,
\end{aligned}
$$

which is a contradiction. Therefore there is a path through all the vertices of $W$ if $w \leq \lceil n/3 \rceil + 1$, completing the first part.

(2) For this part we simply remove $w - \lceil n/3 \rceil - 1$ vertices from $W$ and apply part (1). By Theorem 6.5, this is the best we can do, completing part (2).

(3) Suppose $w \geq \lfloor 2n/3 \rfloor + 1$, $n \geq 5$, and there is no path through $w - n + 2\lceil n/3 \rceil + 1$ vertices of $W$. So $w_C + 1 = w_P \leq w - n + 2\lceil n/3 \rceil$, by Theorem 6.2(1). Therefore $w - w_C \geq n - 2\lceil n/3 \rceil + 1 \geq 2$, since $n \geq 5$.

There are at least $d_{G \setminus C}(w_i) + 1$ vertices in the component of $G \setminus C$ containing the weighted vertex $w_i$, for each $w_i$, and each component contains at most one weighted vertex.

Set $x = |X(W \setminus C)|$. We first consider the rather easy case where $x = 0$, that is, no vertex of $W \setminus C$ has a neighbour on $C$. Let $x_0$ be a vertex on $C$ with a neighbour $u$ in some weighted component of $G \setminus C$. Then we have an independent set $S = (W \setminus C) \cup \{x_0^{+W}\}$, with at most one shared neighbour $u$, as no other vertex of a weighted component can be adjacent to $x_0^{+W}$ without forming a path of weight $w_C + 2$, which contradicts Theorem 6.2(1). We note that $|S| \geq 3$. So each of vertex $v$ of $S$ has at least $d(v) - 1$ neighbours unshared with other vertices of $S$. Therefore, counting the neighbours of vertices of $S$, and the vertices in $S$ itself, we obtain

$$\begin{aligned} n &\geq \sum_{v \in S}(d(v) - 1) + |S| + |\{u\}| \\ &\geq d(w_1) + d(w_2) + d(w_3) + \sum_{v \in S \setminus \{w_1, w_2, w_3\})} d(v) + 1 \end{aligned}$$

for $\{w_1, w_2, w_3\} \subseteq S$, which is an independent set. So $d(w_1) + d(w_2) + d(w_3) \geq n$, and we obtain

$$n \geq n + (w - w_C) + \sum_{u \in S \setminus \{w_1, w_2, w_3\})} d(u)$$

which is a contradiction as $w - w_C \geq 2$ and the sum is non-negative, so the right hand side becomes $n + 2$. This completes the $x = 0$ case.

Now we may assume that $x \geq 1$. Let $m$ be the maximum degree on $C$ of a vertex $u$ in $W \setminus C$. Since $C$ is a maximum cycle there is at least one weighted vertex between every pair of neighbours on $C$ of $u$. Therefore $|C| \geq 2m$. Let $t$ and $q \leq t$ be the degrees in $G \setminus C$ of two vertices in $W \setminus C$ such that $d_{G \setminus C}(v) \leq q$ for all other $v \in W \setminus C$.

Now we consider the case where there are two vertices in $W \setminus C$ with degree sum at least $2\lceil n/3 \rceil$. Therefore $2m + q + t \geq 2\lceil n/3 \rceil$. We count the vertices on $C$ and in each of the components of $G \setminus C$, applying Lemma 5.7 with $i = 0$, to obtain

$$\begin{aligned} n &= |C| + |G \setminus C| \\ &\geq 2m + \sum_{u \in W \setminus C} (d_{G \setminus C}(u) + 1) \\ &\geq 2m + \sum_{u \in W \setminus C} d_{G \setminus C}(u) + n - 2\lceil n/3 \rceil + 1 \\ &\geq 2m + q + t + n - 2\lceil n/3 \rceil + 1. \end{aligned}$$

Now, as stated above $2m + q + t \geq 2\lceil n/3 \rceil$ so we obtain

$$n \geq n + 1,$$

which is a contradiction so no two vertices of $W \setminus C$ have degree sum at least $\lceil n/3 \rceil$.

Now we will consider under which conditions this can occur. Note that $w - w_C \leq 3$, for if not there are two vertices in $W \setminus C$ of degree at least $\lceil n/3 \rceil$, giving a degree sum of at least $2\lceil n/3 \rceil$. This can be seen by considering two subsets $A$ and $B$ of $W \setminus C$, where $B$ does not contain a vertex $v_{max}$ of maximum degree in $A$. Further if $n \equiv 0$ or $2 \pmod 3$, then $w - w_C = 2$, as $\lceil 2n/3 \rceil = 2\lceil n/3 \rceil$.

Again let $S = (W \setminus C) \cup x_0^{+W}$ where $x_0 \in X(W \setminus C)$. Then $S$ is an independent set as a vertex of $W \setminus C$ adjacent to a vertex of $X^{+W}(W \setminus C)$ gives a path of weight $w_C + 2$, which contradicts Theorem 6.2(1). Therefore the sum of degrees of any subset $S_i \in S$ of order three is at least $n$ and so there are two vertices $w_a$ and $w_b$ of $S$ which have degree sum at least $2\lceil n/3 \rceil$, when $|S| = 4$ and $n = 3d + 1$, or when $|S| = 3$ and $n = 3d$ or $n = 3d + 2$. Now we know that not both $w_a$ and $w_b$ are in $W \setminus C$, so therefore, without loss of generality $w_a = x_0^{+W}$. Further $d(x_0^{+W}) \geq \lceil n/3 \rceil$.

We now deal with the cases where either (i) $|S| = 4$ and $n = 3d + 1$, or (ii) $|S| = 3$ and $n = 3d$ or $n = 3d + 2$. In both of these cases, the vertex $x_0^{+W}$ has degree at least $\lceil n/3 \rceil$. Also $c \geq d_C(x_0^{+W}) + m$, where $m$ is again the maximum degree on $C$ of a vertex of $W \setminus C$, as no neighbour of $x_0^{+W}$ on $C$ can be adjacent to a vertex of $X^{+W}(W \setminus C)$. Let $t = \max_{u \in S} d_{G \setminus C}(u)$. Then Now $d_C(x_0^{+W}) + t \geq d(x_0^{+W}) \geq \lceil n/3 \rceil$.

So counting the vertices we have

$$\begin{aligned}
n &= |C| + |G \setminus C| \\
&\geq (m + d_C(x_0^{+W})) + \left( \sum_{u \in S} d_{G \setminus C}(u) + w - w_C \right) \\
&\geq d(x_0^{+W}) + \left( m + \sum_{u \in W \setminus C} d_{G \setminus C}(u) \right) + (n - 2\lceil n/3 \rceil + 1) \\
&\geq (d(x_0^{+W}) - \lceil n/3 \rceil) + \left( m + \sum_{u \in W \setminus C} d_{G \setminus C}(u) - \lceil n/3 \rceil \right) + n + 1,
\end{aligned}$$

but each bracket is positive, so we have $n \geq n + 1$, which is a contradiction. Therefore the only remaining case is when $n = 3d + 1$ and $w - w_C = 2$. Since we have $n - 2\lceil n/3 \rceil \leq w - w_C - 1 = 1$, this implies that $n = 7$.

So we are in the case where $n = 7$ and we have two weighted vertices not on the cycle. So $c \leq 5$. If $x \geq 3$ then we have a contradiction as $c \geq 2x \geq 6$. Therefore $x = 1$ or $x = 2$ as we have already dealt with the case $x = 0$.

We deal with $x = 1$ first. Let $w_1$ and $w_2$ be the vertices of $W \setminus C$, and $x_0 \in X(W \setminus C)$. Then $\{w_1, w_2, x_0^{+W}\}$ is an independent set. As before, $c \geq x + d_C(x_0^{+W})$. We count the vertices on $C$ and the neighbours of $w_1$, $w_2$ and $x_0^{+W}$ which are not on $C$.

Therefore we have

$$\begin{aligned} n &= |C| + |G \setminus C| \\ &\geq x + d_C(x_0^{+W}) + d_{G \setminus C}(w_1) + 1 + d_{G \setminus C}(w_2) + 1 + d_{G \setminus C}(x_0^{+W}) \\ &\geq x + d(w_1) - x + d(w_2) - x + 2 + d(x_0^{+W}) \\ &\geq n + 2 - x \end{aligned}$$

which is a contradiction as $x = 1$.

Now if $x = 2$, there are two vertices $x_0$ and $x_1$ in $X(W \setminus C)$. Similarly to above, we have

$$\begin{aligned} n &= |C| + |G \setminus C| \\ &\geq x + d_C(x_0^{+W}) + d_{G \setminus C}(x_1^{+W}) + d_{G \setminus C}(w_1) + 1 + d_{G \setminus C}(w_2) + 1 + d_{G \setminus C}(x_0^{+W}) \\ &\geq x + d_C(x_0^{+W}) + d(w_1) - x + d(w_2) - x + 2 + d(x_0^{+W}) + d_{G \setminus C}(x_1^{+W}) \\ &\geq n + 2 - x + d_{G \setminus C}(x_1^{+W}). \end{aligned}$$

Since $x = 2$, this implies that $d_{G \setminus C}(x_1^{+W}) = 0$, and we have equality. It also implies that $w_1$ and $w_2$ have two neighbours on $C$. Therefore $c \geq 4$. So $d_C(x_0^{+W}) = 2$. By symmetry, $d_C(x_1^{+W}) = 2$, and $d_{G \setminus C}(x_1^{+W}) = 0$. Therefore $d(x_0^{+W}) = d(x_1^{+W})$. But $7 = n \leq d(w_1) + d(x_1^{+W}) + d(x_0^{+W}) \leq 2 + d_{G \setminus C}(w_1) + 2 + 2 = 6 + d_{G \setminus C}(w_1)$. Therefore $w_1$ has a neighbour not on $C$, and by symmetry, so does $w_2$. Therefore, counting again, we have

$$\begin{aligned} 7 = n &= |C| + |G \setminus C| \\ &\geq 4 + d_{G \setminus C}(x_1^{+W}) + d_{G \setminus C}(w_1) + 1 + d_{G \setminus C}(w_2) + 1 + d_{G \setminus C}(x_0^{+W}) \\ &\geq 4 + 1 + 1 + 2 \\ &= 8, \end{aligned}$$

which is a contradiction, completing the case where $n = 7$, and thus the proof of the theorem. $\square$

Theorem 4.5 follows as a corollary to this result. It will be used in the proof of Theorem 2.6.

**Corollary 6.6 (Theorem 4.5)** *Let $G$ be a connected graph on $n \leq 3d$ vertices where $d$ is the minimum degree of a subset $W$ of vertices. Suppose $|W| = w \leq 2d$. Then $\mu(n, w, d) = \min\{w, d+1\}$.*

**Proof.** Applying Theorem 6.1 with $n = 3d$ gives $\mu(n, w, d) \geq \min\{w, d+1\}$. The bipartite graph $B(n, w, d)$ contains no path of weight more than $d+1$, so $\mu(n, w, d) \leq \min\{w, d+1\}$, giving the result. $\square$

# 7 More Lemmas

We have now proved one of the base cases for the proof of Theorem 4.4, that is Theorem 4.5. Since we have already proved the case where $k = 2$ in Theorem 4.5, we may assume that $k \geq 3$ in the following lemmas. The next lemma we prove shows that there is a path of weight three, as required in the case where $l = 1$. So we will then also take $l \geq 2$.

**Lemma 7.1** *If $n \leq wd$ and $w \geq 3$, then there is a path of weight three in $G$.*

**Proof.** Suppose not. Then no two weighted vertices are adjacent, since if they were there would be a path of weight three, by connectedness. This means that each edge has a weighted and a non-weighted endpoint so there are at least $wd$ edges in $G$.

Now, if there is a cycle in $G$ then we are done by connectedness. So $G$ is a tree and has $n - 1 \leq wd - 1$ edges, which is a contradiction. Therefore there is a path of weight three in $G$. $\square$

Let $C$ be a maximum cycle. The next easy lemma gives results for the weight of a path obtained by combining $C$ with a path $P$ or cycle $C'$ in $G \setminus C$. The technique of finding a maximum cycle and then combining it with a path or cycle in $G \setminus C$ forms a main part of the proof of Theorem 4.4. Lemma 7.2 says that a path $P$ in $G \setminus C$ can be combined with $C$ to form a path $P'$ which contains all the weighted vertices in $C$ and at least half those in $P$. It also says that $C$ can be combined with a cycle $C'$ in $G \setminus C$ to form a path through all the weighted vertices in both cycles. (See Figure 15.)

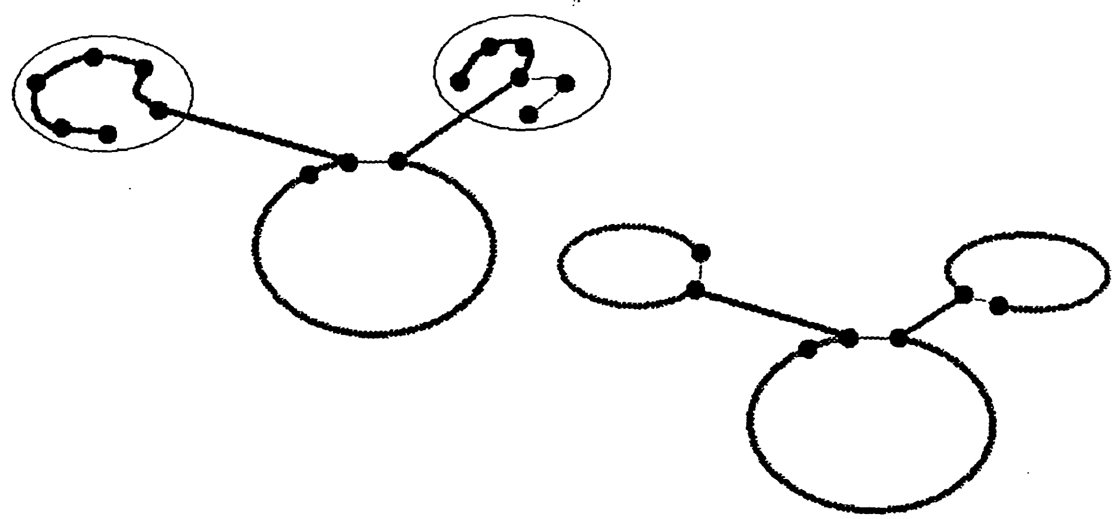

Figure 15: Combining paths and cycles

**Lemma 7.2** *Let $C$ be a cycle in $G$ of weight $w_C$.*

1. *Let $P'$ be a path in $G \setminus C$ of weight $w_{P'}$. Then there is a path $P$ in $G$ of weight*

$$w_P \geq w_C + \lceil w_{P'}/2 \rceil.$$

*Furthermore, if an endpoint of $P'$ is adjacent to $C$ then there is a path $P$ in $G$ of weight*

$$w_P \geq w_C + w_{P'}.$$

2. *Suppose we have a cycle $C'$ of weight $w_{C'}$ in $G \setminus C$. Then there is a path $P$ in $G$ of weight*

$$w_P \geq w_C + w_{C'}.$$

**Proof.** Let $P'$ be the path $p_1 p_2 \ldots p_t$, and $C$ be the cycle $v_1 v_2 \ldots v_c v_1$. Since $G$ is connected, there is a path from some vertex $p_i$ in $P'$ to some vertex $v_j$ in $C$. Without loss of generality, suppose the section $p_1 p_2 \ldots p_i$ of $P'$ has at least the weight of the section $p_i p_{i+1} \dots p_t$. Then the path $v_{j+1}v_{j+2} \dots v_1 \dots v_{j-1}v_j \dots p_i p_{i-1} \dots p_1$ has weight at least $w_C + \lceil w_{P'}/2 \rceil$, as required. If an endpoint of $P'$, say $p_1$, is adjacent to a vertex of $C$, say $v_j$, the path $P'' = v_{j+1}v_{j+2} \dots v_1 \dots v_{j-1}v_j p_1 p_2 \dots p_t$ has weight $w_C + w_{P'}$. Let $C'$ be a cycle $u_1 u_2 \dots u_{c'} u_1$. As $G$ is connected there is a path in $G \setminus (C' \cup C)$ between some vertex $u_i$ and some $v_j$. So there is a path $v_{j+1}v_{j+2} \dots v_{j-1}v_j \dots u_i u_{i+1} \dots u_{i-2}u_{i-1}$ of weight $w_C + w_{C'}$ as required. □

We next prove a corollary to Lemma 7.2 which we use to prove the base case where $k = 3$ of Theorem 4.4.

**Corollary 7.3** *Let $C$ be a maximum cycle.*

1. *If there is a path in $G \setminus C$ which has weight at least $\lceil (w - w_C - 1)/2 \rceil$ with an endpoint adjacent to $C$, then there is a path of weight at least $2l + 1$ in $G$.*
2. *Suppose there are two disjoint cycles $C_1$ and $C_2$ in $G \setminus C$ such that $w_{C_1} + w_{C_2} \geq \lceil (w - w_C - 1)/2 \rceil$. Also suppose that there is a vertex of $C_2$ adjacent to a vertex of $X^{+W}(C_1)$. Then there is a path in $G$ of weight at least $2l + 1$.*

**Proof.**

(1) By substituting $n \leq (k+1)d$ and $w \geq kl + 1$ into Theorem 2.2 we see that we are guaranteed a cycle of weight at least $l + 1$. Then applying Lemma 7.2(1) with $w_{P'} = \lceil (w - w_C - 1)/2 \rceil$ and $w_C \geq l + 1$ gives a path $P$ of weight

$$\begin{aligned} w_P &\geq \lceil (w - w_C - 1)/2 \rceil + w_C \\ &= \lceil (w + w_C - 1)/2 \rceil \\ &\geq \lceil (3l + 1 + l + 1 - 1)/2 \rceil \\ &= \lceil (4l + 1)/2 \rceil \\ &= 2l + 1. \end{aligned}$$

(2) First we form a graph $G^*$ by contracting $C_1$ to a single vertex $u$. By the remark following Lemma 5.3 for all $v \in X(C_1)$, there is a path $P^* = uu_1u_2....u_\gamma v^{+W}$ in $G^*$ through all the weighted vertices of $C$ with endpoints $u$ and $v^{+W}$. Therefore, in $G$, there is a path $P' = P^* \setminus \{u\}$ containing all the weighted vertices of $C$, which has one endpoint $u_1$ adjacent to a vertex of $C_1$, and the other, $v^{+W}$ adjacent to a vertex of $C_2$. Furthermore, $P'$ includes no vertices of $C_1$ or $C_2$. Let $C_1 = a_1a_2 \dots a_\gamma a_1$, $C_2 = b_1b_2 \dots b_\delta b_1$. Suppose that $u$ is adjacent to $a_i \in C_1$ and $v^{+W}$ to $b_j \in C_2$. Then there is a path

$$P = a_{i+1}a_{i+2} \dots a_i u_1 P' v^{+W} b_j b_{j-1} \dots b_{j+1}$$

of weight

$$\begin{aligned} w_P &= w_C + w_{C_1} + w_{C_2} \\ &\geq \lceil (w - w_C - 1)/2 \rceil + w_C \\ &= 2l + 1 \end{aligned}$$

just as in (1). □

The next lemma gives a lower bound for the size of a component when it does not contain a path with certain properties.

**Lemma 7.4** *Let $Q$ be a component of $G \setminus C$ of weight at least two, and let* $x_Q = \max_{v \in W \cap Q} d_C(v)$.

1. *Let $P$ be a maximum path in $Q$. Suppose there is no path of weight $w_P$ in $Q$ with an endpoint adjacent to $C$. Then $|Q| \geq 2d + 1$.*

2. *Suppose there is no path of weight two in $Q$ with an endpoint adjacent to $C$. Then $|Q| \geq w_Q d + 1$.*

3. *If there is no path through all the weighted vertices in $Q$ and $w_Q \leq d - x_Q + 1$, then $|Q| \geq 3(d - x_Q) + 1$.*

4. *If there is no cycle through all the weighted vertices in a component $Q$, then $|Q| \geq 2(d - x_Q) + 1$.*

**Proof.**
(1) Let $P = p_1p_2 \ldots p_t$ be a maximum path in $Q$. Suppose there is no path of weight $w_P$ in $Q$ with an endpoint adjacent to $C$. Suppose there is a cycle $C_0$ through all the weighted vertices of $P$. Then, by connectedness, there is a path in $Q \setminus C_0$ from a vertex of $C_0$ to $C$. This gives a path of weight at least $w_P$ with an endpoint adjacent to $C$ which is a contradiction. Therefore there is no cycle of weight $w_P$ in $Q$. So the sets $D_1 = \Gamma_P(p_1) \cup \{p_1\}$, $D_2 = \Gamma_P^+(p_t)$, $D_3 = \Gamma_{G \setminus P}(p_1)$ and $D_4 = \Gamma_{G \setminus P}(p_t)$ are pairwise disjoint subsets of $Q$. Thus

$$\begin{aligned} |Q| &\geq (|D_1| + |D_3|) + (|D_2| + |D_4|) \\ &\geq (d+1) + d \\ &= 2d + 1, \end{aligned}$$

as required.

(2) Suppose there is no path of weight two in $Q$ with an endpoint adjacent to $C$. Then no weighted vertex of $Q$ is adjacent to $C$, as by connectivity that would give a path of weight two with an endpoint adjacent to $C$, which is a contradiction.

If $w_Q = 2$ then, as there is no path of weight two with an endpoint on $C$, on applying part (1), we have $|Q| \geq 2d + 1$, as required.

Now suppose that $w_Q \geq 3$ and there is a path $P = p_1p_2 \ldots p_t$ of weight at least three in $Q$. By connectivity there is a path $P'$ from $P$ to $C$. Let $p_i$ be the vertex of $P$ nearest to $C$ on $P'$. Then the section $[p_1, p_i]$ or $[p_i, p_t]$ has weight at least two. Without loss of generality, suppose $[p_1, p_i]$ has weight at least two. Then the path $P' = p_1 p_2 \ldots p_i$ has weight at least two and an endpoint on $C$, which is a contradiction. Therefore there is no path of weight three in $Q$. So, by Lemma 7.1 applied to $Q$, $|Q| \geq w_Q d + 1$ as required.

(3) Applying Theorem 4.5, with $d - x_Q$ replacing $d$, we have a path through $w_Q$ weighted vertices of the component if $|Q| \leq 3(d - x_Q)$.

(4) From Theorem 2.2 with $d - x_Q$ replacing $d$, there is a cycle through all the weighted vertices of $Q$ if $|Q| \leq 2(d - x_Q)$. □

Let $P = p_1 p_2 \ldots p_t$ be a maximum path. Suppose $u \in \Gamma_P(p_1)$ such that there is a neighbour $v$ of $p_t$ in $[(u^{-W})^{-W}, u)$. Then there is a cycle $C = p_1 p_2 \ldots v p_t p_{t-1} \ldots u p_1$ of weight $w_C \geq w_P - 1$ in $G$. (See Figure 16.) If $w_C = w_P$, then either all the

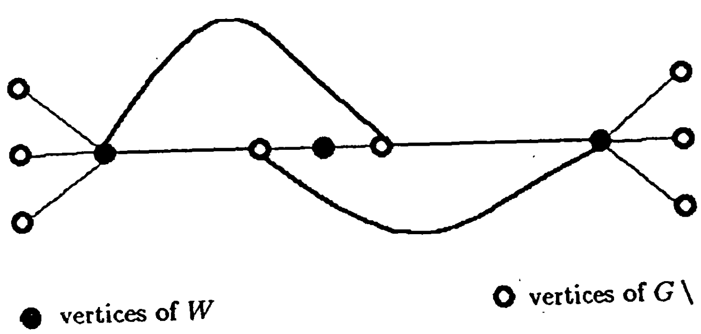


Figure 16: Cycle weight is one less than path weight

weighted vertices lie on $P$, in which case we are done, or there is a path of weight $w_P + 1$, contradicting the maximality of $P$. Therefore we need to deal with the case $w_C = w_P - 1$. The following lemma will be used in the proof of the induction step of Theorem 4.4.

For the remainder of this chapter and the next, let $\alpha$ be the number of weighted components in $G \setminus C$ and $\beta$ the number of components of weight at least two.

**Lemma 7.5** *Let $P$ be a maximum path and $C$ a maximum cycle. Suppose that $C$ has weight $w_C = w_P - 1$. Then $w_P \geq \min\{2l+1, d+1\}$.*

**Proof.** Let $P$ be a maximum path of weight $w_P \leq \min\{2l, d\}$. Let $C$ be a maximum cycle of weight $w_C = w_P - 1$ in $G$. We consider a weighted component $Q$ in $G \setminus C$. Suppose $Q$ has weight $w_Q \geq 2$. Then there is a path $P'$ in $Q$ of weight at least two. Suppose $P'$ has an endpoint adjacent to $C$. Then by Lemma 7.2(1) there is a path in $G$ of weight $w_P + 1$ contradicting the maximality of $P$. Therefore there is no path of weight two with an endpoint adjacent to $C$. So by Lemma 7.4(2), $Q$ has order at least $w_Q d + 1$.

Let $\mathcal{B}$ be the set of components of weight at least two. We define $w_{\mathcal{B}}$ to be the sum of the weights of the components in $\mathcal{B}$, and $\beta = |\mathcal{B}|$. Therefore there are $w - w_C - w_{\mathcal{B}}$ components of weight one. The number of vertices in $\mathcal{B}$ is at least $\sum\limits_{Q_i \in \mathcal{B}} (w_{Q_i} d + 1) = w_{\mathcal{B}} d + \beta$.

Let $x = |X(W \setminus C)|$. We consider the cases $x = 0$, $x = 1$ and $x \geq 2$.

(1) Suppose that $x = 0$. We consider a vertex $u \in C$ which has a neighbour $v$ in some component $Q$. Now $u^{+W}$ is adjacent to at most one vertex in a weighted component of $G \setminus C$, namely $v$, since if it were adjacent to another vertex $v^*$ there would be a cycle $uu^- \dots u^{+W} v^* P^* vu$ of greater weight than $C$, where $P^*$ is a path in $Q$ between $v$ and $v^*$.

So there are at least $d(u^{+W}) - 1 + 1 \geq d$ vertices not in weighted components. Let $\alpha$ be the number of components in $G \setminus C$.Since no weighted vertex in $G \setminus C$

is adjacent to $C$, all components of weight one contain at least $d+1$ vertices. So there are at least $(\alpha-\beta)(d+1)$ vertices in components of weight one.

Now $\alpha - \beta + w_B = w - w_C$, so the number of vertices in the weighted components is at least

$$\begin{aligned}(\alpha-\beta)(d+1) + w_B d + \beta &= (\alpha - \beta + w_B)d + \alpha \\ &= (w - w_C)d + \alpha.\end{aligned}$$

So counting the vertices in the weighted components and the neighbours of $u^{+W}$, we obtain

$$(k+1)d \ \geq \ (w - w_C)d + \alpha + d.$$

But $\alpha \geq 1$, $w - w_C \geq kl + 1 - (2l-1) \geq (k-2)l + 2 \geq k$, as $l \geq 1$. Therefore the above inequality becomes $(k+1)d \geq (k+1)d + 1$, which is a contradiction.

(2) Suppose $x = 1$. Let $u \in X(W \setminus C)$. Then $u^{+W}$ is not adjacent to any vertex in a weighted component of $G \setminus C$, by Corollary 5.4(4). Therefore there are at least $d+1$ vertices in the union of $C$ and the set of neighbours of $X^{+W}(W \setminus C)$, so not in weighted components. Now, using Lemma 5.7, with $i = 1$, we obtain the following inequality.

$$\begin{aligned}(k+1)d \ &\geq \ n \geq (w - w_C - w_B)(d-1+1) + (w_B d + \beta) + (d+1) \\ &\geq \ (w - w_C)d + d + 1 + \beta.\end{aligned}$$

But $\beta \geq 0$, $w - w_C \geq kl + 1 - (2l-1) \geq (k-2)l + 2 \geq k$, since $k \geq 2$ and $l \geq 1$.

Therefore the above inequality becomes $0 \geq 2$, which is a contradiction.

(3) Let $x \geq 2$. We count the vertices in the disjoint union of the sets of neighbours of $X^{+W}(W \setminus C)$ in $G \setminus C$ and vertices in weighted components of $G \setminus C$. As $w_C \leq \min\{2l-1, d-1\}$, we use Corollary 5.10 with $j = 1$ and Lemma 5.7 with $i = 1$ to obtain

$$(k+1)d \ \geq \ n \geq (w - w_C - w_B)(d - x + 1) + (w_B d + \beta) + x - 2 + 2d.$$

Rearranging and putting $w_B \geq 0$, we have

$$\begin{aligned}(-k+1+w-w_C)d &\leq (w-w_C)(x-1)-x+2 \\ (-k+1+w-w_C)(d-x+1) &\leq (x-1)(k-2)+1.\end{aligned}$$

Now, $w - w_C \geq kl - 2l + 2$ and $x \leq \min\{2l-1, d-1\}$, so

$$\begin{aligned}2(-k+3+kl-2l) &\leq (2l-2)(k-2)+1 \\ 0 &\geq 1,\end{aligned}$$

which is a contradiction. This completes the proof. □

In the proof of Theorem 4.4 in chapter 8, the following corollary allows us to assume that if a maximum path $P$ is not of the required weight, the weight of a maximum cycle $C$ is $w_C \leq \min\{2l-2, d-2\}$.

**Corollary 7.6** *If $w_C \geq \min\{2l-1, d-1\}$, then there is a path of weight $\min\{2l+1, d+1\}$ in $G$.*

**Proof.** Let $P$ be a maximum path. Since we have a cycle of weight at least $\min\{2l-1, d-1\}$, then $w_P \geq \min\{2l, d\}$ by connectedness. If $P$ has weight $\min\{2l, d\}$ we have a contradiction to Lemma 7.5. Therefore $P$ has weight at least $\min\{2l+1, d+1\}$. □

The next lemma provides an upper bound on the number of neighbours on $C$ of a vertex in $W$ of a component $Q$ where $w_Q \geq 2$. Let $s = \max_Q \min_{v_j \in Q \cap W} d_C(v_j)$ where the maximum is taken over all components $Q$ of weight at least two. Let $x$ be the maximum number of neighbours on $C$ of a weighted vertex in $G \setminus C$.

**Lemma 7.7**

$$s \leq \min\{x, \lfloor w_C/2 \rfloor, \lfloor c/3 \rfloor\}.$$

**Proof.** Let $Q$ be a component of weight at least two, such that $s = \min_{v_j \in Q \cap W} d_C(v_j)$, and let $u$ and $v$ be weighted vertices of $Q$ such that $v$ has the maximum degree $x_0$ on $C$ and $u$ has the minimum. We need to prove that

$$s = d_C(u) \leq \min\{x, \lfloor w_C/2 \rfloor, \lfloor c/3 \rfloor\}.$$

Certainly $s \leq x$.

Consider the sets $V_1 = \Gamma_C(v)$, $V_2 = \Gamma_C^-(u)$ and $V_3 = \Gamma_C^+(v)$. These are disjoint subsets of $C$, else there is a cycle of greater weight in $G$, and each set has size at least $s$. So $3s \leq c$.

Suppose the vertices of $W \cap C$, read consecutively around $C$, are $v_1, v_2, \ldots, v_\omega$, where $\omega = w_C$. Consider the sections of the cycle $[v_i, v_{i+1}]$, where the subscripts are to be interpreted cyclically. Let $M$ and $N$ be the set of sections which contain a neighbour of $u$, and $v$ respectively. We consider $N$, and the set of sections $M^+ = \{[v_{i+1}, v_{i+2}], [v_i, v_{i+1}] \in M\}$. These are disjoint sets as otherwise there would be a cycle of greater weight than $C$. Now $w_C \geq |N| + |M^+| \geq s + x_0 \geq 2s$, which completes the proof. $\square$

The following results are for the case where $n \leq 4d$ and $w \geq 3l + 1$, and $l \geq 2$ (as the case $l = 1$ was proved in Lemma 7.1.)

The next lemma gives sufficient conditions for there to be a path through all the weighted vertices in a component of $G \setminus C$.

**Lemma 7.8** *Let $Q$ be a component of $G \setminus C$. Let $A$ be either a weighted vertex in $Q$, or the set of all weighted vertices in $Q$. Suppose $x = |X(A)| \geq 2$. Then the following hold.*

1. *If* $w_C \leq \min\{2l-3, d-3\}$ *then there is a path through all the weighted vertices in* $Q$.

2. *There is a maximum weight path in* $Q$ *with an end point adjacent to* $C$.

**Proof.** We prove part (2) first and use it in the proof of part (1).

(2) Suppose that no path of weight $w_P$ in $Q$ has an endpoint adjacent to $C$. Then by Lemma 7.4(1), $|Q| \geq 2d+1$. Let $X = X(W \cap Q)$ be generated by all the weighted vertices in $Q$. As $x \geq 2$ we can apply Lemma 5.7 with $i = 1$, and Corollary 5.10 with $j = 0$, to obtain,

$$4d \geq n \geq 2d + 1 + 2d = 4d + 1,$$

which is a contradiction. This completes the proof.

(1) If there is no path through all weighted vertices in a component $Q$, then, by Lemma 7.4(3), $|Q| \geq 3(d - x_Q) + 1$, or there is a path of weight at least $d - x_Q + 1$ in $Q$, where $x_Q$ is the maximum degree on $C$ of a weighted vertex in $Q$. First we consider the case where we have a path $P$ of maximum weight at least $d - x_Q + 1$ with an endpoint adjacent to $C$. So there is a path in $G$ of weight at least $d - x_Q + 1 + w_C \geq d + 1$ as $x_Q \leq w_C$.

Now, we come to the other case, where there is no path of weight $d - x_Q + 1$ in $Q$. Putting $3(d - x_Q) + 1$ into Lemma 5.7 with $i = 1$ and Corollary 5.10 with $j = 3$, we obtain

$$\begin{aligned} 4d &\geq n \geq 3(d - x_Q) + 1 + 3(x - 2) + 2d \\ &\geq 4d + d - 5. \end{aligned}$$

So $d \leq 5$, and the cycle $C$ has weight at most $d-3 \leq 2$. By substituting $n \leq (k+1)d$ and $w \geq kl+1$ into Theorem 2.2 we obtain a cycle of weight at least $l+1$ for $l \geq 2$.

Therefore we have a contradiction, so there is a path through all the weighted vertices in $Q$. $\square$

The following corollary shows that there is a path of the required weight if there are at least two edges from a weighted vertex in $Q$ to $C$ and $Q$ contains at least half the weighted vertices in $G \setminus C$.

**Corollary 7.9** *Let* $n \leq 4d$*,* $w \geq 3l + 1$ *and* $w_C \leq \min\{2l-3, d-3\}$*. Suppose there is a component* $Q$ *in* $G \setminus C$ *such that* $w_Q \geq \lceil (w - w_C - 1)/2 \rceil$*. Suppose* $x = |X(Q \cap W)| \geq 2$*. Then there is a path in* $G$ *of weight at least* $2l + 1$*.*

**Proof.** By Lemma 7.8(1), there is a path through all the weighted vertices of $Q$. Therefore there is a path in $Q$ of weight $w_Q$ with an endpoint adjacent to $C$, by Lemma 7.8(2). By Corollary 7.3, there is a path of weight at least $2l + 1$ as required. $\square$

# 8 Proof of Theorem 4.4

We now have all the results we require to complete the proof of Theorem 4.4. We restate the theorem here for convenience.

**Theorem 4.4** *If $n$, $k$, $l$, $w$, $d$ are natural numbers such that $n \leq (k+1)d$, $w \geq kl+1$ and $k \geq 2$, then $\mu(n,w,d) \geq \min\{2l+1, d+1\}$.*

The proof of Theorem 4.4 is by induction on $k$. The induction has two base cases which are when $k = 2$, given by Theorem 4.5, and when $k = 3$. This is because the induction step involves removing at least $2d$ vertices and at most $2l$ weighted vertices. Thus $k$ decreases by two.

## 8.1 Induction

**Proof of Theorem 4.4.** We have already proved, in Theorem 4.5, the result for the base case $k = 2$. So we have two tasks remaining, namely the induction step and the base case where $k = 3$. We do the induction step first as it is fairly simple.

Assume the result is true for $n = kd$ and $w = (k-1)l + 1$. Suppose $n = (k+1)d$ and $w = kl+1$. Let $P = p_1p_2 \ldots p_t$ be a maximum path. If $w_P \geq \min\{2l+1, d+1\}$ then we are done, so we suppose that $w_P \leq \min\{2l, d\}$. Now $\Gamma^+_P(p_t) \cap \Gamma_P(p_1) = \emptyset$. Therefore $|P \cup \Gamma_{G\setminus P}(p_1) \cup \Gamma_{G\setminus P}(p_t)| \geq 2d+1$. We consider two cases.

1. $\Gamma^+_P(p_t) \cap \Gamma^-_P(p_1) = \emptyset$. Consider the sets $\Gamma^+_P(p_t)$, $\Gamma^-_P(p_1)$, $\Gamma_{G\setminus P}(p_t)$ and $\Gamma_{G\setminus P}(p_1)$. Let $D$ be the union of these sets. The sets in $D$ are pairwise disjoint, or there would be a path of greater weight, and so $|D| \geq 2d$. If $|D| > 2d$, we form a

new graph $G'$ by replacing all the vertices of $D$ by a single vertex $u$, making $u$ adjacent to a vertex $v$ if and only if $v$ is adjacent to some vertex $z \in D$ in $G$. So $G'$ is connected, and has at most $n - 2d$ vertices. If $|D| = 2d$, there is a vertex $p_i$, say, on $P$ which is not in $D$. We contract $D \cup \{p_i\}$ to a single vertex. Again this decreases $n$ by at least $2d$. Now, no weighted vertex not on $P$ is adjacent to any of the vertices of $D$, as again that would give a path of greater weight. So, contracting these vertices does not decrease the degree of any of the weighted vertices in $G \setminus (P \cup D)$. We write $W' = W \setminus P$. All the vertices of $W'$ have degree at least $d$ and so we define $W'$ to be the set of weighted vertices in $G' = G \setminus (P \cup D)$. Now $W'$ has size at least $w - 2l$. Thus $n' \leq (k-1)d$ and $w' \geq (k-2)l + 1$ which gives a path of weight $\min\{2l+1, d+1\}$ in $G'$, by the induction hypothesis, and therefore in $G$.

2. $\Gamma_P^+(p_t) \cap \Gamma_P^-(p_1) \neq \emptyset$. This gives us a cycle of weight $w_P - 1$ and so by Lemma 7.5 we have a path of weight $\min\{2l+1, d+1\}$ as required.

This completes the induction step.

## 8.2 Preliminaries

In this and the following sections we deal with the proof of the $k = 3$ case which is where $n = 4d$ and $w = 3l + 1$.

We split the proof into a number of cases, depending on the weight of a maximum cycle $C$, the total number $\alpha$ of weighted components in $G \setminus C$ and the number $\beta$ of components of weight at least two. We show that either (1) there are not enough vertices to form the required number of components, or (2) there are paths or cycles that can be added on to the cycle to form a path of weight at least $\min\{2l+1, d+1\}$.

Throughout we assume that there is no path of weight $\min\{2l+1, d+1\}$ and derive contradictions.

Where we introduce some notation or give a condition that is general to several cases, we highlight it by using a bullet •. We finish each case with □.

We start by giving some results which we will use through the remaining proof. The majority of the proofs of these results are by contradiction

**Result 8.1** *Suppose there is no path $P$ of weight $w_P \geq \min\{d+1, 2l+1\}$. Let $C$ be a maximum cycle. Then $l+1 \leq w_C \leq \min\{2l-2, d-2\}$. Furthermore $l \geq 3$ and $d \geq 6$.*

**Proof.** By substituting $n \leq (k+1)d$ and $w \geq kl+1$ into Theorem 2.2 we have $w_C \geq l+1$, and by Corollary 7.6, $w_C \leq \min\{2l-2, d-2\}$. Since $l+1 \leq \min\{2l-2, d-2\}$, $l \geq 3$ and $d \geq 6$. □

### Notation

We introduce some notation, which we will use throughout the rest of the proof.

- We use the notation $Q_i$, for $1 \leq i \leq \alpha$, to represent weighted components of $G \setminus C$, where $w_{Q_i} \geq w_{Q_j}$ whenever $i > j$.
- Let $Q'$ be the component of $G \setminus C$ such that $|X(Q' \cap W)|$ is maximised over all $Q_i$. We write $x' = |X(Q' \cap W)|$.
- We define $Q''$ to be a component of $G \setminus C$ where $|X(Q'' \cap W)|$ is maximised over all components $Q_i$ of weight at least two. We write $x'' = |X(Q'' \cap W)|$.
- For a component $Q_i$, we write $x_{Q_i} = |X(Q_i \cap W)|$.

- We write $y = |Y|$, where $Y = Y(A)$ is the set corresponding to $X(A)$, where $A$ is a valid set, the particular set being obvious from the context.

- Let $y_0$ be the number of vertices of $Y$ which do not have any neighbours in weighted components.

- Let $x_0$ be the number of vertices of $X^{+W}(A)$ with no neighbours in weighted components.

- Let $s_i$ be the minimum degree on $C$ of a weighted vertex in $Q_i$, where $Q_i$ has weight at least two. Then we define $s$ to be the maximum $s_i$ over components $Q_i$ of weight at least two. That is, as in Chapter 7, $s = \max_{Q_i} \min_{v \in Q_i \cap W} d_C(v)$, where the $Q_i$ have weight $w_{Q_i} \geq 2$.

Using this notation, we have the following bounds on component sizes.

**Result 8.2** *Let $Q_i$ be a component of $G \setminus C$.*

1. $|Q_i| \geq d - x' + 1$.

2. *If $w_{Q_i} \geq 2$, then $|Q_i| \geq d - s + 1$.*

**Proof.** In (1), $Q_i$ contains at least one weighted vertex, which has degree at least $d - x'$ in $Q_i$ and in (2), a weighted vertex of degree at least $d - s_i \geq d - s$ in $Q_i$. □

Next we have a result which shows that $x' \geq 1$.

**Result 8.3** *There is at least one weighted vertex in $G \setminus C$ adjacent to $C$, that is, $x' \geq 1$.*

**Proof.** We assume there are no edges from weighted vertices in $G \setminus C$ to $C$. Therefore all weighted components contain at least $d+1$ vertices. Let $m$ be the number of vertices not in weighted components of $G \setminus C$.

Counting the vertices in and not in the weighted components, we have $4d \geq n \geq \alpha(d+1)+m$, so $\alpha \leq 3$.

First we consider the case when there are three components. From the previous inequality we see that $m \leq d-3$. Suppose there is no cycle through all the weighted vertices in some component $Q$. Then by Theorem 2.2, $|Q| \geq 2d+1$. The other two components have size at least $d+1$ each, giving a total of at least $2(d+1)+2d+1 = 4d+3$ vertices in the components, which is a contradiction. Therefore each component contains a cycle through all its weighted vertices. As there is a cycle through all the vertices in each component, no component has weight more than $(w-w_C)/2-1$. Indeed, such a component $Q$ of weight $(w-w_C)/2$ would be joined to $C$ by a path, and so there would be a path of weight at least $w_C + w_Q \geq (w+w_C)/2 \geq ((3l+1)+(l+1))/2 = 2l+1$, as $w_C \geq l+1$ by Theorem 2.2. Next we consider a weighted vertex $u$ on $C$. Now $u$ is adjacent to at least one vertex $v$ in some component $Q$. Suppose $u^{+W}$ is adjacent to another component $Q_1$. The weight of the two components together is $w_Q + w_{Q_1} \geq w - w_C - ((w-w_C)/2-1) = (w-w_C)+1$. So, by Corollary 7.2(2), there is a path of weight at least $w_C + (w-w_C)/2+1 = (w+w_C)/2+1 \geq 2l+2$, as in previous proofs. Therefore $u^{+W}$ is not adjacent to any vertex other than $v$ in the weighted components. This is a contradiction as $u^{+W}$ has at least degree $d$ and there are at most $d-3$ vertices not in weighted components.

Now we move to the case where there are only two weighted components. If one of the components does not have a path through all the weighted vertices, then Theorems 4.1 and 4.5 tell us that either there is a path in the component of weight at least $d+1$, contradicting our assumptions, or the component has size at least $3d+1$, which with the component of size at least $d+1$ leads to a contradiction. Let $P^*$ be a path through all the weighted vertices in $Q'$ where $w_{Q'} \geq \lceil \frac{w-w_C}{2} \rceil$. Suppose $P^*$ has an endpoint adjacent to $C$. Then there is a path in $G$ of weight at least $2l+1$, by Corollary 7.3, and we are done. Therefore $|Q'| \geq 2d+1$ by Lemma 7.4(1).

Let $u$ be a weighted vertex in $C$, and suppose $u$ is adjacent to some vertex $v$ in a weighted component $Q$. Suppose $u^{+W}$ is adjacent to a vertex in $Q$ other than $v$, then we are done, as there are no edges between nonweighted vertices so there is a cycle of greater weight. If $u^{+W}$ is adjacent to a vertex in a different component then there is a path through half the weight in each component and $C$, which gives a path of weight at least $2l+1$, a contradiction. Therefore $u^{+W}$ has at least $d-1$ neighbours not in weighted components. So counting the vertices, we obtain $4d \geq (2d+1) + (d+1) + d - 1$ which is a contradiction.

Finally we come to the case with one weighted component $Q$. Now if there is a path through all the weighted vertices in $Q$, we can find a path through half the weighted vertices in $Q$ and all the weighted vertices on $C$ which gives a path of weight $2l+1$, again a contradiction. Therefore there is no path of weight $w_Q$ in $Q$, and so $|Q| \geq 3d+1$. Again we suppose there is a weighted vertex $u$ on $C$ that is adjacent to a vertex of $Q$. Then, as before, $u^{+W}$ is adjacent to at most one vertex of $Q$, so $m \geq d+1$. Counting the vertices, $4d \geq n \geq 3d+1+d+1 \geq 4d+2$, a contradiction, which completes the proof. □

The next two results deal with the relationship between $x$, $y$, $c$ and $\alpha$.

**Result 8.4** *Let $x = X(A)$ and $y = Y(A)$.*

1. *If $x = \lfloor c/2 \rfloor$, then $y = x$ if $c$ is even and $y = x-1$ if $c$ is odd.*

*2. If* $x = w_C$ *then* $y = x$.

**Proof.** By Lemma 5.2, $x \leq \lfloor c/2 \rfloor$, as all vertices of $X$ must be separated on $C$ by at least one (weighted) vertex. Therefore at most alternate vertices on $C$ can be in $X$.

(1) Hence if $x = \lfloor c/2 \rfloor$, $C$ is divided into $\lfloor c/2 \rfloor$ sections each containing at least one vertex. If $c$ is odd, there is one section containing two vertices, while the rest contain one. Therefore $x = y + 1$. If $c$ is even then each section contains exactly one vertex, which gives $x = y$.

(2) As there are no edges between unweighted vertices, $c \leq 2w_C$. Therefore if $x = w_C$, then $c$ is even and $x = y$, by (1). □

**Result 8.5** *Let $Q$ be a weighted component. Suppose* $y = |Y(Q \cap W)|$.

*1.* $y \leq y_0 + \alpha - 1$.

*2.* $x_Q \leq x_0 + \alpha - 1$.

**Proof.** By Corollary 5.5(6), no two vertices of $X^{+W}(Q \cap W)$ and therefore $Y$ (as $Y \subseteq X^{+W}(Q \cap W)$), have neighbours in the same component of $G \setminus C$. Now $y_0$ vertices of $Y$ do not have any neighbours in weighted components. Therefore there are $y - y_0$ weighted components adjacent to vertices of $Y$. At least one weighted component, namely $Q$, is not adjacent to any vertex of $Y$, by Corollary 5.5(3). So $y - y_0 \leq \alpha - 1$. Exactly similar, there are $x_Q - x_0$ vertices of $X^{+W}(Q \cap W)$ which are adjacent to at least one weighted component each. None of them are adjacent to $Q$, so $x_Q - x_0 \geq \alpha - 1$, which completes the proof. □

In the case where $x' = \lfloor c/2 \rfloor$, the following result gives a lower bound on the number of neighbours of the vertices of $X^{+W}(Q' \cap W)$.

**Result 8.6** *Suppose $x' = \lfloor c/2 \rfloor$. Then the number of neighbours of $X^{+W}(Q' \cap W)$ in $G \setminus C$ is at least $(x' - x_0)(d - x' + 2)$ in the $x' - x_0$ weighted components and $x_0(d - x')$ not in weighted components.*

**Proof.** Now, each vertex $z$ of $Y = Y(Q' \cap W)$ has at most $x'$ neighbours on $C$, by the definition of $Y$. So there are at least $d - x'$ neighbours of $z$ in $G \setminus C$. Suppose a vertex $v \in Q \cap W$ is adjacent to a vertex $u \in Y(Q' \cap W)$. Then $v$ is not adjacent to any other vertex of $Y$, by Corollary 5.5(6), nor to the vertices of $X(Q' \cap W)$ on either side of $u$, by Corollary 5.5(3). As $v$ can not be adjacent to consecutive vertices on $C$, and it is adjacent to a vertex of $Y$, it can not be adjacent to the vertices of $X$ on either side of this vertex of $Y$. It is also not adjacent to any other vertex of $Y$, leaving at most $1 + (x' - 2)$ vertices of $C$ than $v$ can be adjacent to. Therefore $v$ has at most degree $x' - 1$ on $C$, so $|Q| \geq d - (x' - 1) + 1 \geq d - x' + 2$. We recall that $y_0$ is the number of vertices of $Y$ that have no neighbours in weighted components. As no two vertices of $Y$ have neighbours in the same component, by Corollary 5.5(6), there are at least $y - y_0$ weighted components adjacent to vertices of $Y$. So the number of neighbours of vertices of $Y$ is at least $(y - y_0)(d - x' + 2) + y_0(d - x')$.

If $c$ is even, then $Y = X^{+W}(Q' \cap W)$ because alternate vertices belong to $X$ so each weighted vertex separating consecutive pairs of vertices of $X$ is a vertex of $Y$. Therefore $y = x'$ and $x_0 = y_0$, and we are done.

If $c$ is odd, there is a vertex $z$ in $Z^+(Q' \cap W)$, that is $x' = y + 1$. Suppose $z$ is adjacent to a weighted component $Q_0$. Therefore $x_0 = y_0$, that is all the vertices of $X^{+W}(Q' \cap W)$ that have no neighbours in weighted components are vertices of $Y$, since the only vertex of $X^{+W}(Q' \cap W)$ not in $Y$ is $z$. Then, again, $Q_0$ has at most degree $x' - 1$ on the cycle, so $|Q_0| \geq d - x' + 2$. Therefore the number of vertices in weighted components adjacent to vertices of $X^{+W}(Q' \cap W)$ is at least

$(y-y_0)(d-x'+2)+(d-x'+2) = (x-x_0)(d-x'+2)$, and in unweighted components is at least $x_0(d-x')$.

If $z$ is not adjacent to any weighted component, then $x_0 = y_0 + 1$. So $z$ has $d - x'$ neighbours not in weighted components or on the cycle. These neighbours are not shared with any vertices of $Y$, by Corollary 5.5(6). Therefore the number of vertices in weighted components adjacent to vertices of $X^{+W}(Q' \cap W)$ is at least $(y - y_0)(d - x' + 2) = (x - x_0)(d - x' + 2)$, and in unweighted components is at least $y_0(d - x') + (d - x') = x_0(d - x')$, as required. □

Now we will show that there is at least one component of weight two.

**Lemma 8.7** *Let $C$ be a maximum cycle and suppose that $w_C \leq \min\{2l-2, d-2\}$. Then $\beta \geq 1$, that is, there is at least one component in $G \setminus C$ of weight at least two.*

**Proof.** In all cases, we suppose there are no components of weight two and obtain a contradiction. As all the weighted components have weight one, $\alpha = w - w_C \geq (3l+1)-(2l-2) = l+3$. First we deal with the cases for which $x' \leq \min\{2l-3, d-3\}$.

(1) Now $w_C \leq \{2l-2, d-2\}$ and $x' \leq \min\{2l-3, d-3\}$. We count the vertices in each of the $w - w_C$ components and on $C$ to obtain

$$\begin{aligned} 4d &\geq n \geq (w - w_C)(d - x' + 1) + c \\ 0 &\geq (w - w_C - 4)(d - x' + 1) + 4 - 4x' + c. \end{aligned}$$

But $w - w_C \geq l + 3$, $c \geq 2x'$, $x' \leq 2l - 3$ and $d - x' \geq 3$ so we have

$$\begin{aligned} 0 &\geq 4(l-1) + 4 - 2(2l-3) + (c - 2x') \\ &\geq 6 + (c - 2x'), \end{aligned}$$

which is a contradiction. This completes the first case.

(2) Suppose $x' = \min\{2l-2, d-2\}$. Then by Result 8.4(2), $x' = y = w_C$. Therefore $x' = \lfloor c/2 \rfloor$ and we can apply Result 8.6, which guarantees that there are $(x'-x_0)(d-x'+2)+x_0(d-x')$ neighbours of vertices of $X^{+W}(Q' \cap W)$ in $G \setminus C$. In the remaining $\alpha - (x'-x_0)$ weighted components of $G \setminus C$, there are at least $(\alpha-(x'-x_0))(d-x'+1)$ vertices, as some vertex in each component has degree at least $d-x'$. So counting the vertices on $C$, in the components and the neighbours of $X^{+W}(Q' \cap W)$ we obtain

$$\begin{aligned} 4d &\geq n \geq (\alpha - x' + x_0)(d-x'+1) + (x'-x_0)(d-x'+2) + x_0(d-x') + c \\ &\geq (\alpha + x_0)(d-x') + \alpha + (x'-x_0) + c. \end{aligned}$$

Now $\alpha \geq l+3$, so we have

$$0 \geq (l+3+x_0-4)(d-x') + l + 3 + x' - x_0 + c - 4x'.$$

As $d - x' \geq 2$, and $l \geq 3$ we deduce

$$0 \geq 2(l-1+x_0) + l + 3 - x_0 + (c-2x') - x'.$$

But $x' \leq 2l-2$, so we obtain

$$\begin{aligned} 0 &\geq 3l + 1 + x_0 - (2l-2) + (c-2x') \\ &\geq l + 3 + x_0, \end{aligned}$$

which is a contradiction. This completes the proof of the lemma. □

**Lemma 8.8** *Suppose $x_Q = 0$ for some weighted component $Q$ in $G \setminus C$. Then there is a weighted vertex $u$ on $C$ with at most one neighbour in $Q$.*

**Proof.** Since $G$ is connected, there is an edge between some $v \in Q$ and $v_i \in C$. Suppose $v_0 \in Q \setminus \{v\}$ is adjacent to $v_i^{+W}$. Now there is a path $P$ between $v$ and

$v_0$ in $Q$, by connectivity. As there are no edges between unweighted vertices, $P$ contains at least one weighted vertex. Combining this with $C$, we have a cycle of weight at least $w_C + 1$, contradicting the maximality of $C$. So $v_i^{+W}$ is adjacent to no vertex of $Q$, except possibly $v$. $\square$

We split the remainder of the proof into two sections, the first dealing with $w_C = \min\{2l-2, d-2\}$, and the second with $w_C \leq \min\{2l-3, d-3\}$.

## 8.3 $w_C = \min\{2l-2, d-2\}$

Suppose $w_C = \min\{2l-2, d-2\}$. Let $Q$ and $Q_0$ be weighted components in $G \setminus C$. The method of proof in this case is as follows. We assume that there is no path of weight $w_P \geq \min\{2l+1, d+1\}$. Since we have a cycle of weight $\min\{2l-2, d-2\}$, this implies, by Lemma 7.2(1), that (i) there is no path of weight three in a component of $G \setminus C$, which has an endpoint adjacent to $C$, see Figure 17(1), and (ii) if there is a path of weight two with an endpoint $p_t$ adjacent to a vertex $u$ of $C$, then there is no edge from any component to vertices of $X^{+W}(p_t)$. This is because, by Lemma 7.2(1), combining the path of weight two with $C$ gives a path $P'$ of weight $w_C + 2$, with endpoint $u^{+W}$. If any vertex $z$ of a weighted component is adjacent to $u^{+W}$ then, by connectedness, a weighted vertex $v$ can be joined to an endpoint of $P'$, giving a path $P'zv$ of weight $w_C+3 = \min\{2l+1, d+1\}$. (See Figure 17(2).)

Unfortunately, the proof splits into case after case after case.

**Case 1:** $x' = \lfloor c/2 \rfloor$. Let $\theta$ be the number of components of weight at least three. We divide this case into two, depending on the value of $\theta$.

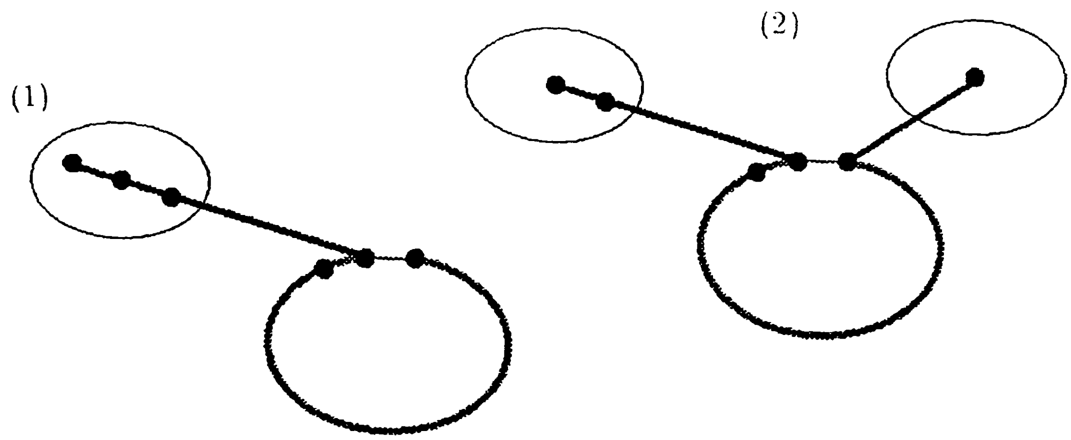

Figure 17: Combination of paths with $C$

(1) $\theta \geq 1$. Let $Q$ be a component of weight three.

(1.1) First we consider the cast where $x_Q \leq 1$. Then either (a) there is a path of weight three in $Q$, in which case neither endpoint is adjacent to $C$ or (b) there is no path of weight three in $Q$. In case (a), $|Q| \geq 2d+1$ by Lemma 7.4, and in case (b), $|Q| \geq w_Q(d-1)+1 \geq 3d-2 \geq 2d+1$, Lemma 7.1.

Now $|Q| \geq 2d+1$. Counting the vertices in each of the weighted components, the neighbours of $X^{+W}(Q' \cap W)$, by Result 8.6, and vertices on $C$, we obtain

$$\begin{aligned} 4d &\geq n \geq 2d+1+(\alpha-1)(d-x'+1)+c+x_0(d-x') \\ &\geq 2d+1+(\alpha-1+x_0)(d-x')+\alpha-1+c. \end{aligned}$$

Now $\alpha-1+x_0 \geq x'$, by Result 8.5, so we have

$$\begin{aligned} &\geq 2d+x'(d-x')+\alpha+c \\ 0 &\geq \alpha+(x'-2)(d-x')+(c-2x'), \end{aligned}$$

which is a contradiction, as $x' = w_C \geq l+1 \geq 4$. Therefore there are no components $Q$ of weight at least three for which $x_Q \leq 1$.

(1.2) Now $x_Q \geq 2$. Suppose $x_Q \geq 2$. Then some maximum path in $Q$ has an endpoint adjacent to $C$, by Lemma 7.8. Therefore $Q$ does not contain a path of weight three, so by Lemma 7.1, $|Q| \geq w_Q(d - x_Q) + 1$. Let $\Theta$ be the set of components of weight at least three. Then $|\Theta| = \theta$. Let $w_\Theta = \sum_{Q\in\Theta} w_Q$. Then there are at least $\sum_{Q\in\Theta}(w_Q(d-x_Q)+1) \geq w_\Theta(d-x'')+\theta$ vertices in the components of $\Theta$. Suppose that $x' = x''$. Then no vertex of any component is adjacent to $X^{+W}(Q'' \cap W)$, as there is a path of weight $w_C + 2$ with one endpoint in $Q''$, and the other, any vertex of $X^{+W}(Q''\cap W)$. So there are at least $x''(d-x'')$ neighbours of vertices of $X^{+W}(Q''\cap W)$ in $G\backslash C$ which are not in weighted components. Counting the vertices in weighted components, on $C$ and the neighbours of $X^{+W}(Q'' \cap W)$, we obtain

$$\begin{aligned}
4d &\geq n \\
&\geq w_\Theta(d-x') + \theta + (\alpha-\beta)(d-x'+1) + (\beta-\theta)(d-s+1) + x'(d-x') + c \\
4d &\geq (w_\Theta + \alpha - \theta + x')\,(d-x') + \theta + \alpha - \beta + (\beta-\theta)(x'-s+1) + c \\
0 &\geq (w_\Theta + \alpha - \theta + x' - 4)\,(d-x') + \alpha + (\beta-\theta)(x'-s) + c - 4x'.
\end{aligned}$$

But $w_\Theta = w - w_C - \alpha - \beta + 2\theta$, so this becomes

$$0 \geq (w - w_C - \beta + \theta + x' - 4)\,(d-x') + \alpha + (\beta-\theta)(x'-s) + c - 4x'.$$

Now $d - x' \geq 2$, and $x' - s \geq \lfloor c/2 \rfloor - \lfloor c/3 \rfloor \geq 1$, so we have

$$\begin{aligned}
0 &\geq 2(w - w_C - \beta + \theta + x' - 4) + \alpha + \beta - \theta + c - 4x' \\
&\geq (w - w_C) - 8 + (w - w_C - \beta + \theta) + \alpha + c - 2x'.
\end{aligned}$$

We have $w - w_C \geq l + 3 \geq 6$ and $\beta \leq \lfloor (w - w_C)/2 \rfloor$ so

$$\begin{aligned} 0 &\geq 6 - 8 + \lceil (w - w_C)/2 \rceil + \theta + \alpha + (c - 2x') \\ &\geq 1 + \theta + \alpha + (c - 2x'), \end{aligned}$$

which is a contradiction.

Therefore $Q'$ has weight one and $x'' \leq x' - 1$. There are at least $w_\Theta(d - x' + 1) + \theta$ vertices in the components of weight at least three, and $(\beta - \theta)(d - x' + 2)$ in the components of weight two. We count the vertices in each of the components, both weighted and unweighted, and on $C$ to obtain

$$\begin{aligned} 4d &\geq n \\ &\geq w_\Theta(d - x' + 1) + \theta + (\alpha - \beta)(d - x' + 1) + (\beta - \theta)(d - x' + 2) \\ &\qquad + x_0(d - x') + c \\ &\geq (w_\Theta - \theta + \alpha + x_0)(d - x') + w_\Theta - \theta + \alpha + \beta + c \\ 0 &\geq (w_\Theta - \theta + \alpha + x_0)(d - x') + w_\Theta - \theta + \alpha + \beta + c - 4x'. \end{aligned}$$

Now, $w_\Theta - \theta \geq 2$, $\alpha + x_0 \geq x' + 1 \geq 2$ and $d - x' \geq 2$, so we have

$$0 \geq 3(w_\Theta - \theta) + \alpha - 8 + \beta + 2(x_0 + \alpha) + c - 4x'$$

But $w_\Theta \geq w - w_C - \alpha - \beta + 2\theta$ and $\alpha + x_0 \geq x' + 1$, so this becomes

$$\begin{aligned} 0 &\geq 3(w - w_C - \alpha - \beta + \theta) + \alpha - 6 + \beta + (c - 2x') \\ &\geq 2(w - w_C - \alpha - \beta) + 3\theta + w - w_C - 6 + (c - 2x'). \end{aligned}$$

Now $w - w_C \geq l + 3 \geq 6$ so we obtain

$$0 \geq 3\theta + 2(w - w_C - \alpha - \beta) + (c - 2x'),$$

which is a contradiction as $\theta \geq 1$ and each term in brackets is non-negative.

(2) $\theta = 0$. Now all the components of $G \setminus C$ have weight at most two. Therefore

$w - w_C = \alpha + \beta$. Again, we suppose $x' = x''$. Then we count the vertices in the weighted components, the neighbours of $X^{+W}(Q' \cap W)$ and vertices on $C$. We obtain

$$\begin{aligned}
4d &\geq n \geq \alpha(d - x' + 1) + \beta(x' - s) + x'(d - x') + c \\
&\geq (w - w_C - \beta + x')(d - x') + \alpha + \beta(x' - s) + c \\
0 &\geq (w - w_C - \beta + x' - 4)(d - x') + \alpha + \beta(x' - s) + c - 4x' \\
&\geq 2(w - w_C - \beta) - 8 + \alpha + \beta + (c - 2x') \\
&\geq 2(w - w_C) - 8 + (w - w_C - 2\beta) + (c - 2x').
\end{aligned}$$

Now $w - w_C \geq l + 3 \geq 6$ so we have

$$0 \geq 4 + (w - w_C - 2\beta) + (c - 2x'),$$

which is a contradiction. Therefore $Q'$ has weight one. Again, $x'' \leq x' - 1$. Since all components have weight at most two and there are $w - w_C \geq l + 3 \geq 6$ weighted vertices in $G \setminus C$, there are at least three components.

Suppose also that $\alpha \geq 4$. Counting the vertices on $C$, the weighted components and the neighbours of vertices of $X^{+W}(Q' \cap W)$, we have

$$\begin{aligned}
4d &\geq n \geq (\alpha - (x' - x_0))(d - x' + 1) + (x' - x_0)(d - x' + 2) + x_0(d - x') + c \\
&\geq (\alpha + x_0)(d - x') + x' - x_0 + \alpha + c \\
0 &\geq (\alpha + x_0 - 4)(d - x') + x' - x_0 + \alpha + c - 4x' \\
&\geq (\alpha + x_0 - 4)(d - x') - x_0 + \alpha + (c - 2x') - x' \\
&\geq 2\alpha + (\alpha + x_0) - 8 + (c - 2x') - x' \\
&\geq 2\alpha - 7 + (c - 2x')
\end{aligned}$$

which is a contradiction as $\alpha \geq 4$. Therefore $\alpha = 3 = \beta$ as $w - w_C \geq l + 3 \geq 6$. This contradicts the assumption that $Q'$ has weight one, thus completing the proof in the case where $x' = \lfloor c/2 \rfloor = \min\{2l - 2, d - 2\}$. □

Case 2: $x' \leq \lfloor c/2 \rfloor - 1$. Again we split this case into two parts depending upon the value of $\theta$.

(1) $\theta \geq 1$. Let $Q$ be a component of weight at least three in $G \setminus C$. We suppose there is no path of weight three in $Q$ with an endpoint adjacent to $C$, and obtain a contradiction.

Let $q = \max_{v \in Q \cap W} d_C(v)$. Then, by Lemma 7.1, $|Q| \geq w_Q(d-q)+1$. Now, a vertex $v_i$ of $X^{+W}(Q \cap W)$ is not adjacent to any vertex in $Q$, by Corollary 5.5. We consider three cases: $x_Q \geq 2$, $x_Q = 1$ and $x_Q = 0$, and obtain contradictions for each, which will complete the proof of (1).

(1.1) First we consider the case where $x_Q \geq 2$. Let $P'$ be a maximal path in $Q$. As $x_Q \geq 2$, by Lemma 7.8(2), we have an endpoint of $P'$ adjacent to $C$, so we only need to prove there is a path of weight three in $Q$ and we are done. So we assume there is no path of weight three in $Q$. By Corollary 5.9, we count the neighbours of vertices of $X^{+W}(Q \cap W)$ and the vertices on $C$, giving $|\Gamma\left(X^{+W}(Q \cap W)\right) \cup C| \geq (x_Q - 2)(d - \lfloor c/2 \rfloor) + y(\lfloor c/2 \rfloor - x) + 2d$. Now, counting these vertices, and the vertices in $Q$ by applying Lemma 5.7, with $i = 1$, we obtain

$$
\begin{aligned}
4d &\geq n \geq w_Q(d-q) + 1 + (x_Q - 2)(d - \lfloor c/2 \rfloor) + y(\lfloor c/2 \rfloor - x) + 2d \\
0 &\geq (w_Q - 3)(d-q) + d - q - 3 + 2(x_Q - q) + y(\lfloor c/2 \rfloor - x).
\end{aligned}
$$

Since $x_Q \leq \lfloor c/2 \rfloor - 1$, we have

$$
0 \geq 2(w_Q - 3) + (d - q - 3) + y + 2(x_Q - q).
$$

Now each of the three terms in brackets, and $y$, are non-negative, as $q \leq x_Q \leq w_C - 1 \leq d-3$. Suppose $d-q-3 = 0$. Then $q = w_C - 1$, and so $y \geq 1$. So the right hand side of the inequality is positive, and we have a contradiction, completing the $x_Q \geq 2$ case.

(1.2) Now we consider $x_Q = 1$. We assume there is no path of weight three in $Q$ with an endpoint adjacent to $C$. We consider the cases where $\alpha = 1$ and $\alpha \geq 2$.

(i) Suppose there is only one weighted component. Let $v \in X(Q \cap W)$. Suppose we remove $\Gamma(v^{+W}) \cup C \setminus \{v\}$ from $G$ to obtain $G'$, with specified set $W' = W \setminus C$. We are removing at most weight $w_C$ and at least $d$ vertices. So $G'$ has $n' \leq 3d$ vertices and $w' = w - w_C$ specified vertices of minimum degree $d$. Therefore by Theorem 2.5, there is a path in $G'$, and therefore in $Q \cup \{v\}$, of weight at least $\min\{w - w_C, d+1\}$. Now if there is a path of weight $d+1$, we have a contradiction. As $w_Q = w - w_C \geq l + 3 \geq 6$, there is a path of weight at least three in $Q$ with an endpoint adjacent to $C$ which is a contradiction. This completes (i).

(ii) Now $\alpha \geq 2$. We assume there is no path of weight three in $Q$. We consider two subcases where $w_Q \geq 4$ and $w_Q = 3$.
(a) Suppose $w_Q \geq 4$. Then $|Q| \geq w_Q(d-q)+1 \geq w_Q(d-1)+1$, by Lemma 7.1. Again, there are no neighbours of weighted components adjacent to a vertex of $X^{+W}(Q \cap W)$, as this would give a path of weight at least $\min\{2l+1, d+1\}$. Let $v \in X^{+W}(Q \cap W)$. Since $v \in W$, it has degree at least $d$. The neighbours of $v$ are not in $Q$, so there are at least $d+1$ vertices in $\Gamma\left(X^{+W}(Q \cap W)\right) \cup C$. We count the vertices in $Q$ and in $\Gamma\left(X^{+W}(Q \cap W)\right) \cup C$, to obtain

$$\begin{aligned} 4d &\geq n \geq w_Q(d-1)+1+d+1 \\ 0 &\geq (w_Q-3)(d-1)-1. \end{aligned}$$

Since $w_Q \geq 4$ and $d-1 > 0$, we have a contradiction.
(b) Now $w_Q = 3$. Let $Q_0 \neq Q$ be a component in $G \setminus C$. Then $Q_0$ contains at least $d - x' + 1$ vertices, and $Q$ at least $3(d-1)+1 = 3d-2$ vertices. Counting the vertices in the components and on $C$, we have

$$4d \geq n \geq (3d-2)+(d-x'+1)+c$$

$$0 \geq c - x' - 1.$$

Now $c - x' \geq x'$. If $x' \geq 2$, then we have a contradiction. If $x' = 1$, then $c - x' - 1 \geq c - 2 \geq (l + 1) - 2 \geq 2$, which, again, is a contradiction.

Let $P'$ be a path of weight three in $Q$. We assume that $P'$ does not have an endpoint adjacent to $C$. Therefore $|Q| \geq 2d + 1$ by Lemma 7.4(1). As before, there is a path in $Q$ of weight at least two with an endpoint adjacent to $C$. Now there are at least two components $Q$ and $Q_0$ in $G \setminus C$. No vertex of $Q$ can be adjacent to a vertex of $X^{+W}(Q_0 \cap W)$ as this would give a path of the required length, by Lemma 7.2(1). Suppose $x_{Q_0} \geq 2$. Then, counting the vertices in $Q$, $Q_0$ and $C$ and the neighbours of $X^{+W}(Q_0 \cup C)$, we have

$$\begin{aligned} 4d &\geq n \geq (2d + 1) + (d - x_{Q_0} + 1) + 2(x_{Q_0} - 2) + 2d \\ 0 &\geq 2 + (d - x_{Q_0}) + 2(x_{Q_0} - 2), \end{aligned}$$

which is a contradiction, so $x_{Q_0} = 1$.

Again $|Q| \geq 2d + 1$, $|Q_0| \geq d$ and $\left|\Gamma\left(X^{+W}(Q_0 \cap W)\right) \cup C\right| \geq d + 1$. So we have

$$4d \geq n \geq (2d + 1) + d + (d + 1),$$

which is a contradiction. This completes (1.2).

(1.3) Now $x_Q = 0$. By Result 8.3, $x' \geq 1$ so there are at least two weighted components in $G \setminus C$. Now $w_Q \geq 3$ so $|Q| \geq w_Q d + 1 \geq 3d + 1$.
(i) Suppose $x' \geq 2$. Then counting the vertices in $Q$ and $Q'$ and on $C$ we obtain $4d \geq 3d+1+d-x'+1+c \geq 4d+2+c-x'$ which is a contradiction. Therefore there is a path of weight three in $Q$. Now $Q$ is not adjacent to any vertex in $X^{+W}(Q' \cap W)$ as that would give a path of weight $1 + w_C + 2$, which is a contradiction.

Therefore counting the vertices in $Q$, $Q'$ and $|\Gamma(X^{+W}(Q' \cap W))| \cup C$, we obtain

$$\begin{aligned} 4d &\geq n \geq 2d + 1 + d - x' + 1 + 2(x' - 2) + 2d \\ &\geq 4d + 2 + d - x' + 2(x' - 2) \end{aligned}$$

which is a contradiction.

(ii) So $x' = 1$. No vertex of $Q'$ or $Q$ is adjacent to the vertex $v$ in $X^{+W}(Q' \cap W)$. Now $d(v) \geq d$, so $|\Gamma(v) \cup \{v\}| \geq d+1$. So there are at least $d+1$ vertices which are not in $Q \cup Q'$. There is no path of weight three in $Q$ with an endpoint adjacent to $C$, so $|Q| \geq 2d+1$. Counting the vertices in $Q$, $Q'$ and $X^{+W}(Q' \cap W)$, we obtain

$$4d \geq n \geq (2d+1) + d + (d+1),$$

which is a contradiction.

(2) $\theta = 0$. Now all the components are of weight at most two. There are at least $w - w_C \geq (3l+1) - (2l-2) = l + 3 \geq 6$ weighted vertices in $G \setminus C$. As each component has weight at most two, there are at least three components in $G \setminus C$. We deal with the case where $x'' \geq 2$ first.

(2.1) Suppose $x'' \geq 2$. Now there is a path of weight two with an endpoint adjacent to $C$ for both weighted vertices in $Q''$ generating $X(Q'' \cap W)$. Therefore there are no vertices of any of the other components adjacent to $X^{+W}(Q'' \cap W)$. We count the vertices in $Q''$, the neighbours of $X^{+W}(Q'' \cap W)$, the vertices in all the other weighted components, and on $C$. The number of vertices in each component of weight one is at least $d - x' + 1$. There are $\alpha - \beta$ such components, so the total number of vertices in components of weight one is at least $(\alpha - \beta)(d - x' + 1)$. For components of weight two, excluding $Q''$, there are at least $(\beta - 1)(d - s + 1)$ vertices. The number of vertices in $\Gamma\left(X^{+W}(Q'' \cap W)\right) \cup C$ is at least $2d + j(x-2)$, where $j = d - \lfloor c/2 \rfloor \geq 2$, by Corollary 5.10. So, by Lemma 5.7, we have

$$\begin{aligned}
(\dagger)\quad 4d &\geq n \geq (\alpha-\beta)(d-x'+1)+(\beta-1)(d-s+1) \\
&\quad +(d-x''+1)+j(x''-2)+2d \\
&\geq (\alpha-\beta)(d-x'+1)+(\beta-2)(d-s+1)+(d-s+1) \\
&\quad +(d-x''+1)+2x''-4+2d \\
0 &\geq (\alpha-\beta)(d-x'+1)+(\beta-2)(d-s+1)-s+x''-2.
\end{aligned}$$

Since $s \leq x''$, by Lemma 7.7, this becomes

$$(\star)\quad 0 \geq (\alpha-\beta)(d-x'+1)+(\beta-2)(d-s+1)-2,$$

which gives a contradiction if $\beta \geq 3$, as $d-s \geq 3$. Therefore $\beta \leq 2$. Suppose first that $\beta = 2$. Then there are at least four components in $G \setminus C$, that is $\alpha \geq 4$. We recall that $x' \leq w_C$. Substituting for $\alpha$ and $\beta$ in the last inequality $(\star)$, we have

$$0 \geq 2(d-x'+1)-2.$$

But $d-x' \geq 2$, so we obtain $0 \geq 4$, which is a contradiction.

So $\beta = 1$. We note that $\alpha = w-w_C-2 \leq (3l+1)-(2l-2)-2 = l+1$. Substituting for $\alpha$ and $\beta$ in inequality $(\dagger)$, with $j \geq 2$, we obtain

$$\begin{aligned}
4d &\geq l(d-x'+1)+(d-x''+1)+2(x''-2)+2d \\
&\geq (l-1)(d-x'+1)+(d-x'+1)+3d-3+x'' \\
0 &\geq (l-1)(d-x'+1)-x'-2+x''.
\end{aligned}$$

Now, $d-x'+1 \geq 3$, as $x' \leq w_C \leq \min\{2l-2, d-2\}$, so

$$\begin{aligned}
0 &\geq 3(l-1)-(2l-2)-2+x'' \\
&\geq 3l-3-2l+x'' \\
&\geq l-3+x'',
\end{aligned}$$

which is a contradiction as $l \geq 3$ and $x'' \geq 2$. This completes the case $x'' \geq 2$.

(2.2) Now we deal with $x'' \leq 1$. Unfortunately, we have to split this case based on the value $\beta$ and $x'$. First we consider

(i) $\beta = 1$ and $x' \leq 3$.

As $\beta = 1$, $\alpha = w - w_C - 2 \geq l + 1$. Therefore the number of vertices in components of weight one is at least $(l+1)(d - x' + 1)$, and at least $d$ in the component of weight two. We can now count the vertices in each of the components and on the cycle, to obtain

$$\begin{aligned} 4d &\geq n \geq (l+1)(d - x' + 1) + d + c \\ &\geq (l-2)(d - x' + 1) + 4d + (c - 2x') + (3 - x'). \end{aligned}$$

Now $c \geq 2x'$ and $x' \leq 3$, so we have

$$0 \geq (l-2)(d - x' + 1),$$

which is a contradiction as $d \geq x' + 2$ and $l \geq 3$.

(ii) Now, we deal with the cases (a) $\beta \geq 2$ and $x' \geq 2$, and (b) $x' \geq 4$. As $x'' \leq 1$ then $x_{Q_i} \leq 1$ for all components $Q_i$ of weight two. Therefore $Q'$ has weight one. The vertices of $W$ in a component of weight two are not adjacent to vertices of $X^{+W}(Q' \cap W)$. Indeed, if one of them was adjacent to a vertex of $X^{+W}(Q' \cap W)$, there would be a path of weight $\min\{2l+1, d+1\}$ in $G$. So the set of vertices of $Q'$ and components $Q_1 \ldots Q_\beta$ of weight two, and $\Gamma\left(X^{+W}(Q' \cap W)\right) \cup C$ are disjoint. Each component of weight two contains at least $d - x'' + 1 \geq d$ vertices, and $|Q'| \geq d - x' + 1$. Again, by Corollary 5.10, there are at least $2(x-2) + 2d$ vertices in $X^{+W}(Q' \cap W) \cup C$. Now, applying Lemma 5.7 with $i = 1$, we have

$$\begin{aligned} 4d &\geq (d - x' + 1) + \beta d + 2(x' - 2) + 2d \\ 0 &\geq (\beta - 1)d + x' - 3, \end{aligned}$$

which is a contradiction as $d \geq 4$ and either (a) $\beta \geq 2$ and $x' \geq 2$, or (b) $x' \geq 4$.

(iii) Now we deal with the remaining case, where $x' = 1$ and $\beta \geq 2$. Again, we count the vertices in each of the weighted components and on the cycle, and neighbours of $X^{+W}(Q' \cap W)$. Each component, regardless of weight, contains at least $d$ vertices. Considering the vertex in $X^{+W}(Q' \cap W)$, we again see that it is not adjacent to any vertex in a weighted component, so $\left|\Gamma\left(X^{+W}(Q' \cap W)\right) \cup C\right| \geq d+1$. So counting the vertices in the components and on $C$, we have

$$4d \;\geq\; n \geq (\alpha-\beta)d + \beta d + (d+1).$$

If $\beta \geq 3$ this is a contradiction, so $\beta = 2$ and

$$4d \;\geq\; (\alpha-2)d + 3d + 1.$$

Now $\alpha = w - w_C - \beta \geq 3l + 1 - (2l-2) - 2 \geq l+1 \geq 4$, so we obtain

$$0 \;\geq\; d+1,$$

which is also a contradiction. This completes the case $w_C = \min\{2l-2, d-2\}$. □

## 8.4 $w_C \leq \min\{2l-3, d-3\}$

We begin this section with a result which gives lower bounds for $d$ and $l$.

**Result 8.9** *For $w_C \leq \min\{2l-3, d-3\}$, $l \geq 4$ and $d \geq 8$.*

**Proof.** As $w_C \geq l+1$, $2l-3 \geq l+1$, giving $l \geq 4$. Now $5 \leq l+1 \leq d-3$ so $d \geq 8$. □

Now we intoduce some notation.

- Let $f = \min\{2l, d\}$. Then $w_C \leq f-3$.

We prove a lemma which gives an upper bound for each of $x'$, $x''$ and $s$.

**Lemma 8.10** *Suppose $w_C \le \min\{2l-3, d-3\}$. Then*

1. $s \le \min\left\{l-2, \left\lfloor \frac{d-3}{2} \right\rfloor\right\}$,
2. $x' \le \min\left\{l+2, \left\lfloor \frac{d+5}{2} \right\rfloor\right\}$,
3. $x'' \le \min\left\{l+1, \left\lfloor \frac{d}{2} \right\rfloor + 1\right\}$.

**Proof.** Putting $w_C \le \min\{2l-3, d-3\}$ into Lemma 7.7 gives part (1), so we move on immediately to part (2). Since $l+2$ and $\lfloor (d+5)/2 \rfloor$ are both at least three, we may consider $x' \ge 3$. Then by Corollary 5.5, there are no vertices of $Q'$ adjacent to vertices of $X^{+W}(Q' \cap W)$. By Corollary 5.9 $\left|\Gamma\left(X^{+W}(Q' \cap W)\right) \cup C\right| \ge (x'-2)(d - \lfloor c/2 \rfloor) + 2d$. Also, $|Q'| \ge d - x' + 1$. So counting the vertices in the disjoint sets $Q'$ and $\Gamma\left(X^{+W}(Q' \cap W)\right) \cup C$, that is, applying Lemma 5.7 with $i = 1$, we obtain

$$\begin{aligned} 4d &\ge n \ge d - x' + 1 + (x'-2)(d - \lfloor c/2 \rfloor) + 2d \\ 4d &\ge 4d - x' - \lfloor c/2 \rfloor + 1 + (x'-3)(d - \lfloor c/2 \rfloor) \\ 0 &\ge -x' - \lfloor c/2 \rfloor + 1 + (x'-3)(d - \lfloor c/2 \rfloor). \end{aligned}$$

Now, $\lfloor c/2 \rfloor \le w_C \le f - 3$ and we have

$$\begin{aligned} 0 &\ge -x' - f + 4 + 3(x'-3) \\ f + 5 &\ge 2x', \end{aligned}$$

which gives the required result.

For part (3), we have a similar argument to part (2), with $Q''$ replacing $Q'$. Again, we assume $x'' \ge 3$, and count as in part (2) with $|Q''| \ge d - s + 1$, and as before, $\left|\Gamma\left(X^{+W}(Q'' \cap W)\right) \cup C\right| \ge (x''-2)(d - \lfloor c/2 \rfloor) + 2d$. So, applying

Lemma 5.7, with $i = 0$, we have

$$\begin{aligned} 4d &\geq n \geq d - s + 1 + (x'' - 2)(d - \lfloor c/2 \rfloor) + 2d \\ &\geq 4d - s - \lfloor c/2 \rfloor + 1 + (x'' - 3)(d - \lfloor c/2 \rfloor) \\ 0 &\geq -s - \lfloor c/2 \rfloor + 1 + (x'' - 3)(d - \lfloor c/2 \rfloor). \end{aligned}$$

As $f = \min\{2l, d\}$, $\lfloor c/2 \rfloor \leq w_C \leq f - 3$, and, by part (1), $s \leq \lfloor (f-3)/2 \rfloor$, we obtain

$$\begin{aligned} 0 &\geq -\lfloor (f-3)/2 \rfloor - f + 3 + 1 + 3(x'' - 3) \\ &\geq -\lfloor (3f-3)/2 \rfloor + 3x'' - 5 \\ x'' &\leq \left\lfloor \frac{\lfloor (3f-3)/2 \rfloor + 5}{3} \right\rfloor \\ &\leq \min\{l + 1, \left\lfloor \frac{d}{2} \right\rfloor + 1\}, \end{aligned}$$

as required. □

Now we come to a lemma which gives a bound on $x'$ in terms of $c$.

**Lemma 8.11** *Suppose $w_C \leq f - 3$. Then $x' \leq \lfloor c/3 \rfloor + 1$, or $x' = \lfloor c/2 \rfloor$.*

**Proof.** We assume that $x' = \lfloor c/3 \rfloor + m \leq \lfloor c/2 \rfloor - 1$, for $m \geq 2$, and obtain a contradiction. As $x' \geq \lfloor c/3 \rfloor$, $y \geq 3x' - c \geq 3m - 2$. Now $x' \geq 3$, as $c \geq l + 1 \geq 5$. By Corollary 5.9, $\left|\Gamma\left(X^{+W}(Q' \cap W)\right) \cup C\right| \geq x(d - \lfloor c/2 \rfloor) + y(\lfloor c/2 \rfloor - x) + c$. Now $|Q'| \geq d - x' + 1$. Counting the vertices in these two sets, we obtain

$$\begin{aligned} 4d &\geq d - x' + 1 + y(\lfloor c/2 \rfloor - x') + (x' - 3)(d - \lfloor c/2 \rfloor) - \lfloor c/2 \rfloor \\ 0 &\geq -x' + 1 + (y - 2)(\lfloor c/2 \rfloor - x') + 2\lfloor c/2 \rfloor - 2x' \\ &\quad +(x' - 3)(d - \lfloor c/2 \rfloor) - \lfloor c/2 \rfloor. \end{aligned}$$

Now, $d - \lfloor c/2 \rfloor \geq 3$, $y \geq 3m - 2$ and $\lfloor c/2 \rfloor - x' \geq 1$. So we have

$$
\begin{aligned}
0 &\geq -3x' + 1 + 3m - 4 + \lfloor c/2 \rfloor + 3(x' - 3) \\
0 &\geq -12 + 3m + \lfloor c/2 \rfloor \\
12 - 3m &\geq \lfloor c/2 \rfloor .
\end{aligned}
$$

So $\lfloor c/2 \rfloor \leq 6$ as $m \geq 2$. Now, $\lfloor c/2 \rfloor - 1 \geq \lfloor c/3 \rfloor + m$. Therefore $\lfloor c/2 \rfloor - \lfloor c/3 \rfloor \geq m + 1 \geq 3$ and so $c \geq 14$, which is a contradiction. This completes the proof. □

**Lemma 8.12** *Suppose $\alpha \geq 4$. Then $x' \leq \lfloor c/2 \rfloor - 1$.*

**Proof.** We assume $x' = \lfloor c/2 \rfloor$ and obtain a contradiction. By Result 8.6, there are $(x' - x_0)(d - x' + 1) + x_0(d - x')$ neighbours of vertices of $X^{+W}(Q' \cap W)$ in $G \setminus C$. There are also $(\alpha - (x - x_0))(d - x' + 1)$ vertices in weighted components not adjacent to $X^{+W}(Q' \cap W)$. Counting the vertices on $C$ and in $G \setminus C$, we obtain

$$
\begin{aligned}
4d &\geq n \geq (\alpha - (x' - x_0))(d - x' + 1) + (x - x_0)(d - x' + 1) + x_0(d - x') + c \\
&\geq (\alpha + x_0)(d - x') + 1 + \alpha + x - x_0 + c \\
0 &\geq (\alpha + x_0 - 4)(d - x') + 1 + \alpha - x_0 - x' + (c - 2x') \\
&\geq 3(\alpha + x_0 - 4) + 1 + \alpha - x_0 - x' + (c - 2x') \\
&\geq 3\alpha + x_0 + (\alpha + x_0 - x' - 1) - 10 + (c - 2x') \\
&\geq 2 + 3(\alpha - 4) + x_0 + (\alpha + x_0 - x' - 1) + (c - 2x')
\end{aligned}
$$

which is a contradiction. □

We move on to the case where we have a large number of components.

**Case 1:** $\alpha \geq 5$. The method we use in this case is to count the vertices in all the components and on $C$. When we do this, we find that, for the values of $x'$ and $x''$ given in Lemma 8.10, we do not have enough vertices in our graph.

We count the vertices in the components of $G \setminus C$, and on $C$. There are $\alpha - \beta$ components of weight one which have size at least $d - x' + 1$, and $\beta$ of weight two or more which contain at least $d - s + 1$ vertices. We recall that $s \leq \lfloor c/3 \rfloor$, and $x' \geq s$, by Lemma 7.7, and $c + 3 \geq x'$, by Lemma 8.11. Now, applying Lemma 5.7 with $i = 0$, we obtain

$$\begin{aligned}
4d &\geq n \geq (\alpha - \beta)(d - x' + 1) + \beta(d - s + 1) + c \\
&\geq \alpha(d - x' + 1) + \beta(x' - s) + c \\
&\geq (\alpha - 5)(d - x' + 1) + 5d - 5x' + 5 + (\beta - 1)(x' - s) + x' - s + c \\
0 &\geq (\alpha - 5)(d - x' + 1) + d - x' + 5 - s + (\beta - 1)(x' - s) \\
&\quad +(c - 3x' + 3) - 3.
\end{aligned}$$

By Lemma 8.10, $x' \leq \lfloor (d + 5)/2 \rfloor$, and $s \leq \lfloor (d - 3)/2 \rfloor$. So we have

$$\begin{aligned}
0 &\geq (\alpha - 5)(d - x' + 1) + d - \lfloor (d + 5)/2 \rfloor - \lfloor (d - 3)/2 \rfloor \\
&\quad +2 + (\beta - 1)(x' - s) \\
&\geq (\alpha - 5)(d - x' + 1) + (d + 1) - (\lfloor (d + 1)/2 \rfloor + 2) - (\lfloor (d + 1)/2 \rfloor - 2) \\
&\quad +1 + (\beta - 1)(x' - s) \\
&\geq 1 + (\alpha - 5)(d - x' + 1) + (\beta - 1)(x' - s),
\end{aligned}$$

which is a contradiction, as $\alpha \geq 5$ and $\beta \geq 1$. This completes the case. □

**Lemma 8.13** *Suppose $\alpha \geq 4$ and $\beta \geq 2$. Then, in all components $Q$ such that $w_Q \geq 2$, there is a cycle of weight $w_Q$.*

**Proof.** Let $Q$ be a component in $G \backslash C$ of weight $w_Q \geq 2$. Suppose there is no cycle through all the weighted vertices of $Q$. Then $|Q| \geq 2(d - x_Q) + 1 \geq 2(d - x'') + 1$, by Lemma 7.4(4). There are $\beta - 1$ other components of weight at least two, containing at least $d - s + 1$ vertices each. The $\alpha - \beta$ components of weight one are of size at least $d - x' + 1$, giving $(\alpha - \beta)(d - x' + 1)$ vertices. Counting these vertices and the vertices on $C$, by Lemma 5.7 with $i = 0$, we have

$$\begin{aligned} 4d &\geq n \geq 2(d - x'') + 1 + (\beta - 1)(d - s + 1) + (\alpha - \beta)(d - x' + 1) + c \\ 0 &\geq -x'' + (\beta - 1)(x' - s) + (\alpha - 3)(d - x' + 1) + (c - 2x' - x'' + 3). \end{aligned}$$

Now $x'' \leq x' \leq \lfloor c/2 \rfloor + 1$, by Lemma 8.11, and $\alpha \geq 4$, $\beta \geq 2$ giving

$$0 \geq (d - s - x'') + 1.$$

Using Lemma 8.10, we substitute upper bounds for $x''$ and $s'$ to obtain

$$\begin{aligned} 0 &\geq 1 + (d - (\lfloor (d-1)/2 \rfloor - 1) - (\lfloor d/2 \rfloor + 1)) \\ &\geq 1, \end{aligned}$$

which is a contradiction. This completes the proof of the lemma. □

**Lemma 8.14** *Suppose $\alpha \geq 3$. Let $Q$ be a component of weight at least two in $G \backslash C$ such that either $x_Q \leq 1$, or for a path $p_1 \ldots p_t$ of weight $w_Q$ in $Q$, $\max\{x_{p_t}, x_{p_1}\} \leq 1$. Then there is a cycle of weight $w_Q$ in $Q$.*

**Proof.** Suppose there is no cycle in $Q$ of weight $w_Q$. Then $|Q| \geq 2(d-1) + 1 = 2d - 1$, by Lemma 7.4. The other $\alpha - 1$ components contain at least $d - x' + 1$ vertices. We count the vertices in each of the components and on $C$, to obtain

$$\begin{aligned} 4d &\geq n \geq (2d - 1) + 2(d - x' + 1) + c \\ 0 &\geq 1 + (c - 2x'), \end{aligned}$$

which is a contradiction as $c \geq 2x'$. Therefore there is a cycle through all the weighted vertices in $Q$. □

**Case 2:** $3 \leq \alpha \leq 4$, $\beta = 1$. Here, we find a path (or cycle) through all the weighted vertices in the component of largest weight, with an endpoint on $C$ and show that this is sufficient for a path of the required weight.

Now, $Q_1$ is the component of weight two. Suppose $x_{Q_1} \geq 2$. Since the other components have weight one, their combined weight is $\alpha - 1$. Therefore, $w_{Q_1} = w - w_C - (\alpha - 1) \geq w - w_C - 4$. There is a path with an endpoint on $C$ through all the weighted vertices in $Q_1$ by Lemma 7.8. So, by Corollary 7.3(1), there is a path in $G$ of weight at least $w - w_C - 4 + w_C = w - 4 = 3l - 3 \geq 2l + 1$ since $l \geq 4$. This is a contradiction, so $x_{Q_1} = 1$. By Lemma 8.14, there is a cycle through all the weighted vertices in $Q_1$, and so by Corollary 7.3(2), there is a path of weight at least $2l + 1$, as above. This completes the proof for $\alpha \geq 3$ and $\beta = 1$. □

The cases remaining are when $\alpha \leq 4$ and $\beta \geq 2$, and when $\alpha \leq 2$. As before, $w_C \leq \min\{2l - 3, d - 3\}$. We continue with the case where $\alpha = 4$ and $\beta \geq 2$.

**Case 3:** $\alpha = 4$, $\beta \geq 2$. By Lemma 8.13, there is a cycle in each of the components of weight at least two. We show that two of these cycles, including a component $Q_1$ of greatest weight, can be combined with $C$ to give a path of weight at least $\min\{2l+1, d+1\}$, by Corollary 7.3. We first show that the combined weights of two of the components satisfies the conditions for Corollary 7.3. The components are $Q_1$, $Q_2$, $Q_3$ and $Q_4$ where $w_{Q_1} \geq w_{Q_2} \geq w_{Q_3} \geq w_{Q_4}$. Therefore $w_{Q_1} \geq \lceil (w - w_C - w_{Q_3} - w_{Q_4})/2 \rceil$. Now, for $g = 2$ or $3$, $w_{Q_1} + w_{Q_g} + w_C \geq \lceil (w + w_C + (w_{Q_3} - w_{Q_4}))/2 \rceil \geq 2l + 1$. So, by Corollary 7.3, no vertex of $Q_2$ or $Q_3$ is adjacent to a vertex of $X^{+W}(Q_1 \cap W)$. We will obtain a contradiction.

We count the vertices in the components $Q_1$, $Q_2$ and $Q_3$, on the cycle $C$ and in the set of neighbours of $X^{+W}(Q_1 \cap W)$. Now, $|Q_1| \geq d - x_{Q_1} + 1$, $|Q_2| \geq d - s + 1$, and $Q_3 \geq d - x' + 1$. We consider the two cases $x_{Q_1} \geq 2$, and $x_{Q_1} \leq 1$.

(1) We deal with $x_{Q_1} \geq 2$ first. Now, $|\Gamma(X^{+W}(Q_1 \cap W)) \cup C| \geq 3(x_{Q_1} - 2) + 2d$, by Corollary 5.10 with $j \geq 3$. Using Lemma 5.7 with $i = 1$ we obtain

$$\begin{aligned} 4d &\geq n \geq (d - x_{Q_1} + 1) + (d - s + 1) + (d - x' + 1) + 3(x_{Q_1} - 2) + 2d \\ 0 &\geq 1 + (d - s - x') + 2(x_{Q_1} - 2), \end{aligned}$$

which is a contradiction, as $d - s - x' \geq 0$, and $x_{Q_1} \geq 2$. This completes the case where $x_{Q_1} \geq 2$.

(2) As $x_{Q_1} \leq 1$, we have $|Q_1| \geq d$. Also $|Q_2| \geq d - s + 1$ and $|Q_g| \geq d - x' + 1$ for $g = 3$ or 4. We count the vertices in all the components, and on $C$, noting that $s \leq \lfloor c/3 \rfloor$ by Lemma 7.7, and $x' \leq \lfloor c/3 \rfloor + 1$ by Lemma 8.11. Applying Lemma 5.7, with $i = 0$, we have

$$\begin{aligned} 4d &\geq n \geq d + (d - s + 1) + 2(d - x' + 1) + c \\ 0 &\geq 1 + (c - s - 2x' + 2), \end{aligned}$$

which is a contradiction. This completes the proof of the $\alpha = 4$ and $\beta \geq 2$ case. □

**Case 4:** $\alpha = 3$, $\beta \geq 2$. In this case we show that either $w_{Q_1} \geq \lceil (w - w_C - 1)/2 \rceil$, and there is a path of weight at least $w_{Q_1} + w_C$ or we find a path through all the weight in one component, the cycle $C$ and half the weight in either of the other two components.

(1) For $\beta = 2$, $w_{Q_1} \geq \lceil (w - w_C - 1)/2 \rceil$. Now suppose, $w_{Q_1} \geq \lceil (w - w_C - 1)/2 \rceil$ for $\beta = 2$. Suppose $x_{Q_1} \geq 2$. Then, by Corollary 7.9, there is a path of weight at least $2l + 1$, giving a contradiction. Therefore $x_{Q_1} \leq 1$. By Lemma 8.14, there is a cycle in $Q_1$ through all the weighted vertices, which again gives a contradiction. This completes the case for (i) $\alpha = \beta = 3$ and $w_{Q_1} \geq \lceil (w - w_C - 1)/2 \rceil$, and (ii) $\alpha = 3$, $\beta = 2$.

(2) Now, we consider the remaining cases for $\alpha = 3 = \beta$. We note that for $g = 2$ or 3, we have $w_{Q_1} + \lceil w_{Q_g}/2 \rceil \geq \lceil (w + w_C)/2 \rceil$. This weight combined with $w_C$ is at least $\min\{2l+1, d+1\}$. Now $|Q_1| \geq d - x_{Q_1} + 1$ and $|Q_g| \geq d - s + 1$ for $g = 2$ or 3. Let $P = p_1 p_2 \ldots p_t$ be a maximum path in $Q_1$ such that $x_{p_1}$ is the maximum over all endpoints of maximum paths in $Q_1$.

(2.1) Suppose $x_{p_1} \geq 2$. Then there is a path through all the weighted vertices in $Q$. with an endpoint adjacent to $C$, that is, $w_P = w_Q$, by Lemma 7.8(1). No vertex of $Q_2 \cup Q_3$ is adjacent to a vertex of $X^{+W}(p_1 \cap W)$. So we have $\left|\Gamma\left(X^{+W}(p_1)\right) \cup C\right| \geq 3(x_{p_1} - 2) + 2d$ by Corollary 5.10. Counting the vertices in each of the components. on $C$ and neighbours of $X^{+W}(p_1)$, we obtain

$$\begin{aligned} 4d &\geq n \geq (d - x_{p_1} + 1) + 2(d - s + 1) + 3(x_{p_1} - 2) + 2d \\ 0 &\geq 1 + (d - s - x_{p_1}) + 2(x_{p_1} - 2). \end{aligned}$$

By Lemma 8.10, we have upper bounds for $x'' \geq x_{p_1}$ and $s$. So we have

$$\begin{aligned} 0 &\geq 1 + (d - (\lfloor (d-1)/2 \rfloor - 1) - (\lfloor d/2 \rfloor + 1)) + 2(x_{p_1} - 2) \\ &\geq 1 + 2(x_{p_1} - 2), \end{aligned}$$

which is a contradiction as $x_{p_1} \geq 2$. This completes the case for $x_{p_1} \geq 2$.

(2.2) Now $x_{p_1} \leq 1$. The weight of $Q_2$ and $Q_3$ combined is

$$w_{Q_2} + w_{Q_3} \geq w - w_C - w_{Q_3} \geq \lceil (w - w_C)/2 \rceil.$$

As $x_{p_1} \leq 1$, there is a cycle through all the weighted vertices in $Q_1$ by Lemma 8.14. Also we have $|Q_1| \geq d$.

Now if $x_{Q_g} \leq 1$, there is a cycle through all the weighted vertices in $Q_g$, by Lemma 8.14. Now suppose $x_{Q_g} \geq 2$. Suppose there is no cycle of weight $w_{Q_g}$ in $Q_g$, for $g = 1$ or 2. Then $|Q_g| \geq 2(d - x_{Q_i}) + 1$. Now no vertex of $Q_1$ is adjacent to a neighbour of $Q_3$. So counting the vertices in $Q_1$, $Q_g$ and $C$ and the neighbours of $X^{+W}(Q_g \cap W)$, we have

$$\begin{aligned} 4d &\geq n \geq 2(d - x_{Q_g}) + 1 + d + 3(x_{Q_g} - 2) + 2d \\ 0 &\geq d - 2 + x_{Q_g}, \end{aligned}$$

which is a contradiction as $d \geq 8$. Therefore there is a cycle through all the weighted vertices in each of the components.

So no vertex in any of the weighted components is adjacent to the vertex of $X^{+W}(Q_i \cap W)$, for $i \in \{1,2,3\}$ as $w_{Q_2} + w_{Q_1} \geq w_{Q_3} + w_{Q_1} \geq w_{Q_3} + w_{Q_2} \geq \lceil (w - w_C)/2 \rceil$.

Let $x_g = \max\{x_{Q_2}, x_{Q_3}\}$
(i) Suppose $x_g \geq 2$. Then counting the vertices in all the components, on $C$ and the neighbours of $X^{+W}(Q_g \cap W)$, we obtain

$$\begin{aligned} 4d &\geq n \geq d + 2(d - s + 1) + 3(x_g - 2) + 2d \\ 0 &\geq d - 2s + 2 + 3(x_g - 2) \end{aligned}$$

But by Lemma 8.10, $s \leq \lfloor (d-1)/2 \rceil$, so we have

$$0 \geq 3 + (d - 1 - 2\lfloor (d-1)/2 \rfloor) + 3(x_g - 2),$$

which is a contradiction.

(ii) Suppose $x_g \leq 1$. By Result 8.3, we have $x' \geq 1$. So for some $i \in \{1,2,3\}$, $x_{Q_i} = 1$. Then all the components have size at least $d$ and there are at least $d+1$ vertices in $\Gamma(X^{+W}(Q_i \cap W)) \cup X^{+W}(Q_i \cap W)$. Therefore counting the vertices in each of the components and these neighbours, we have $4d \geq n \geq 4d + 1$ which is a contradiction.

This completes the case. □

Case 5: $\alpha = 2 = \beta$. (1) Suppose $x_{Q_1} \geq 2$. Then there is a path through all the weighted vertices in $Q_1$ with an endpoint adjacent to $C$. This gives a contradiction as $Q_1$ contains at least half the weighted vertices in $G \setminus C$.

(2) Therefore $x_{Q_1} \leq 1$. Now, suppose there is no path through all the weighted vertices in $Q_1$. Then $|Q_1| \geq 3(d-1)+1 = 3d-2$, by Theorem 2.5. Counting the vertices in each of the two components and on $C$, we have

$$\begin{aligned} 4d &\geq n \geq (3d-2)+(d-s+1)+c \\ 0 &\geq c-s-1 \\ &\geq w_C - \lfloor w_C/2 \rfloor - 1 \\ &\geq \lceil (l+1)/2 \rceil - 1 \\ &\geq 1, \end{aligned}$$

since $l \geq 4$. So there is a path through all the weighted vertices in $Q_1$. Suppose a path through all the weighted vertices in $Q_1$ has an endpoint adjacent to $C$. Then we are done. So suppose not. Then $|Q_1| \geq 2d+1$. Now we consider the other component $Q_2$. Let $P$ be a path of maximum weight in $Q$ where $p_t$, an endpoint of $P$ is such that $x_{p_t}$ is maximised over all paths of weight $w_{Q_2}$ in $Q_2$.

(i) Suppose $x_{p_t} \geq 2$. Then there is a path $P$ through all the weighted vertices in $Q_2$ with an endpoint adjacent to $C$. Now, suppose a vertex of $Q_1$ is adjacent to a vertex of $X^{+W}(p_t)$. Then there is a path in $G$ through all the weighted vertices in $Q_2$ and $C$, and half the weighted vertices in $Q_1$. This gives a path of weight at least $2l+1$. Therefore $\Gamma\left(X^{+W}(p_t)\right) \cap Q_1 = \emptyset$. So, counting the vertices in $Q_1$, $Q_2$ and $\Gamma\left(X^{+W}(p_t)\right) \cup C$, we obtain

$$\begin{aligned} 4d &\geq n \geq (2d+1)+(d-s+1)+3(x_{p_t}-2)+2d \\ 0 &\geq 1+(d-s)+(c-2x'), \end{aligned}$$

which is a contradiction.

(ii) Suppose $x_{p_t} = 1$, then the vertex $u \in X^{+W}(p_t)$ is not adjacent to any vertex in either component, as before, so we can count all the neighbours of $u$, plus $u$ itself, as well as the vertices in both components. We have

$$4d \geq n \geq (2d+1) + d + (d+1),$$

which gives a contradiction.

(iii) Now $x_{p_1} = 0$. Then $|Q_2| \geq d+1$. By Lemma 8.8, there is a weighted vertex $v^{+W}$ on $C$ which has at most one neighbour in $Q_2$, where $v \in \Gamma(Q_2)$. Now $v$ is not adjacent to a vertex of $Q_1$, so counting the vertices in $Q_1$, $Q_2$ and $\{v^{+W}\} \cup \Gamma(v^{+W}) \setminus Q_2$, we have

$$4d \geq (2d+1) + (d+1) + d,$$

which is a contradiction. □

**Case 6:** $\alpha \leq 2$, $\beta = 1$. We consider the component $Q_1$ of weight two. Suppose $x_{Q_1} \geq 2$. Then there is a path of weight $w_{Q_1}$ in $Q_1$ with an endpoint adjacent to $C$. Hence we are done. So $x_{Q_1} \leq 1$. If there is a path through all the weighted vertices in $Q_1$, then there is a path in $G$ of weight at least $w_C + \lceil w_{Q_1}/2 \rceil \geq 2l+1$. So we assume not. We find a set $D$ for the two cases $x_{Q_1} = 1$ and $x_{Q_1} = 0$ which we remove from $G$. If $x_{Q_1} = 1$, let $u$ be a vertex in $X(Q_1 \cap W)$. Then $u^{+W}$ is not adjacent to any vertex in $Q_1$, so there are at least $d$ vertices not in $Q_1$, nor adjacent to weighted vertices of $Q$, namely $D = \Gamma(u^{+W}) \setminus \{u\}$. If $x_{Q_1} = 0$, there is a vertex $v$ on $C$ which does not have more than one neighbour $p$, say, in a weighted component, by Lemma 8.8. Then the set $D = C \cup \Gamma(v) \setminus \{p\}$ contains no vertices of $Q_1$, nor any neighbours of weighted vertices of $Q_1$. In both cases, we remove $D$ from $G$. Then we are removing at least $d$ vertices, so $|Q_1| \leq 3d$. We also do not change the degree of any vertex in $Q_1$. So the minimum degree of the vertices of $Q_1 \cap W$ in $G \setminus D$ is $d$. Therefore, by Theorem 2.5, there is a path through $\min\{d+1, w_{Q_1}\}$ weighted vertices in $Q_1$, and we are done. □

This completes the proof of the second base case which completes the proof of Theorem 4.4. $\square$

# 9 Degree Sums and Neighbourhood Unions

In this chapter, which concludes the work on graphs, we generalise some results which have degree sum conditions. These are known as *Ore-like* results. We also discuss some results concerning the number of vertices in the set containing all the neighbours of vertices in an independent set.

## 9.1 Ore-like Results

We start our discussion with the following well known result of Ore.

**Theorem 9.1 (Ore)** *Let $G$ be a graph of order $n \geq 3$. If for every pair of non-adjacent vertices $x$ and $y$, $d(x) + d(y) \geq n$, then $G$ is Hamiltonian.*

Bollobás and Brightwell [3] extend Theorem 9.1 to a subset of specified vertices for which the degree sum of non-adjacent vertices is known. This result is given in the following theorem.

**Theorem 9.2 (Bollobás and Brightwell)** *Let $G$ be a graph on $n$ vertices, containing a set $W$ of $w$ vertices of $G$ such that each pair $w_1$ and $w_2$ of non-adjacent vertices of $W$ satisfies $d(w_1) + d(w_2) \geq c$. If $c = 2d$ is even, set $k = \lceil n/d \rceil$: if $c = 2d + 1$ is odd, set $k = \lceil (n+1)/(d+1) \rceil$. Now set $s = \lceil w/(k-1) \rceil$. If $s \geq 2$, then there is a cycle through at least $s$ vertices of $W$.*

As we saw earlier (Theorem 3.3), the analogue of Theorem 9.1 for paths is easily generalised.

We now turn to some generalisations of other results which are useful tools in problems concerning paths and cycles. The proofs in this section are very similar to standard proofs of the corresponding results for $W = V(G)$, unless otherwise stated.

**Lemma 9.3** *Let $G$ be a graph on $n$ vertices, and $W$ a subset of vertices. Suppose that $d(u) + d(v) \geq n$ for a pair of non-adjacent vertices $u$, $v$ in $W$. Then there is a cycle through all the vertices of $W$ in $G + (u, v)$ if and only if there is in $G$.*

**Proof.** Suppose there is a cycle $C$ through all the vertices of $W$ in $G$, that is, a cycle of weight $w_C = w$. Then obviously there is a cycle of weight $w$ in $G + (u, v)$.

Now suppose there is a cycle of weight $w$ in $G \cup (u, v)$, but no cycle of weight $w$ in $G$. Therefore there is a path $P$ of weight $w$ in $G$, with endpoints $u$ and $v$. Let $D_1 = \Gamma(u) \cup \{u\}$ and $D_2 = \Gamma_P^+(v) \cup \Gamma_{G \setminus P}(v)$. So $|D_1| = d(u)+1$ and $|D_2| = d(v)$ and, as usual, $D_1 \cap D_2 = \emptyset$. Therefore $n \geq |D_1 \cup D_2| = |D_1| + |D_2| = d(u)+d(v)+1 \geq n+1$ which is a contradiction. So there is a cycle in $G$ of weight $w$ as required. $\square$

For a graph $G$ and subset $W$, we define the *closure* $C(W)$ of $W$ in $G$ to be the graph obtained from $G$ by repeatedly adding edges $(u, v)$ whenever $d(u) + d(v) \geq n$, for two non-adjacent vertices $u$ and $v$ of $W$.

**Corollary 9.4** *Let $G$ be a graph containing a subset of vertices $W$. Then there is a cycle through all the vertices of $W$ in $C(W)$ if and only if there is in $G$.*

**Proof.** This follows immediately from repeated application of Lemma 9.3. $\square$

We now come to another property of graphs. A graph is *Hamiltonian connected* if there is a Hamiltonian path between every pair of vertices in $G$. The following result gives a sufficient condition for a graph to be Hamiltonian connected. (See, for example, [14].)

**Theorem 9.5** *Let $G$ be a graph on $n$ vertices. Suppose that $d(u) + d(v) \geq n + 1$ for all non-adjacent vertices $u$ and $v$. Then $G$ is Hamiltonian connected.* $\square$

The next theorem is an analogue of Theorem 9.5 for specified vertices where we are interested in paths through all the specified vertices between all pairs of specified vertices. We define an $(x, y)$-full weight path to be a path of weight $w$ between the vertices $x$ and $y$ of $W$.

**Theorem 9.6** *Let $G$ be a graph and $W$ a subset of the vertices of $G$ such that $d(u) + d(v) \geq n + 1$ for every pair of independent vertices $u$ and $v$ of $W$. Then, for any pair of vertices $a$ and $b$ of $W$, there is an $(a, b)$-full weight path.*

**Proof.** We may assume there are no edges between vertices not in $W$. Let $a$ and $b$ be vertices of $W$ and suppose that there is no $(a, b)$-full weight path. Since $d(u) + d(v) > n$ for all non-adjacent vertices $u$, $v$ in $W$, there is a cycle through all the vertices of $W$, by Theorem 9.2. Therefore there is a path with an endpoint $a$. Let $P = a \dots zb \dots t$ be the shortest path through all the vertices of $W$ with endpoint $a$. We consider the neighbours of $z = b^{-W}$, that is the weighted predecessor of $b$ on the path, and $t$. Suppose $z$ and $t$ are adjacent. Then there is a path $a \dots ztt^{-} \dots b$ which is a contradiction. Therefore $d(z) + d(t) \geq n + 1$. We consider the neighbours of $z$ and $t$ on the sections of the path $[a, z]$ and $(z, t]$. We consider the sets, $\Gamma_{[a,z^-]}(z)$, $\Gamma_{[z^+,t]}(z)$, $\Gamma^{-}_{[a,z]}(t)$, $\Gamma^{+}_{[z^+,t]}(t)$, $\Gamma_{G\setminus P}(z)$ and $\Gamma_{G\setminus P}(t)$. We claim that these sets are pairwise disjoint. (See Figure 18.) The three sets of neighbours of $z$, namely $\Gamma_{[a,z^-]}(z)$, $\Gamma_{[z^+,t]}(z)$ and $\Gamma_{G\setminus P}(z)$, are certainly pairwise disjoint, as are

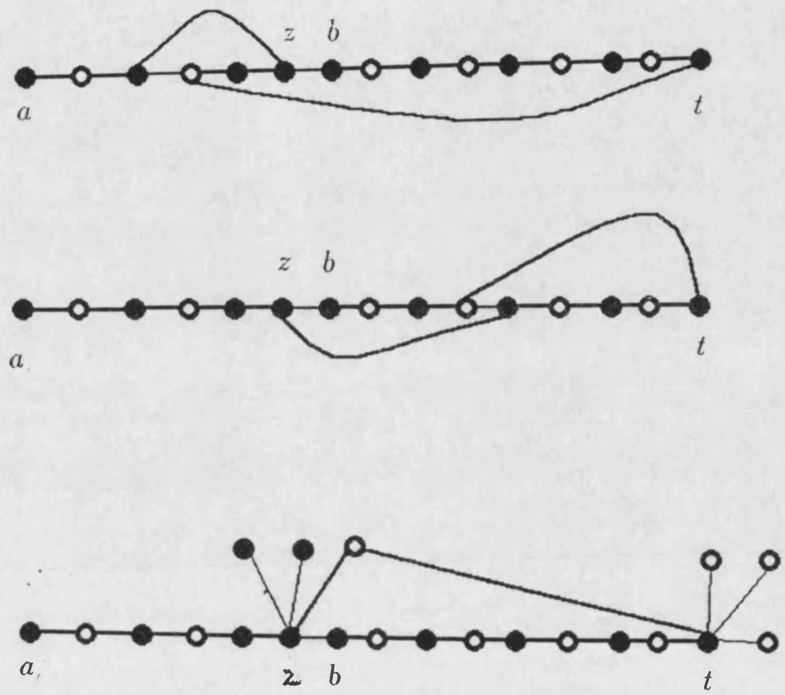


Figure 18: Disjoint sets of vertices

$\Gamma^-_{[a,z]}(t)$, $\Gamma^+_{[z^+,t]}(t)$ and $\Gamma_{G\backslash P}(t)$. We consider $\Gamma_{[a,z^-]}(z) \cap \Gamma^-_{[a,z]}(t)$. Suppose there is a vertex $s$, say, in this intersection. Then the path

$$a \ldots szz^- \ldots s^+tt^- \ldots b$$

is a $(a,b)$-full weight path, which is a contradiction. Therefore $\Gamma_{[a,z^-]}(z) \cap \Gamma^-_{[a,z]}(t) = \emptyset$. Similarly $\Gamma_{[z^+,t]}(z) \cap \Gamma^+_{[z^+,t]}(t) = \emptyset$. Also the sets $\Gamma_{G\backslash P}(z)$ and $\Gamma_{G\backslash P}(t)$ are disjoint as a shared neighbour $q$ off the path would result in the path $a \ldots zqtt^- \ldots b^+b$ which is again a contradiction. Now we consider the size of the union of all these sets. We have

$$
\begin{aligned}
n &\geq |\Gamma_{[a,z^-]}(z) \cup \Gamma_{[z^+,t]}(z) \cup \Gamma_{G\setminus P}(z) \cup \Gamma^-_{[a,z]}(t) \cup \Gamma^+_{[z^+,t]}(t) \cup \Gamma_{G\setminus P}(t)| \\
&= |\Gamma_{[a,z^-]}(z) \cup \Gamma_{[z^+,t]}(z) \cup \Gamma_{G\setminus P}(z)| + | \cup \Gamma^-_{[a,z]}(t) \cup \Gamma^+_{[z^+,t]}(t) \cup \Gamma_{G\setminus P}(s)| \\
&= |\Gamma(z)| + |\Gamma(t)| \\
&= d(z) + d(t) \\
&\geq n + 1,
\end{aligned}
$$

which is a contradiction. Therefore, there is an $(a, b)$-full weight path, and therefore between any pair of vertices in $W$. □

An obvious corollary to this for a subset $W$ of vertices of minimum degree $(n+1)/2$ is given below.

**Corollary 9.7** *Let $G$ be a graph on $n$ vertices containing a subset of vertices $W$ of minimum degree $(n+1)/2$. Then for any pair $a$, $b$ of vertices of $W$, there is a $(a, b)$-full weight path.* □

## 9.2 Neighbourhood Unions

Now, instead of looking at the sum of the degrees of independent vertices, we will look at the union of their neighbour sets. We start with a theorem of Faudree, Gould, Jacobson and Schelp [11] which considers the neighbourhood sets of a pair of non-adjacent vertices.

**Theorem 9.8 (Faudree, Gould, Jacobson and Schelp)** *Suppose that $G$ is a 2-connected graph on $n \geq 3$ vertices. If for every pair of non-adjacent vertices $x$ and $y$, we have*

$$|\Gamma(x) \cup \Gamma(y)| \geq \frac{(2n-1)}{3},$$

*then $G$ is Hamiltonian.* □

A generalisation of this theorem was conjectured by Faudree, Gould, Jacobson and Schelp [11] and proved by Fraisse [13].

**Theorem 9.9 (Fraisse)** *Let a graph $G$ of order $n \geq 3$ be $k$-connected. Suppose there exists some $t \leq k$ such that for every set $S$ of $t$ mutually non-adjacent vertices, $|\Gamma(S)| > \frac{t(n-1)}{t+1}$. Then $G$ is Hamiltonian.* □

This is easily generalised for specified vertices. The proof is included for completeness, although it is very similar to that given in [14].

**Theorem 9.10** *Let $G$ be a $k$-connected graph of order $n \geq 3$ and $W$ a subset of the vertices of $G$. Suppose there exists some $t \leq k$ such that for every set $S$ of $t$ mutually non-adjacent vertices of $W$, $|\Gamma(S)| > \frac{t(n-1)}{t+1}$. Then there is a cycle through all the vertices of $W$.*

**Proof.** Let $C = v_1 v_2 \ldots v_c$ be a maximum cycle in $G$. Suppose $C$ does not contain all the vertices of $W$. Then there is a weighted vertex $v$ in $G \setminus C$. As $G$ is $k$-connected, and $t \leq k$, there is a family of $t$ vertex disjoint paths from $v$ to $C$ (Menger [18]). Let $x_1, x_2, \ldots, x_t \in C$ be the endpoints of the paths, labelled cyclically around $C$. Let $X = \{x_1, \ldots, x_t\}$, and let $U = X^{+W} \cup \{v\}$. Then we define the following sets:

$$\begin{aligned} A &= G \setminus \Gamma(U) \\ B &= \{v \in V : (u,v) \in E(G), \text{ for exactly one } u \in U\} \\ D &= \Gamma(U) \setminus B \\ A_C &= A \cap C \\ A_R &= A \cap (G \setminus C) \end{aligned}$$

We note that $A$ is the set of vertices which are not neighbours of vertices of $X^{+W}$, the weighted successors of vertices of $X$, or of $v$. Also $A_C$ are the vertices which are not neighbours of vertices of $U$ on $C$, and $A_R$, the rest of the non-neighbours of vertices of $U$. Now, no vertices of $U$ are adjacent, by Lemma 5.2, and so $U \subseteq A$. Also no two vertices of $U$ share a neighbour in $G \setminus C$.

**Claim** *There are at most t vertices of D between any two consecutive vertices of A on C.*

Let $a_1$ and $a_2$ be two consecutive vertices of $A$ on $C$. Since $X^{+W} \subseteq A$, we may assume that $a_1$ and $a_2$ belong to some segment of $C$ between consecutive vertices of $X^{+W}$, $[x_l^{+W}, x_{l+1}^{+W}]$. Suppose $c_s \in \Gamma_{[a_1,a_2]}(x_i^{+W})$ and $c_{s+1} \in \Gamma_{[a_1,a_2]}(x_j^{+W})$. Then the vertices $x_j$, $x_l$ and $x_i$ cannot occur in that order on $C$ for that would produce a cycle through more vertices of $W$. (See Figure 19.)

Now, by definition of $x_i$ and $x_{i+1}$ as consecutive neighbours of $v$ on $C$, there is at most one neighbour of $v$ in $[a_1, a_2]$, namely $x_{l+1}$. Every vertex of $(a_1, a_2)$ is in $\Gamma(x_i^{+W})$ for some $i$, since $a_1$ and $a_2$ are consecutive non-neighbours of $U$ on $C$. Therefore the neighbours in $[a_1, a_2]$ of consecutive vertices of $X^{+W}$ and of $v$, that is, $\Gamma_{[a_1,a_2]}(x_l^{+W}), \Gamma_{[a_1,a_2]}(x_{l-1}^{+W}), \ldots, \Gamma_{[a_1,a_2]}(x_{l+1}^{+W}), \Gamma(v)$ form consecutive segments of $[a_1, a_2]$ since $a_1$ and $a_2$ are consecutive vertices in $A$. These segments have at most their endpoints in common. There are $t+1$ segments and so at most $t$ elements of $D$ between $a_1$ and $a_2$, which justifies our claim.

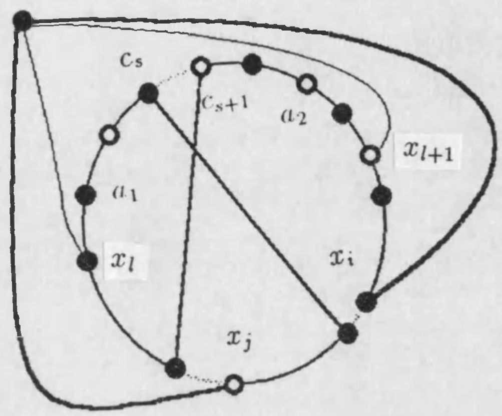


Figure 19: Prohibited pairs of neighbours

To complete the proof, let $p = |\Gamma_B(x_j)|$, where $x_j \in U$ gives the maximum value of $p$. Then $p \geq \dfrac{|B|}{t+1}$. As vertices of $U$ have no common neighbours in $G \setminus C$, the total number of such unshared neighbours of $U$ is $|B \setminus C| = n - c - |A_R|$. From the claim, the number of vertices of $B$ on $C$ is at least $c - (t+1)|A_C|$. Therefore

$$\begin{aligned}
|B| &= |B \cap C| + |B \setminus C| \\
&\geq n - c - |A_R| + c - (t+1)|A_C| \\
&= n - |A_R| - (t+1)|A_C|
\end{aligned}$$

and

$$p \geq \frac{|B|}{t+1} \geq \frac{n}{t+1} - \frac{|A_R|}{t+1} - |A_C|.$$

Now, $U \setminus \{x_j\}$ is an independent set of $t$ vertices, and $|A_R| \geq 1$ since $v \in A_r$, so

$$|\Gamma(X^{+W} \setminus \{x_j\})| \geq n - |A| - p$$

$$
\begin{aligned}
&= n - |A_R| - |A_C| - p \\
&\le \frac{t(n - |A_R|)}{t+1} \\
&\le \frac{t(n-1)}{t+1},
\end{aligned}
$$

which is a contradiction, thus proving the result. □

# 10 Posets

Now we come to the other topic of the thesis, partial orders. We begin with some definitions and notation. A *partially ordered set* or *poset* is a pair $(Z, <)$ where $Z$ is a set, called the *ground set*, and $<$ is an order relation on $Z$. Let $x$ and $y$ be distinct elements of $Z$. If $x < y$ or $y < x$ in $P$ then $x$ and $y$ are *comparable.* Otherwise $x$ and $y$ are *incomparable*, and we write $x||y$. (See Figure 20.) A *chain* is a poset where each pair of distinct points is comparable, and an *antichain* is a poset where each pair of distinct points is incomparable. The *width* of a poset is the maximum size of an antichain. (See Figure 20.)

The following theorem of Dilworth [7] gives an equivalent definition of the width of a partial order.

**Theorem 10.1 (Dilworth)** *Let $P = (Z, <)$ be a poset of width $k$. Then there is a partition of $P$ into $k$ chains.* □

So a poset $P$ of width two can be partitioned into two chains $X_1$ and $X_2$. We call $(X_1, X_2, <)$ a *two-chain covering* of $P$. (See Figure 20.)

Let $P = (Z, <)$ be a poset on the set $Z = \{1, 2, \ldots, N\}$. Then we call $P$ a *labelled poset.* Let $C(P)$ be the class of all posets isomorphic to $P$, that is, $P' \in C(P)$ if and only if there is a bijection between $P$ and $P'$ that preserves all the order relations. The classes $C(P)$ form a partition of the labelled posets. We define an *unlabelled poset* to be an isomorphism class of labelled posets. Where no confusion arises, we will refer to the class as a poset.

Our aim is to count, asymptotically, the number of posets of width two. We ask

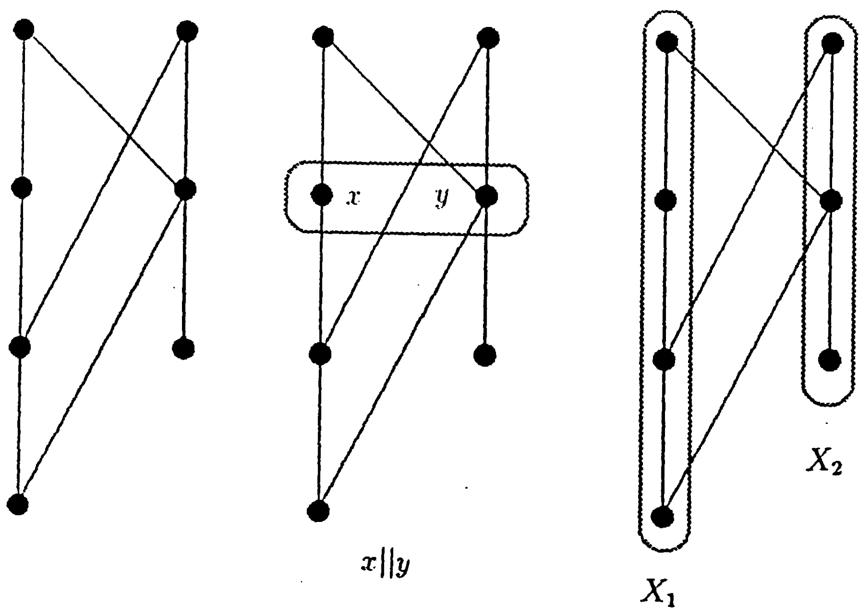


Figure 20: Chains, antichains and posets of width two

how many labelled and how many unlabelled posets there are. We also consider the relationship between the asymptotic numbers of these two sets of posets.

We introduce some notation for the number of width two posets.

- Let $\Omega_2(N)$ be the number of labelled width two posets on $\{1, 2, \ldots, N\}$.
- Let $\Omega_2^u(N)$ be the number of unlabelled width two posets on $N$ vertices.

We will prove the following two results.

**Theorem 10.2** *The number of width two posets with vertex set* $\{1, 2, \ldots, N\}$ *is*

$$\Omega_2(N) = \frac{(2N+1)!}{(N+2)!}\left(\frac{4}{25} + O\left(N^{-\frac{3}{8}}\right)\right).$$

Now we consider the relationship between the number of unlabelled and the number of labelled posets. The maximum size of an equivalence class of labelled posets corresponding to an unlabelled poset is $N!$. This occurs when a poset has a trivial automorphism group. So we observe that $\Omega_2^u(N) \geq \frac{\Omega_2(N)}{N!}$. This would be asymptotically sharp if almost every width two partial order has a trivial automorphism group, as, for example, is the case with graphs [2]. However, it is easy to see that this is not the case here. For example, consider the partial orders which have two incomparable elements $x$ and $y$ which are comparable to all the other elements. Simply exchanging $x$ and $y$ gives a non-trivial automorphism.

We compare Theorem 10.2 with the following theorem.

**Theorem 10.3** *The number of unlabelled posets on $N$ vertices is*

$$\begin{aligned}\Omega_2^u(N) &= \frac{1}{N!}\frac{(2N+1)!}{(N+2)!}\left(\frac{4}{25}\left(\frac{1}{1-\frac{4}{5}c}\right)^2 + O\left(N^{-\frac{3}{8}}\right)\right)\\ &\approx \frac{1}{N!}\frac{(2N+1)!}{(N+2)!}\left(\frac{4}{25}1.11645 + O\left(N^{-\frac{3}{8}}\right)\right),\end{aligned}$$

*where*

$$c = \sum_{k\geq 1}\frac{1}{2^{4k}k}\binom{2k-2}{k-1} \approx 0.0669873.$$

So

$$\Omega_2^u(N) \approx \frac{1.12}{N!}\Omega_2(N).$$

We initially consider the problem of counting labelled width two posets by considering their relationship with unlabelled posets of width two.

## 10.1 Two Chain Coverings

The first step is to consider the natural mapping $\gamma$ from the set $\mathcal{P}$ of labelled posets on the set $\{1, 2, \ldots, N\}$ to the set $\mathcal{C}$ of unlabelled posets, where $\gamma(P)$ is the isomorphism class containing the labelled poset $P$. We define a *two chain covering of an unlabelled poset* $C$ to be an isomorphism class of two chain coverings of the labelled posets in the isomorphism class $C$. The *incomparability graph* $G(P)$ of a poset $P$ is a graph $G(P)$ on the ground set $Z$ of $P$ such that $(x, y)$ is an edge of $G(P)$ if and only if $x||y$ in $P$. Let $A_1, A_2, \ldots, A_k$ be the components of the incomparability graph, $G(P)$. We define $A_i < A_j$ to mean that $a_i < a_j$ in $P$ for all $a_i \in A_i$ and $a_j \in A_j$. Then there is a total ordering of the components where $A_1 < A_2 < \cdots < A_k$. (See Figure 21.) Let $P(A_i)$ be the poset $P$ restricted to the vertices of $A_i$. We call $P(A_i)$ a *factor* of $C$. The automorphism groups of $P(A_i)$ are either trivial or

contain just the identity and the mapping which swaps the chains $X_i$ and $Y_i$. A factor $P(A_i)$ with a non-trivial automorphism group is called *symmetric*. Let $C$ be an unlabelled poset and suppose there are $s$ symmetric factors in $G(C)$. Then the automorphism group $\mathrm{Aut}(C)$ has order $|\mathrm{Aut}(C)| = 2^s$.

Now we consider the mapping $\gamma : \mathcal{P} \to \mathcal{C}$, as defined above. Let $C \in \mathcal{C}$. Suppose there are $s$ symmetric factors in $P$. Then $\gamma$ maps $\dfrac{N!}{\mathrm{Aut}(C)} = \dfrac{N!}{2^s}$ labelled posets to $C(P)$.

**Lemma 10.4**

$$\Omega_2(N) = \sum_{C \in \mathcal{C}} |\gamma^{-1}(C)| = \sum_{C \in \mathcal{C}} \frac{N!}{2^s},$$

*where $s = s(C)$ is the number of symmetric factors of $C$ and the sum is taken over all the isomorphism classes of labelled posets.*

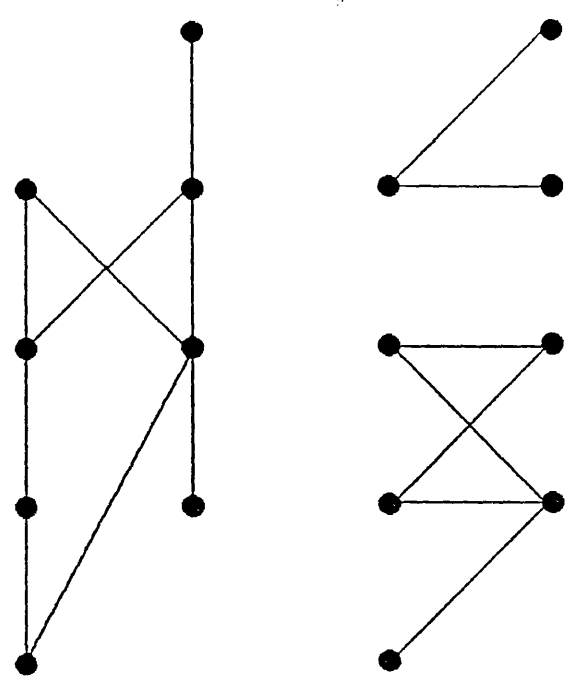

Figure 21: A poset of width two and its incomparability graph

Each factor $P(A_i)$ of $P$ has a two chain covering $(X_i, Y_i)$. To obtain a two chain covering $(X, Y, <)$, we order the factors $P(A_i)$ in the order they occur in $P$, and select either $X_i$ or $Y_i$ to be in $X$ and the other to be in $Y$. Now, how many such two chain coverings of an isomorphism class of posets $C$ are there? For each non-symmetric factor, we can choose one of two chains $X_i$ or $Y_i$. For a symmetric factor choosing $X_i$ is equivalent to choosing $Y_i$. So, letting $k$ be the number of factors and $s$ the number of symmetric factors, there are $2^{k-s}$ two chain coverings of $C$. (See Figure 22.)

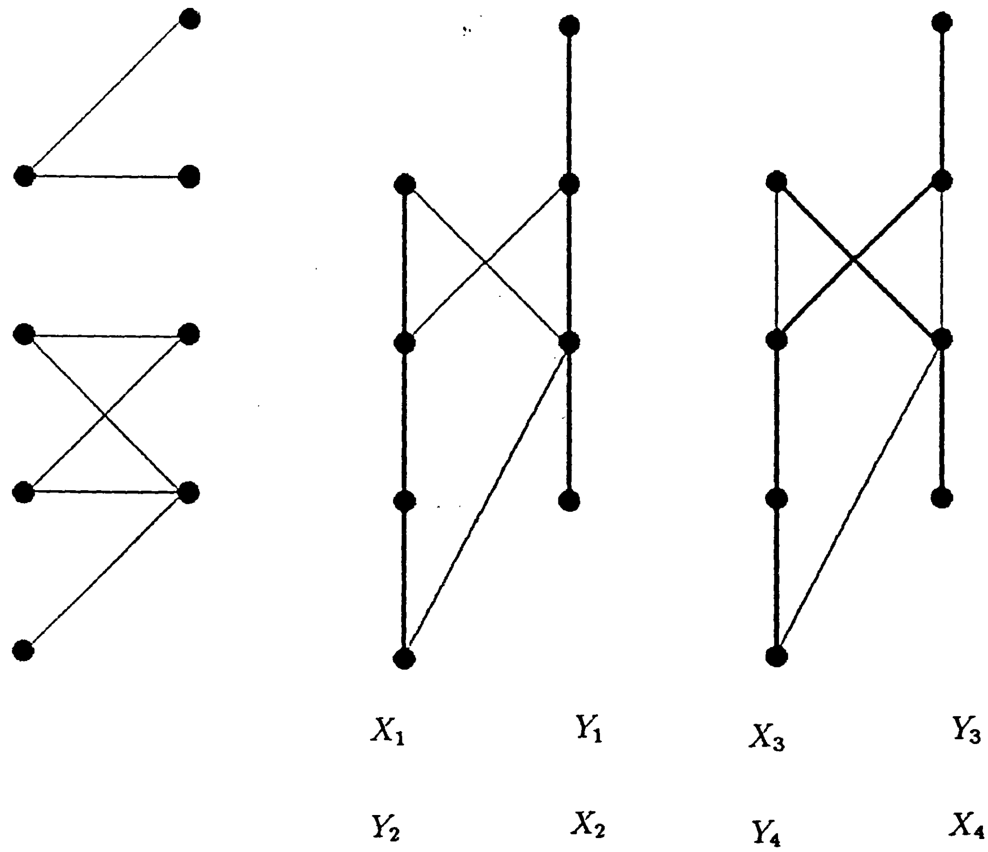


Figure 22: An incomparability graph $G(P)$ and the four two chain coverings of $P$

Let $\mathcal{T}$ be the set of isomorphism classes of two chain coverings. We consider the mapping $\omega : \mathcal{T} \rightarrow \mathcal{C}$, where $\omega$ maps a two chain covering $(X, Y, <)$ to an isomorphism class $C$ of labelled posets. Let $T(C) = \omega^{-1}(C)$, the set of two chain coverings of $C$. Suppose there are $s$ symmetric factors and $k$ factors in $G(C)$. Then $|T| = 2^{k-s}$. Therefore, we can count the number of posets in terms of their two chain coverings. (See Figure 23.)

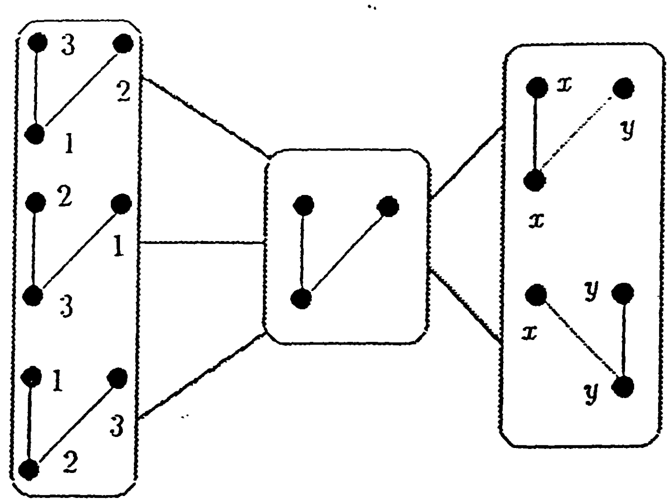


Figure 23: The correspondence between labelled posets, unlabelled posets and two chain coverings

**Lemma 10.5** *The number of labelled posets on $\{1, 2, \ldots, N\}$ is*

$$\Omega_2(N) = \sum_{T \in \mathcal{T}} \frac{N!}{2^k},$$

*where $k = k(T)$ is the number of factors in the incomparability graph of $\omega(T)$. The sum is taken over all isomorphism classes $T$ of two chain coverings $(X, Y, <)$ of unlabelled posets.*

**Proof.** By Lemma 10.4,

$$\Omega_2(N) = \sum_{C \in \mathcal{C}} \frac{N!}{2^s},$$

where $s = s(C)$ is the number of symmetric factors in the incomparability graph of $C$. There are $2^{k-s}$ two chain coverings of each $C \in \mathcal{C}$. So $\dfrac{|T(C)|}{2^{k-s}} = 1$ for each set $T(C)$ of two chain coverings.

Therefore

$$\begin{aligned}\Omega_2(N) &= \sum_{C\in\mathcal{C}} \frac{N!}{2^s}\frac{|T(C)|}{2^{k-s}} \\ &= \sum_{T\in\mathcal{T}} \frac{N!}{2^k},\end{aligned}$$

as required. □

**Lemma 10.6** *The number of unlabelled posets on $N$ vertices is*

$$\Omega_2^u(N) = \sum_{T\in\mathcal{T}} \frac{1}{2^{k-s}},$$

*where $k = k(T)$ is the number of factors and $s = s(T)$ is the number of symmetric factors in the incomparability graph of the unlabelled poset $\omega(T)$. The sum is taken over all isomorphism classes $T$ of two chain coverings $(X, Y, <)$ of unlabelled posets.*

**Proof.** For the unlabelled case, we have

$$\begin{aligned}\Omega_2^u(N) &= \sum_{C\in\mathcal{C}} 1 \\ &= \sum_{C\in\mathcal{C}} \frac{|T(C)|}{2^{k-s}}\end{aligned}$$

as again, $\frac{|T(C)|}{2^{k-s}} = 1$. So $\Omega_2(N) = \sum_{T\in\mathcal{T}} \frac{1}{2^{k-s}}$. □

## 10.2 Linear Extensions

Before we continue our discussion, we need a definition. A *linear extension* of a poset $P$ is a total ordering of the elements of $P$ which is consistent with their ordering in $P$. To count the numbers of posets, we consider a mapping which transforms our two counting problems into one more standard counting problem. We give an outline of this process before going into detail. We start with a poset $P = (Z, <)$ where $|Z| = N$, and a two chain covering $(X, Y, <)$ of $P$. We form two linear extensions $\lambda$ and $\mu$ of $P$ from the two chains $X$ and $Y$ (this section). Then we interleave the linear extensions $\lambda$ and $\mu$ to form a walk of length $2N$, (section 10.3). (See Figure 24.) Finally (Chapters 11 and 12) we count the walks generated in this way.

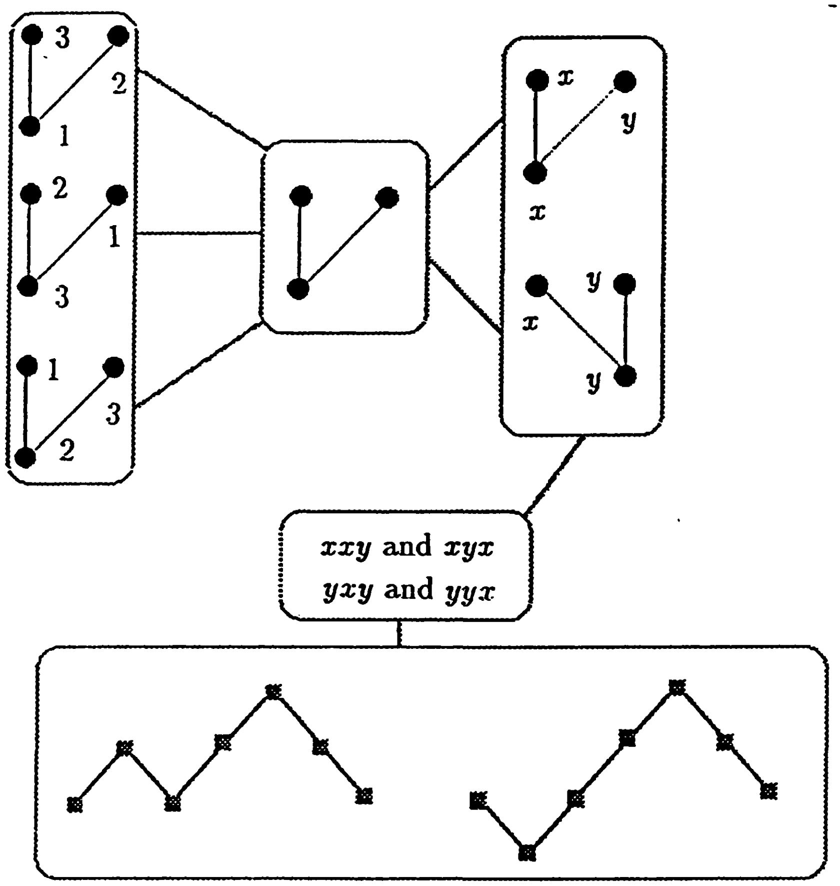


Figure 24: From posets to walks

Suppose the ordered pair $(X, Y, <)$ is a two-chain covering of some poset $P$ where

$X$ is the chain $(x_i)$, with $x_1 < x_2 < \cdots < x_n$, and $Y$ is the chain $(y_i)$ with $y_1 < y_2 < \cdots < y_m$. We say a linear extension $\lambda$ of $P$ is *left-greedy* if $\lambda x_i < \lambda y_j$ whenever $x_i || y_j$. (See Figure 25.) Similarly we say a linear extension $\lambda$ of $P$ is *right-greedy* if $\lambda y_j < \lambda x_i$ whenever $x_i || y_j$.

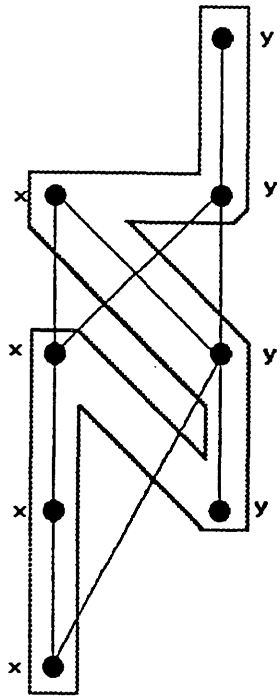


Figure 25: The construction of a left-greedy linear extensions

Given a partition of the ground set $Z$ into two chains, $X$ and $Y$, a width two partial order $P$ on $Z$ is determined by its left-greedy and right-greedy linear extensions. Indeed, they form a *realiser* of $P$, which is a set of linear extensions of $P$, the intersection of which is $P$. We will call an ordered pair $(\lambda, \mu)$, where $\lambda$ is the left-greedy linear extension of $P$ and $\mu$ is the corresponding right-greedy linear extension of $P$, a *greedy pair*. When do two linear orders on $\{x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_m\}$ make up a greedy pair? A necessary and sufficient condition is that if $x_i < y_j$ in the right-greedy linear extension, then $x_i < y_j$ in the left-greedy linear extension.

We call such a pair of linear extensions *allowable*. Let the set of allowable pairs of linear extensions obtained from chains $x_1 < x_2 < \cdots < x_n$ and $y_1 < y_2 < \cdots < y_m$ be $\mathcal{A}$. Define $\phi : \mathcal{T} \to \mathcal{A}$, where $\phi$ maps a two chain covering $(X, Y, <)$ to the greedy pair generated from it as above. Then $\phi$ is a bijection.

## 10.3 Walks

Let $(\lambda, \mu)$ be a greedy pair obtained from a two chain covering $(X, Y, <)$ of a poset $P$. Let $\lambda$ be the linear extension $l_1 < l_2 < \cdots < l_N$ and $\mu$ be $r_1 < r_2 < \ldots < r_N$. We construct a walk based on the order the elements of the chains $X = (x_i)$ and $Y = (y_j)$ occur in $(\lambda, \mu)$. We shall use geometric terminology and refer to the

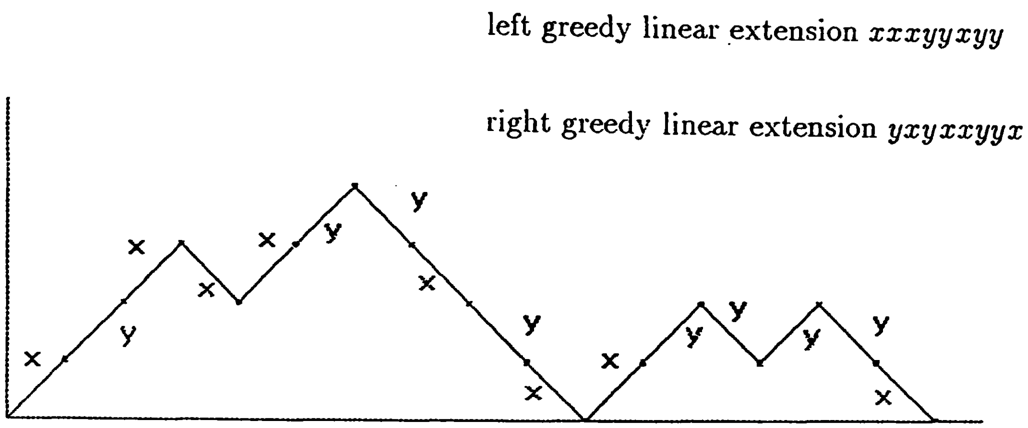


Figure 26: The construction of a walk from a greedy pair

rectangular coordinates $p, q$. We define $W = \theta(\lambda, \mu)$ as follows:

$$(p_0, q_0) = (0, 0),$$

and for $n \geq 1$

$$(p_{2n-1}, q_{2n-1}) = \begin{cases} (2n-1, q_{2n-2}+1) & \text{if } l_n = x_s \text{ for some } s; \\ (2n-1, q_{2n-2}-1) & \text{if } l_n = y_t \text{ for some } t, \end{cases}$$

$$(p_{2n}, q_{2n}) = \begin{cases} (2n, q_{2n-1}-1) & \text{if } r_n = x_s \text{ for some } s; \\ (2n, q_{2n-1}+1) & \text{if } r_n = y_t \text{ for some } t. \end{cases}$$

Therefore our walk $W$ starts at the point $(0,0)$ and ends at the point $(2N,0)$. Next we will consider which walks can arise from allowable greedy pairs. Let a *good walk* be a walk that starts at $(0,0)$ and ends at $(2N,0)$, and never falls below $p = -1$. (See Figure 27). Let $\mathcal{W}$ be the set of good walks. We consider the

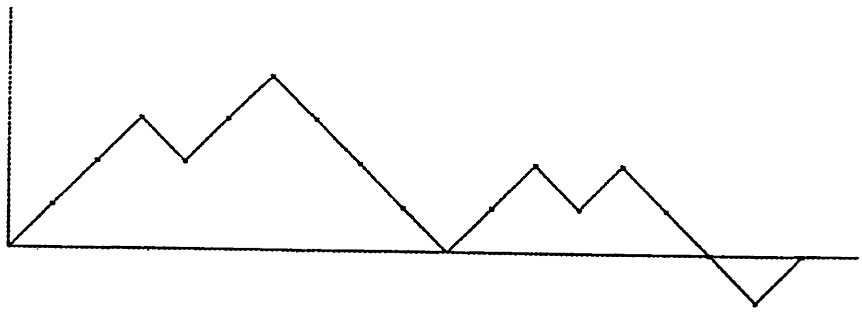

Figure 27: A good walk

mapping $\theta : \mathcal{A} \to \mathcal{W}$ between the pairs of allowable linear extensions and good walks. We claim that $\theta$ is a bijection. Every walk arising from an allowable pair starts at the point $(0,0)$ and ends at $(2N,0)$. This is because, in the left-greedy linear extension, every $x_i$ adds one to the $y$-coordinate and in the right-greedy linear extension, subtracts one. Suppose the walk $\theta(\lambda, \mu)$ falls below $-1$ at the point $(2\alpha, -2)$. Consider the first $\alpha$ elements of $\lambda$ and $\mu$. Then there is some $x_i$,

$i \leq \alpha$ in the first $\alpha$ elements of $\mu$ that is not in the first $\alpha$ elements of $\lambda$. This contradicts the definition of an allowable pair. Therefore $\theta$ is an injection from the allowable linear extensions to the good walks. Let $W$ be a good walk. Since $W$ does not go below $-1$, no vertex $x_i$ occurs before $y_j$ in $\mu$ but not in $\lambda$. Hence we have an allowable pair. Therefore $\theta$ is a surjection and thus a bijection. As there is a bijection between $\mathcal{A}$ and $\mathcal{W}$, and between $\mathcal{T}$ and $\mathcal{A}$, there is a bijection between $\mathcal{T}$ and $\mathcal{W}$.

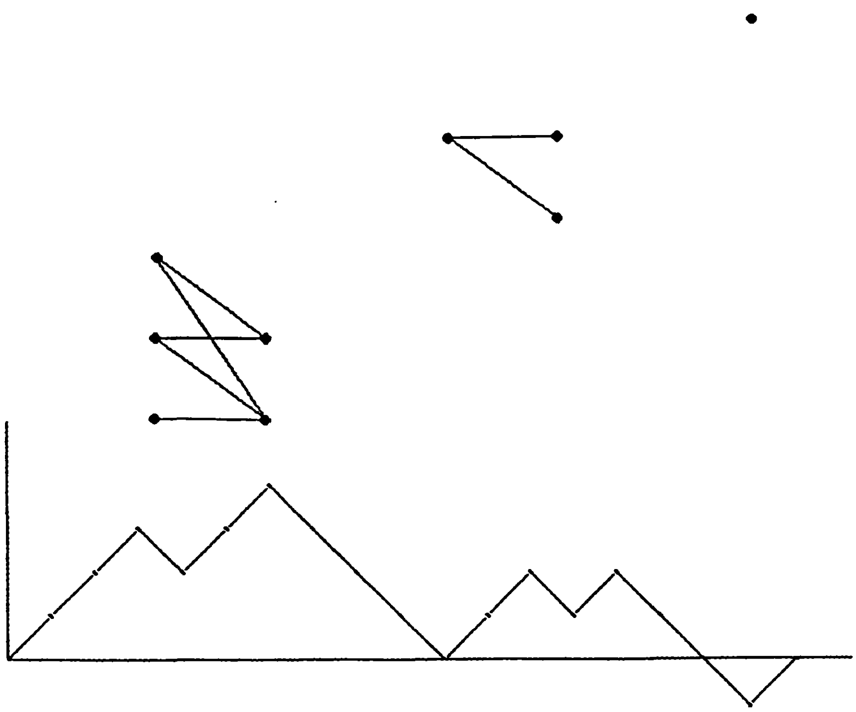

Figure 28: A good walk and the corresponding incomparability graph

We define a *hit* to be any point $(p, 0)$ where $p \neq 0$ or $2N$. Let $h$ be the number of hits. Now, we will consider the relationship between hits in a good walk and multiple two-chain coverings of a poset. Suppose a walk hits at $(2a, 0)$. Then at this hit the same $x_i$ and $y_j$ in both the left and right linear extensions have been represented in the walk from $(0,0)$ to $(2a, 0)$. The remaining $x_i$ and $y_j$ are the

$x_i$ and $y_j$ which are in higher component of the incomparability graph. So, the number $k$ of components in the incomparability graph is the number of times the walk returns to the $x$-axis, that is $h+1$, as we do not count the final return to $x=0$ as a hit, so $k=h+1$. (See Figure 28.)

Therefore from Lemma 10.5 in the labelled case we have the following lemma.

**Lemma 10.7** *The number of labelled posets on* $\{1, 2, \ldots, N\}$ *is*

$$\Omega_2(N) = \sum_{W \in \mathcal{W}} \frac{N!}{2^{h+1}},$$

*where* $h$ *is the number of hits of a good walk* $W$.

For the unlabelled case we need another definition. We define a *symmetric hit* to be a hit which occurs after a section of the walk which corresponds to a symmetric factor.

Let $s$ be the number of symmetric hits. (See Figure 29.) So we have the following lemma for unlabelled posets.

**Lemma 10.8** *The number of unlabelled posets on* $N$ *vertices is*

$$\Omega_2^u(N) = \sum_{W \in \mathcal{W}} \frac{N!}{2^{h-s+1}},$$

*where* $h$ *is the number of hits, and* $s$ *the number of symmetric hits of a good walk* $W$*, and the sum is over all good walks.*

We now introduce some definitions and notation.

- $G(A, B)$ is the number of good walks from $(0,0)$ to $(A, B)$.

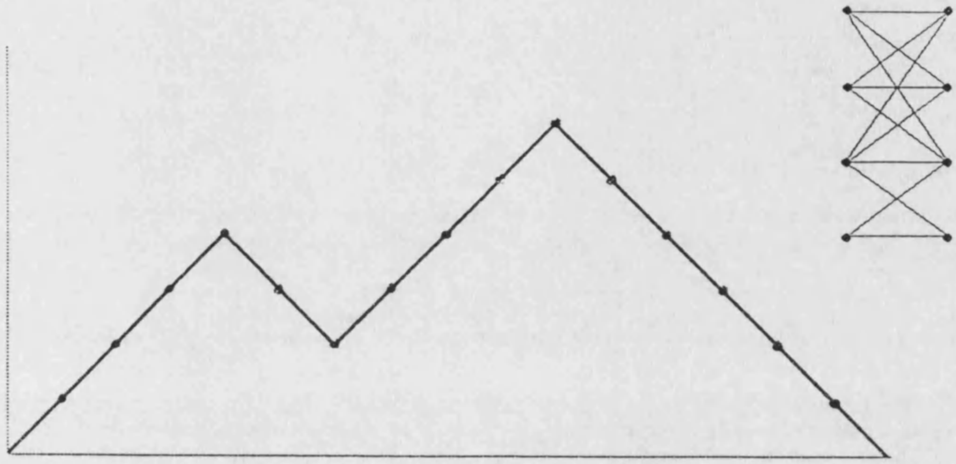

Figure 29: A component corresponding to a symmetric factor and corresponding symmetric hit

- $G^{r}(A, B)$ is the number of good walks from $(A, B)$ to $(2N, 0)$.
- $G^{\rightarrow}(A, B)$ is the number of good walks from $(0, 0)$ to $(2N, 0)$ which go through $(A, B)$.
- $G_i(N, b)$ is the number of good walks to $(N, b)$ which hit exactly $i$ times.
- $G_{i,t}(N, b)$ is the number of good walks to $(N, b)$ which hit exactly $i$ times and have $t$ symmetric hits.
- $G^{r}_{j}(N, b)$ is the number of good walks from $(N, b)$ to $(2N, 0)$ which hit exactly $j$ times.
- $G^{r}_{j,u}(N, b)$ is the number of good walks from $(N, b)$ to $(2N, 0)$ which hit exactly $j$ times and have $u$ symmetric hits.
- We define a *very good walk* to be a good walk that does not have any hits. Let $V(A, B)$ be the number of very good walks from $(0, 0)$ to $(A, B)$.

Rewriting Lemma 10.7 using this notation, we obtain the following lemma.

**Lemma 10.9** *The number of labelled posets of width two on* $\{1, 2, \ldots, N\}$ *is*

$$\Omega_2(N) = \frac{N!}{2} \sum_{b,j,i}^{N} \frac{G_i(N,b)}{2^i} \frac{G_j^r(N,b)}{2^j}.$$

□

For the unlabelled case, Lemma 10.8 becomes Lemma 10.10.

**Lemma 10.10** *The number of unlabelled posets of width two on* $N$ *vertices is*

$$\Omega_2^u(N) = \frac{1}{2} \sum_{b,j,i,t,u=0}^{N} \frac{G_{i,t}(N,b)}{2^{i-t}} \frac{G_{j,u}^r(N,b)}{2^{j-u}},$$

*where* $t$ *is the number of symmetric hits of good walks to* $(N, b)$ *with* $i$ *hits and* $s$ *is the number of symmetric hits of good walks from* $(N, b)$ *to* $(2N, 0)$ *with* $j$ *hits.* □

Now we consider what a section of a walk before a symmetric hit looks like . (See Figure 29.) We know that it corresponds to a symmetric factor, by definition. So let $S$ be a symmetric factor of a poset $P$. Let $(X, Y, <)$ be a two chain covering of the elements of $S$, where $X = x_1x_2 \ldots x_l$ and $Y = y_1y_2 \ldots y_l$. Let $(\lambda, \mu)$ be the greedy pair corresponding to $(X, Y, <)$. Then there is a bijection $\kappa : \lambda \rightarrow \mu$ such that $\kappa(x_j) = y_j$ and $\kappa(y_k) = x_k$. Suppose we construct a walk $W$ from these two linear extensions. We consider the walk from $(2n, b)$ to $(2n + 2, B)$, for integers $n$ such that $0 \leq n \leq 2l - 2$, where $b$ is the position of the walk at $2n$ and $B$ is to be determined. Now, suppose next element of $\lambda$ is $x_k$. Then the walk will be at the point $(2n+1, b+1)$. Due to the symmetry, the next element of $\mu$ is $y_k$, so the walk is at $(2n + 2, b + 2)$. So the two steps are in the same direction. Similarly if $y_m$ is the $(n+1)^{\text{st}}$ element in $\lambda$ and $x_m$ in $\mu$, the walk will be at $(2n+2, b-2)$. So again, both the steps are in the same direction. This characterises a section of the walk corresponding to a symmetric factor, namely a section ending with a symmetric hit.

In the next chapter we will count the number of labelled posets, and in the following chapter the number of unlabelled posets.

# 11 Labelled Posets

We treat the set $\mathcal{W}$ of good walks as a probability space with each good walk being equally likely. Then quantities such as the number of hits are random variables within this probability space. Given a good walk $W$, we consider the number of hits and where they occur. We show that there are almost surely no hits except near the ends of the walk. Let $b$ be the position of the walk after $N$ steps. We call the part of the walk from $(0,0)$ to $(N,b)$ the *left* and from $(N,b)$ to $(2N,0)$ the *right*. We will also prove that the number $h$ of hits on the left is asymptotically a geometric random variable with $\Pr(h \text{ hits on left}) \approx \frac{3^h}{4^{h+1}}$. By symmetry, the number of hits on the right has the same distribution, and the number of hits on the left is asymptotically independent of the number of hits on the right. We define a *left very good walk* to be a walk that is very good to $(N,b)$ and good from $(N,b)$ to $(2N,0)$. We define $LV$ to be the number of left very good walks. To show the probability of $h$ hits on the left is approximately $\frac{3^h}{4^{h+1}}$, we first show the probability that a good walk is very good on the left is about $\frac{1}{4}$. This gives the probability of at least one hit on the left to be about $\frac{3}{4}$.

The number of left very good walks can be found by considering the number of left very good walks that go through $(N,b)$ and summing over $b$. The number of left very good walks through $(N,b)$ is the product of the number of walks that are very good to $(N,b)$, that is $V(N,b)$, times the number of walks that are good from $(N,b)$ to $(2N,0)$, that is $G^{\rightarrow}(N,b)$. Now by symmetry, $G(N,b) = G^{\rightarrow}(N,b)$. So the number of left very good walks is

$$\begin{aligned} LV &= \sum_{b=0}^{N} V(N,b)G^{\rightarrow}(N,b) \\ &= \sum_{b=0}^{N} \frac{V(N,b)}{G(N,b)} G^{\rightarrow}(N,b). \end{aligned}$$

We will prove that $G^{\rightarrow}(N,b)$ is relatively small except for $b$ close to $N^{\frac{1}{2}}$, and, for such values of $b$, $\dfrac{V(N,b)}{G(N,b)}$ is close to $\frac{1}{4}$. This implies that $LV$ is about $\frac{1}{4}G(2N,0)$. So, to calculate $LV$ we find $G(2N,0)$ and $G(N,b)$, and bounds for them. From these we obtain $G^{\rightarrow}(N,b)$ and $V(N,b)G^{\mathrm{T}}(N,b)$. To obtain the bounds for $G(2N,0)$ and $G(N,b)$, we note the following inequalities from Stirling's formula.

**Lemma 11.1**

$$\begin{aligned} 2^{\frac{1}{2}}n^{\frac{1}{2}}\left(\frac{n}{e}\right)^{n} &\leq n! \leq 2^{\frac{3}{2}}n^{\frac{1}{2}}\left(\frac{n}{e}\right)^{n}, \\ \frac{2^{2n-1}}{n^{\frac{1}{2}}} &\leq \binom{2n}{n} \leq \frac{2^{2n+1}}{n^{\frac{1}{2}}}. \end{aligned}$$

□

First we consider the number of good walks between $(0,0)$ and $(2N,0)$. We use the solution to the parenthesis problem , which is called the *Catalan number.* Let $\mathcal{B}$ be the set of strings of $2n$ parentheses, $n$ of which are '(' and $n$ are ')', such that in any string, there are at least as many '(' as ')'. Then the number of these strings is given by the following theorem. (See, for example [15].)

**Theorem 11.2 (Parenthesis Problem)**

$$|\mathcal{B}| = \frac{1}{n+1}\binom{2n}{n}.$$

□

**Lemma 11.3** *The number of good walks from* $(0,0)$ *to* $(2N,0)$ *is*

$$G(2N,0) = \frac{1}{N+2}\binom{2N+2}{N+1},$$

*and lies in the range*

$$\frac{2^{2N}}{N^{\frac{3}{2}}} \leq G(2N,0) \leq \frac{2^{2N+3}}{N^{\frac{3}{2}}}.$$

**Proof.** Let $\mathcal{W}$ be the set of good walks between $(0,0)$ and $(2N,0)$. A good walk does not go below $-1$. Let us consider the set of walks $\mathcal{W}'$ formed from $\mathcal{W}$ by the addition of the steps from $(-1,-1)$ to $(0,0)$ and $(2N,0)$ to $(2N+1,-1)$. We translate $\mathcal{W}'$ to form $\mathcal{W}''$ by the addition of 1 to both coordinates. So $\mathcal{W}''$ is the set of walks between $(0,0)$ and $(2N+2,0)$ which do not go below 0. There is a bijection between $\mathcal{W}$ and $\mathcal{W}''$. We now show that counting the number of walks in $W''$ is equivalent to the parenthesis problem on $N+1$ pairs of brackets. Let an edge from $(a,b)$ to $(a+1,b+1)$ represents a left bracket and to $(a+1,b-1)$ a right bracket. Since the walk never drops below the $x$-axis, there are never more right brackets than left brackets. Also, the added first and last edge ensure that we start with a left bracket and end with a right bracket. This gives us the parenthesis problem on $N+1$ pairs of brackets, so by Theorem 11.2 the number of good walks is

$$\frac{1}{N+2}\binom{2N+2}{N+1}.$$

By applying Lemma 11.1 to the above formula for $G(2N,0)$ we obtain the required inequalities. $\square$

To obtain the number of good walks to the point $(N,b)$, we use the reflection principle [12], which is a more general version of Theorem 11.2.

**Lemma 11.4** *Suppose* $0 \le b \le N$. *Then the number of good walks from* $(0,0)$ *to* $(N,b)$ *is*

$$G(N,b) = \frac{(N+1)!(b+2)}{\left(\frac{N+b+4}{2}\right)!\left(\frac{N-b}{2}\right)!}.$$

**Proof.** The number of good walks to $(N,b)$ is the total number of walks from $(0,0)$ to $(N,b)$ minus the number that go below $-1$. There are $\binom{N}{\frac{N+b}{2}}$ walks from

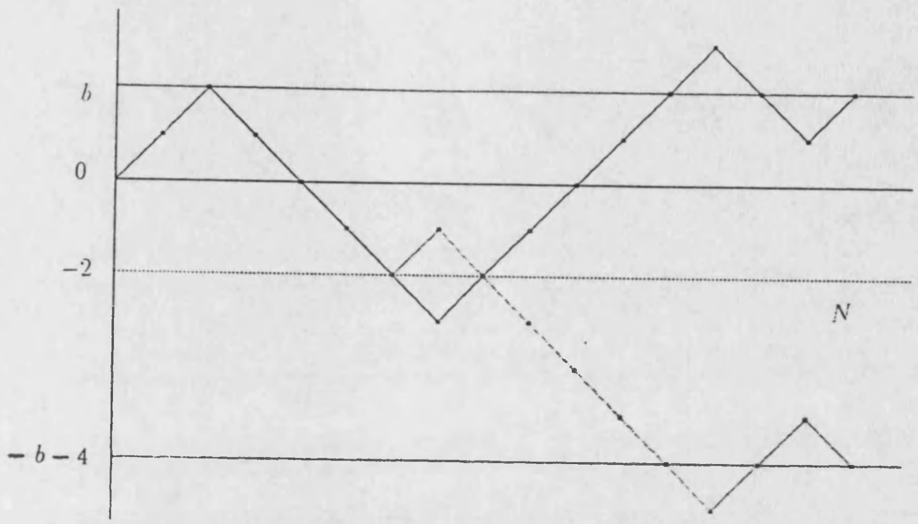


Figure 30: The reflection principle

(0, 0) to $(N, b)$ of which $\binom{N}{\frac{N-b-4}{2}}$ fall below $-1$, by the reflection principle. (See Figure 30.)

Therefore we have

$$
\begin{aligned}
G(N,b) &= \binom{N}{\frac{N+b}{2}} - \binom{N}{\frac{N-b-4}{2}} \\
&= N!\left(\frac{1}{\left(\frac{N+b}{2}\right)!\left(\frac{N-b}{2}\right)!} - \frac{1}{\left(\frac{N+b+4}{2}\right)!\left(\frac{N-b-4}{2}\right)!}\right) \\
&= N!\frac{(N+b+4)(N+b+2)-(N-b)(N-b-2)}{4\left(\frac{N+b+4}{2}\right)!\left(\frac{N-b}{2}\right)!} \\
&= N!\frac{4(N+1)(b+2)}{4\left(\frac{N+b+4}{2}\right)!\left(\frac{N-b}{2}\right)!} \\
&= \frac{(N+1)!(b+2)}{\left(\frac{N+b+4}{2}\right)!\left(\frac{N-b}{2}\right)!}.
\end{aligned}
$$

This completes the proof. □

So far, we have expressions for the number of good walks $G(2N,0)$ and the number of good walks to $(N, b)$. Now we calculate what proportion of walks which are good to $(N, b)$ are very good to $(N, b)$.

**Lemma 11.5**

$$\frac{V(N,b)}{G(N,b)} = \frac{b(N+b+4)(N+b+2)}{4(b+2)(N+1)N}$$

**Proof.** First we need to find $V(N, b)$. Again, we use the reflection principle. We count the number of walks from $(1,1)$ to $(N, b)$ and subtract the number of walks which fall below height 1.

$$\begin{aligned}
V(N,b) &= \binom{N-1}{\frac{N+b-2}{2}} - \binom{N-1}{\frac{N+b}{2}} \\
&= (N-1)!\left(\frac{1}{\left(\frac{N+b-2}{2}\right)!\left(\frac{N-b+2}{2}\right)!} - \frac{1}{\left(\frac{N+b}{2}\right)!\left(\frac{N-b-2}{2}\right)!}\right) \\
&= (N-1)!\frac{(N+b)-(N-b)}{2\left(\frac{N+b}{2}\right)!\left(\frac{N-b}{2}\right)!} \\
&= \frac{(N-1)!b}{\left(\frac{N+b}{2}\right)!\left(\frac{N-b}{2}\right)!}.
\end{aligned}$$

Using the expression for $G(N, b)$ in Lemma 11.4, we now have

$$\begin{aligned}
\frac{V(N,b)}{G(N,b)} &= \frac{(N-1)!b\left(\frac{N+b+4}{2}\right)!\left(\frac{N-b}{2}\right)!}{\left(\frac{N+b}{2}\right)!\left(\frac{N-b}{2}\right)!(N+1)!(b+2)} \\
&= \frac{b(N+b+4)(N+b+2)}{4(b+2)(N+1)N}.
\end{aligned}$$

□

We use the expression for the number of good walks to $(N, b)$ to obtain upper bounds for the number of good walks to $(2N, 0)$ which pass through $(N, b)$, and the number of walks which are very good to $(N, b)$ and then good from $(N, b)$ to $(2N, 0)$. We note that $V(N, b)$ is certainly smaller than $G(N, b)$ and so $V(N, b)G(N, b) \leq G^{\rightarrow}(N, b)$. Therefore we only need to find an upper bound for $G^{\rightarrow}(N, b) = G(N, b)^2$.

**Lemma 11.6** *For all $b$ with $0 \le b \le N$,*

$$\frac{G^{\rightarrow}(N,b)}{G(2N,0)} \le 2^{10} N^{-\frac{3}{2}} b^2 e^{-\frac{b^2}{2N^2}}$$

**Proof.** First we note that we can write

$$\frac{G^{\rightarrow}(N,b)}{G(2N,0)} = \frac{G(N,b)^2}{G(2N,0)}.$$

We find an upper bound for the numerator,

$$G(N,b)^2 = \frac{(N+1)!^2(b+2)^2}{\left(\frac{N+b+4}{2}\right)!^2\left(\frac{N-b}{2}\right)!^2},$$

by applying Lemma 11.1.

$$\begin{aligned}
G(N,b)^2 &\le \frac{4\left(\frac{N+1}{e}\right)^{2N+3}(b+2)^2 e}{2e\left(\frac{N+b+4}{2e}\right)^{N+b+5} 2e\left(\frac{N-b}{2e}\right)^{N-b+1}} \\
&\le \frac{(N+1)^{2N+3}(b+2)^2 e^2}{\left(\frac{N+b+4}{2}\right)^{N+b+5}\left(\frac{N-b}{2}\right)^{N-b+1}} \\
&\le \frac{2^{2N+6}e^2(N+1)^{-3}(b+2)^2}{\left(1+\frac{b+3}{N+1}\right)^{N+b+5}\left(1-\frac{b+1}{N+1}\right)^{N-b+1}} \\
&\le \frac{2^{2N+10}N^{-3}b^2}{\left(1+\frac{b}{N}\right)^{N+b}\left(1-\frac{b}{N}\right)^{N-b}} \\
&\le 2^{2N+10}N^{-3}b^2 e^{(N+b)\left(-\frac{b}{N}+\frac{b^2}{2N^2}\right)+(N-b)\left(\frac{b}{N}+\frac{b^2}{N^2}\right)} \\
&\le 2^{2N+10}N^{-3}b^2 e^{-\frac{b^2}{2N}-\frac{b^3}{2N^2}} \\
&\le 2^{2N+10}N^{-3}b^2 e^{-\frac{b^2}{2N}}.
\end{aligned}$$

We use the above bound and Lemma 11.3 to obtain an upper bound for $\dfrac{G^{\rightarrow}(N,b)}{G(2N,0)}$.

$$\begin{aligned}
\frac{G^{\rightarrow}(N,b)}{G(2N,0)} &= \frac{G(N,b)^2}{G(2N,0)} \\
&\le \frac{2^{2N+10}N^{-3}b^2e^{-\frac{b^2}{2N}}}{2^{2N}N^{-\frac{3}{2}}} \\
&\le 2^{10}N^{-\frac{3}{2}}b^2e^{-\frac{b^2}{2N}}.
\end{aligned}$$

This completes the proof. $\square$

Now we have all the information we need to calculate how many good walks there are which are very good to $N$ and good from $N$ to $2N$, that is, to calculate $LV$. We divide our sum into three parts which we deal with separately for clarity.

$$\begin{aligned} LV &= \sum_{b=0}^{N} V(N,b)G^{\tau}(N,b) \\ &= \sum_{b=0}^{N^{\frac{3}{8}}} V(N,b)G^{\tau}(N,b) + \sum_{b=N^{\frac{3}{8}}}^{N^{\frac{5}{8}}} V(N,b)G^{\tau}(N,b) + \sum_{b=N^{\frac{5}{8}}}^{N} V(N,b)G^{\tau}(N,b). \end{aligned}$$

Denote the three sums by $S_1$, $S_2$, $S_3$ respectively. We shall prove that $S_1$ and $S_3$ are small and $S_2$ is approximately $\frac{1}{4}G(2N,0)$.

**Lemma 11.7**

$$\begin{aligned} S_1 + S_3 &\leq G(2N,0)\left(\sum_{b=0}^{N^{\frac{3}{8}}} \frac{G^{\rightarrow}(N,b)}{G(2N,0)} + \sum_{b=N^{\frac{5}{8}}}^{N} \frac{G^{\rightarrow}(N,b)}{G(2N,0)}\right) \\ &= O\left(N^{-\frac{3}{8}}\right) G(2N,0). \end{aligned}$$

**Proof.** We rewrite the expressions for $S_1$ and $S_3$ as follows:

$$\begin{aligned} S_1 + S_3 &= \sum_{b=0}^{N^{\frac{3}{8}}} V(N,b)G^{\tau}(N,b) + \sum_{b=N^{\frac{5}{8}}}^{N} V(N,b)G^{\tau}(N,b) \\ &= \sum_{b=0}^{N^{\frac{3}{8}}} \frac{G^{\rightarrow}(N,b)}{G(2N,0)}G(2N,0) + \sum_{b=N^{\frac{5}{8}}}^{N} \frac{G^{\rightarrow}(N,b)}{G(2N,0)}G(2N,0). \end{aligned}$$

Now we apply Lemma 11.6 to $\dfrac{G^{\rightarrow}(N,b)}{G(2N,0)}$ to obtain an upper bound for $S_1 + S_3$.

$$
\begin{aligned}
S_1 + S_3 &\leq \sum_{b=0}^{N^{\frac{3}{8}}} 2^{10} N^{-\frac{3}{2}} b^2 e^{-\frac{b^2}{2N}} G(2N,0) + \sum_{b=N^{\frac{5}{8}}}^{N} 2^{10} N^{-\frac{3}{2}} b^2 e^{-\frac{b^2}{2N}} G(2N,0) \\
&\leq 2^{10} N^{-\frac{3}{2}} G(2N,0) \left( \sum_{b=0}^{N^{\frac{3}{8}}} b^2 + \sum_{b=N^{\frac{5}{8}}}^{N} b^2 e^{-\frac{b^2}{2N}} \right).
\end{aligned}
$$

Since the terms in the second sum are decreasing as $b$ increases to $N$, we can approximate it using the following integral:

$$
\begin{aligned}
N^{-\frac{3}{2}} \sum_{b=N^{\frac{5}{8}}}^{N} b^2 e^{-\frac{b^2}{2N}} &\leq N^{-\frac{3}{2}} \int_{N^{\frac{5}{8}}}^{N} b^2 e^{-\frac{b^2}{2N}} db \\
&\leq N^{-\frac{3}{2}} \int_{N^{\frac{5}{8}}}^{N} b^3 e^{-\frac{b^2}{2N}} db \\
&= \left(2N^{\frac{1}{2}} + N^{\frac{3}{4}}\right) e^{-\frac{N^{\frac{1}{4}}}{2}} - \left(2N^{\frac{1}{2}} + N^{\frac{3}{2}}\right) e^{-\frac{N}{2}} \\
&\leq 2N^{\frac{3}{4}} e^{-\frac{N^{\frac{1}{4}}}{2}}.
\end{aligned}
$$

Similarly, the first summation approximates to the following integral.

$$
\begin{aligned}
\sum_{b=0}^{N^{\frac{3}{8}}} N^{-\frac{3}{2}} b^2 &\leq N^{-\frac{3}{2}} \int_0^{N^{\frac{3}{8}}} b^2 db \\
&= N^{-\frac{3}{2}} \frac{N^{\frac{9}{8}}}{3} \\
&= \frac{N^{-\frac{3}{8}}}{3}.
\end{aligned}
$$

Therefore we have an upper bound for $S_1 + S_3$.

$$
\begin{aligned}
S_1 + S_3 &\leq 2^{10} G(2N,0) N^{-\frac{3}{8}} + G(2N,0) 2^{11} N^{\frac{3}{4}} e^{-\frac{N^{\frac{1}{4}}}{2}} \\
&\leq 2^{11} G(2N,0) N^{-\frac{3}{8}},
\end{aligned}
$$

for sufficiently large $N$, as required. □

Now we come to the central sum $S_2$.

**Lemma 11.8**

$$\sum_{b=N^{\frac{3}{8}}}^{N^{\frac{5}{8}}} V(N,b)G^{\text{r}}(N,b) = \left(\frac{1}{4} + O\left(N^{-\frac{3}{8}}\right)\right) G(2N,0).$$

**Proof.** We have

$$\begin{aligned} S_2 &= \sum_{b=N^{\frac{3}{8}}}^{N^{\frac{5}{8}}} V(N,b)G^{\text{r}}(N,b) \\ &= \sum_{b=N^{\frac{3}{8}}}^{N^{\frac{5}{8}}} \frac{V(N,b)}{G(N,b)} G^{\rightarrow}(N,b). \end{aligned}$$

Applying Lemma 11.5 this becomes

$$\begin{aligned} S_2 &= \sum_{b=N^{\frac{3}{8}}}^{N^{\frac{5}{8}}} \frac{b(N+b+4)(N+b+2)}{4(b+2)(N+1)N} G^{\rightarrow}(N,b) \\ &= \sum_{b=N^{\frac{3}{8}}}^{N^{\frac{5}{8}}} \frac{1}{4}\left(1-\frac{2}{b+2}\right)\left(1+O\left(\frac{b}{N}\right)\right) G^{\rightarrow}(N,b) \\ &= \frac{1}{4}\sum_{b=N^{\frac{3}{8}}}^{N^{\frac{5}{8}}} \left(1+O\left(\frac{b}{N}\right)\right) G^{\rightarrow}(N,b). \end{aligned}$$

Now, $\sum_{0}^{N} G^{\rightarrow}(N,b) = G(2N,0)$, so

$$S_2 = \frac{1}{4}G(2N,0)\left(1 - \sum_{b=0}^{N^{\frac{3}{8}}} \frac{G^{\rightarrow}(N,b)}{G(2N,0)} - \sum_{b=N^{\frac{5}{8}}}^{N} \frac{G^{\rightarrow}(N,b)}{G(2N,0)} + O\left(N^{-\frac{3}{8}}\right)\right).$$

We have upper bounds for the second and third terms which are $S_1$ and $S_3$ respectively, so

$$
\begin{aligned}
S_2 &= \frac{1}{4}G(2N,0)\left(1+O\left(N^{-\frac{3}{8}}\right)+O\left(N^{-\frac{3}{8}}\right)\right)\\
&= \frac{1}{4}G(2N,0)\left(1+O\left(N^{-\frac{3}{8}}\right)\right).
\end{aligned}
$$

□

Now, combining the expressions for $S_2$ and $S_1 + S_3$ given by Lemma 11.8 and Lemma 11.7, we obtain the following lemma.

**Lemma 11.9** *There exists a constant $c_0$ such that, for all $n$,*

$$
\left|\frac{LV}{G(2N,0)} - \frac{1}{4}G(2N,0)\right| \leq c_0 N^{-\frac{3}{8}}.
$$

□

So, we see that the probability of a very good walk to $(N, b)$ and then a good walk to $(2N, 0)$, given that it is a good walk from $(0, 0)$ to $(2N, 0)$, is about $\frac{1}{4}$. Next we show that the probability of a hit in the centre of the walk is small, namely $O\left(\omega^{-\frac{1}{2}}\right)$ in the range between $2\omega$ and $2(N-\omega)$ where $0 < \omega < N-\omega$. This means that we can deal with the two extremes separately. We calculate this probability by looking at the number of good walks that hit at $(2k, 0)$. That is, we count the good walks to $(2k, 0)$ and from $(2k, 0)$ to $(2N, 0)$.

**Lemma 11.10** *The probability $H$ of a hit in the centre of the walk is*

$$
H \leq \sum_{k=\omega}^{N-\omega} \frac{2^5 N^{\frac{3}{2}}}{k^{\frac{3}{2}}(N-k)^{\frac{3}{2}}} \leq 2^9 \omega^{-\frac{1}{2}}.
$$

**Proof.** The proportion of good walks that hit at $(2k, 0)$, where $\omega \leq k \leq N-\omega$ is $\dfrac{G^{\rightarrow}(2k,0)}{G(2N,0)}$. So summing over $k$ and replacing $G^{\rightarrow}(2k, 0)$ by $G(2k, 0)G^{r}(2k, 0)$, we obtain

$$H = \sum_{k=\omega}^{N-\omega} \frac{G(2k,0)G^{r}(2k,0)}{G(2N,0)}.$$

Applying Lemma 11.3 and Lemma 11.4, this becomes

$$H = \sum_{k=\omega}^{N-\omega} \frac{\frac{2}{2k+4}\binom{2k+2}{k+1}\binom{2(N-k+1)}{N-k+1}\frac{2}{2(N-k+2)}}{\frac{2}{2N+4}\binom{2N+2}{N+1}}.$$

Now applying Lemma 11.1,we obtain

$$\begin{aligned} H &\leq \sum_{k=\omega}^{N-\omega} \frac{(N+2)2^{2k+3+2N-2k+3-2N-1}(N+1)^{\frac{1}{2}}}{(k+2)(k+1)^{\frac{1}{2}}(N-k+2)(N-k+1)^{\frac{1}{2}}} \\ &\leq \sum_{k=\omega}^{N-\omega} \frac{(N+2)2^{5}(N+1)^{\frac{1}{2}}}{(k+2)(k+1)^{\frac{1}{2}}(N-k+2)(N-k+1)^{\frac{1}{2}}} \\ &\leq \sum_{k=\omega}^{N-\omega} \frac{2^{5}N^{\frac{3}{2}}}{k^{\frac{3}{2}}(N-k)^{\frac{3}{2}}}. \end{aligned}$$

Now, the sum is symmetric about $\frac{N}{2}$ so

$$H \leq 2^{6}\sum_{k=\omega}^{\frac{N}{2}} \frac{N^{\frac{3}{2}}}{k^{\frac{3}{2}}(N-k)^{\frac{3}{2}}}.$$

Since the function $k^{-\frac{3}{2}}(N-k)^{-\frac{3}{2}}$ is symmetric about $b = \frac{N}{2}$ and decreasing as $b$ increases to $\frac{N}{2}$ we can approximate the sum by the following integral.

$$\begin{aligned} H &\leq 2^{6}\sum_{k=\omega}^{\frac{N}{2}} \frac{N^{\frac{3}{2}}}{k^{\frac{3}{2}}(N-k)^{\frac{3}{2}}} \\ &\leq 2^{6}\int_{\omega-1}^{\frac{N}{2}} \frac{N^{\frac{3}{2}}}{k^{\frac{3}{2}}(N-k)^{\frac{3}{2}}}dk \end{aligned}$$

$$
\begin{aligned}
H &\leq 2^6 N^{-\frac{1}{2}} \int_{\frac{\omega-1}{N}}^{\frac{1}{2}} x^{-\frac{3}{2}}(1-x)^{-\frac{3}{2}} dx \\
&\leq 2^6 N^{-\frac{1}{2}} \left[ \frac{4x-2}{x^{\frac{1}{2}}(1-x)^{\frac{1}{2}}} \right]_{\frac{\omega-1}{N}}^{\frac{1}{2}} \\
&\leq 2^6 N^{-\frac{1}{2}} \frac{-4\frac{\omega-1}{N}+2}{\left(\frac{\omega-1}{N}\right)^{\frac{1}{2}}\left(1-\frac{\omega-1}{N}\right)^{\frac{1}{2}}} \\
&\leq 2^7 \frac{1-2\frac{\omega-1}{N}}{(\omega-1)^{\frac{1}{2}}\left(1-\frac{\omega-1}{N}\right)^{\frac{1}{2}}} \\
&\leq \frac{2^8}{(\omega-1)^{\frac{1}{2}}} \\
&\leq 2^9 \omega^{-\frac{1}{2}}.
\end{aligned}
$$

□

To summarise so far, we have found the probability of a very good walk on the left followed by a good walk on the right. Therefore we know the probability of a hit in the extreme regions and also the central region. We now need to estimate the probability of exactly $h$ hits on the left and a good walk on the right. This will enable us to find how many of the good walks have exactly $h$ hits on the left (and hence, by symmetry, on the right).

Considering the probability of a walk having $h$ hits on the left, we prove the next lemma.

**Lemma 11.11**

$$
\Pr(h \text{ hits on left}) = \frac{3^h}{4^{h+1}} + O\left(hN^{-\frac{3}{8}}\right).
$$

**Proof.** For an event $A$, let $P_{\geq h}(A)$ be the probability of $A$ given at least $h$ hits on the left and let $P_k(A)$ be the probability of $A$ given a hit at $k$. Let $K_s(k)$ be the event '$s^{\text{th}}$ hit at $k$'. Using this notation we write the probability of at least $h$ hits. We split it into two sums $P_1$ and $P_2$ over $k$, the position at which the $h^{\text{th}}$ hit occurs.

$$\begin{aligned}
\Pr(\geq h \text{ hits}) &= P_{\geq h-1}(\text{at least } h \text{ hits})\Pr(\text{at least } h-1 \text{ hits}) \\
&= \sum_{k<N} P_{\geq h-1}(K_{h-1}(k))P_k(\geq 1 \text{ more hit})\Pr(\geq h-1 \text{ hits}) \\
&= \sum_{k<\frac{N}{\ln N}} P_{\geq h-1}(K_{h-1}(k))P_k(\geq 1 \text{ more hit})\Pr(\geq h-1 \text{ hits}) \\
&\quad + \sum_{k\geq\frac{N}{\ln N}} P_{\geq h-1}(K_{h-1}(k))P_k(\geq 1 \text{ more hit})\Pr(\geq h-1 \text{ hits}).
\end{aligned}$$

We shall next prove that the second term, $P_2$, is small and the first term, $P_1$. is close to $\left(\frac{3}{4}\right)^h$.

$$\begin{aligned}
P_2 &= \sum_{k\geq\frac{N}{\ln N}} P_{\geq h-1}(K_{h-1}(k))P_k(\geq 1 \text{ more hit})\Pr(\geq h-1 \text{ hits}) \\
&\leq \sum_{k\geq\frac{N}{\ln N}} \Pr(\text{hit at } k).
\end{aligned}$$

Applying Lemma 11.10, we have

$$P_2 \leq 2^9 \left(\frac{\ln N}{N}\right)^{\frac{1}{2}}.$$

We now deal with $P_1$. We will prove, by induction, that the probability of at least $h$ hits in the region from 0 to $\frac{N}{\ln N}$ lies in the range $\left[\left(\frac{3}{4}\right)^h - c_0 h N^{-\frac{3}{8}}, \left(\frac{3}{4}\right)^h + c_0 h N^{-\frac{3}{8}}\right]$, where $c_0$ is as in Lemma 11.9. From Lemma 11.9, we have that the probability of at least one hit on the left, is in the range $\left[\frac{3}{4} - c_0 h N^{-\frac{3}{8}}, \frac{3}{4} + c_0 h N^{-\frac{3}{8}}\right]$, for a fixed contant $c_0$, independent of $N$. First we will obtain the upper bound and then the lower bound.

$$
\begin{aligned}
&P_1 - \left(\frac{3}{4}\right)^h \\
&= \sum_{k<\frac{N}{\ln N}} P_{\geq h-1}(K_{h-1}(k))P_k(\geq 1 \text{ more hit})\Pr(\geq h-1 \text{ hits}) - \left(\frac{3}{4}\right)^h \\
&\leq \sum_{k<\frac{N}{\ln N}} P_{\geq h-1}(K_{h-1}(k))\left(\frac{3}{4}+\frac{c}{(N-k)^{\frac{3}{8}}}\right)\left(\left(\frac{3}{4}\right)^{h-1}+\frac{c(h-1)}{N^{\frac{3}{8}}}\right) - \left(\frac{3}{4}\right)^h \\
&\leq \left(1-\sum_{k=\frac{N}{\ln N}}^{N} P_{\geq h-1}(K_{h-1}(k))\right)\left(\frac{3}{4}+\frac{c}{(N-N/\ln N)^{\frac{3}{8}}}\right)\left(\left(\frac{3}{4}\right)^{h-1}+\frac{c(h-1)}{N^{\frac{3}{8}}}\right) \\
&\qquad - \left(\frac{3}{4}\right)^h \\
&\leq \left(\frac{3}{4}+\frac{c}{(N-N/\ln N)^{\frac{3}{8}}}\right)\left(\left(\frac{3}{4}\right)^{h-1}+\frac{c(h-1)}{N^{\frac{3}{8}}}\right) - \left(\frac{3}{4}\right)^h \\
&\leq chN^{-\frac{3}{8}}.
\end{aligned}
$$

Similarly for the lower bound,

$$
\begin{aligned}
&P_1 - \left(\frac{3}{4}\right)^h \\
&= \sum_{k<\frac{N}{\ln N}} P_{\geq h-1}(K_{h-1}(k))P_k(\geq 1 \text{ more hit})\Pr(\geq h-1 \text{ hits}) - \left(\frac{3}{4}\right)^h \\
&\geq \sum_{k<\frac{N}{\ln N}} P_{\geq h-1}(K_{h-1}(k))\left(\frac{3}{4}-\frac{c}{(N-k)^{\frac{3}{8}}}\right)\left(\left(\frac{3}{4}\right)^{h-1}-\frac{c(h-1)}{N^{\frac{3}{8}}}\right) - \left(\frac{3}{4}\right)^h \\
&\geq \left(1-\sum_{k=\frac{N}{\ln N}}^{N} P_{\geq h-1}(K_{h-1}(k))\right)\left(\frac{3}{4}-\frac{c}{N^{\frac{3}{8}}}\right)\left(\left(\frac{3}{4}\right)^{h-1}-\frac{c(h-1)}{N^{\frac{3}{8}}}\right) - \left(\frac{3}{4}\right)^h \\
&\geq (1-2^9N^{-1}\ln N)^{-\frac{1}{2}}\left(\frac{3}{4}-\frac{c}{N^{\frac{3}{8}}}\right)\left(\left(\frac{3}{4}\right)^{h-1}-\frac{c(h-1)}{N^{\frac{3}{8}}}\right) - \left(\frac{3}{4}\right)^h \\
&\geq -chN^{-\frac{3}{8}}.
\end{aligned}
$$

Therefore

$$\left| P_1 - \left(\frac{3}{4}\right)^h \right| \leq chN^{-\frac{3}{8}}.$$

So, combining our estimates for $P_1$ and $P_2$ we have

$$\Pr(\geq h \text{ hits}) = \left(\frac{3}{4}\right)^h + O\left(hN^{-\frac{3}{8}}\right).$$

Now we will show that the probability for a good walk to have exactly $h$ hits on the left is about $\dfrac{3^h}{4^{h+1}}$.

$$\begin{aligned}
\Pr(\text{exactly } h \text{ hits on left}) &= \Pr(\text{at least } h \text{ hits on left}) - \Pr(\text{at least } h+1 \text{ hits on left}) \\
&\geq \left(\left(\frac{3}{4}\right)^h + chN^{-\frac{3}{8}} - \left(\frac{3}{4}\right)^{h+1} - chN^{-\frac{3}{8}}\right) \\
&\geq \frac{1}{4}\left(\frac{3}{4}\right)^h + 2chN^{-\frac{3}{8}}
\end{aligned}$$

This completes the proof. □

Now we bring everything together by using Lemma 10.9. We count the number of walks going through $(N, b)$, weighting each one by $2^{-l}$ where $l$ is the number of hits.

**Proof of Theorem 10.2.** We use Lemma 10.9 to obtain the number of labelled posets of width two on $\{1, 2, \ldots, N\}$.

$$\begin{aligned}
\Omega_2(N) &= \frac{N!}{2} \sum_{b,j,i=0}^{N} \frac{G_i(N,b)}{2^i} \frac{G_j^{\mathrm{r}}(N,b)}{2^j} \\
&= \frac{N!}{2} \sum_{b=0}^{N} \left( \sum_{i=0}^{N} \frac{G_i(N,b)}{2^i} \right)^2 .
\end{aligned}$$

Applying Lemma 11.11, we obtain

$$\begin{aligned}
\Omega_2(N) &= \frac{N!}{2}\sum_{b=0}^{N}\left(\sum_{i=0}^{N}\frac{G(N,b)}{2^i}\left(\frac{3^i}{4^{i+1}}+O\left(iN^{-\frac{3}{8}}\right)\right)\right)^2 \\
&= \frac{N!}{2}\sum_{b=0}^{N}\left(\sum_{i=0}^{N}\frac{3^i}{4^{i+1}2^i}\right)^2 G(N,b)^2 + O\left(N^{-\frac{3}{8}}\right)\frac{N!}{2}G(2N,0) \\
&= \frac{N!}{2}G(2N,0)\left(\frac{4}{25}+O\left(N^{-\frac{3}{8}}\right)\right).
\end{aligned}$$

Replacing $G(2N,0)$, by Lemma 11.3,

$$\begin{aligned}
\Omega_2(N) &= \frac{N!}{2(N+2)}\binom{2N+2}{N+1}\left(\frac{4}{25}+O\left(N^{-\frac{3}{8}}\right)\right) \\
&= \frac{(2N+1)!}{(N+2)!}\left(\frac{4}{25}+O\left(N^{-\frac{3}{8}}\right)\right).
\end{aligned}$$

□

Now we come to a result which can be easily obtained from the previous calculations. One such result is the expected number of components of an incomparability graph of a poset. This is obtained as follows. We find the number of small components on the left, which by symmetry is the same as the number of small components on the right. We then sum these, plus one for the central component. The expected number of small components on the left is

$$\frac{25}{2}\sum_{i=0}^{N} i\frac{3^i}{2^{3i+2}} = \frac{25}{8}\sum_{i=0}^{N} i\frac{3^i}{8^i} = \frac{25}{8}\frac{3/8}{(5/8)^2} = 3.$$

Therefore the expected number of components is seven.

# 12 Unlabelled Posets

Now we conclude the section on posets with this chapter in which we will prove Theorem 10.3. Recalling Lemma 10.10, we see that we need to calculate the number of walks which have $s$ symmetric hits and $i$ hits on the left. First we consider the number of good walks that have a symmetric first hit. Let $L(4k,0)$ be the number of very good walks from $(0,0)$ to $(4k,0)$ which are symmetric.

First we note that
$$\binom{2n}{n} = \frac{2^{2n}}{\sqrt{n\pi}} e^{\alpha_{2n}-2\alpha_n},$$
where $1/(12j+1) < \alpha_j < 1/(12j)$, by Stirling's formula. Now we consider $L(4k,0)$. We can count the set $\mathcal{S}$ of symmetric very good walks from $(0,0)$ to $(4k,0)$ by comparing it with the set $\mathcal{V}$ of very good walks from $(0,0)$ to $(2k,0)$. There is a bijection between $\mathcal{S}$ and $\mathcal{V}$ as each consecutive pair of edges of the symmetric walk corresponds to one edge of the very good walk. (See Figure 31.) We multiply

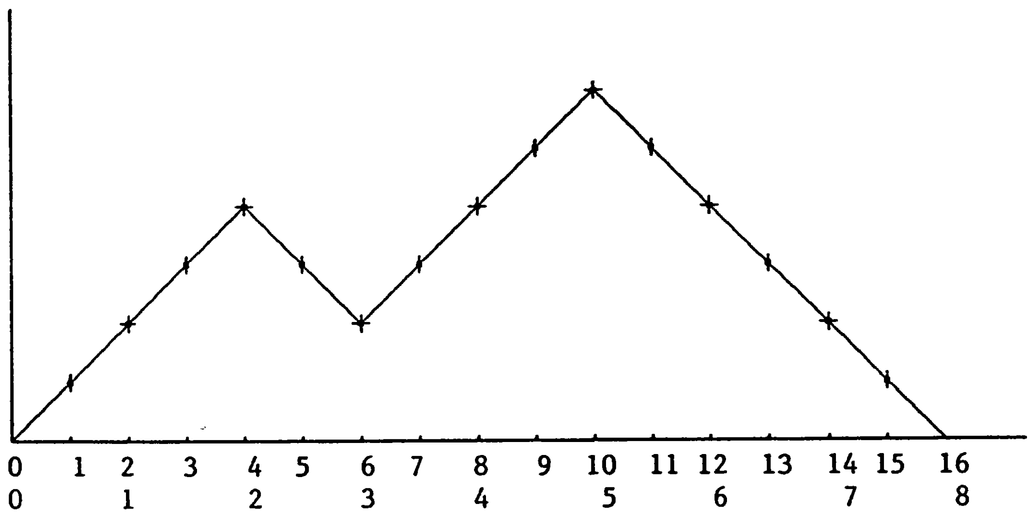


Figure 31: $L(4k,0)$ and $VG(2k,0)$.

$L(4k,0)$ by the number of good walks from $(4k,0)$ to $(2N,0)$, to give the total number of good walks that start with a symmetric hit at $4k$. To find the probability of this we then divide by $G(2N,0)$. Let $L$ be the probability of the first hit of a good walk being symmetric hit. We obtain the following lemma.

**Lemma 12.1**

$$\begin{aligned} L &= \sum_{k\geq 1} \frac{1}{2^{4k}k}\binom{2k-2}{k-1}\left(1+O\left(N^{-\frac{1}{2}}\ln N\right)\right) \\ &\approx 0.0669873 + O\left(N^{-\frac{1}{2}}\ln N\right). \end{aligned}$$

**Proof.** As we have already discussed,

$$\begin{aligned} L &= \sum_{k\geq 1} \frac{L(2k,0)G(2N-4k,0)}{G(2N,0)} \\ &= \sum_{k\geq 1} \frac{\frac{1}{k}\binom{2k-2}{k-1}\frac{1}{N-2k+2}\binom{2N-4k+2}{N-2k+1}}{\frac{1}{N+2}\binom{2N+2}{N+1}} \\ &= \sum_{k\geq 1} \frac{N+1}{k(N-2k+2)}\binom{2k-2}{k-1}\frac{2^{2N-4k+2}}{\sqrt{\pi(N-2k+1)}}\frac{\sqrt{\pi(N+1)}}{2^{2N+2}}e^{\beta} \end{aligned}$$

where $\beta = \alpha_{2N-4k+2} - 2\alpha_{N-2k+1} + \alpha_{2N+2} - 2\alpha_{N+1}$. Continuing,

$$\begin{aligned} L &= \sum_{k\geq 1} \frac{N+1}{k(N-2k+2)}\binom{2k-2}{k-1}2^{-4k}(N-2k+1)^{-1/2}(N+1)^{1/2}e^{\beta} \\ &= \sum_{k\geq 1} \frac{1}{2^{4k}k}\binom{2k-2}{k-1}\left(1+O\left(N^{-\frac{1}{2}}\ln N\right)\right) \\ &\approx 0.0669873 + O\left(N^{-\frac{1}{2}}\ln N\right). \end{aligned}$$

Now we consider the probability that the first hit is symmetric given that there is a hit.

**Lemma 12.2**

$$\Pr(\text{first hit on left is a symmetric hit} \mid \text{hit}) = \frac{4}{3}L + O(N^{-\frac{3}{8}}).$$

**Proof.** Now,

$$\begin{aligned}&\Pr\text{ (first hit on left is symmetric} \mid \text{hit)}\\ &= \frac{\Pr\text{ (first hit on the left is symmetric)}}{\Pr\text{ (hit)}}.\end{aligned}$$

So the probability that the first hit on the left is a symmetric hit given that there is a hit on the left is

$$\frac{L}{3/4} + O(N^{-\frac{3}{8}}) = \frac{4}{3}L + O(N^{-\frac{3}{8}}).$$

□

We require the probability of $s$ symmetric hits given $i$ hits on the left. There are $\binom{i}{s}$ ways of choosing the $s$ symmetric hits. The probability of a symmetric hit is $\frac{4}{3}L$ and of a non symmetric hit is $1 - \frac{4}{3}L$. Therefore the probability is given by the next lemma.

**Lemma 12.3** *The probability of $s$ symmetric hits given $i$ hits on the left is*

$$\binom{i}{s}\left(1-\frac{4}{3}L\right)^{i-s}\left(\frac{4}{3}L\right)^{s}\left(1+O\left(N^{-\frac{3}{8}i}\right)\right).$$

□

Now we come to the proof of Theorem 10.3.

**Proof of Theorem 10.3.** We recall the notation $G_{a,b}(A,B)$ which is the number of good walks to $(A,B)$ with $a$ hits and $b$ symmetric hits. Also $G^{\pi}_{a,b}(A,B)$ is the number of good walks from $(A, B)$ to $(2N, 0)$ with $a$ hits and $b$ symmetric hits. By Lemma 10.10, the number of unlabelled posets of width on $N$ vertices is

$$\Omega_2^u(N) = \frac{1}{2} \sum_{b,j,i,t,u=0}^{N} \frac{G_{i,t}(N,b)}{2^{i-t}} \frac{G^r_{j,u}(N,b)}{2^{j-u}}.$$

By symmetry, $G^r_{j,u}(N,b) = G_{j,u}(N,b)$, so

$$\Omega_2^u(N) = \frac{1}{2} \sum_b \left( \sum_i \sum_s \frac{G_{i,s}(N,b)}{2^{i-s}} \right)^2.$$

Now, applying Lemma 12.3, we have

$$\begin{aligned}
&\Omega_2^u(N) \\
&= \frac{1}{2} \sum_b \left( \sum_i \sum_{s \leq i} \binom{i}{s} \frac{G_i(N,b)}{2^{i-s}} \left(1 - \frac{4}{3}L\right)^{i-s} \left(\frac{4}{3}L\right)^s \left(1 + O\left(N^{-\frac{3}{8}}\right)\right) \right)^2 \\
&= \frac{1}{2} \sum_b \left( \sum_i \frac{G_i(N,b)}{2^i} \sum_{s \leq i} \binom{i}{s} \left(1 - \frac{4}{3}L\right)^{i-s} \left(\frac{8}{3}L\right)^s \right)^2 + O(N^{-\frac{3}{8}}) G(2N,0) \\
&= \frac{1}{2} \sum_b \left( \sum_i \frac{G_i(N,b)}{2^i} \left(1 + \frac{4}{3}L\right)^i \right)^2 + O(N^{-\frac{3}{8}}) G(2N,0).
\end{aligned}$$

By Lemma 11.11 this becomes

$$\begin{aligned}
\Omega_2^u(N) &= \frac{1}{2} \sum_b G(N,b)^2 \left( \sum_i \frac{3^i}{4^{i+1}} 2^{-i} \left(1 + \frac{4}{3}L\right)^i \right)^2 + O(N^{-\frac{3}{8}}) G(2N,0) \\
&= \frac{1}{2} G(2N,0) \left( \sum_i \left(\frac{3}{8}\right)^i \frac{1}{4} \left(1 + \frac{4}{3}L\right)^i \right)^2 + O(N^{-\frac{3}{8}}) G(2N,0) \\
&= \frac{1}{32} G(2N,0) \left( \sum_i \left(\frac{3}{8}\right)^i \left(1 + \frac{4}{3}L\right)^i \right)^2 + O(N^{-\frac{3}{8}}) G(2N,0)
\end{aligned}$$

$$\begin{aligned}
\Omega_2^u(N) &= \frac{1}{32}G(2N,0)\left(\sum_{i=0}^{\infty}\left(\frac{3}{8}+\frac{L}{2}\right)^i\right)^2 + O(N^{-\frac{3}{8}})G(2N,0)\\
&= \frac{1}{32}G(2N,0)\left(\frac{1}{\frac{5}{8}-\frac{L}{2}}\right)^2 + O(N^{-\frac{3}{8}})G(2N,0).
\end{aligned}$$

Using Lemma 11.3, we replace $G(2N,0)$ to obtain

$$\begin{aligned}
\Omega_2^u(N) &= \left(\frac{2}{25}\left(\frac{1}{1-\frac{4L}{5}}\right)^2 + O\left(N^{-\frac{3}{8}}\right)\right)G(2N,0)\\
&= \frac{1}{N!}\frac{(2N+1)!}{(N+2)!}\left(\frac{4}{25}\left(\frac{1}{1-\frac{4L}{5}}\right)^2 + O\left(N^{-\frac{3}{8}}\right)\right)\\
&\approx \frac{1}{N!}\frac{(2N+1)!}{(N+2)!}\left(\frac{4}{25}1.11645 + O\left(N^{-\frac{3}{8}}\right)\right)
\end{aligned}$$

which completes the proof. □

Again, we consider the expected number of components in an incomparability graph, this time of an unlabelled poset. We use a similar calculation to that in the previous chapter. The expected number of small components on the left is

$$\begin{aligned}
32\left(\frac{5}{8}-\frac{L}{2}\right)^2\sum_{i=0}^{N}\frac{i}{4}\left(\frac{3}{8}+\frac{L}{2}\right)^i &= 32\left(\frac{5}{8}-\frac{L}{2}\right)^2\frac{\frac{3}{8}+\frac{L}{2}}{4\left(\frac{5}{8}-\frac{L}{2}\right)^2}\\
&= 3+4L
\end{aligned}$$

where $L = \sum_{k\geq 1}\frac{1}{2^{4k}k}\binom{2k-2}{k-1} \approx 0.669873$. So the expected number of components is $7+8L \approx 7.5359$, compared with 7 for the labelled case.

# References


[1] N. Alon. The largest cycle of a graph with a large minimal degree. *J. Graph Theory*, 10:123–127, 1986.

[2] B. Bollobás. *Random Graphs.* Academic Press, London, 1985.

[3] B. Bollobás and G. Brightwell. Cycles through specified vertices. *Combinatorica*, 13:147–155, 1993.

[4] B. Bollobás and R. Häggkvist. The circumference of a graph with given minimal degree. In A. Baker, B. Bollobás, and A. Hajnal, editors, *A Tribute to Paul Erdős.* Cambridge University Press, 1989.

[5] J. A. Bondy and M. Kouider. Hamilton cycles in regular 2-connected graphs. *J. Combinatorial Theory B*, 44:177–186, 1988.

[6] J. A. Bondy and U. S. R. Murty. *Graph Theory with Application.* The Macmillan Press Ltd, London, Basingstoke, 1976.

[7] R. P. Dilworth. A decomposition theorem for partially ordered sets. *Ann. Math.*, 51:161–165, 1950.

[8] G. A. Dirac. Some theorems on abstract graphs. *Proc. London Math. Soc.*, 2:69–81, 1952.

[9] Y. Egawa and T. Miyamoto. The longest cycles in a graph $G$ with minimum degree at least $|G|/k$. *J. Combinatorial Theory B*, 46:356–362, 1989.

[10] H. Enomoto, A. Kaneko, and Zs. Tuza. $P_3$–factors and covering cycles in graphs of minimum degree $n/3$. In A. Hajnal, L. Lovász, and V. T. Sós, editors, *Combinatorics, Eger (Hungary).* Colloquia Mathematica Societas János Bolyai, 1987.

[11] R. J. Faudree, R. J. Gould, M. S. Jacobson, and R. H. Schelp. Neighbourhood unions and hamiltonian properties of graphs. *J. Combinatorial Theory B*, 47:1–9, 1989.

[12] W. Feller. *An Introduction to Probability Theory and its Applications*, volume 1. John Wiley & Sons, New York, London, Sidney, 1968.

[13] P. Fraisse. A new sufficient condition for hamiltonian graphs. *J. Graph Theory*, 10:405–409, 1986.

[14] R. Gould. *Graph Theory.* The Benjamin/Cummings Publishing Company, Inc., 1988.

[15] R. L. Graham, D. E. Knuth, and O. Patashnik. *Concrete Mathematics.* Addison-Wesley, Reading, Mass., 1989.

[16] W. Jackson. Hamilton cycles in regular 2-connected graphs. *J. Combinatorial Theory B*, 29:27–46, 1980.

[17] W. Jackson. Cycles through vertices of large maximum degree. *J. Graph Theory*, 1994. To appear.

[18] K. Menger. Zur allgemeinen Kurventheorie. *Fund. Math.*, 10:96–115, 1927.

[19] J. van den Heuvel. *Degree and Toughness Conditions for Cycles in Graphs.* PhD thesis, University of Twente, 1993.

[20] D. R. Woodall. The binding number of a graph and its anderson number. *J. Combinatorial Theory B*, 15:225–255, 1973.