Essays on Macro-finance

Pawel Zabczyk

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Abstract

This thesis is a collection of four essays dealing with issues on the verge of macroeconomics and finance. The first two chapters are joint with Bianca De Paoli. In the first one, we try to analyze factors causing risk premia to vary over time. The second one is an attempt to understand how these factors – manifesting themselves through swings in the desire to save for precautionary reasons – affect monetary policy.

We show analytically that in endowment economies, procyclical recession expectations can 'outweigh' countercyclical changes in 'risk-aversion' – generating counterfactual risk-premium behavior. However, allowing shocks or habits to be sufficiently persistent, or explicitly accounting for the impact of habits on consumption, suffices to generate countercyclical recession risks and risk premia. We also show that taking note of precautionary saving motives justifies an accommodative policy bias in the face of persistent, adverse disturbances. Equally, policy should be more restrictive – i.e. 'lean against the wind' – following positive shocks.

Both of these essays rely on approximate solutions to a simple, external habit model. In the third chapter, I derive closed form formulae for the model's solution. I then use these formulae to estimate the model and analyze its ability to jointly fit consumption growth and asset price data. I find no specification capable of simultaneously matching these data and argue that 'exotic' shock distributions are an unlikely panacea.

In general, however, closed form formulae for asset prices cannot be derived analytically and need to be approximated. The final chapter proposes a method of doing exactly that. In contrast to several alternative approaches, the approximating function is not restricted to be a polynomial in state variables. This flexibility and efficient use of nested solutions can allow 'low-order' formulae to exceed the accuracy of higher-order perturbation approximations.

Declaration

I certify that this thesis is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

I warrant that this authorization does not, to the best of my belief, infringe the rights of any third party.

Pawel Zabczyk

Signature ....

Date ....

Statement Concerning Conjoint Work

Chapters 1 and 2 in my thesis are based on unpublished research with Bianca De Paoli (Bank of England). Both of us worked on all aspects of the papers and, while hard to quantify exactly, my personal share equalled approximately 50%.

Pawel Zabczyk

Signature ....

Date ....

Acknowledgements

I would have never finished this thesis alone. It is equally clear, that I will never be able to fully express my gratitude to those who helped me complete it.

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Introduction

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Both of these essays rely on approximate solutions to a simple, external habit model. In the third chapter, I derive closed form formulae for the model's solution. I then use these formulae to estimate the model and analyze its ability to jointly fit consumption growth and asset price data. In general, however, closed form formulae for asset prices cannot be derived analytically and need to be approximated. The final chapter proposes a method of doing exactly that and applies it to five popular models.

In the remainder of the introduction I discuss each of the four contributions in a bit more detail.

Why do risk premia vary over time? A theoretical investigation under habit formation

In this essay we study the dynamics of risk premia in a model with external habit formation and highlight the significance of 'recession predictability'. While under the specification of Campbell and Cochrane (1999) the equity risk premium is countercyclical because increases in risk aversion are reinforced by rising recession risks – this need not be the case more generally. We show analytically that in endowment economies, procyclical recession expectations can 'outweigh' countercyclical changes in 'risk-aversion' – generating counterfactual risk-premium behavior. However, allowing shocks or habits to be sufficiently persistent, or explicitly accounting for the impact of habits on consumption suffices to generate countercyclical recession risks and risk premia.

Cyclical Risk Aversion, Precautionary Saving and Monetary Policy

This chapter analyzes the conduct of monetary policy in an environment in which cyclical swings in risk appetite affect households' propensity to save. It uses a New-Keynesian model featuring external habit formation to show that taking note of precautionary saving motives justifies an accommodative policy bias in the face of persistent, adverse disturbances. Equally, policy should be more restrictive – i.e. 'lean against the wind' – following positive shocks. Since the size of these 'risk-adjustments' is increasing in the degree of macroeconomic volatility, ignoring this channel could lead to larger policy errors in turbulent periods.

Asset Prices Under Persistent Habits and Arbitrary Shocks to Consumption Growth

The third essay derives closed-form solutions for the equity price-dividend ratio and equity risk-premium in a model in which agents have difference-form external habits. The setup allows for arbitrary shock distributions, correlated consumption growth and persistent extensions of the keeping-up and catching-up with the Joneses specifications. We use the exact solutions to study the ability of alternative estimated models – including one capturing rare events – to simultaneously account for consumption, equity and bond returns in the UK and US. We find no specification capable of simultaneously matching these data and argue that 'exotic' shock distributions are an unlikely panacea.

Approximating Solutions of Asset Pricing Models: The Implicit Function Approach

The final chapter proposes a new method of approximating solutions of asset pricing models. It shows how such models can be restricted and solved exactly and how these exact formulae can be used to approximate solutions of the original, unrestricted problems. It also demonstrates how approximation errors can be assessed. In contrast to several alternative approaches, the approximating function is not restricted to be a polynomial in state variables. This flexibility and efficient use of nested solutions can allow 'low-order' formulae to exceed the accuracy of higher-order perturbation approximations. The approach is illustrated through application to five asset pricing models.

Chapter 1

Why do risk premia vary over time? A theoretical investigation under habit formation

1.1 Introduction

Existing empirical evidence suggests that risk premia vary countercyclically over time (Harvey (1989) and Li (2001) focus on equities; Campbell and Shiller (1991) and Cochrane and Piazessi (2005) present evidence for the term premium; Lustig and Verdelhan (2007) document strong countercyclicality in the exchange rate risk premium). The two most seminal, representative-agent models proposed in this context seem to rely on very different mechanisms of accounting for the empirical regularities. In Campbell and Cochrane (1999), risk aversion fluctuates countercyclically as agents' consumption moves relative to a reference habit level. In Bansal and Yaron (2004), shifts in premia are driven by predictable components in consumption as well as time-varying volatility. In this paper we focus on the habit mechanism and show that it also heavily relies on cyclical variation in underlying state variables. In particular, it depends on the fact that recession risks change predictably over the cycle.[1]

While Campbell and Cochrane (2000) have also stressed the role of predictability in the context of consumption based asset pricing models, the quantitative

[1]Our use of the terms 'recession risks' and 'risk aversion' follows Campbell and Cochrane (1999). Both terms are further disambiguated in Section 1.2.

importance of the recession risk channel has not been studied extensively.[2] The likely reason is that under the original specification of Campbell and Cochrane (1999), slow-moving habits ensure that recession expectations vary countercyclically. Accordingly, the 'recession risk' channel reinforces the 'risk-aversion' channel and amplifies countercyclical variation in risk premia. Our first contribution is to document that under simpler, but commonly used habit specifications, this need not be the case. In particular, we show that in a model with *non-persistent* habits and trend-stationary consumption – recession risks vary procyclically, offsetting the impact of countercyclical risk-aversion and generating *procyclical* premia. We also demonstrate how increasing the degree of persistence in consumption and habits can overturn this result.

The benchmark endowment-economy framework we adopt is that of Lucas' (1978) and Mehra and Prescott's (1985) asset pricing model. While the parsimony of our setup keeps the analysis tractable and enables us to prove most results analytically, the danger is that misspecifying the process for consumption might invalidate the conclusions. To address this point, we verify our findings under a wide range of specifications – including the standard 'finance' assumption of i.i.d. consumption growth, persistent consumption growth dynamics (Carroll et al., 2008) and allowing for alternative trend specifications (Den Haan, 1995). The finding that increasing shock and habit persistence changes the cyclical properties of 'recession risks' and is more likely to make risk-premia vary countercyclically survives irrespective of the exact set of underlying assumptions.

Since a slowly time-varying, countercyclical risk premium is key for matching asset pricing data (Campbell and Cochrane, 1999, p. 207) any model with procyclical recession risks might fail along the asset pricing dimension. This finding could potentially have important implications for dynamic macroeconomic models which rely on *non-persistent* habit specifications (Christiano et al., 2005; Smets and Wouters, 2007; Uhlig, 2007) and which are increasingly frequently used to address asset pricing questions (Jermann, 1998; De Paoli et al., 2007; Hordahl et al., 2008; Rudebusch and Swanson, 2008).[3] In the remainder of this paper we thus try to clarify risk-premium implications of habit formation in production-economy models.

[2] Collard et al. (2006) examine the implications of consumption predictability in an *internal* habit model where risk aversion is, by construction, constant. Thus, implicitly, the variation in equity risk premia in their model is driven entirely by changing recession risks.

[3] Both Uhlig (2007) and Smets and Wouters (2007) consider consumption-leisure nonseparabilities, which are absent from our setup.

An important complication which arises in non-endowment economy models is that habits – in addition to affecting the degree of fluctuations in risk aversion and recession risks – also have an endogenous impact on the dynamics of consumption (in particular, they have a well documented role in helping to generate hump-shaped consumption impulse responses; see also Fuhrer (2000); Christiano et al. (2005); Smets and Wouters (2007)). To avoid obfuscating these different channels, we start by proving results analytically in an endowment economy setup under the assumption of 'hump-shaped' consumption dynamics (proxy for the general equilibrium impact of habits). We then verify these results numerically in a fully specified dynamic general equilibrium model. In summary, our second contribution is to show that allowing habits to endogenously feed back on consumption is likely to make recession risks vary countercyclically regardless of the degree of persistence in the habit process. This suggests a potentially important role for habit formation in helping to match risk-premium dynamics in production-economy models.

Several existing contributions explore issues related to those we analyze. Li (2007) documents that premia in the framework of Campbell and Cochrane (1999) are not robustly countercyclical - a point similar to the one we make, though in a different setup. Den Haan (1995) considers a related set of driving processes for consumption, but focuses on their implications for the slope of the yield curve. In addition, there are many papers showing how habits in the utility help match empirical properties of anything from yield curve dynamics (Wachter, 2006) to exchange rate risk premia (Verdelhan, 2009), though we are unaware of any which explicitly investigate the implications of time-varying recession-risks in a general equilibrium context.

In the remainder, we set up the model, present the analytical results and discuss the underlying intuition, before summarizing and concluding.

1.2 Model and Notation

Our analysis proceeds in the simplest possible setup - that of Lucas' (1978) asset pricing model augmented by allowing for external habits.[4] Agents, indexed by

[4]Several studies argue that disaggregated data offer little empirical support for the external habit specification (Dynan, 2000; Chen and Ludvigson, 2009). Despite this, aggregate data can still behave as if generated in a representative agent habit model. The study of Garleanu and Panageas (2008) demonstrates this point clearly.

$i \in [0,1]$ choose consumption C_t^i, investment in riskless bonds B_t^i and investment in risky assets S_t^i to maximize expected lifetime utility

$$\max_{C_t^i,\, B_t^i,\, S_t^i} \mathrm{E}\left(\sum_{t=0}^{+\infty} \beta^t \frac{\left(C_t^i - hX_t\right)^{1-\rho} - 1}{1-\rho}\right) \tag{1.1}$$

$$C_t^i + V_t^f B_t^i + V_t^r S_t^i = B_{t-1}^i + S_{t-1}^i\left(V_t^r + D_t\right) \tag{1.2}$$

where X_t denotes the external habit, V_t^f is the time t price of a one-period bond, paying a unit of the consumption-good next period and V_t^r is the price of a perfectly divisible risky asset entitling its owner to the stream of dividends $D_{t+1}, D_{t+2} \ldots$[5]

The way in which habits X_t are defined has important implications for the dynamics of asset prices and risk premia. We choose not to adopt the non-linear specification of Campbell and Cochrane (1999) because the latter hasn't been used in any larger, production economy DSGE models, isn't particularly tractable or tightly parameterized[6] and finally – in contrast to simpler, linear specifications – the rationale for referring to X_t as 'habits' is not immediately clear.

Instead, our approach is to posit that habits are an average of past levels of aggregate consumption C_t, with a single parameter $\phi \in [0,1]$ controlling persistence

$$X_t := (1-\phi)C_{t-1} + \phi X_{t-1}. \tag{1.3}$$

By deliberately keeping the setup simple, our results can shed light on the relative importance of various channels driving the results in Campbell and Cochrane (1999) who use an equation like (1.3) as a foundation of their model. Furthermore, when $\phi = 0$ the specification simplifies to one in which habits are purely a function of last period's aggregate consumption, which has been frequently used in the macro literature. Finally, despite the simplicity, the implied coefficient

[5]There are a few reasons why we choose the external rather than internal habit specification. As noted in Campbell and Cochrane (1999), external habits simplify the analysis by eliminating terms in marginal utility by which extra consumption today raises habits tomorrow. Crucially, even though marginal utility under the two specifications differs, what matters for asset prices is the ratio of marginal utilities. As the NBER working paper version of the Campbell and Cochrane article suggests – under linear habits, similar to those we consider – the ratios behave very much alike. Accordingly, we chose to focus on the more tractable, external habit setup.

[6]Strictly speaking the Campbell and Cochrane (1999) parameterization is 'infinitely dimensional' as, other coefficients aside, it depends on a whole sensitivity function $\lambda(\cdot)$.

of risk aversion varies countercyclically, and so we retain crucial features of the successful nonlinear specification.[7]

The standard first order conditions with respect to asset holdings are

$$R^f_{t+1} \cdot \mathrm{E}_t \mathcal{M}^i_{t+1} = 1 \qquad \mathrm{E}_t \mathcal{M}^i_{t+1} \, R^r_{t+1} = 1 \tag{1.4}$$

where the stochastic discount factor $\mathcal{M}^i_t$, the marginal utility of consumption Λ^i_t and gross returns on bonds R^f_t and risky assets R^r_t are given by[8]

$$\mathcal{M}^i_{t+1} := \beta \cdot \frac{\Lambda^i_{t+1}}{\Lambda^i_t} \qquad \Lambda^i_{t+1} := \left(C^i_{t+1} - hX_{t+1}\right)^{-\rho}$$

$$R^f_{t+1} := \frac{1}{V^f_t} \qquad R^r_{t+1} := \frac{V^r_{t+1} + D_{t+1}}{V^r_t}.$$

We then let excess consumption C^e_t and the surplus ratio S_t be defined as

$$C^e_t := C_t - hX_t \qquad S_t := \frac{C_t - hX_t}{C_t} = \frac{C^e_t}{C_t}. \tag{1.5}$$

As in Campbell and Cochrane (1999) (p.220), we shall refer to periods with low values of the surplus consumption ratio as 'recessions' and the term $\mathrm{E}_t 1/S_{t+1}$ (or $S_t \, \mathrm{E}_t 1/S_{t+1}$) will be used to characterize the 'fear of recessions' channel.[9]

The textbook definition of the equity risk premium rp_t is

$$rp_t := \mathrm{E}_t \left(\log\left(R^r_{t+1}\right)\right) - \log\left(R^f_{t+1}\right) = \mathrm{E}_t r^r_{t+1} - r^f_{t+1} \tag{1.6}$$

where lower-case letters denote logs. Notably, our measure of the risk premium corresponds to a generic asset held for one period. Thus, since neither the length of that period nor the asset payoffs have been specified, the subsequent

[7]The parsimony of our setup comes at a cost. For example, to ensure that habits never exceed consumption and so utility remains well-defined we would need restrictions on the support of shocks driving consumption growth. While such conditions are derived in the third chapter, in the remainder we ignore the problem and proceed under the assumption of normality. Furthermore, unlike Campbell and Cochrane (1999), we have no means of using precautionary saving motives to exactly offset the intratemporal substitution effect implying that our risk-free rate is going to be volatile. Given that the stated goal of this paper is analyzing premium cyclicality rather than exactly matching asset price dynamics, we consider the implied detriment in fit a small price to pay for tractability.

[8]For analytical tractability, we shall restrict attention to a symmetric equilibrium, i.e. we assume that individual consumption C^i_t as well as marginal utility Λ^i_t and $\mathcal{M}^i_t$ – the stochastic discount factor – equal their respective aggregate equivalents C_t, Λ_t and $\mathcal{M}_t$.

[9]High values correspond to expectations of deteriorating living conditions. In terms of interpretation, the mechanism is similar to the one in Kandel and Stambaugh (1990).

exposition applies to premia on any risky asset.

Finally, we can define the coefficient of relative risk aversion as[10]

$$\eta(C_t, X_t) := -C_t \cdot \frac{U_{cc}(C_t, X_t)}{U_c(C_t, X_t)}$$

where $U_y(\cdot,\cdot)$ denotes the partial derivative of utility function $U(\cdot,\cdot)$ with respect to y. Since this coefficient measures agents' willingness to enter pure consumption gambles, given habits fixed at reference level X_t, this can be referred to as *consumption* risk aversion. It is easy to show that if the utility function and external habits are as in equations (1.1) - (1.3) then the coefficient of consumption risk aversion is countercyclical.[11]

1.3 Results

To analyze the determinants of risk premium dynamics in the model, we can derive a second order approximation to the first order conditions. This approximation implies

$$rp_t + \frac{1}{2} var_t\, r^r_{t+1} \approx \rho\, cov_t(c^e_{t+1}, r^r_{t+1}). \tag{1.7}$$

Jensen's inequality term aside, the risk premium is proportional to *excess consumption* relative risk aversion ρ and the conditional covariance of returns r^r_{t+1} with excess consumption c^e_{t+1}. Following Li (2001), and under the assumptions discussed therein (see Appendix A for a brief discussion), we can apply Stein's lemma and express $cov_t(c^e_{t+1}, r^r_{t+1})$ as

$$cov_t(c^e_{t+1}, r^r_{t+1}) = cov_t(c_{t+1}, r^r_{t+1})\, \mathbf{E}_t \frac{1}{S_{t+1}}. \tag{1.8}$$

Mechanically, equation (1.8) demonstrates that agents' expectations about recessions matter because they affect the covariance of excess consumption and returns. Combined with equation (1.7) this shows that if $cov_t(c_{t+1}, r^r_{t+1})$ is time invariant, then only changes in these expectations are going to affect risk pre-

[10] As noted in Campbell et al. (1997) "*risk aversion may also be measured by the normalized curvature of the value function [...] or by the volatility of the stochastic discount factor [...] While these measures of risk aversion are different from each other in this model, they all move inversely with S_t.*"

[11] Even though risk aversion η_t and recession risks, as summarized by $\mathbf{E}_t 1/S_{t+1}$, are both functions of the surplus ratio S_t, their dynamics and cyclicality can be very different.

mium cyclicality.[12]

Crucially, equation (1.8) can further be rewritten as

$$rp_t + \frac{1}{2} var_t \, r^r_{t+1} \approx \eta_t \, cov_t(c_{t+1}, r^r_{t+1}) \, \mathrm{E}_t \frac{S_t}{S_{t+1}} \tag{1.9}$$

which demonstrates that the risk premium is determined by the coefficient of risk aversion η_t, the covariance of consumption and returns as well as the 'fear or recessions' variable $\mathrm{E}_t(1/S_{t+1})/(1/S_t)$ - which determines whether agents expect their living conditions to improve or deteriorate next period. Since the coefficient of relative risk aversion η_t is countercyclical and the covariance term approximately constant therefore the only factor with the potential to distort the cyclicality of rp_t is the recession expectation term.

In the remainder of this paper we shall therefore study how assumptions made about consumption and habits affect the cyclicality of $\mathrm{E}_t S_t/S_{t+1}$. In the first subsection we shall implicitly ignore the fact that habits have the potential to endogenously affect the dynamics of consumption. In the second part we explicitly try and account for such feedbacks and investigate the impact of hump shaped consumption responses on the cyclicality of recession expectations (in an endowment and production economy).

1.3.1 No Endogenous Feedback of Habits on Consumption

In this section we investigate factors driving risk premium cyclicality under different specifications for the endowment process. Following Den Haan (1995), we examine the case in which consumption is stationary around a linear trend and one in which it is growth-rate stationary (the case of i.i.d. consumption growth is nested by both of these specifications). We shall show that if consumption is not sufficiently persistent then habits need to adjust slowly, as otherwise procyclical changes in recession expectations $\mathrm{E}_t S_t/S_{t+1}$ could more than offset the impact of countercyclical risk aversion leading to procyclical risk premium dynamics.

[12]Many partial-equilibrium finance papers assume that the covariance of consumption and returns is constant. Numerical simulations conducted on our model suggest that fluctuations in these covariances are small and for this reason we impose the assumption of time-invariance in subsequent propositions.

Trend-Stationary Consumption Process

Proposition 1. If the conditional variance of returns $\mathrm{var}_t(r^r_{t+1})$ and their conditional covariance with consumption $\mathrm{cov}_t(r^r_{t+1}, c_{t+1})$ are constant and log-consumption follows

$$c_t = \kappa_0 + \kappa_1 t + \gamma c_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim i.i.d.(0, \sigma^2), \qquad \gamma \in [0, 1] \tag{1.10}$$

then the derivative of the risk premium rp_t with respect to the current shock realization can be expressed as[13,14]

$$\frac{\partial rp_t}{\partial \varepsilon_t} \approx E_t S^{-2}_{t+1} C^{-1}_{t+1} h(1-\phi)\left[C_t - \gamma \sum_{s=0}^{t} \phi^s C_{t-s}\right].\square \tag{1.11}$$

The proof of this and all subsequent propositions can be found in Appendix A. Since, in general, the sign of the risk premium is ambiguous, in order to build some intuition we now focus on two popular nested specifications – one in which habits fully adjust in a single period and another in which log-consumption is a random walk.

Corollary 1. Under the assumptions of Proposition 1, if habits only depend on past periods' consumption ($\phi = 0$) then the premium is procyclical

$$\frac{\partial rp_t}{\partial \varepsilon_t} = E_t S^{-2}_{t+1} C^{-1}_{t+1} h(1-\gamma) \geq 0.\square \tag{1.12}$$

Under the assumptions of Corollary 1, following an adverse shock, recession risks abate and agents expect future conditions to improve (given the AR(1) nature of the endowment process, consumption is expected to increase while habits will unambiguously fall). Since habits adjust fully in a single period, excess consumption, which is all agents care about, is expected to *increase* following the negative shock. Thus, even more risk averse agents will require lower compensation for bearing risk and consequently risk premia fall.

[13]In the absence of output in our simple model, we use consumption as a measure of cyclical stance. Hence if variables positively co-move with consumption - as measured by their derivative with respect to shocks to consumption growth ε_t - we say they are procyclical.

[14]Equality (1.11) holds exactly under the additional assumption that excess consumption and risky returns are jointly conditionally log-normal and that consumption is also conditionally log-normal. See Appendix A for details.

On the face of it, the fact that the risk premium is procyclical if $\phi = 0$ could be important as that assumption is frequently used in macro-models. As demonstrated in the following subsections, however, this conclusion does not survive extensions to production economies – where the internal propagation mechanism starts playing a role – or alternative assumptions about consumption dynamics.

Corollary 2. Under the assumptions of Proposition 1 if log-consumption follows a random walk ($\kappa_0 = \kappa_1 = 0, \gamma = 1$) and habits are persistent ($\phi > 0$) then the risk premium is countercyclical as

$$\frac{\partial rp_t}{\partial \varepsilon_t} = -E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1-\phi) \sum_{s=1}^{t} \phi^s C_{t-s} < 0. \square \tag{1.13}$$

Under the assumptions of Corollary 2, shocks to consumption are permanent and habits adjust gradually. In this setting, after adverse shocks – as expected excess consumption falls – expectations of future conditions deteriorate (recession risks become larger) leading to an increase in the risk premium. This shows that a combination of permanent shocks and persistent habits generates countercyclically varying risk premia.

Equation (1.11) generalizes this point and shows that a sufficiently persistent shock yields countercyclical premium variation. While the effect of increasing habit persistence ϕ might seem less clear cut, evaluating expression (1.11) for plausible parameter values suggests that raising ϕ has a similar effect.

Growth-Stationary Consumption Process

The final specification we focus on assumes that log consumption *growth* is an AR(1) process.

$$\Delta c_t = \kappa + \delta \Delta c_{t-1} + \varepsilon_t \qquad \varepsilon_t \sim \mathcal{N}.i.d.(0, \sigma^2), \qquad \delta \in [0,1]. \tag{1.14}$$

This formulation is supported by recent empirical evidence on sticky consumption growth - see e.g. Carroll et al. (2008).

Proposition 2. If the conditional variance of returns $\text{var}_t(r_{t+1}^r)$ and their conditional covariance with consumption $\text{cov}_t(r_{t+1}^r, c_{t+1})$ are constant and consumption follows a stochastic process as in equation (1.14), then

$$\frac{\partial rp_t}{\partial \varepsilon_t} = -\mathrm{E}_t S_{t+1}^2 (1-\phi) \cdot \frac{C_t}{C_{t+1}} \cdot \left(\delta + (1+\delta) \cdot \sum_{s=1}^{+\infty} \phi^s \frac{C_{t-s}}{C_t} \right) < 0.\square \tag{1.15}$$

Under this endowment specification, in contrast to the case of autoregressive detrended consumption (see Corollary 1), risk premia are countercyclical even under a non-persistent habit specification ($\phi = 0$). [15]

1.3.2 Allowing for Endogenous Feedback of Habits on Consumption

As argued previously, more sophisticated, production-economy general equilibrium models are likely to introduce a variety of state-dependencies into consumption's law of motion.[16] Importantly, Fuhrer (2000) documented that habits help generate hump-shaped consumption impulse responses we now inspect the 'recession risk' channel under hump-shaped consumption dynamics.

In what follows we show that – relative to the specifications of the previous subsection – the perceived risk of recession under hump-shaped consumption dynamics is more likely to be countercyclical, amplifying the countercyclical risk-aversion channel. So, in a fully general equilibrium setting, habits help generate countercyclically varying risk-premia not only because they imply a countercyclical risk aversion, but also because they generate countercyclical 'recession risks'.

To illustrate this point, we first derive analytical results in an endowment economy in which the *exogenous* law of motion for consumption implies hump-shaped impulse responses. Arguably, this analysis has a partial equilibrium flavor, as habit persistence does not map into the shape of the 'hump' – which is controlled by different parameters. Accordingly, we conclude by numerically analyzing risk premium dynamics in a fully-fledged production economy model. The latter confirms that *endogenous* hump-shaped consumption responses are associated with countercyclical risk premium variation, even for non-persistent habits.

[15]Note that, irrespective of the consumption specification, cyclicality of premia is defined as their 'on-impact' response to the white-noise disturbance term ε.

[16]Even though the dynamics of aggregate consumption might, particularly given limited data, be indistinguishable from that generated by the processes considered above, the asset pricing implications of different specifications could very well differ.

Endowment Economy

To capture the idea that the peak response of consumption is not attained immediately after the shock hits (i.e. to model 'hump-shaped' consumption responses) we now assume that the detrended endowment follows an ARMA(1,1) process[17]

$$c_t = \kappa_0 + \kappa_1 t + \gamma c_{t-1} + \varepsilon_t + \theta \varepsilon_{t-i}, \qquad \varepsilon_t \sim \mathcal{N}.i.d.(0, \sigma^2), \qquad \gamma \in [0, 1]. \tag{1.16}$$

Proposition 3. If the conditional variance of returns $\text{var}_t(r^r_{t+1})$ and their conditional covariance with consumption $\text{cov}_t(r^r_{t+1}, c_{t+1})$ are constant and log-consumption follows the specification in equation (1.16) then

$$\frac{\partial rp_t}{\partial \varepsilon_t} \approx E_t S^{-2}_{t+1} C^{-1}_{t+1} h(1-\phi) \left[C_t - (\gamma + \theta) \sum_{s=0}^{t} \phi^s C_{t-s} \right]. \tag{1.17}$$

In particular, if habits only depend on past periods' consumption ($\phi = 0$) and $\gamma + \theta \geq 1$ then

$$\frac{\partial rp_t}{\partial \varepsilon_t} \approx E_t S^{-2}_{t+1} C^{-1}_{t+1} h C_t \left[1 - (\gamma + \theta) \right] \leq 0.\square \tag{1.18}$$

Equation (1.17) demonstrates that θ – the parameter controlling the size of the 'hump' – can play a similar role to γ – i.e. consumption persistence. It suggests that models with hump-shaped consumption responses may be able to generate countercyclical premia without persistent habits or shocks. In the case of non-persistent habits, described in equation (1.18), if $\gamma + \theta > 1$ then the risk premium is unambiguously countercyclical.

Under this specification log-consumption decreases further in the period after an adverse shock, following which it converges back to its steady state (i.e. it bottoms out in period two). That is, after a bad shock agents expect the future to get worse (i.e. recession risks increase) and therefore require a higher compensation for bearing risk.

[17]The results discussed in this section naturally generalize to arbitrary ARMA(1,K) processes.

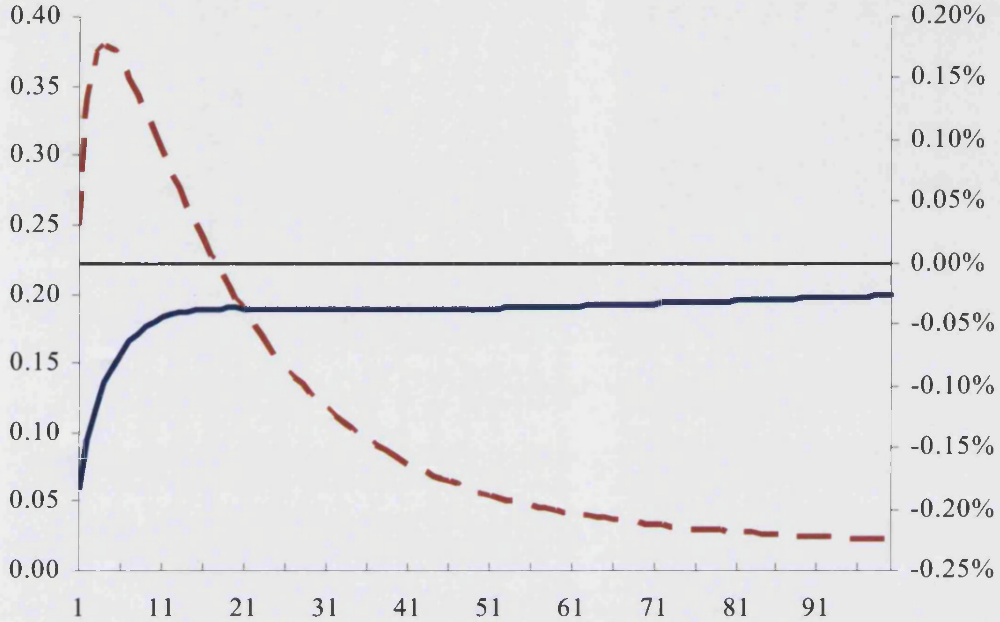

Figure 1.1: Response of Consumption (Solid Line, LHS) and the Equity Risk Premium (dashed line, RHS) to a Productivity Shock in the Model of De Paoli et al. (2007)

Production Economy

We now scrutinize the dynamics of the equity risk premium in a production economy with real rigidities. In particular, we use the model documented in De Paoli et al. (2007).[18] The setup features non-persistent habit formation (i.e. $h \neq 0$ but $\phi = 0$) and explicitly models capital and capital adjustment costs. While the structural shocks are AR(1) the model's internal propagation mechanism generates a hump-shaped response of consumption to productivity shocks – see also Figure 1.

To compute the response of the equity risk premium to a productivity shock, we used a third-order perturbation approximation to the model's policy function.[19] Figure 1 confirms that in a general equilibrium framework with habit formation

[18] We use the flexible price version of the model. The calibration is detailed in Table A, p.40.

[19] Third order is the lowest which allows for time variation in risk premia. As discussed in Schmitt-Grohe and Uribe (2004) — a first-order approximation would imply that premia are zero at all times, while a second-order approximation would only allow for constant premia. The approximation was computed using *Dynare++* [*www.dynare.org*] and Perturbation AIM [*http://www.ericswanson.us/perturbation.html*].

consumption has a sluggish response to the shock, generating countercyclical recession risks and thus countercyclical risk premia. This suggests that the findings of Proposition 3 also apply to a production-economy model.

1.4 Summary and Conclusions

We have used the simplest possible framework to analyze the determinants of risk premium dynamics. We demonstrated that due to changing recession risks, risk premia can be *procyclical* even though the volatility of consumption is *constant* and despite a *countercyclically* varying risk aversion coefficient. We have also documented that persistent habits, shocks or features generating hump shaped consumption responses are all likely to make the premium countercyclical.

Fundamentally, the countercyclicality of the premium rests on agents' belief that changes in economic conditions are persistent. In other words, after an adverse shock, more risk-averse agents will only require a larger premium on risky assets if they don't expect their future conditions to improve massively. Expressed alternatively, our work explicitly explores the role of countercyclical recession risks – a feature that is implicit in Campbell and Cochrane (1999), and similar in spirit to the mechanism driving the results in Bansal and Yaron (2004). Our results suggest that factors which help match activity data - i.e. allowing for consumption habits and persistent shocks - are also likely to help along the asset pricing dimension.

Changes in premia substantially contribute to asset price volatility and so having a good understanding of factors driving them is crucial for modeling asset prices. Given the increasing frequency with which macroeconomic models are being used to address asset pricing puzzles, it is key to clarify how and why changes in standard modeling assumptions translate into different dynamics of premia. While this study attempts to clarify these issues, further analysis of risk premium dynamics could still be undertaken. For example, in production-economy models the dynamics of term-premia or the inflation risk premium would depend on other aspects of the transmission mechanism. We believe that examining these aspects would be of interest.

1.A Appendix

Proof of Proposition 1

Li (2001) demonstrates that under certain distributional assumptions on c_t^e, r_t^r and C_t

$$rp_t = -\frac{1}{2} var_t \left(r_{t+1}^r\right) + \delta \, cov_t \left(c_{t+1}, r_{t+1}^r\right) + \lambda_t \delta \, cov_t \left(c_{t+1}, r_{t+1}^r\right) \tag{1.19}$$

where $\lambda_t = E_t(\partial s_{t+1}/\partial c_{t+1}) = E_t(1/S_{t+1}) - 1$. Repeated use of the definition of habits - equation (1.3) - and the endowment specification - equation (1.10) - makes it possible to express S_t as

$$S_{t+1} = \left[1 - \left(C_t^{\gamma} e^{\kappa_0 + \kappa_1 (t+1) + \varepsilon_{t+1}}\right)^{-1} h(1-\phi) \sum_{s=0}^{+\infty} \phi^s C_{t-s}\right]. \tag{1.20}$$

Thus, computing the derivative of the above expression we have

$$\frac{\partial \lambda_t}{\partial C_t} = E_t - S_{t+1}^{-2} \frac{\partial S_{t+1}}{\partial C_t} = E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1-\phi) \left[1 - \gamma C_t^{-1} \sum_{s=0}^{+\infty} \phi^s C_{t-s}\right]$$

which, can be plugged into the chain rule $\partial \lambda_t / \partial \varepsilon_t = \partial \lambda_t / \partial C_t \cdot \partial C_t / \partial c_t \cdot \partial c_t / \partial \varepsilon_t$ to yield

$$\frac{\partial \lambda_t}{\partial \varepsilon_t} \approx E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1-\phi) \left[C_t - \gamma \sum_{s=0}^{+\infty} \phi^s C_{t-s}\right]. \blacksquare \tag{1.21}$$

Proof of Proposition 2

Denoting gross consumption growth by $c^g_t := C_t/C_{t-1}$ and defining $x_t := X_t/C_t$ equations (1.14) and (1.3) can be written as

$$c^g_{t+1} = \exp\left(\kappa + \varepsilon_{t+1}\right) \cdot \left(c^g_t\right)^{\delta} \tag{1.22}$$

$$x_t = \left((1-\phi) + \phi x_{t-1}\right) \cdot \left(c^g_{t-1}\right)^{-1}. \tag{1.23}$$

Iterating on (1.23) we can express x_t as

$$x_t = (1-\phi)\sum_{i=1}^{+\infty}\phi^{i-1}\left\{\prod_{j=0}^{i-1}\left(c^g_{t-j}\right)^{-1}\right\} = (1-\phi)\sum_{s=0}^{+\infty}\phi^{s}\left\{\prod_{j=0}^{s}\left(c^g_{t-j}\right)^{-1}\right\} \tag{1.24}$$

which implies that

$$S_{t+1} = (1 - hx_{t+1}) = \left(1 - h(1-\phi)\sum_{s=0}^{+\infty}\phi^{s}\left\{\prod_{j=0}^{s}\left(c^g_{t+1-j}\right)^{-1}\right\}\right). \tag{1.25}$$

Similarly as in the case of a deterministic trend in consumption we can now compute the derivative

$$\frac{\partial S_{t+1}}{\partial c^g_t} = \frac{\partial}{\partial c^g_t}\left(1 - h(1-\phi)\left(c^g_{t+1}\right)^{-1} - h(1-\phi)\left(c^g_{t+1}c^g_t\right)^{-1}\sum_{s=1}^{+\infty}\phi^{s}\left\{\prod_{j=2}^{s}\left(c^g_{t+1-j}\right)^{-1}\right\}\right)$$

where $\prod_{k=2}^{1} a_k = 1$. Exploiting equation (1.22) we know that

$$\frac{\partial\left(c^g_{t+1}\right)^{-1}}{\partial c^g_t} = -\left(c^g_{t+1}\right)^{-2}\delta\frac{c^g_{t+1}}{c^g_t} = -\delta\left(c^g_{t+1}c^g_t\right)^{-1} \tag{1.26}$$

$$\frac{\partial\left(c^g_{t+1}c^g_t\right)^{-1}}{\partial c^g_t} = -\left(c^g_{t+1}c^g_t\right)^{-2}\left(c^g_{t+1} + c^g_t\delta\frac{c^g_{t+1}}{c^g_t}\right) = -(1+\delta)\left(c^g_{t+1}\right)^{-1}\left(c^g_t\right)^{-2} \tag{1.27}$$

and so

$$\frac{\partial S_{t+1}}{\partial c^g_t} = h(1-\phi)\left(c^g_{t+1}c^g_t\right)^{-1}\left(\delta + (1+\delta)\cdot\sum_{s=1}^{+\infty}\phi^{s}\left\{\prod_{j=1}^{s}\left(c^g_{t+1-j}\right)^{-1}\right\}\right). \tag{1.28}$$

Accordingly

$$\frac{\lambda_t}{\varepsilon_t} = \mathrm{E}_t - S^2_{t+1}(1-\phi)\left(c^g_{t+1}\right)^{-1}\left(\delta + (1+\delta)\cdot\sum_{s=1}^{+\infty}\phi^{s}\left\{\prod_{j=1}^{s}\left(c^g_{t+1-j}\right)^{-1}\right\}\right). \blacksquare$$

Proof of Proposition 3

Given that

$$\lambda_t = E_t\left[1 - C_{t+1}^{-1}h(1-\phi)\sum_{s=0}^{+\infty}\phi^s C_{t-s}\right]^{-1} - 1 \tag{1.29}$$

and treating λ_t as a function of two variables $\lambda(C_t, C_{t+1})$

$$\frac{\partial \lambda_t}{\partial C_t} = E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1-\phi), \qquad \frac{\partial \lambda_t}{\partial C_{t+1}} = -E_t S_{t+1}^{-2} C_{t+1}^{-2} h(1-\phi)\sum_{s=0}^{+\infty}\phi^s C_{t-s}$$

we can apply the chain rule

$$\frac{\partial \lambda_t}{\partial \varepsilon_t} = \frac{\partial \lambda_t}{\partial C_t}\cdot\frac{\partial C_t}{\partial c_t}\cdot\frac{\partial c_t}{\partial \varepsilon_t} + \frac{\partial \lambda_t}{\partial C_{t+1}}\cdot\frac{\partial C_{t+1}}{\partial c_{t+1}}\cdot\left(\frac{\partial c_{t+1}}{\partial \varepsilon_t} + \frac{\partial c_{t+1}}{\partial c_t}\frac{\partial c_t}{\partial \varepsilon_t}\right)$$

to obtain that

$$\frac{\partial \lambda_t}{\partial \varepsilon_t} \approx E_t S_{t+1}^{-2} C_{t+1}^{-1} h(1-\phi)\left[C_t - (\theta+\gamma)\sum_{s=0}^{+\infty}\phi^s C_{t-s}\right].\blacksquare \tag{1.30}$$

Chapter 2

Cyclical Risk Aversion, Precautionary Saving and Monetary Policy

2.1 Introduction

A lot of modern policy analysis is conducted using linear, or linearized, models. While these may be able to replicate salient features of macroeconomic dynamics, there are important areas where their ability to 'match data' is less satisfactory. In particular, all linear models are 'certainty-equivalent', which implies that coefficients of their policy-rules are independent of the level of shock volatility. A striking consequence of this fact is that, absent ad-hoc adjustments, expected returns on all assets are identical – i.e. risk-premia are restricted to counterfactually equal zero. To the extent that asset prices reflect agents' attitudes towards risk, using models so badly misspecified along this dimension could result in systematically biased policy recommendations. This paper investigates the issue in more depth.

Rather than trying to analyze many aspects of uncertainty, we focus on just one – precautionary savings.[1] Given that some studies (Carroll and Samwick, 1998) estimate that in excess of 40% of all wealth accumulation has to do with

[1]The fact that households save for 'a rainy day', also referred to as precautionary or buffer stock saving, has long been well recognized and documented (see e.g. Leland (1968), Sandmo (1970), Carroll (1992), Kazarosian (1997), Carroll and Samwick (1998) or Ludvigson and Michaelides (2001)).

precautionary motives, incorporating this channel is likely to be important. At the same time, focusing on a single aspect of risk makes it easier to establish traction with standard New-Keynesian models and allows us to derive our results analytically – retaining some of the linearized framework's appeal.

Since we are motivated by the benchmark model's failure to match asset price data, we extend the setup by allowing for persistent external habits – whose appeal in the asset pricing context was demonstrated by Campbell and Cochrane (1999). At the same time, we retain standard features such as monopolistic competition and staggered Calvo (1983) price setting. In our framework, external habits generate cyclical swings in risk aversion, which translate into fluctuations in the desire to save for precautionary reasons. Crucially, to give this channel bite, we consider a nonlinear approximation to the consumption-Euler equation – explicitly accounting for a state-dependant precautionary saving motive.[2]

Incorporating cyclical risk aversion has clear implications for the propagation mechanism of shocks and consequently for the appropriate policy response. Consider, for example, the case in which a negative demand disturbance hits the economy. Increases in risk aversion and the desire to save for precautionary reasons would likely magnify its negative impact. Consequently, if policy-makers ignored these extra 'precautionary' effects, then they would under-react to the shock. By the same token, low risk aversion and reduced desire to accumulate buffer-stock savings associated with economic booms would strengthen the case for using policy relatively more aggressively – subsequently referred to as 'leaning against the wind'.

Our first contribution is to characterize conditions under which models with habits generate countercyclical variation in the desire to save for precautionary reasons. We show that a countercyclical coefficient of risk aversion, which is a standard feature of all habit models, might not be sufficient to generate realistic dynamics. What is necessary is that the persistence of shocks and habits is sufficiently high – i.e. agents must expect a fall in living conditions to persist in order for higher risk aversion to translate into a greater desire to save.[3]

We then analyze policy implications of such swings in precautionary saving motives. We derive expressions for the 'natural' rate of interest – i.e. the one that

[2]But, as stressed previously, this is the only deviation from linearity which we consider.

[3]Arguably, since these conditions closely mirror those for risk premium countercyclicality derived in De Paoli et al. (2007), our model is likely to inherit the desirable asset pricing properties of Campbell and Cochrane (1999).

would prevail if prices were fully flexible – both in a linear, 'certainty-equivalent' setup and in a world in which agents save for precautionary reasons. Since in our framework the policy of complete price stability mimics the flexible price allocation (Woodford, 2003), therefore, by describing the implications of buffer stock savings for the level of the natural rate, we also characterize monetary policy consistent with price stability. In a habit model, as shown by Amato and Laubach (2004), such a policy is also close to the social optimum, which is why we focus on it in the remainder.

We find that properly accounting for swings in risk appetite and the desire to save reduces the magnitude of monetary policy responses to productivity shocks. Following a positive productivity shock, central bankers striving to maintain price stability cut rates to boost demand and prevent falls in the price level. However, since a persistent positive productivity shock also reduces agents' desire to save, the cut in rates required to boost demand is smaller – i.e. the intertemporal substitution effect is partially offset by swings in the precautionary motive. Conversely, given that a positive demand shock merits interest rate hikes – and since associated falls in precautionary motives exacerbate the increases in demand – policy needs to have a 'contractionary bias' during booms, and an accommodative slant during downturns, once changes in precautionary motives are accounted for.

Our analytical expressions show that the size of the 'precautionary' correction is increasing in the degree of shock volatility. This has two important implications. First, it stresses the dependence of the policy multiplier on underlying uncertainty – i.e. once the impact of uncertainty on agents' behavior is factored in, the output or consumption implications of a 25bp policy cut could be very different during a period of 'great moderation' than in a financial crisis. Second, higher volatility translates into greater 'precautionary' corrections – meaning that ignoring the impact of swings in 'buffer-stock' motives would tend to lead to largest systematic policy mistakes in highly turbulent periods.

The remainder of the paper is structured as follows. In the next section we present the model. We also characterize the linearized system of equilibrium conditions and the corresponding natural rate of interest. In Section 3, we incorporate the precautionary savings channel and analyze its implications for the natural rate of interest and thus monetary policy. We then use simulations to illustrate our results and inspect their robustness before concluding by summarizing and highlighting possible extensions.

2.2 Model

Our model economy is inhabited by a continuum of consumer-producers living on the unit interval (and indexed by $j \in [0,1]$). Agents are assumed to maximize expected utility, which is given by

$$U^j = E\sum_{t=0}^{\infty}\beta^{-t}\left(\frac{\xi_{d,t}\left(C_t^j - hX_t\right)^{1-\rho} - 1}{1-\rho} - \frac{\xi_{y,t}^{-\eta}y_t(j)^{\eta+1}}{\eta+1}\right) \tag{2.1}$$

where C_t^j denotes agent j's consumption, X_t is the level of habits and $\xi_{d,t}$ is a preference shock. The second term in the large bracket captures the disutility of producing $y_t(j)$ units of the differentiated output good given productivity denoted by $\xi_{y,t}$.[4]

We define the coefficient of relative risk aversion as[5]

$$\vartheta(C_t, X_t) := -C_t \cdot \frac{U_{cc}(C_t, X_t)}{U_c(C_t, X_t)} = \frac{\rho}{S_t} \tag{2.2}$$

where surplus consumption S_t is given by

$$S_t := \frac{C_t - hX_t}{C_t}. \tag{2.3}$$

and $U_y(\cdot,\cdot)$ denotes the partial derivative of utility function $U(\cdot,\cdot)$ with respect to y. Since this coefficient measures agents' willingness to enter pure consumption gambles, given habits equal to X_t, it can be referred to as consumption risk aversion. It is easy to show that $\vartheta(C_t, X_t)$ is countercyclical, when – as in Campbell and Cochrane (1999) – S_t is used as a measure of cyclical stance.

We assume that habits X_t are 'external' - i.e. individual agents treat them as exogenous. We adopt a slow-moving habit specification under which

$$x_t = (1-\phi)c_{t-1} + \phi x_{t-1} \tag{2.4}$$

[4]Given the Calvo price setting specification that we subsequently adopt, households' production income could be different depending on the type of good produced. In the remainder, as in Woodford (2003), we assume that there exist competitive financial markets in which these risks are efficiently shared.

[5]As noted in Campbell et al. (1997) 'risk aversion may also be measured by the normalized curvature of the value function [...] or by the volatility of the stochastic discount factor [...] While these measures of risk aversion are different from each other in this model, they all move inversely with S_t.'

where ϕ controls the persistence of the habit process and small letters denote logs. We further assume that both preference and productivity shocks are autoregressive processes given by

$$\varepsilon_{d,\,t+1} = \gamma_{dem}\varepsilon_{d,\,t} + \epsilon_{d,\,t+1} \qquad\qquad \varepsilon_{y,\,t+1} = \gamma_{prod}\varepsilon_{y,\,t} + \epsilon_{y,\,t+1}$$

with $\varepsilon_{x,t} \equiv \log(\xi_{x,t})$ and the disturbances $\epsilon_{x,t+1}$ being mean zero *i.i.d.* random variables with variance given by σ_x^2, $x \in \{d, y\}$.

Aggregate consumption and price indices, C_t and P_t, are defined as

$$C_t = \left[\int_0^1 c_t\,(z)^{\frac{\sigma-1}{\sigma}}\,dz\right]^{\frac{\sigma}{\sigma-1}} \quad P_t = \left[\int_0^1 p\,(z)^{1-\sigma}\,dz\right]^{\frac{1}{1-\sigma}}$$

where $\sigma > 0$ is the elasticity of substitution between the differentiated varieties. Conditional on the specification above, we can characterize agents' intratemporal and intertemporal decisions. Optimality implies, respectively

$$y_t(j) = \left(\frac{p_t(j)}{P_t}\right)^{-\sigma} Y_t \tag{2.5}$$

$$1 = R_t E_t\left[\beta\frac{\xi_{d,t+1}(C_{t+1} - hX_{t+1})^{-\rho}}{\xi_{d,t}\,(C_t - hX_t)^{-\rho}}\right]. \tag{2.6}$$

Alternatively, we could rewrite the consumption Euler equation as

$$1 = R_t E_t \mathcal{M}_{t+1} \tag{2.7}$$

where the stochastic discount factor $\mathcal{M}_{t+1}$ is defined as

$$\mathcal{M}_{t+1} \equiv \beta\,\frac{\xi_{d,t+1}(C_{t+1} - hX_{t+1})^{-\rho}}{\xi_{d,t}\,(C_t - hX_t)^{-\rho}}. \tag{2.8}$$

Prices are assumed to follow a partial adjustment rule à la Calvo (1983). Producers of differentiated goods know the form of their individual demand functions, given by (2.5), and maximize profits taking aggregate demand Y_t and price level P_t as given. In each period, a fraction $\alpha \in [0, 1)$ of randomly chosen producers is not allowed to change the nominal price of their output. The remaining fraction of firms, given by $(1 - \alpha)$, chooses prices optimally by maximizing the expected discounted value of profits. The optimal choice of producer j allowed to reset his price at time t can be shown to satisfy

$$E_t \sum_{T=t}^{+\infty} \frac{y_{t,T}(j)}{(\alpha\beta)^{t-T}} \left[\frac{\widetilde{p}_t(j)}{P_T} U_c(C_T, X_T, \xi_{d,T}) - \frac{\sigma}{(\sigma-1)} V_y \left(y_{t,T}(j), \xi_{y,T} \right) \right] = 0 \qquad (2.9)$$

where $y_{t,T}(j)$ is producer j's time t estimate of demand for his good at time T, should he be unable to reset his price $\widetilde{p}_t(j)$ before period T. It can be proved that equation (2.9) implies that the price index evolves according to

$$(P_t)^{1-\sigma} = \alpha P_{t-1}^{1-\sigma} + (1-\alpha) \left(\widetilde{p}_t \right)^{1-\sigma} \qquad (2.10)$$

where we exploit the fact that all producers who reset prices at time t equate them to $\widetilde{p}_t$. Thus, using the market clearing condition $C_t = Y_t$, we can summarize the log-linearized system of equilibrium conditions by[6]

$$\begin{cases} r_t = & E_t(\rho\,(1-h)^{-1}\,(\Delta y_{t+1} - h\Delta x_{t+1}) - \Delta\varepsilon_{d,t+1}) \\ \pi_t = & k\left(\kappa_0\,(1-h)^{-1}\,y_t - \rho\,(1-h)^{-1}\,hx_t - \eta\varepsilon_{y,t} - \varepsilon_{d,t}\right) + \beta E_t\pi_{t+1} \\ x_t = & (1-\phi)y_{t-1} + \phi\,x_{t-1}. \end{cases}$$

From the system above we can derive the equilibrium interest rate consistent with price stability (i.e. $\pi_t \equiv \log(P_{t+1}/P_t) = 0$ for every t) in a linear world

$$r_t^* = \kappa_1 E_t(\Delta\varepsilon_{y,t+1} - (1-h)\Delta\varepsilon_{d,t+1} - h\Delta x_{t+1}^*) \qquad (2.11)$$

where x^* is the flexible-price level of habits and where

$$\kappa_0 = (1-h)\eta + \rho \qquad \text{and} \qquad \kappa_1 = \rho\,\eta\,\kappa_0^{-1}.$$

Note that r_t^* is also the equilibrium interest rate that would prevail if prices were perfectly flexible.

Expression (2.11) shows that the interest rate consistent with full price stability falls [rises] following a positive supply [demand] shock – with the magnitude of the response, on impact, given by κ_1 $[(1-h)\kappa_1]$.

[6]Since in the simple setup we use there is no aggregate saving, therefore we refer to the 'precautionary saving motive' (rather than changes in actual savings). That motive would be reflected in the dynamics of market-clearing asset prices – e.g. the riskless rate.

2.3 Cyclical Risk Aversion and Precautionary Saving

We now consider the minimum departure from a linear model, in which we can analyze the impact of cyclical swings in risk aversion and precautionary saving motives on economic dynamics. While retaining the linear specification of all equations other than the Euler condition (2.7), we exploit the fact that under conditional log-normality of $\mathcal{M}_t$ the latter becomes[7]

$$-r_t = \underbrace{E_t(m_{t+1})}_{\text{Intertemporal substitution effect}} + \frac{1}{2} \underbrace{var_t(m_{t+1})}_{\text{Precautionary savings effect}} \tag{2.12}$$

where $m_{t+1} \equiv \log(\mathcal{M}_{t+1})$. Reiterating footnote 6, our framework abstracts from investment and, thus, there are also no savings. Nevertheless, the interest rate that clears the bond market is affected by agents' willingness to save both for precautionary and intertemporal smoothing reasons.

While linear models capture the intertemporal substitution effect, they ignore the term $var_t(m_{t+1})$. This term summarizes how uncertainty affects interest rates through changes in agents' willingness to amass precautionary savings.[8] Accordingly, to analyze how the precautionary savings channel affects the transmission mechanism of shocks, we need to understand the determinants of $var_t(m_{t+1})$. In particular, we would like to evaluate how such precautionary savings behave over the cycle. Defining

$$\widetilde{\mathcal{M}}_{t+1} = \beta \frac{(C_{t+1} - hX_{t+1})^{-\rho}}{(C_t - hX_t)^{-\rho}} \tag{2.13}$$

we can write

$$var_t(m_{t+1}) = var_t(\tilde{m}_{t+1}) + cov_t(\tilde{m}_{t+1}, \Delta\varepsilon_{d,t+1}) + \sigma_d^2. \tag{2.14}$$

Note that absent preference shocks, only the first term would be present in the expression above. To analyze the cyclical properties of $var_t(\tilde{m}_{t+1})$, we approxi-

[7]This equation holds up to second order without any distributional assumptions on the stochastic discount factor.

[8]There has been some ambiguity as to what exactly precautionary savings are – see also Floden (2008). Our usage of the term is closest to that in Kimball (1990) and implies that, absent uncertainty, there would be no precautionary savings, unlike in Huggett and Ospina (2001).

mate it to third order (details can be found in the Appendix)

$$var_t(\tilde{m}_{t+1}) = \kappa_1^2(\eta^{-2}\sigma_d^2 + \sigma_y^2)(1 - \kappa_y\varepsilon_{y,t} - \kappa_d\varepsilon_{d,t} + \kappa_x x_t) \tag{2.15}$$

where

$$\kappa_{y/d} = \frac{2h\eta(h\gamma(1-\phi) + \kappa_0(\gamma_{prec/dem} + \phi - 1))}{\kappa_0^2} \tag{2.16}$$

$$\kappa_x = \frac{2h(\rho+\eta)(\kappa_0 + \rho h(1-\phi))}{\kappa_0^2}. \tag{2.17}$$

Equation (2.15) highlights three channels through which uncertainty affects investors' behavior (and through which the policy multiplier would be affected):

- the overall level of macroeconomic volatility - given by σ_y^2 and σ_d^2;
- investors' risk aversion ρ - which in turn determines κ_1;
- current and past economic conditions - as summarized by the state variable x_t and shocks $\varepsilon_{y,t}$, and $\varepsilon_{d,t}$.

Equation (2.15) demonstrates that as long as investors are risk averse ($\rho > 0 \Rightarrow \kappa_1 > 0$) uncertainty affects their consumption decisions. It also illustrates that without habit formation ($h = 0$) the strength of the precautionary saving motive would not vary over the cycle ($\kappa_y = \kappa_d = \kappa_x = 0 \Rightarrow var_t(\tilde{m}_{t+1})$ is constant). Furthermore, inspecting expression (2.16) reveals that

$$\gamma_{prod} + \phi > 1 \Rightarrow \kappa_y > 0 \text{ and } \gamma_{dem} + \phi > 1 \Rightarrow \kappa_d > 0 \tag{2.18}$$

which means that if shocks affecting economic activity are sufficiently persistent and habits adjust slowly, then $var_t(\tilde{m}_{t+1})$ changes countercyclically. Actually, since the other terms in equation (2.14) are either constant or necessarily countercyclical (as shown in the Appendix), condition (2.18) suffices to guarantee countercyclicality of $var_t(m_{t+1})$. Accordingly, investors will increase their willingness to engage in precautionary saving following bad shocks if they expect future economic conditions to remain poor (consumption to persistently undershoot the habit level).

If, on the other hand, the median expectation is for an improvement in economic prospects, then negative shocks might not translate into higher precautionary savings – even if the coefficient of risk aversion given by (2.2) increases. This is because if habits are fast moving and consumption recovers quickly, investors faced with the bad shock will quickly get used to lower levels of consumption while at the same time, the latter quickly recovers. This means that investors actually expect consumption to be above their habit level in the future and therefore might be less inclined to engage in precautionary savings.[9]

Note that in this exercise, we are only interested in understanding the determinants of precautionary savings and not in capturing the effect of all the other nonlinearities in the model and/or their interplay. For this reason, we only derive higher order approximations to the volatility of the stochastic discount factor as, in our framework, it is the only term capturing the precautionary channel.[10]

2.4 Precautionary Saving and Monetary Policy

The implications of precautionary saving for interest rates will, therefore, depend on the structural characteristics of the economy. Absent consumption habits, with time invariant risk aversion, the presence of uncertainty will affect the average level of the natural interest rate, but not its business cycle properties. In this case, interest rate responses consistent with price stability would not be affected by buffer-stock saving motives. In the general case, however, changes in perceived uncertainty ($var_t(m_{t+1})$) would generate fluctuations in the equilibrium interest rate - with ramifications for the conduct of monetary policy.

If the central bank's goal is to maintain price stability and it uses interest rates as an instrument to achieve this goal, then knowing the behavior of the natural rate of interest is crucial. In fact, as alluded to in the introduction, policy rates that ensure price stability would track the natural rate of interest. So how do precautionary savings affect the natural rate and the appropriate policy response to shocks?

Equation (2.11) implies that the magnitude of responses of the natural rate to a productivity shock in a 'linear' world is given by κ_1. When accounting for uncertainty, the size of these responses also depends on the cyclicality of pre-

[9] As discussed in De Paoli and Zabczyk (2009), similar conditions are necessary to ensure that risk premia are countercyclical. The paper highlights the role played by the persistence of the consumption surplus ratio.

[10] That is, we are explicitly dismissing the effects of nonlinearities present in the other equilibrium conditions and other higher order terms in the Euler equation, which would affect the intertemporal substitution effect.

cautionary savings. If shocks and habits are persistent, and thus precautionary savings are countercyclical, then the response to shocks is dampened. A negative productivity shock increases perceived uncertainty, which raises investors' willingness to save and puts downward pressure on interest rates. As a result, the equilibrium interest rate that is consistent with stable prices will be lower than in a linear economy. Thus, these results suggest that interest rates should respond less to supply shocks when precautionary savings are taken into account.

The condition $\gamma_{dem} + \phi > 1$ implies that $\kappa_d > 0$ and thus also guarantees that precautionary savings are countercyclical in the face of preference shocks. So, when uncertainty is introduced in a model that features persistent shocks and habits, negative preference shocks also lead to higher precautionary savings. In other words, incorporating uncertainty magnifies the impact of the shock. As a result, the response of the natural rate of interest to such shocks should exceed the one, when precautionary motives are dismissed. Accordingly, in such settings, policymakers striving for price stability should respond more aggressively to demand shocks. This is in contrast to the case of productivity shocks analyzed previously.

These results suggest that policy implications of precautionary savings depend crucially on the source of the shock hitting the economy. In other words, higher precautionary savings can be thought of as introducing an extra negative demand shock - both following negative productivity and preference shocks. Since productivity and demand shocks call for opposite interest rate reactions – at least when the policymakers' aim is to maintain price stability – therefore, depending on the source of the shock, policy that ignores precautionary savings (and hence the extra negative demand shock) will either undershoot or overshoot its' appropriate level.

2.5 Quantitative Analysis

The model developed so far offered a stylized, qualitative representation of the monetary policy transmission mechanism. However, a quantitative illustration of differences in policy responses is of independent interest. In this section we provide one, stressing two important caveats at the outset:

- Our model is stylized. For example – the assumption of price level targeting implies that inflation volatility is, by construction, zero. Since it wasn't clear what weight to put on deviations of simulated moments from their data counterparts, we opted to calibrate rather than estimate the model. To verify the robustness of our findings, we then conducted a thorough sensitivity analysis – documenting the implications of varying key parameters.

- While we chose to focus on the habit mechanism, there are alternatives – e.g. Epstein-Zin preferences (Epstein and Zin, 1989) – which could also generate swings in precautionary saving motives. While these alternative frameworks would likely have similar qualitative implications, the quantitative ones could very well differ.

For our calibration we define one period as a quarter and set $\beta = 0.99$ to yield a 4% steady-state real interest rate. As in Campbell and Cochrane (1999) the coefficient of risk aversion ρ is equal to 2.37 and the degree of habit persistence ϕ is set to 0.97. Following Canzoneri et al. (2007) we assume a value of 6 for the inverse of the elasticity of labor supply η and set $\alpha = 0.66$ to obtain an average length of price contracts of 3 quarters. The elasticity of substitution between differentiated goods σ is assumed to take the value of 10 in line with Benigno and Woodford (2005). Similar to Juillard et al. (2006) and Banerjee and Batini (2003) we calibrate the habit size parameter to $h = 0.85$. As in Smets and Wouters (2003, 2007), the persistence of productivity and preference shocks is set to 0.997 and 0.9 respectively, and the variance of productivity shocks is 3.5 times higher than that of preference shocks. Finally, the overall level of shock volatility is calibrated to match the standard deviation of consumption growth equal to 0.75% (consistent with official UK Office for National Statistics quarterly data for consumption of non-durables and services from 1976 Q1 to 2007 Q3). These values are all summarized in Table 2.1.

We begin the quantitative part of our investigation by comparing the level of the natural rate of interest in a linear world with the one that would prevail if the precautionary savings channel was additionally taken into account. In line with the theoretical part, we consistently maintain a linearized version of the Phillips curve while alternating between first and third order approximations to the Euler equation to switch the precautionary channel off and on respectively.[11]

[11]In particular, we compute a third order approximation of the Euler equation using perturbation methods as implemented in *Dynare++* and *Perturbation AIM*. As mentioned previously, third order is the lowest which allows us examine changes in the precautionary saving motive.

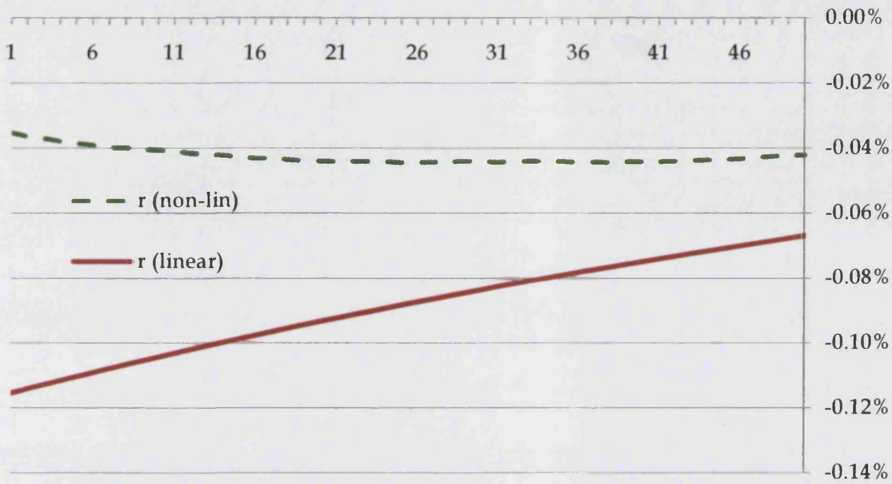

Figure 2.1: Natural Rate of Interest Following a Positive Productivity Shock

Figure 2.1 illustrates how the natural rate of interest responds to a positive productivity shock. The chart shows that the fall in the natural rate is smaller once the precautionary saving motive is incorporated. More specifically, once the decreased desire to save is taken into account, the magnitude of the change in interest rates required to boost demand sufficiently to prevent falls in prices is more than halved (from roughly 50bp (annualized) to 15bp on impact). The simulation results thus suggest that a central bank following an interest rate rule should be less aggressive in the face of productivity shocks – confirming our qualitative story and additionally suggesting that these effects can be quantitatively relevant.

Figure 2.2 demonstrates that the analytical results for preference shocks are also confirmed in simulation. In particular, the response of the natural rate to a negative preference shock is magnified – agents' increased desire to save exacerbates the initial shock and calls for more accommodative policy – i.e. bigger cuts in rates. But the quantitative impact of the precautionary saving effect on the natural rate is smaller than in the case of productivity shocks. We now briefly investigate why, inspecting the sensitivity of the reported differences to changes in various model parameters.

The first group of parameters we focus on are those controlling the size (Figure 2.3) and persistence of habits (Figure 2.4). Under the benchmark calibration

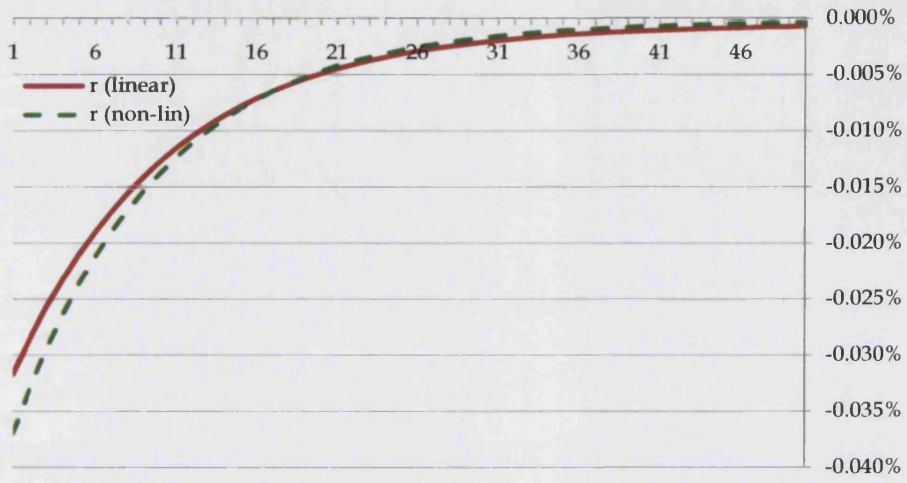

Figure 2.2: Natural Rate of Interest Following a Negative Preference Shock

– at 0.85 and 0.97 respectively – these are, arguably, on the high side.[12] As panels A and B in figures 2.3 and 2.4 show, increasing their values magnifies differences between the response of 'linear' and 'precautionary-adjusted' rates to both productivity and demand shocks.[13] Panels C and E demonstrate, however, that the mean difference in rates (denoted by the 'prec' green line) seems to respond much more strongly to changes in the habit size parameter h. Notably, comparing the slope of the 'prec' line with that showing the mean of the (non-annualized) interest rate r, suggests that this is because of falls in the 'precautionary' rate rather than by changes in the 'linear' one.

The second group of parameters whose significance we scrutinize, are those controlling the persistence of productivity shocks - γ_{prod} (Figure 2.5) and demand shocks γ_{dem} (Figure 2.6). Perhaps surprisingly, while they affect the impulse responses of the 'precautionary correction' to the respective shocks (Panels A and B) they have negligible impact on the mean value of the difference between 'linear' and 'risk-adjusted' policy rates.

The above conclusions are definitely not true of the risk-aversion/intratemporal elasticity of substitution parameter ρ (Figure 2.7) or the volatilities of productivity σ_y and demand shocks σ_d (Figures 2.8 and 2.9 respectively). Increasing

[12] As discussed previously, high habit persistence helps ensure that the precautionary savings motive is countercyclical and – as emphasized by Campbell and Cochrane (1999) and De Paoli and Zabczyk (2009) – helps along the asset pricing dimension.

[13] Reducing the value of these parameters would thus undoubtedly reduce the quantitative relevance of our results. Most likely, however, it would also adversely affect our model's ability to match asset pricing data – which was our original motivation to study this channel.

their values magnifies differences between 'linear' rates and those accounting for changes in agents' desire to save for precautionary reasons – both in response to shocks (panels A and B) and overall means (panels C and E). Again – these differences are mainly driven by changes in the 'precautionary' rather than 'linear' rate. Somewhat strikingly, however, boosting the volatility of demand shocks seems to have a much smaller impact than changing the variance of productivity shocks. Most likely this reflects differences in the underlying shock persistence – suggesting that what is necessary to induce significant changes in the dynamics of the policy rate is a simultaneous combination of high shock volatility and high persistence.

We conclude the quantitative section by investigating policy errors which a central bank would make if it incorrectly ignored changes in the strength of agents' precautionary savings motive when setting interest rates. More specifically, we assume that the central bank follows a Taylor rule given by

$$r_t^n = r_t^* + \phi_\pi \pi_t + \phi_\pi (y_t - y_t^*) \tag{2.19}$$

where r_t^n is the nominal interest rate, y_t^* is the flexible price allocation of output, and r_t^* is the natural rate of interest defined in Equation (2.11) – i.e. one consistent with price stability in a 'linear', risk-free world.

Table 2 shows the implications of this policy for inflation and the output gap.[14] We see that whereas the Taylor rule given by Equation 2.19 ensures zero inflation and output gap volatility in a linear world – where the natural rate is driven purely by the 'intertemporal-substitution' channel – this is no longer the case when uncertainty influences agents' behavior. More specifically, in that case, the wrong policy increases the standard deviation of the output gap and inflation by 0.1pp.

While our numerical results suggest that implications of 'policy mistakes' are quite small, this is partially driven by consumption in our model being very insensitive to changes in the interest rate. If we were to reduce the elasticity of intertemporal substitution and consider the case of log utility, this sensitivity would increase and with it the standard deviation of the output gap and inflation.[15] Furthermore, in our calibration the Phillips curve is extremely flat, so

[14]Note that in this exercise we use a nominal version of the Euler equation, given that the central bank is assumed to control the level of the nominal interest rate.

[15]Epstein-Zin preferences would allow us to illustrate this easily - as we could reduce the elasticity of substitution without changing the coefficient of risk aversion.

even if the Taylor rule produces movements in the output gap, this does not translate into a volatile inflation rate (see fifth column of Table 2.2) – i.e. under a slightly changed calibration these policy errors could become much more relevant.

The concluding observation we make is that decreasing the level of uncertainty would also lower the size of the inflation and output gap volatility. That is, lower uncertainty would decrease the size of policy mistakes. Thus, in these settings if central banks have good luck (i.e. they confront a stable economic environment), this will also translate into good policy (i.e. low policy mistakes). This result, which is consistent with Figure 2.8, is illustrated in column 6 of Table 2.2.

2.6 Conclusions

Our results show that, following persistent adverse shocks, policy-makers might be well advised to steer off predictions of linear models and conduct more accommodative policy. Equally, when demand and supply conditions are improving, taking note of the precautionary saving motives justifies 'leaning against the wind'. Since the size of the precautionary correction is increasing in the degree of volatility, mistakenly ignoring this channel would be most costly during highly turbulent periods. We believe that formally accounting for stochastic volatility and enriching the framework by considering Epstein-Zin preferences would both make for interesting extensions.

2.A Appendix

The logarithm of the stochastic discount factor is given by

$$\begin{aligned}\log(\mathcal{M}_{t+1}) &= \log\left(\frac{(C_{t+1}-X_{t+1})^{-\rho}}{(C_t-X_t)^{-\rho}}\right)\\ &= \log\left(\frac{(C_{t+1}-X_{t+1})^{-\rho}}{C_{t+1}^{-\rho}}\frac{C_t^{-\rho}}{(C_t-X_t)^{-\rho}}\frac{C_{t+1}^{-\rho}}{C_t^{-\rho}}\right) = \log\left(S_{t+1}^{-\rho}S_t^{\rho}C_{t+1}^{-\rho}C_t^{\rho}\right)\\ &= -\rho\big(\log C_{t+1} - \log C_t + \log S_{t+1} - \log S_t\big) \quad (2.20)\end{aligned}$$

and so

$$m_{t+1} = -\rho\big(c_{t+1} - c_t + s_{t+1} - s_t\big).$$

It thus follows that

$$\begin{aligned}\mathrm{var}_t m_{t+1} = \mathrm{E}_t\left(m_{t+1} - \mathrm{E}_t m_{t+1}\right)^2 &= \rho^2 \mathrm{E}_t\Big((c_{t+1} - \mathrm{E}_t c_{t+1}) + (s_{t+1} - \mathrm{E}_t s_{t+1})\Big)^2\\ &= \rho^2\Big(\mathrm{var}_t c_{t+1} + 2\mathrm{cov}_t(c_{t+1}, s_{t+1}) + \mathrm{var}_t s_{+1}\Big) \quad (2.21)\end{aligned}$$

as the conditional expectations of all t-dated variables can be eliminated.

Up to a second order approximation (which is all we need to compute a third-order accurate expression for $\mathrm{var}_t m_{t+1}$) we get

$$s_{t+1} = \Psi_1\big(c_{t+1} - \frac{1}{2}(1-h)^{-1}c_{t+1}^2 - \widetilde{x}_t + c_{t+1}\widetilde{x}_t(1-h)^{-1} - \frac{1}{2}(1-h)^{-1}\widetilde{x}_t^2\big) - \log(1-h)$$

where we used the fact that the habit at time $t+1$ depends only on measurable variables and so we denoted x_{t+1} by $\widetilde{x}_t$ and defined $\Psi_1 := h/(1-h)$.

We can now compute a third order approximation to the $\mathrm{var}_t s_{t+1}$ and to that of $\mathrm{cov}_t(c_{t+1}, s_{t+1})$. From the definition

$$\mathrm{var}_t s_{t+1} = \Psi_1^2 \mathrm{var}_t\big(c_{t+1} - \frac{1}{2}(1-h)^{-1}c_{t+1}^2 + c_{t+1}\widetilde{x}_t(1-h)^{-1}\big) \quad (2.22)$$

where again expectations of all t-dated variables were eliminated. It is easy to see that

$$\mathrm{var}_t s_{t+1} = \frac{\Psi_1^2}{(1-h)^2}\bigg((1-h)^2\mathrm{var}_t\big(c_{t+1}\big) + \frac{1}{4}\mathrm{var}_t(c_{t+1}^2) + \widetilde{x}_t^2\mathrm{var}_t\big(c_{t+1}\big)$$
$$-(1-h)cov_t(c_{t+1}, c_{t+1}^2) + 2\widetilde{x}_t(1-h)\mathrm{var}_t c_{t+1} - \widetilde{x}_t\mathrm{cov}_t(c_{t+1}, c_{t+1}^2)\bigg). \quad (2.23)$$

Similarly

$$\mathrm{cov}_t(c_{t+1}, s_{t+1}) = \Psi_1\mathrm{cov}_t\big(c_{t+1}, c_{t+1} - \frac{1}{2}(1-h)^{-1}c_{t+1}^2 + c_{t+1}\widetilde{x}_t(1-h)^{-1}\big)$$
$$= \Psi_1\big(\mathrm{var}_t c_{t+1} - \frac{1}{2}(1-h)^{-1}\mathrm{cov}_t\big(c_{t+1}, c_{t+1}^2\big) + \widetilde{x}_t(1-h)^{-1}\mathrm{var}_t c_{t+1}\big). \quad (2.24)$$

Consider the case in which shocks follow an AR(1) process, i.e.

$$\varepsilon_{y,t+1} = \gamma_{prod}\varepsilon_{y,t} + \epsilon_{y,t+1} \qquad \varepsilon_{d,t+1} = \gamma_{dem}\varepsilon_{d,t} + \epsilon_{d,t+1}.$$

where ε_y and ε_d are independent (cross-sectionally and inter-temporally). As shown above, to compute the variance of m_{t+1} we need expressions for

$$\mathrm{var}_t c_{t+1} \qquad \text{and} \qquad cov_t(c_{t+1}, c_{t+1}^2).$$

We know that

$$c_t = y_t = \big(\rho(1-h)^{-1} + \eta\big)^{-1}\big(\rho(1-h)^{-1}h\widetilde{x}_{t-1} + \varepsilon_{d,t} + \eta\varepsilon_{y,t}\big) \quad (2.25)$$

and so

$$\mathrm{var}_t c_{t+1} = \mathrm{var}_t\big(\rho(1-h)^{-1}h\widetilde{x}_t + \epsilon_{d,t+1} + \gamma_{dem}\varepsilon_{d,t} + \eta\epsilon_{y,t+1} + \eta\gamma_{prod}\varepsilon_{y,t}\big)$$
$$\cdot\big(\rho(1-h)^{-1} + \eta\big)^{-2} = \Psi_2^{-2}\big(\mathrm{var}_t\epsilon_{d,t+1} + \eta^2\mathrm{var}_t\epsilon_{y,t+1}\big) = \Psi_2^{-2}\big(\sigma_d^2 + \eta^2\sigma_y^2\big) \quad (2.26)$$

where $\Psi_2 = \big(\rho(1-h)^{-1} + \eta\big)$ and where $\mathrm{cov}_t(\epsilon_{d,t+1}, \epsilon_{y,t+1}) = 0$.

By a similar token

$$\mathrm{cov}_t(c_{t+1}, c_{t+1}^2) = \Psi_2^{-3}\mathrm{cov}_t\bigg(\big(\rho\,\Psi_1\,\widetilde{x}_t + \epsilon_{d,t+1} + \gamma_{dem}\varepsilon_{d,t} + \eta\epsilon_{y,t+1} + \eta\gamma_{prod}\varepsilon_{y,t}\big),$$
$$\big(\rho\,\Psi_1\,\widetilde{x}_t + \epsilon_{d,t+1} + \gamma_{dem}\varepsilon_{d,t} + \eta\epsilon_{y,t+1} + \eta\gamma_{prod}\varepsilon_{y,t}\big)^2\bigg)$$

Since the shocks are assumed Gaussian and uncorrelated, we can write

$$\mathrm{cov}_t(c_{t+1}, c_{t+1}^2) = 2\Psi_2^{-3}\Big(\rho\,\Psi_1\,\widetilde{x}_t + \gamma_{dem}\varepsilon_{d,t} + \eta\gamma_{prod}\varepsilon_{y,t}\Big)\Big(\sigma_d^2 + \sigma_y^2\eta^2\Big). \quad (2.27)$$

Using equalities (2.26) and (2.27) in equation (2.23) yields

$$\begin{aligned}\mathrm{var}_t s_{t+1} &= \Psi_1^2\Big(\Psi_2^{-2}(\sigma_d^2 + \eta^2\sigma_y^2) - (1-h)^{-1}2\Psi_2^{-3}(\sigma_d^2 + \sigma_y^2\eta^2) \\ &\quad \cdot \big(\rho\,\Psi_1\,\widetilde{x}_t + \gamma_{dem}\varepsilon_{d,t} + \eta\gamma_{prod}\varepsilon_{y,t}\big) + 2\widetilde{x}_t(1-h)^{-1}\Psi_2^{-2}(\sigma_d^2 + \eta^2\sigma_y^2)\Big) \\ &= \frac{2\Psi_3^2(\sigma_d^2 + \eta^2\sigma_y^2)}{(1-h)}\left(\frac{(1-h)}{2} + (1-\rho\Psi_3)\widetilde{x}_t - \frac{\big(\gamma_{dem}\varepsilon_{d,t} + \eta\gamma_{prod}\varepsilon_{y,t}\big)}{\Psi_2}\right) \quad (2.28)\end{aligned}$$

where $\Psi^3 := \Psi_1\Psi_2^{-1} = h/\big(\rho + (1-h)\eta\big)$. Similarly, plugging (2.26) and (2.27) into equation (2.24) and denoting $\Psi_4 := \Psi_1\Psi_2^{-2} = \big(h(1-h)\big)/\big(\rho + \eta(1-h)\big)^2$ we can write down

$$\begin{aligned}\mathrm{cov}_t(c_{t+1}, s_{t+1}) &= \frac{\Psi_4\big(\sigma_d^2 + \eta^2\sigma_y^2\big)}{(1-h)} \\ &\quad \cdot \left(1 - h + (1-\rho\,\Psi_3)\widetilde{x}_t - \Psi_2^{-1}\Big(\gamma_{dem}\varepsilon_{d,t} + \eta\gamma_{prod}\varepsilon_{y,t}\Big)\right). \quad (2.29)\end{aligned}$$

We can then use equations (2.26), (2.28) and (2.29) in (2.21) to obtain

$$\mathrm{var}_t m_{t+1} = \frac{\rho^2\big(\sigma_d^2 + \eta^2\sigma_y^2\big)}{(\rho + \eta(1-h))^2}\left(1 + \frac{2h(\rho+\eta)}{(\rho + \eta(1-h))}\widetilde{x}_t - \frac{2h\big(\gamma_{dem}\varepsilon_{d,t} + \eta\gamma_{prod}\varepsilon_{y,t}\big)}{(\rho + \eta(1-h))}\right).$$

Recalling the fact that $\widetilde{x}_t = x_{t+1}$ and so, $\widetilde{x}_t = c_t(1-\phi) + \phi x_t$ we get

$$\begin{aligned}\mathrm{var}_t m_{t+1} = \frac{\rho^2\big(\sigma_d^2 + \eta^2\sigma_y^2\big)}{(\rho + \eta(1-h))^2}\Bigg(1 &- \frac{2h\gamma_{dem}}{(\rho + \eta(1-h))}\varepsilon_{d,t} - \frac{2h\eta\gamma_{prod}}{(\rho + \eta(1-h))}\varepsilon_{y,t} \\ &+ \frac{2h(\rho+\eta)(1-\phi)}{(\rho + \eta(1-h))}c_t + \frac{2h(\rho+\eta)\phi}{(\rho + \eta(1-h))}x_t\Bigg).\end{aligned}$$

Recalling the definition of c_t - equation (2.25)

$$c_t = \big(\rho(1-h)^{-1} + \eta\big)^{-1}\big(\rho(1-h)^{-1}hx_t + \varepsilon_{d,t} + \eta\varepsilon_{y,t}\big) \quad (2.30)$$

and plugging it into the expression derived above yields, after simplifying

$$\begin{aligned}\mathrm{var}_t m_{t+1} = \Bigg(1 &- \frac{2h((1-h)(\rho+\eta)(\phi-1)+(\rho+\eta(1-h))\gamma_{dem})}{(\rho+\eta(1-h))^2}\varepsilon_{d,t} \\ &- \frac{2h\eta((1-h)(\rho+\eta)(\phi-1)+(\rho+\eta(1-h))\gamma_{prod})}{(\rho+\eta(1-h))^2}\varepsilon_{y,t} \\ &+ \frac{2h(\rho+\eta)((1-h)\eta+\rho(1-h(\phi-1)))}{(\rho+\eta(1-h))^2}x_t\Bigg)\cdot\frac{\rho^2(\sigma_d^2+\eta^2\sigma_y^2)}{(\rho+\eta(1-h))^2}.\end{aligned}$$

which is the expression reported in the body of the text.[16]

The covariance term $\mathrm{cov}_t(m_{t+1}, \Delta\varepsilon_{d,t+1})$

In line with the reasoning of the previous section, we can write

$$\begin{aligned}\mathrm{cov}_t(m_{t+1}, \Delta\varepsilon_{d,t+1}) &= -\frac{\rho\sigma_d^2}{(\rho+\eta(1-h))}\Bigg(1+\frac{h(1-h)(\rho+\eta)(1-\phi)}{(\rho+\eta(1-h))^2}\cdot\varepsilon_{d,t} \\ &+\frac{h(1-h)(\rho+\eta)(1-\phi)\eta}{(\rho+\eta(1-h))^2}\cdot\varepsilon_{y,t}+\frac{h(\rho+\eta)\big((\rho+\eta)\phi+h(\rho^2(1-\phi)-\eta\phi)\big)}{(\rho+\eta(1-h))^2}\cdot x_t\Bigg).\end{aligned}$$

Note that the coefficients on $\varepsilon_{y,t}$ and $\varepsilon_{d,t}$ are negative, so the covariance term always moves countercyclically. The coefficient multiplying x_t is negative when $\rho^2(1-\phi) > \eta\phi$, but given that x_t is predetermined, this would not affect the countercyclicality of the covariance term.[17]

[16]Note that, taking the limit of this expression as $\eta \to +\infty$ gives

$$\mathrm{var}_t m_{t+1} = \frac{\rho^2\sigma_y^2}{(1-h)^2} + \frac{2h\rho^2(1-\phi-\gamma_{prod})\sigma_y^2}{(1-h)^3}\varepsilon_{y,t} + \frac{2h\rho^2\sigma_y^2}{(1-h)^3}x_t.$$

[17]Note that, again computing the limit of the coefficients as $\eta \to +\infty$ we get

$$r_t = -E_t(m_{t+1}-\Delta\varepsilon_{d,t+1}) - \left(\frac{1}{2}\left(\sigma_d^2+\frac{\rho^2\sigma_y^2}{(1-h)^2}\right)+\frac{h\rho^2\sigma_y^2}{(1-h)^3}\cdot x_t+\frac{h\rho^2\sigma_y^2(1-\phi-\gamma_{prod})}{(1-h)^3}\cdot\varepsilon_{y,t}\right).$$

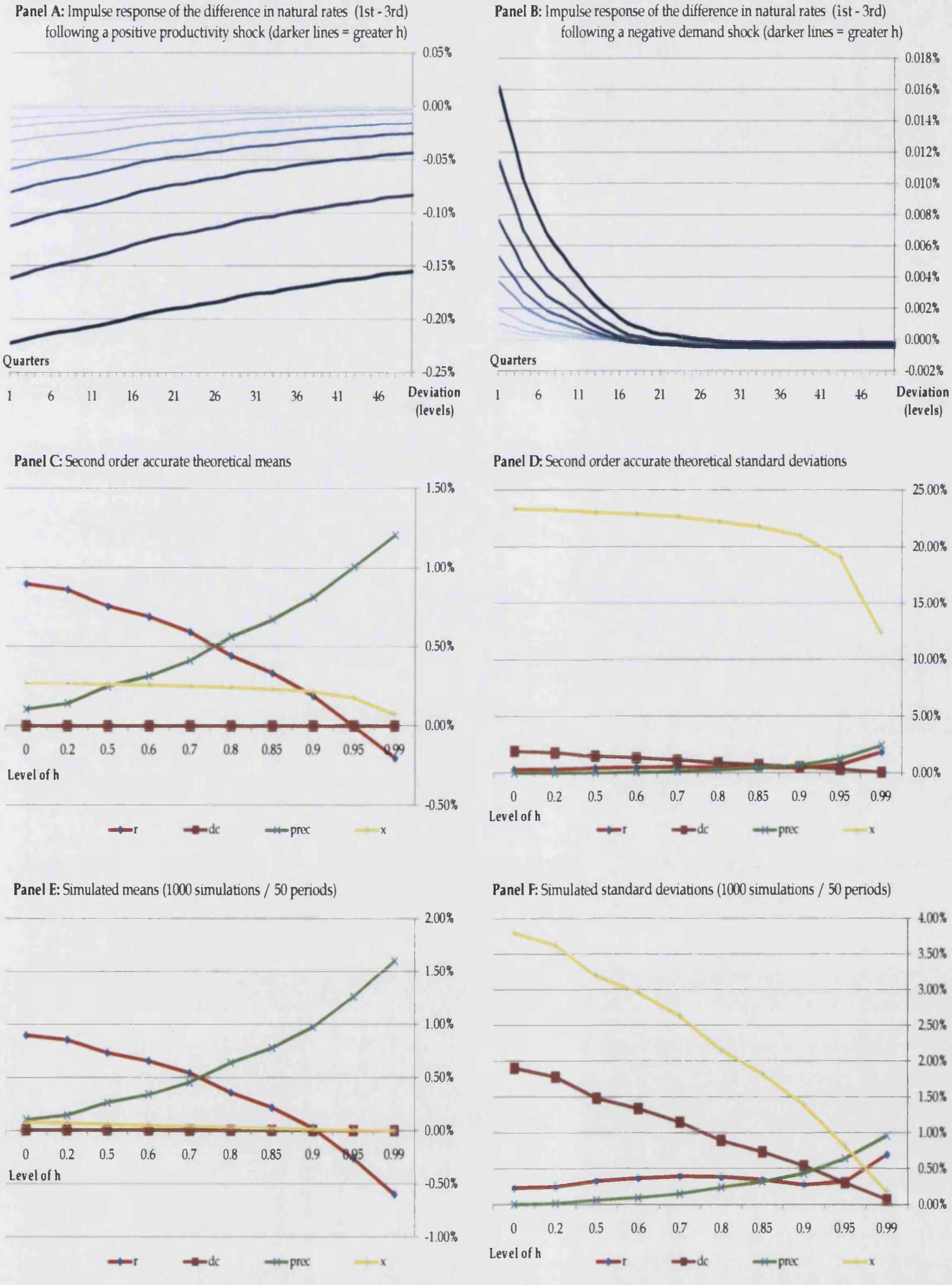

Figure 2.3: Sensitivity to Changes in the Habit Size Parameter h

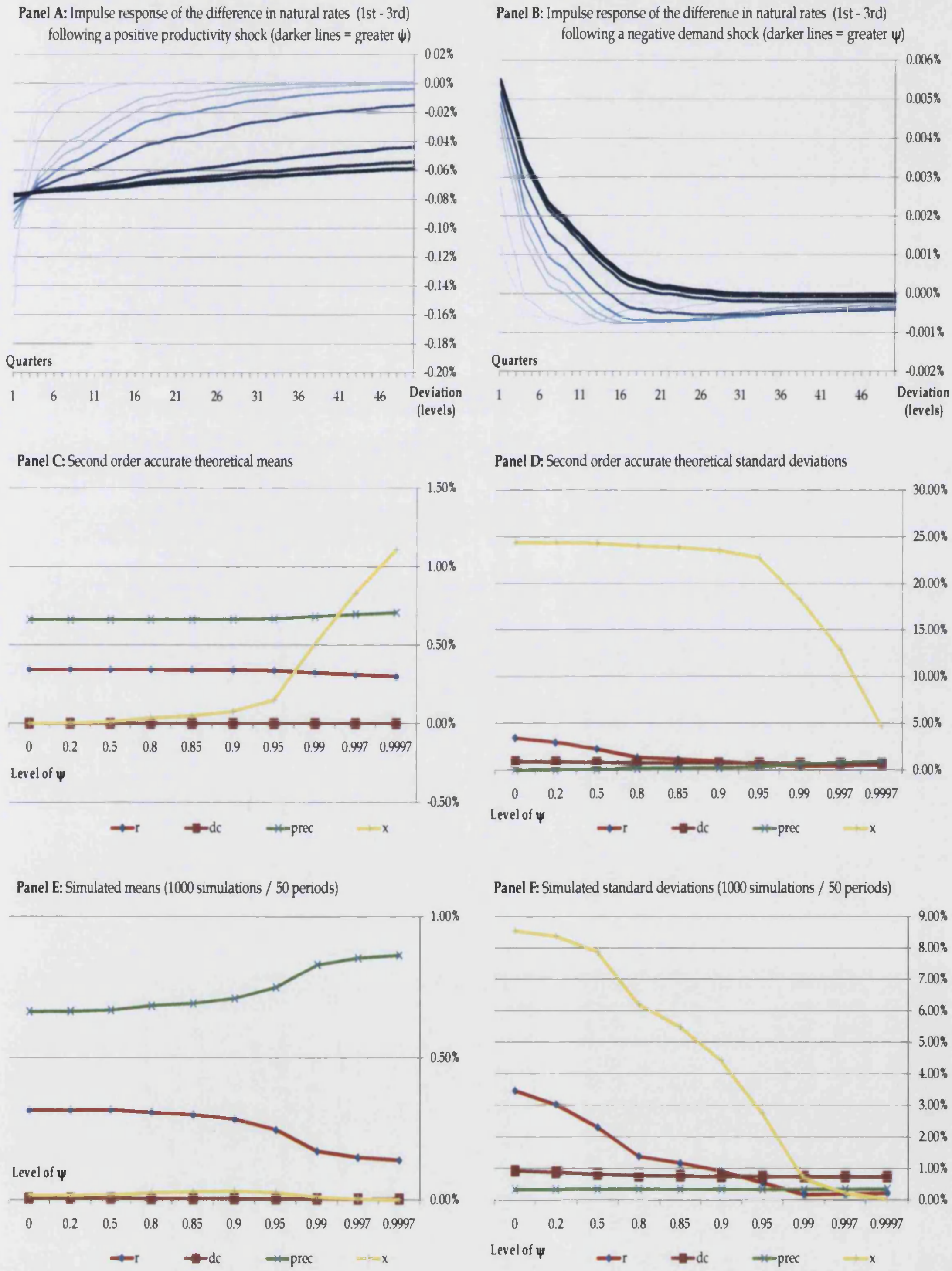

Figure 2.4: Sensitivity to Changes in the Habit Persistence Parameter ψ

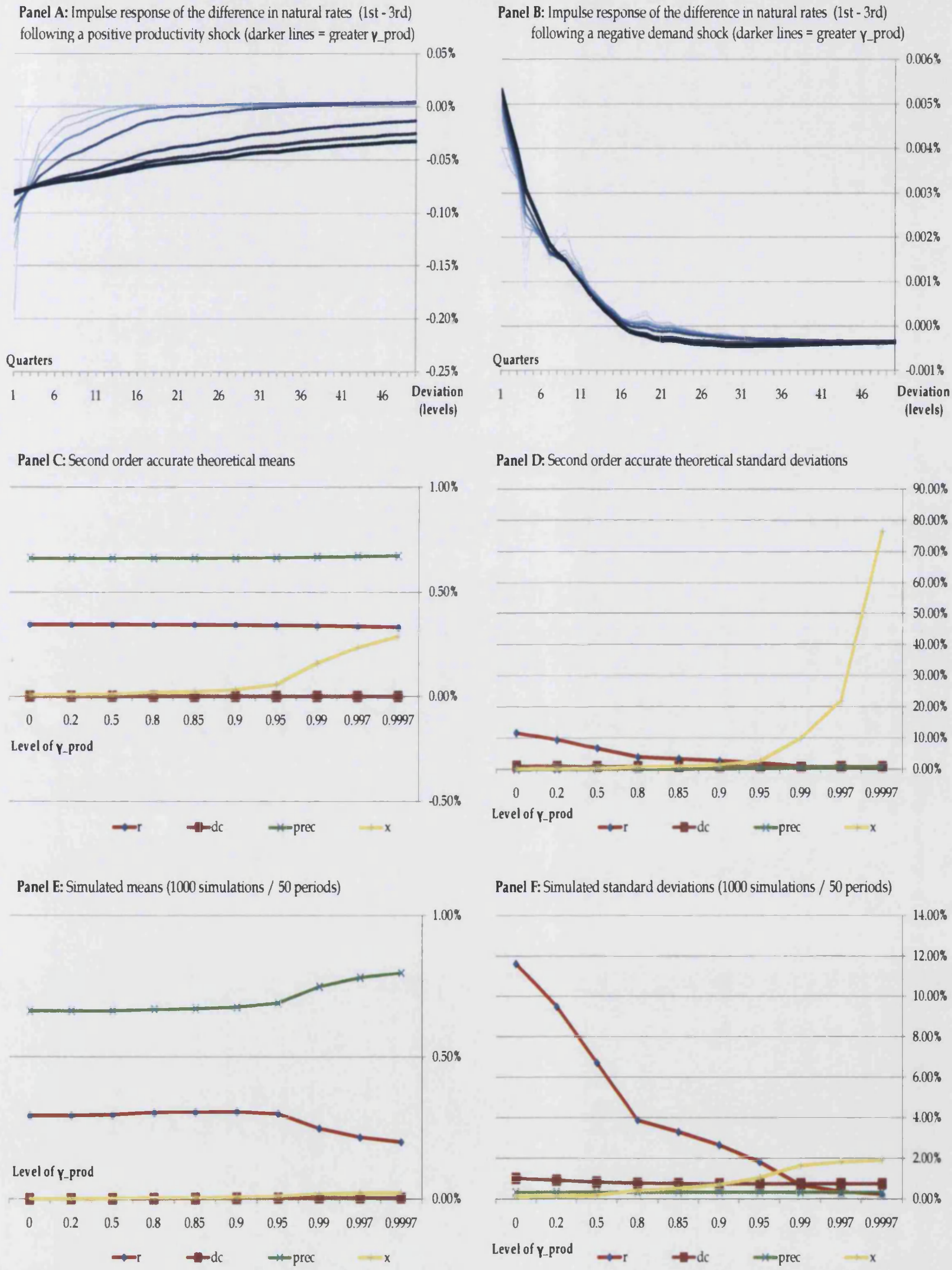

Figure 2.5: Sensitivity to Changes in Productivity Shock Persistence γ_{prod}

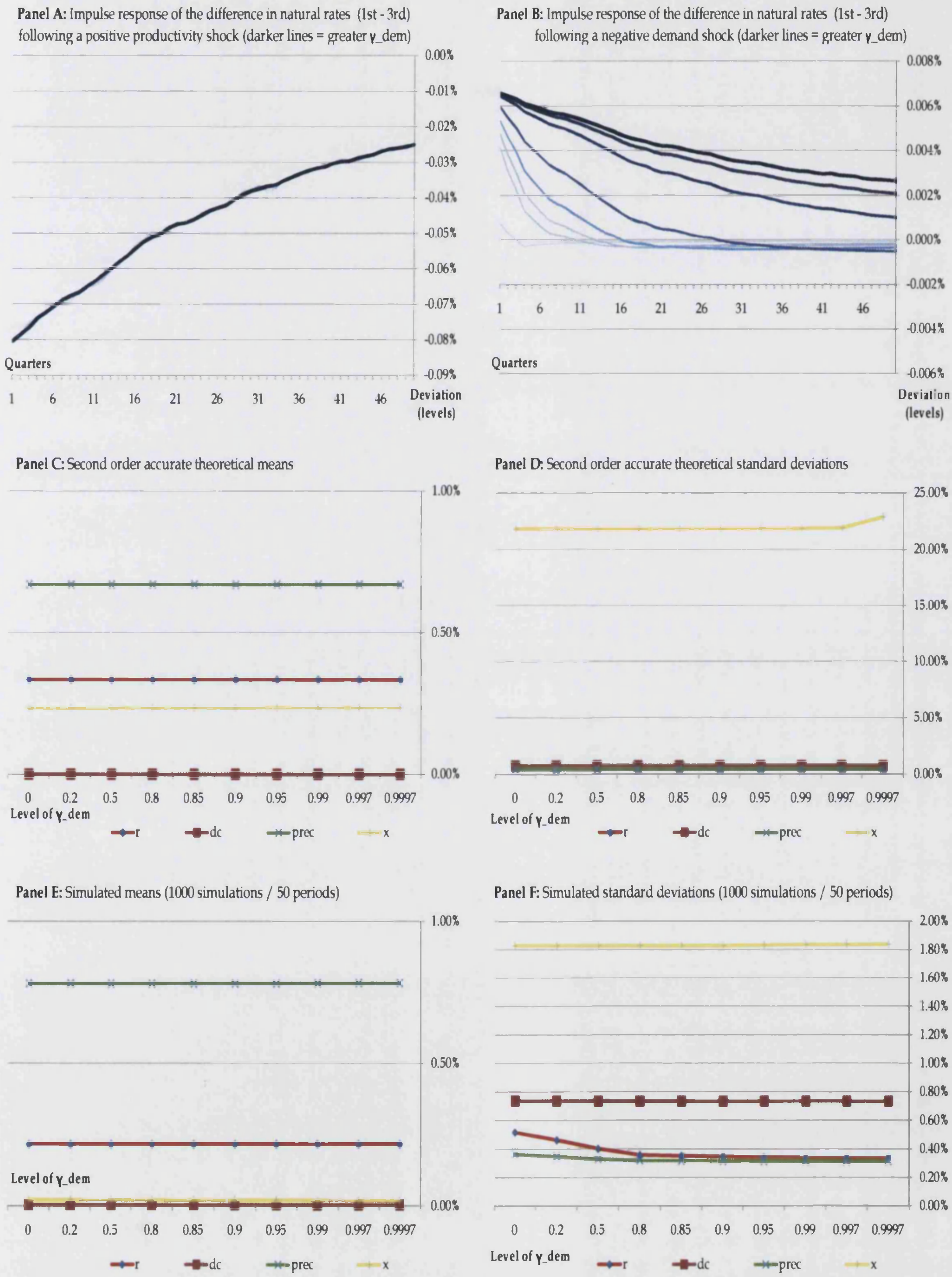

Figure 2.6: Sensitivity to Changes in Demand Shock Persistence γ_{dem}

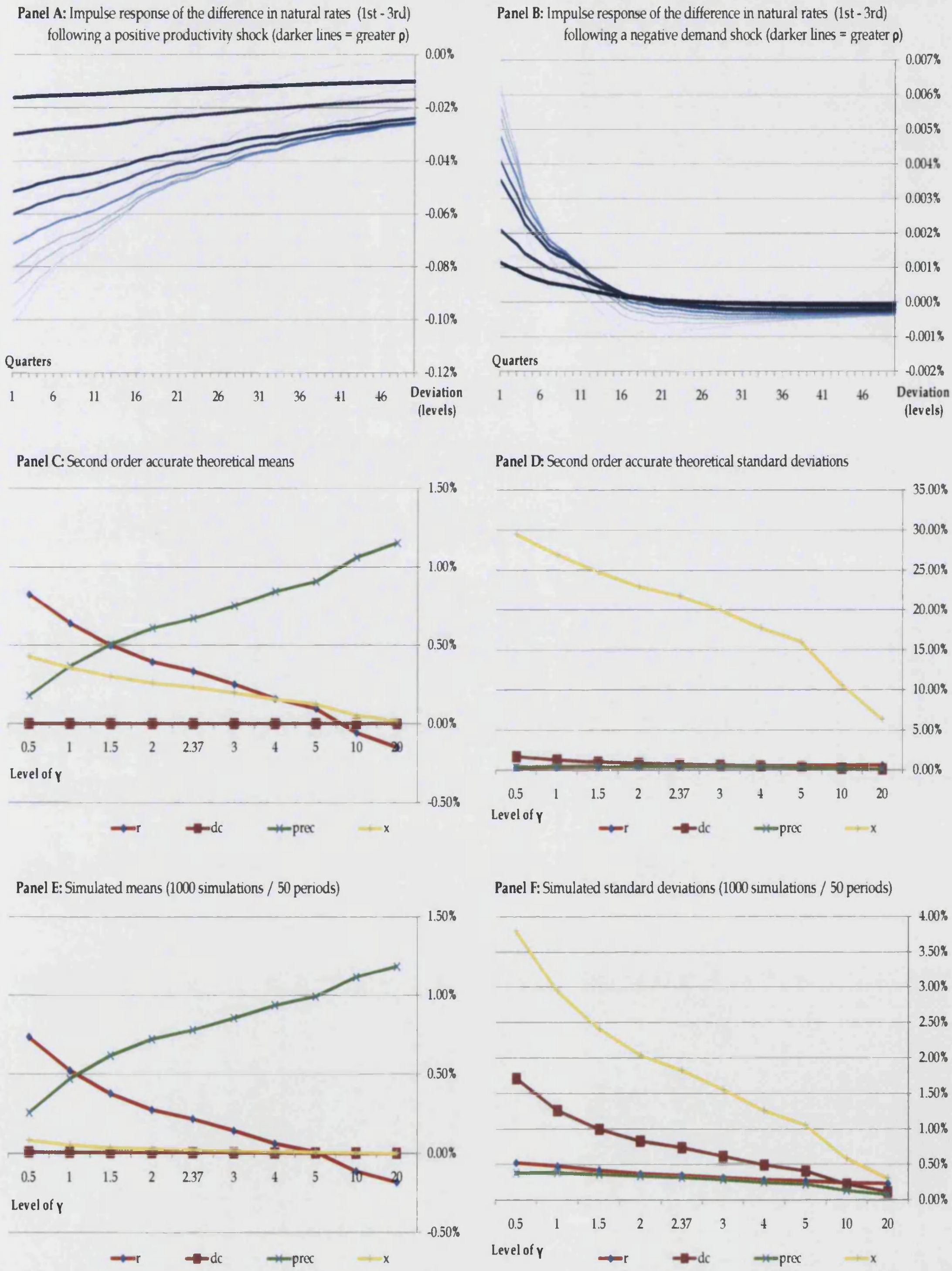

Figure 2.7: Sensitivity to Changes in the Coefficient of Risk Aversion ρ

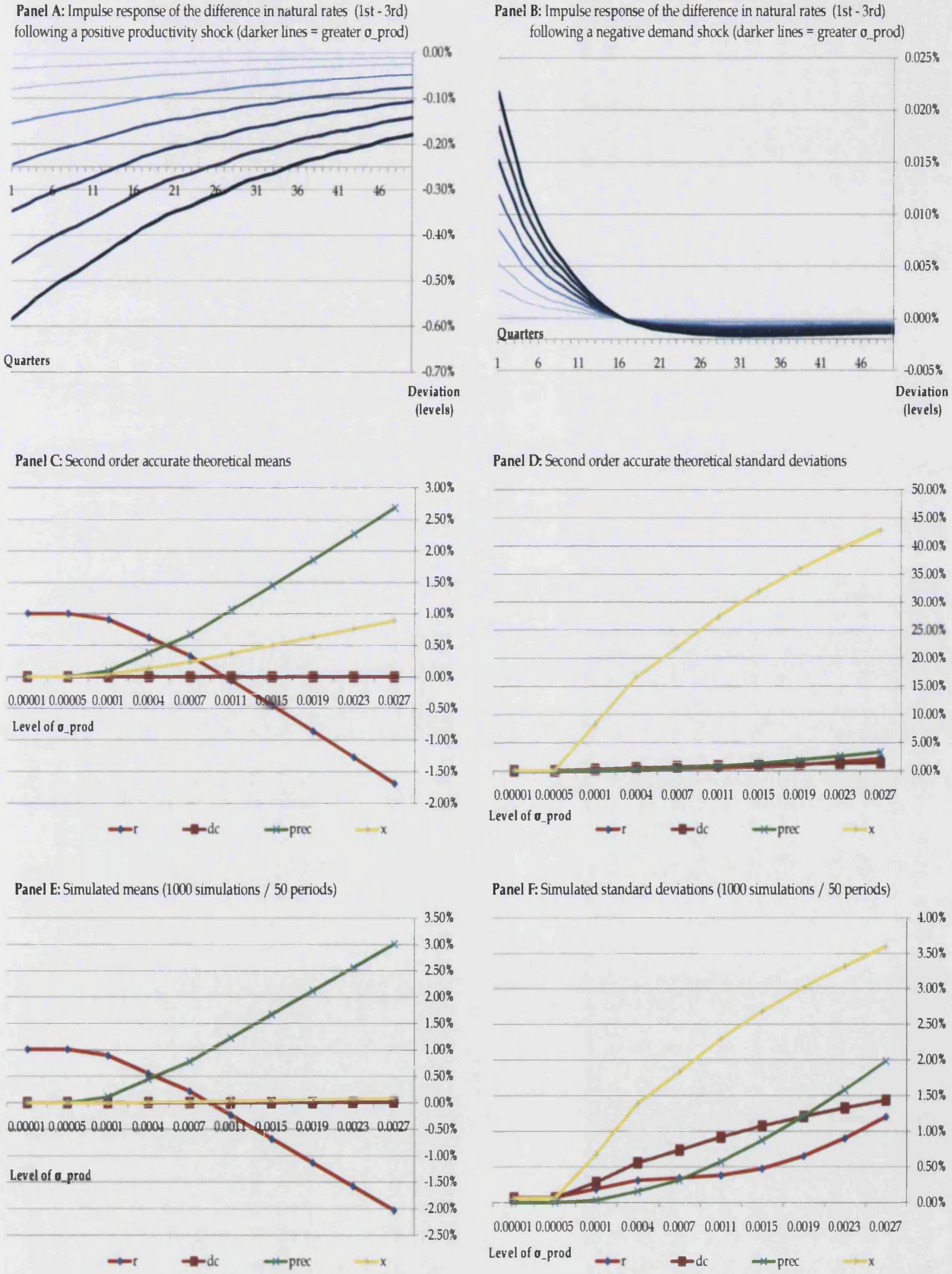

Figure 2.8: Sensitivity to Changes in Productivity Shock Variance σ_{prod}

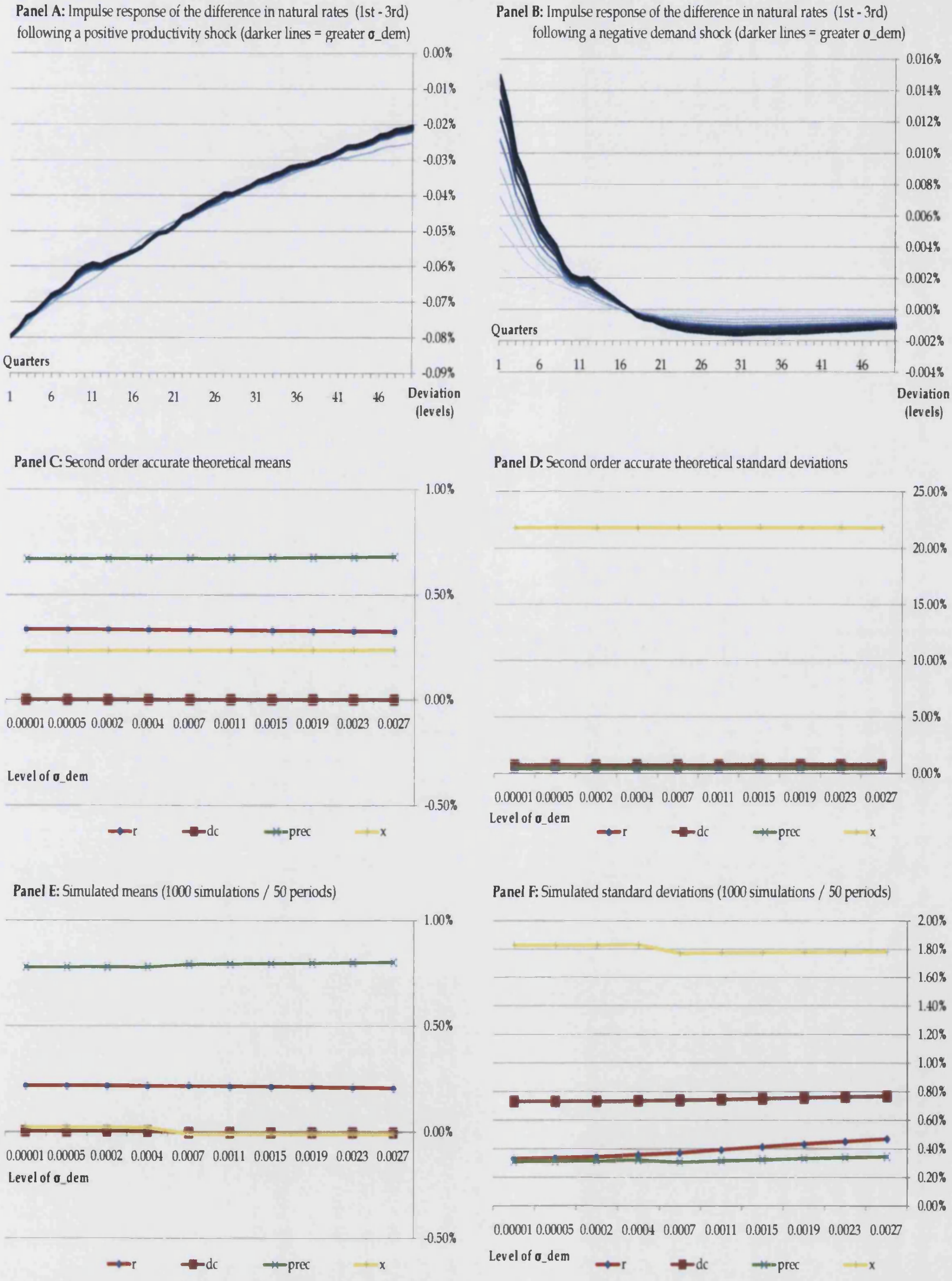

Figure 2.9: Sensitivity to Changes in Demand Shock Variance σ_{dem}

Table 2.1: Parameter Values Used in the Quantitative Analysis

Parameter	Value	Notes:
β	0.99	To yield a 4% steady-state real interest rate
η	6	As in Canzoneri et al. (2007)
ρ	2.37	Following Campbell and Cochrane (1999)
α	0.66	Length of average price contract 3 quarters
σ	10	Following Benigno and Woodford (2005)
h	0.85	Juillard et al. (2006) and Banerjee and Batini (2003)
ϕ	0.97	Following Campbell and Cochrane (1999)
γ_{dem}	0.9	Following Smets and Wouters (2003)
γ_{prod}	0.997	Following Smets and Wouters (2007)
σ_y^2/σ_d^2	3.5	Following Smets and Wouters (2003)
$\sigma_{\Delta c}$	0.75%	UK ONS data from 1976 Q1 to 2007 Q3

Table 2.2: Policy Exercise Parameters

Moment	Linear model	Incorporating precationary saving Benchmark	$\rho=1$	$\&\kappa=0.1$	$\&\sigma_{\Delta c}$= 1.5%
σ_π	0	0.1%	0.14%	0.2%	0.2%
σ_{ygap}	0	0.1%	0.24%	0.1%	0.4%

Chapter 3

Asset Prices Under Persistent Habits and Arbitrary Shocks to Consumption Growth

3.1 Introduction

The idea that habits affect human behavior was already present in the writings of Smith (1776) ('customary' consumption) and Marshall (1898). The notion was formalized by Pigou (1903) and resurrected in the works of Duesenberry (1949) and more recently Muellbauer (1988), Abel (1990) and Constantinides (1990). With time, the original habit concept became referred to as 'internal' to distinguish it from reference levels in utility 'external' to the consumer – such as aggregate consumption in Gali's (1994) 'keeping-up-with-the-Joneses' specification. Habits entered utility either in differences (Constantinides, 1990; Heaton, 1995) or ratios (Abel, 1990). 'Persistent' extensions – in which the entire history of aggregate consumption determined current habits – were also considered (Campbell and Cochrane, 1999; Abel, 2008). Our first contribution is to provide analytical, closed-form solutions for the equity price-dividend ratio and equity risk premium in a difference-form, external habit model with persistent habits and under arbitrary shocks to auto-correlated consumption growth.[1] We also derive restrictions on shock support, which ensure that utility remains well-

[1]We assume a Constant Relative Risk Aversion (CRRA) utility function. Alessie and Lusardi (1997) have shown how to solve the additive model when instantaneous utility is of the Constant Absolute Risk Aversion (CARA) form.

defined, and characterize necessary and sufficient conditions for unconditional k-th order moments of the price dividend ratio to be finite.

Habit-based specifications are prevalent, which is our main reason for focusing on them. For example, in finance habits have been used to 'solve' the equity premium puzzle and match expected stock return volatility (Constantinides, 1990; Campbell and Cochrane, 1999; Tallarini and Zhang, 2005).[2] In the foreign exchange literature (Verdelhan, 2009; De Paoli and Sondergaard, 2009) habits were utilized to address the Fama puzzle and to generate a countercyclical FX premium, while Wachter (2006) and Gallmeyer et al. (2009) used them to match yield curve properties. Ljungqvist and Uhlig (2000) and Fuhrer (2000) discussed the implications of habits for tax and monetary policy and Christiano et al. (2005) and Smets and Wouters (2007) showed that they play an important role in large scale macroeconomic models.[3] Carroll et al. (2000) proposed that habits may explain why high growth causes saving, while Polkovnichenko (2007), Jaccard (2007) and Uhlig (2007) analyzed the interplay of habits and labor market risks.

We proceed within the external difference-form habit framework, which lies at the heart of the Campbell and Cochrane (1999) model (inspect, for example, Equation (1) p. 208) and which, unlike its multiplicative habit counterparts (Carroll, 2000; Collard et al., 2006; Abel, 2008), generates time-varying risk aversion and risk premia. Importantly – relative to Campbell and Cochrane (1999) – we assume a simpler, linear specification for X_t, one which is governed by a single parameter ϕ controlling habit persistence (see also Equations (3.3) and (3.5)). The relative parsimony of our habit setup implies that unbounded shocks – e.g. drawn from the normal distribution – could drive current consumption below the reference level hX_t, pushing utility into regions where it is ill-defined.[4] This is one reason why we explicitly consider non-Gaussian shock

[2] Mehra and Prescott (1985) demonstrated that even standard CRRA utility functions can generate realistic equity risk premia if the coefficient of risk aversion is counterfactually high (> 10). Since Kocherlakota (1996) shows that Constantinides' (1990) specification implies very high levels of risk aversion and given that Campbell and Cochrane (1999) admit to that as well, we thus use inverted commas around the word solve. See also Mehra and Prescott (2004) for a good discussion.

[3] Boldrin et al. (2001) and Uhlig (2007) demonstrate, however, that simply assuming habits in a production economy does little to generate plausible asset price dynamics because agents have many opportunities of smoothing consumption risks.

[4] There are at least two ways of ensuring that the representative agents' utility function is always well-defined. One, pursued by Campbell and Cochrane (1999), is to hard-wire the constraint that habits can never exceed consumption directly into the definition of X_t. The alternative – which we follow – is to characterize and impose constraints on the white-

distributions. Another is to establish contact with the recently resurgent 'rare events' literature (Rietz, 1988; Barro, 2006; Barro and Ursua, 2008; Barro et al., 2009), which clearly demonstrated the important quantitative role which non-normalities can play.[5]

The final dimension along which our setup expands on popular alternatives, is that it allows for persistence in consumption growth – nesting the 'finance' random-walk assumption. We do that for three main reasons. First, a number of studies – see e.g. Carroll et al. (2008) and the references therein – suggest that the growth rate of consumption is, in fact, serially correlated. Second, allowing for two sources of persistence makes the model much more likely to avoid the pitfalls discussed in Campbell and Cochrane (2000) – though at the cost of a larger state vector. Finally, and related to the previous point, allowing for persistence in consumption growth makes the asset pricing problem more interesting and harder to solve.

To put our results in perspective, note that following the seminal contribution of Lucas (1978) many authors – e.g. Labadie (1989), Burnside (1998), Tsionas (2003) or Bidarkota and McCulloch (2003) – have analytically characterized the behavior of asset prices under different assumptions on the dividend process. The introduction of habits led to a renewed interest in closed-form formulae – as exemplified by contributions of Carroll (2000), Collard et al. (2006) or Abel (2008). This paper fills a gap in the literature, however, as existing analytical work mainly focuses on the 'ratio' specification of Abel (1990). The proofs build on and extend those in Burnside (1998) and Tsionas (2003). In particular, we obtain their results by respectively setting the weight on habits h to zero and restricting shocks to be normal. Notably, we consider persistent extensions of both the 'catching-up' and 'keeping-up with the Joneses' (Gali, 1994) formulations.

The second contribution of this paper is to use the exact, closed-form solutions to investigate the ability of the underlying model – possibly driven by non-normal shocks – to simultaneously match consumption growth, bond return and equity return data in the UK and US. Since Campbell and Cochrane (1999) showed

noise process driving consumption such that $C_t \geq hX_t$ with probability one. We show that under the assumption that consumption growth and habits are drawn from their stationary distributions, this implies that shocks have to be appropriately bounded from below when consumption growth is positively auto-correlated and bounded both from below and above when consumption growth is negatively auto-correlated.

[5]'Gaussianity' might not be a good assumption also due to cross-sectional violations of the Lindeberg-Levy condition often invoked to justify it.

that their model is capable of doing precisely that – even under random-walk log-consumption – the rationale for conducting such an exercise might not seem particularly clear. We reiterate, however, that our specification is much more tightly parameterized than that of Campbell and Cochrane (1999). While this parsimony has advantages – e.g. it allows us to derive the formulae and makes it simple to embed such habit processes in standard, easily-solvable DSGE models – it is also associated with costs. One such cost has already been noted – the shock distributions we use have to be bounded from below. Given the importance of disasters (Rietz, 1988; Barro, 2006), limiting downside risks this way could adversely affect the fit of the model. Furthermore, our habit specification, while persistent, is much less 'nonlinear' than that of Campbell and Cochrane (1999) potentially compromising the models' ability to generate large average premia. Finally, and unlike Campbell and Cochrane (1999), we have no way of ensuring that precautionary savings exactly offset 'intertemporal smoothing' – possibly generating volatile risk-free rates.[6]

Since under difference-form external habits, shocks to consumption growth cannot be normal, we need to take a stand on what alternative distribution to assume. We use this as an opportunity to compare the performance of models based on several simple, bounded shock distributions. To create a level playing field, we impose the assumption that shocks are mean zero and choose distributions with continuous densities parameterized by a single coefficient.[7] Because the normal distribution obtains as the limit of sums of i.i.d. random variables we chose to scrutinize the performance of normed sums of independent uniform shocks (Irwin-Hall distributions, after Irwin (1927) and Hall (1927)) and normed sums of independent exponential random variables (gamma distributions). By increasing the number of summed components we can thus reduce the 'distance' from the standard Gaussian benchmark. We also considered two bimodal distributions – the quadratic and inverse triangular – to investigate the impact of putting a lot of probability mass in the 'tails' of the distribution.

In order to estimate the model we used a two-stage procedure. In the first step we estimated coefficients of the consumption growth process using maximum likelihood and exploiting the distributional assumptions.[8] In the second step we

[6] Many papers – including ones allegedly successful at matching data e.g. Boldrin et al. (2001) – conveniently overlook this aspect.

[7] In all the cases there is a '1-1' mapping between that coefficient and shock variance.

[8] Since sequences of consumption growth map 1-1 into asset prices, it is impossible to estimate the entire model by maximum likelihood as the joint distribution of consumption,

found all four remaining parameters using GMM, choosing the same weighing matrix as Boldrin et al. (2001).[9] The only restrictions we imposed were those ensuring that utility is well-defined and guaranteeing that the resulting model is stable and generates finite unconditional moments of the price dividend ratio.

Our first finding may appear unsurprising. Trying to use shock distributions which differ too much from the Gaussian produces a consumption growth process which looks nothing like the data. This may happen when the shocks are bimodal, or because their probability mass is distributed either very asymmetrically or too 'uniformly' around the mean. Accordingly, while such models are capable of producing big and volatile risk premia, we believe this is not a particularly fruitful research avenue. To an extent, these conclusions also apply to our estimated proxy of a 'rare disaster' specification. However hard we tried, and despite the use of several alternative data sets, the estimates always said 'no thanks' (yielding support to the findings of Julliard and Ghosh (2009)).

The shock distributions that yield ML estimates of mean annual consumption growth within 20bp of the sample mean are limited to Irwin Hall and gamma distributions.[10, 11] All the asset pricing models estimated by GMM conditional on these consumption specifications (the formulae rely on the Laplace transform of the underlying shocks) can match the level of mean bond returns and mean equity returns almost exactly (up to 10bp). Importantly, however – and despite too volatile underlying consumption growth – all of them do not generate sufficiently volatile equity returns. They also tend to overshoot bond return volatility – particularly in the models estimated on US data. One finding which is of interest, and robust to the exact specification, is that the standard deviation of the equity risk premium is around two orders of magnitude smaller than that of excess returns (the means are roughly in line).

The mechanisms which the most successful specifications rely on to fit asset pricing data seem sample-specific. In particular, estimates based on UK data show very low values of habit size (no greater than 0.25 and typically smaller) and small values of habit persistence (≤ 0.3). To generate the large UK equity

bond returns and equity returns would be singular.

[9]In particular, we minimized the deviation of predicted mean bond and equity returns, weighted by their variances, from sample means.

[10]To generate sufficient density curvature, however, at least five uniform random variables are necessary – otherwise the estimates of mean consumption growth are poor.

[11]Notably, these tend to imply a more volatile consumption process than the one found in the data, though that volatility seems to converge to the sample average as the number of summed random variables increases.

premium, the models favor the well-trodden path of very high risk aversion – greater than ten and typically around 15 – and high values of β – frequently in excess of 1 (Kocherlakota (1996) provides a lucid account of why the combination works). A similar combination also works for some of the models estimated on US data. There however, two of the models based on gamma distributions work better when risk aversion and habit persistence are moderate (3 – 9 and 0.44 – 0.49 respectively) and when they are combined with a high weight of habits in the utility function (0.5 – 0.7).

To summarize, our results highlight tensions which exist when one tries to simultaneously match the properties of consumption growth, bond returns and equity returns. While the habit specification proposed by Campbell and Cochrane (1999) is capable of cutting the umbilical cord linking all three, the same cannot be achieved with the more parsimonious model which this papers solves in closed-form. We document the dimensions in which the model fails and argue that more exotic shock distributions are unlikely to solve the problems outlined here.

In the remainder, we set up the model, characterize restrictions on the underlying shocks, which ensure that the problem is well-defined, and solve it. We come up with additional restrictions guaranteeing that the price dividend ratio remains finite and that all of its unconditional moments exist. We estimate the model using asset pricing formulae derived and subject to the restrictions characterized. We present the results of the estimation and conclude.

3.2 The Asset Pricing Model

We assume there is a single agent who optimally chooses contingency plans for consumption C and investment in bonds B and equities S so as to maximize the stream of expected discounted utility

$$\max_{C_t,\, S_t,\, B_t} \mathrm{E}\left(\sum_{t=0}^{+\infty} \beta^t \frac{\left(C_t - hX_t\right)^{1-\gamma} - 1}{1-\gamma}\right) \tag{3.1}$$

$$\text{s.t. } C_t + P_t^b B_t + P_t^{eq} S_t = S_{t-1}\left(P_t^{eq} + D_t\right) + B_{t-1} \tag{3.2}$$

where X stands for the reference 'habit' level, which the agent takes as independent of her choices – hence habits are 'external', P^{eq} denotes the price of an equity share which entitles its owner to the stream of dividends D while P^b is the price of a zero-coupon bond paying a unit of consumption in the next period. The habit share h lies in $[0, 1]$, while the subjective discount factor β and the coefficient γ are non-negative. Setting $h = 0$ reduces equation (3.1) to a time-separable, no-habit specification analyzed in Burnside (1998), Bidarkota and McCulloch (2003) and Tsionas (2003), while $\gamma = 1$ corresponds to log-utility.

Since we major on the difference-habit specification, it might be worth addressing three claims sometimes leveled against it:

- that it finds little support in disaggregated data (Heien and Durham, 1991; Dynan, 2000; Chen and Ludvigson, 2009);
- that habit-based solutions of asset pricing puzzles rely on implausibly high coefficients of risk aversion (inspect also Footnote (2));
- and that the difference-form specification can easily lead to infinite negative utility (Carroll, 2000; Uhlig, 2007).

On the first two points, note that Guvenen (2003) and Garleanu and Panageas (2008) present heterogenous agent models which do not rely on habit in the utility but behave as if aggregate data came from a habit model. Accordingly – even if habits are not a feature of micro data – the mechanism can still provide a good description of aggregate dynamics.[12] As to the final point, the conditions we subsequently derive explicitly rule such problems out.

In what follows, in line with much of the equity premium literature, we specialize by assuming that the equity share pays out consumption as dividends ($\forall t : D_t = C_t$).[13] Furthermore, defining x_t to be the continuously compounded growth rate of consumption and z_t to be the log-ratio of habits to consumption, i.e.

$$x_{t+1} := \log\big(C_{t+1}/C_t\big), \qquad z_{t+1} := \log\big(X_{t+1}/C_{t+1}\big) \tag{3.3}$$

[12]These models also suggest that there might be no simple mapping from individual agents' coefficients of risk aversion to that of the representative consumer – and so high risk aversion in aggregate might be less damning than previously thought.

[13] Defining equities this way eliminates none of the real difficulties inherent in asset pricing. Notably, under the standard assumptions that equities are the only assets in non-zero net supply (implying $B \equiv 0$) and that equity markets clear ($S \equiv 1$), the identity $C_t \equiv D_t$ emerges straight from the aggregate budget constraint – i.e. equation (3.2).

we shall analyze asset prices under the assumption that x_t follows

$$x_t = (1-\rho)\mu + \rho x_{t-1} + \xi_t \tag{3.4}$$

where ξ is a mean zero, *i.i.d.* process.[14] As discussed, we focus on the AR(1) specification because it is typically used in the literature (Burnside, 1998; Collard et al., 2006), it appears in line with recent empirical estimates (Carroll et al., 2008) and nests the most popular alternative of uncorrelated consumption growth (Constantinides, 1990; Campbell and Cochrane, 1999).

To proceed, we need to make some assumptions about the dynamics of habits X_t. We work under two alternative specifications – the default of 'catching-up with the Joneses' (Gali, 1994)

$$X_{t+1} := C_t^{\phi} X_t^{1-\phi} \Rightarrow \left(\frac{X_{t+1}}{C_{t+1}}\right) = \left(\left(\frac{X_t}{C_t}\right)^{1-\phi} \cdot \frac{C_t}{C_{t+1}}\right)$$

which, in terms of x_t and z_t, simplifies to

$$z_{t+1} = (1-\phi)z_t - x_{t+1} \tag{3.5}$$

and the alternative of 'keeping-up with the Joneses'

$$X_{t+1} := C_{t+1}^{\phi} X_t^{1-\phi} \Rightarrow z_{t+1} = (1-\phi)\big(z_t - x_{t+1}\big). \tag{3.6}$$

By setting ϕ to one in the default specification (3.5), habits become dependant only on last period's level of consumption (as in Smets and Wouters (2007) or Uhlig (2007)).[15] For values of ϕ different from one, habits are an infinite weighted average of past levels of consumption with $(1-\phi)$ denoting the geometrically decaying weight put on past values of consumption

$$\log(X_{t+1}) = \phi \sum_{i=0}^{+\infty} (1-\phi)^i \log(C_{t-i}).$$

[14]We stress that the assumption of mean zero noise ξ is introduced merely to give the parameter μ a clear interpretation – that of mean consumption growth. However, all the subsequent formulae remain well-defined if the mean of $\xi \neq 0$ and sometimes also if the mean of ξ does not exist.

[15]Though in both of these papers utility also depends on leisure – an extension which we don't analyze.

3.3 Conditions for Well-Defined Utility

As discussed in the introduction, one complication related to the difference-form habit specification – stressed (though not addressed) e.g. by Carroll (2000) or Uhlig (2007), is that utility can become ill-defined if current consumption falls below the reference level hX.[16] Accordingly, the condition which we would like to impose is $\forall t : C_t \geq hX_t$. Dividing both sides through by X_t and taking logs gives an equivalent formulation $z_t \leq -\log h$. Lemma 1 below is useful, because it shows that this condition is equivalent to $x_t \geq \phi \log h$. Lemmas 2 and 3 then characterize conditions on shock support – for positive and negative shock autocorrelation ρ respectively – which ensure that consumption growth is indeed bounded from below by $\phi \log h$.

Lemma 1. If $[x_l, x_h]$ is invariant for x_t, $t \in \{0, 1, \ldots\}$ and $z_l \leq -x_h/\phi$, $z_h \geq -x_l/\phi$ then $[z_l, z_h]$ is invariant for z_t, $t \in \{0, 1, \ldots\}$.[17]

Proof of Lemma 1. The proof can be found in the Appendix. □

Lemma 2. If $[\xi_l, \xi_h]$ is invariant for ξ_t, $0 \leq \rho < 1$ and $x_h \geq \mu + \xi_h/(1-\rho)$, $x_l \leq \mu + \xi_l/(1-\rho)$ then $[x_l, x_h]$ is invariant for (x_t).

Proof of Lemma 2. The proof can be found in the Appendix. □

Importantly, Lemma 2 remains true if $\xi_h = +\infty$ when all intervals $[x_l, +\infty)$ such that $x_l \leq \mu + \xi_l/(1-\rho)$ are invariant for (x_t). This implies that for non-negative shock autocorrelation ρ, a necessary and sufficient condition for $x_t \geq \phi \log h$ is $\xi_l \geq (1-\rho)\big(\phi \log h - \mu\big)$ – i.e. shocks need to be bounded from below.

Lemma 3. If $[\xi_l, \xi_h]$ is invariant for $\xi_t, \rho \in [-1, 0]$ and

$$\rho x_l + (1-\rho)\mu + \xi_h \leq x_h \tag{3.7}$$

$$x_h \leq x_l/\rho - (1-\rho)/\rho\mu - \xi_l/\rho \tag{3.8}$$

then $x_l \leq (\rho\xi_h + \xi_l)/(1-\rho^2) + \mu$ and $[x_l, x_l/\rho - (1-\rho)/\rho\mu - \xi_l/\rho]$ is invariant for x_t.

[16] As alluded to, there are two different ways of addressing this problem. The first – taken for example by Campbell and Cochrane (1999) – is to allow for Gaussian shocks but modify the specification of habits to ensure that they can don't exceed current consumption. The second, followed here, is to retain a simple habit specification – like the one in (3.5) or (3.6) – but restrict the support of the shock process.

[17] We shall say that $[\xi_l, \xi_h]$ is invariant for ξ_t, if $\forall t \in \{0, 1, \ldots\} : \xi_l \leq \xi_t \leq \xi_h$.

Proof of Lemma 3. The proof can be found in the Appendix. □

From Lemma 1, $x_l \geq \phi \log(h/(1-\rho)) \Rightarrow z_t \leq -\log(h/(1-\rho)$. Thus, Lemma 3 implies that the constraint, which the invariant noise has to satisfy is

$$\phi \log(h/(1-\rho)) \leq (\rho\xi_h + \xi_l)/(1-\rho^2) + \mu. \tag{3.9}$$

This means that ξ_l cannot be minus infinity – as otherwise the inequality would not be satisfied – and, because $\rho \leq 0$, the upper bound ξ_h cannot be plus infinity either. In other words, for a negative ρ the support of the invariant noise distribution has to be bounded from below and above.

We can go further, however, and use equation (3) to derive necessary conditions which ξ_l and ξ_h satisfy. In particular it has to hold that (set ξ_l and ξ_h to zero)

$$\xi_l \geq (1-\rho^2)\big(\phi \log(h/(1-\rho)) - \mu\big), \quad \xi_h \leq (1-\rho^2)/\rho \cdot \big(\phi \log(h/(1-\rho)) - \mu\big).$$

To prove convergence of asset pricing formulae we shall sometimes require slightly stronger conditions, namely that $C_t \geq (h/(1-\delta))X_t$ where δ is an arbitrarily small, positive constant. For reference, we thus introduce assumption (**A**δ).[18]

$$(\mathbf{A}\delta): \begin{cases} \xi_l \geq (1-\rho)\big(\phi \log(h/(1-\delta)) - \mu\big) & \rho \geq 0 \\ \begin{cases} \xi_l \geq (1-\rho^2)\big(\phi \log(h/((1-\rho)(1-\delta))) - \mu\big) \\ \xi_h \leq (1-\rho^2)/\rho \cdot \big(\phi \log(h/((1-\rho)(1-\delta))) - \mu\big) \end{cases} & \rho < 0 \end{cases}$$

3.4 Asset Prices and the Equity Risk Premium

3.4.1 Definitions

To avoid ambiguities, we begin by defining several important concepts.

Definition 1. All conditional expectation operators are always taken with respect to the *natural filtration* $\mathcal{F}$ defined as

$$\mathcal{F}_t := \sigma\big(\xi_s : s \leq t\big). \tag{3.10}$$

Definition 2. Given $\mathcal{F}_t$, there will typically be a continuum of processes satisfying the first order conditions with respect to asset prices. We shall therefore focus

[18] Assumption (**A**δ) guarantees that $\big(1 - h\exp(z_i)\big)^{-\gamma}$ is bounded from above by $\delta^{-\gamma}$.

on the (unique) *fundamental solution* which additionally sates the transversality conditions. For example in the case of equation (3.14) the fundamental solution v_t would need to satisfy

$$\lim_{s\to+\infty} \mathrm{E}_t \left(\prod_{i=1}^{s} \beta \exp\big((1-\gamma)\, x_{t+i}\big) \right) \big(1 - h\exp(z_{t+s})\big)^{-\gamma} v_{t+s} =_{a.s.} 0. \quad (3.11)$$

Definition 3. The *Laplace transform* of random variable ξ is defined as[19]

$$\mathcal{L}_\xi(\lambda) := \mathrm{E}\exp(-\lambda\,\xi).$$

Since the only source of uncertainty in our model is ξ, we shall frequently omit the subscript where no ambiguity can arise.

Definition 4. *Generalized binomial coefficients* $\binom{\alpha}{n}, \alpha \in \mathbb{R}, n \in \mathcal{N}$ equal

$$\binom{\alpha}{n} := \prod_{k=1}^{n} (\alpha - k + 1)/k, \;\; n > 0, \qquad \binom{\alpha}{0} := 1. \quad (3.12)$$

Note that all generalized binomial coefficients are strictly positive if $\alpha > 0$.

Definition 5. *Definition of r_t^{eq} and r_t^b and the risk premium rp_t*

$$r_t^b = \frac{1}{P_{t-1}^b} \qquad r_t^{eq} = \frac{P_t^{eq} + C_t}{P_{t-1}^{eq}} \qquad rp_t = \mathrm{E}_t r_{t+1}^{eq} - r_{t+1}^b. \quad (3.13)$$

3.4.2 The Asset Pricing Equation and its Fundamental Solutions

Defining the equity share's price – consumption ratio $v_t := P_t^{eq}/C_t$ and exploiting $D_t \equiv C_t$ one can show that the first order condition of problem (3.1) – (3.2) with respect to equity share holdings is (see also Footnote 13 for a brief discussion of the $D_t \equiv C_t$ condition)

$$\frac{P_t^{eq}}{C_t} = \mathrm{E}_t \beta \left(\frac{1 - hX_{t+1}/C_{t+1}}{1 - hX_t/C_t} \right)^{-\gamma} \cdot \left(\frac{C_{t+1}}{C_t} \right)^{-\gamma} \cdot \frac{C_{t+1}}{C_t} \cdot \left(\frac{P_{t+1}^{eq} + C_{t+1}}{C_{t+1}} \right)$$

[19]Note that there is a simple relationship between the moment generating function $M(\cdot)$ referred to in Tsionas (2003) or Collard et al. (2006) and the Laplace transform. In particular, $M(\lambda) = \mathcal{L}_\xi(-\lambda)$.

or, using the definitions in (3.3) express it in terms of v_t, z_t and x_t as

$$v_t = \mathrm{E}_t \beta \left(\frac{1 - h\exp(z_{t+1})}{1 - h\exp(z_t)} \right)^{-\gamma} \cdot \exp\big((1-\gamma)\, x_{t+1}\big) \cdot \big(v_{t+1} + 1\big). \tag{3.14}$$

Theorem 1. *Under assumption (Aδ) and conditional on the dynamics of consumption growth and habits – as specified in equations (3.4) and (3.5) respectively – the fundamental solution of equation (3.14) is finite if and only if*

$$\beta \exp\big((1-\gamma)\mu\big) \mathcal{L}\left(\frac{\gamma - 1}{1 - \rho} \right) < 1. \tag{3.15}$$

When both (Aδ) and (3.15) hold, the price dividend ratio v_t is given by

$$v_t = \big(1 - h\exp(z_t)\big)^{\gamma} \sum_{i=1}^{+\infty} \sum_{n=0}^{+\infty} d_{i,n} \exp\Big(z_t\, a_{i,n} + (x_t - \mu)\, b_{i,n} + c_{i,n} \Big) \tag{3.16}$$

where

$$\begin{aligned}
a_{i,n} :=& n(1-\phi)^i \\
b_{i,n} :=& \rho \left((1-\gamma) \frac{(1-\rho^i)}{(1-\rho)} - n \frac{(1-\phi)^i - \rho^i}{(1-\phi) - \rho} \right) \\
c_{i,n} :=& \mu \left((1-\gamma) i - n \frac{1 - (1-\phi)^i}{\phi} \right) \\
d_{i,n} :=& \beta^i h^n \binom{\gamma - 1 + n}{n} \cdot \prod_{j=1}^{i} \mathcal{L}\left((\gamma - 1) \frac{(1-\rho^j)}{(1-\rho)} + n \frac{(1-\phi)^j - \rho^j}{(1-\phi) - \rho} \right).
\end{aligned}$$

Note that for $\gamma \geq 1$ the argument at which the Laplace transform $\mathcal{L}$ is evaluated is positive, and so $\mathcal{L}(\cdot)$ is well-defined.[20] Additionally, $\gamma > 1$ implies that all the terms of series (3.16) defining v_t are positive and thus, conditional on convergence, good estimates of v_t can be obtained by truncating sufficiently 'far'.

Corollary 1. For $\phi = 1$, equation (3.5) implies that $X_t = C_{t-1}$. In this simpler

[20]To see this, note that

$$\frac{(1-\phi)^j - \rho^j}{(1-\phi) - \rho} > 0 \tag{3.17}$$

as $(1-\phi) > \rho$ in which case both the numerator and the denominator are positive or $(1-\phi) < \rho$ in which case both are negative but the fraction remains positive.

case, formula (3.16) reduces to

$$v_t = \left(1 - h\exp(z_t)\right)^{\gamma} \sum_{i=1}^{+\infty} \sum_{n=0}^{+\infty} d_{i,n} \exp\Big((x_t - \mu)\, b_{i,n} + c_{i,n} \Big) \tag{3.18}$$

where

$$b_{i,n} := \rho\left((1-\gamma)\frac{(1-\rho^i)}{(1-\rho)} - n\rho^{i-1} \right) \qquad c_{i,n} := \mu\Big((1-\gamma)i - n \Big)$$

$$d_{i,n} := \beta^i h^n \binom{\gamma - 1 + n}{n} \cdot \prod_{j=1}^{i} \mathcal{L}\left((\gamma-1)\frac{(1-\rho^j)}{(1-\rho)} + n\rho^{j-1} \right).$$

Corollary 2. For $\rho = 0$ – i.e. in the case of *i.i.d.* consumption growth considered e.g. in Campbell and Cochrane (1999) – equation (3.16) simplifies to

$$v_t = \left(1 - h\exp(z_t)\right)^{\gamma} \sum_{i=1}^{+\infty} \sum_{n=0}^{+\infty} d_{i,n} \exp\Big(z_t\, a_{i,n} + c_{i,n} \Big) \tag{3.19}$$

where

$$a_{i,n} := n(1-\phi)^i \qquad c_{i,n} := \mu\left((1-\gamma)i - n\frac{1-(1-\phi)^i}{\phi} \right)$$

$$d_{i,n} := \beta^i h^n \binom{\gamma - 1 + n}{n} \cdot \prod_{j=1}^{i} \mathcal{L}\left((\gamma-1) + n(1-\phi)^{j-1} \right).$$

Corollary 3. The price dividend ratio under the contemporaneous / 'keeping-up with the Joneses' habit specification (3.6) is given by

$$v_t = \left(1 - h\exp(z_t)\right)^{\gamma} \sum_{i=1}^{+\infty} \sum_{n=0}^{+\infty} d_{i,n} \exp\Big(z_t\, a_{i,n} + (x_t - \mu)\, b_{i,n} + c_{i,n} \Big) \tag{3.20}$$

where

$$
\begin{aligned}
a_{i,n} :=& n(1-\phi)^i \\
b_{i,n} :=& \rho\left((1-\gamma)\frac{(1-\rho^i)}{(1-\rho)} - n(1-\phi)\frac{(1-\phi)^i-\rho^i}{(1-\phi)-\rho}\right) \\
c_{i,n} :=& \mu\left((1-\gamma)i - n(1-\phi)\frac{1-(1-\phi)^i}{\phi}\right) \\
d_{i,n} :=& \beta^i h^n \binom{\gamma-1+n}{n} \cdot \prod_{j=1}^{i} \mathcal{L}\left((\gamma-1)\frac{(1-\rho^j)}{(1-\rho)} + n(1-\phi)\frac{(1-\phi)^j-\rho^j}{(1-\phi)-\rho}\right).
\end{aligned}
$$

The following theorem characterizes conditions under which unconditional moments of arbitrary powers of the price-dividend ratio exist.

Theorem 2. *Under assumption (Aδ) and conditional on the dynamics of consumption growth and habits – as specified in equations (3.4) and (3.5) – the unconditional k-th moment of the price-dividend ratio v_t exists if and only if*

$$
\beta \exp\big((1-\gamma)\mu\big)\mathcal{L}\left(\frac{\gamma-1}{1-\rho}\right) < 1. \tag{3.21}
$$

Note that as γ converges to 1, the left-hand side converges to $\beta < 1$. Accordingly, we can always find a γ sufficiently close to 1 such that condition (3.15) is satisfied and equity share prices are finite.

Theorem 3. *Under the assumptions of Theorem 2, the riskless rate of return equals*

$$
\begin{aligned}
r^b_{t+1} =& \big(1 - h\exp(z_t)\big)^{-\gamma} \\
&\cdot \left\{\sum_{n=0}^{+\infty} d_{1,n}\frac{\mathcal{L}(\gamma+n)}{\mathcal{L}(\gamma+n-1)}\exp\Big(a_{1,n}z_t + (b_{1,n}-\rho)(x_t-\mu) + (c_{1,n}-\mu)\Big)\right\}^{-1}
\end{aligned}
$$

while the equity risk premium is given by

$$rp_t = \frac{\exp(\mu + \rho(x_t - \mu))}{(1 - h\exp(z_t))^\gamma} \cdot \left\{ \left[\sum_{i=1}^{+\infty} \sum_{n=0}^{+\infty} d_{i,n} \exp\Big(z_t\, a_{i,n} + (x_t - \mu)\, b_{i,n} + c_{i,n} \Big) \right]^{-1} \cdot \right.$$
$$\left[\mathcal{L}(-1) + \sum_{n,m=0}^{+\infty} \sum_{i=1}^{+\infty} d_{i,n,m} \cdot \mathcal{L}\big(-b_{i,n,m} - 1\big) \cdot \exp\Big(z_t\, a_{i,n,m} + \rho\,(x_t - \mu)\, b_{i,n,m} + c_{i,n,m} \Big) \right]$$
$$\left. - \left[\sum_{n=0}^{+\infty} d_{1,n} \frac{\mathcal{L}(\gamma + n)}{\mathcal{L}(\gamma + n - 1)} \exp\Big(z_t a_{1,n} + (x_t - \mu)\, b_{1,n} + c_{1,n} \Big) \right]^{-1} \right\} \tag{3.22}$$

with coefficients defined in Theorem 2 and additionally given by

$$a_{i,n,m} = (1 - \phi)\,(a_{i,n} + m) \qquad b_{i,n,m} = b_{i,n} - \frac{a_{i,n,m}}{(1 - \phi)}$$
$$c_{i,n,m} = c_{i,n} - \frac{\mu\, a_{i,n,m}}{(1 - \phi)} \qquad d_{i,n,m} = d_{i,n} \binom{-\gamma}{m} (-h)^m.$$

Corollary 4. When consumption growth is *i.i.d.* ($\rho = 0$) and habits are purely a function of last period's consumption ($\phi = 1$), then the equity risk premium rp_t is given by

$$rp_t = \frac{\exp(\mu)\mathcal{L}(-1)}{(1 - h\exp(-x_t))^\gamma} \left\{ - \left[\sum_{n=0}^{+\infty} d_{1,n} \exp\big(c_{1,n}\big) \frac{\mathcal{L}(\gamma + n)\mathcal{L}(-1)}{\mathcal{L}(\gamma + n - 1)} \right]^{-1} \right.$$
$$\left. + (1 - h)^{-\gamma} + \left[\sum_{n=0}^{+\infty} d_{1,n} \exp\big(c_{1,n}\big) \big(1 - \beta \exp(\mu(1 - \gamma))\mathcal{L}(\gamma - 1)\big)^{-1} \right]^{-1} \right\}. \tag{3.23}$$

Notably, the term in the wiggly brackets is a constant. Since $(1 - h)^{-\gamma} > 0$, therefore a sufficient condition for the equity risk premium to be positive (though only in the restricted case of $\rho = 0$, $\phi = 1$) is that the inverse of the first square bracket is smaller than the inverse of the second one. Given that $\forall n : d_{1,n}, \exp\big(c_{1,n}\big) > 0$, therefore this condition follows from

$$\forall n : \frac{\mathcal{L}(\gamma + n)\mathcal{L}(-1)}{\mathcal{L}(\gamma + n - 1)} \geq \big(1 - \beta \exp(\mu(1 - \gamma))\mathcal{L}(\gamma - 1)\big)^{-1}.$$

Inspecting expression (3.23) we see that since

$$\frac{d}{dx}\big(1 - he^{-x}\big)^{-\gamma} = -\gamma h e^{-x}\big(1 - he^{-x}\big)^{-(\gamma+1)} \leq 0$$

therefore, if the term in the wiggly brackets in (3.23) is positive (i.e. if bonds are perceived to be less risky than equities and the *level* of the premium is positive), then a high contemporaneous realization of x_t tends to lower both the risk premium and the riskless rate of return. Expressed alternatively, under these assumptions, both the interest rate which clears the bond market and the equity risk premium are countercyclical.[21]

3.5 Can the Model Fit the Data?

In this section we use the exact, closed-form solutions derived above to investigate the ability of the underlying external habit specification – possibly driven by non-normal shocks – to simultaneously match consumption growth, bond return and equity return data in the UK and US. We shall also study the evolution of model-implied equity risk premia.

There are three main reasons why the empirical performance or our model warrants inspection – even though Campbell and Cochrane (1999) argue that their specification – which is closely related to ours – is capable of satisfactorily fitting the data. To begin with, our setup is much more tightly parameterized. One consequence of this is that the shock processes we rely on to drive consumption have to be bounded, with unclear implications for fit (we are decreasing 'downside' risks in consumption growth). Furthermore, we cannot build as much 'nonlinearity' into our setup, possibly compromising the models' ability to generate large average premia. Finally, and unlike Campbell and Cochrane (1999), we have no way of ensuring that precautionary savings exactly offset 'intertemporal smoothing' – which could detriment fit by generating an excessively volatile risk-free rate.

The inability to proceed under the standard assumption of Gaussian shocks to consumption growth provides an opportunity to document the impact of alter-

[21]The magnitude of these shifts would likely be very small, however. To understand why, recall an alternative, 'continuously-compounded' definition of the risk premium

$$\widetilde{rp}_t := \mathrm{E}_t \log\left(r^{eq}_{t+1}\right) - \log\left(r_{t+1}\right). \tag{3.24}$$

It is clear, that under this definition the term $(1 - h\exp(z_t))$ drops out and the risk premium becomes a state-invariant constant – i.e. in particular it is (counterfactually) acyclical. Since both premium definitions are first-order equivalent, therefore terms driving changes in the premium are, at best, 'second-order' (in interest rate changes). For an analysis of factors affecting premium cyclicality in external habit models with persistent habits see also De Paoli and Zabczyk (2009).

native distributional assumptions on model fit. We take that opportunity for two main reasons. One is to establish contact with the 'consumption disaster' literature of (Rietz, 1988; Barro, 2006).[22] The second is to verify which assumptions work well, and which don't – possibly preventing future research in unfruitful directions. Fundamentally, most tensions which arise are due to the fact that in our setup consumption growth and asset prices are driven by the same shock, and so assumptions made about its distribution have implications for both.[23] To establish which shock distributions to consider we now undertake a preliminary data analysis based solely on consumption growth data.

3.5.1 Preliminary Consumption Data Analysis

In this section we scrutinize UK and US consumption growth data. We utilize two sources:

1. The quarterly and annual data set underlying Campbell and Cochrane (1999) containing series for: real consumption (non-durable where available) as well as bond returns and dividends.[24]
2. The annual data set used in Barro and Ursua (2008) providing consumption (total) and GDP series.[25]

We use both quarterly and annual consumption growth data to verify whether there aren't any marked differences in the properties of residuals – i.e. whether some shock distributions fit better at particular frequencies.

We begin by documenting differences between the properties of real, annual, per-capita consumption growth series derived from the Campbell and Cochrane (1999) and Barro and Ursua (2008) data sets – plotted in Figures 3.1 and 3.2 for the US and UK respectively (for the sub-sample for which both sets contain

[22]For other papers studying the asset pricing implications of different consumption growth and/or utility specifications see also Nason (1988), Kandel and Stambaugh (1989), Kandel and Stambaugh (1991), Bansal and Yaron (2004), Tsionas (2005) or Barro et al. (2009).

[23]We are conscious that the properties of dividends and aggregate consumption differ in the data and we follow the standard, though internally inconsistent, procedure of estimating the model using both (as done e.g. in Collard et al. (2006)). The discussion in Campbell and Cochrane (1999) – who also proceed along these lines – provides hope that the distortions introduced by this procedure might not be quantitatively significant.

[24]Address: http://dvn.iq.harvard.edu/dvn/dv/jcampbell. Visited: 19 February 2009.

[25]Address: http://www.economics.harvard.edu/faculty/barro/data_sets_barro. Visited: 14 June 2009. File: MacroCrisesSince1870_08_0614.xls.

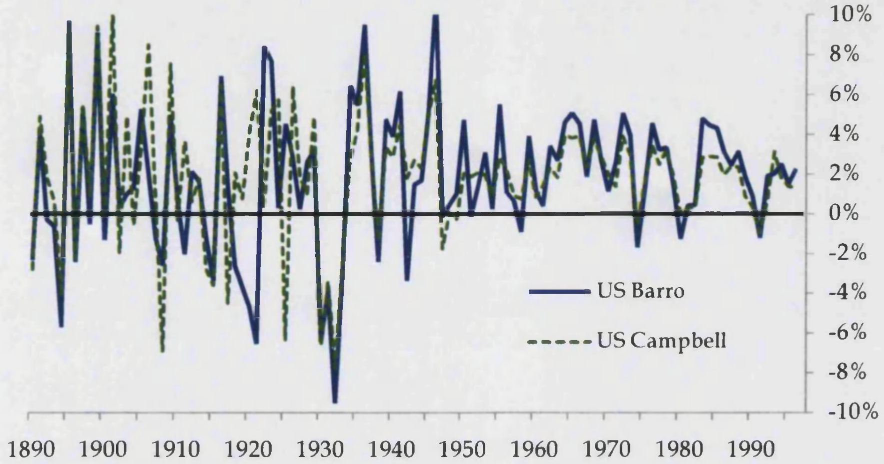

Figure 3.1: Comparison of US Annual Consumption Growth Series Used in Barro (2006) and Campbell and Cochrane (1999)

values). In particular the correlation coefficient for the respective series equals 0.84 in the UK, but only 0.66 in the US (it should be one if both accurately measured the same variable). While part of the low US correlation is attributable to differences in the underlying concept measured (total vs non-durable consumption) the key is a different account of developments before 1930 (in particular, if we stripped the pre-1930 data out of the sample, the correlation coefficient would increase to 0.91).

Interestingly, and with potentially important asset pricing implications, even though the means of the annual US series are almost identical at 1.85% the (point estimates of) first order autocorrelations of detrended consumption growth appear very different and equal -16.2% in the data set of Campbell and Cochrane and 7% in the data set constructed by Barro and Ursua.[26] While we leave these discrepancies largely unexplained, we note that restricting the sample to post-WWII data reduces these differences, with the corresponding point estimates equal to 27.5% and 13.5% respectively. Since in the asset pricing section we shall only use post-WWII quarterly data we believe that such discrepancies – and the potentially deeper factors they may reflect – are unlikely to have a marked impact on our results. Nonetheless, given the documented differences, in this section we shall separately plot histograms based on both series.

[26]Given the lower correlation coefficient of the two US series, we only focus on US data in this paragraph.

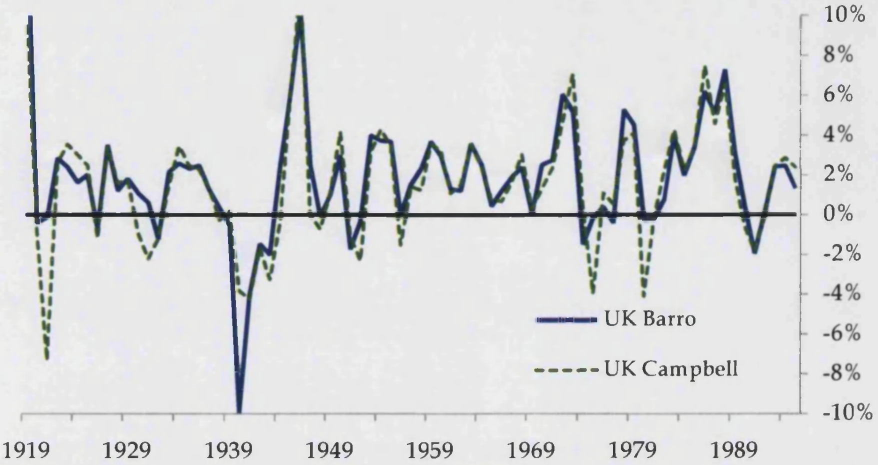

Figure 3.2: Comparison of UK Annual Consumption Growth Series Used in Barro (2006) and Campbell and Cochrane (1999)

We predicate the entire subsequent analysis on the assumption that consumption growth follows

$$x_t = (1-\rho)\mu + \rho x_{t-1} + \xi_t \tag{3.25}$$

where periods correspond to either quarters or years, depending on the data set used. To focus on the shape of the distribution of consumption growth shocks – in this section only – we fix μ and ρ equal to their method of moments' estimates – i.e. sample mean and correlation of detrended growth. Figures 3.3 and 3.4 plot the time series and empirical density estimates based on the residuals from specification 3.25 – at annual and quarterly frequency respectively.[27] Inspecting the figures, we see that despite differences in the properties of consumption growth reported above, the empirical densities of residuals appear reasonably in line. The histograms are fairly symmetrical, tend to have one dominant mode and moderate amount of mass distributed in the tails.

In the remainder of this paper, in large part motivated by Charts 3.3 and 3.4, we consider the shock distributions depicted in Figures 3.5 and 3.6, and summarized in Table 3.5.1:

- **Exponential** (Panel A Figure 3.5): has a single dominant mode, no mass in the left tail – potentially useful in quantifying the asset pricing role of the left tail;

[27]The density estimates are simply the smoothed histograms (based on different numbers of underlying 'bins' – to minimize the risk of introducing distortions this way) of the residuals, rescaled to ensure the density integrates to one.

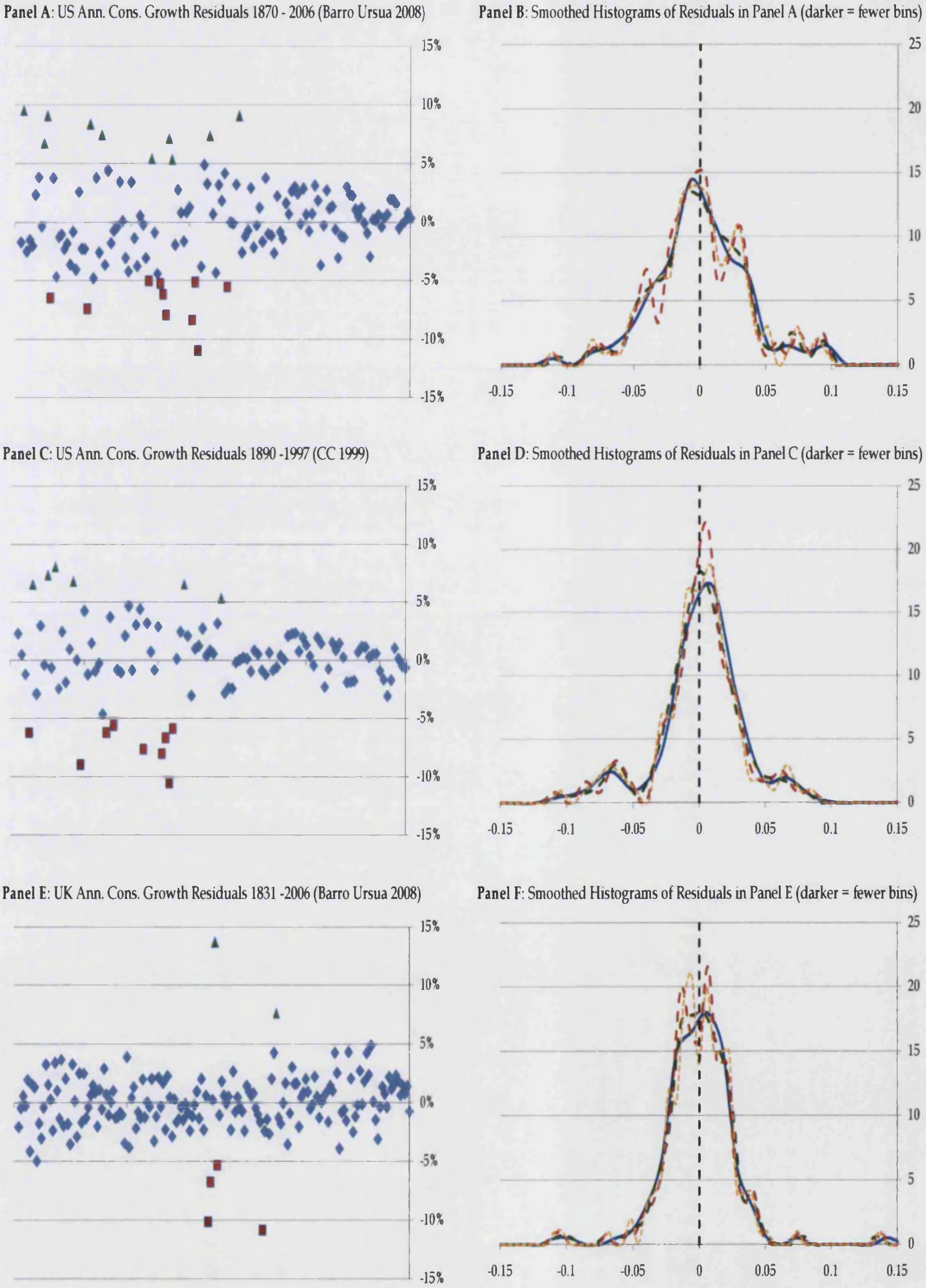

Figure 3.3: Overview of Annual Residuals and Empirical Distribution Functions; Green Triangles/Red Squares denote residuals in excess of 5%/smaller than -5%

- **Quadratic and Inverse Triangular** (Panels B and C Figure 3.5): both of these are symmetric but bimodal; could provide a very rough proxy for the bimodalities in some of the empirical histograms in Figures 3.3 Panel F or 3.4 Panel B;

- **Uniform and Triangular** (Panels D and E Figure 3.5): the triangular could proxy for Panel D in Figure 3.4 while the uniform – bar the few outliers – captures some features of Panels E/F in Chart 3.3;

- **Irwin Hall (3) and (5)** (Panel F, Figure 3.5 and Panel A, Figure 3.6): named after Irwin (1927) and Hall (1927) these are distributions of normed sums of three and five uniform random variables respectively; they are symmetric, bounded and as the number of summed components increase, they approach the normal density;

- **Gamma 2, 5 and 10** (Panels B,C and D, Figure 3.6): these are distributions of normed sums of 2, 5, and 10 exponential random variables; they also converge to the normal as the number of summed components increases but they are all somewhat skewed to the left (median $<$ mean) in line with Panel B in Chart 3.3;

- **Normal** (Panel E, Figure 3.6): while the Gaussian distribution is not bounded it is included here for reference;

- **Rare Events Distribution** (Panel F, Figure 3.6): inspired by Panel D in Charts 3.3 and 3.4 this is a mixture of two Irwin Hall (5) distributions with an explicitly modelled thick left tail;

To create a level playing field, we shall subsequently impose the assumption that shocks are mean zero. We also note that all of the distributions reported above – with the exception of the 'Rare Events' distribution – are parameterized by a single coefficient (which maps 1-1 into variance), so, in a loose sense, none of them will have an unfair 'fit' advantage at the estimation stage.

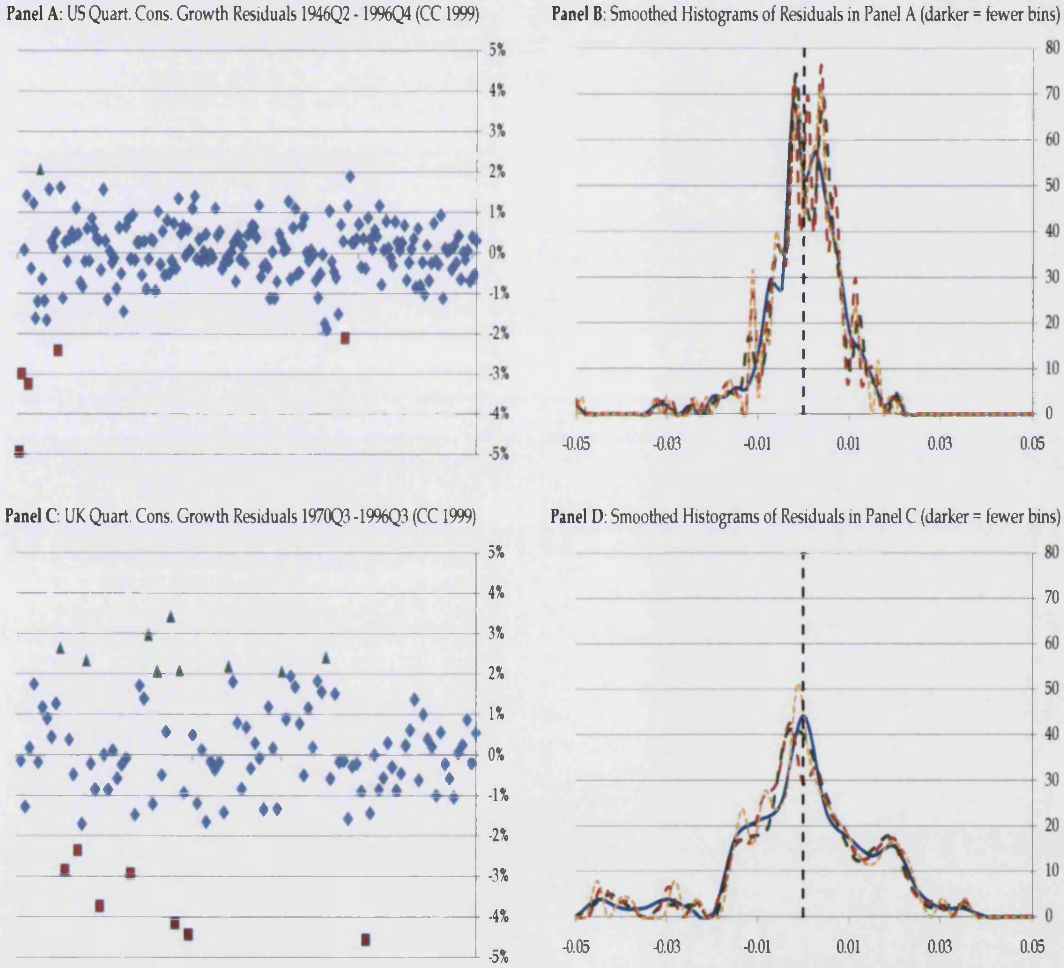

Figure 3.4: Overview of Quarterly Residuals and Empirical Distributions; Green Triangles/Red Squares denote residuals in excess of 5%/smaller than −5%

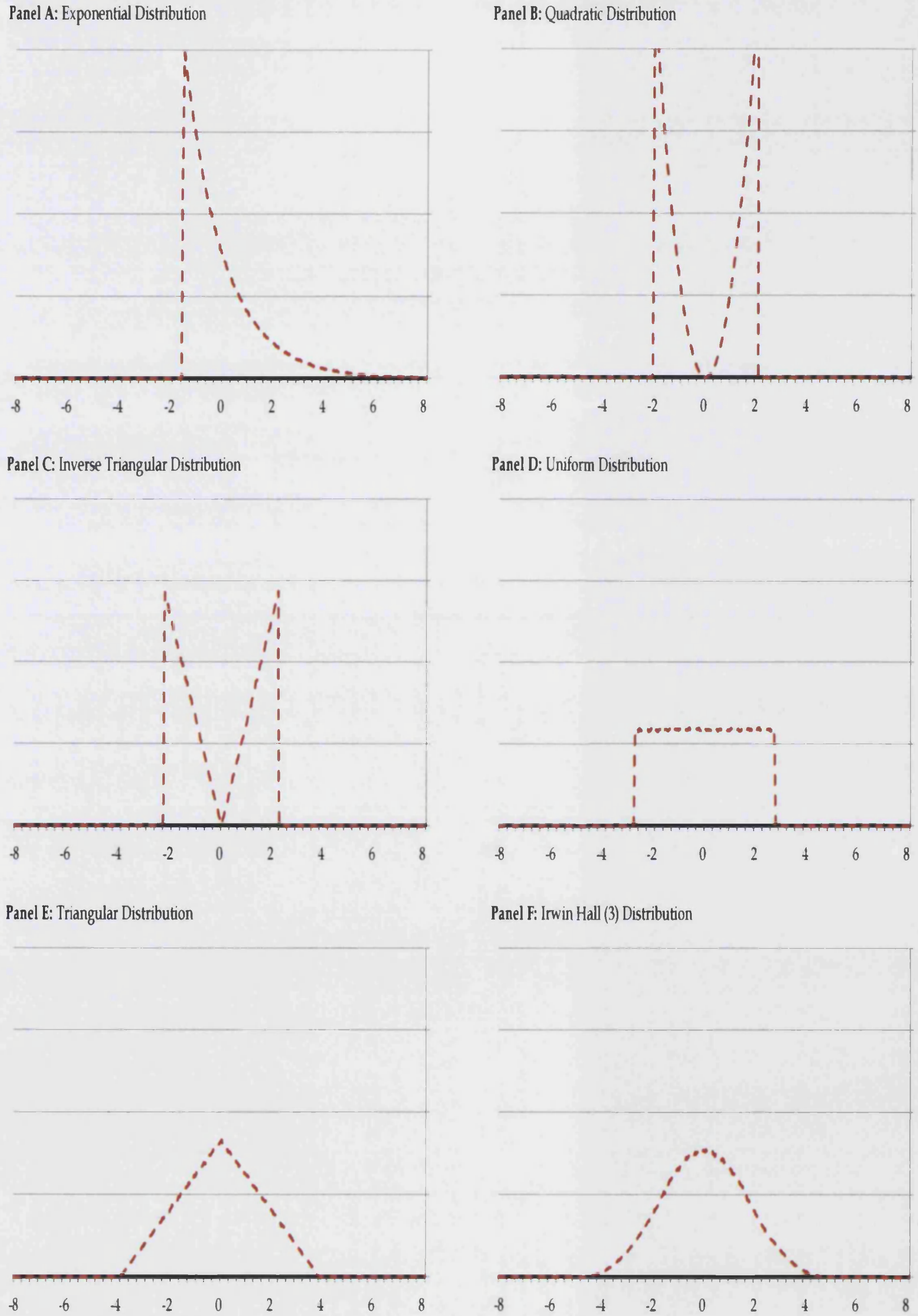

Figure 3.5: Overview of Distributions Analyzed (I)

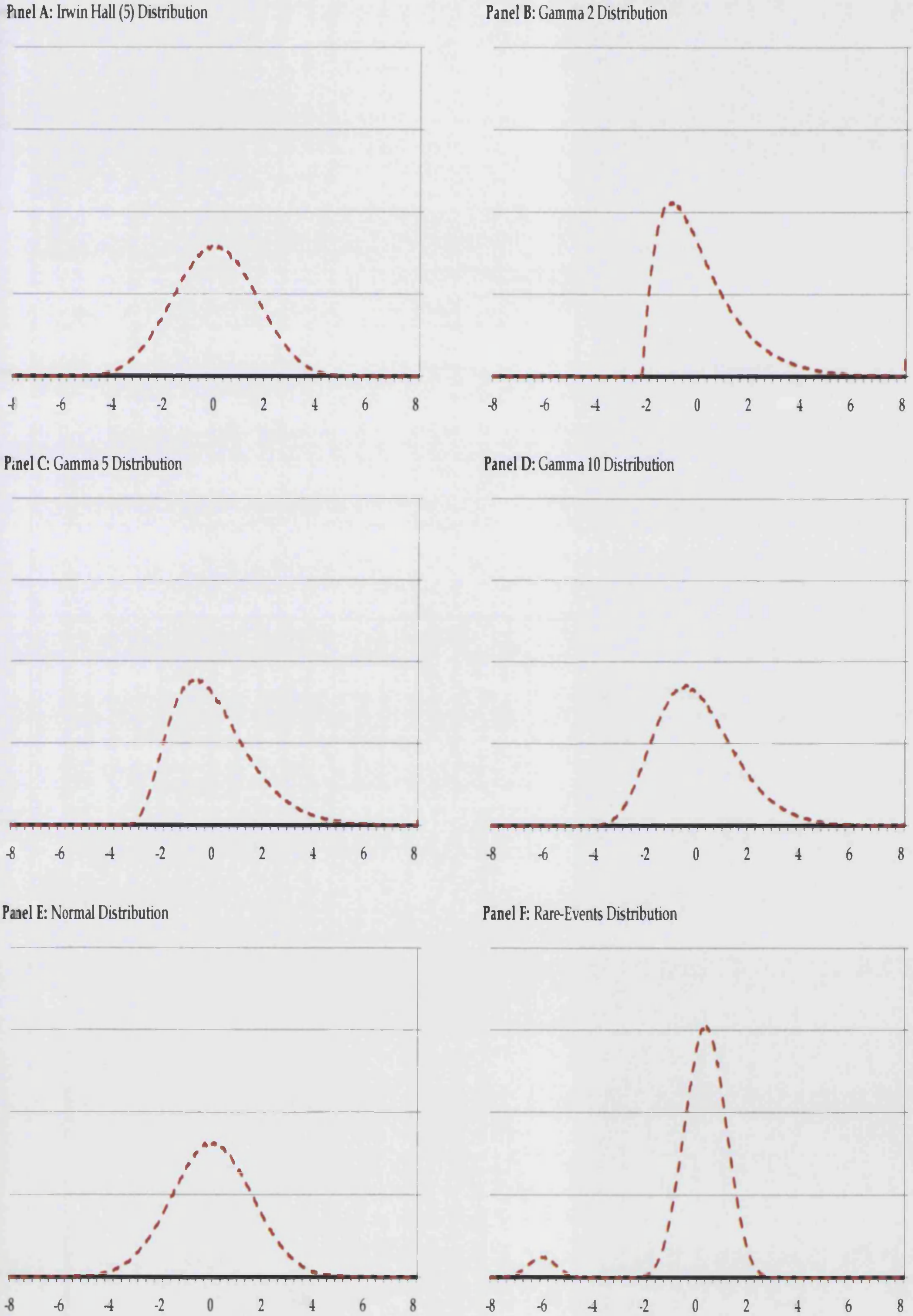

Figure 3.6: Overview of Distributions Analyzed (II)

Distribution	Density	Laplace Transform
Exponential	$\lambda \cdot \exp(-\lambda x)$	$(1+t/\lambda)^{-1}$
Quadratic	$(3x^2)/(2\lambda^3) \cdot \mathbf{1}_{[-\lambda,\lambda]}(x)$	$3\big(\exp(\lambda t)-\exp(-\lambda t)\big)\big(\frac{1}{(\lambda t)^3}+\frac{1}{2\lambda t}\big)-3\big(\exp(\lambda t)+\exp(-\lambda t)\big)\frac{1}{(\lambda t)^2}$
Inv. Triangular	$\lvert x\rvert/\lambda^2 \cdot \mathbf{1}_{[-\lambda,\lambda]}(x)$	$\exp(-\lambda t)\big(\exp(\lambda t)-1\big)\big(1-\exp(\lambda t)+\lambda t(1+\exp(\lambda t))\big)/(\lambda^2 t^2)$
Uniform	$1/\lambda \cdot \mathbf{1}_{[-\lambda/2,\lambda/2]}(x)$	$\big(\exp(\lambda t/2)-\exp(-\lambda t/2)\big)/(\lambda t)$
Triangular	$(\lambda-\lvert x\rvert)/\lambda^2 \cdot \mathbf{1}_{[-\lambda,\lambda]}(x)$	$\exp(-\lambda t)\big(\exp(\lambda t)-1\big)^2/(\lambda t)^2$
Irwin Hall (3)	$\begin{cases} (3\lambda+2x)^2/(8\lambda^3)\cdot & \mathbf{1}_{[-3/2\lambda,-1/2\lambda]}(x) \\ (3/(4\lambda)-x^2/\lambda^3)\cdot & \mathbf{1}_{[-1/2\lambda,\ 1/2\lambda]}(x) \\ (3\lambda-2x)^2/(8\lambda^3)\cdot & \mathbf{1}_{[\ 1/2\lambda,\ 3/2\lambda]}(x) \end{cases}$	$\big(\exp(\lambda t/2)-\exp(-\lambda t/2)\big)^3/(\lambda t)^3$
Irwin Hall (5)	$\begin{cases} (5\lambda+2x)^4/(384\lambda^5)\cdot & \mathbf{1}_{[-5/2\lambda,-3/2\lambda]}(x) \\ -\big(-55\lambda^4+20\lambda^3x+120\lambda^2x^2+80\lambda x^3+16x^4\big)/(96\lambda^5)\cdot & \mathbf{1}_{[-3/2\lambda,-1/2\lambda]}(x) \\ (115\lambda^4-120\lambda^2x^2+48x^4)/(192\lambda^5)\cdot & \mathbf{1}_{[-1/2\lambda,\ 1/2\lambda]}(x) \\ \big(55\lambda^4+20\lambda^3x-120\lambda^2x^2+80\lambda x^3-16x^4\big)/(96\lambda^5)\cdot & \mathbf{1}_{[\ 1/2\lambda,\ 3/2\lambda]}(x) \\ (5\lambda-2x)^4/(384\lambda^5)\cdot & \mathbf{1}_{[\ 3/2\lambda,\ 5/2\lambda]}(x) \end{cases}$	$\big(\exp(\lambda t/2)-\exp(-\lambda t/2)\big)^5/(\lambda t)^5$
Gamma 2	$x\exp(-x/\lambda)/(\Gamma(2)\lambda^2)\mathbf{1}_{[0,+\infty)}(x)$	$(1+\lambda t)^{-2}\mathbf{1}_{[-\infty,1/\lambda]}(t)$
Gamma 5	$x^4\exp(-x/\lambda)/(\Gamma(5)\lambda^5)\mathbf{1}_{[0,+\infty)}(x)$	$(1+\lambda t)^{-5}\mathbf{1}_{[-\infty,1/\lambda]}(t)$
Gamma 10	$x^9\exp(-x/\lambda)/(\Gamma(10)\lambda^{10})\mathbf{1}_{[0,+\infty)}(x)$	$(1+\lambda t)^{-10}\mathbf{1}_{[-\infty,1/\lambda]}(t)$
Normal	$\exp\big(-x/(2\lambda^2)\big)$	$\exp(\sigma^2t^2/2)$
Rare Event	$u_1 g_{IH(5)}(x-\mu)+(1-u_1)g_{IH(5)}(x+\mu u_1/(1-u_1))$	$u_1\exp(t\mu)\mathcal{L}_{IH5}(t,\sigma_1^2)+(1-u_1)\exp(-t\mu u_1/(1-u_1))\mathcal{L}_{IH5}(t,\sigma_2^2)$

Table 3.1: Overview of Distributions Used; Actual Gamma Distributions Were Centered to Ensure the Mean of $\xi = 0$; For Reference, the Variances of All These Distributions Equal Respectively: $1/\lambda^2$, $(3\lambda^2)/5, \lambda^2/2$, $\lambda^2/12, \lambda^2/6$, $\lambda^2/3$, $5\lambda^2/12$, $2\lambda^2$, $5\lambda^2$, $10\lambda^2$, and $\lambda^2, u_1\sigma_1^2+(1-u1)\sigma_2^2+\mu^2u_1/(1-u_1)$

3.5.2 Estimating Consumption Growth Parameters

Now that we have narrowed down our choice of distributions for consumption growth to twelve, we discuss the details of the estimation process. In total our model has seven free parameters. We divide them into 'consumption growth' parameters: μ, ρ and σ^2 and 'asset pricing' parameters: γ, ϕ, h and β. To explicitly account for the distributional assumptions we estimate these coefficients by maximizing the value of the log-likelihood corresponding to the particular shock distributions and observed data.

In principle, having chosen the shock distribution, we could try to estimate all of these coefficients simultaneously. This runs into 'singularity' problems, however. As formula (3.16) makes clear, consumption growth x_t and the habit/consumption ratio z_t uniquely pin down the level of any asset price. Accordingly, unless we found a combination of parameters such that the observed equity and bond returns were simultaneously functions of observed consumption growth and the habit/consumption ratio, the log-likelihood would be minus infinity. Given that in each sample we have more than one hundred consumption growth observations but only seven parameters, the probability of finding a combination of coefficients under which the observed paths of equity and bond returns conditional on the observed path of consumption growth were not probability zero events would be zero.

To circumvent this issue we follow a two stage procedure. In the first step we use maximum likelihood together with our distributional assumptions to estimate μ, ρ and σ^2. In the second step we find the remaining parameters using GMM as done in Boldrin et al. (2001) and using the closed form solutions for asset prices derived earlier to compute implied returns. Notably, due to the dependency of asset pricing formulae on the Laplace transforms of the underlying shocks to growth (and because the second stage of the estimation is conditional on the estimated $\hat{\mu}, \hat{\rho}$ and $\hat{\sigma}^2$) the distribution assumed has an impact during both stages of estimation.

Since we don't know the properties of the log-likelihood corresponding to the two data sets – from now on we only rely on quarterly US (1946 Q2 – 1996 Q4) and UK (1970 Q3 – 1996 Q3) consumption and asset pricing data underlying Campbell and Cochrane (1999) – to eliminate the possible impact of starting conditions and Matlab's optimizer performance – for each distribution and each

	Consumption Parameters			Asset Price Parameters			
Distribution	μ	ρ	σ2 (*1000)	γ	φ	h	β
1 Exponential	0.41%	1.00	2.21				
2 Quadratic	-4.52%	0.15	2.28	5.55	0.37	0.02	0.78
3 Inv. Triangular	-2.53%	0.15	1.00	3.85	0.72	0.04	0.91
4 Uniform	-1.34%	0.15	1.59	1.24	0.27	0.16	0.98
5 Triangular	-0.64%	0.49	0.32	3.52	0.58	0.54	0.97
6 IH 3	-0.23%	0.36	0.23	2.83	0.55	0.56	0.98
7 IH 5	0.13%	0.30	0.16	35.97	0.10	0.33	0.85
8 Gamma 2	0.41%	1.00	1.12				
9 Gamma 5	0.42%	0.56	0.49	2.78	0.44	0.70	0.99
10 Gamma 10	0.41%	0.39	0.27	8.62	0.49	0.50	1.00
11 Normal	0.41%	0.23	0.08	35.00	0.05	0.35	1.05
12 GMM	**0.41%**	**0.23**	**0.08**				

Table 3.2: Overview of Final Parameter Estimates – US Data Set

	Consumption Parameters			Asset Price Parameters			
Distribution	μ	ρ	σ2 (*1000)	γ	φ	h	β
1 Exponential	0.52%	0.07	1.99	5.39	0.11	0.26	0.98
2 Quadratic	-9.91%	0.26	7.30	0.00	0.90	0.49	1.06
3 Inv. Triangular	-3.02%	0.26	1.74	0.00	0.21	0.34	1.00
4 Uniform	-0.07%	0.07	2.05	0.00	0.52	0.03	0.98
5 Triangular	0.18%	0.01	0.31	41.78	0.11	0.03	0.81
6 IH 3	0.36%	0.00	0.26	23.77	0.36	0.07	1.00
7 IH 5	0.46%	-0.02	0.23	34.84	0.02	0.06	1.00
8 Gamma 2	0.52%	0.04	1.05	16.51	0.13	0.09	0.95
9 Gamma 5	0.52%	-0.02	0.51	10.72	0.09	0.23	1.01
10 Gamma 10	0.52%	-0.04	0.36	14.16	0.35	0.17	1.02
11 Normal	0.52%	-0.02	0.22	14.09	0.28	0.25	1.03
12 GMM	**0.52%**	**-0.02**	**0.22**				

Table 3.3: Overview of Final Parameter Estimates – UK Data Set

data set we used 1 000 000 different (admissible) random starting points.[28] The final estimated values of parameters (both 'consumption growth' and 'asset pricing' discussed subsequently) are reported in Table 3.2 for the US and Table 3.3 for the UK. Charts 3.7 – 3.12 then plot one step ahead consumption growth forecasts derived under different distributional assumptions.

Inspecting the first three columns of Tables 3.2 and 3.3 reveals that imposing distributional assumptions which differ too much from the empirical ones plotted in Charts 3.3 – 3.4 – either because they are too asymmetric (exponential), bimodal (quadratic or inverse triangular) or because the probability mass is distributed too 'uniformly' (uniform) – produces estimates which are very counterintuitive.

[28]We also explicitly restricted ρ to lie in the 'stable' $[-1, 1]$ interval, imposed the assumption that the variance is non-negative and additionally imposed the finite k-th order moment condition (3.15).

For example, if one was convinced that the true shocks are uniform then that implies ML estimates of mean quarterly consumption growth of -1.34% in the US and -0.07% in the UK – significantly different from the sample means of 0.41% and 0.52%. Effectively what is happening, is that mean consumption growth μ, persistence ρ and variance σ^2 are chosen so as to make the resulting consumption growth residuals look as close to 'uniform' (or 'exponential' or 'quadratic') as possible. As made clear by the table, sometimes that can lead to massive differences from classical method of moment estimates. The full absurdity of the resulting estimates is perhaps best seen in Charts 3.7 and 3.10 where the one step ahead forecasts corresponding to say the quadratic distribution never intersect with the actual data (very high values of in-sample root mean square errors (RMSE) and very low values of the log-likelihood reported in Table 3.7 are other manifestations of the same phenomenon).[29]

[29]There are other pathologies associated with the ML estimates. For example, one can show formally that the ML estimate of ρ corresponding to the exponential distribution is always equal to one – irrespective of the data (or more precisely that the first order condition associated with the likelihood maximization problem is always satisfied for $\rho = 1$; sometimes $\rho = 1$ might be outside the range of admissible parameters – as is the case in the UK sample; our estimates of ρ based on US data and the Gamma 2 distribution also converge to one – though stop marginally short of it due to the imposed stability constraints – which could reflect a similar issue). Clearly, an integrated $I(1)$ process for consumption growth implies that consumption is $I(2)$ and that asset prices explode (which is also why we don't estimate or report 'asset pricing' parameters corresponding to these distributions in Table 3.2).

Annualized Risk Premium

	Mean		SD	
Distribution	UK	US	UK	US
1 Exponential	9.63%		0.09%	
2 Quadratic	-23.47%	1.86%	0.00%	0.00%
3 Inv. Triangular	0.00%	1.10%	0.00%	0.00%
4 Uniform	6.73%	1.11%	0.00%	0.00%
5 Triangular	6.89%	2.37%	0.04%	0.02%
6 IH 3	7.16%	2.07%	0.04%	0.01%
7 IH 5	8.19%	5.82%	0.05%	0.46%
8 Gamma 2	8.88%		0.06%	
9 Gamma 5	9.40%	7.78%	0.10%	0.12%
10 Gamma 10	9.31%	7.58%	0.09%	0.07%
11 Normal	8.99%	7.29%	0.12%	0.20%

Annualized Real Bond Returns

	Mean		SD	
Distribution	UK	US	UK	US
1 Exponential	1.21%		0.73%	
2 Quadratic	1.19%	-0.38%	0.00%	0.60%
3 Inv. Triangular	1.09%	-0.38%	0.00%	0.36%
4 Uniform	1.21%	-0.39%	0.00%	0.12%
5 Triangular	1.21%	-0.39%	0.82%	1.33%
6 IH 3	1.21%	-0.40%	1.31%	0.84%
7 IH 5	1.21%	-0.33%	1.06%	11.21%
8 Gamma 2	1.21%		0.94%	
9 Gamma 5	1.21%	-0.40%	1.04%	2.29%
10 Gamma 10	1.21%	-0.40%	2.84%	2.70%
11 Normal	1.21%	-0.35%	3.36%	8.83%
12 Data	**1.21%**	**-0.40%**	**1.21%**	**0.86%**

Summary Statistics of Consumption Fit

	RMSE		Likelihood	
Distribution	UK	US	UK	US
1 Exponential	0.15	0.16	219.37	415.40
2 Quadratic	0.81	0.61	193.52	479.49
3 Inv. Triangular	0.31	0.38	203.67	501.57
4 Uniform	0.16	0.25	264.76	539.60
5 Triangular	0.15	0.15	285.14	583.43
6 IH 3	0.15	0.14	287.92	610.94
7 IH 5	0.15	0.13	289.85	639.16
8 Gamma 2	0.15	0.16	245.22	482.15
9 Gamma 5	0.15	0.13	268.52	556.03
10 Gamma 10	0.15	0.13	278.30	600.84
11 Normal	0.15	0.12	291.37	671.88
12 GMM	**0.15**	**0.12**	-	-

Annualized Excess Returns

	Mean		SD	
Distribution	UK	US	UK	US
1 Exponential	8.66%		2.68%	
2 Quadratic	14.88%	8.07%	2.12%	0.66%
3 Inv. Triangular	13.70%	8.07%	2.12%	0.37%
4 Uniform	8.66%	8.07%	1.64%	0.80%
5 Triangular	8.66%	8.07%	1.51%	1.27%
6 IH 3	8.66%	8.08%	4.94%	2.37%
7 IH 5	8.66%	8.06%	4.83%	14.48%
8 Gamma 2	8.66%		1.85%	
9 Gamma 5	8.66%	8.08%	6.18%	2.10%
10 Gamma 10	8.66%	8.08%	8.60%	2.84%
11 Normal	8.66%	8.09%	10.61%	9.55%
12 Data	**8.66%**	**8.00%**	**12.07%**	**7.66%**

Annualized Equity Returns

	Mean		SD	
Distribution	UK	US	UK	US
1 Exponential	9.87%		3.22%	
2 Quadratic	16.08%	7.69%	2.12%	0.66%
3 Inv. Triangular	14.80%	7.69%	2.12%	0.63%
4 Uniform	9.87%	7.69%	1.64%	0.91%
5 Triangular	9.87%	7.68%	2.16%	2.00%
6 IH 3	9.87%	7.68%	4.13%	2.76%
7 IH 5	9.87%	7.73%	3.81%	11.32%
8 Gamma 2	9.87%		2.56%	
9 Gamma 5	9.87%	7.68%	5.57%	3.82%
10 Gamma 10	9.87%	7.67%	6.54%	3.86%
11 Normal	9.87%	7.74%	8.56%	9.28%
12 Data	**9.87%**	**7.60%**	**11.89%**	**7.75%**

Annualized Real Consumption Growth

	MEAN		SD	
Distribution	UK	US	UK	US
1 Exponential	2.07%	1.86%	4.62%	
2 Quadratic	-28.96%	-15.15%	8.83%	4.83%
3 Inv. Triangular	-8.47%	-8.34%	4.19%	3.19%
4 Uniform	-0.11%	-4.30%	4.53%	4.04%
5 Triangular	0.75%	-0.38%	1.77%	2.04%
6 IH 3	1.46%	0.09%	1.62%	1.63%
7 IH 5	1.84%	0.92%	1.52%	1.32%
8 Gamma 2	2.06%	1.86%	3.23%	
9 Gamma 5	2.06%	1.77%	2.26%	2.68%
10 Gamma 10	2.06%	1.73%	1.89%	1.78%
11 Normal	2.06%	1.70%	1.47%	0.89%
12 Data	**2.04%**	**1.83%**	**1.49%**	**0.77%**

Table 3.4: Estimated Model Properties – by Distribution of Shock Driving Consumption Growth

In light of the above, assuming exponential, quadratic, inverse triangular or uniform distributions is unlikely to constitute a fruitful avenue, as the underlying consumption process would be very hard to defend. We stress also, that while such models may be capable of generating large and volatile risk premia, these will, at least in part, reflect unrealistically large consumption growth volatility, as the associated consumption growth variance exceeds the GMM estimate by more than an order of magnitude (stacking the odds in these models' favor). So, while for completeness (and where available), we report the asset pricing properties of these specifications, we do not discuss them in any great detail.

While some of the other distributional assumptions for shocks suffered from related types of issues – the triangular and Irwin Hall 3 distributions being borderline cases – at least the associated variance estimates were not massively out of line, making it instructive to scrutinize these models' asset pricing performance.

Clearly, however, of the underlying shock distributions considered, the normal assumption seems to work best for consumption growth – as summarized by the highest likelihood values (see also Table 3.4) in both the US and UK sample. While a Gaussian consumption growth driving process is hard to square with the external habit assumption, our results suggest that in terms of obtaining the best fit to consumption growth data, the chosen replacement should be unimodal and should not be too asymmetric. Finite normed sums of either uniform or exponential random variables seem like good alternatives as they are bounded and display reasonable properties. Importantly – and reflecting these conclusions – one probably needs to consider normed sums of at least five to ten such variables as fit tends to improve in the number of summed components.

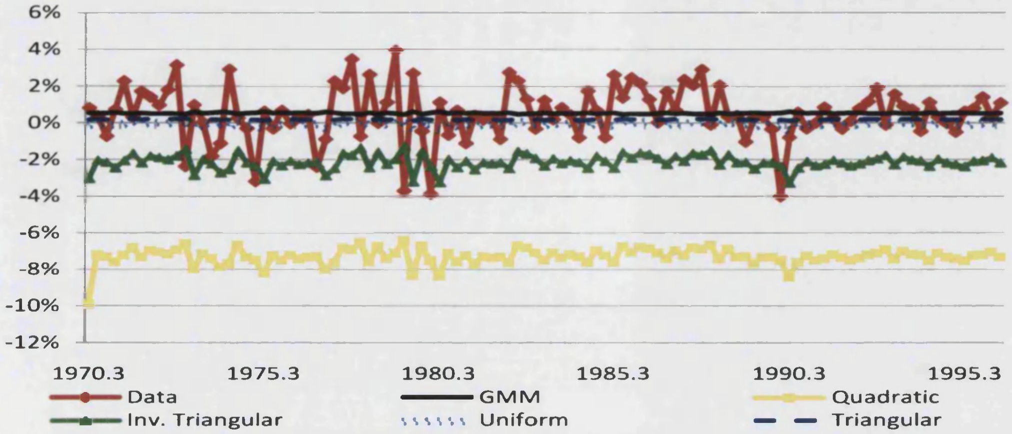

Figure 3.7: One Step Ahead UK Consumption Growth Forecasts (I)

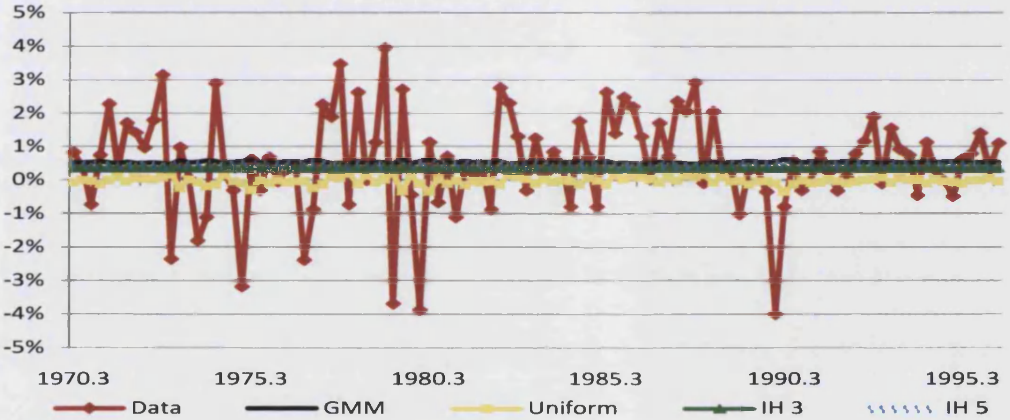

Figure 3.8: One Step Ahead UK Consumption Growth Forecasts (II)

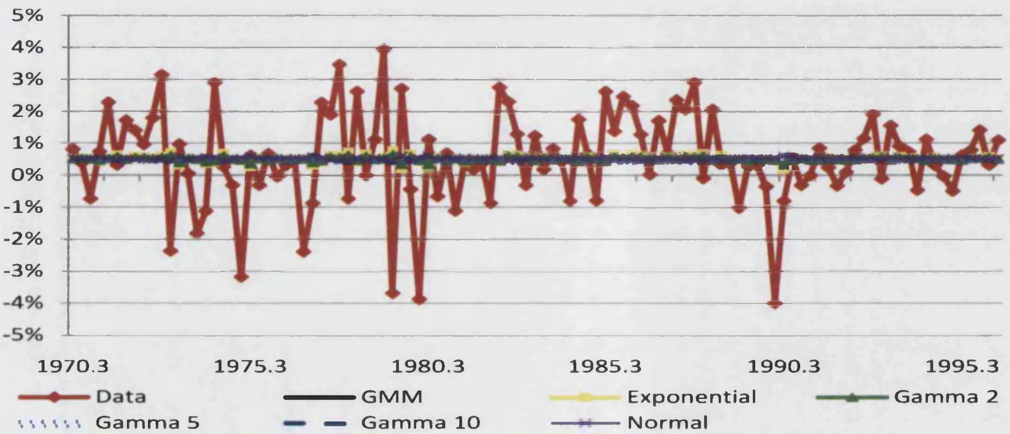

Figure 3.9: One Step Ahead UK Consumption Growth Forecasts (III)

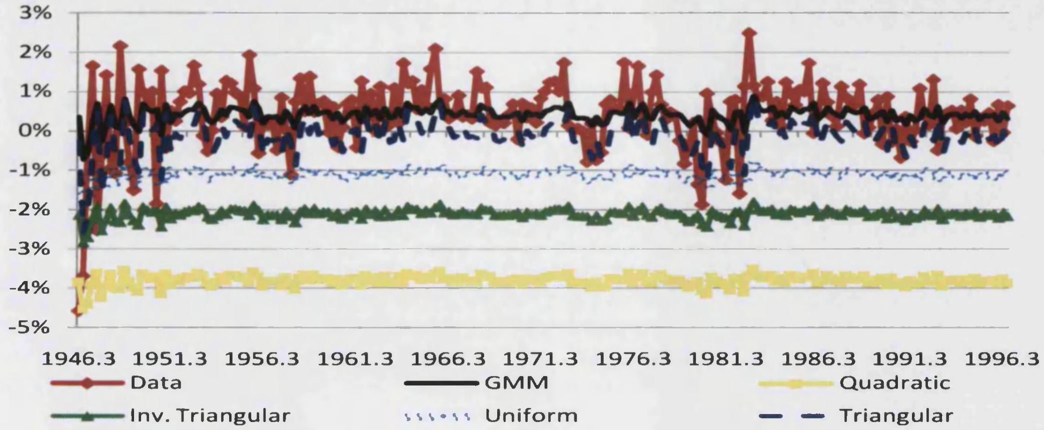

Figure 3.10: One Step Ahead US Consumption Growth Forecasts (I)

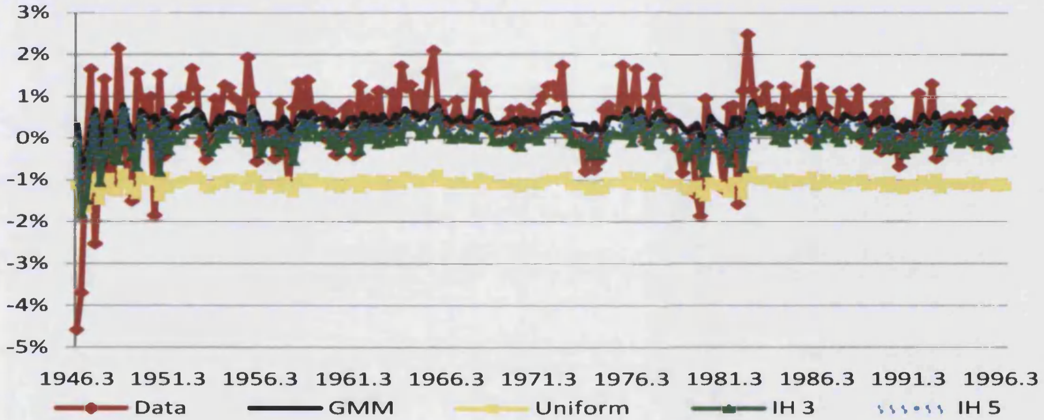

Figure 3.11: One Step Ahead US Consumption Growth Forecasts (II)

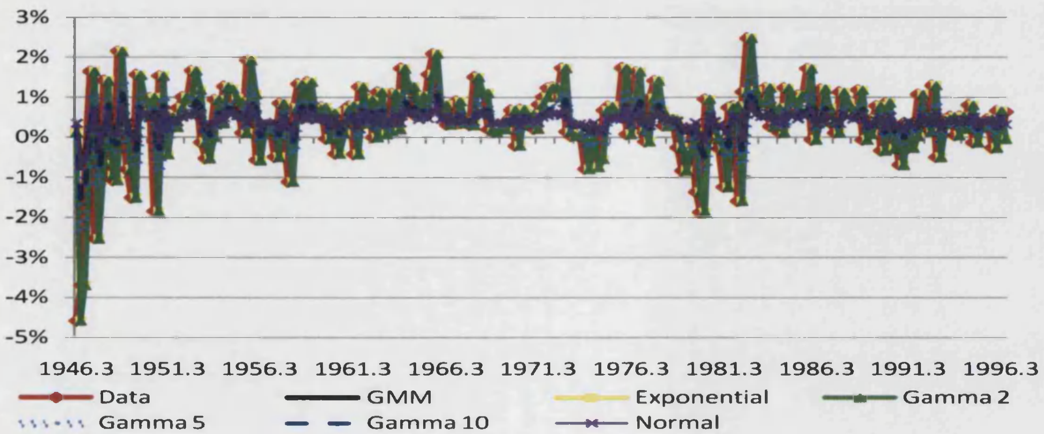

Figure 3.12: One Step Ahead US Consumption Growth Forecasts (III)

3.5.3 What Consumption Disasters?

Initially the purpose of this study was to investigate the interplay of habits are 'rare event' / 'consumption disaster' type mechanisms and to document the extent to which moderate amounts of 'tail-risks' in consumption growth allow one to reduce the role played by habits (while still retaining desirable asset pricing properties). Unfortunately, the fact that our analysis was data-driven made such an exercise impossible. In short, despite the use of several alternative data sets (we tried both the quarterly and annual data sets including those underlying Barro and Ursua (2008)) and alternative ways of trying to account for 'rare events' the estimates always converged to a 'no-disaster' consumption growth specification.

For example, while it is very difficult to embed elaborate mechanisms like those considered in Barro (2006) or Barro et al. (2009) in our framework, one way in which we could try to account for disasters is by means of a (bounded) shock distribution like that in Figure 3.6, Panel F. There, shocks are assumed to come from a mixture of two $IH(5)$ distributions – with most of the probability mass in a large, fairly central hump, and the remainder in a small tail hump – mean to reflect 'normal' and 'disaster' periods respectively. Clearly, if the weight put on the tail is a free parameter that specification nests the simple IH(5) one. For all the data sets considered, ML estimates conditional on this distribution robustly returned a weight on the small hump of zero.[30]

Hence, taken at face value, our results in this section yield support to those of Julliard and Ghosh (2009) and make it clear why a complex Bayesian framework, and rather heroic cross-country priors (Barro et al., 2009) are necessary to find evidence of 'rare events' in actual data.

3.5.4 Estimating Asset Pricing Parameters

Conditional on the estimates of μ, ρ and σ^2 we now estimated values of $(\gamma, \phi, h$ and $\beta)$. In particular, denoting time-series of actual bond and equity returns by r^b, r^{eq} and those implied by the model conditional on the actual time-series of consumption growth (the whole path matters when habits are persistent) by $\hat{r}^b(\gamma, \phi, h, \beta; x)$ and $\hat{r}^{eq}(\gamma, \phi, h, \beta; x)$ we followed Boldrin et al. (2001) and

[30]This could reflect the fact that the data are fairly symmetric and the bimodal 'rare event' distribution is not – something we have not pursued further.

minimized

$$\min_{\gamma,\phi,h,\beta} M(\gamma,\phi,h,\beta) \cdot \left(\hat{\text{VAR}}([r^b, r^{eq}])\right)^{-1} \cdot M(\gamma,\phi,h,\beta)'$$

where

$$M(\gamma,\phi,h,\beta) \equiv \left[\hat{\text{E}}\left(r^b - \hat{r}^b(\gamma,\phi,h,\beta;x)\right), \hat{\text{E}}\left(r^{eq} - \hat{r}^{eq}(\gamma,\phi,h,\beta;x)\right)\right]$$

and where the hatted expectation operator and variance simply denote their empirical equivalents. In short, we seek for (γ, ϕ, h and β) which ensure that our models match the mean level of excess returns and penalize deviations by taking into account differences in the variances of bond and equity returns (i.e. deviations in bond returns are penalized more heavily).

To compute bond and equity returns corresponding to (γ, ϕ, h, β) we used the exact formulae derived in previous sections – i.e. those characterized in Theorem 1 and Theorem 3. Two issues arose. The first one was how to compute the (log) habit / consumption ratio. Here we simply assumed that at the start of the sample z_t is at the steady state and allowed five periods of data (which were subsequently truncated) for z_t to converge to its 'true' level. The second issue was how to truncate the nested infinite sums reported in Theorems 1 and 3. The problem we faced was that the convergence criteria might depend on both the distribution used, sample data and the unknown parameters which we were trying to find. We experimented with many different cut-off criteria – inspecting the speed accuracy trade-off before settling on a rule-of-thumb that work well for all the distributions considered – i.e. to compute each equity return, we summed 1 500 000 terms ($i = 3000, n = 500$ in the asset pricing formulae presented).[31] Again, given the unknown properties of the minimized function and Matlab's optimizer we randomly choose 500 000 starting points for each distribution and sample.

The estimated values of coefficients found using this procedure are reported in Tables 3.2 and 3.3 for the US and UK respectively. The corresponding properties of model implied bond prices, equity prices and risk premia (based on simulated samples of 5 000 000 points) can be found in Table 3.4 and the respective series implied by various distributions and sets of estimated parameters are plotted

[31]The only restrictions we imposed were those ensuring that utility is well-defined and guaranteeing that the resulting model is stable and generates finite unconditional moments of the price dividend ratio.

in Figures 3.13 – 3.24 in the Appendix. Tables 3.5 – 3.9 also in the Appendix zoom in on asset pricing moments, while Figures 3.25 – 3.37 compare the joint distributions of equity returns and consumption growth implied by the model to that found in the data.

Inspecting these results, we see that all the asset pricing models estimated conditional on plausible consumption growth processes can match the level of mean bond returns and mean equity returns almost exactly (up to 10bp). Importantly, however – and despite too volatile underlying consumption growth – all of them do not generate sufficiently volatile equity returns and tend to undershoot bond return volatility – particularly in the models estimated on US data.[32] One finding which is of interest, and robust to the exact specification, is that the standard deviation of the equity risk premium is around two orders of magnitude smaller than that of excess returns (the means are roughly in line).

The mechanisms which the most successful specifications rely on to fit asset pricing data seem sample-specific. In particular, estimates based on UK data show very low values of habit size (no-greater than 0.25 and typically smaller) and small values of habit persistence (≤ 0.3). To generate the large UK equity premium, the models favor the well-trodden path of very high risk aversion – greater than ten and typically around 15 – and high values of β – frequently in excess of 1 (Kocherlakota (1996) provides a lucid account of why the combination works). A similar combination also works for some of the models estimated on US data. There however, two of the models based on gamma distributions work better when risk aversion and habit persistence are moderate (3 – 9 and 0.44 – 0.49 respectively) and when they are combined with a high weight of habits in the utility function (0.5 – 0.7).

Summing all this information up: while the equity premium puzzle can be accounted for within this framework, this is often accomplished by relying on high risk aversion and high values of the discount factor β. The very high values of habit size / persistence crucial for the story of Campbell and Cochrane (1999) to work, do not seem to endogenously emerge from the estimation process – most likely due to the excessive risk-free rate volatility they would imply. In fact, our estimates point to the tension between matching equity return volatility – which our models typically undershoot, despite consumption growth volatility

[32]The model implied Sharpe ratios were below those found in the data while at the same time, price dividend ratios were more volatile. In sample, higher values of distribution skew were associated with higher Sharpe ratio volatility.

often in excess of that in the data – and matching bond return volatility which they frequently overshoot.

3.6 Conclusions

In this paper we derive closed-form solutions for the equity price-dividend ratio and equity risk-premium in a model in which agents have difference-form external habits. The setup allows for arbitrary shock distributions, correlated consumption growth and nests extensions of the keeping-up and catching-up with the Joneses specifications. We use the exact solutions to study the ability of alternative estimated models – including one capturing rare events – to simultaneously account for consumption, equity and bond returns in the UK and US.

Our results highlight tensions which exist when one tries to simultaneously match the properties of consumption growth, bond returns and equity returns. While the habit specification proposed by Campbell and Cochrane (1999) is capable of cutting the umbilical-cord linking all three, the same cannot be achieved with the more parsimonious model which this papers solves in closed-form. We document the dimensions in which the model fails and argue that more exotic shock distributions are unlikely to solve the problems outlined here. However, explicitly accounting for differences between consumption and dividends would provide for an interesting extension and could potentially help tackle some of the problems highlighted here.

3.A Appendix

Proof of Lemma 1. Assume that $z_t \in [z_l, z_h]$, then $z_{t+1} = (1-\phi)z_t - x_{t+1} \leq (1-\phi)z_h - x_l$ and $z_{t+1} \geq (1-\phi)z_l - x_h$. But

$$(1-\phi)z_h - x_l \leq z_h, \qquad (1-\phi)z_l - x_h \leq z_l \tag{3.26}$$

and therefore $z_l \leq z_{t+1} \leq z_h$ as required. To show that both inequalities in (3.26) hold, note that

$$\begin{aligned}(1-\phi)z_h - x_l \leq z_h &\Leftrightarrow -\phi z_h \leq x_l \Leftrightarrow z_h \geq -\frac{x_l}{\phi}\\ (1-\phi)z_l - x_h \geq z_l &\Leftrightarrow -\phi z_l \geq x_h \Leftrightarrow z_l \leq -\frac{x_h}{\phi}. \ \blacksquare\end{aligned}$$

Proof of Lemma 2. If $x_t \in [x_l, x_h]$, then

$$x_{t+1} \leq \rho x_h + (1-\rho)\mu + \xi_h, \qquad x_{t+1} \geq \rho x_l + (1-\rho)\mu + \xi_l.$$

But $\rho x_h + (1-\rho)\mu + \xi_h \leq x_h \Leftrightarrow x_h \geq \mu + \xi_h/(1-\rho)$ and $\rho x_l + (1-\rho)\mu + \xi_l \geq x_l \Leftrightarrow x_l \leq \mu + \xi_l/(1-\rho)$. ■

Proof of Lemma 3. Combining equations (3.7) – (3.8) and simplifying gives the desired inequality: $x_l \leq (\rho\xi_h + \xi_l)/(1-\rho^2) + \mu$. To prove that $[x_l, x_l/\rho - (1-\rho)/\rho\mu - \xi_l/\rho]$ is invariant for x_t, note that if $x_l \leq x_t \leq x_h$ then $x_{t+1} = \rho x_t + (1-\rho)\mu + \xi_{t+1} \leq \rho x_h + (1-\rho)\mu + \xi_h \leq x_h$ by assumption. Similarly, we show that $x_{t+1} \geq x_l$ if $x_t \in [x_l, x_h]$, which completes the proof. ■

Proof of Theorem 1. Iterating forward on equation (3.14) – i.e. substituting in for v_{t+1} which satisfies a similar equation, we obtain

$$\begin{aligned} v_t =& \mathrm{E}_t\beta\left(\frac{1-h\exp(z_{t+1})}{1-h\exp(z_t)}\right)^{-\gamma}\cdot\exp\big((1-\gamma)\,x_{t+1}\big) + \mathrm{E}_t\beta^2\Bigg\{\left(\frac{1-h\exp(z_{t+1})}{1-h\exp(z_t)}\right)^{-\gamma}\\ &\cdot\exp\big((1-\gamma)\,x_{t+1}\big)\cdot\mathrm{E}_{t+1}\left(\frac{1-h\exp(z_{t+2})}{1-h\exp(z_{t+1})}\right)^{-\gamma}\cdot\exp\big((1-\gamma)\,x_{t+2}\big)(1+v_{t+2})\Bigg\}\\ =& \mathrm{E}_t\beta\left(\frac{1-h\exp(z_{t+1})}{1-h\exp(z_t)}\right)^{-\gamma}\cdot\exp\big((1-\gamma)\,x_{t+1}\big)\\ &+\mathrm{E}_t\beta^2\left(\frac{1-h\exp(z_{t+2})}{1-h\exp(z_t)}\right)^{-\gamma}\cdot\exp\big((1-\gamma)(x_{t+1}+x_{t+2})\big)(1+v_{t+2}) \end{aligned}$$

where we have relied on the fact that for any random variable Y

$$\beta^2 \cdot \left(\frac{1 - h\exp(z_{t+1})}{1 - h\exp(z_t)}\right)^{-\gamma} \cdot \exp\big((1-\gamma)\, x_{t+1}\big) \cdot \mathrm{E}_{t+1} Y$$
$$= \mathrm{E}_{t+1}\left\{\beta^2 \cdot \left(\frac{1 - h\exp(z_{t+1})}{1 - h\exp(z_t)}\right)^{-\gamma} \cdot \exp\big((1-\gamma)\, x_{t+1}\big) \cdot Y\right\} \quad (3.27)$$

and then used the law of iterated expectations to replace $\mathrm{E}_t \mathrm{E}_{t+1} Y = \mathrm{E}_t Y$. We can repeat this procedure and, invoking the transversality condition, write

$$v_t = \sum_{i=1}^{+\infty} \mathrm{E}_t \beta^i \left(\frac{1 - h\exp(z_{t+i})}{1 - h\exp(z_t)}\right)^{-\gamma} \cdot \exp\Big((1-\gamma)\sum_{j=1}^{i} x_{t+j}\Big)$$
$$= \big(1 - h\exp(z_t)\big)^{\gamma} \sum_{i=1}^{+\infty} \mathrm{E}_t \beta^i \big(1 - h\exp(z_{t+i})\big)^{-\gamma} \cdot \exp\Big((1-\gamma)\sum_{j=1}^{i} x_{t+j}\Big). \quad (3.28)$$

We first investigate when the infinite series in (3.28) converges. Since x satisfies (3.4) therefore, as in Burnside (1998), we can express x_{t+s} in terms of the 'primitive' shocks ξ and the $t-$measurable variable x_t as

$$x_{t+s} - \mu = \xi_{t+s} + \rho\,\xi_{t+s-1} + \ldots + \rho^{s-1}\,\xi_{t+1} + \rho^s\,(x_t - \mu)$$

which implies that

$$\sum_{j=1}^{i} x_{t+j} = i\mu + \big(x_t - \mu\big)\frac{\rho(1-\rho^i)}{(1-\rho)}$$
$$+ (1-\rho)^{-1}\big(\xi_{t+1}(1-\rho^i) + \xi_{t+2}(1-\rho^{i-1}) + \ldots + \xi_{t+i}(1-\rho)\big). \quad (3.29)$$

Plugging this into equation (3.28) we obtain that

$$v_t = \big(1 - h\exp(z_t)\big)^{\gamma} \sum_{i=1}^{+\infty} \beta^i \exp\left((1-\gamma)\cdot\Big(i\mu + \big(x_t - \mu\big)\frac{\rho(1-\rho^i)}{(1-\rho)}\Big)\right)$$
$$\cdot\, \mathrm{E}_t \big(1 - h\exp(z_{t+i})\big)^{-\gamma} \cdot \exp\left(\frac{(1-\gamma)}{(1-\rho)}\big(\xi_{t+1}(1-\rho^i) + \ldots + \xi_{t+i}(1-\rho)\big)\right).$$

Observe that

$$\exists\,\delta:\ \ 1 \leq (1 - h\exp(z_{t+i}))^{-\gamma} \leq \delta^{-\gamma},\ \ i = 0, 1, 2, \ldots \quad (3.30)$$

where the first inequality is trivially true and the second holds as a consequence of assumption (Aδ). Accordingly, denoting

$$\mathcal{K}(x_t, z_t) := \big(1 - h\exp(z_t)\big)^{\gamma} \sum_{i=1}^{+\infty} \left[\beta^i \exp\left(\left(1-\gamma\right) \cdot \left(i\mu + \left(x_t - \mu\right)\frac{\rho(1-\rho^i)}{(1-\rho)}\right)\right) \cdot \mathrm{E}_t \exp\left(\frac{(1-\gamma)}{(1-\rho)}\left(\xi_{t+1}(1-\rho^i) + \ldots + \xi_{t+i}(1-\rho)\right)\right)\right]. \tag{3.31}$$

we can exploit the observation that all terms in the infinite sum in (3.31) are non-negative and then use (3.30) to write

$$\mathcal{K}(x_t, z_t) \leq v_t \leq \delta^{-\gamma} \cdot \mathcal{K}(x_t, z_t). \tag{3.32}$$

Since all ξ_{t+j}'s are identically distributed and independent of $\mathcal{F}_t$, we can replace the conditional expectation operator in (3.31) with the unconditional one and then substitute in for the Laplace transform of ξ, $\mathcal{L}(\cdot)$

$$\mathcal{K}(x_t, z_t) = \big(1 - h\exp(z_t)\big)^{\gamma} \cdot \sum_{i=1}^{+\infty} \left[\beta^i \exp\left(\left(1-\gamma\right) \cdot \left(i\mu + \left(x_t - \mu\right)\frac{\rho(1-\rho^i)}{(1-\rho)}\right)\right) \cdot \prod_{j=1}^{i} \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right)\right]. \tag{3.33}$$

Inequality (3.32) is important. It shows that v_t is finite if and only if $\mathcal{K}(x_t, z_t)$ is finite. Accordingly, to establish when v_t is finite it suffices to focus on $\mathcal{K}(x_t, z_t)$. To verify when (3.33) converges, we can use the ratio test. Denoting the term in the square brackets by w_i we have that

$$\begin{aligned}\lim_{i\to+\infty}\left|\frac{w_{i+1}}{w_i}\right| &= \lim_{i\to+\infty} \beta\exp\big((1-\gamma)\left[\mu + (x-\mu)\rho^{i+1}\right]\big)\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^{i+1})\right)\\ &= \beta\exp\big((1-\gamma)\,\mu\big)\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right)\end{aligned}$$

where we exploited the fact that $\rho < 1 \Rightarrow \rho^i \to 0$. Thus, directly from the ratio test, $\mathcal{K}(x_t, z_t)$ – and accordingly also v_t – is finite if

$$\beta\exp\big((1-\gamma)\,\mu\big)\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right) < 1$$

and it is infinite when the expression is greater than one.

Limiting Case

Since the test is inconclusive when $\beta \exp\big((1-\gamma)\,\mu\big)\mathcal{L}\big((\gamma-1)/(1-\rho)\big) = 1$, we now analyze this case separately. First, rewrite $\mathcal{K}(x_t, z_t)$ as

$$\mathcal{K}(x_t, z_t) = \big(1 - h\exp(z_t)\big)^{\gamma} \cdot \sum_{i=1}^{+\infty} \Bigg[\left\{ \beta \exp\big((1-\gamma)\cdot\mu\big) \cdot \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right) \right\}^{i} \cdot \exp\!\left((1-\gamma)\,(x_t-\mu)\frac{\rho(1-\rho^i)}{(1-\rho)} \right) \cdot \prod_{j=1}^{i} \left\{ \frac{\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right)}{\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right)} \right\} \Bigg]. \tag{3.34}$$

Given that $\beta \exp\big((1-\gamma)\,\mu\big)\mathcal{L}\big((\gamma-1)/(1-\rho)\big) = 1$, therefore

$$\mathcal{K}(x_t, z_t) = \big(1 - h\exp(z_t)\big)^{\gamma} \cdot \sum_{i=1}^{+\infty} \Bigg[\exp\!\left((1-\gamma)\,(x_t-\mu)\frac{\rho(1-\rho^i)}{(1-\rho)} \right) \cdot \prod_{j=1}^{i} \left\{ \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right) / \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right) \right\} \Bigg]. \tag{3.35}$$

Since $\lim_{i\to+\infty}(1-\gamma)(x_t-\mu)\rho(1-\rho^i)/(1-\rho) = (1-\gamma)(x_t-\mu)\rho/(1-\rho)$ therefore, to show that $\mathcal{K}(x_t, z_t)$ is infinite, it is enough to show that

$$\sum_{i=1}^{+\infty} \prod_{j=1}^{i} \left\{ \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right) / \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right) \right\} = +\infty. \tag{3.36}$$

To show that the sum (3.36) diverges it suffices to demonstrate that its elements converge to a constant $b > 0$. To show that, we can use the fact that if $\forall j$: $0 \le a_j < 1$ then $\prod_{j=1}^{+\infty}(1-a_j)$ is convergent and strictly positive if and only if $\sum_{j=1}^{+\infty} a_j < +\infty$. To apply this fact to (3.36) define a_j as

$$a_j := 1 - \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right) / \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right)$$

and note that since the denominator $\mathcal{L}\big((\gamma-1)/(1-\rho)\big)$ is a constant

$$\sum_{j=1}^{+\infty} \left| \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right) - \mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right) \right| < +\infty \Rightarrow \sum_{j=1}^{+\infty} a_j < +\infty.$$

Given that the Laplace transform is differentiable the mean value theorem implies that $\exists\, \zeta_j \in \big[(\gamma-1)(1-\rho^j)/(1-\rho), (\gamma-1)/(1-\rho)\big]$ such that

$$\left|\mathcal{L}\Big(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\Big)-\mathcal{L}\Big(\frac{(\gamma-1)}{(1-\rho)}\Big)\right| = |\mathcal{L}'(\zeta_j)|\cdot\left|\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)-\frac{(\gamma-1)}{(1-\rho)}\right|.$$

Accordingly

$$\sum_{j=1}^{+\infty}\left|\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right)-\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right)\right|$$
$$=\left|\frac{(\gamma-1)}{(1-\rho)}\right|\cdot\sum_{j=1}^{+\infty}\left|\mathcal{L}'(\xi_j)\right|\cdot\left|(1-\rho^j)-1\right| \le \rho\cdot\left|(\gamma-1)\right|\cdot\left|\max_{j\ge1}\mathcal{L}'(\xi_j)\right|.$$

Since $(\gamma-1)(1-\rho^j)/(1-\rho)$ is increasing in j and equals $(\gamma-1)$ for $j=1$, therefore $\max_{j\ge1}\mathcal{L}'(\xi_j)\le\max_{\lambda\in[(\gamma-1),(\gamma-1)/(1-\rho)]}\left|\mathcal{L}'(\lambda)\right|$. Accordingly

$$\max_{\lambda\in[(\gamma-1),(\gamma-1)/(1-\rho)]}\left|\mathcal{L}'(\lambda)\right|<+\infty\Rightarrow\sum_{j=1}^{+\infty}\left|\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right)-\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right)\right|<+\infty.$$

But $\mathcal{L}'(\lambda) = -\mathrm{E}(\exp(-\lambda\xi)\cdot\xi)$ and $\forall\lambda>0, \exists\bar{\lambda}\in(0,\lambda) : \exp(-\lambda\xi)\cdot\xi = \exp(-(\lambda-\bar{\lambda})\,\xi)\cdot\exp(-\bar{\lambda}\,\xi)\cdot\xi$. Given assumption (A$\delta$), $\exists\kappa\ge0 : \exp(-\bar{\lambda}\,\xi)\cdot\xi\le\kappa$ and so $\exp(-\lambda\xi)\cdot\xi\le\kappa\cdot\exp(-(\lambda-\bar{\lambda})\,\xi)\Rightarrow|\mathrm{E}(\exp(-\lambda\xi)\cdot\xi)|\le\kappa\cdot\mathcal{L}(\lambda-\bar{\lambda})$. Continuity of the Laplace transform $\mathcal{L}$ therefore implies that for any bounded set $\mathcal{A}$: $\max_{\lambda\in\mathcal{A}}\mathcal{L}'(\lambda)\le\bar{\kappa}\max_{\lambda\in\mathcal{A}}\max_{\bar{\lambda}\in(0,\lambda)}\mathcal{L}(\lambda-\bar{\lambda})\le+\infty$ – where the last inequality follows from (Aδ) which implies that ξ is bounded from below and so $\mathcal{L}(\cdot)$ is finite for positive arguments. Hence $\max_{\lambda\in\mathcal{A}}|\mathcal{L}'(\lambda)|<+\infty\Rightarrow\sum_{j=1}^{+\infty}a_j=\rho\cdot|(\gamma-1)|\cdot\max_{\lambda\in\mathcal{A}}|\mathcal{L}'(\lambda)|<+\infty\Rightarrow\prod_{j=1}^{+\infty}(1-a_j)=b>0\Rightarrow\sum_{i=1}^{+\infty}\prod_{j=1}^{i}\left\{\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right)/\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}\right)\right\}=+\infty$ i.e. $\mathcal{K}(x_t,z_t)<+\infty$ if and only if $\beta\exp\big((1-\gamma)\,\mu\big)\cdot\mathcal{L}\big((\gamma-1)/(1-\rho)\big)<1$.

Having found necessary and sufficient conditions under which formula (3.28) converges, we now transform its right-hand side to derive an expression for the price dividend ratio v_t only in terms of the state variables x_t, z_t, model parameters and the Laplace transform of shocks to consumption growth ξ. To proceed, we shall use the following relationship

$$\big(1-h\exp(z_{t+i})\big)^{-\gamma}=\sum_{n=0}^{+\infty}\left(\begin{array}{c}-\gamma\\ n\end{array}\right)\big(-h\exp(z_{t+i})\big)^{n} \tag{3.37}$$

where generalized binomial coefficients are given in Definition 4 and where the right-hand side converges for $h\exp(z_{t+i}) < 1$. Clearly, assumption (Aδ) $\Rightarrow$ $h\exp(z_{t+i}) < 1$ and so under (Aδ) equation (3.37) is always satisfied (and all series converge under condition (3.15)). Accordingly, plugging (3.37) back into (3.28) we have that

$$v_t = \big(1-h\exp(z_t)\big)^\gamma \sum_{i=1}^{+\infty} \mathrm{E}_t\beta^i \sum_{n=0}^{+\infty} \binom{-\gamma}{n} \big(-h\exp(z_{t+i})\big)^n \exp\Big((1-\gamma)\sum_{j=1}^{i} x_{t+j}\Big)$$
$$= \big(1-h\exp(z_t)\big)^\gamma \sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} \beta^i(-h)^n \binom{-\gamma}{n} \mathrm{E}_t \exp\Big(nz_{t+i} + (1-\gamma)\sum_{j=1}^{i} x_{t+j}\Big).$$

Plugging equation (3.29) into the expression for v_t gives

$$v_t = \big(1-h\exp(z_t)\big)^\gamma \sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} \beta^i(-h)^n \binom{-\gamma}{n} \mathrm{E}_t\Bigg[\exp\Bigg((1-\gamma)\Bigg\{i\mu$$
$$+ (x_t-\mu)\frac{\rho(1-\rho^i)}{(1-\rho)} + (1-\rho)^{-1}\big(\xi_{t+1}(1-\rho^i) + \ldots + \xi_{t+i}(1-\rho)\big)\Bigg\} + nz_{t+i}\Bigg)\Bigg].$$

Since equation (3.5) implies that $z_{t+i} = (1-\phi)z_{t+i-1} - x_{t+i}$ therefore

$$z_{t+i} = -\sum_{s=0}^{i-1}(1-\phi)^s x_{t+i-s} + (1-\phi)^i z_t. \tag{3.38}$$

To express $\sum_{s=0}^{i-1}(1-\phi)^s x_{t+i-s}$ in terms of the underlying shocks, we can plug in for x_{t+1} and write the sum as

$$\begin{array}{ll}
(1-\phi)^{i-1}\big(\mu + \rho\ (x_t-\mu) + \xi_{t+1}\big) + \\
(1-\phi)^{i-2}\big(\mu + \rho^2\ (x_t-\mu) + \rho\ \xi_{t+1} + \xi_{t+2}\big) + \\
(1-\phi)^{i-3}\big(\mu + \rho^3\ (x_t-\mu) + \rho^2\ \xi_{t+1} + \rho\ \xi_{t+2} + \xi_{t+3}\big) + \\
\cdots\cdots\cdots\cdots\cdots\cdots \\
(1-\phi)\ \big(\mu + \rho^{i-1}(x_t-\mu) + \rho^{i-2}\xi_{t+1} + \rho^{i-3}\xi_{t+2} + \ldots + \xi_{t+i-1}\big) + \\
\qquad\quad \big(\mu + \rho^i\ (x_t-\mu) + \rho^{i-1}\xi_{t+1} + \rho^{i-2}\xi_{t+2} + \ldots + \rho\,\xi_{t+i-1} + \xi_{t+i}\big).
\end{array}$$

Summing terms up, we can compute the coefficients on μ, $(x_t-\mu), \xi_{t+1}, \ldots, \xi_{t+i}$

$$
\begin{aligned}
\mu: &\qquad \left(1+(1-\phi)+\ldots+(1-\phi)^{i-1}\right)=\frac{1-(1-\phi)^i}{1-(1-\phi)} \\
(x_t-\mu): &\qquad \rho^i\left(1+\frac{(1-\phi)}{\rho}+\ldots+\frac{(1-\phi)^{i-1}}{\rho^{i-1}}\right)=\frac{1-(1-\phi)^i/\rho^i}{1-(1-\phi)/\rho}\rho^i \\
\xi_{t+1}: &\qquad \rho^{i-1}\left(1+\frac{(1-\phi)}{\rho}+\ldots+\frac{(1-\phi)^{i-1}}{\rho^{i-1}}\right)=\frac{1-(1-\phi)^i/\rho^i}{1-(1-\phi)/\rho}\rho^{i-1} \\
\xi_{t+2}: &\qquad \rho^{i-2}\left(1+\frac{(1-\phi)}{\rho}+\ldots+\frac{(1-\phi)^{i-2}}{\rho^{i-2}}\right)=\frac{1-(1-\phi)^{i-1}/\rho^{i-1}}{1-(1-\phi)/\rho}\rho^{i-2} \\
\ldots &\qquad \ldots \\
\xi_{t+i-1}: &\qquad \rho\left(1+\frac{(1-\phi)}{\rho}\right)=\frac{1-(1-\phi)^2/\rho^2}{1-(1-\phi)/\rho}\rho
\end{aligned}
$$

and 1 for ξ_{t+i}, so that after simplifying, we obtain

$$
\begin{aligned}
\sum_{s=0}^{i-1}(1-\phi)^s x_{t+i-s} = {}& \mu\frac{1-(1-\phi)^i}{\phi}+(x_t-\mu)\rho\frac{\rho^i-(1-\phi)^i}{\rho-(1-\phi)}+\xi_{t+1}\frac{\rho^i-(1-\phi)^i}{\rho-(1-\phi)} \\
&+\xi_{t+2}\frac{\rho^{i-1}-(1-\phi)^{i-1}}{\rho-(1-\phi)}+\ldots+\xi_{t+i-1}\frac{\rho^2-(1-\phi)^2}{\rho-(1-\phi)}+\xi_{t+i}. \qquad (3.39)
\end{aligned}
$$

Accordingly, since $nz_{t+i} = -n\sum_{s=0}^{i-1}(1-\phi)^s x_{t+i-s}+n(1-\phi)^i z_t$, we can plug (3.39) into the expression for v_t

$$
\begin{aligned}
v_t = {}& \left(1-h\exp(z_t)\right)^\gamma \sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty}\beta^i(-h)^n\begin{pmatrix}-\gamma\\ n\end{pmatrix}\mathbf{E}_t\Bigg[\exp\Bigg((1-\gamma)\Big\{i\mu \\
&+(x_t-\mu)\frac{\rho(1-\rho^i)}{(1-\rho)}+(1-\rho)^{-1}\left(\xi_{t+1}(1-\rho^i)+\ldots+\xi_{t+i}(1-\rho)\right)\Big\} \\
&-n\mu\frac{1-(1-\phi)^i}{\phi}-n(x_t-\mu)\rho\frac{\rho^i-(1-\phi)^i}{\rho-(1-\phi)}-n\xi_{t+1}\frac{\rho^i-(1-\phi)^i}{\rho-(1-\phi)} \\
&-n\xi_{t+2}\frac{\rho^{i-1}-(1-\phi)^{i-1}}{\rho-(1-\phi)}+\ldots-n\xi_{t+i-1}\frac{\rho^2-(1-\phi)^2}{\rho-(1-\phi)}-n\xi_{t+i}+n(1-\phi)^i z_t\Bigg)\Bigg]
\end{aligned}
$$

to yield after ordering and simplifying

$$
\begin{aligned}
v_t = & \left(1 - h\exp(z_t)\right)^{\gamma} \sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} \beta^i (-h)^n \binom{-\gamma}{n} \mathrm{E}_t\Bigg[\exp\Bigg(\mu\left\{(1-\gamma)i - n\frac{1-(1-\phi)^i}{\phi}\right\} \\
& + (x_t - \mu)\rho\left\{(1-\gamma)\frac{(1-\rho^i)}{(1-\rho)} + n\frac{\rho^i - (1-\phi)^i}{1-\rho-\phi}\right\} + \xi_{t+1}\left\{(1-\gamma)\frac{(1-\rho^i)}{(1-\rho)}\right. \\
& \left. + n\frac{\rho^i - (1-\phi)^i}{1-\rho-\phi}\right\} + \xi_{t+2}\left\{(1-\gamma)\frac{(1-\rho^{i-1})}{(1-\rho)} + n\frac{\rho^{i-1} - (1-\phi)^{i-1}}{1-\rho-\phi}\right\} + \xi_{t+i-1} \\
& \cdot \left\{(1-\gamma)\frac{(1-\rho^2)}{(1-\rho)} + n\frac{\rho^2 - (1-\phi)^2}{1-\rho-\phi}\right\} + \xi_{t+i}\left\{(1-\gamma) - n\right\} + n(1-\phi)^i z_t\Bigg)\Bigg].
\end{aligned}
$$

Since ξ is an *i.i.d.* process, therefore

$$
\mathrm{E}_t \exp\left(\sum_{j=1}^{i} \lambda_j \cdot \xi_{t+j}\right) = \prod_{j=1}^{i} \mathrm{E}\exp(\lambda_j \cdot \xi) = \prod_{j=1}^{i} \mathcal{L}(-\lambda_j) \tag{3.40}
$$

where ξ is a random variable drawn from the common distribution of all ξ's, λ_j are arbitrary constants. Note also that

$$
\begin{aligned}
(-h)^n \binom{-\gamma}{n} &= h^n (-1)^n \frac{(-\gamma-1+1)(-\gamma-2+1)\dots(-\gamma-n+1)}{n!} \\
&= h^n \frac{(\gamma-1+1)(\gamma-1+2)\dots(\gamma-1+n)}{n!} = h^n \binom{\gamma-1+n}{n}.
\end{aligned} \tag{3.41}
$$

Accordingly, exploiting both these identities we finally get

$$
\begin{aligned}
v_t = & \left(1 - h\exp(z_t)\right)^{\gamma} \sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} \Bigg[\beta^i h^n \binom{\gamma-1+n}{n} \\
& \cdot \exp\Bigg(n(1-\phi)^i z_t + (x_t-\mu)\rho\left\{(1-\gamma)\frac{(1-\rho^i)}{(1-\rho)} + n\frac{\rho^i-(1-\phi)^i}{1-\rho-\phi}\right\} \\
& + \mu\left\{(1-\gamma)i - n\frac{1-(1-\phi)^i}{\phi}\right\}\Bigg) \cdot \prod_{j=1}^{i} \mathcal{L}\left((\gamma-1)\frac{(1-\rho^j)}{(1-\rho)} + n\frac{(1-\phi)^j - \rho^j}{(1-\phi)-\rho}\right)\Bigg]
\end{aligned}
$$

which exactly matches formula (3.16) and so completes the proof. ■

Proof of Corollaries 1, 2 and 3. To prove corollaries 2 and 3 simply plug in $\phi = 1$ and $\rho = 0$ respectively into formula (3.16). To prove corollary 3, note that under the alternative / 'contemporaneous' specification $X_{t+1} = C_{t+1}^{\phi} X_t^{1-\phi}$

implies that $z_{t+1} = (1-\phi)(z_t - x_{t+1})$ which in turn yields

$$z_{t+i} = (1-\phi)(z_{t+i-1} - x_{t+i}) = -(1-\phi)\sum_{s=0}^{i-1}(1-\phi)^s x_{t+i-s} + (1-\phi)^i z_t.$$

Exploiting the above relationship, rather than equation (3.38), and following the steps in the proof of Theorem 2 gives the final formula. ■

Proof of Theorem 2. If (3.15) is violated, then the price dividend ratio diverges and so in particular all its unconditional moments are ill defined. This proves the 'only if' part of the theorem. To show that under (3.15) unconditional k-th order moments of v_t exist, note that equation (3.32) implies that

$$v_t \leq \delta^{-\gamma} \cdot \mathcal{K}(x_t, z_t) \tag{3.42}$$

where equation (3.33) implies that $\mathcal{K}(x_t, z_t)$ can be expressed as

$$\mathcal{K}(x_t, z_t) = \left(1 - h\exp(z_t)\right)^{\gamma} \cdot \sum_{i=1}^{+\infty} s_i(x_t)$$

$$s_i(\mathsf{x}) := \left[\beta^i \exp\left((1-\gamma)\cdot\left(i\mu + (\mathsf{x}-\mu)\frac{\rho(1-\rho^i)}{(1-\rho)}\right)\right)\cdot\prod_{j=1}^{i}\mathcal{L}\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right)\right].$$

Accordingly

$$v_t^k \leq \left(\frac{(1-h\exp(z_t))}{\delta}\right)^{k\gamma} \sum_{i_1}\cdots\sum_{i_k} s_{i_1}(x_t)\ldots s_{i_k}(x_t)$$
$$\Rightarrow \mathrm{E}v_t^k \leq \vartheta^k \sum_{i_1}\cdots\sum_{i_k} \mathrm{E}s_{i_1}(x_t)\ldots s_{i_k}(x_t)$$

where the constant $\vartheta := (1-h\exp(z_t))^{\gamma}/\delta^{\gamma}$. From the generalized Schwarz inequality we know that

$$\mathrm{E}s_{i_1}(x_t)\ldots s_{i_k}(x_t) \leq \left(\mathrm{E}s_{i_1}^k(x_t)\right)^{\frac{1}{k}}\ldots\left(\mathrm{E}s_{i_k}^k(x_t)\right)^{\frac{1}{k}}$$

and so, applying this inequality, we have that

$$\mathrm{E}v_t^k \leq \vartheta^k \sum_{i_1}\left(\mathrm{E}s_{i_1}^k(x_t)\right)^{\frac{1}{k}}\ldots\sum_{i_k}\left(\mathrm{E}s_{i_k}^k(x_t)\right)^{\frac{1}{k}} = \vartheta^k\left(\sum_{i=1}^{+\infty}\left(\mathrm{E}s_i^k(x_t)\right)^{\frac{1}{k}}\right)^k$$

where we have relied on the fact that $\sum_{i_j}\left(\mathrm{E}s_{i_j}^k(x_t)\right)^{\frac{1}{k}}$ are identically distributed random variables and replaced their product by the k-th power of its element. Accordingly

$$\sum_{i=1}^{+\infty}\left(\mathrm{E}s_i^k(x_t)\right)^{\frac{1}{k}} < +\infty \Rightarrow \mathrm{E}v_t^k < +\infty. \tag{3.43}$$

To complete the proof, we shall now show that the infinite sum on the left-hand side converges under assumption (3.15). To do that, note that

$$\left(\mathrm{E}s_i^k(x_t)\right)^{\frac{1}{k}} = \beta^i \cdot \exp\left((1-\gamma)\, i\mu\right) \cdot \left[\prod_{j=1}^{i}\mathcal{L}_\xi\left(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^j)\right)\right] \cdot \left[\mathrm{E}\exp\left(k\cdot(1-\gamma)\cdot(x_t-\mu)\cdot\frac{\rho(1-\rho^i)}{(1-\rho)}\right)\right]^{\frac{1}{k}}. \tag{3.44}$$

First we need to check that $\left(\mathrm{E}s_i^k(x_t)\right)^{\frac{1}{k}}$ is finite. Iterating on equation (3.4) we know that if $\lim_{i\to+\infty}\rho^i(x_{t-i}-\mu) =_{a.s} 0$ then[33]

$$x_t - \mu = \sum_{i=0}^{+\infty}\rho^i\,\xi_{t-i}. \tag{3.45}$$

Since ξ_s are *i.i.d.* mean zero and bounded from below (by assumption Aδ), therefore $\mathrm{E}|\xi_s| = m < +\infty$ and so[34]

$$\mathrm{E}\sum_{i=0}^{+\infty}|\rho^i\xi_{t-i}| = \sum_{i=0}^{+\infty}\rho^i\mathrm{E}|\xi_{t-i}| = \frac{m}{1-\rho} \leq +\infty \Rightarrow \sum_{i=0}^{+\infty}\rho^i\xi_{t-i} < +\infty \tag{3.46}$$

i.e. $x_t-\mu$ is a well-defined random variable, and crucially - one which is bounded from below. The latter implies that $x-\mu$'s Laplace transform is well-defined for all positive arguments, yielding

$$\mathrm{E}\exp\left(k\cdot(1-\gamma)\cdot(x_t-\mu)\cdot\frac{\rho(1-\rho^i)}{(1-\rho)}\right) = \mathcal{L}_{x-\mu}\left(k\cdot\frac{(\gamma-1)\rho(1-\rho^i)}{(1-\rho)}\right) < +\infty.$$

Having shown that the terms of the infinite series $\sum_{i=1}^{+\infty}(\mathrm{E}s_i^k(x_t))^{1/k}$ are all finite, we can now apply the ratio test to determine when the series itself converges.

[33]This condition obviously holds when x are drawn from a stationary distribution.

[34]Extending the proof to the case of ξ with finite, but non-zero mean is straightforward. Notably, the proof can also be extended to the case of shocks ξ s.t $\mathrm{E}\xi = +\infty$ though extra conditions are then necessary to ensure that $x_t-\mu$ is still well-defined.

From equation (3.44) we obtain

$$\lim_{i\to+\infty}\frac{(\mathrm{E}s_{i+1}^k(x_t))^{1/k}}{(\mathrm{E}s_i^k(x_t))^{1/k}}=\lim_{i\to+\infty}\Bigg[\beta\cdot\exp\big((1-\gamma)\mu\big)\cdot\mathcal{L}_\xi\Big(\frac{(\gamma-1)}{(1-\rho)}(1-\rho^{i+1})\Big)$$
$$\cdot\frac{\mathcal{L}_{x-\mu}\big(k\frac{(\gamma-1)\rho(1-\rho^{i+1})}{(1-\rho)}\big)}{\mathcal{L}_{x-\mu}\big(k\frac{(\gamma-1)\rho(1-\rho^{i})}{(1-\rho)}\big)}\Bigg]=\beta\cdot\exp\big((1-\gamma)\mu\big)\cdot\mathcal{L}_\xi\Big(\frac{(\gamma-1)}{(1-\rho)}\Big)\quad(3.47)$$

where in the last equality we have exploited the continuity of the Laplace transform $\mathcal{L}_{x-\mu}(\cdot)$. Accordingly, when

$$\beta\cdot\exp\big((1-\gamma)\mu\big)\cdot\mathcal{L}_\xi\Big(\frac{(\gamma-1)}{(1-\rho)}\Big)<1$$

then the ratio test is conclusive and $\sum_{i=1}^{+\infty}(\mathrm{E}s_i^k(x_t))^{1/k}$ converges. By virtue of equation (3.43) this ensures that $\mathrm{E}v_t^k<+\infty$ and so concludes the proof. ■

Proof of Theorem 3. Let P_t^b denote the price of a zero-coupon bond and let $r_{t+1}^b=1/P_t^b$ be the gross rate of return on bonds held over $[t,t+1]$. Then, directly from the first order conditions

$$P_t^b=\mathrm{E}_t\beta\left(\frac{1-h\,X_{t+1}/C_{t+1}}{1-h\,X_t/C_t}\right)^{-\gamma}\cdot\left(\frac{C_{t+1}}{C_t}\right)^{-\gamma}\Rightarrow$$
$$r_{t+1}^b=\big(1-h\exp(z_t)\big)^{-\gamma}\cdot\Big[\mathrm{E}_t\beta\big(1-h\exp(z_{t+1})\big)^{-\gamma}\exp(-\gamma\,x_{t+1})\Big]^{-1}.$$

Using (3.37) to express $\big(1-h\exp(z_{t+1})\big)^{-\gamma}$ as the infinite sum and plugging in for z_{t+1} from equation (3.3) we obtain

$$r_{t+1}^b=\left[\big(1-h\exp(z_t)\big)^{\gamma}\cdot\sum_{n=0}^{+\infty}\beta(-h)^n\begin{pmatrix}-\gamma\\ n\end{pmatrix}\mathrm{E}_t\exp\big(n(1-\phi)z_t-(\gamma+n)\,x_{t+1}\big)\right]^{-1}.$$

We can then use equation (3.4) to express x_{t+1} in terms of x_t and finally obtain

$$r_{t+1}^b=\big(1-h\exp(z_t)\big)^{-\gamma}\cdot\Bigg\{\sum_{n=0}^{+\infty}\Bigg[\beta(-h)^n\cdot\begin{pmatrix}-\gamma\\ n\end{pmatrix}\cdot\mathcal{L}(\gamma+n)$$
$$\cdot\exp\Big(n(1-\phi)z_t-\rho(\gamma+n)(x_t-\mu)-\mu(\gamma+n)\Big)\Bigg]\Bigg\}^{-1}$$

or, in terms of the symbols defined in Theorem 2

$$
\begin{aligned}
r^b_{t+1} =& \big(1-h\exp(z_t)\big)^{-\gamma} \\
&\cdot\left\{\sum_{n=0}^{+\infty} d_{1,n}\frac{\mathcal{L}(\gamma+n)}{\mathcal{L}(\gamma+n-1)}\exp\Big(a_{1,n}z_t+(b_{1,n}-\rho)(x_t-\mu)+(c_{1,n}-\mu)\Big)\right\}^{-1}
\end{aligned}
$$

with $(\gamma-1)+n\geq 0$ implying that both Laplace transforms are well-defined. Similarly, the realized rate of return on equity r^{eq}_{t+1} over $[t,t+1]$ is given by

$$
r^{eq}_{t+1}=\frac{P_{t+1}+C_{t+1}}{P_t}=\frac{P_{t+1}+C_{t+1}}{C_{t+1}}\frac{C_{t+1}}{C_t}\frac{C_t}{P_t}=\frac{1+v_{t+1}}{v_t}\exp(x_{t+1}).
$$

Accordingly

$$
\mathbf{E}_t r^{eq}_{t+1}=v_t^{-1}\big(\mathbf{E}_t\exp(x_{t+1})+\mathbf{E}_t v_{t+1}\exp(x_{t+1})\big).
$$

Of course, from equation (3.4), $\mathbf{E}_t\exp(x_{t+1})=\exp\big((1-\rho)\mu+\rho x_t\big)\cdot\mathcal{L}(-1)$ - i.e. a necessary condition for the risk premium to be well-defined is $\mathcal{L}(-1)<+\infty$. We can then compute $\mathbf{E}_t v_{t+1}\exp(x_{t+1})$. From (3.16) and (3.37) we find that

$$
\begin{aligned}
v_{t+1}\exp(x_{t+1}) =& \left(\sum_{m=0}^{+\infty}(-h)^m\binom{-\gamma}{m}\exp(mz_{t+1})\right) \\
&\cdot\left(\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} d_{i,n}\exp\Big(z_{t+1}\,a_{i,n}+(x_{t+1}-\mu)\,(b_{i,n}+1)+(c_{i,n}+\mu)\Big)\right).
\end{aligned}
$$

Using the fact that $z_{t+1}=(1-\phi)z_t-x_{t+1}$ we obtain

$$
\begin{aligned}
v_{t+1}\exp(x_{t+1}) =& \sum_{m=0}^{+\infty}\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty}(-h)^m\binom{-\gamma}{m}\cdot d_{i,n}\exp\Big(z_t(1-\phi)\,(a_{i,n}+m) \\
&+(x_{t+1}-\mu)\,\big(b_{i,n}+1-(a_{i,n}+m)\big)+\big(c_{i,n}+\mu(1-(a_{i,n}+m))\big)\Big)
\end{aligned}
$$

and so, since (3.4) implies that $x_{t+1}-\mu=\rho\,(x_t-\mu)+\xi_{t+1}$, we can then compute the conditional expectation of $v_{t+1}\exp(x_{t+1})$ as

$$\mathbf{E}_t v_{t+1}\exp(x_{t+1}) = \Bigg\{\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}$$
$$d_{i,n,m}\cdot\exp\Bigg(z_t\, a_{i,n,m} + \rho\,(x_t-\mu)\,(b_{i,n,m}+1) + (c_{i,n,m}+\mu)\Bigg)\cdot\mathcal{L}\big(-b_{i,n,m}-1\big)\Bigg\}$$

where

$$a_{i,n,m} = (1-\phi)\,(a_{i,n}+m) \qquad b_{i,n,m} = b_{i,n} - \frac{a_{i,n,m}}{(1-\phi)}$$
$$c_{i,n,m} = c_{i,n} - \frac{\mu\, a_{i,n,m}}{(1-\phi)} \qquad d_{i,n,m} = d_{i,n}\binom{-\gamma}{m}(-h)^m.$$

One question to answer is: when does the formula above converge - i.e. when is $\mathbf{E}|v_{t+1}\exp\big(x_{t+1}\big)| < +\infty$? Note that for $1 < \mathsf{p},\mathsf{q} < +\infty$ we get from the Hölder inequality that

$$\mathbf{E}_t v_{t+1}\exp\big(x_{t+1}\big) \le \big(\mathbf{E}_t v_{t+1}^{\mathsf{p}}\big)^{\frac{1}{\mathsf{p}}}\cdot\big(\mathbf{E}_t\exp(\mathsf{q}\cdot x_{t+1})\big)^{\frac{1}{\mathsf{q}}}. \tag{3.48}$$

Since, under the conditions derived earlier $\mathbf{E}v_{t+1}^{\mathsf{p}} < +\infty$ for arbitrary $\mathsf{p} > 1$, therefore to prove finiteness it suffices to show that there exists $\mathsf{q} > 1$ such that $\mathbf{E}\exp\big(\mathsf{q}x_{t+1}\big) < 1$. We show the latter for the exponential distribution with parameter λ. Note that similar proofs for other distributions could also be formulated.

$$\mathbf{E}\left(\exp\big(\mathsf{q}(\mathsf{p}x+\mu(1-\rho)+\xi)\big)\right) < +\infty \Leftrightarrow \mathbf{E}\exp(\mathsf{q}\xi) < +\infty \tag{3.49}$$

but

$$\mathbf{E}\big(\exp(\mathsf{q}\xi)\big) = \lambda\int_{-\frac{1}{\lambda}}^{+\infty}\exp\big(\mathsf{q}x - \lambda(x-\frac{1}{\lambda})\big)dx \sim \int_0^{+\infty}\exp\big((\mathsf{q}-\lambda)x\big)dx < +\infty$$

and so the integral is finite as long as $\mathsf{q} < \lambda$. Of course, we can always find $1 > \mathsf{q} < \lambda$ as long as $\lambda > 1$ - i.e. the proof follows for all $\lambda > 1\square$.

Combining all these bits of information

$$
\begin{aligned}
rp_t &= \mathbf{E}_t r^{eq}_{t+1} - r^b_{t+1} = v_t^{-1}\big(\mathbf{E}_t \exp(x_{t+1}) + \mathbf{E}_t v_{t+1}\exp(x_{t+1})\big) - r^b_{t+1} \\
&= \frac{\exp(\mu+\rho(x_t-\mu))}{(1-h\exp(z_t))^{\gamma}} \cdot \Bigg\{ \Bigg[\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} d_{i,n}\exp\Big(z_t\, a_{i,n} + (x_t-\mu)\, b_{i,n} + c_{i,n}\Big)\Bigg]^{-1}
\end{aligned}
$$

$$
\Bigg[\mathcal{L}(-1) + \sum_{n,m=0}^{+\infty}\sum_{i=1}^{+\infty} d_{i,n,m}\cdot\mathcal{L}(-b_{i,n,m}-1)\cdot\exp\Big(z_t\, a_{i,n,m} + \rho\,(x_t-\mu)\, b_{i,n,m} + c_{i,n,m}\Big)\Bigg]
$$

$$
- \Bigg[\sum_{n=0}^{+\infty} d_{1,n}\frac{\mathcal{L}(\gamma+n)}{\mathcal{L}(\gamma+n-1)}\exp\Big(z_t a_{1,n} + (x_t-\mu)\, b_{1,n} + c_{1,n}\Big)\Bigg]^{-1}\Bigg\}. \quad \blacksquare
$$

Notably under the alternative habit specification $z_{t+1} = (1-\phi)(z_t - x_{t+1})$ we get

$$
v_{t+1}\exp(x_{t+1}) = \sum_{m=0}^{+\infty}\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty}(-h)^m \binom{-\gamma}{m} \cdot d_{i,n}\exp\Big(z_t(1-\phi)\,(a_{i,n}+m)
$$

$$
+ (x_{t+1}-\mu)\,\big(b_{i,n} + (1-(1-\phi)(a_{i,n}+m))\big) + \big(c_{i,n} + \mu(1-(1-\phi)(a_{i,n}+m))\big)\Big)
$$

and so, since (3.4) implies that $x_{t+1}-\mu = \rho\,(x_t-\mu) + \xi_{t+1}$, we can then compute the conditional expectation of $v_{t+1}\exp(x_{t+1})$ as

$$
\mathbf{E}_t v_{t+1}\exp(x_{t+1}) = \Bigg\{\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty}\sum_{m=0}^{+\infty}
$$

$$
d_{i,n,m}\cdot\exp\Big(z_t\, a_{i,n,m} + \rho\,(x_t-\mu)\,(b_{i,n,m}+1) + (c_{i,n,m}+\mu)\Big)\cdot\mathcal{L}(-b_{i,n,m}-1)\Bigg\}
$$

where

$$
\begin{aligned}
a_{i,n,m} &= (1-\phi)\,(a_{i,n}+m) & \qquad b_{i,n,m} &= b_{i,n} - a_{i,n,m} \\
c_{i,n,m} &= c_{i,n} - \mu\, a_{i,n,m} & \qquad d_{i,n,m} &= d_{i,n}\binom{-\gamma}{m}(-h)^m.
\end{aligned}
$$

Combining all these bits of information

$$rp_t = \mathrm{E}_t r^{eq}_{t+1} - r^b_{t+1} = v_t^{-1}\big(\mathrm{E}_t \exp(x_{t+1}) + \mathrm{E}_t v_{t+1}\exp(x_{t+1})\big) - r^b_{t+1}$$
$$= \frac{\exp(\mu+\rho(x_t-\mu))}{(1-h\exp(z_t))^\gamma}\cdot\left\{\left[\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} d_{i,n}\exp\Big(z_t\,a_{i,n}+(x_t-\mu)\,b_{i,n}+c_{i,n}\Big)\right]^{-1}\right.$$
$$\left[\mathcal{L}(-1)+\sum_{m,n=0}^{+\infty}\sum_{i=1}^{+\infty} d_{i,n,m}\cdot\mathcal{L}(-b_{i,n,m}-1)\cdot\exp\Big(z_t\,a_{i,n,m}+\rho\,(x_t-\mu)\,b_{i,n,m}+c_{i,n,m}\Big)\right]$$
$$\left.-\left[\sum_{n=0}^{+\infty} d_{1,n}\frac{\mathcal{L}(\gamma+n)}{\mathcal{L}(\gamma+n-1)}\exp\Big(z_t a_{1,n}+(x_t-\mu)\,b_{1,n}+c_{1,n}\Big)\right]^{-1}\right\}. \quad \blacksquare$$

Proof of Corollary 5. If $\rho = 0$ and $\phi = 1$ then

$$\begin{aligned} a_{i,n} &:= 0 & c_{i,n} &:= \mu\big((1-\gamma)i-n\big) \\ b_{i,n} &:= 0 & d_{i,n} &:= \beta^i h^n \binom{\gamma-1+n}{n}\cdot\mathcal{L}((\gamma-1)+n)\cdot\big(\mathcal{L}(\gamma-1)\big)^{i-1} \end{aligned}$$

furthermore

$$\begin{aligned} a_{i,n,m} &= 0 & b_{i,n,m} &= b_{i,n} = 0 \\ c_{i,n,m} &= c_{i,n} & d_{i,n,m} &= d_{i,n}\binom{-\gamma}{m}(-h)^m. \end{aligned}$$

Plugging this into the formula for the risk premium yields

$$rp_t = \exp(\mu)\cdot(1-h\exp(z_t))^{-\gamma}\cdot\left\{-\left[\sum_{n=0}^{+\infty} d_{1,n}\frac{\mathcal{L}(\gamma+n)}{\mathcal{L}(\gamma+n-1)}\exp\big(c_{1,n}\big)\right]^{-1}\right.$$
$$\left.+\left[\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} d_{i,n}\exp\big(c_{i,n}\big)\right]^{-1}\cdot\left[\mathcal{L}(-1)+(1-h)^{-\gamma}\mathcal{L}(-1)\cdot\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} d_{i,n}\exp\big(c_{i,n}\big)\right]\right\}$$

where we have used (3.37) to express $\sum_{m=0}^{+\infty}\binom{-\gamma}{m}(-h)^m$ as $(1-h)^{-\gamma}$. Simplifying, we get

$$rp_t = \exp(\mu)\cdot(1-h\exp(z_t))^{-\gamma}\cdot\mathcal{L}(-1)\cdot\left\{(1-h)^{-\gamma}\right.$$
$$\left.+\left[\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} d_{i,n}\exp\big(c_{i,n}\big)\right]^{-1}-\left[\sum_{n=0}^{+\infty} d_{1,n}\exp\big(c_{1,n}\big)\cdot\frac{\mathcal{L}(\gamma+n)\mathcal{L}(-1)}{\mathcal{L}(\gamma+n-1)}\right]^{-1}\right\}.$$

Exploiting the respective definitions

$$\begin{aligned}\sum_{i=1}^{+\infty}\sum_{n=0}^{+\infty} d_{i,n}\exp\big(c_{i,n}\big) &= \left\{\sum_{i=0}^{+\infty}\Big(\beta\cdot\mathcal{L}(\gamma-1)\cdot\exp\big(\mu(1-\gamma)\big)\Big)^{i}\right\}\\ &\cdot\left\{\sum_{n=0}^{+\infty}\beta h^{n}\binom{\gamma-1+n}{n}\cdot\mathcal{L}((\gamma-1)+n)\cdot\exp\big(\mu(1-\gamma)-\mu\cdot n\big)\right\}\\ &=\Big(1-\beta\cdot\mathcal{L}(\gamma-1)\cdot\exp\big(\mu(1-\gamma)\big)\Big)^{-1}\cdot\sum_{n=0}^{+\infty} d_{1,n}\exp\big(c_{1,n}\big) \qquad (3.50)\end{aligned}$$

we finally arrive at

$$\begin{aligned} rp_t = \frac{\exp(\mu)\cdot\mathcal{L}(-1)}{(1-h\exp(z_t))^{\gamma}}\cdot\Bigg\{-\left[\sum_{n=0}^{+\infty} d_{1,n}\exp\big(c_{1,n}\big)\frac{\mathcal{L}(\gamma+n)\mathcal{L}(-1)}{\mathcal{L}(\gamma+n-1)}\right]^{-1}\\ +(1-h)^{-\gamma}+\big(1-\beta\cdot\mathcal{L}(\gamma-1)\cdot\exp\big(\mu(1-\gamma)\big)\big)\left[\sum_{n=0}^{+\infty} d_{1,n}\exp\big(c_{1,n}\big)\right]^{-1}\Bigg\}.\blacksquare\end{aligned}$$

	Ex (annual)		σX*100		Erb (annual)		SR(Req-Rb)*100		E(Req - Rb) (annual)		σ(Req-Rb)*100		σP/D	
Distribution	UK	US	UK	US	UK	US	UK	US	UK	US	UK	US	UK	US
1 Exponential	2.17%	-	4.58	-	2.88%	-	0.14	-	11.25%	-	19.97	-	5.81	-
2 Quadratic	-39.73%	-18.32%	8.75	4.92	1.47%	-0.43%	-0.07	0.01	-3.15%	0.19%	10.98	5.60	0.31	0.18
3 Inv. Triangular	-11.75%	-8.56%	4.32	2.94	1.09%	-3.35%	0.02	0.10	0.37%	0.67%	5.60	1.74	0.46	0.33
4 Uniform	-1.83%	-5.91%	5.46	4.13	1.29%	0.77%	0.06	-0.01	1.65%	-0.22%	7.42	6.15	2.47	1.13
5 Triangular	3.12%	2.29%	1.50	1.78	-2.32%	-0.41%	0.45	0.40	12.78%	4.91%	7.16	3.10	5.69	3.65
6 IH 3	1.50%	-0.88%	1.60	1.63	1.77%	-0.38%	0.02	0.01	0.61%	0.05%	7.33	2.41	26.54	9.10
7 IH 5	1.90%	0.39%	1.51	1.31	1.87%	0.64%	0.08	0.00	8.26%	0.02%	24.76	2.70	19.48	41.46
8 Gamma 2	2.32%	-	3.28	-	4.93%	-	0.12	-	9.83%	-	20.57	-	9.79	-
9 Gamma 5	1.97%	1.82%	2.29	2.74	2.18%	1.54%	0.15	0.09	9.88%	0.61%	17.04	1.65	7.54	13.29
10 Gamma 10	2.25%	1.46%	1.85	1.77	1.95%	1.82%	0.14	0.01	5.58%	0.09%	9.75	4.24	6.98	80.71
11 Normal	2.23%	1.72%	1.52	0.89	-0.06%	0.95%	0.11	0.11	10.53%	6.53%	24.47	14.63	11.91	25.50
12 Data	2.07%	1.64%	1.47	0.89	1.00%	-0.78%	0.19	0.25	8.92%	7.62%	11.97	7.67	4.88	6.85

Table 3.5: Means and Standard Deviations of Simulated and Historical Data

	ACF P/D													
	UK							US						
Distribution	1Q	2Q	4Q	2Y	3Y	5Y	7Y	1Q	2Q	4Q	2Y	3Y	5Y	7Y
1 Exponential	0.45	0.19	0.09	0.03	0.02	-0.03	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2 Quadratic	0.26	0.05	0.02	0.00	0.02	-0.01	-0.01	0.16	0.02	0.00	-0.03	0.02	0.03	-0.01
3 Inv. Triangular	0.23	0.05	0.01	-0.01	0.02	-0.01	-0.01	0.13	0.02	0.05	0.01	-0.01	-0.02	0.02
4 Uniform	0.30	0.08	0.02	0.01	0.00	0.01	0.02	0.20	0.07	0.01	0.06	-0.01	-0.01	0.00
5 Triangular	0.84	0.70	0.49	0.26	0.15	0.06	-0.01	0.53	0.30	0.08	0.03	0.05	-0.08	-0.02
6 IH 3	0.04	-0.01	-0.02	-0.01	0.01	0.02	0.01	0.41	0.15	0.02	0.02	0.00	-0.03	0.00
7 IH 5	0.19	0.04	0.00	0.00	-0.05	-0.01	-0.01	0.52	0.23	0.00	-0.06	-0.01	-0.05	0.03
8 Gamma 2	0.68	0.45	0.21	0.02	0.00	0.01	-0.06	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9 Gamma 5	0.58	0.38	0.15	0.00	-0.01	-0.06	0.00	0.58	0.32	0.13	0.04	0.01	0.01	-0.02
10 Gamma 10	0.52	0.30	0.11	-0.02	-0.01	0.01	0.02	0.55	0.26	0.03	0.02	0.05	-0.03	-0.02
11 Normal	0.00	-0.02	0.00	0.01	-0.01	0.00	-0.01	0.69	0.40	0.11	-0.03	-0.03	0.01	0.00
12 Data	0.92	0.83	0.68	0.49	0.29	0.25	0.15	0.95	0.89	0.77	0.63	0.62	0.38	0.10

	ACF (Req - Rb)													
	UK							US						
Distribution	1Q	2Q	4Q	2Y	3Y	5Y	7Y	1Q	2Q	4Q	2Y	3Y	5Y	7Y
1 Exponential	-0.09	-0.07	0.05	0.01	0.01	-0.02	-0.02	-	-	-	-	-	-	-
2 Quadratic	0.01	-0.01	0.02	0.01	0.03	0.00	-0.01	0.01	0.00	0.01	-0.03	0.03	0.03	-0.01
3 Inv. Triangular	-0.02	0.01	0.02	-0.01	0.02	-0.01	-0.02	0.38	0.05	0.05	0.01	0.00	-0.02	0.02
4 Uniform	0.01	-0.01	0.01	0.02	0.00	0.01	0.02	-0.09	0.02	0.01	0.06	0.00	-0.01	-0.01
5 Triangular	-0.01	-0.05	-0.03	-0.01	-0.04	0.04	0.00	0.00	0.05	-0.01	0.01	0.02	-0.04	0.00
6 IH 3	-0.31	-0.01	-0.01	-0.01	0.02	0.01	0.01	0.02	-0.02	-0.01	0.02	0.01	-0.02	0.02
7 IH 5	-0.30	-0.06	0.00	-0.01	-0.05	0.02	0.00	0.02	-0.04	-0.04	-0.03	0.00	-0.05	0.02
8 Gamma 2	-0.05	-0.07	-0.01	-0.01	-0.02	-0.02	-0.02	-	-	-	-	-	-	-
9 Gamma 5	-0.14	-0.03	-0.02	-0.01	0.01	-0.02	0.00	0.37	0.19	0.09	0.03	0.01	0.00	-0.01
10 Gamma 10	-0.19	-0.09	0.01	-0.03	0.00	0.01	0.01	0.02	-0.03	-0.04	-0.01	0.02	-0.01	0.00
11 Normal	-0.38	-0.02	0.00	0.00	-0.02	-0.01	0.01	0.05	-0.09	-0.05	-0.02	0.01	0.00	0.00
12 Data	0.07	-0.07	-0.18	0.05	-0.10	0.03	0.15	0.09	-0.09	-0.04	-0.04	-0.01	-0.08	-0.06

Table 3.6: Autocorrelations of Simulated and Historical Data (I)

	Σ ACF(Req - Rb)													
	UK							US						
Distribution	1Q	2Q	4Q	2Y	3Y	5Y	7Y	1Q	2Q	4Q	2Y	3Y	5Y	7Y
1 Exponential	-0.09	-0.16	-0.15	-0.16	-0.15	-0.13	-0.14	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2 Quadratic	0.01	-0.01	-0.01	-0.02	0.00	-0.03	-0.11	0.01	0.01	-0.02	-0.06	-0.07	-0.02	0.00
3 Inv. Triangular	-0.02	-0.02	-0.04	-0.06	-0.07	-0.09	-0.10	0.38	0.42	0.47	0.59	0.58	0.51	0.44
4 Uniform	0.01	0.00	0.00	0.05	0.01	0.02	0.00	-0.09	-0.07	-0.09	-0.05	-0.03	-0.09	-0.11
5 Triangular	-0.01	-0.06	-0.09	-0.19	-0.19	-0.14	-0.19	0.00	0.05	0.03	0.04	0.12	0.06	-0.04
6 IH 3	-0.31	-0.32	-0.35	-0.35	-0.35	-0.33	-0.35	0.02	0.00	-0.02	-0.03	-0.05	-0.10	-0.10
7 IH 5	-0.30	-0.36	-0.37	-0.37	-0.39	-0.38	-0.38	0.02	-0.02	-0.08	-0.18	-0.17	-0.25	-0.25
8 Gamma 2	-0.05	-0.13	-0.16	-0.22	-0.26	-0.24	-0.31	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9 Gamma 5	-0.14	-0.17	-0.20	-0.22	-0.25	-0.25	-0.30	0.37	0.55	0.76	0.93	0.98	0.97	0.96
10 Gamma 10	-0.19	-0.28	-0.32	-0.41	-0.38	-0.39	-0.37	0.02	-0.01	-0.09	-0.10	-0.01	-0.07	-0.02
11 Normal	-0.38	-0.40	-0.39	-0.36	-0.38	-0.38	-0.40	0.05	-0.05	-0.16	-0.22	-0.25	-0.27	-0.28
12 Data	0.07	0.00	-0.20	-0.29	-0.49	-0.54	-0.30	0.09	0.00	-0.08	-0.38	-0.29	0.05	-0.05

	ACF|Req|													
	UK							US						
Distribution	1Q	2Q	4Q	2Y	3Y	5Y	7Y	1Q	2Q	4Q	2Y	3Y	5Y	7Y
1 Exponential	-0.20	-0.13	0.04	0.01	0.01	-0.02	-0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2 Quadratic	0.01	-0.01	0.02	0.01	0.03	0.00	-0.01	0.01	0.00	0.01	-0.03	0.03	0.03	-0.01
3 Inv. Triangular	-0.02	0.01	0.02	-0.01	0.02	-0.01	-0.02	0.30	0.04	0.05	0.01	-0.01	-0.02	0.02
4 Uniform	0.01	-0.01	0.01	0.02	0.00	0.01	0.02	-0.12	0.02	0.01	0.05	0.00	-0.01	-0.01
5 Triangular	-0.06	-0.09	-0.05	-0.02	-0.05	0.04	0.00	-0.01	0.04	-0.01	0.01	0.02	-0.04	0.00
6 IH 3	-0.43	-0.01	0.00	-0.01	0.02	0.01	0.01	0.02	-0.02	-0.01	0.02	0.01	-0.02	0.02
7 IH 5	-0.39	-0.07	0.00	-0.02	-0.05	0.02	0.00	-0.06	-0.09	-0.05	-0.02	0.00	-0.04	0.02
8 Gamma 2	-0.13	-0.13	-0.03	-0.01	-0.02	-0.02	-0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9 Gamma 5	-0.23	-0.08	-0.05	-0.02	0.02	-0.02	0.00	0.46	0.24	0.11	0.03	0.01	0.01	-0.01
10 Gamma 10	-0.26	-0.11	0.00	-0.03	0.00	0.00	0.01	-0.06	-0.08	-0.05	-0.02	0.01	-0.01	0.00
11 Normal	-0.48	-0.02	0.00	0.00	-0.02	-0.01	0.01	-0.05	-0.17	-0.08	-0.02	0.01	0.00	0.00
12 Data	0.09	-0.05	-0.18	0.05	-0.10	0.01	0.14	0.12	-0.08	-0.02	-0.03	-0.03	-0.07	-0.06

Table 3.7: Autocorrelations of Simulated and Historical Data (II)

	CCF(P/D, Req - Rb)													
	UK							US						
Distribution	1Q	2Q	4Q	2Y	3Y	5Y	7Y	1Q	2Q	4Q	2Y	3Y	5Y	7Y
1 Exponential	0.38	0.15	0.07	0.02	0.01	-0.02	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2 Quadratic	0.25	0.05	0.02	0.00	0.03	-0.01	-0.01	0.15	0.02	0.00	-0.03	0.03	0.03	-0.01
3 Inv. Triangular	0.22	0.06	0.01	-0.01	0.01	0.00	-0.01	-0.13	-0.01	-0.05	0.00	0.01	0.02	-0.02
4 Uniform	0.29	0.08	0.01	0.02	0.00	0.02	0.01	0.19	0.07	0.00	0.05	0.00	-0.01	0.00
5 Triangular	0.41	0.34	0.25	0.10	0.05	0.04	-0.01	0.44	0.26	0.06	0.01	0.04	-0.05	-0.01
6 IH 3	0.04	0.00	-0.02	-0.01	0.01	0.02	0.01	0.38	0.14	0.02	0.02	0.00	-0.02	0.00
7 IH 5	0.16	0.03	0.01	0.00	-0.04	0.00	0.00	0.46	0.22	0.02	-0.04	0.00	-0.04	0.02
8 Gamma 2	0.45	0.28	0.15	0.02	-0.01	0.00	-0.04	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9 Gamma 5	0.38	0.25	0.10	0.01	-0.01	-0.01	-0.01	0.56	0.31	0.12	0.04	0.01	0.01	-0.01
10 Gamma 10	0.32	0.17	0.08	-0.02	0.00	0.00	0.01	0.48	0.24	0.03	0.00	0.04	-0.03	0.00
11 Normal	0.01	-0.02	0.00	0.01	-0.01	0.00	0.00	0.53	0.32	0.10	0.00	-0.01	0.01	-0.01
12 Data	0.11	0.06	0.05	0.06	-0.01	0.00	0.04	0.10	0.08	0.05	-0.05	-0.05	0.09	0.09

CCF(Req - Rb, |Req - Rb|)

	UK							US						
Distribution	1Q	2Q	4Q	2Y	3Y	5Y	7Y	1Q	2Q	4Q	2Y	3Y	5Y	7Y
1 Exponential	-0.06	-0.06	0.02	0.02	0.00	-0.01	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2 Quadratic	-0.02	0.01	-0.01	0.00	0.00	0.01	-0.03	-0.04	-0.01	0.00	-0.02	0.01	0.04	0.00
3 Inv. Triangular	0.01	-0.04	0.02	0.01	-0.03	-0.02	-0.02	0.06	0.02	0.01	-0.01	0.01	-0.02	0.00
4 Uniform	-0.03	-0.01	-0.01	0.00	0.00	0.02	0.00	0.00	0.01	0.01	0.03	-0.02	-0.01	-0.03
5 Triangular	-0.01	-0.02	0.02	-0.01	-0.03	0.03	0.01	0.00	0.00	0.00	0.00	-0.02	-0.02	-0.01
6 IH 3	0.13	0.02	0.00	0.01	-0.01	-0.01	-0.04	0.01	-0.02	0.01	0.01	-0.01	0.00	-0.01
7 IH 5	0.06	0.04	0.01	-0.03	0.01	0.01	-0.04	0.02	0.01	-0.02	-0.01	0.03	-0.01	0.00
8 Gamma 2	0.00	-0.03	0.00	-0.03	0.01	-0.03	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9 Gamma 5	-0.08	-0.03	0.00	-0.02	0.01	-0.01	-0.03	0.16	0.07	0.04	0.03	0.01	-0.01	0.01
10 Gamma 10	0.05	-0.01	0.00	-0.04	0.02	-0.01	0.01	0.03	-0.01	-0.02	0.00	0.01	0.00	-0.01
11 Normal	-0.02	-0.04	-0.01	0.01	0.01	0.01	0.01	0.07	0.03	0.00	0.00	-0.01	0.00	-0.02
12 Data	0.03	0.15	0.09	0.12	0.03	0.03	0.04	0.18	0.16	-0.01	-0.10	-0.01	0.03	-0.15

Table 3.8: Cross-Correlations of Simulated and Historical Data (I)

	CCF(P/D, |Req - Rb|)													
	UK							US						
Distribution	1Q	2Q	4Q	2Y	3Y	5Y	7Y	1Q	2Q	4Q	2Y	3Y	5Y	7Y
1 Exponential	0.32	0.13	0.05	0.01	0.00	-0.01	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2 Quadratic	-0.07	-0.01	0.00	0.00	0.00	0.01	-0.03	-0.04	-0.02	0.00	-0.02	0.00	0.04	-0.01
3 Inv. Triangular	0.01	-0.03	0.02	0.01	-0.04	-0.02	-0.02	-0.02	-0.02	-0.01	0.02	-0.01	0.02	-0.01
4 Uniform	0.01	0.00	-0.01	0.00	0.00	0.03	-0.01	-0.02	0.00	0.01	0.03	-0.02	-0.02	-0.02
5 Triangular	-0.01	-0.01	0.02	0.01	-0.01	0.02	-0.01	0.10	0.05	0.00	0.00	-0.01	-0.03	-0.01
6 IH 3	-0.02	0.01	0.01	0.01	-0.01	0.00	-0.02	0.02	-0.01	0.00	0.02	-0.02	-0.01	0.01
7 IH 5	0.00	0.04	0.02	-0.01	0.02	0.00	-0.01	-0.03	-0.01	-0.03	-0.02	0.03	-0.03	-0.02
8 Gamma 2	0.15	0.08	0.04	0.00	-0.01	-0.02	-0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
9 Gamma 5	0.20	0.11	0.06	0.00	-0.02	0.00	-0.02	0.21	0.11	0.05	0.03	0.01	-0.02	0.00
10 Gamma 10	0.04	0.01	0.00	-0.01	-0.01	0.02	0.00	0.10	0.04	-0.01	0.01	0.02	-0.01	0.00
11 Normal	0.01	-0.04	-0.02	0.01	0.02	-0.01	0.01	-0.06	-0.03	-0.02	-0.01	0.00	-0.01	0.00
12 Data	-0.18	-0.16	-0.14	-0.06	-0.14	-0.21	-0.24	-0.05	-0.02	-0.01	-0.03	-0.08	0.00	-0.13

	CCF(Req - Rb, Δc(t+i))									
	UK					US				
Distribution	i=-2	i=-1	i=0	i=1	i=2	i=-2	i=-1	i=0	i=1	i=2
1 Exponential	0.00	0.06	0.92	-0.28	-0.16	0.00	0.00	0.00	0.00	0.00
2 Quadratic	0.05	0.25	0.97	0.00	-0.02	0.02	0.15	0.99	0.01	0.00
3 Inv. Triangular	0.06	0.22	0.97	-0.02	0.00	0.01	0.13	0.98	0.31	0.04
4 Uniform	0.08	0.29	0.96	0.00	-0.01	0.07	0.15	0.96	-0.11	0.01
5 Triangular	-0.02	0.02	0.96	-0.12	-0.14	0.25	0.42	0.87	0.01	0.03
6 IH 3	0.02	0.03	0.82	-0.50	-0.03	0.11	0.35	0.93	0.00	-0.03
7 IH 5	0.00	-0.03	0.81	-0.58	-0.12	0.08	0.28	0.91	-0.11	-0.12
8 Gamma 2	-0.02	0.03	0.90	-0.22	-0.19	0.00	0.00	0.00	0.00	0.00
9 Gamma 5	0.01	-0.06	0.93	-0.32	-0.14	0.31	0.57	0.99	0.46	0.25
10 Gamma 10	-0.02	-0.06	0.88	-0.32	-0.15	0.14	0.33	0.86	-0.12	-0.12
11 Normal	-0.02	0.01	0.75	-0.66	-0.01	0.04	0.20	0.80	-0.26	-0.31
12 Data	0.19	0.09	0.08	-0.09	-0.03	0.07	0.26	0.32	0.02	-0.03

Table 3.9: Cross-Correlations of Simulated and Historical Data (II)

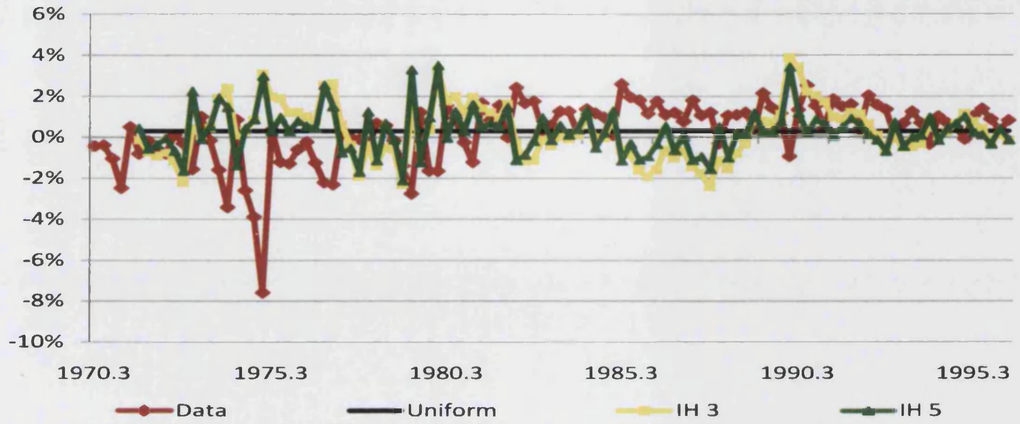

Figure 3.13: Actual vs Model Implied UK Bond Returns (I)

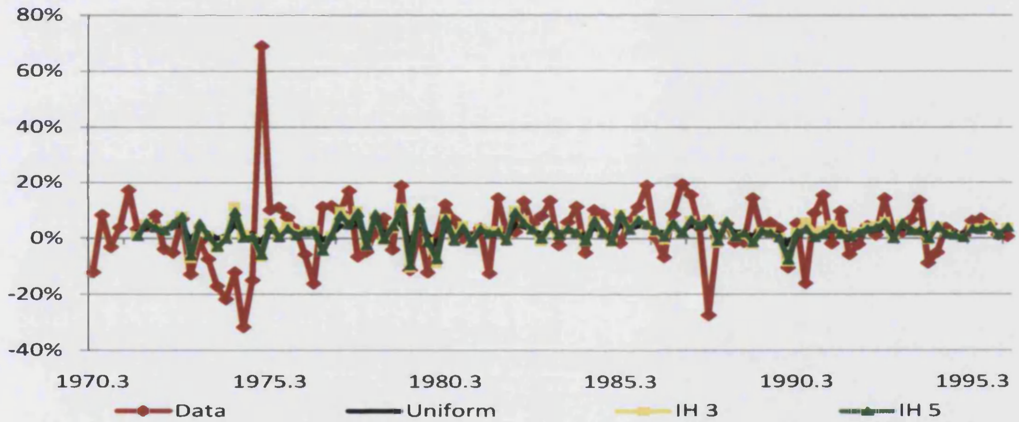

Figure 3.14: Actual vs Model Implied UK Equity Returns (I)

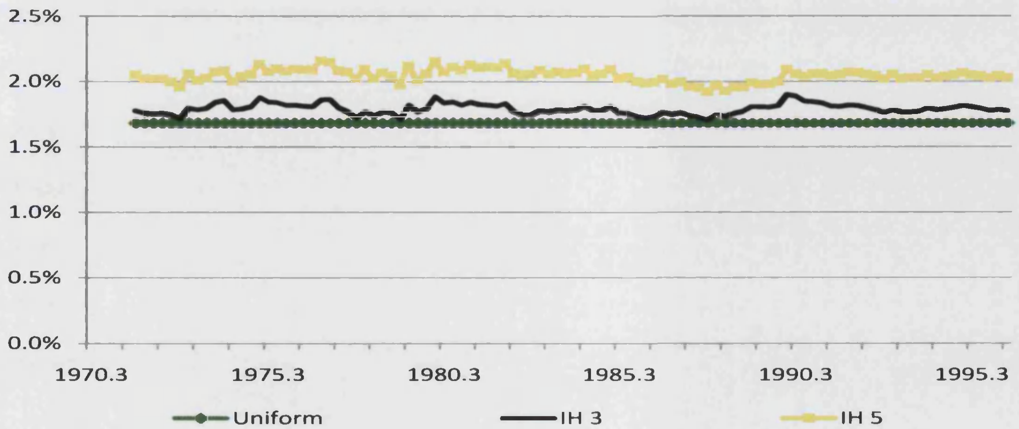

Figure 3.15: Model Implied UK Risk Premia (I)

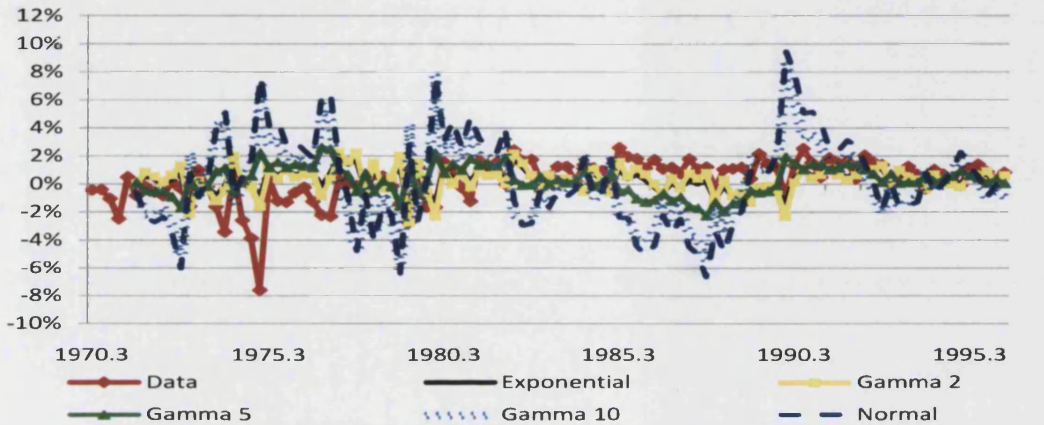

Figure 3.16: Actual vs Model Implied UK Bond Returns (II)

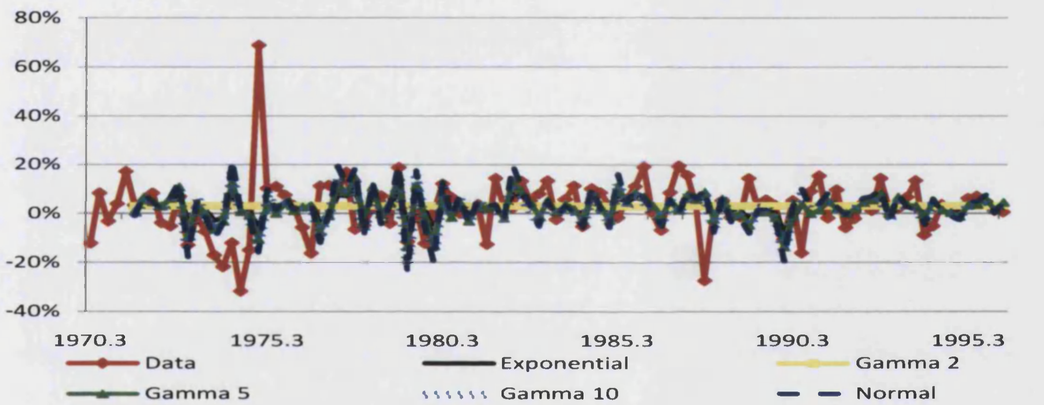

Figure 3.17: Actual vs Model Implied UK Equity Returns (II)

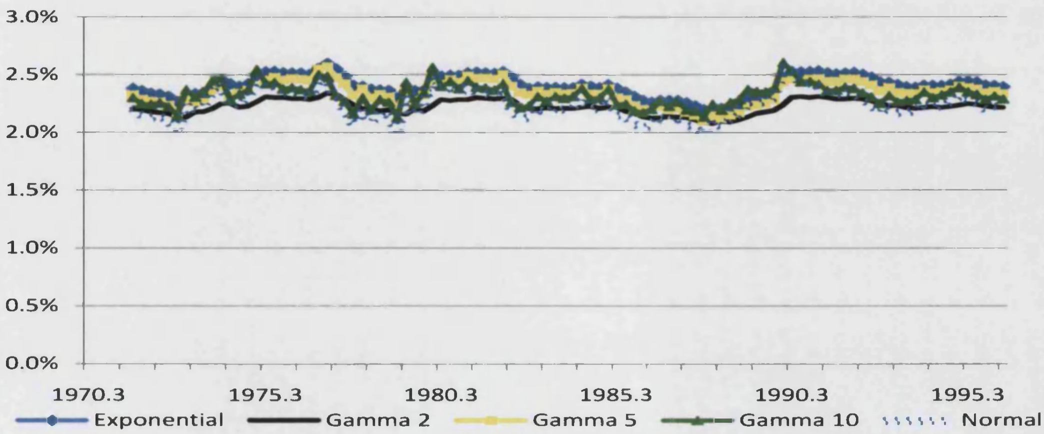

Figure 3.18: Model Implied UK Risk Premia (II)

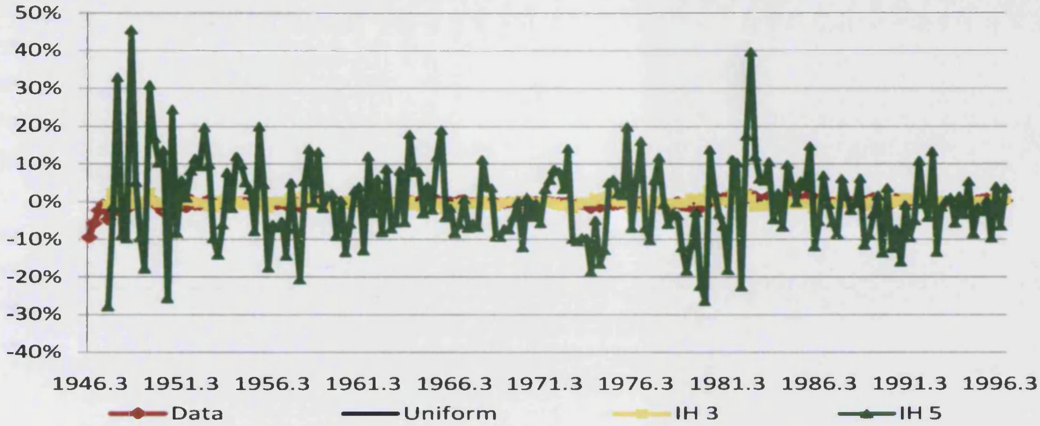

Figure 3.19: Actual vs Model Implied US Bond Returns (I)

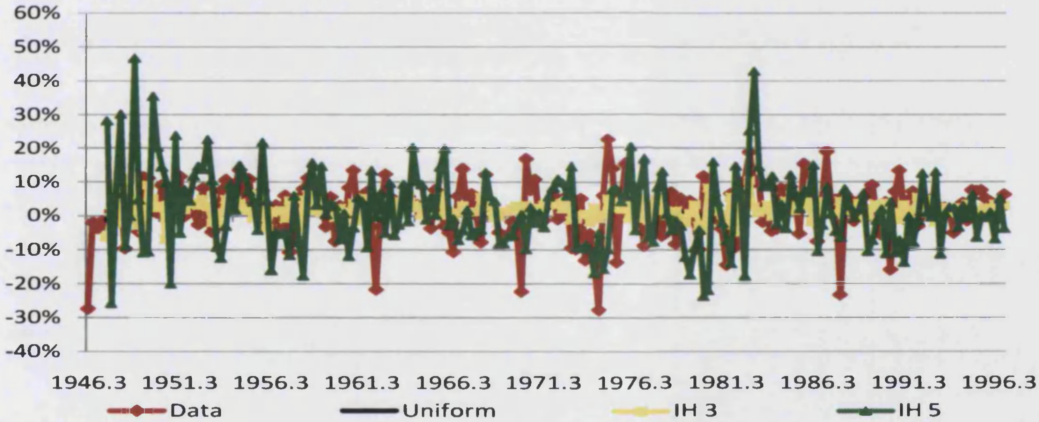

Figure 3.20: Actual vs Model Implied US Equity Returns (I)

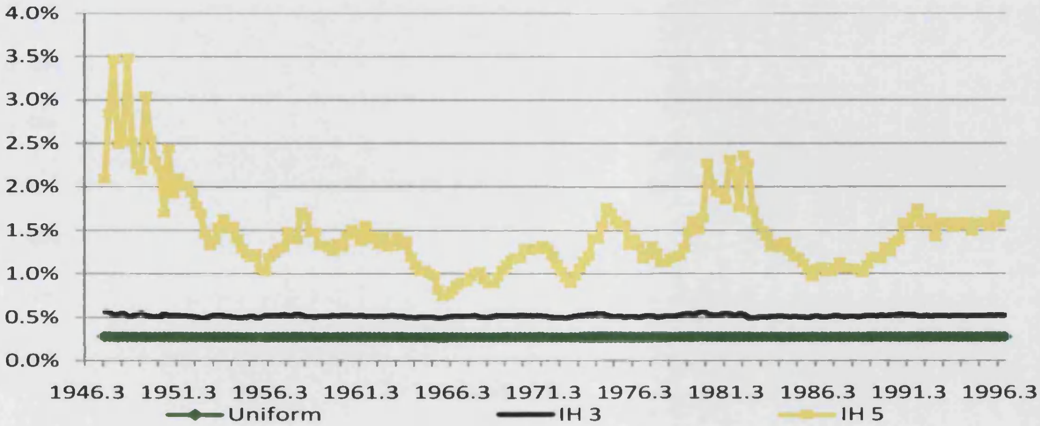

Figure 3.21: Model Implied US Equity Risk Premia (I)

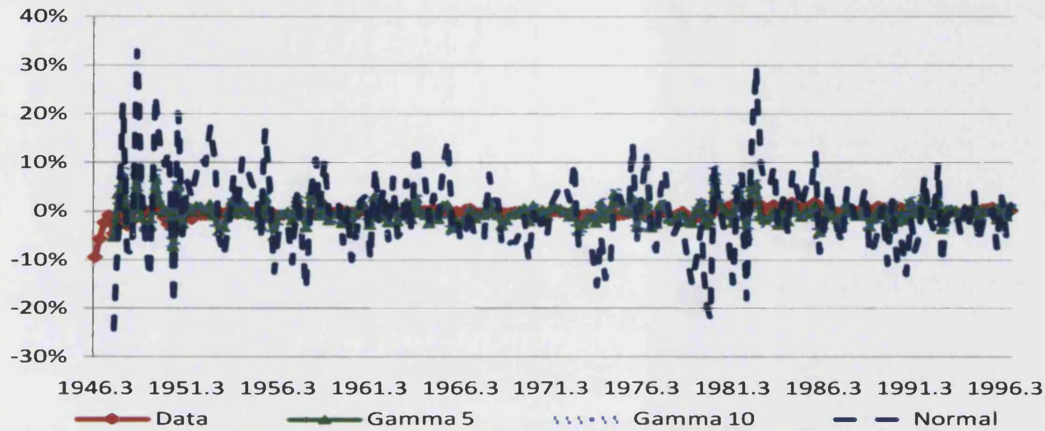

Figure 3.22: Actual vs Model Implied US Bond Returns (II)

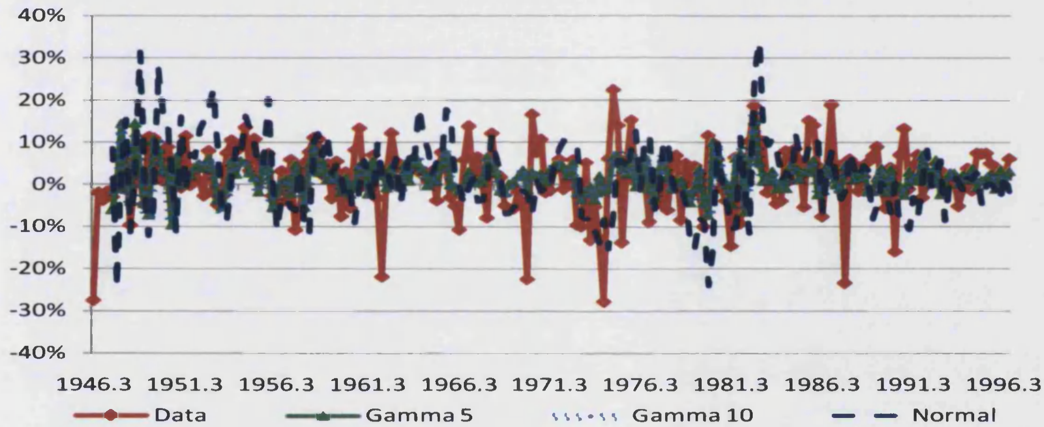

Figure 3.23: Actual vs Model Implied US Equity Returns (II)

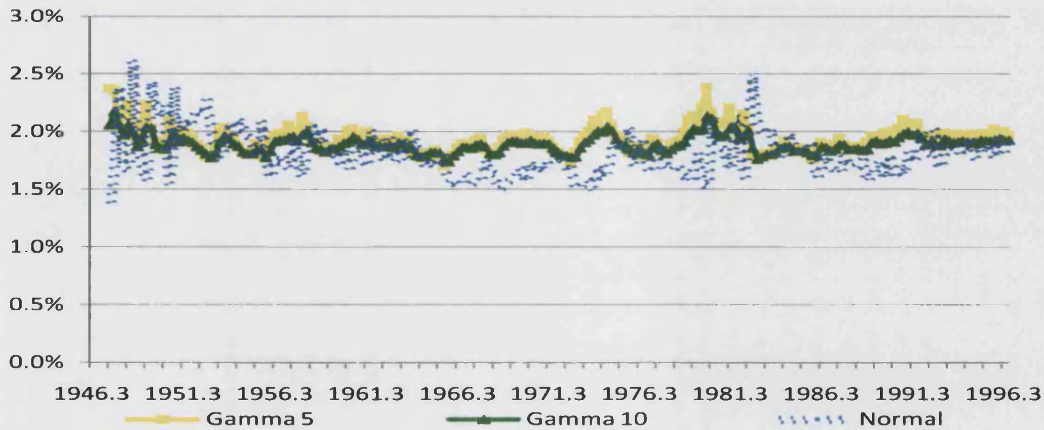

Figure 3.24: Model Implied US Equity Risk Premia (II)

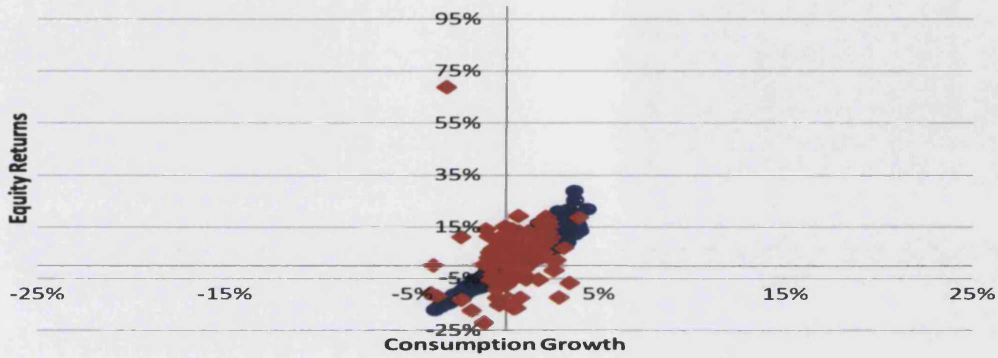

Figure 3.25: Scatter Plot of UK Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Triangular Shocks (blue)

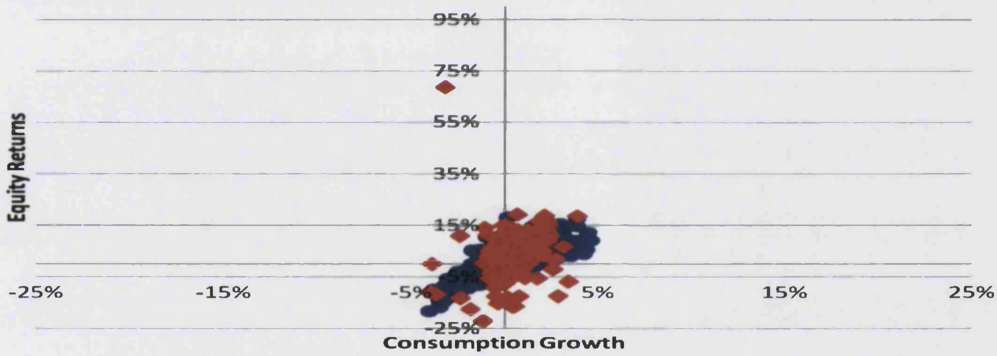

Figure 3.26: Scatter Plot of UK Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With 'Irwin Hall 3' Shocks (blue)

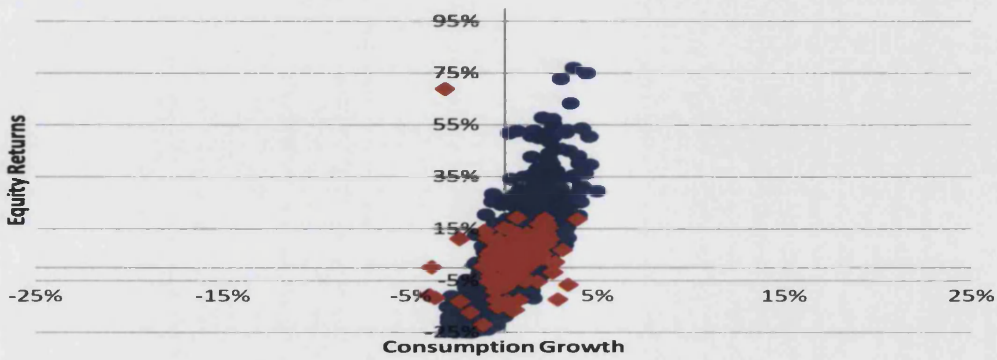

Figure 3.27: Scatter Plot of UK Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With 'Irwin Hall 5' Shocks (blue)

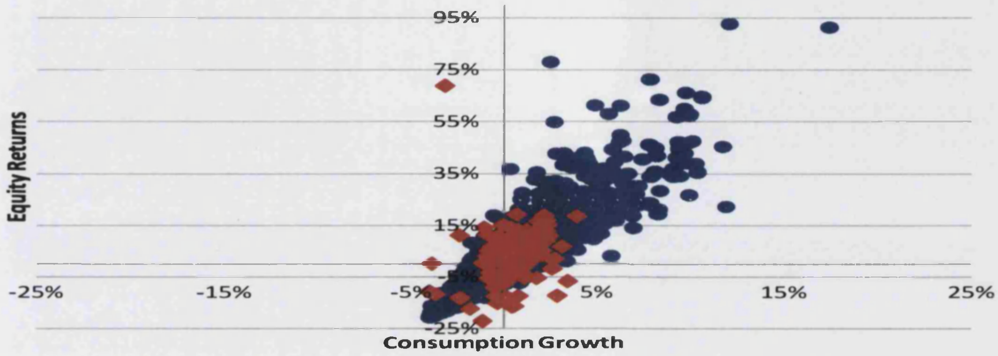

Figure 3.28: Scatter Plot of UK Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Gamma 2 Shocks (blue)

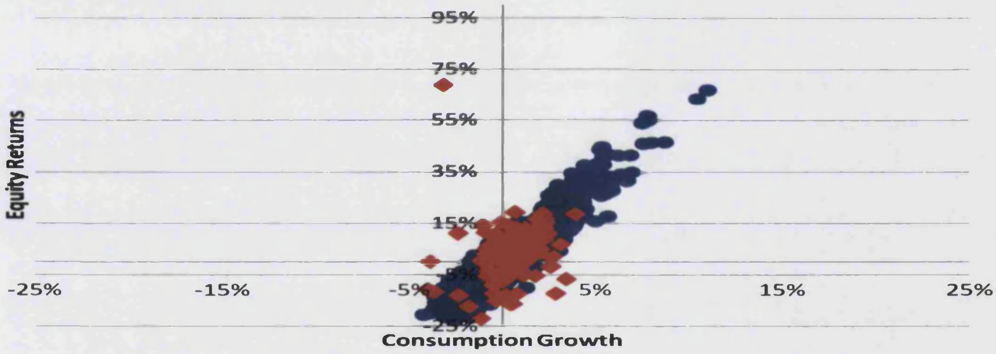

Figure 3.29: Scatter Plot of UK Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Gamma 5 Shocks (blue)

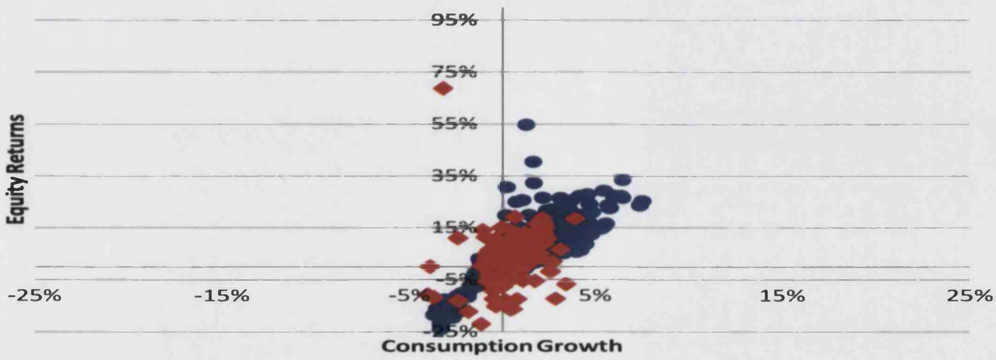

Figure 3.30: Scatter Plot of UK Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Gamma 10 Shocks (blue)

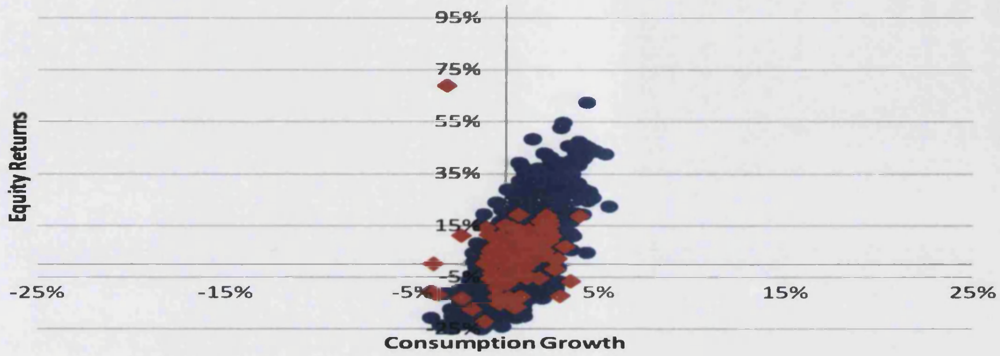

Figure 3.31: Scatter Plot of UK Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Normal Shocks (blue)

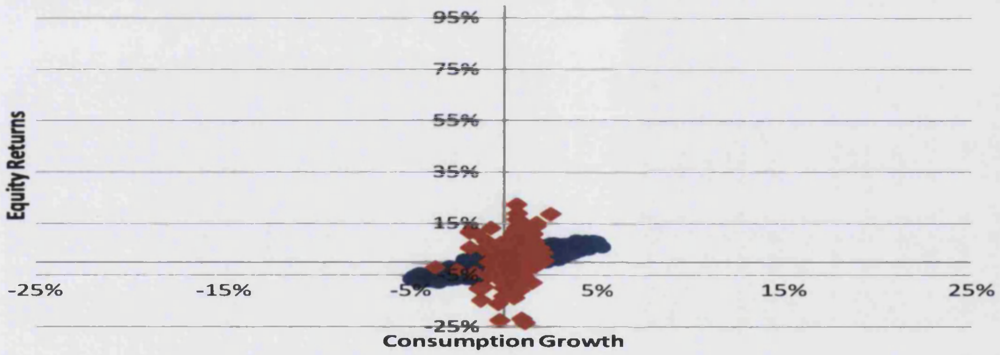

Figure 3.32: Scatter Plot of US Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Triangular Shocks (blue)

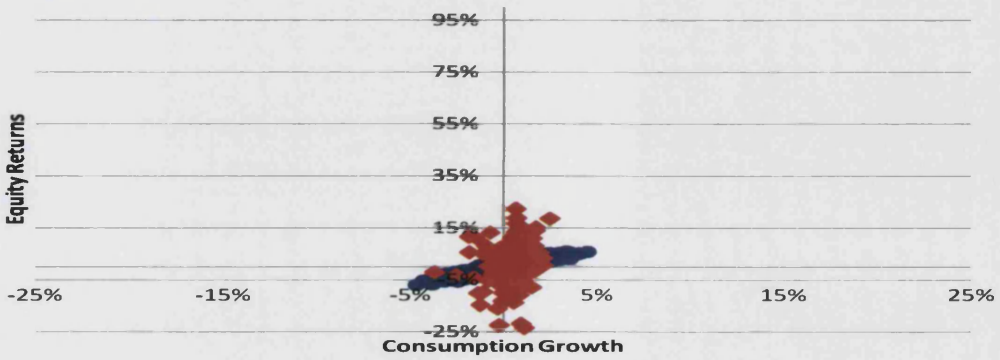

Figure 3.33: Scatter Plot of US Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With 'Irwin Hall (3)' Shocks (blue)

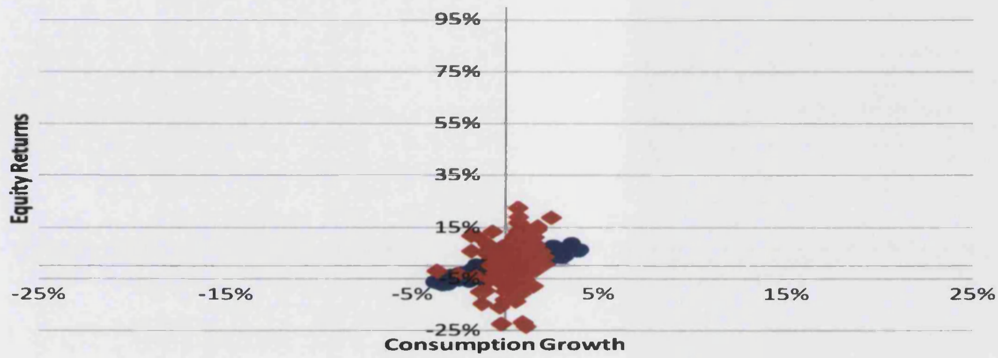

Figure 3.34: Scatter Plot of US Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With 'Irwin Hall (5)' Shocks (blue)

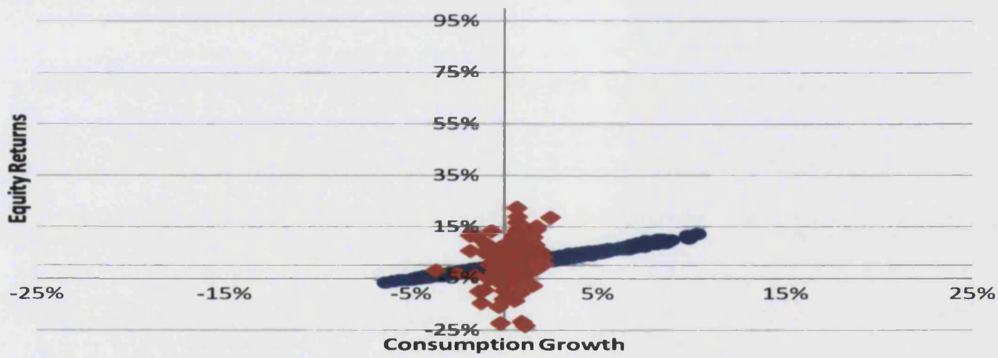

Figure 3.35: Scatter Plot of US Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Gamma 5 Shocks (blue)

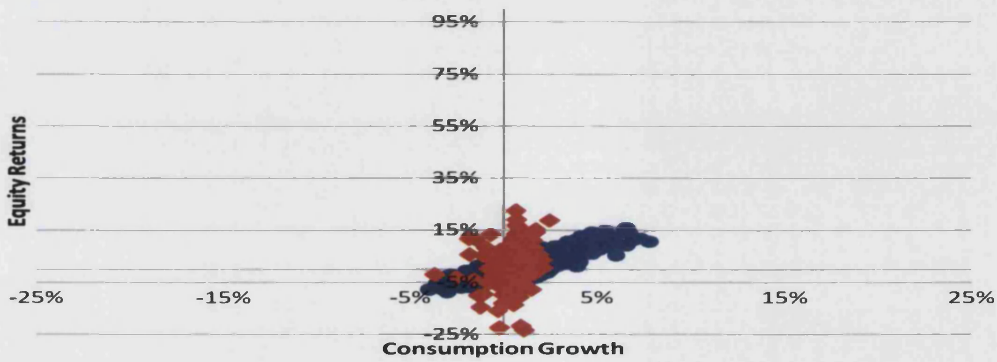

Figure 3.36: Scatter Plot of US Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Gamma 10 Shocks (blue)

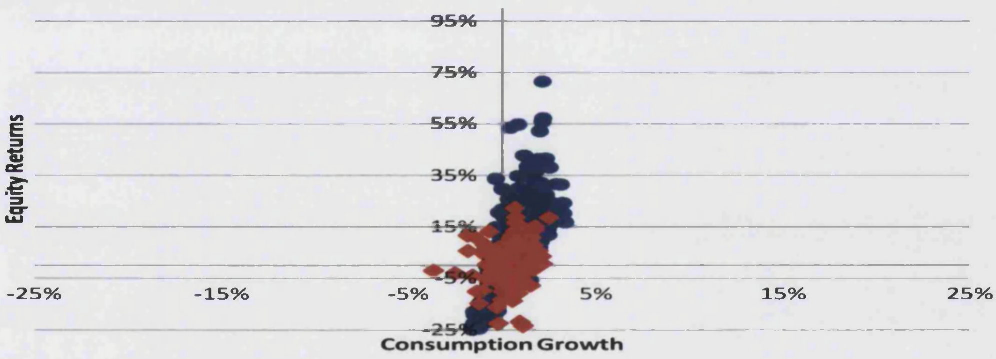

Figure 3.37: Scatter Plot of US Consumption Growth and Equity Returns: Data (red) vs Data Simulated From a Model With Normal Shocks (blue)

Chapter 4

Approximating Solutions of Asset Pricing Models: The Implicit Function Approach

4.1 Introduction

Conditions under which consumption-based asset pricing models can be solved in 'closed-form' are relatively restrictive.[1] Absent exact solutions, functions mapping state variables into asset prices need to be approximated. For simpler models one can use grid-search (e.g. Campbell and Cochrane (1999)) or exploit distributional assumptions – e.g. that of log-normality (as done in Campbell and Shiller (1988) or Bansal and Yaron (2004)). Other alternatives include finite-difference (Tauchen and Hussey, 1991) and projection methods (Judd, 1998, Ch. 11) as well as 'local', perturbation (Judd, 1998, Ch. 13, 14) or complex analysis approaches (Calin et al., 2005).[2] The main contribution of this paper is to add an extra method to the list above and demonstrate its applicability by approximating solutions of five popular asset pricing models.

[1]Lucas (1978); Mehra and Prescott (1985); Abel (1990, 2008) have shown how to solve such models when state variables are temporally independent or drawn from simple Markov chains. The independence assumption has been generalized by assuming AR(1) dynamics – as in Labadie (1989), Burnside (1998), Bidarkota and McCulloch (2003), Tsionas (2003), Collard et al. (2006) or chapter 3.

[2]Perturbation and complex-analysis methods pin down coefficients of the Taylor series approximation to the unknown solution. Clearly, Taylor series approximations can be 'global' – that is converge to the true function everywhere, as the order of approximation increases – as long as the approximated function is analytic.

Having precise and tractable estimates of the mapping between state variables and asset prices is clearly important as it can further our understanding of asset price dynamics. What is perhaps less clear, particularly given the large number of existent methods, is whether any new one is truly necessary. To address this point, we now highlight how our approach works, position it within the literature and clarify what 'extra' it brings to the table.

To illustrate the underlying idea, consider the simplest real business cycle framework. Under log-utility and full capital depreciation the solution can be characterized analytically. We show that once the model is re-expressed as a fixed-point problem in Banach spaces, the implicit function theorem can be invoked to approximate the solution for general coefficients of risk aversion and depreciation 'around' that corresponding to log-utility and full-depreciation. The approximation would be a polynomial in parameters measuring risk aversion and depreciation but the polynomial coefficients would be nonlinear functions of state variables and other parameters.

First, note that the procedure we advocate fits exactly into the 'perturbation' framework as we 'formulate a general problem, find a particular case that has a known solution, and then use that particular case and its solution as a starting point for computing approximate solutions to nearby problems' (Judd, 1998, p.447). Three important features distinguish our approach from 'textbook' perturbation methods. In particular:

1. approximations are not restricted to be polynomials in state variables;[3]
2. we do not approximate around solutions of deterministic sub-models;
3. we use the fixed-point formulation to find parameter restrictions ensuring solution uniqueness and show how to use the Lipschitz constant of that representation to put bounds on approximation errors.

The benefits of the first two are clear: by moving away from the deterministic steady-state and relying on nonlinear approximations, even parsimonious, low-order approximations can capture the effect of uncertainty – something particularly relevant in an asset pricing context. We stress that the same is not true

[3]The method proposed here can be seen as 'dual' to 'standard' perturbations. In the 'textbook' case, the resulting approximation is a polynomial in state variables and the coefficients of that polynomial are nonlinear functions of model parameters. In our approach, the approximation is a polynomial in a subset of model parameters, with the coefficients being nonlinear functions of state variables and remaining coefficients.

of standard perturbation approximations. As discussed in Schmitt-Grohe and Uribe (2004), there, up to second-order, risk only has an impact on the constant of the approximate policy rule. Expressed alternatively, if one is interested in changes over time in risk-premia, then a third-order approximation is the lowest at which these are not zero 'by construction'. Even with only two state variables, a third order polynomial has ten distinct coefficients, making it difficult to glean economic insights.

Other important problems with 'standard' perturbation methods have been emphasized in Den Haan and De Wind (2009). The authors note, in particular, that 'outside the radius of convergence higher-order Taylor series approximations behave extremely badly'. This is an important drawback as it is often not clear whether the entire probability-mass of the 'true' ergodic distribution lies within that radius.[4] Furthermore, Den Haan and De Wind (2009) also stress that even within the radius of convergence, polynomial approximations can display 'wild swings in numerical solutions'.

Since our proposed approximations are not restricted to be polynomials in state variables, they are likely to be less susceptible to some of these criticisms.[5] More generally, however, in light of questionable performance of finite-order perturbation approximations, the issue of assessing approximation accuracy becomes of paramount importance. In this context, an advantage of the approach advocated here is that the Lipschitz constant of the related fixed-point representation makes it possible to put bounds directly on the errors of the policy function approximation (as opposed to the indirect, 'Euler-equation' approach to solution accuracy proposed by Santos (2000)).[6]

[4] In general, 'standard' perturbation approximations are only valid locally, i.e. in the vicinity of the deterministic steady state. Since in asset pricing applications the mean and median of simulated series can differ markedly from their deterministic steady-state counterparts, it is not even clear whether they lie in the radius of convergence and / or are representative of the actual mean/median. Partially addressing that point, Juillard and Kamenik (2005) propose (and implement in *Dynare++*) a method of approximating solutions around what they refer to as the 'stochastic steady state'. While the latter typically won't equal the mean of the ergodic distribution it could lie 'closer' to it than the standard, deterministic steady state.

[5] Further to footnote 3, our approximations are not immune to these criticisms as they are polynomials in selected model *parameters*. Hence, their performance outside the radius of convergence *for these parameters* could also be poor. Conditional on these parameters lying within the convergence radius, however, approximations are unlikely to display such instabilities as they are arbitrary, nonlinear functions of state-variables.

[6] Formulas for Taylor series approximation errors always depend on values of derivatives in some *neighborhood* of the approximation point. While standard perturbation methods can be used to compute derivatives of an arbitrarily high order in the deterministic steady state, they are usually mute about values of derivatives away from it.

We stress that while the focus in this paper is on asset pricing applications, the framework, as presented here, can be directly applied to any model with known state variable dynamics. In particular, no extra distributional assumptions on the dynamics of the stochastic discount factor (SDF) or about joint conditional distributions of the SDF and endogenous returns are necessary.[7] A potential constraint, however, is that one has to be able to solve a nested problem exactly. To prove that this constraint is unlikely to bind, we demonstrate how generic problems can be restricted and solved and how these solutions can be used to approximate those of 'unrestricted' problems.

We apply the general principles to five popular asset pricing models. In particular, we approximate the function mapping state variables into the equity-price/consumption ratio in the models of Abel (1990), Campbell and Cochrane (1999), Bansal and Yaron (2004) and under two simple difference-form habit specifications related to those used in Uhlig (2004) and Smets and Wouters (2007). To restrict attention, we assume – in line with much of the equity premium literature (cf. Kocherlakota (1996); Mehra and Prescott (2004)) – that equities entitle their owner to a stream of consumption. Our choice of models was motivated both by popularity and the fact that between them, they allow for nonlinear habits, catching and keeping-up with the Joneses', recursive utility and long-run growth risks. Importantly, in some of the models we consider the state variable is multi-dimensional, showing that our methods can easily be applied in such settings as well.

The remainder of this paper is structured as follows. First, we present the theoretical underpinnings. We briefly discuss how the asset pricing equation can be recast as a fixed point problem and how the contraction mapping theorem can be used to characterize sufficient conditions for the existence of solutions. We then show how the implicit function theorem can be exploited to find Taylor series approximations around known solutions of nested problems and how the Lipschitz constant can be used to provide upper bounds on approximation errors. In the following section we apply these methods to the models listed above, briefly comparing their accuracy to exact solutions (where available) and standard perturbation approximations. We conclude by summarizing and discussing possible extensions. Most of the proofs can be found in the appendix.

[7] In fact, it is fairly straightforward to compare solutions corresponding to different distributions of underlying shocks.

4.2 The Theoretical Underpinnings

4.2.1 Recasting the Asset Pricing Equation as a Fixed Point Problem

The problem we consider is that of solving for the price P_t of an equity share. To simplify, and in line with much of the literature, we assume that equities pay the stream of consumption C_t as dividends – though we note that the methods presented could easily be applied to price other assets as well. From the fundamental asset pricing equation, we know that

$$P_t = \mathbf{E}_t \mathcal{M}_{t+1} \left(P_{t+1} + C_{t+1}\right) \tag{4.1}$$

where $\mathcal{M}_{t+1}$ is the stochastic discount factor. Letting v_t be the price to consumption ratio, i.e. $v_t = P_t/C_t$, equation (4.1) can be expressed in terms of stationary variables as[8]

$$v_t = \mathbf{E}_t \mathcal{M}_{t+1} \frac{P_{t+1} + C_{t+1}}{C_t} = \mathbf{E}_t \mathcal{M}_{t+1} C^g_{t+1} \left(v_{t+1} + 1\right) \tag{4.2}$$

where C^g_t denotes gross consumption growth. Let the state vector x take values in $\mathcal{D} \subset \mathbb{R}^{d_1}$ and let the parameter vector λ be an element of $\Lambda \in \mathbb{R}^{d_2}$. The exact definitions of both these vectors will depend on the specific model considered. We seek a solution

$$v_t = g(X_t, \lambda) \tag{4.3}$$

where $g : \mathcal{D} \times \Lambda \to \mathcal{D}$. Accordingly, $g(\cdot, \cdot)$ has to satisfy

$$\forall t : g(X_t, \lambda) = \mathbf{E}_t \mathcal{M}_{t+1} C^g_{t+1} \left(g(X_{t+1}, \lambda) + 1\right) \tag{4.4}$$

where $X_t : \Omega \to \mathcal{D}$ is the state process.[9] We further assume that the state process is Markov, i.e. that it's dynamics is described by

$$X_{t+1} := \mathbb{F}(X_t, \lambda, \xi_{t+1}) \tag{4.5}$$

[8]In models with stationary consumption - e.g. that of Lucas (1978) - we could directly approximate the function mapping states into prices (rather than into P_t/C_t).

[9]Though we do not explicitly stress the dependance in this equation, the dynamics of $\mathcal{M}$ and C shall typically depend on λ.

where $\xi_t : \Omega \to \Xi \subset \mathbb{R}^{d_3}$ are i.i.d. shocks and where $\mathbb{F} : \mathcal{D} \times \Lambda \times \Xi \to \mathcal{D}$ is a known function. To proceed, let $\mathcal{G}$ be a closed subset of a Banach space of functions on $\mathcal{D}$ with norm $|| \cdot ||$.[10] To simplify the algebra we first rewrite equation (4.4) as

$$G(x) = \mathbb{U}\big(x, \mathcal{P}(G, \lambda)(x), \lambda\big) \tag{4.6}$$

where $\mathbb{U} : \mathcal{D} \times \mathcal{D} \times \Lambda \to \mathcal{D}, G \in \mathcal{G}$ and the operator $\mathcal{P} : \mathcal{G} \to \mathcal{G}$ is defined as

$$\mathcal{P}(G, \lambda)(x) = \mathrm{E}\big(G(\mathbb{F}(x, \lambda, \xi))\big). \tag{4.7}$$

Here $G(\cdot)$ is a new unknown function, which is introduced for convenience, to simplify the fixed-point representation of the original problem. The exact definition of $G(\cdot)$ will depend on the problem in question. We deliberately refrain from putting any restrictions on the functional form of $\mathbb{U}(\cdot, \cdot, \cdot)$ to make the current exposition applicable to the widest possible range of models. In general, $\mathbb{U}$ will be a known, possibly nonlinear function and it will uniquely map $G(x, \lambda)$ into the solution $g(x, \lambda)$. Notably, defining the operator $\mathcal{Z} : \mathcal{G} \times \Lambda \to \mathcal{G}$ as $\mathcal{Z}(G, \lambda)(x) := \mathbb{U}\big(x, \mathcal{P}(G, \lambda)(x), \lambda\big)$, equation (4.6) can be written as a fixed point problem in $\mathcal{G}$

$$G = \mathcal{Z}(G, \lambda). \tag{4.8}$$

As mentioned in the introduction, specification (4.8) has three main advantages over the original equation (4.2).

1. The contraction mapping principle can be used to establish sufficient conditions for the existence of a solution, which will be unique in $\mathcal{G}$.

2. It will frequently be easy to characterize parameter restrictions under which equation (4.8) can be solved exactly. Letting $\bar{\lambda}_1$ denote 'restricted' elements of the parameter vector λ, the implicit function theorem can then be applied to compute Taylor series approximations of $G(\cdot, \cdot)$

$$G(x, \{\lambda_1, \lambda_{-1}\}) = G(x, \{\bar{\lambda}_1, \lambda_{-1}\}) + \sum_{i=1}^{+\infty} \frac{(\lambda_1 - \bar{\lambda}_1)^i}{i!} \cdot \frac{\partial^i}{\partial \lambda_1^i} G(x, \{\bar{\lambda}_1, \lambda_{-1}\})$$

where $\lambda = \{\lambda_1, \lambda_{-1}\}$ with λ_1, λ_{-1} possibly multi-dimensional vectors such that $\lambda_1 \in \Lambda^1, \lambda_{-1} \in \Lambda^{-1}$ and $\Lambda^1 \times \Lambda^{-1} = \Lambda$.[11] Notably, coefficients of the

[10] In practice, the choice of $\mathcal{G}$ and the definition of the norm $|| \cdot ||$ will depend on the model considered.

[11] To simplify notation, we assume – without loss of generality – that λ_1 is one-dimensional.

Taylor series expansion – i.e. $(\partial^i/\partial\lambda_1^i)G(x,\lambda)|_{\lambda=\{\bar{\lambda}_1,\lambda_{-1}\}}$ – can be nonlinear functions of x.[12]

3. Finally, estimates of $\mathcal{Z}$'s norm in $\mathcal{G}$ can be used to derive bounds on the Taylor series approximation errors: $|\,G(x,\{\lambda_1,\lambda_{-1}\}) - G(x,\{\bar{\lambda}_1,\ \lambda_{-1}\})|$.

Further to points (1) and (3), sufficient conditions for the existence of solutions to equation (4.8) as well as bounds on the norms of derivatives, can be obtained by applying the implicit function theorem to the operator $\mathcal{Z} : \mathcal{G} \longrightarrow \mathcal{G}$. For completeness, we restate the theorem here in a form tailored to subsequent applications.

Assumption A1. *There exists* $\alpha \in (0,1)$ *such that*

$$\forall G_1, G_2 \in \mathcal{G}, \lambda \in \Lambda : \ \ ||\mathcal{Z}(G_1,\lambda) - \mathcal{Z}(G_2,\lambda)|| \leq \alpha \cdot ||G_1 - G_2||. \qquad (4.9)$$

Theorem 3. Implicit Function Theorem *Under Assumption A1 equation (4.8) has a unique solution in* $\mathcal{G}$ *(Contraction Principle). We shall denote that solution by* $G(\lambda), \lambda \in \Lambda$.[13] *Furthermore:*

- *if* $\mathcal{Z}$ *is continuous in* λ *(for any* $G \in \mathcal{G}$*) then* $G(\lambda)$ *is also continuous;*

- *if* $\mathcal{Z}$ *is differentiable with respect to* G *and* λ *then* $G(\lambda)$ *is differentiable and*

$$\frac{\partial G}{\partial \lambda}(\lambda) = (I - \frac{\partial \mathcal{Z}}{\partial G})^{-1}(\frac{\partial \mathcal{Z}}{\partial \lambda}), \qquad ||\frac{\partial G}{\partial \lambda}(\lambda)|| \leq \left(\frac{1}{1-\alpha}\right) ||\frac{\partial \mathcal{Z}}{\partial \lambda}||;$$

- *if* $\mathcal{Z}$ *is twice differentiable with respect to* G *and* λ*, then* $G(\lambda)$ *is also twice differentiable with respect to* λ *and*

$$\frac{\partial^2 G}{\partial \lambda^2}(\lambda) = (I - \frac{\partial \mathcal{Z}}{\partial G})^{-1}(\frac{\partial^2 \mathcal{Z}}{\partial \lambda^2} + 2\frac{\partial^2 \mathcal{Z}}{\partial G \partial \lambda} \cdot \frac{\partial G}{\partial \lambda} + \frac{\partial^2 \mathcal{Z}}{\partial G^2}(\frac{\partial G}{\partial \lambda}, \frac{\partial G}{\partial \lambda})),$$
$$||\frac{\partial^2 G}{\partial \lambda^2}(\lambda)|| \leq \left(\frac{1}{1-\alpha}\right)\left[||\frac{\partial^2 \mathcal{Z}}{\partial \lambda^2}|| + 2||\frac{\partial^2 \mathcal{Z}}{\partial G \partial \lambda}|| \cdot ||\frac{\partial G}{\partial \lambda}|| + ||\frac{\partial^2 \mathcal{Z}}{\partial G^2}|| \cdot ||\frac{\partial G}{\partial \lambda}||^2\right]$$

[12] As mentioned previously, exact solutions of nested problems – i.e. the $G(x,\{\bar{\lambda}_1,\lambda_{-1}\})$ are important and serve as the zeroth order approximation. Since in many cases this nested solution is that of a stochastic, rather than deterministic model therefore – and unlike low-order perturbation approximations – it will not satisfy certainty equivalence.

[13] See also Stokey, N. L. and Lucas, R. E. with Prescott (1989), Ch. 17, pp. 501 - 542 for a discussion of contraction mapping theorem applications.

where all the derivatives of $\mathcal{Z}$ are evaluated at $(G(\lambda), \lambda)$. Similar types of estimates can also be derived for higher order derivatives.

Proof of Theorem 3. For a general formulation and proof of the Implicit Function Theorem see also (Dieudonné, 1960, p.265, (10.2.1)). □

The Taylor series formula makes it clear that bounds on the norm of derivatives of function G can be directly used to assess the error of the Taylor series approximation.[14] Theorem 4 makes that link explicit.

Assumption A2. Differentiability *G is continuously differentiable $n+1$ times in the ball $I := \{\{\lambda_1, \lambda_{-1}\} : |\{\lambda_1, \lambda_{-1}\} - \{\bar{\lambda}_1, \lambda_{-1}\}| < r\} \subset \Lambda^1 \times \Lambda^{-1}$.*

Theorem 4. Taylor Series Expansion *Under Assumption A2, for all $\{\lambda_1, \lambda_{-1}\} \in I$*

$$||\, G(\{\lambda_1, \lambda_{-1}\}) - \sum_{j=0}^{n} \frac{(\lambda_1 - \bar{\lambda}_1)^j}{j!} \frac{dG}{d\lambda_1^j}(\{\bar{\lambda}_1, \lambda_{-1}\})|| \leq \frac{|\lambda_1 - \bar{\lambda}_1|^{n+1}}{(n+1)!} \sup_{|\nu - \{\bar{\lambda}_1, \lambda_{-1}\}| \leq r} ||\frac{d^{n+1}G}{d\lambda_1^{n+1}}(\nu)||. \quad (4.10)$$

Proof of Theorem 4. For a general formulation and proof of the Taylor formula see also (Dieudonné, 1960, p.186, (8.14.2)). □

While Theorem 3 gives formulae for derivatives of $\mathcal{G}$ it does not specify how to find the operator $(I - (\partial/\partial G)\mathcal{Z})^{-1}$. In practice, solving for $(I - (\partial/\partial G)\mathcal{Z})^{-1}$ is often the most difficult part of the approximation process.[15] The subsequent theorems – which form the main analytical contribution of this paper – characterize conditions under which the implicit relationship satisfied by G can be exploited to solve for $(d/d\lambda_1^j)G(\{\bar{\lambda}_1, \lambda_{-1}\})$, $j \in \{0, \ldots\}$. Armed with these, one can write formulae for the first order Taylor series approximation to $G(\lambda)$ around the nested solution $G(\lambda_0)$ solely in terms of known functions $\mathbb{U}(\cdot, \cdot, \cdot)$ and $\mathbb{F}(\cdot, \cdot)$. General formulae for higher order derivatives could be computed analogously.[16]

[14]Crucially, the Lipschitz constant can be used to bind values of the solution function derivatives for arbitrary model parameters (i.e. not only for the restricted coefficients).

[15]Expressed alternatively, the implicit function theorem reduces the problem of finding derivatives $(\partial^j/\partial\lambda^j)G(x, \lambda)$ to that of solving a nonlinear equation in function spaces. We shall characterize conditions under which the latter can be done in closed-form.

[16]And we provide formulae for fifth order approximations in one of the examples.

Assumption A3. Sufficient Parameter Restrictions *There exists* $\bar{\lambda}_1 \in \Lambda^1$ *such that* $\mathbb{F}$ *evaluated at* $\bar{\lambda}_1$ *becomes independent of* x *i.e. that*

$$\forall x: \ \mathbb{F}(x, \{\bar{\lambda}_1, \lambda_{-1}\}, \xi) = \mathbb{F}_0(\lambda_{-1}, \xi).$$

Theorem 5. Benchmark Solution *Under Assumptions A1 and A3 the unique solution* G_0 *to the fixed point equation*

$$G = \mathcal{Z}(G, \{\bar{\lambda}_1, \lambda_{-1}\})$$

is given by

$$G_0(x, \lambda_{-1}) = \mathbb{U}\big(x, \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}\big) \tag{4.11}$$

where $\Psi(\lambda_{-1})$ *is a solution to*

$$\Psi(\lambda_{-1}) = \mathbf{E}\mathbb{U}\big(\mathbb{F}_0(\lambda_{-1}, \xi), \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}\big). \tag{4.12}$$

Proof of Theorem 5. The proof can be found in the Appendix. □

Theorem 6. First Order Derivative *Under assumptions A1 and A3 the first order derivative of* $G(\cdot, \cdot)$ *with respect to* λ_1 *evaluated at* $\bar{\lambda}_1$ *equals*

$$\begin{aligned}\frac{\partial G}{\partial \lambda_1}(x, \{\bar{\lambda}_1, \lambda_{-1}\}) = \frac{\partial \mathbb{U}}{\partial \lambda_1}(x, \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}) + \frac{\partial \mathbb{U}}{\partial x_2}(x, \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}) \\ \cdot \mathbf{E}\left[\frac{\partial \mathbb{F}}{\partial \lambda_1}(x, \{\bar{\lambda}_1, \lambda_{-1}\}, \xi) \frac{\partial G_0}{\partial x}(\mathbb{F}_0(\lambda_{-1}, \xi), \lambda_{-1}) + \kappa(\lambda_{-1})\right]\end{aligned} \tag{4.13}$$

where the constant $\kappa(\lambda_{-1})$ *is given by*

$$\begin{aligned}\kappa(\lambda_{-1}) \equiv \big(1 - \bar{\mathbf{E}}(\partial/\partial x_2)\mathbb{U}(\mathbb{F}_0(\lambda_{-1}, \bar{\xi}), \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\})\big)^{-1} \cdot \\ \cdot \bar{\mathbf{E}}\left\{\frac{\partial \mathbb{U}}{\partial \lambda_1}(\mathbb{F}_0(\lambda_{-1}, \bar{\xi}), \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}) + \frac{\partial \mathbb{U}}{\partial x_2}(\mathbb{F}_0(\lambda_{-1}, \bar{\xi}), \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\})\right. \\ \left. + \mathbf{E}\left[\frac{\partial}{\partial x} G(\mathbb{F}_0(\lambda_{-1}, \xi), \{\bar{\lambda}_1, \lambda_{-1}\}) \cdot \frac{\partial}{\partial \lambda_1}\mathbb{F}(\mathbb{F}_0(\lambda_{-1}, \bar{\xi}), \{\bar{\lambda}_1, \lambda_{-1}\}, \xi)\right]\right\}\end{aligned}$$

and where the expectation of $\bar{\xi}$*, distributed identically as* ξ*, is denoted by* $\bar{\mathbf{E}}$.

Proof of Theorem 6. The proof can be found in the Appendix. □

Corollary 3. *It follows directly from Theorems 5 and 6 that the first order Taylor series approximation to function $G(\cdot,\cdot)$ is given by*

$$G(x,\{\lambda_1,\lambda_{-1}\}) = G_0(x,\lambda_{-1}) + \big(\lambda_1 - \bar{\lambda}_1\big) \cdot \frac{\partial G}{\partial \lambda_1}(x,\{\bar{\lambda}_1,\lambda_{-1}\}) \tag{4.14}$$

where formulae for $G_0(x,\lambda_{-1})$ and $(\partial/\partial\lambda_1)(x,\{\bar{\lambda}_1,\lambda_{-1}\})$ are given in equations (4.11) and (4.13).

4.3 Applications

4.3.1 A Difference-Form External Habit Model

We first apply the techniques described above to approximate solutions of a difference-form external habit model. Initially we focus on the case in which habits are purely a function of last period's consumption before demonstrating how the approach can be applied when habits are more persistent. Since this is the first model we discuss, the exposition shall be more detailed and shall refer to the theoretical section more frequently that those accompanying subsequent models.

We assume a representative agent model in which instantaneous utility satisfies

$$U(C_t, C_{t-1}) = \frac{(C_t - h\,C_{t-1})^{1-\gamma} - 1}{1-\gamma}$$

where $\gamma \geq 1$ and $h \in [0,1)$. Notably, for $h = 0$ this utility specification is identical to the ones considered by Lucas (1978) or Mehra and Prescott (1985). The equity share's fundamental asset pricing equation, which is a first order condition of the related utility maximization problem, can then be written as[17]

$$\begin{aligned} v_t = \frac{P_t}{C_t} &= \mathrm{E}_t\beta\left(\frac{(C_{t+1} - hC_t)/C_t}{(C_t - hC_{t-1})/C_t}\right)^{-\gamma} \cdot \frac{C_{t+1}}{C_t} \cdot \left(\frac{P_{t+1} + C_{t+1}}{C_{t+1}}\right) \\ &= \mathrm{E}_t\beta\frac{(\exp(c^g_{t+1}) - h)^{-\gamma}}{(1 - h\exp(-c^g_t))^{-\gamma}} \exp(c^g_{t+1})\big(v_{t+1} + 1\big) \end{aligned} \tag{4.15}$$

where $c^g_{t+1} := \log\big(C_{t+1}/C_t\big)$ denotes continuously compounded consumption growth. We proceed under the assumption that consumption growth follows

[17]Note that habits 'hC_{t-1}' are considered exogenous when deriving the first order conditions, hence the name 'external habit model'.

an AR(1) process

$$(c^g_{t+1} - \mu) = \rho(c^g_t - \mu) + \xi_{t+1} \tag{4.16}$$

where $0 \leq \rho < 1$ and ξ is an *i.i.d.* sequence.

Fixed point representation

Since equation (4.15) shows that c^g is the only state variable, therefore in terms of the previous section's notation we have $x := \{c^g\}$ and $X_t := c^g_t$. Accordingly, letting $\lambda := \{\rho, \mu, \gamma, \beta, h\}$ be the parameter vector, equation (4.16) defines the function $\mathbb{F}$ such that $X_{t+1} = \mathbb{F}(X_t, \lambda, \xi_{t+1})$. In particular

$$\mathbb{F}(x\,, \lambda\,, \xi) := (1 - \lambda_1)\,\lambda_2 + \lambda_1\,x + \xi \tag{4.17}$$

where λ_k denotes the k-th coordinate of vector λ. To transform equation (4.15) into fixed-point form (4.8) define

$$H(x, \lambda) := \lambda_4(\exp(x) - \lambda_5)^{-\lambda_3} \cdot \exp(x) \quad \text{and} \quad W(x, \lambda) := (1 - \lambda_5 \exp(-x))^{-\lambda_3}.$$

In light of the definition (4.3), equation (4.15) can immediately be rewritten as

$$g(X_t, \lambda) = W^{-1}(X_t, \lambda) \cdot \mathbf{E}_t \left[H(X_{t+1}, \lambda) \cdot \big(1 + g(X_{t+1}, \lambda)\big)\right].$$

Adding one to both sides and multiplying by $H(X_t, \lambda)$, the fixed point form becomes

$$G(x\,, \lambda) = H(x\,, \lambda) + I(x\,, \lambda) \cdot \mathbf{E}\,G(\mathbb{F}(x\,, \lambda\,, \xi)\,, \lambda) \tag{4.18}$$

where the functions $G(\cdot, \cdot)$ and $I(\cdot, \cdot)$ are defined as

$$G(x\,, \lambda) := H(x\,, \lambda) \cdot (1 + g(x\,, \lambda)), \quad I(x, \lambda) := H(x, \lambda) \cdot W^{-1}(x, \lambda). \tag{4.19}$$

Accordingly, in terms of the notation introduced in equations (4.6) and (4.8)

$$\mathbb{U}(x\,, z\,, \lambda) := H(x\,, \lambda) + I(x\,, \lambda) \cdot z \tag{4.20}$$

$$\mathcal{Z}(G, \lambda)(x) := \mathbb{U}\big(x, \mathcal{P}(G, \lambda)(x), \lambda\big) \tag{4.21}$$

with $\mathcal{P}(G, \lambda)(x) = \mathbf{E}\big(G(\mathbb{F}(x, \lambda, \xi))\big)$ and equation (4.15) reduced to $G = \mathcal{Z}(G, \lambda)$.

Restrictions on shock support

Before proceeding, we need to ensure that $C_t \geq hC_{t-1} \Leftrightarrow c_t^g \geq \log(h)$ and so the utility remains well-defined. The following proposition characterizes the necessary and sufficient conditions.

Proposition 4. Invariance of c_t^g. *If c_t^g follows (4.17), $c_t^g \in (c_l^g, c_h^g)$ and $\xi_{t+1} \in (\xi_l, \xi_h)$ then $c_{t+1}^g \in (c_l^g, c_h^g) \iff \xi_i = (1-\rho)(c_i^g - \mu), i \in \{l, h\}$.*

Proof of Proposition 4. See the Appendix. □

Accordingly, $c_t^g > \log(h)$ if and only if the support of ξ is a subset of $((1-\rho)(\log(h) - \mu), +\infty)$ which we shall subsequently assume.

Existence and uniqueness of solutions

We now use the contraction mapping principle – as outlined in Theorem 3 – to characterize sufficient conditions for (4.18) to have a unique solution.

Proposition 5. *Under the assumptions of Proposition 4, the fixed point equivalent of equation (4.15) i.e. $G = \mathcal{Z}(G, \lambda)$ has a unique solution in $\mathcal{G}$ if*[18]

$$\exp\big((1-\gamma)\cdot((1-\rho)\mu + a\rho)\big)\cdot \mathcal{L}_\xi(\gamma - 1) < \beta^{-1} \tag{4.22}$$

where $\mathcal{L}_\xi$ denotes ξ's Laplace transform – i.e. $\mathcal{L}_\xi(u) = \mathbf{E}\exp(-u\,\xi)$ – and a is the lower bound of log consumption growth's (c^g) support.

Proof of Proposition 5. See the Appendix. □

Benchmark solutions

To find a benchmark solution recall that $\mathbb{F}(x, \{\rho, \mu, \gamma, \beta, h\}, \xi) := (1-\rho)\,\mu + \rho x + \xi$. Accordingly $\mathbb{F}(\cdot,\cdot,\cdot)$ becomes independent of x – which is the condition underlying assumption A3 – when $\rho = 0$. In terms of previous notation $\lambda_1 =$

[18]Since $\gamma \geq 1$ and $\rho \in [0,1]$, therefore this condition implies that

$$\begin{aligned}\mathbf{E}_t \exp\big((1-\gamma)\,c_{t+1}^g\big) &= \mathbf{E}_t \exp\big((1-\gamma)\cdot((1-\rho)\,\mu + c_t^g\rho + \xi_{t+1})\big)\\ &\leq \mathbf{E}\exp\big((1-\gamma)\cdot((1-\rho)\,\mu + a\rho + \xi)\big) < \beta^{-1}.\end{aligned}$$

ρ, $\bar{\lambda}_1 = 0$ and $\lambda_{-1} = \{\mu, \gamma, \beta, h\}$. The following Proposition can thus be seen as an application of Theorem 5 to the simple habit model and characterizes the nested solution corresponding to $\rho = 0$.

Proposition 6. *Under the assumptions of Proposition 4 and 5, when ρ in (4.17) is set to zero - i.e. when $x := \mu + \xi$ - then the unique solution of equation (4.15) is given by*

$$g(x\,, \{0, \lambda_{-1}\}) = \frac{I(x\,, \{0, \lambda_{-1}\})/H(x\,, \{0, \lambda_{-1}\}) \cdot \mathbf{E}\, H(\mu + \xi, \{0, \lambda_{-1}\})}{1 - \mathbf{E}\, W^{-1}(\mu + \xi, \{0, \lambda_{-1}\})\, H(\mu + \xi,\, \{0, \lambda_{-1}\})}. \quad (4.23)$$

Equation (4.23) is equivalent to $P_t/C_t = \kappa(\lambda_{-1}) \cdot (C_t - hC_{t-1})^{\gamma}/C_t^{-\gamma}$ where $\kappa(\lambda_{-1}) := \mathbf{E}\big(1/\beta - \exp((\mu + \xi) \cdot (1 - \gamma))\big)^{-1} \cdot \mathbf{E}\exp\big((\mu + \xi) \cdot (1 - \gamma)\big) \cdot \big(1 - h\exp(-(\mu + \xi))\big)^{-\gamma}$.[19]

Proof of Proposition 6. See the Appendix. □

Proposition 6 serves to illustrate an important point. Substituting in for $I(\cdot, \cdot)$ and $H(\cdot, \cdot)$ from (4.19) into equation (4.23) and simplifying, we can write the zeroth order approximation as

$$g^0(x, \{\rho, \mu, \gamma, \beta, h\}) = (1 - h\exp(-x))^{\gamma} \frac{\mathbf{E}\, H(\mu + \xi, \{0, \lambda_{-1}\})}{1 - \mathbf{E}\, I(\mu + \xi, \{0, \lambda_{-1}\})}. \quad (4.24)$$

Since $H(x, \lambda)$ and $I(x, \lambda)$ are nonlinear functions of x, therefore changes in the volatility of ξ are typically going to affect $\mathbf{E}\, H(\mu + \xi, \{0, \lambda_{-1}\})$ and $\mathbf{E}\, I(\mu + \xi, \{0, \lambda_{-1}\})$. This shows that our zeroth order approximation is **not** certainty-equivalent as the implied policy function can change in response to changes in distributional properties of shocks. This is in stark contrast to standard perturbation methods (Schmitt-Grohe and Uribe, 2004) where even the first order approximation is certainty-equivalent - i.e. it is totally unsuitable for the analysis of asset price dynamics.

Furthermore, we know that even at second order changes in shock volatility only affect the constant of the 'standard' perturbation approximation. In order for 'risk' to have an impact on the slope of $g^k(x, \lambda)$ one needs to go to third-order or higher. This is important as, arguably, it is precisely why researchers interested

[19] Some of the coefficients in equation (4.23) as well as those listed in Table 1, equal expected values of functions of ξ. Since the distribution of ξ and all the functions are known (the functions equal $H(\cdot, \cdot)$ or $I(\cdot, \cdot)$ or their derivatives) therefore these constants can easily be evaluated, for example using Monte-Carlo methods.

in time-variation in risk premia (or any other variable which reflects 'risk') are forced to derive third order polynomial approximations to the policy function. Equation (4.24) shows, however, that in contrast to 'standard' perturbations, changes in shock volatility will affect the slope of our zeroth-order approximation. Hence even our lowest order approximations do not inherit the undesirable properties of 'standard' perturbations and can be better suited for asset pricing applications.

Approximations around benchmark solutions

We now show how to find an approximation of the solution corresponding to $\rho \neq 0$ - i.e. how to use $g(x\,, \{0, \lambda_{-1}\})$ to approximate $g(x, \{\rho, \lambda_{-1}\})$.

Proposition 7. *Under the assumptions of Proposition 4 and 5, the 'fifth order' approximation $g^5(x, \lambda)$ to $g(x, \{\rho, \lambda_{-1}\})$ 'around' $g(x, \{0, \lambda_{-1}\})$ is*

$$
\begin{aligned}
g^5(x, \{\rho, \mu, \gamma, \beta, h\}) = (1 - h\exp(-x))^\gamma \cdot \Bigg(& \frac{C^H}{1-C^I} + \rho\left(U_{1,1}(x-\mu) + U_{2,1}\right) \\
& + \frac{\rho^2}{2}\left(U_{1,2}(x-\mu)^2 + U_{2,2}(x-\mu) + U_{3,2}\right) \\
& + \frac{\rho^3}{3!}\left(U_{1,3}(x-\mu)^3 + U_{2,3}(x-\mu)^2 + U_{3,3}(x-\mu) + U_{4,3}\right) \\
& + \frac{\rho^4}{4!}\left(U_{1,4}(x-\mu)^4 + U_{2,4}(x-\mu)^3 + U_{3,4}(x-\mu)^2 + U_{4,4}(x-\mu) + U_{5,4}\right) \\
& + \frac{\rho^5}{5!}\left(U_{1,5}(x-\mu)^5 + U_{2,5}(x-\mu)^4 + U_{3,5}(x-\mu)^3 + U_{4,5}(x-\mu)^2 + U_{5,5}(x-\mu) + U_{6,5}\right)\Bigg)
\end{aligned}
$$

where $C^H = \mathbf{E}\, H(\mu + \xi, \{0, \lambda_{-1}\}), C^I = \mathbf{E}\, I(\mu + \xi, \{0, \lambda_{-1}\})$ and the coefficients $U_{i,j}$ are defined in Table 1 and can easily be computed.

Proof of Proposition 7. See the Appendix. □

Proposition 7 shows that $g^5(x,\, \lambda)$ is a fifth-order polynomial in the parameter ρ. It also demonstrates that the approximation is **not** a polynomial in the state-variable x, as the latter enters via products of $(1 - h\exp(-x))^\gamma$ and $(x - \mu)^k$. Both of these reflect general properties of our approximate formulae – i.e. x will typically enter nonlinearly and the approximation will be a polynomial in parameters which needed to be restricted to find a 'nested' solution (in this case ρ had to be set to zero).[20]

[20]What is not a general property of the method, however, is that x enters as a product of

$U_{i,j}$	$i=1$	$i=2$	$i=3$
$j=1$	$\mathcal{C}^{\mathcal{H}(1)}+\mathcal{C}^{G}\mathcal{C}^{\mathcal{I}(1)}$	See below for general formula	–
$j=2$	$\mathcal{C}^{\mathcal{H}(2)}+\mathcal{C}^{G}\mathcal{C}^{\mathcal{I}(2)}$	$2\big((\mathcal{C}^{\mathcal{I}}+\mathcal{C}^{\mathcal{I}(1)\xi})U_{1,1}+\mathcal{C}^{\mathcal{I}(1)}U_{2,1}\big)$	See below for general formula
$j=3$	$\mathcal{C}^{\mathcal{H}(3)}+\mathcal{C}^{G}\mathcal{C}^{\mathcal{I}(3)}$	$3\big((2\mathcal{C}^{\mathcal{I}(1)}+\mathcal{C}^{\mathcal{I}(2)\xi})U_{1,1}+\mathcal{C}^{\mathcal{I}(2)}U_{2,1}\big)$	$3\big((2\mathcal{C}^{\mathcal{I}\xi}+\mathcal{C}^{\mathcal{I}(1)\xi^2})U_{1,2}+(\mathcal{C}^{\mathcal{I}}+\mathcal{C}^{\mathcal{I}(1)\xi})U_{2,2}+\mathcal{C}^{\mathcal{I}(1)}U_{3,2}\big)$
$j=4$	$\mathcal{C}^{\mathcal{H}(4)}+\mathcal{C}^{G}\mathcal{C}^{\mathcal{I}(4)}$	$4\big((3\mathcal{C}^{\mathcal{I}(2)}+\mathcal{C}^{\mathcal{I}(3)\xi})U_{1,1}+\mathcal{C}^{\mathcal{I}(3)}U_{2,1}\big)$	$6\big((2\mathcal{C}^{\mathcal{I}}+4\mathcal{C}^{\mathcal{I}(1)\xi}+\mathcal{C}^{\mathcal{I}(2)\xi^2})U_{1,2}+(2\mathcal{C}^{\mathcal{I}(1)}+\mathcal{C}^{\mathcal{I}(2)\xi})U_{2,2}+\mathcal{C}^{\mathcal{I}(2)}U_{3,2}\big)$
$j=5$	$\mathcal{C}^{\mathcal{H}(5)}+\mathcal{C}^{G}\mathcal{C}^{\mathcal{I}(5)}$	$5\big((4\mathcal{C}^{\mathcal{I}(3)}+\mathcal{C}^{\mathcal{I}(4)\xi})U_{1,1}+\mathcal{C}^{\mathcal{I}(4)}U_{2,1}\big)$	$10\big((6\mathcal{C}^{\mathcal{I}(1)}+6\mathcal{C}^{\mathcal{I}(2)\xi}+\mathcal{C}^{\mathcal{I}(3)\xi^2})U_{1,2}+(3\mathcal{C}^{\mathcal{I}(2)}+\mathcal{C}^{\mathcal{I}(3)\xi})U_{2,2}+\mathcal{C}^{\mathcal{I}(3)}U_{3,2}\big)$
	$i=4$		
$j=4$	$4\big((3\mathcal{C}^{\mathcal{I}\xi^2}+\mathcal{C}^{\mathcal{I}(1)\xi^3})U_{1,3}+(2\mathcal{C}^{\mathcal{I}\xi}+\mathcal{C}^{\mathcal{I}(1)\xi^2})U_{2,3}+(\mathcal{C}^{\mathcal{I}}+\mathcal{C}^{\mathcal{I}(1)\xi})U_{3,3}+\mathcal{C}^{\mathcal{I}(1)}U_{4,3}\big)$		
$j=5$	$10\big((6(\mathcal{C}^{\mathcal{I}\xi}+\mathcal{C}^{\mathcal{I}(1)\xi^2})+\mathcal{C}^{\mathcal{I}(2)\xi^3})U_{1,3}+(2\mathcal{C}^{\mathcal{I}}+4\mathcal{C}^{\mathcal{I}(1)\xi}+\mathcal{C}^{\mathcal{I}(2)\xi^2})U_{2,3}+(2\mathcal{C}^{\mathcal{I}'}+\mathcal{C}^{\mathcal{I}(2)\xi})U_{3,3}+\mathcal{C}^{\mathcal{I}(2)}U_{4,3}\big)$		
	$i=5$		
$j=5$	$5\big((4\mathcal{C}^{\mathcal{I}\xi^3}+\mathcal{C}^{\mathcal{I}(1)\xi^4})U_{1,4}+(3\mathcal{C}^{\mathcal{I}\xi^2}+\mathcal{C}^{\mathcal{I}(1)\xi^3})U_{2,4}+(2\mathcal{C}^{\mathcal{I}\xi}+\mathcal{C}^{\mathcal{I}(1)\xi^2})U_{3,4}+(\mathcal{C}^{\mathcal{I}}+\mathcal{C}^{\mathcal{I}(1)\xi})U_{4,4}+\mathcal{C}^{\mathcal{I}(1)}U_{5,4}\big)$		

And where $\forall i \geq 1 : U_{i+1,i} := \big(\mathcal{C}^{\mathcal{I}\xi^i}U_{1,i}+\mathcal{C}^{\mathcal{I}\xi^{(i-1)}}U_{2,i}+\ldots+\mathcal{C}^{\mathcal{I}\xi}U_{i,i}\big)/(1-\mathcal{C}^{\mathcal{I}})$ and the other constants referred to above are given by:

$\mathcal{C}^{\mathcal{H}}=\mathbf{E}\mathcal{H}(\mu+\xi)$	$\mathcal{C}^{\mathcal{H}(1)}=\mathbf{E}\mathcal{H}'(\mu+\xi)$	$\mathcal{C}^{\mathcal{H}(2)}=\mathbf{E}\mathcal{H}''(\mu+\xi)$	$\mathcal{C}^{\mathcal{H}(3)}=\mathbf{E}\mathcal{H}^{(3)}(\mu+\xi)$	$\mathcal{C}^{\mathcal{H}(4)}=\mathbf{E}\mathcal{H}^{(4)}(\mu+\xi)$	$\mathcal{C}^{\mathcal{H}(5)}=\mathbf{E}\mathcal{H}^{(5)}(\mu+\xi)$
$\mathcal{C}^{\mathcal{I}}=\mathbf{E}\mathcal{I}(\mu+\xi)$	$\mathcal{C}^{\mathcal{I}(1)}=\mathbf{E}\mathcal{I}'(\mu+\xi)$	$\mathcal{C}^{\mathcal{I}(2)}=\mathbf{E}\mathcal{I}''(\mu+\xi)$	$\mathcal{C}^{\mathcal{I}(3)}=\mathbf{E}\mathcal{I}^{(3)}(\mu+\xi)$	$\mathcal{C}^{\mathcal{I}(4)}=\mathbf{E}\mathcal{I}^{(4)}(\mu+\xi)$	$\mathcal{C}^{\mathcal{I}(5)}=\mathbf{E}\mathcal{I}^{(5)}(\mu+\xi)$
$\mathcal{C}^{\mathcal{I}\xi}=\mathbf{E}\mathcal{I}(\mu+\xi)\xi$	$\mathcal{C}^{\mathcal{I}(1)\xi}=\mathbf{E}\mathcal{I}'(\mu+\xi)\xi$	$\mathcal{C}^{\mathcal{I}(2)\xi}=\mathbf{E}\mathcal{I}''(\mu+\xi)\xi$	$\mathcal{C}^{\mathcal{I}(3)\xi}=\mathbf{E}\mathcal{I}^{(3)}(\mu+\xi)\xi$	$\mathcal{C}^{\mathcal{I}(4)\xi}=\mathbf{E}\mathcal{I}^{(4)}(\mu+\xi)\xi$	–
$\mathcal{C}^{\mathcal{I}\xi^2}=\mathbf{E}\mathcal{I}(\mu+\xi)\xi^2$	$\mathcal{C}^{\mathcal{I}(1)\xi^2}=\mathbf{E}\mathcal{I}'(\mu+\xi)\xi^2$	$\mathcal{C}^{\mathcal{I}(2)\xi^2}=\mathbf{E}\mathcal{I}''(\mu+\xi)\xi^2$	$\mathcal{C}^{\mathcal{I}(3)\xi^2}=\mathbf{E}\mathcal{I}^{(3)}(\mu+\xi)\xi^2$	–	–
$\mathcal{C}^{\mathcal{I}\xi^3}=\mathbf{E}\mathcal{I}(\mu+\xi)\xi^3$	$\mathcal{C}^{\mathcal{I}(1)\xi^3}=\mathbf{E}\mathcal{I}'(\mu+\xi)\xi^3$	$\mathcal{C}^{\mathcal{I}(2)\xi^3}=\mathbf{E}\mathcal{I}''(\mu+\xi)\xi^3$	–	–	–
$\mathcal{C}^{\mathcal{I}\xi^4}=\mathbf{E}\mathcal{I}(\mu+\xi)\xi^4$	$\mathcal{C}^{\mathcal{I}(1)\xi^4}=\mathbf{E}\mathcal{I}'(\mu+\xi)\xi^4$	–	–	–	–
$\mathcal{C}^{\mathcal{I}\xi^5}=\mathbf{E}\mathcal{I}(\mu+\xi)\xi^5$	–	–	–	–	$\mathcal{C}^{G}=\mathcal{C}^{\mathcal{H}}/(1-\mathcal{C}^{\mathcal{I}})$

Table 1. Coefficients of Perturbation Approximations to the Price Dividend Ratio Function in the difference-form External Habit Model.

Deriving Expressions for Error Bounds

We shall now demonstrate how bounds can be put on the approximation error by estimating the relevant norms appearing in the theoretical section. To fix attention, we asses how closely the zeroth order formula approximates the true solution. Errors of higher order approximations can be computed analogously.

Proposition 8. *Under the assumptions of Proposition 4 and 5, letting* $\lambda = \{\lambda_1, \lambda_{-1}\}$ *with* $\lambda_1 = \{\rho\}, \lambda_{-1} = \{\mu, \gamma, \beta, h\}$*, bounds on the error made approximating* $G(x, \{\lambda_1, \lambda_{-1}\})$ *by* $G(x, \{0, \lambda_{-1}\})$ *are given by*

$$\begin{aligned} &\| G(\cdot, \{\rho, \mu, \gamma, \beta, h\}) - G(\cdot, \{0, \mu, \gamma, \beta, h\})\|^{1-\gamma} \\ &\qquad \leq \rho \cdot \kappa_a \cdot \left(\frac{\beta}{1-\alpha}\right)^2 \cdot [1 - \gamma\alpha] \cdot \left(1 - \frac{h}{\exp(a)}\right)^{-\gamma} \end{aligned}$$

where $\kappa_a := (\sup_{x \geq a} |x - \mu| \exp((1-\gamma)x))$*, a is the lower bound of the shock support and the Lipschitz constant* α *is given by* $\alpha = \beta \exp((1-\gamma) \cdot ((1-\rho)\mu + a\rho))\mathcal{L}_\xi(\gamma - 1)$ *where* $\mathcal{L}_\xi$ *is the Laplace transform of* ξ*'s distribution.*

Proof of Proposition 8. See the Appendix. □

A Comparison of Approximate Policy Functions

We conclude the discussion of the simple, difference-form external habit model by comparing the approximate policy functions found using our proposed approach to those obtained using 'standard' perturbation methods as well as the exact solution derived in chapter 3. We stress that since the charts reported here are specific to the underlying calibration, they are simply meant to give an idea of how the approximations compare and are not necessarily indicative of the superiority of one method over another.

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a nonlinear function and a polynomial. Courtesy of the exact solutions to this model derived in Corollary 1 of Chapter 3, we know that this simply reflects the form of the true solution. In particular, given that the Taylor expansion to $\exp(x) = \sum_i x^i/i!$ and in light of the exact formula

$$v_t = \left(1 - h\exp(-x_t)\right)^\gamma \sum_{i=1}^{+\infty} \sum_{n=0}^{+\infty} d_{i,n} \exp\Big((x_t - \mu)\, b_{i,n} + c_{i,n}\Big) \tag{4.25}$$

where $b_{i,n} := \rho\left((1-\gamma)\frac{(1-\rho^i)}{(1-\rho)} - n\rho^{i-1}\right)$, the form of the approximation should come as no surprise (see also Chapter 3 for definitions of $c_{i,n}$ and $d_{i,n}$).

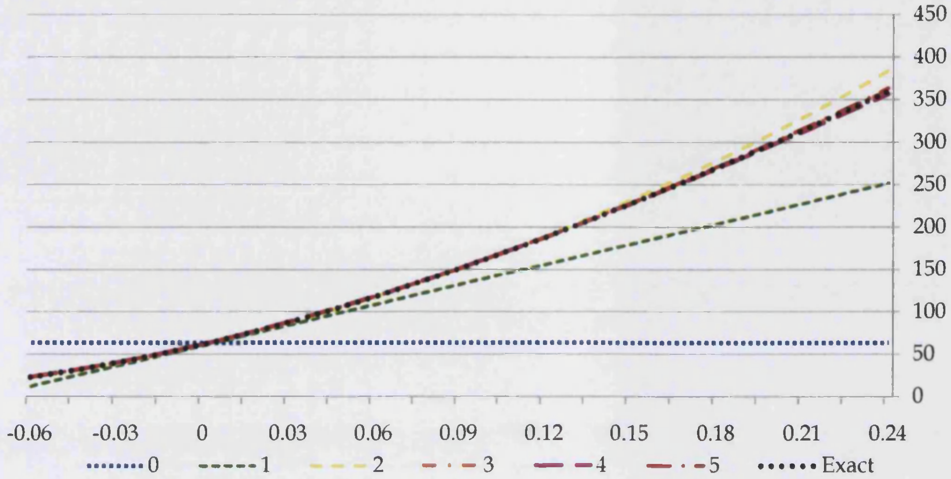

Figure 4.1: Approximate Policy Functions Derived Using Standard Perturbation Methods by Approximation Order (0 - 5) - Case of the Difference-Form External Habit Model

To fix attention we let β equal 0.99, set the coefficient γ to 2.37 (Campbell and Cochrane, 1999), use an h of 0.85 (Juillard et al., 2006) and fix coefficients for mean quarterly consumption growth μ and consumption growth persistence ρ equal to their GMM estimates of 0.41% and 0.23 respectively (based on quarterly US data 1946 Q1 – 1996 Q4, the same sample period and series used in Campbell and Cochrane (1999)). Policy function approximations derived using standard perturbation methods are plotted in Figure 4.1, those obtained using the approach proposed here (see also Proposition 7) are presented in Figure 4.2, while Figure 4.3 shows a 'side-by-side' comparison.

In light of the fact that 'standard' perturbation approximations returned by *dynare++* implicitly assume normally distributed shocks, therefore – to make the resulting approximations comparable – we assumed that ξ is a Gaussian white-noise process.[21,22] In line with the previous parametrization we set the standard deviation of log-consumption growth equal to the GMM estimate of 0.87% (this coincided with the value of the gaussian-ML estimate). Finally, given that *dynare++* returns the perturbation approximation as a function of c^g_{t-1} and ξ_t we also needed to transform the resulting polynomial into one, which

[21]For more information on *dynare++* see also *http://www.cepremap.cnrs.fr/dynare*.

[22]Notably, this assumption is incompatible with the conditions derived in Proposition (4) – as the normal distribution is not bounded from below. While this means that C_t is not guaranteed to stay above hC_{t-1}, and limits the 'economic' interpretation of the resulting formulae, we can still formally compare the resulting approximating functions.

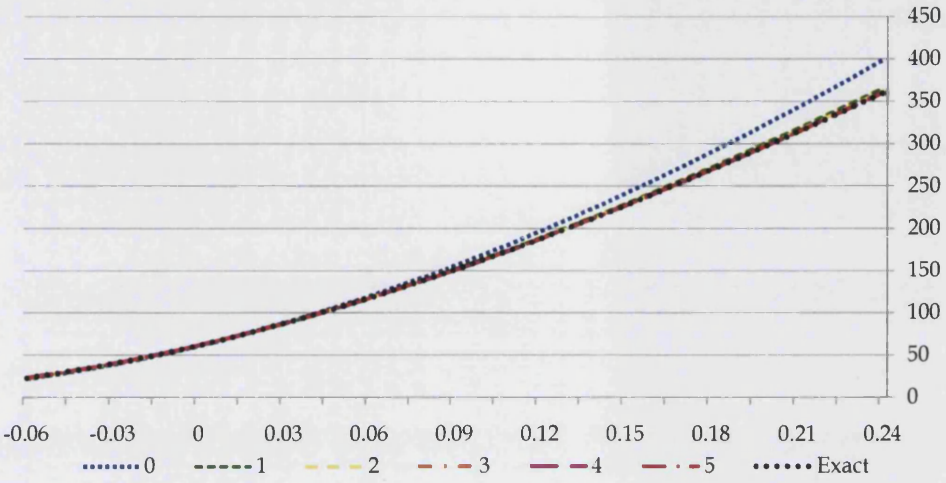

Figure 4.2: Approximate Policy Functions Derived Using the Implicit Function Approach by Approximation Order (0 - 5) - Case of the Difference-Form External Habit Model

was purely a function of c_t^g. We used standard Matlab optimization routines to find the required coefficients.

Inspecting figures 4.1 – 4.3 reveals that, for a wide range of values of log-consumption growth – ranging from -24% to 96% in annualized terms – 3rd (and higher) order approximations derived using both methods are virtually indistinguishable from the exact solution. The same is no longer true of lower order approximations, where differences in performance are clearly visible. In particular, even the first order approximation found using the Banach / 'implicit theorem' approach does a very good job of matching the policy function – easily exceeding the accuracy of second order perturbation approximations for high values of log-consumption growth. We take this as suggestive evidence that if one is interested in a parsimonious and accurate characterization of the policy function then the 'implicit theorem' method advocated here has the potential to deliver.

4.3.2 A Persistent Habit Extension

We now consider an extension of the previous model in which habits display some persistence – i.e. depend not only on last period's consumption but also

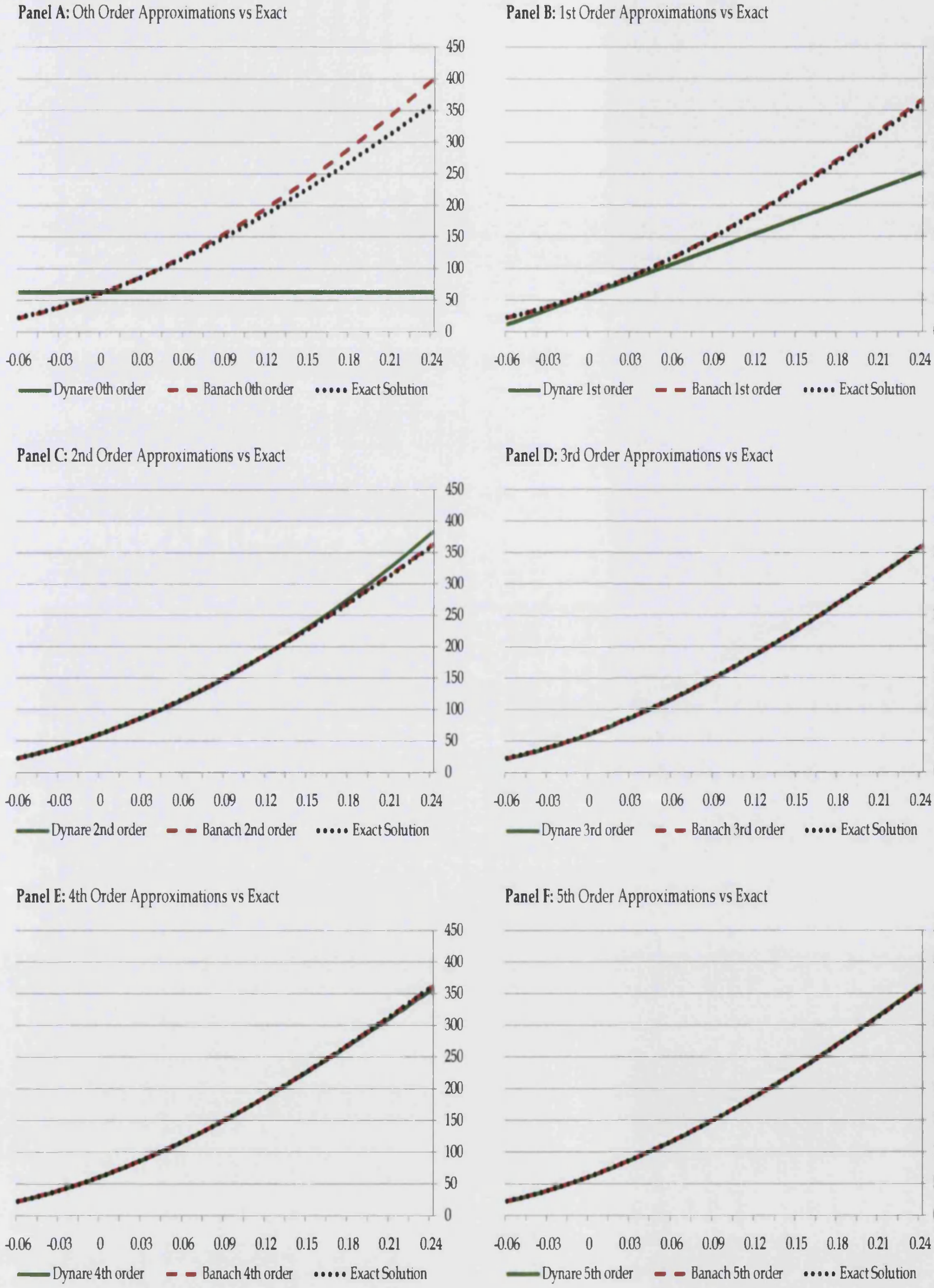

Figure 4.3: Standard Perturbation vs Approximations Derived Using the Implicit Function Approach (Banach, red-dashed line) – Case of the Difference-Form External Habit Model

on consumption two periods ago. Let

$$U(C_t, C_{t-1}, C_{t-2}) = \frac{\left(C_t - h((1-\phi)\, C_{t-1} + \phi\, C_{t-2})\right)^{1-\gamma} - 1}{1-\gamma}$$

where $\gamma \geq 1$ and $h, \phi \in [0,1)$. Clearly, if $\phi = 0$ we recover the previous specification. The equity share's fundamental asset pricing equation can be written as[23]

$$\begin{aligned} v_t = \frac{P_t}{C_t} &= \mathrm{E}_t\beta\left(\frac{(C_{t+1} - h((1-\phi)C_t + \phi C_{t-1}))/C_t}{(C_t - h((1-\phi)C_{t-1} + \phi C_{t-2}))/C_t}\right)^{-\gamma} \cdot \frac{C_{t+1}}{C_t} \cdot \left(\frac{P_{t+1} + C_{t+1}}{C_{t+1}}\right) \\ &= \mathrm{E}_t\beta\frac{(\exp(c^g_{t+1}) - h((1-\phi) + \phi\exp(-c^g_t)))^{-\gamma}\exp(c^g_{t+1})}{(1 - h((1-\phi)\exp(-c^g_t) + \phi\exp(-c^g_t - c^g_{t-1})))^{-\gamma}}(v_{t+1} + 1) \end{aligned} \tag{4.27}$$

and again we proceed under the assumption that c^g_t follows (4.16).

Fixed point representation

Inspecting (4.27) we see that the state variable x is now two dimensional and depends on both current and past consumption growth, i.e. $X_{t+1} = (c^g_{t+1}, c^g_t)$. Accordingly

$$X_{t+1} = \mathbb{F}(X_t, \lambda, \xi_{t+1}) \tag{4.28}$$

where the parameter vector is given by $\lambda := \{\rho, \mu, \gamma, \beta, h, \phi\}$ and $\mathbb{F}$ is implicitly defined in equation (4.16) as $\mathbb{F}(x, \lambda, \xi) := \{\mathbb{F}_1(x, \lambda, \xi), \mathbb{F}_2(x, \lambda, \xi)\} = \{(1 - \lambda_1)\,\lambda_2 + \lambda_1 x_1 + \xi, x_1\}$ and x_i denotes the i-th coordinate of vector x.

To transform equation (4.27) into fixed-point form (4.8) define

$$\begin{aligned} H(x, \lambda) &:= \lambda_4(\exp(x_1) - \lambda_5((1-\lambda_6) + \lambda_6\exp(-x_2)))^{-\lambda_3}\exp(x_1) \\ W(x, \lambda) &:= (1 - \lambda_5((1-\lambda_6)\exp(-x_1) + \lambda_6\exp(-x_1 - x_2)))^{-\lambda_3}. \end{aligned}$$

Letting $g(\cdot,\cdot)$ map the state into asset prices equation (4.27) becomes

$$g(X_t, \lambda) = W^{-1}(X_t, \lambda) \cdot \mathrm{E}_t\left[H(X_{t+1}, \lambda)\big(1 + g(X_{t+1}, \lambda)\big)\right].$$

[23] We use the fact that

$$\frac{C_{t+k}}{C_t} = \frac{C_{t+k}}{C_{t+k-1}} \cdot \frac{C_{t+k-1}}{C_{t+k-2}} \cdot \ldots \cdot \frac{C_{t+1}}{C_t} = \exp\left(c^g_{t+k} + c^g_{t+k-1} + \ldots + c^g_{t+1}\right). \tag{4.26}$$

Adding one to both sides, multiplying by $H(X_t, \lambda)$ and defining

$$G(x\,,\lambda) := H(x\,,\lambda) \cdot (1 + g(x\,,\lambda)), \quad I(x\,,\lambda) := H(x\,,\lambda) \cdot W^{-1}(x\,,\lambda) \tag{4.29}$$

we arrive at the fixed-point form equivalent of equation (4.27)

$$G(x\,,\lambda) = H(x\,,\lambda) + I(x\,,\lambda) \cdot \mathrm{E}\, G(\mathbb{F}(x\,,\lambda\,,\xi)\,,\lambda) \tag{4.30}$$

or in terms of the general notation introduced in equations (4.6) and (4.8)

$$\mathbb{U}(x\,,z\,,\lambda) := H(x\,,\lambda) + I(x\,,\lambda) \cdot z \tag{4.31}$$

$$\mathcal{Z}(G,\lambda)(x) := \mathbb{U}\big(x, \mathcal{P}(G,\lambda)(x), \lambda\big) \tag{4.32}$$

with $\mathcal{P}(G,\lambda)(x) = \mathrm{E}\big(G(\mathbb{F}(x,\lambda,\xi))\big)$ and equation (4.27) reduced to $G = \mathcal{Z}(G,\lambda)$. Notably, the only change – relative to the previous example – is in the definitions of $H(\cdot,\cdot)$ and $I(\cdot,\cdot)$ in equation (4.31).

Restrictions on shock support

As in the previous example, before proceeding, we first need to characterize conditions which ensure that $C_t \geq h\big((1-\phi)\, C_{t-1} + \phi\, C_{t-2}\big)$ and so utility remains well-defined.

Proposition 9. Invariance of c_t^g. *If c_t^g follows (4.17), $c_t^g \geq c_l^g$ and*

$$c_l^g \in \left(\log\big(h(1-\phi) - \sqrt{\frac{h^2(1-\phi)^2}{4} + h\phi}\big), \log\big(h(1-\phi) + \sqrt{\frac{h^2(1-\phi)^2}{4} + h\phi}\big)\right)$$

then $c_{t+1}^g \geq c_l^g \iff \exp(\xi_l) \geq h\big((1-\phi) + \phi\exp(-c_l^g)\big)\exp(-\mu(1-\rho) - \rho c_l^g)$. *Accordingly,* $c_t^g > h\big((1-\phi) + \phi\exp(-c_l^g)\big)$ *if and only if the support of* ξ *is a subset of* $\big(\log\big(h((1-\phi) + \phi\exp(-c_l^g))\exp(-\mu(1-\rho) - \rho c_l^g)\big), +\infty\big)$.

Proof of Proposition 9. See the Appendix. □

As in the previous example, in the remainder of this section we assume that the initial conditions and shock support satisfy the restrictions above.

Existence and uniqueness of solutions

Having rewritten equation (4.27) in fixed point form, we can now use the contraction mapping principle – as outlined in Theorem 3 – to characterize sufficient conditions for $G = \mathcal{Z}(G, \lambda)$ to have a unique solution in $\mathcal{G}$.

Proposition 10. *Under the assumptions of Proposition 9, the fixed point equivalent of equation (4.27), i.e. $G = \mathcal{Z}(G, \lambda)$ has a unique solution in $\mathcal{G}$ if*

$$\exp\big((1-\gamma)\cdot((1-\rho)\mu + c\rho)\big)\cdot \mathcal{L}_\xi(\gamma-1) < \beta^{-1} \tag{4.33}$$

where $\mathcal{L}$ denotes ξ's Laplace transform and c is the lower bound of c^g's support.

Proof of Proposition 10. See the Appendix. □

Benchmark solutions

Inspecting the definition of function $\mathbb{F}(x, \lambda, \xi)$ reveals that there is no parameter constellation under which $\mathbb{F}$ becomes independent of x.[24] Despite that, it is straightforward to generalize the problem – in the true spirit of perturbation methods – in order to ensure that such restrictions can be introduced. In particular, we can introduce a new parameter κ such that

$$\mathbb{F}_1(\{x_1, x_2\}, \{\kappa, \rho, \mu, \gamma, \beta, h, \phi\}, \xi) = (1-\rho)\,\mu + \kappa\rho x_1 + \xi \tag{4.34}$$

$$\mathbb{F}_2(\{x_1, x_2\}, \{\kappa, \rho, \mu, \gamma, \beta, h, \phi\}, \xi) = \kappa\, x_1. \tag{4.35}$$

[24]Even though setting $\rho = 0$ in the original problem does not make $\mathbb{F}(\cdot,\cdot,\cdot)$ independent of x – which is the condition underlying Assumption A3 – we can still find a corresponding 'nested' solution. While Proposition 11 gives the formula, it is not straightforward to use it to derive higher order approximations.

Proposition 11. *Under the assumptions of Proposition 4 and 5, when ρ is set to zero and so $x := \mu + \xi$, then the unique solution of equation (4.27) is given by*

$$g(\{x_1, x_2\}, \{0, \lambda_{-1}\}) = \frac{\beta \mathrm{U}(x_1) + \kappa_g}{\big(1 - h((1-\phi)\exp(-x_1) + \phi\exp(-x_1 - x_2))\big)^{\gamma}}.$$

where $U(y) := \mathbf{E}\exp(x)(\exp(x) - h((1-\phi) + \phi\exp(-y)))^{-\gamma}$ and the constant κ_g is given by $\kappa_g := \big[\mathbf{E}\exp\big((1-\gamma)\,x\big)\cdot\beta U(x)/\,\big(1/\beta - \mathbf{E}\exp\big((1-\gamma)\,x\big)\big)\big]$. This implies that

$$P_t = C_t \cdot \frac{\beta\mathbf{E}\exp(x)(\exp(x) - h((1-\phi) + \phi C_{t-1}/C_t))^{-\gamma} + \kappa_g}{(1 - h((1-\phi)C_{t-1}/C_t + \phi C_{t-2}/C_t))^{\gamma}}.$$

Proof of Proposition 11. See the Appendix. □.

Clearly, enriching the model in this fashion affects the definition of the parameter vector λ, which now becomes $\lambda = \{\kappa, \rho, \mu, \gamma, \beta, h, \phi\}$. Crucially, however, setting $\lambda_1 = \kappa = 1$ recovers the original transition function, while setting $\kappa = 0$ implies that $\mathbb{F}(\cdot,\cdot,\cdot)$ becomes independent of x. Accordingly, we can use Theorem 5 to find a solution for $\kappa = 0$, which can then be used to construct approximate solutions corresponding to arbitrary κ. Setting $\kappa = 1$ in these formulae will yield approximations to the solution of the original problem (the one without the artificially introduced parameter κ).

Proposition 12. *Under the assumptions of Proposition 4 and 5, when κ in (4.34) - (4.35) is set to zero, then the unique solution of equation (4.27) is*

$$G(x, \{0, \lambda_{-1}\}) := H(x\,, \{0, \lambda_{-1}\}) + I(x\,, \{0, \lambda_{-1}\})\Psi(\lambda_{-1})$$

where $\Psi(\lambda_{-1})$ equals

$$\left(1 - \mathbf{E}I(\{(1-\rho)\,\mu + \xi, 0\}\,, \{0, \lambda_{-1}\})\right)^{-1} \cdot \mathbf{E}H(\{(1-\rho)\,\mu + \xi, 0\}\,, \{0, \lambda_{-1}\}).$$

Proof of Proposition 12. The proof follows directly from Theorem 5. □

Corollary 4. *It follows from Proposition 12 that*

$$g(x, \{0, \lambda_{-1}\}) := I(x\,, \{0, \lambda_{-1}\})/H(x\,, \{0, \lambda_{-1}\}) \cdot \Psi(\lambda_{-1}) \tag{4.36}$$

and so, after simplifying, we obtain

$$P_t/C_t = \Psi(\lambda_{-1}) \cdot \left(1 - h\exp(-c_t^g\,)\right)^{\gamma}$$

where $\Psi(\lambda_{-1}) = \left(1/\beta - \mathbf{E}\exp\left((1-\gamma)((1-\rho)\mu + \xi)\right)\right)^{-1} \cdot \mathbf{E}\left\{\exp\left((1-\gamma)\right.\right.$ $\left.\left.\cdot\,((1-\rho)\mu + \xi)\right) \cdot \left(1 - h\exp(-(1-\rho)\mu - \xi)\right)^{-\gamma}\right\}$.

Notably, even though the state-variable is two dimensional, the nested solution only depends on c_t^g – mirroring properties of the *dynare++* perturbation approximation (transformed to eliminate ξ_t – as done previously).

Approximations around benchmark solutions

Inspecting equation (4.30) reveals that even though the asset pricing equation (4.27) is more complicated than (4.15) and even though the dimensionality of the state variable x is different, the corresponding fixed point equations can in both cases be written as

$$G(x\,,\lambda) = H(x\,,\lambda) + I(x\,,\lambda)\cdot \mathrm{E}\,G(\mathbb{F}(x\,,\lambda\,,\xi)\,,\lambda)$$

where, as stressed previously, the definitions of functions $H(x\,,\lambda), I(x\,,\lambda)$ and $\mathbb{F}(x, \lambda, \xi)$ are different (in particular, $\mathbb{F}$ is now given by (4.34) – (4.35), and λ is correspondingly expanded). That observation implies that approximations to solutions expressed in terms of functions $H(x\,,\lambda)$ and $I(x\,,\lambda)$ are going to be identical – of which Proposition 12 and Corollary 4 are particular examples.[25] Accordingly, to obtain a fifth order approximation to $g(x, \{\lambda_1, \lambda_{-1}\})$ – which would now be a fifth order polynomial in κ – we would simply need to adapt the derivations behind Proposition 7. Due to space constraints, we do not report the formulae or the modified derivations.

4.3.3 The Model of Abel (1990)

We now consider an extension of the catching up with the Joneses habit model discussed in Abel (1990).[26] Abel (1990) assumes that instantaneous utility is given by[27]

$$U(c_t, \nu_t) := \frac{(c_t/\nu_t)^{1-\gamma}}{1-\gamma} \qquad \text{where} \qquad \nu_t := \left(c_{t-1}^{D}\cdot C_{t-1}^{1-D}\right)^{\eta}$$

and $\gamma, \eta, D \geq 0$ are constants, c_t is the agent's own consumption level and C_t is aggregate per capita consumption, both in period t. Abel demonstrates (direct implication of his equation (8)) that the price dividend ratio satisfies

$$v_t = \mathrm{E}_t \beta \frac{\partial \mathcal{U}_{t+1}/\partial c_{t+1}}{\mathrm{E}_t\, \partial \mathcal{U}_t/\partial c_t}\cdot\left(\frac{C_{t+1}}{C_t}\right)\cdot(1+v_{t+1}) \tag{4.37}$$

where $\mathcal{U}_t = \sum_{j=0}^{+\infty}\beta^j U(c_{t+j}, \nu_{t+j})$ and where his equation (5) states that

$$\frac{\partial \mathcal{U}_t}{\partial c_t} = \left(1 - \beta\eta D\left(\frac{c_{t+1}}{c_t}\right)^{1-\gamma}\cdot\left(\frac{\nu_t}{\nu_{t+1}}\right)^{1-\gamma}\right)\cdot\left(\frac{c_t}{\nu_t}\right)^{1-\gamma}\cdot\left(\frac{1}{c_t}\right).$$

[25] Note, in particular, the formula for the nested solution $g(x, \{0, \lambda_{-1}\})$ in terms of functions $H(\cdot,\cdot), I(\cdot,\cdot)$ and $W(\cdot,\cdot)$ is identical in equations (4.23) and (4.36).

[26] It is an extension because we allow for a more general consumption growth specification, which nests the i.i.d. assumption of the original model.

[27] To ensure that parameters retain their meaning throughout this paper, our notation deviates somewhat from that of the original paper.

Denoting $c_t^g := \log(C_t/C_{t-1})$ and assuming that in equilibrium $c_t = C_t$ yields

$$\frac{\partial \mathcal{U}_t}{\partial c_t} C_t = \Big(\exp\big(c_t^g(1-\gamma)\eta\big) - \beta\,\eta\,D\,\exp\big(c_{t+1}^g(1-\gamma)\big)\Big) \cdot C_t^{(1-\eta)(1-\gamma)}. \quad (4.38)$$

where we have used the fact that $\nu_t = C_{t-1}^{\eta}$. Accordingly (4.37) becomes

$$v_t = \mathbf{E}_t \beta \frac{\exp\big((1-\gamma)c_{t+1}^g\big) - \beta\eta D \exp\big((1-\gamma)(c_{t+2}^g + (1-\eta)c_{t+1}^g)\big)}{\big[\exp\big((1-\gamma)\eta c_t^g\big) - \beta\eta D \mathbf{E}_t \exp\big((1-\gamma)c_{t+1}^g\big)\big]\,[1+v_{t+1}]^{-1}}. \quad (4.39)$$

As before (e.g. equation (4.16)) we assume that consumption growth follows

$$\big(c_{t+1}^g - \mu\big) = \rho\big(c_t^g - \mu\big) + \xi_{t+1} \quad (4.40)$$

where $0 \le \rho < 1$ and ξ is an *i.i.d.* sequence.

Fixed point representation

Plugging in for c_{t+2}^g from (4.40), equation (4.39) can be rewritten as

$$v_t = \mathbf{E}_t \beta \frac{\exp\big((1-\gamma)c_{t+1}^g\big) - \beta\eta D\zeta \exp\big((1-\gamma)(1-\eta+\rho)c_{t+1}^g\big)}{\exp\big((1-\gamma)\eta c_t^g\big) - \beta\eta D\zeta \exp\big((1-\gamma)\rho c_t^g\big)} \big(1 + v_{t+1}\big) \quad (4.41)$$

where ζ is a constant equal to $\zeta := \mathbf{E}\exp\big((1-\gamma)((1-\rho)\mu+\xi)\big)$.[28] This suggests that the state variable x is one dimensional with $X_t = c_t^g$.

To express equation (4.41) in fixed point form we can add one to both sides and then multiply by the numerator lagged one period i.e. $\exp\big((1-\gamma)c_t^g\big) - \beta\eta D\zeta \exp\big((1-\gamma)(1-\eta+\rho)c_t^g\big)$. Exploiting the fact that the denominator is $\mathcal{F}_t$ measurable the fixed point form of equation (4.41) becomes

$$G(x,\lambda) = H(x,\lambda) + I(x,\lambda)\,\mathbf{E}G(\mathbb{F}(x,\lambda,\xi),\lambda) \quad (4.42)$$

where λ is defined as $\lambda := \{\rho, \beta, D, \eta, \gamma, \mu, \zeta\}$, and – similarly as in previous examples

$$G(x\,,\lambda) := H(x\,,\lambda) \cdot (1 + g(x\,,\lambda))$$

[28] Where we have applied the law of iterated expectations and expressed $\mathbf{E}_t RHS = \mathbf{E}_t\mathbf{E}_{t+1} RHS$ and then exploited the fact that everything on the RHS, with the exception of ξ_{t+2}, is $\mathcal{F}_{t+1}$ measurable. Combining that with the fact that ξ_{t+2} is independent of $\mathcal{F}_{t+1}$ and therefore $\mathbf{E}_{t+1}\exp\big((1-\gamma)\xi_{t+2}\big) = \mathbf{E}_t \exp\big((1-\gamma)\xi_{t+1}\big)$ and both are equal to $\mathbf{E}\exp\big((1-\gamma)\xi\big)$ i.e. a constant we obtain (4.41).

where the transition equation is given by $\mathbb{F}(x\,,\{\rho,\beta,D,\eta,\gamma,\mu,\zeta\}\,,\xi) := (1-\rho)\,\mu + \rho x + \xi$ and the functions $H(x,\lambda)$ and $I(x,\lambda))$ are defined as

$$H(x,\lambda) := \beta \exp\big((1-\gamma)(1-\eta+\rho)x\big)\cdot\big(\exp\big((1-\gamma)(\eta-\rho)x\big) - \beta\eta D\zeta\big) \quad (4.43)$$

$$I(x,\lambda) := \beta \exp\big((1-\gamma)(1-\eta)x\big). \quad (4.44)$$

Restrictions on shock support

Before proceeding, we need to ensure that marginal utility remains positive and the utility maximization problem is well-defined. The following proposition characterizes necessary and sufficient conditions for this to be the case.

Proposition 13. Invariance of c_t^g and positive marginal utility. *Assume c_t^g follows (4.17), $c_t^g \in (c_l^g, c_h^g)$ and $\xi_{t+1} \in (\xi_l, \xi_h)$ where $\xi_i = (1-\rho)(c_i^g - \mu), i \in \{l, h\}$. Under these conditions, marginal utility remains positive if c_l^g, c_h^g are such that*

$$(\xi_l, \xi_h) \subset \begin{cases} [\max\{(1-\rho)c_l^g, c_h^g(\eta-\rho) + C\}, c_h^g] & \eta > \rho \\ [\max\{(1-\rho)c_l^g, c_l^g(\eta-\rho) + C\}, c_h^g] & \eta \le \rho. \end{cases} \quad (4.45)$$

where the constant $C = -[(1-\rho)\mu + \log(\beta\,\eta\,D)/(1-\gamma)]$.

Proof of Proposition 13 See the Appendix. □

Existence and uniqueness of solutions

We can now use the contraction mapping principle – as outlined in Theorem 3 – to characterize sufficient conditions for (4.41) to have a solution.

Proposition 14. *Under the assumptions of Proposition 13, the fixed point equivalent of equation (4.41), i.e. $G = \mathcal{Z}(G, \lambda)$ has a unique solution in $\mathcal{G}$ if*

$$\exp\big((1-\gamma)\cdot((1-\rho)\mu + (\rho-\eta)\kappa_\eta)\big)\cdot \mathcal{L}_\xi(\gamma-1) < \beta^{-1}$$

where $\kappa_\eta = c^g_{\max}$ if $\eta > \rho$, $\kappa_\eta = 0$ if $\eta = \rho$ and $\kappa_\eta = c^g_{\min}$ if $\eta < \rho$ and where $\mathcal{L}_\xi$ denotes ξ's Laplace transform.

Proof of Proposition 14. See the Appendix. □

Benchmark solutions

To find a benchmark solution recall that $\mathbb{F}(x\,,\{\rho,\beta,D,\eta,\gamma,\mu,\zeta\}\,,\xi) := (1-\rho)\,\mu+\rho x+\xi$. Accordingly $\mathbb{F}(\cdot,\cdot,\cdot)$ becomes independent of x – which is the condition underlying Theorem 5 – when $\rho = 0$.[29] Proposition (15) – an application of Theorem 5 – gives the formula for the nested solution corresponding to $\rho = 0$.

Proposition 15. *Under the assumptions of Proposition 13 and 14, when ρ in (4.40) is set to zero then the unique solution of equation (4.15) is*

$$g(x\,,\{0,\lambda_{-1}\}) = \frac{I(x\,,\{0,\lambda_{-1}\})/H(x\,,\{0,\lambda_{-1}\})\cdot \mathbf{E}\,H(\mu+\xi,\{0,\lambda_{-1}\})}{1-\mathbf{E}\,I(\mu+\xi,\ \{0,\lambda_{-1}\})}$$

which, after plugging in for the functions $I(\cdot,\cdot)$ and $H(\cdot,\cdot)$ implies that

$$\begin{aligned} &g(x\,,\{0,\lambda_{-1}\}) \\ &\quad= \frac{\mathbf{E}\,\beta\exp\big((1-\gamma)(1-\eta)(\mu+\xi)\big)\big(\exp\big((1-\gamma)\eta(\mu+\xi)\big)-\beta\eta D\zeta\big)}{\big(\exp\big((1-\gamma)\eta x\big)-\beta\eta D\zeta\big)\cdot\big(1-\mathbf{E}\,\beta\exp\big((1-\gamma)(1-\eta)(\mu+\xi)\big)\big)}. \end{aligned}$$

Proof of Proposition 15. See the Appendix. □

Approximations around benchmark solutions

As before, the form of the fixed point equation (4.42) is identical as in the case of the two previous models. The key difference, however, is that $H(x,\{\rho,\beta,D,\eta,\gamma,\mu,\zeta\})$ now depends on ρ – i.e. the parameter around which approximations are derived. This implies that the generic, approximate formulae for $g(x,\{\lambda_1,\lambda_{-1}\})$ will no longer match those in the previous two models. The following theorem formalizes this observation.

Proposition 16. *Under the assumptions of Proposition 13 and 14, the formula for the 'second order' approximation to $g(x,\{\rho,\lambda_{-1}\})$ around $g(x,\{0,\lambda_{-1}\})$ is*

[29] In terms of previous notation $\lambda_1 = \rho, \bar{\lambda}_1 = 0$ and $\lambda_{-1} = \{\mu,\gamma,\beta,h\}$.

given by

$$g^2(x,\{\rho,\beta,D,\eta,\gamma,\mu,\zeta\}) = \left(\exp((1-\gamma)\eta x) - \beta\eta D\zeta\right)^{-1}$$
$$\cdot \left(\frac{\mathcal{C}^H}{1-\mathcal{C}^I} + \rho\left(V_{1,1}(x-\mu)+V_{2,1}\right) + \frac{\rho^2}{2}\left(V_{1,2}(x-\mu)^2 + V_{2,2}(x-\mu) + V_{3,2}\right)\right)$$
$$+ H(x,\{0,\lambda_{-1}\})\left(\rho H_\rho(x,\{0,\lambda_{-1}\}) + \rho^2/2\cdot H_{\rho\rho}(x,\{0,\lambda_{-1}\})\right)$$

where $\mathcal{C}^H := \mathrm{E}\,H(\mu+\xi,\{0,\lambda_{-1}\})$, $\mathcal{C}^I := \mathrm{E}\,I(\mu+\xi,\{0,\lambda_{-1}\})$ *and* $V_{1,1} = U_{1,1}, V_{1,2} = U_{1,2}$ *with* $V_{2,1} = U_{2,1} + \ EH_\rho(\mu+\xi,\{0,\lambda_{-1}\})/(1-\mathcal{C}^I)$, $V_{2,2} = U_{2,2} + 2\mathrm{E}H_{\rho x}(\mu+\xi,\{0,\lambda_{-1}\})$, $V_{3,2} = U_{3,2} + \ EH_{\rho\rho}(\mu+\xi,\{0,\lambda_{-1}\})/(1-\mathcal{C}^I)$ *and formulae for all the* $U_{i,j}$ *given in Table 1.*

Proof of Proposition 16. See the Appendix. □

Formulae for approximations of order higher than 2 can be computed analogously but are not listed here due to space constraints.

A Comparison of Approximate Policy Functions

As in the case of the previous model we can compare the resulting approximations to the corresponding 'standard' perturbation ones given by *dynare++*. For the purpose of the exercise we calibrate the parameters as follows: $\beta = 0.99$, $D = 0.75$, $\eta = 0.5$, $\gamma = 2.37$ with the coefficients governing the consumption growth process unchanged from those in the simple habit model. The charts display a similar pattern to that in the previous model - i.e. they are virtually indistinguishable from order 3 upwards, and the 'implicit function' approximations seem to stabilize faster. The two types of approximations do not converge even at higher order, which could reflect the fact that values of states or parameters lie outside the radius of approximation convergence.

4.3.4 The Model of Campbell and Cochrane (1999)

In the model of Campbell and Cochrane (1999) instantaneous utility is given by

$$U(C_t, X_t) = \frac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}$$

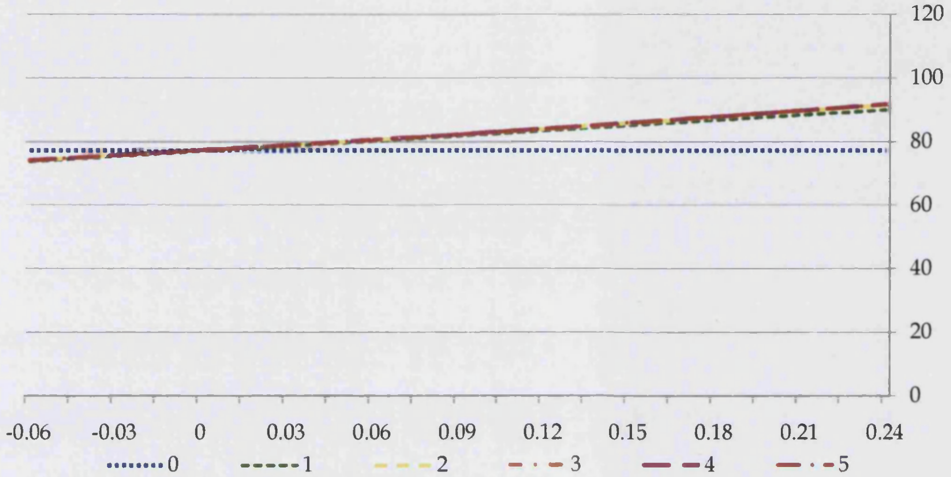

Figure 4.4: Approximate Policy Functions Derived Using Standard Perturbation Methods by Approximation Order (0 - 5) – Case of the Abel (1990) Model

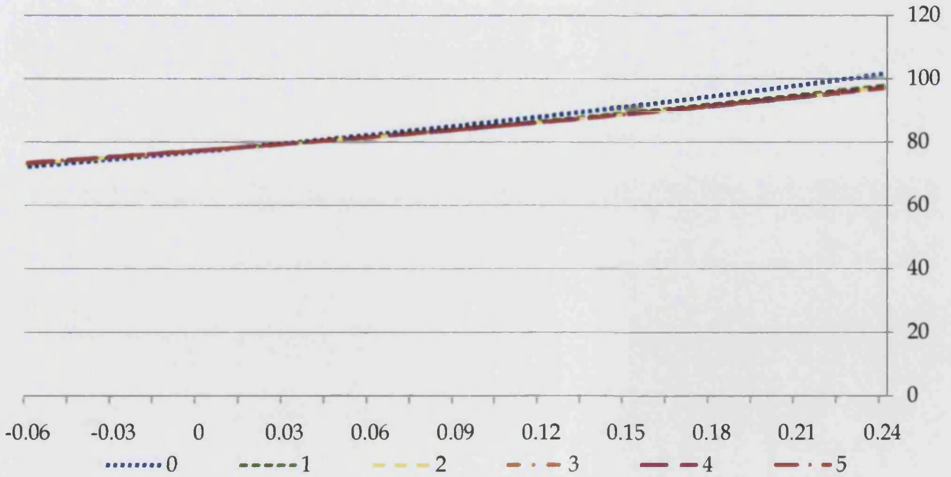

Figure 4.5: Approximate Policy Functions Derived Using the Implicit Function Approach by Approximation Order (0 - 5) – Case of the Abel (1990) Model

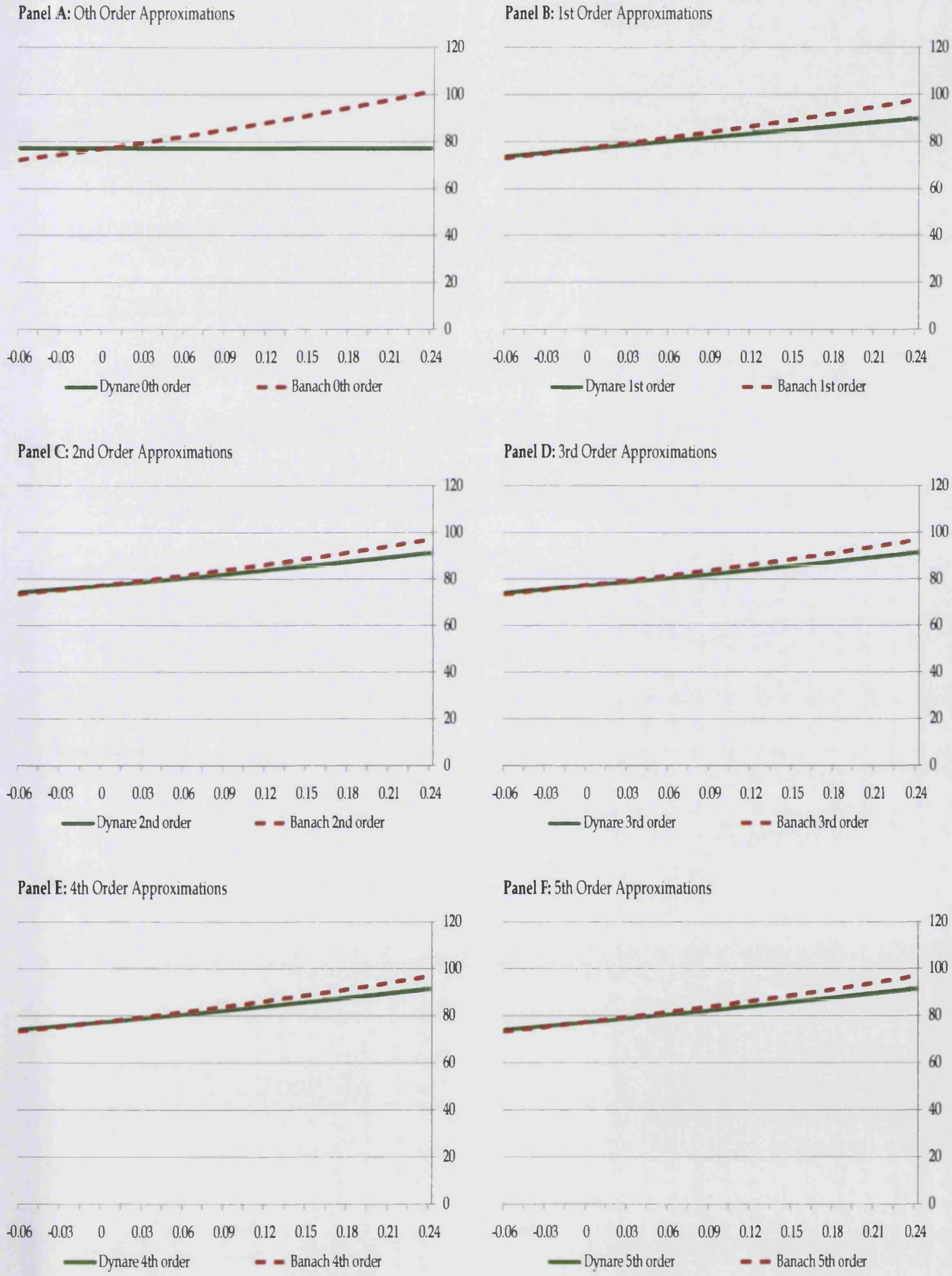

Figure 4.6: Standard Perturbation vs Approximations Derived Using the Implicit Function Approach (Banach, red-dashed line) – Case of the Abel (1990) Model

and, as in the simple habit model, the equity price-dividend ratio follows

$$v_t = \mathrm{E}_t\beta\left(\frac{1 - X_{t+1}/C_{t+1}}{1 - X_t/C_t}\right)^{-\gamma} \cdot \left(\frac{C_{t+1}}{C_t}\right)^{1-\gamma} \cdot (v_{t+1} + 1)\,. \tag{4.46}$$

Campbell and Cochrane (1999) posit that $s_t := \log\bigl(1 - X_t/C_t\bigr)$ follows

$$s_{t+1} = \bar{s} + \kappa\bigl(\phi\,(s_t - \bar{s}) + \Lambda(s_t)(c_{t+1} - c_t - \widetilde{g})\bigr) \tag{4.47}$$

where $\kappa, \phi, \widetilde{g}$ and $\bar{s}$ are parameters, lowercase letters indicate logs and the sensitivity function $\Lambda(\mathsf{s})$ is defined as[30]

$$\Lambda(\mathsf{s}) := \left(\bar{S}^{-1}\sqrt{1 - 2(\mathsf{s} - \bar{s})} - 1\right) \cdot \chi_{\{\mathsf{s}\,:\,\mathsf{s} \le s_{\max}\}}(\mathsf{s}) \tag{4.48}$$

where $s_{\max}$ is a constant equal to $\bar{s} + 1/2 \cdot (1 - \bar{S}^2)$ and χ is the indicator function, i.e. $\chi_A(x)$ equals 1 if $x \in A$ and zero otherwise. Campbell and Cochrane (1999) close the model by assuming *i.i.d.* consumption growth dynamics, i.e. that

$$c_{t+1} - c_t = \widetilde{g} + \xi_{t+1}$$

where ξ is a white-noise process. We shall further posit that ξ is bounded from below and above. This easily implies that there exist numbers $a < 0 < b$ such that the set $\mathcal{D} = [a, b]$ is invariant for the state process s_t.

Fixed point representation

Inspecting the equations above we see that the parameter vector equals $\lambda = \{\kappa, \gamma, \beta, \bar{s}, \phi, \tilde{g}, s_{\max}\}$ and the fundamental asset pricing equation becomes

$$\begin{aligned} v_t = \beta \exp\bigl((1-\gamma)\tilde{g} + \gamma(1 - \kappa\phi)\,(s_t - \bar{s})\bigr) \\ \cdot\, \mathrm{E}_t \exp\bigl(((1-\gamma) - \gamma\kappa\Lambda(s_t))\xi_{t+1}\bigr)\,(v_{t+1} + 1)\,. \end{aligned}$$

Adding one to both sides gives us the fixed point form

$$g(x, \lambda) = 1 + f(x, \lambda) \cdot \mathrm{E}h(x, \xi, \lambda) \cdot g(\mathbb{F}(x, \lambda, \xi), \lambda) \tag{4.49}$$

[30]Note that we slightly modifying the original notation in Campbell and Cochrane (1999). In particular, we use the tilde to distinguish the constant $\widetilde{g}$ from function $g(\cdot)$, we replace the function $\lambda(\cdot)$ with $\Lambda(\cdot)$ to avoid confusion with the parameter vector λ and we extend the original specification by introducing a parameter κ - which shall help us find 'nested' solutions.

where $g(x\,,\lambda) := 1 + v(x\,,\lambda)$ and

$$\begin{aligned}
f(x,\lambda) :=& \beta \exp\big((1-\gamma)\tilde{g} + \gamma(1-\kappa\phi)\,(x-\bar{s})\big) \\
h(x,\lambda,\xi) :=& \;\; \exp\big(((1-\gamma) - \gamma\kappa\Lambda(x))\xi\big) \\
\mathbb{F}(x,\lambda,\xi) :=& \bar{s} + \kappa\big(\phi\,(x-\bar{s}) + \Lambda(x)\,\xi\big).
\end{aligned}$$

To clarify, the fixed point problem (4.49) can be considered in the space of all bounded functions on $\mathcal{D}$ with the sup norm (i.e. $||g|| = \sup_{x\in\mathcal{D}}|g(x)|$). Under the assumptions of bounded shock support it is then not difficult to show that the transformation $g = \mathcal{Z}(g)$ is Lipschitz and the corresponding Lipschitz constant α can be characterized analytically. Further analysis proceeds under the assumption that $\alpha \in (0,1)$ which implies that the fixed-point problem (4.49) has a well-defined solution $g(\cdot,\lambda)$ – unique in $\mathcal{G}$.

Benchmark solutions

The important thing to note is that by setting κ to one we exactly recover the original problem in Campbell and Cochrane (1999), while letting $\kappa = 0$ eliminates the dependance of the term under the expectation operator on the state s_t - i.e. by Theorem 5 we can hope to solve the problem exactly.

Proposition 17. *The nested solution of equation (4.49) corresponding to $\kappa = 0$ is given by*

$$g(x,\{0,\lambda_{-1}\}) = 1 + \frac{f(x,\{0,\lambda_{-1}\})\cdot \mathrm{E}h(x,\xi,\{0,\lambda_{-1}\})}{1-\beta\cdot\exp((1-\gamma)\,\tilde{g})\cdot \mathrm{E}\exp((1-\gamma)\,\xi)}. \tag{4.50}$$

Proof of Proposition 17. See the Appendix. □

Approximations around benchmark solutions

We now show how to find an approximation of the solution corresponding to $\kappa \neq 0$ - i.e. how to use $g(x\,,\{0,\lambda_{-1}\})$ to approximate $g(x,\{\kappa,\lambda_{-1}\})$.

Proposition 18. *The 'first order' approximation to $g(x,\{\kappa,\lambda_{-1}\})$ 'around'*

$g(x, \{0, \lambda_{-1}\})$ *is given by*

$$g(x,\lambda)-1\approx\big(g(x,\{0,\lambda_{-1}\})-1\big)+\kappa g_\kappa(x,\{0,\lambda_{-1}\})=\exp\big(\gamma(x-\bar{s})\big)\cdot\beta$$
$$\exp((1-\gamma)\tilde{g})\cdot\mathrm{E}\exp((1-\gamma)\,\xi)\Big(1-\beta\cdot\exp((1-\gamma)\,\tilde{g})\cdot\mathrm{E}\exp((1-\gamma)\,\xi)-\kappa\gamma\phi(x-\bar{s})$$
$$+\gamma\Lambda(x)\mathrm{E}\,\xi\cdot\exp\big((1-\gamma)\,\xi\big)\cdot\big[1+\frac{\Lambda(\bar{s})/\Lambda(x)\cdot\beta\exp\big((1-\gamma)\tilde{g}\big)}{\big(1-\beta\exp((1-\gamma)\tilde{g})\mathrm{E}\exp((1-\gamma)\,\xi)\big)}\big]\Big).$$

Proof of Proposition 18. See the Appendix. □

4.3.5 The Model of Bansal and Yaron (2004)

We conclude by approximating the solution to the model of Bansal and Yaron (2004). Under their specification, utility is defined recursively as in Epstein and Zin (1989, 1991) and the SDF equals

$$\mathcal{M}_{t+1}:=\beta^\theta\left(\frac{C_{t+1}}{C_t}\right)^{-\frac{\theta}{\psi}}\left(\frac{P_{t+1}+C_{t+1}}{P_t}\right)^{\theta-1} \tag{4.51}$$

where P_t denotes the price of an equity share and β, θ, ψ are parameters.[31] Accordingly, the fundamental asset pricing equation for the equity share is

$$v_t^\theta=\mathrm{E}_t\left(\beta\left(\frac{C_{t+1}}{C_t}\right)^{(1-\frac{1}{\psi})}\left(1+v_{t+1}\right)\right)^\theta. \tag{4.52}$$

The key difference between equation (4.52) and the asset pricing equations we have considered previously is that the expression under the expectation operator is not linear in v_{t+1}. Bansal and Yaron (2004) further assume

$$c_{t+1}^g=\mu+\kappa z_t+\sigma_t\eta_{t+1} \tag{4.53}$$
$$z_{t+1}=\kappa\rho z_t+\phi_e\sigma_t e_{t+1} \tag{4.54}$$
$$\sigma_{t+1}^2=\sigma^2+\kappa\big(\nu_1(\sigma_t^2-\sigma^2)+\sigma_w w_{t+1}\big) \tag{4.55}$$

where $e_{t+1}, \eta_{t+1}, w_{t+1} \sim i.i.d.$ and are mean zero and bounded and z_t is a 'small persistent predictable component' of consumption growth.[32] Notably, as before,

[31] We use a notation consistent with Bansal and Yaron (2004) with the exception of replacing δ by β and x_t by z_t.

[32] Again, the assumption of bounded – rather than normal – shocks is introduced for technical convenience.

we have introduced an extra constant $\kappa \geq 0$ such that when $\kappa = 1$ we recover the original specification of Bansal and Yaron (2004) (i.e. this model nests theirs). Under the assumption that $\nu_1, \rho \in (0,1)$ and $\kappa \in [0,1]$ it is not difficult to show that the set $\mathcal{D} = [-a_1, b_1] \times [-a_2, b_2] \times [-a_3, b_3]$ is invariant for the dynamics (4.53) - (4.55) for sufficiently large positive numbers a_i, b_i where $i \in \{1,2,3\}$.

Fixed point representation

As the equations above suggest, the state-variable x is now equal to $x_t = \{c_t^g, z_t, \sigma_t^2\}$, the shock process is three dimensional as well and given by $\xi_t = \{\eta_t, e_t, w_t\}$ and we can define the parameter vector λ as $\lambda := \{\kappa, \psi, \theta, \mu, \rho, \phi_e, \sigma^2, \sigma_w, \nu_1, \beta\}$. The transition function $\mathbb{F}(\cdot,\cdot,\cdot)$ then is

$$\mathbb{F}(x,\lambda,\xi) = \begin{pmatrix} \mathbb{F}_1(x,\lambda,\xi) \\ \mathbb{F}_2(x,\lambda,\xi) \\ \mathbb{F}_3(x,\lambda,\xi) \end{pmatrix} = \begin{pmatrix} \lambda_4 + \lambda_1 x_2 + \sqrt{x_3}\xi_1 \\ \lambda_1\lambda_5 x_2 + \lambda_6\sqrt{x_3}\xi_2 \\ \lambda_7 + \lambda_1\big(\lambda_9(x_3 - \lambda_7) + \lambda_8\xi_3\big) \end{pmatrix} \tag{4.56}$$

while letting $g(x,\lambda)$ denote the solution, the fundamental asset pricing equation can be rewritten as

$$g(x,\lambda)^{\lambda_3} = \mathbf{E}\Big(\lambda_{10}\big(\exp((1 - 1/\lambda_2)\mathbb{F}_1(x,\lambda,\xi))\big)\big(1 + g(\mathbb{F}(x,\lambda,\xi),\lambda)\big)\Big)^{\lambda_3}.$$

Raising both sides to the power $1/\lambda_3$, adding one and then multiplying by $\lambda_{10}\exp((1 - 1/\lambda_2)x_1)$ and raising both sides to the power λ_3 yields

$$G(x,\lambda) = \lambda_{10}^{\lambda_3}\exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big)\left(1 + \big(\mathbf{E}\, G(\mathbb{F}(x,\lambda,\xi),\lambda)\big)^{1/\lambda_3}\right)^{\lambda_3} \tag{4.57}$$

where

$$G(x,\lambda) := \big(\lambda_{10}\big(\exp((1 - 1/\lambda_2)x_1)\big)\big(1 + g(x,\lambda)\big)\big)^{\lambda_3}. \tag{4.58}$$

We have thus rewritten equation (4.52) as $G = \mathcal{Z}(G,\lambda)$ where $\mathcal{Z}(G,\lambda)(x) := \mathbb{U}\big(x, \mathcal{P}(G,\lambda)(x), \lambda\big)$ with $\mathcal{P}(G,\lambda)(x) = \mathbf{E}\big(G(\mathbb{F}(x,\lambda,\xi))\big)$ and $\mathbb{U}(x,z,\lambda)$ given by

$$\mathbb{U}(x,z,\lambda) := \lambda_{10}^{\lambda_3}\exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big)\left(1 + z^{1/\lambda_3}\right)^{\lambda_3}.$$

Under the assumption of bounded shock support if A, B are constants such that $0 \leq A \leq \lambda_{10}^{\lambda_3}\exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big) \leq B$ then $G(x,\lambda) \geq A$. Further, the function $h(z) := \left(1 + z^{1/\lambda_3}\right)^{\lambda_3}$ is Lipschitz on $[A, +\infty)$ with constant C being the supremum of $|h'(z)|$, i.e.

$$C = \sup_{z \geq A} \left| \left(\frac{1 + z^{1/\lambda_3}}{z^{1/\lambda_3}} \right)^{\lambda_3 - 1} \right|.$$

Since the shocks are assumed bounded, therefore one can always apply the fixed point theorem in the space of bounded functions on $\mathcal{D}$ with supremum norm $||\cdot||$. To calculate the Lipschitz constant of $\mathcal{Z}$ we can use the observations above

$$\begin{aligned} |\mathcal{Z}(G_1)(x) - \mathcal{Z}(G_2)(x)| &= |\lambda_{10}^{\lambda_3} \exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big)\big(1 + \mathcal{P}(G_1, \lambda)(x)^{1/\lambda_3}\big)^{\lambda_3} \\ - \lambda_{10}^{\lambda_3} \exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big)&\big(1 + \mathcal{P}(G_2, \lambda)(x)^{1/\lambda_3}\big)^{\lambda_3}| \leq |\lambda_{10}^{\lambda_3} \exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big)| \\ &\cdot C \cdot |\mathcal{P}(G_1 - G_2)(x)| \leq |\lambda_{10}^{\lambda_3} \exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big)| \cdot C \cdot ||G_1 - G_2||. \end{aligned}$$

where the latter inequality follows from the fact that the Lipschitz constant associated with the transformation $\mathcal{P}$ equals one. Accordingly $||\mathcal{Z}(G_1) - \mathcal{Z}(G_2)|| \leq \sup_{x \in \mathcal{D}} |\lambda_{10}^{\lambda_3} \exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big)| \cdot C \cdot ||G_1 - G_2||$ where C is defined above. Thus, $\mathcal{Z}$ will be a contraction if $|\lambda_{10}^{\lambda_3} \exp\big(\lambda_3(1 - 1/\lambda_2)x_1\big)| \cdot C < 1$ i.e. for example, for sufficiently small values of λ_{10}. Assuming this condition is satisfied we can proceed to inspect approximations to the unique solution.

Benchmark solutions

To find a benchmark solution recall that equation (4.56) implies that $\mathbb{F}(\cdot, \cdot, \cdot)$ becomes independent of x – which is the condition underlying assumption A3 – when $\kappa = 0$.[33] Theorem 5 thus implies that we can characterize a solution of equation (4.52) when $\kappa = 0$. Proposition 19 gives the formula.[34]

[33]Accordingly $\lambda_1 = \kappa, \bar{\lambda}_1 = 0$ and $\lambda_{-1} = \{\psi, \theta, \mu, \rho, \phi_e, \sigma^2, \sigma_w, \nu_1, \beta\}$.

[34]Alternatively, letting $\kappa = 1$ and setting ρ, ν_1 and σ_w equal to zero we would obtain

$$x_{t+1} = \mu + z_t + \sigma\eta_{t+1} \tag{4.59}$$

where $z_t \sim \mathcal{N}.i.i.d(0, \sigma^2\phi_e^2)$. Defining $\mathcal{H}(x, \lambda) := \beta \exp\big(x((1 - \frac{1}{\psi}))\big)$ and noting that $\mathbf{E}_t G(x_{t+1}, z_{t+1}) = \mathbf{E} G(\mu + z_t + \sigma\eta, \phi_e\sigma e) := \mathsf{F}(z_t)$ the fixed point equation becomes

$$G(x_1, x_2) = \mathcal{H}^\theta(x_1) \cdot \big(1 + \big(\mathbf{E} G(\mu + x_2 + \sigma\eta, \phi_e\sigma e)\big)^{\frac{1}{\theta}}\big)^\theta. \tag{4.60}$$

To fully characterize the solution we need to find $\mathsf{F}(\mathsf{y}) = \mathbf{E} G(\mu + \mathsf{y} + \sigma\eta, \phi_e\sigma e)$. Plugging in $(x_1, x_2) = (\mu + \mathsf{y} + \sigma\eta, \phi_e\sigma e)$ and taking expectations we find that

$$\mathsf{F}(\mathsf{y}) = \mathbf{E}\mathcal{H}^\theta(\mu + \mathsf{y} + \sigma\eta) \cdot \big(1 + \big(\mathsf{F}(\phi_e\sigma e)\big)^{\frac{1}{\theta}}\big)^\theta = \kappa \cdot \exp\big(\mathsf{y}(\theta - \frac{\theta}{\psi})\big) \tag{4.61}$$

Proposition 19. *Under the assumptions of Proposition 4 and 5, when λ_1 (corresponding to κ) in equation (4.56) is set to zero then*

$$g(x,\{0,\psi,\theta,\mu,\rho,\phi_e,\sigma^2,\sigma_w,\nu_1,\beta\})=\frac{\beta\big(\mathbf{E}\exp(\phi(1-1/\psi)(\mu+\sigma\xi_1))\big)^{1/\phi}}{1-\beta\big(\mathbf{E}\exp(\phi(1-1/\psi)(\mu+\sigma\xi_1))\big)^{1/\phi}}$$

i.e. the price dividend ratio under these parameter restrictions is constant.

Proof of Proposition 19. See the Appendix. □

Approximations around benchmark solutions

We now show how to find a perturbation approximation of the solution corresponding to $\kappa \neq 0$ - i.e. how to use $g(x\,,\{0,\lambda_{-1}\})$ to approximate $g(x,\{\kappa,\lambda_{-1}\})$.

Proposition 20. *Under the assumptions of Proposition 4 and 5, the 'first order' approximation to $g(x,\{\kappa,\lambda_{-1}\})$ around $g(x,\{0,\lambda_{-1}\})$ is given by*

$$g(x,\lambda)\approx\frac{\big(1+\kappa(\nu(\lambda_{-1})^{1/\lambda_3-1})/(1+\nu(\lambda_{-1})^{1/\lambda_3})\cdot(\iota(x,\lambda_{-1})+\vartheta(\lambda_{-1}))\big)^{1/\lambda_3}}{1-\lambda_{10}\big(\mathbf{E}\exp(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1))\big)^{1/\lambda_3}}-1$$

---

where the constant κ is given by

$$\kappa:=\mathbf{E}\beta^{\theta}\exp\big((\mu+\sigma\eta)(\theta-\frac{\theta}{\psi})\big)\cdot\big(1+(\mathsf{F}(\phi_e\sigma e))^{\frac{1}{\theta}}\big)^{\theta}. \tag{4.62}$$

To find κ we can evaluate (4.61) at $y=\phi_e\sigma e$ and plug the RHS into (4.62)

$$\kappa=\mathbf{E}\beta^{\theta}\exp\big((\mu+\sigma\eta)(\theta-\frac{\theta}{\psi})\big)\cdot\big(1+(\kappa\cdot\exp((\phi_e\sigma e)(\theta-\frac{\theta}{\psi})))^{\frac{1}{\theta}}\big)^{\theta}.$$

Denoting the solution of this nonlinear equation by κ^* (we do not analyze the existence and uniqueness of κ^* here) we could write

$$G(x_1,x_2)=\beta^{\theta}\cdot\exp\big(x_1(\theta-\frac{\theta}{\psi})\big)\cdot\big(1+(\kappa^*)^{\frac{1}{\theta}}\cdot\exp\big(x_2(1-\frac{1}{\psi})\big)\big)^{\theta}$$

$$\Rightarrow g(x_1,x_2)=(\kappa^*)^{\frac{1}{\theta}}\cdot\exp\big(x_2(1-\frac{1}{\psi})\big) \tag{4.63}$$

which given that $v_t=g(x_t,z_t)$ finally implies that equity share prices evolve according to

$$P_t=C_t\cdot(\kappa^*)^{\frac{1}{\theta}}\cdot\exp\big(z_t(1-\frac{1}{\psi})\big) \tag{4.64}$$

i.e. they are only a function of current consumption levels and the predictable consumption growth component z_t. Again, however, as in the case of the external habit model with persistent habits, it is challenging to construct approximations around this solution, which is why we continue with the more restricted solution of proposition 19.

where the functions $\iota(x,\lambda_{-1})$ *and* $\vartheta(\lambda_{-1})$ *are given by*

$$\begin{aligned}\iota(x,\lambda_{-1}) &= x_2\lambda_3(1-1/\lambda_2)\lambda_{10}^{\lambda_3}\cdot\exp\bigl(\lambda_3(1-1/\lambda_2)x_1\bigr)\\ &\qquad\cdot\Bigl(1-\lambda_{10}\bigl(\mathbf{E}\exp(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1))\bigr)^{1/\lambda_3}\Bigr)^{-\lambda_3};\\ \vartheta(\lambda_{-1}) &= \nu(\lambda_{-1})^{1/\lambda_3-1}\cdot\mathbf{E}\Bigl(G(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\{0,\lambda_{-1}\})\\ &\qquad\qquad\cdot\iota(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\lambda_{-1})\Bigr);\end{aligned}$$

and $\nu(\lambda_{-1})=\lambda_{10}^{\lambda_3}\left(1+\bigl(\nu(\lambda_{-1})\bigr)^{1/\lambda_3}\right)^{\lambda_3}\mathrm{E}\exp\bigl(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\bigr).$

Proof of Proposition 20. See the Appendix. □

4.4 Conclusions

This paper proposed a new method of approximating solutions of models with known endogenous state-variable dynamics. Focusing on asset pricing models, it re-expressed the Euler equation as a fixed-point problem in Banach spaces. This was key to finding solutions of nested problems, which were then used to approximate the full solution. It was shown that even low-order approximations were not 'certainty equivalent' and that the zeroth order approximation could allow for time-variation in risk premia. Arguably – by allowing the approximation to be an arbitrary, nonlinear function of the state – the method made the approximations more parsimonious while retaining high precision. The paper also clarified how to use the Lipschitz constant to provide upper bounds on the resulting approximation errors. Generalizing the methodology to models in which the dynamics of state variables is not known and a more thorough investigation of its accuracy and asymptotic properties would both make for worthwhile extensions.

4.A Appendix

Proof of Theorem 5. The fixed point equation (4.6) corresponding to $\lambda = \{\bar{\lambda}_1, \lambda_{-1}\}$ can be written as

$$G(x, \{\bar{\lambda}_1, \lambda_{-1}\}) = \mathbb{U}\big(x, \mathbf{E}G(\mathbb{F}(x, \{\bar{\lambda}_1, \lambda_{-1}\}, \xi), \{\bar{\lambda}_1, \lambda_{-1}\}), \{\bar{\lambda}_1, \lambda_{-1}\}\big) \quad (4.65)$$

given the respective definitions of G_0 and $\mathbb{F}_0$ this can be rewritten as

$$G_0(x, \lambda_{-1}) = \mathbb{U}\big(x, \mathbf{E}G_0(\mathbb{F}_0(\lambda_{-1}, \xi), \lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}\big).$$

Since $\mathbb{U}(\cdot, \cdot, \cdot)$ is a known function, therefore to fully characterize $G_0(\cdot, \cdot)$ it suffices to find $\Psi(\lambda_{-1}) := \mathbf{E}G_0(\mathbb{F}_0(\lambda_{-1}, \xi), \lambda_{-1})$. Because $G_0(\cdot, \cdot)$ satisfies

$$G_0(x, \lambda_{-1}) = \mathbb{U}\big(x, \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}\big)$$

therefore, plugging in $x = \mathbb{F}_0(\lambda_{-1}, \xi)$, taking expectations of both sides and exploiting the definition of $\Psi(\lambda_{-1})$ this becomes

$$\Psi(\lambda_{-1}) = \mathbf{E}\mathbb{U}\big(\mathbb{F}_0(\lambda_{-1}, \xi), \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}\big). \quad (4.66)$$

Accordingly, if we find $\Psi(\lambda_{-1})$ which solves (4.66) then the solution of equation (4.65) will be given by

$$G_0(x, \lambda_{-1}) = \mathbb{U}\big(x, \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}\big) \quad (4.67)$$

as posited in Theorem 5. ■

Proof of Theorem 6. The fixed point equation is

$$G(x, \{\lambda_1, \lambda_{-1}\}) = \mathbb{U}\big(x, \mathbf{E}G(\mathbb{F}(x, \{\lambda_1, \lambda_{-1}\}, \xi), \{\lambda_1, \lambda_{-1}\}), \{\lambda_1, \lambda_{-1}\}\big). \quad (4.68)$$

Differentiating this with respect to λ_1 we obtain

$$\frac{\partial}{\partial \lambda_1} G(x, \{\lambda_1, \lambda_{-1}\}) = \frac{\partial}{\partial \lambda_1} \mathbb{U}\big(x, \mathbf{E}G(\mathbb{F}(x, \{\lambda_1, \lambda_{-1}\}, \xi), \{\lambda_1, \lambda_{-1}\}), \{\lambda_1, \lambda_{-1}\}\big)$$
$$+ \frac{\partial \mathbb{U}}{\partial x_2}\big(x, \mathbf{E}G(\mathbb{F}(x, \{\lambda_1, \lambda_{-1}\}, \xi), \{\lambda_1, \lambda_{-1}\}), \{\lambda_1, \lambda_{-1}\}\big) \cdot \bigg[\mathbf{E}\frac{\partial \mathbb{F}}{\partial \lambda_1}(x, \{\lambda_1, \lambda_{-1}\}, \xi)$$
$$\cdot \frac{\partial G}{\partial x}\big(\mathbb{F}(x, \{\lambda_1, \lambda_{-1}\}, \xi), \{\lambda_1, \lambda_{-1}\}\big) + \mathbf{E}\frac{\partial G}{\partial \lambda_1}\big(\mathbb{F}(x, \{\lambda_1, \lambda_{-1}\}, \xi), \{\lambda_1, \lambda_{-1}\}\big)\bigg]$$

where $\partial\mathbb{U}/\partial\mathrm{x}_2$ denotes the partial derivative of $\mathbb{U}$ with respect to the second variable. Evaluating this at $\lambda = \{\bar{\lambda}_1, \lambda_{-1}\}$ then yields

$$\frac{\partial G}{\partial \lambda_1}(x, \{\bar{\lambda}_1, \lambda_{-1}\}) = \frac{\partial \mathbb{U}}{\partial \lambda_1}(x, \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}) + \frac{\partial \mathbb{U}}{\partial \mathrm{x}_2}(x, \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\})$$
$$\cdot \mathbf{E}\left[\frac{\partial \mathbb{F}}{\partial \lambda_1}(x, \{\bar{\lambda}_1, \lambda_{-1}\}, \xi)\frac{\partial G}{\partial x}(\mathbb{F}_0(\lambda_{-1}, \xi), \{\bar{\lambda}_1, \lambda_{-1}\}) + \frac{\partial G}{\partial \lambda_1}(\mathbb{F}_0(\lambda_{-1}, \xi), \{\bar{\lambda}_1, \lambda_{-1}\})\right]$$

where we have used the definition of $\Psi(\lambda_{-1}) = \mathbf{E}G_0(\mathbb{F}_0(\lambda_{-1}, \xi), \lambda_{-1})$. Plugging in $x = \mathbb{F}_0(\lambda_{-1}, \xi)$ and taking expectations of both sides yields a linear equation which can be solved for $\mathbf{E}(\partial/\partial\lambda_1)G(\mathbb{F}_0(\lambda_{-1}, \xi), \{\bar{\lambda}_1, \lambda_{-1}\})$

$$\bar{\mathbf{E}}\frac{\partial G}{\partial \lambda_1}(\mathbb{F}_0(\lambda_{-1}, \xi), \{\bar{\lambda}_1, \lambda_{-1}\}) = \left(1 - \bar{\mathbf{E}}\frac{\partial \mathbb{U}}{\partial \mathrm{x}_2}(\mathbb{F}_0(\lambda_{-1}, \bar{\xi}), \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\})\right)^{-1}$$
$$\cdot \bar{\mathbf{E}}\left\{\frac{\partial \mathbb{U}}{\partial \lambda_1}(\mathbb{F}_0(\lambda_{-1}, \bar{\xi}), \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\}) + \frac{\partial \mathbb{U}}{\partial \mathrm{x}_2}(\mathbb{F}_0(\lambda_{-1}, \bar{\xi}), \Psi(\lambda_{-1}), \{\bar{\lambda}_1, \lambda_{-1}\})\right.$$
$$\left.\cdot \mathbf{E}\left(\frac{\partial \mathbb{F}}{\partial \lambda_1}(\mathbb{F}_0(\lambda_{-1}, \bar{\xi}), \{\bar{\lambda}_1, \lambda_{-1}\}, \xi) \cdot \frac{\partial G}{\partial x}(\mathbb{F}_0(\lambda_{-1}, \xi), \{\bar{\lambda}_1, \lambda_{-1}\})\right)\right\}.$$

Exploiting the fact that $G_0(x, \lambda_{-1}) := G(x, \{\bar{\lambda}_1, \lambda_{-1}\})$ yields

$$\frac{\partial G}{\partial x}(\mathbb{F}_0(\lambda_{-1}, \xi), \{\bar{\lambda}_1, \lambda_{-1}\}) = \frac{\partial G_0}{\partial x}(\mathbb{F}_0(\lambda_{-1}, \xi), \lambda_{-1})$$

which completes the proof. ■

Proof of Proposition 4. Since $c_{t+1}^g = \rho c_t^g + (1-\rho)\mu + \xi_{t+1}$, therefore $\rho \geq 0, c_t^g > c_l^g, \xi_t > \xi_l \Rightarrow c_{t+1}^g > \rho c_l^g + (1-\rho)\mu + \xi_l$. But $\rho c_l^g + (1-\rho)\mu + \xi_l > c_l^g \Leftrightarrow \xi_l > (1-\rho)(c_l^g - \mu)$. Similarly, $\rho > 0, c_t^g < c_h^g, \xi_t < \xi_h \Rightarrow c_{t+1}^g < \rho c_h^g + (1-\rho)\mu + \xi_h$. But $\rho c_h^g + (1-\rho)\mu + \xi_h < c_h^g \Leftrightarrow \xi_h < (1-\rho)(c_h^g - \mu)$. Note, $c_l^g, c_h^g \in \mathbb{R} \cup \{-\infty, +\infty\}$.■

Proof of Proposition 5. Let $\tilde{\mathcal{G}}$ be a Banach space with norm $||\cdot||$ whose elements are functions $G : [a, +\infty) \to \mathbb{R}$ and let $\mathcal{G}$ be a closed subset of $\tilde{\mathcal{G}}$ such that $G \in \mathcal{G}$ if and only if $G \in \tilde{\mathcal{G}}$ and $G : [a, +\infty) \to \mathbb{R}^+$. Let the norm on $\tilde{\mathcal{G}}$ be defined as

$$||G|| := \sup_{\mathrm{x}>a} \exp(-\delta \mathrm{x}) \cdot |G(\mathrm{x})| < +\infty \tag{4.69}$$

where a is a constant greater that $\log(h)$ and δ is a positive constant yet to be determined. We shall demonstrate that condition (4.22) guarantees that the operator $\mathcal{Z} : \mathcal{G} \to \mathcal{G}$ defined in equation (4.21) is Lipschitz with a Lipschitz constant smaller than one. By Assumption A1 and Theorem 3 this shall imply that there exists a solution, unique in $\mathcal{G}$, to the fixed point equation $G = \mathcal{Z}(G,\lambda)$ and so also to the corresponding equation (4.15).

To ensure that $\mathcal{Z} : \mathcal{G} \to \mathcal{G}$, we first check whether $H \in \mathcal{G}$. Recalling the definition of H we have that

$$||H|| = \sup_{\mathrm{x}>a} \beta \left(1 - h\exp(-\mathrm{x})\right)^{-\gamma} \cdot \exp\big((1-\gamma-\delta)\cdot \mathrm{x}\big) \tag{4.70}$$

and so the necessary conditions for $||H|| < +\infty \Rightarrow H \in \mathcal{G}$ are $a > \log(h)$ and $\delta \geq 1-\gamma$. We now derive expressions for $\mathcal{Z}$'s Lipschitz constant[35]

$$\begin{aligned} ||\mathcal{Z}(G)|| \leq ||I\cdot\mathcal{P}(G)|| &= \beta \sup_{\mathrm{x}>a} \exp\big(\mathrm{x}(1-\gamma-\delta)\big)\mathrm{E}G\big((1-\rho)\mu+\rho\mathrm{x}+\xi\big) \\ &\leq \beta ||G|| \sup_{\mathrm{x}>a} \exp\big(\mathrm{x}(1-\gamma-\delta)\big)\mathrm{E}\exp\left(\delta\big((1-\rho)\mu+\rho\mathrm{x}+\xi\big)\right) \\ &= ||G||\cdot\beta\cdot\exp\big(\delta(1-\rho)\mu\big)\cdot\mathrm{E}\exp\big(\delta\,\xi\big)\sup_{\mathrm{x}>a}\exp\left(\mathrm{x}\big(1-\gamma-\delta(1-\rho)\big)\right). \end{aligned}$$

We thus see that $\mathcal{Z}$ is Lipschitz with constant α given by[36]

$$\alpha = \beta\cdot\exp\big(\delta(1-\rho)\mu\big)\cdot\mathrm{E}\exp\big(\delta\,\xi\big)\cdot\sup_{\mathrm{x}>a}\exp\left(\mathrm{x}\big(1-\gamma-\delta(1-\rho)\big)\right).$$

Setting δ equal to $(1-\gamma)$ and assuming $\gamma \geq 1$ the Lipschitz constant becomes

$$\alpha = \beta\,\exp\big((1-\gamma)\cdot((1-\rho)\mu + a\rho)\big)\cdot\mathrm{E}\exp\big((1-\gamma)\xi\big) \tag{4.71}$$

where $\gamma \geq 1$ implied $\sup_{\mathrm{x}>a}\exp(\mathrm{x}\,\rho\,(1-\gamma)) = \exp(a\,\rho\,(1-\gamma))$.[37] Accordingly, for this choice of δ, the operator $\mathcal{Z}$ is a contraction mapping, and by Theorem 3 there exists a unique solution of equation (4.15) if and only if

$$\alpha < 1 \Leftrightarrow \mathrm{E}\exp\big((1-\gamma)\cdot((1-\rho)\mu + a\rho + \xi)\big) < \beta^{-1}. \tag{4.72}$$

[35]Where the inequality between the first and second lines follows from

$$\begin{aligned} ||G|| := \sup_{\mathrm{x}>a}\exp(-\delta\mathrm{x})\,G(\mathrm{x}) &\Rightarrow \forall y > a: ||G|| \geq \exp(-\delta y)\,G(y) \\ &\Rightarrow \forall y > a: G(y) \leq \exp(\delta y)||G|| \Rightarrow \mathrm{E}G(\zeta) \leq ||G||\mathrm{E}\exp(\delta\zeta) \end{aligned}$$

where ζ is an arbitrary random variable with support $\subset (a, +\infty)$.

[36]Of course, α is only then a Lipschitz constant if it is finite. Since in this, and all subsequent examples, we characterize sufficient conditions for $\alpha < 1$, which clearly implies $\alpha < +\infty$, therefore we do not separately check finiteness.

[37]As discussed above, our choice of δ implies that $||H|| < +\infty$ as required.

To conclude we note that for $\rho = 0$ equation (4.72) becomes

$$\mathbf{E}\exp\big((1-\gamma)\cdot(\mu+\xi)\big) = \mathbf{E}\exp\big((1-\gamma)\cdot x\big) < \frac{1}{\beta} \tag{4.73}$$

which can be shown directly to be both a sufficient but also *necessary* condition for a well-defined solution. ■

Proof of Proposition 6. The fixed point equation (4.18) corresponding to parameter vector $\lambda_1 \times \lambda_{-1} = \{0, \mu, \gamma, \beta, h\}$ (i.e. one in which $\rho = 0$) is

$$G(\mathsf{x}, \{0, \lambda_{-1}\}) = H(\mathsf{x}, \{0, \lambda_{-1}\}) + I(\mathsf{x}, \{0, \lambda_{-1}\}) \cdot \mathbf{E}\, G(\mu + \xi, \{0, \lambda_{-1}\}) \tag{4.74}$$

Since $H(\cdot,\cdot)$ and $I(\cdot,\cdot)$ are known functions, therefore to find $G(\mathsf{x}, \{0, \lambda_{-1}\})$ all we need is the *constant* $\mathbf{E}G(\mu+\xi, \{0, \lambda_{-1}\})$. To back it out, we can plug in $\mathsf{x} = \mu+\xi$, take expectations of both sides of (4.74) and solve for $\mathbf{E}G(\mu + \xi, \{0, \lambda_{-1}\}) = (1 - \mathbf{E}\, I(\mu + \xi, \{0, \lambda_{-1}\}))^{-1} \cdot \mathbf{E}H(\mu + \xi, \{0, \lambda_{-1}\})$. Plugging this constant back into equation (4.74), defining $x := \mu + \xi$ and using equation (4.19) to back out the value of $g(\mathsf{x}, \{0, \lambda_{-1}\})$ we finally obtain

$$\begin{aligned} g(\mathsf{x}, \{0, \lambda_{-1}\}) &= \big(H(\mathsf{x}, \{0, \lambda_{-1}\})\big)^{-1} G(\mathsf{x}, \{0, \lambda_{-1}\}) - 1 = W^{-1}(\mathsf{x}, \{0, \lambda_{-1}\}) \\ &\quad \cdot \big(1 - \mathbf{E}\, W^{-1}(x, \{0, \lambda_{-1}\})\, H(x, \{0, \lambda_{-1}\})\big)^{-1} \cdot \mathbf{E}H(x, \{0, \lambda_{-1}\}). \blacksquare \end{aligned}$$

Proof of Proposition 7. First note that after plugging in for $\mathbb{F}(\cdot,\cdot,\cdot)$ equation (4.18) becomes

$$G(\mathsf{x}, \mathsf{y}) = H(\mathsf{x}, \mathsf{y}) + I(\mathsf{x}, \mathsf{y}) \cdot \mathbf{E}\, G((1-\mathsf{y}_1)\mathsf{y}_2 + \mathsf{y}_1\mathsf{x} + \xi, \mathsf{y}, \xi), \mathsf{y}) \tag{4.75}$$

where from the proof of Proposition 6 it follows that[38]

$$G(\mathsf{x}, \{0, \mathsf{y}\}) = \frac{\mathcal{C}^H}{1 - \mathcal{C}^I} I(\mathsf{x}, \{0, \mathsf{y}\}) + H(\mathsf{x}, \{0, \mathsf{y}\}) \tag{4.76}$$

and equation (4.19) can be used to express $g(\mathsf{x}, \mathsf{y})$ in terms of $G(\mathsf{x}, \mathsf{y})$. Straight

[38]Definition of all the constants, denoted by $\mathcal{C}$ can be found in Table 1.

from (4.75), letting $\lambda_1 = \{\rho\}, \lambda_{-1} = \{\mu, \gamma, \beta, h\}$ and $\lambda = \lambda_1 \times \lambda_{-1}$ - we get

$$\frac{\partial G(\mathsf{x},\lambda)}{\partial \rho} = I(\mathsf{x},\lambda)\ \mathbf{E}\left(\frac{\partial G\big((1-\rho)\mu+\rho\mathsf{x}+\xi,\lambda\big)}{\partial \mathsf{x}}(\mathsf{x}-\mu) + \frac{\partial G\big((1-\rho)\mu+\rho\mathsf{x}+\xi,\lambda\big)}{\partial \rho}\right) \quad (4.77)$$

Equation (4.76) implies

$$\frac{\partial G(\mathsf{x},\{0,\lambda_{-1}\})}{\partial \mathsf{x}} = \frac{\partial H(\mathsf{x},\{0,\lambda_{-1}\})}{\partial \mathsf{x}} + \frac{\partial I(\mathsf{x},\{0,\lambda_{-1}\})}{\partial \mathsf{x}}\ \frac{\mathcal{C}^H}{1-\mathcal{C}^I}$$

which can be plugged back into equation (4.77) evaluated at $\{0, \lambda_{-1}\}$ to yield

$$\frac{\partial G(\mathsf{x},\{0,\lambda_{-1}\})}{\partial \rho} = I(\mathsf{x},\{0,\lambda_{-1}\})\ \mathbf{E}\left(\frac{\partial G\big(\mu+\xi,\{0,\lambda_{-1}\}\big)}{\partial \rho} + \left[\frac{\partial H(\mu+\xi,\{0,\lambda_{-1}\})}{\partial \mathsf{x}} + \frac{\partial I(\mu+\xi,\{0,\lambda_{-1}\})}{\partial \mathsf{x}}\ \frac{\mathcal{C}^H}{1-\mathcal{C}^I}\right](\mathsf{x}-\mu)\right). \quad (4.78)$$

The terms in the square brackets do not depend on x and consist entirely of derivatives of known functions $H(\cdot,\cdot), I(\cdot,\cdot)$. Accordingly, defining[39]

$$U_{1,1} := \mathbf{E}\left(\frac{\partial H(\mu+\xi,\{0,\lambda_{-1}\})}{\partial \mathsf{x}} + \frac{\partial I(\mu+\xi,\{0,\lambda_{-1}\})}{\partial \mathsf{x}}\ \frac{\mathcal{C}^H}{1-\mathcal{C}^I}\right)$$

we can rewrite equation (4.78) as

$$\frac{\partial G(\mathsf{x},\{0,\lambda_{-1}\})}{\partial \rho} = I(\mathsf{x},\{0,\lambda_{-1}\})\ \mathbf{E}\left[\frac{\partial G\big(\mu+\xi,\{0,\lambda_{-1}\}\big)}{\partial \rho} + U_{1,1}(\mathsf{x}-\mu)\right]. \quad (4.79)$$

We can solve for the value of the constant $U_{1,0} := \mathbf{E}(\partial/\partial\rho)G\big(\mu+\xi,\{0,\lambda_{-1}\}\big)$ by plugging in $\mathsf{x} = \mu + \xi$ and taking expectation of both sides of (4.79)

$$U_{2,1} = \mathbf{E}\big(I(\mu+\xi,\{0,\lambda_{-1}\})\cdot\big(U_{1,1}\xi + U_{2,1}\big)\big) \Rightarrow U_{2,1} = U_{1,1}\big(1-\mathcal{C}^I\big)^{-1}\mathcal{C}^{I\xi}.$$

Plugging this back into equation (4.79) we finally obtain

$$\frac{\partial G(\mathsf{x},\{0,\lambda_{-1}\})}{\partial \rho} = I(\mathsf{x},\{0,\lambda_{-1}\})\Big[U_{1,0} + U_{1,1}\cdot(\mathsf{x}-\mu)\Big] \quad (4.80)$$

[39]Where no ambiguity can arise, to cut on notation, we shall not stress the dependence of $U_{1,1}$ on λ_{-1} and simply write $U_{1,1}$ rather than $U_{1,1}(\lambda_{-1})$.

which, given that from equation (4.19) $g(\mathsf{x},\mathsf{y}) = H^{-1}(\mathsf{x},\mathsf{y})G(\mathsf{x},\mathsf{y}) - 1$ immediately implies that the first order approximation to $g(\mathsf{x},\lambda)$ is given by

$$g^1(\mathsf{x},\lambda) = (1 - h\exp(-x))^\gamma \left(\frac{C^H}{1-C^I} + \rho\cdot\left(U_{2,1} + U_{1,1}\cdot(\mathsf{x}-\mu)\right)\right).$$

Proceeding as above, to compute the second order approximation to $g(x,\lambda)$ we need to find $(\partial^2/\partial\rho^2)G(\mathsf{x},\{0,\lambda_{-1}\})$. From (4.75), this equals

$$\begin{aligned}\frac{\partial^2 G(\mathsf{x},\{0,\lambda_{-1}\})}{\partial\rho\,\partial\rho} &= I(\mathsf{x},\{0,\lambda_{-1}\})\;\mathrm{E}\left(\frac{\partial^2 G(\mu+\xi,\{0,\lambda_{-1}\})}{\partial\mathsf{x}\,\partial\mathsf{x}}(\mathsf{x}-\mu)^2\right.\\ &\left.+2\frac{\partial^2 G(\mu+\xi,\{0,\lambda_{-1}\})}{\partial\mathsf{x}\,\partial\rho}(\mathsf{x}-\mu) + \frac{\partial^2 G(\mu+\xi,\{0,\lambda_{-1}\})}{\partial\rho\,\partial\rho}\right)\end{aligned}\tag{4.81}$$

Again straight from equation (4.76) it follows that

$$\frac{\partial^2 G(\mathsf{x},\{0,\lambda_{-1}\})}{\partial\mathsf{x}\,\partial\mathsf{x}} = H''(\mathsf{x},\{0,\lambda_{-1}\}) + I''(\mathsf{x},\{0,\lambda_{-1}\})\frac{C^H}{1-C^I}\tag{4.82}$$

where H'' and I'' denote derivatives with respect to x. Further, from (4.80)

$$\frac{\partial^2 G(\mathsf{x},\{0,\lambda_{-1}\})}{\partial\rho\,\partial\mathsf{x}} = I'(\mathsf{x},\{0,\lambda_{-1}\})\left(U_{1,1}(\mathsf{x}-\mu) + U_{1,0}\right) + I(\mathsf{x},\{0,\lambda_{-1}\})\cdot U_{1,1}.\tag{4.83}$$

To compute $E(\partial^2/\partial\rho^2)\,G(\mu+\xi,\{0,\lambda_{-1}\})$ first plug in (4.82) and (4.83) back into (4.81), then plug in $\mathsf{x} = \mu+\xi$ and take expectations. This implies

$$\begin{aligned}\mathrm{E}\frac{\partial^2 G(\mu+\xi,\{0,\lambda_{-1}\})}{\partial\rho\,\partial\rho} &= \mathrm{E}I(\mu+\xi,\{0,\lambda_{-1}\})\left(\left[C^{H''} + C^{I''}\frac{C^H}{1-C^I}\right]\xi^2\right.\\ &\left.+2\left[C^{I'\xi}U_{1,1} + C^{I'}U_{2,1} + C^I U_{1,1}\right]\xi + \bar{\mathrm{E}}\frac{\partial^2 G(\mu+\bar\xi,\{0,\lambda_{-1}\})}{\partial\rho\,\partial\rho}\right).\end{aligned}$$

Denoting the constants in the square brackets by $U_{1,2}$ and $U_{2,2}$ respectively and letting $U_{3,2} := \mathrm{E}(\partial^2/(\partial\rho\,\partial\rho))\,G(\mu+\xi,\{0,\lambda_{-1}\})$ we can solve for $U_{3,2}$

$$U_{3,2} = \left(1-C^I\right)^{-1}\mathrm{E}I(\mu+\xi)\left(U_{1,2}\xi^2 + U_{2,2}\xi\right) = \frac{U_{2,2}C^{I\xi} + U_{1,2}C^{I\xi^2}}{1-C^I}\tag{4.84}$$

where, for reference

$$U_{1,2} = C^{H''} + C^{I''}\frac{C^H}{1-C^I} \qquad U_{2,2} = 2\left(\left(C^{I(1)\xi} + C^I\right)U_{1,1} + C^{I(1)}U_{2,1}\right).\tag{4.85}$$

Accordingly, we have found all the terms entering (4.81) and so can use the formula for $(\partial^2/(\partial\rho\,\partial\rho))\,G(\mathsf{x},\{0,\lambda_{-1}\})$ to write down an expression for the second order approximation to $g(\mathsf{x},\{0,\lambda_{-1}\})$

$$g^2(\mathsf{x},\lambda)=g^1(\mathsf{x},\lambda)+(1-h\exp(-\mathsf{x}))^{\gamma}\cdot\frac{\rho^2}{2}\cdot\left[U_{2,0}+(\mathsf{x}-\mu)^2U_{2,1}+(\mathsf{x}-\mu)\,U_{2,2}\right].$$

Higher order derivatives / approximations can be computed analogously though the algebra is messier and so we skip them to save on space. Details are available upon request. ■

Proof of Proposition 8. We know from Theorem 4 that

$$||G(\{\rho,\lambda_{-1}\})-G(\{0,\lambda_{-1}\})||\leq|\rho-0|\sup_{|\nu-\{0,\lambda_{-1}\}|\leq r}||\frac{dG}{d\rho}(\nu)||. \tag{4.86}$$

Straight from (4.75) we get

$$\frac{\partial G(\mathsf{x},\lambda)}{\partial\rho}=I(\mathsf{x},\lambda)\cdot\left(\mathbf{E}\frac{\partial G\big((1-\rho)\mu+\rho\mathsf{x}+\xi,\lambda\big)}{\partial\mathsf{x}}(\mathsf{x}-\mu)+\mathcal{P}\big(\frac{\partial G}{\partial\rho}(\lambda)\big)\right) \tag{4.87}$$

and Theorem 3 further implies that

$$\begin{aligned}||\frac{\partial G}{\partial\rho}||\leq&\frac{1}{1-\alpha}||I(\mathsf{x},\lambda)\cdot|\mathsf{x}-\mu|\cdot\mathbf{E}\left(\frac{\partial G((1-\rho)\mu+\rho\mathsf{x}+\xi,\lambda)}{\partial\mathsf{x}}\right)||\\ \leq&\frac{1}{1-\alpha}\mathbf{E}||I(\mathsf{x},\lambda)\cdot|\mathsf{x}-\mu|\cdot\left(\frac{\partial G((1-\rho)\mu+\rho\mathsf{x}+\xi,\lambda)}{\partial\mathsf{x}}\right)||\\ =&\beta\sup_{\mathsf{x}>a}\exp(-\delta\mathsf{x})|\mathsf{x}-\mu|\exp\big((1-\gamma)\mathsf{x}\big)||\frac{\partial G}{\partial\mathsf{x}}((1-\rho)\mu+\rho\mathsf{x}+\xi,\lambda)||\\ \leq&\beta\left(\sup_{\mathsf{x}\geq a}|\mathsf{x}-\mu|\exp\big((1-\gamma)\mathsf{x}\big)\right)||\frac{\partial G}{\partial\mathsf{x}}||.\end{aligned}$$

To compute the norm of $(\partial/\partial\mathsf{x})\,G$ we differentiate formula (4.18) and obtain

$$\frac{\partial G}{\partial\mathsf{x}}(\mathsf{x},\lambda)=\frac{\partial H(\mathsf{x},\lambda)}{\partial\mathsf{x}}+(1-\gamma)\mathcal{P}\big(G(\mathsf{x},\lambda)\big)+\mathcal{P}\big(\frac{\partial G}{\partial\mathsf{x}}(\mathsf{x},\lambda)\big) \tag{4.88}$$

and so

$$\begin{aligned}||\frac{\partial G}{\partial\mathsf{x}}||\leq&\frac{1}{1-\alpha}||\frac{\partial H}{\partial\mathsf{x}}+(1-\gamma)\mathcal{P}\big(G(\rho)\big)||\leq\frac{1}{1-\alpha}\Big[||\frac{\partial H}{\partial\mathsf{x}}||\\ &+(1-\gamma)||\mathcal{P}\big(G(\rho)\big)||\Big]\leq\frac{1}{1-\alpha}\Big[||\frac{\partial H}{\partial\mathsf{x}}||+(1-\gamma)\frac{\alpha}{1-\alpha}||H||\Big].\end{aligned} \tag{4.89}$$

We can then use the definition of $H(\mathsf{x},\lambda)$ to write

$$H(\mathsf{x},\lambda) = \beta\frac{\exp(\mathsf{x})}{(\exp(\mathsf{x})-h)^{\gamma}} \qquad \frac{\partial H}{\partial \mathsf{x}}(\mathsf{x},\lambda) = H(\mathsf{x},\lambda)\left(1-\frac{\gamma}{\exp(\mathsf{x})-h}\right)$$

which implies that

$$\begin{aligned}||H|| &= \beta\sup_{\mathsf{x}>a}\exp(-\mathsf{x}\,\delta)\frac{\exp(\mathsf{x})}{(\exp(\mathsf{x})-h)^{\gamma}} = \beta\sup_{\mathsf{x}>a}\frac{\exp(\mathsf{x}(1-\delta))}{(\exp(\mathsf{x})-h)^{\gamma}}\\ &= \beta\sup_{\mathsf{x}>a}\exp(\mathsf{x}(1-\delta-\gamma))\left(1-\frac{h}{\exp(\mathsf{x})}\right)^{-\gamma} \leq \beta\cdot\exp(a(1-\delta-\gamma))\left(1-\frac{h}{\exp(a)}\right)^{-\gamma}\end{aligned}$$

and

$$||\frac{\partial H}{\partial \mathsf{x}}|| \leq ||H||\sup_{\mathsf{x}>a}\left(1-\frac{\gamma}{\exp(\mathsf{x})-h}\right) \leq ||H||.$$

Plugging all the estimates above into formula (4.86) we obtain

$$\begin{aligned}&||G(\{\rho,\lambda_{-1}\})-G(\{0,\lambda_{-1}\})|| \leq \rho \sup_{|\nu-\{0,\lambda_{-1}\}|\leq r}||\frac{dG}{d\rho}(\nu)|| \leq \rho\cdot\beta\cdot||\frac{\partial G}{\partial \mathsf{x}}||\\ &\quad\cdot\left(\sup_{\mathsf{x}\geq a}|\mathsf{x}-\mu|\exp((1-\gamma)\mathsf{x})\right) \leq \kappa_a\cdot\frac{\rho\cdot\beta}{1-\alpha}\left[1+(1-\gamma)\frac{\alpha}{1-\alpha}\right]||H||\\ &\qquad\leq \rho\cdot\kappa_a\cdot\left(\frac{\beta}{1-\alpha}\right)^2\cdot\left[1-\gamma\alpha\right]\cdot\exp(a(1-\delta-\gamma))\cdot\left(1-\frac{h}{\exp(a)}\right)^{-\gamma}\end{aligned}$$

where $\kappa_a := \left(\sup_{\mathsf{x}\geq a}|\mathsf{x}-\mu|\exp((1-\gamma)\mathsf{x})\right)$. Setting $\delta=(1-\gamma)$ and plugging in for α from equation (4.71) completes the proof. ■

Proof of Proposition 9. To ensure that the support of c_t^g is invariant - i.e. if $c_{t-1}^g > c_l^g \Rightarrow c_t^g > c_l^g$ we first observe that $\exp(c_t^g) \geq h\big((1-\phi)+\phi\exp(-c_l^g)\big) \geq \exp(c_l^g)$ is an inequality which can be solved for c_l^g

$$c_l^g \in \left(\log\Big(h(1-\phi)-\sqrt{\frac{h^2(1-\phi)^2}{4}+h\phi}\Big), \log\Big(h(1-\phi)+\sqrt{\frac{h^2(1-\phi)^2}{4}+h\phi}\Big)\right).$$

Dividing both sides by $C_{t-1}\geq 0$ we obtain $\exp(c_t^g) \geq h\big((1-\phi)+\phi\exp(-c_{t-1}^g)\big)$. Letting c_l^g be the infimum of x's support and plugging in for $c_t^g = \mu(1-\rho)+\rho c_{t-1}^g+\xi_t$ a sufficient condition for the original inequality to hold is

$$\begin{aligned}&\exp(\mu(1-\rho)+\rho c_{t-1}^g+\xi_t) \geq h\big((1-\phi)+\phi\exp(-c_l^g)\big)\\ &\qquad\Rightarrow \exp(\xi_t) \geq h\big((1-\phi)+\phi\exp(-c_l^g)\big)\exp(-\mu(1-\rho)-\rho c_l^g) \quad (4.90)\end{aligned}$$

where the latter inequality holds if and only if $\rho > 0$ (which we assume). ■

Proof of Proposition 10. Let $\tilde{\mathcal{G}}$ be a Banach space with norm $||\cdot||$ whose elements are functions $G : [c, +\infty)^2 \to \mathbb{R}$ such that $G \in \tilde{\mathcal{G}}$ if and only if

$$||G|| := \sup_{(\mathrm{x}_1,\mathrm{x}_2)\in[c,+\infty)^2} \exp(-\delta_1\mathrm{x}_1 - \delta_2\mathrm{x}_2) \cdot |G(\mathrm{x}_1,\mathrm{x}_2)| < +\infty \tag{4.91}$$

where c is a constant greater than c_l^g and δ_1, δ_2 are positive constants yet to be determined. Further, let $\mathcal{G}$ be a closed subset of $\tilde{\mathcal{G}}$ whose elements are functions $G : [c, +\infty)^2 \to \mathbb{R}^+$. We shall demonstrate that condition (4.33) guarantees that the operator $\mathcal{Z} : \mathcal{G} \to \mathcal{G}$ defined in equation (4.32) is Lipschitz with a Lipschitz constant smaller than one. By Assumption A1 and Theorem 3 this shall imply that there exists a solution, unique in $\mathcal{G}$, to the fixed point equation $G = \mathcal{Z}(G, \lambda)$ and so also to the corresponding equation (4.27).

As in the previous example, to ensure that $\mathcal{Z} : \mathcal{G} \to \mathcal{G}$, we first check whether $H \in \mathcal{G}$. Recalling the definition of H we have that

$$\begin{aligned}||H|| = \sup_{\mathrm{x}_1,\mathrm{x}_2>c} &\beta(1 - h((1-\phi)\exp(-\mathrm{x}_1) + \phi\exp(-\mathrm{x}_1 - \mathrm{x}_2)))^{-\gamma} \\ &\cdot \exp((1-\gamma-\delta_1)\mathrm{x}_1) \cdot \exp(-\delta_2\mathrm{x}_2)\end{aligned}$$

which implies that $||H|| < +\infty \Leftrightarrow \delta_1 \geq 1 - \gamma, \delta_2 \geq 0$.[40] To derive an expression for the Lipschitz constant of operator $\mathcal{Z}$ note that the triangle inequality implies

[40]We use the fact that for $\mathrm{x}_1, \mathrm{x}_2 > c$ the first line in the definition of $||H||$ is bounded.

that $||\mathcal{Z}(G)|| \leq ||H|| + ||I \cdot \mathcal{P}(G)|| \leq ||I \cdot \mathcal{P}(G)||$ and so[41]

$$\begin{aligned}
||\mathcal{Z}(G)|| &\leq \beta \sup_{(x_1,x_2)\in[c,+\infty)^2} \exp\big(x_1(1-\gamma-\delta_1) - \delta_2 x_2\big)|\mathbf{E}G\big((1-\rho)\mu + \rho x_1 + \xi, x_1\big)| \\
&\leq \beta||G|| \sup_{(x_1,x_2)\in[c,+\infty)^2} \exp\big(x_1(1-\gamma-\delta_1) - \delta_2 x_2\big)\mathbf{E}\exp\big(\delta_1((1-\rho)\mu + \rho x_1 + \xi) + \delta_2 x_1\big) \\
&= ||G|| \cdot \beta \cdot \exp\big(\delta_1(1-\rho)\mu\big) \cdot \mathbf{E}\exp\big(\delta_1 \xi\big) \\
&\qquad \cdot \sup_{(x_1,x_2)\in[c,+\infty)^2} \exp\left(x_1\big(1-\gamma-\delta_1(1-\rho)+\delta_2\big) - x_2\delta_2\right).
\end{aligned}$$

We thus see that the Lipschitz constant is given by

$$\alpha = \beta\exp\big(\delta_1(1-\rho)\mu\big)\mathcal{L}_\xi(-\delta_1) \sup_{(x_1,x_2)\in[c,+\infty)^2} \exp\left(x_1\big(1-\gamma-\delta_1(1-\rho)+\delta_2\big) - x_2\delta_2\right).$$

where $\mathcal{L}_\xi(\cdot)$ is the Laplace transform corresponding to random variable ξ. For $\mathcal{Z}$ to be a contraction we set $\{\delta_1, \delta_2\} = \{1-\gamma, 0\}$. The corresponding α is

$$\alpha = \beta \exp\big((1-\gamma)(1-\rho)\mu + (1-\gamma)\rho c\big) \cdot \mathcal{L}_\xi(\gamma - 1). \tag{4.92}$$

Accordingly, for this choice of δ, the operator $\mathcal{Z}$ is a contraction mapping and the price dividend ratio $g(\cdot)$ is a well-defined function of the state if

$$\alpha < 1 \Leftrightarrow \exp\big((1-\gamma)(1-\rho)\mu + (1-\gamma)\rho c\big) \cdot \mathcal{L}_\xi(\gamma-1) < \beta^{-1}. \ \blacksquare \tag{4.93}$$

Proof of Proposition 11. The fixed point equation (4.30) corresponding to parameter vector $\lambda_1 \times \lambda_{-1} = \{0, \mu, \gamma, \beta, h, \phi\}$ (i.e. one in which $\rho = 0$) is

$$G(x, \{0, \lambda_{-1}\}) = H(x, \{0, \lambda_{-1}\}) + I(x, \{0, \lambda_{-1}\}) \cdot \mathbf{E}\, G(\{\mu + \xi, x_2\}, \{0, \lambda_{-1}\})$$

Since $H(\cdot,\cdot)$ and $I(\cdot,\cdot)$ are known functions, therefore to find $G(x, \{0, \lambda_{-1}\})$ all we need is the *function* $u_G(y) := \mathbf{E}G(\{\mu + \xi, y\}, \{0, \lambda_{-1}\})$. To do that, we can

[41]Where, the inequality between the first and second lines follows from

$$\begin{aligned}
||G|| := &\sup_{(x_1,x_2)\in[c,+\infty)^2} \exp(-\delta x_1 - \delta x_2)\,|G(x_1,x_2)| \\
&\Rightarrow \forall (y_1,y_2) \in [c,+\infty)^2 : ||G|| \geq \exp(-\delta_1 y_1 - \delta_2 y_2)\,|G(y_1,y_2)| \\
&\Rightarrow \forall (y_1,y_2) \in [c,+\infty)^2 : |G(y_1,y_2)| \leq \exp(\delta y_1 + \delta y_2)||G|| \\
&\Rightarrow \mathbf{E}|G\big((1-\rho)\mu + \rho x_1 + \xi, x_1\big)| \leq ||G|| \cdot \mathbf{E}\exp\big(\delta_1((1-\rho)\mu + \rho x_1 + \xi) + \delta_2 x_1\big)
\end{aligned}$$

where we set $y_1 = (1-\rho)\mu + \rho x_1 + \xi > c$ and $y_2 = x_2 > c$.

plug in $\mathsf{x} = \{\mu + \xi, \mathsf{y}\}$ and take expectations of both sides to obtain

$$u_G(\mathsf{y}) = \mathbf{E}H(\{\mu + \xi, \mathsf{y}\}, \{0, \lambda_{-1}\}) + \mathbf{E}I(x, \{0, \lambda_{-1}\})u_G(x)$$

where (slightly abusing notation) we have exploited the fact that $I(\{\mathsf{x}_1, \mathsf{x}_2\}, \lambda)$ does not depend on x_2. Denoting

$$u_H(\mathsf{y}) := \mathbf{E}H(\{\mu + \xi, \mathsf{y}\}, \{0, \lambda_{-1}\}) \qquad \kappa := \mathbf{E}I(x, \{0, \lambda_{-1}\})u_G(x) \tag{4.94}$$

and plugging in $u_G(\mu + \xi) = u_H(\mu + \xi) + \kappa$ (definition of $u_G(\cdot)$ evaluated at x) into the definition of κ yields and equation which can be solved for κ. Using the solution $\kappa = \mathbf{E}(I(x)\, u_H(x))/(1 - \mathbf{E}I(x))$ in the definition of $G(\cdot, \cdot)$ gives

$$G(\{\mathsf{x}_1, \mathsf{x}_2\}, \{0, \lambda_{-1}\}) = H(\mathsf{x}_1, \mathsf{x}_2) + I(\mathsf{x}_1)u_H(\mathsf{x}_1) + I(\mathsf{x}_1)\frac{\mathbf{E}(I(x)\, u_H(x))}{(1 - \mathbf{E}I(x))}.$$

where we have omitted the dependence of all the functions other than G on $\{0, \lambda_{-1}\}$. the function $u_H(\mathsf{y})$ was defined in (4.94) above. Plugging in the definitions of H and I allows us to back out $g(\mathsf{x}, \{0, \lambda_{-1}\})$ as

$$g(\{\mathsf{x}_1, \mathsf{x}_2\}, \{0, \lambda_{-1}\}) = \frac{\beta \mathsf{U}(\mathsf{x}_1) + \kappa_g}{(1 - h((1-\phi)\exp(-\mathsf{x}_1) + \phi \exp(-\mathsf{x}_1 - \mathsf{x}_2)))^{\gamma}}.$$

where $\mathsf{U}(\mathsf{y}) := \mathbf{E}\exp(x)(\exp(x) - h((1-\phi) + \phi\exp(-\mathsf{y})))^{-\gamma}$ and the constant κ_g is given by $\kappa_g := \left[\mathbf{E}\exp\big((1-\gamma)\,x\big) \cdot \beta\mathsf{U}(x) / \big(1/\beta - \mathbf{E}\exp\big((1-\gamma)\,x\big)\big)\right]$. Given $v_t = g(c_t^g, c_{t-1}^g)$ this implies $P_t = C_t \cdot g\big(\log(C_t/C_{t-1}), \log(C_{t-1}/C_{t-2})\big)$

$$\Rightarrow P_t = C_t \cdot \frac{\beta\mathbf{E}\exp(x)(\exp(x) - h((1-\phi) + \phi C_{t-1}/C_t))^{-\gamma} + \kappa_g}{(1 - h((1-\phi)C_{t-1}/C_t + \phi\, C_{t-2}/C_t))^{\gamma}}$$

where $x = \mu + \xi$. This concludes the proof. ■

Proof of Proposition 13 Recalling equation(4.38) we know that

$$\frac{\partial \mathcal{U}_t}{\partial c_t} \geq 0 \Leftrightarrow \exp\!\big(c_t^g(1-\gamma)\eta\big) \geq \beta\,\eta\, D \exp\big(((1-\rho)\mu + \rho c_{t-1}^g + \xi_t) \cdot (1-\gamma)\big)$$

where we have plugged in for c_{t+1}^g from equation (4.17). Taking logs

$$c_t^g \cdot (1-\gamma) \cdot \eta \geq \log\!\big(\beta\,\eta\, D\big) \cdot \big(((1-\rho)\mu + \rho c_t^g + \xi_{t+1}) \cdot (1-\gamma)\big)$$

where $\log(\beta \cdot \eta \cdot D) < 0$ as the product of β, η and D is smaller than 1. Proceeding under the assumption that $\gamma > 1$ (if $\gamma = 1$ the inequality is trivially satisfied), this can be rewritten as

$$\xi_{t+1} \geq c_t^g(\eta - \rho) + \mathcal{C}$$

where the constant $\mathcal{C} = -[(1-\rho)\mu + \log(\beta\eta D)/(1-\gamma)]$ is negative. If $\eta > \rho$, this implies that $\xi_{t+1} \geq c_h^g(\eta - \rho) + \mathcal{C}$, which combined with Proposition (4) implies $\xi \in [\max\{(1-\rho)c_l^g, c_h^g(\eta-\rho) + \mathcal{C}\}, c_h^g]$. Alternatively, if $\eta \leq \rho$ then $\xi \in [\max\{(1-\rho)c_l^g, c_l^g(\eta-\rho) + \mathcal{C}\}, c_h^g]$.[42] This will imply that marginal utility is always positive, and by Proposition (4), that the support of $c_t^g = [c_l^g, c_h^g]$ is time-invariant. ■

Proof of Proposition 14. Let $\tilde{\mathcal{G}}$ be a Banach space with norm $||\cdot||$ whose elements are functions $G : [c_l^g, c_h^g] \to \mathbb{R}$ s.t. $G \in \tilde{\mathcal{G}}$ if

$$||G|| := \sup_{x\in[c_l^g, c_h^g]} \exp(-\delta x) \cdot G(x) < +\infty. \tag{4.95}$$

where δ is a positive constant whose value is yet to be determined. Let $\mathcal{G}$ be a closed subset of $\tilde{\mathcal{G}}$ whose elements are functions $G : [c_l^g, c_h^g] \to \mathbb{R}^+$. As before, we shall demonstrate that the operator $\mathcal{Z}(G)(x) := H(x,\lambda) + I(x,\lambda)\,\mathrm{E}\,G(\mathbb{F}((x),\lambda,\xi),\lambda)$ on $\mathcal{G}$ is Lipschitz with a constant smaller than one.

Before proceeding, we first need to verify that $\mathcal{Z} : \mathcal{G} \to \mathcal{G}$ and in particular that $H \in \mathcal{G}$. Straight from the definition

$$||H|| := \sup_{x\in[c_l^g, c_h^g]} \beta\big(\exp([(1-\gamma)-\delta]x) - \beta^2\eta D\zeta \exp([(1-\gamma)(1-\eta-\rho)-\delta]x\big).$$

If both c_l^g and c_h^g are finite then for any choice of δ, $||H|| < +\infty$. From Footnote 42, if $\eta > \rho$ then $c_h^g \neq +\infty$ and a sufficient condition for $||H|| < +\infty$ is $\delta \leq 1-\gamma$. By the same token, if $\eta < \rho$ then $c_l^g \neq -\infty$ and then a sufficient condition for $||H|| < +\infty$ is $\delta \geq 1-\gamma$. If $\eta = \rho$ then a sufficient condition is $\delta = 1-\gamma$. Our choice of δ shall ensure that these restrictions are satisfied.

[42] One implication of this fact is that when $\eta > \rho, c_h^g \neq +\infty$ while when $\eta < \rho, c_l^g \neq -\infty$ i.e. if $\eta \neq \rho$ then the set of admissible x is always bounded from one side.

To prove that $\mathcal{Z}$ is Lipschitz note[43]

$$\begin{aligned}||\mathcal{Z}(G)|| &\leq \beta \sup_{\mathrm{x}\in[c_l^g,c_h^g]} \exp\big(\mathrm{x}((1-\gamma)(1-\eta)-\delta)\big)\mathbf{E}G\big((1-\rho)\mu+\rho\mathrm{x}+\xi\big) \\ &\leq \beta||G|| \sup_{\mathrm{x}\in[c_l^g,c_h^g]} \exp\big(\mathrm{x}((1-\gamma)(1-\eta)-\delta)\big)\mathbf{E}\exp\big(\delta((1-\rho)\mu+\rho\mathrm{x}+\xi)\big) \\ &= ||G||\beta\exp\big(\delta(1-\rho)\mu\big)\mathbf{E}\exp\big(\delta\,\xi\big)\sup_{\mathrm{x}\in[c_l^g,c_h^g]}\exp\big(\mathrm{x}((1-\gamma)(1-\eta)-\delta(1-\rho))\big)\,.\end{aligned}$$

Accordingly, $\mathcal{Z}$'s Lipschitz constant λ is given by

$$\lambda = \beta\cdot\exp\big(\delta(1-\rho)\mu\big)\cdot\mathbf{E}\exp\big(\delta\,\xi\big)\cdot\sup_{\mathrm{x}\in[c_l^g,c_h^g]}\exp\big(\mathrm{x}((1-\gamma)(1-\eta)-\delta(1-\rho))\big)\,.$$

Setting δ equal to $(1-\gamma)$ the Lipschitz constant becomes[44]

$$\lambda=\begin{cases}\beta\exp\big((1-\gamma)\cdot((1-\rho)\mu+c^g_{\max}(\rho-\eta))\big)\cdot\mathcal{L}_\xi(\gamma-1) & \eta>\rho\\ \beta\exp\big((1-\gamma)\cdot((1-\rho)\mu+c^g_{\min}(\rho-\eta))\big)\cdot\mathcal{L}_\xi(\gamma-1) & \eta<\rho\end{cases}$$

Transforming $\lambda<1$ gives the required condition and completes the proof. ■

Proof of Propositions 15 and 16. The proofs are conceptually identical to those of Propositions 6 and 7 respectively and so are skipped to save space. ■

Proof of Proposition 17. To find a benchmark solution set κ equal to zero. Equation (4.49) then becomes

$$g(\mathrm{x},\{0,\lambda_{-1}\}) = 1+f(\mathrm{x},\{0,\lambda_{-1}\})\cdot g(\bar{s},\{0,\lambda_{-1}\})\cdot\mathbf{E}h(\mathrm{x},\xi,\{0,\lambda_{-1}\}) \tag{4.96}$$

where $\mathbf{E}h(\mathrm{x},\xi,\{0,\lambda_{-1}\})$ is known. To compute the value of $g(\bar{s},\{0,\lambda_{-1}\})$ plug in $\mathrm{x}=\bar{s}$ to obtain

$$g(\bar{s},\{0,\lambda_{-1}\}) = \big(1-\beta\exp\big((1-\gamma)\tilde{g}\big)\cdot\mathbf{E}\exp\big((1-\gamma)\,\xi\big)\big)^{-1}$$

[43]Where the inequality between the first and second lines follows from

$$\begin{aligned}||G|| := \sup_{\mathrm{x}\in[c_l^g,c_h^g]}\exp(-\delta\mathrm{x})\,G(\mathrm{x}) &\Rightarrow \forall y\in[c_l^g,c_h^g]:\ ||G||\geq\exp(-\delta y)\,G(y)\\ &\Rightarrow \forall y\in[c_l^g,c_h^g]:\ G(y)\leq\exp(\delta y)||G|| \Rightarrow \mathbf{E}G(\zeta)\leq||G||\mathbf{E}\exp(\delta\zeta)\end{aligned}$$

where ζ is an arbitrary random variable with support $\subset[c_l^g,c_h^g]$.

[44]Our choice of δ implies that $||H||$ is always finite, as required. If $\eta=\rho$ one can use the first formula with $c^g_{\max}=0$.

and so

$$g(\mathrm{x},\{0,\lambda_{-1}\}) = 1 + \frac{f(\mathrm{x},\{0,\lambda_{-1}\})\cdot \mathbf{E}h(\mathrm{x},\xi,\{0,\lambda_{-1}\})}{1-\beta\cdot\exp((1-\gamma)\,\tilde{g})\cdot\mathbf{E}\exp((1-\gamma)\,\xi)} \blacksquare \tag{4.97}$$

Proof of Proposition 18. Differentiating equation (4.49) with respect to κ yields

$$\begin{aligned}&g_\kappa(\mathrm{x},\lambda) = f_\kappa(\mathrm{x},\lambda)\cdot\mathbf{E}h(\mathrm{x},\xi,\lambda)\cdot g(\mathbb{F}(\mathrm{x},\lambda,\xi),\lambda) + f(\mathrm{x},\lambda)\cdot\mathbf{E}\big[h_\kappa(\mathrm{x},\xi,\lambda)\\&\cdot g(\mathbb{F}(\mathrm{x},\lambda,\xi),\lambda) + h(\mathrm{x},\xi,\lambda)\cdot\big(g_\mathrm{x}(\mathbb{F}(\mathrm{x},\lambda,\xi),\lambda)\cdot\mathbb{F}_\kappa(\mathrm{x},\lambda,\xi) + g_\kappa(\mathbb{F}(\mathrm{x},\lambda,\xi),\lambda)\big)\big]\end{aligned}$$

which evaluated at $\{0,\lambda_{-1}\}$ becomes

$$\begin{aligned}g_\kappa(\mathrm{x},\{0,\lambda_{-1}\}) &= -\gamma\phi(\mathrm{x}-\bar{s})f(\mathrm{x},\{0,\lambda_{-1}\})\cdot\mathbf{E}\exp\big((1-\gamma)\,\xi\big)\cdot g(\bar{s},\{0,\lambda_{-1}\})\\&\quad + f(\mathrm{x},\{0,\lambda_{-1}\})\cdot\mathbf{E}\exp\big((1-\gamma)\,\xi\big)\cdot\big[\big(g_\mathrm{x}(\bar{s},\{0,\lambda_{-1}\})\\&\cdot(\phi\,(\mathrm{x}-\bar{s})+\Lambda(\mathrm{x})\,\xi) + g_\kappa(\bar{s},\{0,\lambda_{-1}\})\big) - \gamma\cdot\Lambda(\mathrm{x})\cdot\xi\cdot g(\bar{s},\{0,\lambda_{-1}\})\big].\end{aligned} \tag{4.98}$$

From equation (4.50)

$$g_\mathrm{x}(\mathrm{x},\{0,\lambda_{-1}\}) = f(\mathrm{x},\{0,\lambda_{-1}\})\frac{\gamma\mathbf{E}h(\mathrm{x},\xi,\{0,\lambda_{-1}\}) + \mathbf{E}h_\mathrm{x}(\mathrm{x},\xi,\{0,\lambda_{-1}\})}{1-\beta\cdot\exp((1-\gamma)\,\tilde{g})\cdot\mathbf{E}\exp((1-\gamma)\,\xi)}$$

and so

$$g_\mathrm{x}(\bar{s},\{0,\lambda_{-1}\}) = \gamma\cdot\frac{\beta\exp\big((1-\gamma)\tilde{g}\big)\cdot\mathbf{E}\exp((1-\gamma)\cdot\xi)}{1-\beta\cdot\exp((1-\gamma)\,\tilde{g})\cdot\mathbf{E}\exp((1-\gamma)\,\xi)}$$

Accordingly, plugging in $\mathrm{x}=\bar{s}$ into equation (4.98) we obtain

$$\begin{aligned}g_\kappa(\bar{s},\{0,\lambda_{-1}\}) &= f(\bar{s},\{0,\lambda_{-1}\})\cdot\mathbf{E}\exp\big((1-\gamma)\,\xi\big)\cdot\big[\big(g_\mathrm{x}(\bar{s},\{0,\lambda_{-1}\})\\&\quad\cdot\Lambda(\bar{s})\,\xi + g_\kappa(\bar{s},\{0,\lambda_{-1}\})\big) - \gamma\cdot\Lambda(\bar{s})\cdot\xi\cdot g(\bar{s},\{0,\lambda_{-1}\})\big].\end{aligned} \tag{4.99}$$

which is a linear equation for $g_\kappa(\bar{s},\{0,\lambda_{-1}\})$, the solution of which is given by

$$\begin{aligned}g_\kappa(\bar{s},\{0,\lambda_{-1}\}) &= \Lambda(\bar{s})\cdot\big[g_\mathrm{x}(\bar{s},\{0,\lambda_{-1}\}) - \gamma\cdot g(\bar{s},\{0,\lambda_{-1}\})\big]\\&\quad\cdot\frac{\beta\exp\big((1-\gamma)\tilde{g}\big)\cdot\mathbf{E}\,\xi\cdot\exp\big((1-\gamma)\,\xi\big)}{\big(1-\beta\exp\big((1-\gamma)\tilde{g}\big)\cdot\mathbf{E}\exp\big((1-\gamma)\,\xi\big)\big)}.\end{aligned}$$

Plugging in for g_{x} and g we arrive at

$$g_\kappa(\bar{s},\{0,\lambda_{-1}\}) = -\gamma\cdot\Lambda(\bar{s})\frac{\beta\exp\big((1-\gamma)\tilde{g}\big)\cdot\mathbf{E}\,\xi\cdot\exp\big((1-\gamma)\,\xi\big)}{\big(1-\beta\exp\big((1-\gamma)\tilde{g}\big)\cdot\mathbf{E}\exp((1-\gamma)\,\xi)\big)}.$$

which can be plugged back into equation (4.98) and simplified to obtain

$$g_\kappa(\mathsf{x},\{0,\lambda_{-1}\}) = -\exp\big(\gamma\,(\mathsf{x}-\bar{s})\big)\beta\exp((1-\gamma)\tilde{g})\mathbf{E}\exp((1-\gamma)\,\xi)\cdot\Bigg(\gamma\phi(\mathsf{x}-\bar{s})$$
$$+\gamma\Lambda(\mathsf{x})\mathbf{E}\,\xi\cdot\exp\big((1-\gamma)\,\xi\big)\cdot\big[1+\frac{\Lambda(\bar{s})/\Lambda(\mathsf{x})\cdot\beta\exp\big((1-\gamma)\tilde{g}\big)}{\big(1-\beta\exp((1-\gamma)\tilde{g})\mathbf{E}\exp((1-\gamma)\,\xi)\big)}\big]\Bigg).$$

Combining all this together implies that a first order approximation to $g(\mathsf{x},\lambda)$ around $g(\mathsf{x},\{0,\lambda_{-1}\})$ is given by

$$g(\mathsf{x},\lambda)-1\approx\big(g(\mathsf{x},\{0,\lambda_{-1}\})-1\big)+\kappa g_\kappa(\mathsf{x},\{0,\lambda_{-1}\}) = \exp\big(\gamma(\mathsf{x}-\bar{s})\big)\cdot\beta$$
$$\exp((1-\gamma)\tilde{g})\cdot\mathbf{E}\exp((1-\gamma)\,\xi)\Bigg(1-\beta\cdot\exp((1-\gamma)\,\tilde{g})\cdot\mathbf{E}\exp((1-\gamma)\,\xi)-\kappa\gamma\phi(\mathsf{x}-\bar{s})$$
$$+\gamma\Lambda(\mathsf{x})\mathbf{E}\,\xi\cdot\exp\big((1-\gamma)\,\xi\big)\cdot\big[1+\frac{\Lambda(\bar{s})/\Lambda(\mathsf{x})\cdot\beta\exp\big((1-\gamma)\tilde{g}\big)}{\big(1-\beta\exp((1-\gamma)\tilde{g})\mathbf{E}\exp((1-\gamma)\,\xi)\big)}\big]\Bigg).\ \blacksquare$$

Proof of Proposition 19. Evaluating equation (4.56) at $\lambda_1=\kappa=0$ yields

$$\mathbb{F}(\mathsf{x},\{0,\lambda_{-1}\},\xi) = \big\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\big\}$$

and so equation (4.57) corresponding to $\lambda_1=\rho=0$ becomes

$$G(\mathsf{x},\{0,\lambda_{-1}\}) = \lambda_{10}^{\lambda_3}\exp\big(\lambda_3(1-1/\lambda_2)\mathsf{x}_1\big)$$
$$\cdot\Big(1+\big(\mathbf{E}G(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\{0,\lambda_{-1}\})\big)^{1/\lambda_3}\Big)^{\lambda_3}.$$

Accordingly, the only unknown in the above equation is the expectation term, which is a function of λ_{-1}. To compute that function, we can plug in $\mathsf{x}=\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\}$ and take expectations. Defining $\nu(\lambda_{-1})=\mathbf{E}G(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\{0,\lambda_{-1}\})$ we then obtain

$$\nu(\lambda_{-1}) = \lambda_{10}^{\lambda_3}\Big(1+\big(\nu(\lambda_{-1})\big)^{1/\lambda_3}\Big)^{\lambda_3}\mathbf{E}\exp\big(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\big).$$

Raising both sides to the power $1/\lambda_3$ yields a linear equation for $(\psi(\lambda_{-1}))^{1/\lambda_3}$.

We can solve this equation, back out $\nu(\lambda_{-1})$

$$\nu(\lambda_{-1}) = \frac{\lambda_{10}^{\lambda_3}\left(\mathbf{E}\exp\left(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\right)\right)}{\left(1-\lambda_{10}\left(\mathbf{E}\exp\left(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\right)\right)^{1/\lambda_3}\right)^{\lambda_3}}.$$

and finally plug in for the *known* function $\nu(\lambda_{-1})$ to obtain a closed-form formula for $G(\mathsf{x}, \{0, \lambda_{-1}\})$

$$G(\mathsf{x}, \{0,\lambda_{-1}\}) = \lambda_{10}^{\lambda_3}\exp\left(\lambda_3(1-1/\lambda_2)\mathsf{x}_1\right) \cdot \left(1-\lambda_{10}\left(\mathbf{E}\exp\left(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\right)\right)^{1/\lambda_3}\right)^{-\lambda_3}. \tag{4.100}$$

Since

$$g(\mathsf{x},\lambda) = \frac{G(\mathsf{x},\lambda)^{1/\lambda_3}}{\lambda_{10}\left(\exp((1-1/\lambda_2)\mathsf{x}_1)\right)} - 1 \tag{4.101}$$

therefore

$$g(\mathsf{x}, \{0,\lambda_{-1}\}) = \frac{\lambda_{10}\left(\mathbf{E}\exp\left(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\right)\right)^{1/\lambda_3}}{1-\lambda_{10}\left(\mathbf{E}\exp\left(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\right)\right)^{1/\lambda_3}}. \quad \blacksquare$$

Proof of Proposition 20. Equation (4.57) yields

$$G(\mathsf{x},\{\kappa,\lambda_{-1}\}) = \lambda_{10}^{\lambda_3}\exp\left(\lambda_3(1-1/\lambda_2)\mathsf{x}_1\right)\left(1+\left(\mathbf{E}\,G(\mathbb{F}(\mathsf{x},\{\kappa,\lambda_{-1}\},\xi),\lambda)\right)^{1/\lambda_3}\right)^{\lambda_3}$$

and so

$$G_\kappa(\mathsf{x},\{\kappa,\lambda_{-1}\}) = \frac{G(\mathsf{x},\{\kappa,\lambda_{-1}\})}{1+\left(\mathbf{E}\,G(\mathbb{F}(\mathsf{x},\{\kappa,\lambda_{-1}\},\xi),\lambda)\right)^{1/\lambda_3}}\left(\mathbf{E}\,G(\mathbb{F}(\mathsf{x},\lambda,\xi),\lambda)\right)^{1/\lambda_3-1}$$
$$\cdot\,\mathbf{E}\left(\frac{\partial G}{\partial \mathsf{x}}(\mathbb{F}(\mathsf{x},\{\kappa,\lambda_{-1}\},\xi),\lambda)\frac{\partial \mathbb{F}}{\partial\kappa}(\mathsf{x},\{\kappa,\lambda_{-1}\},\xi)+\frac{\partial G}{\partial\kappa}(\mathbb{F}(\mathsf{x},\{\kappa,\lambda_{-1}\},\xi),\lambda)\right).$$

From equation (4.100) we obtain

$$G_{\mathsf{x}}(\mathsf{x},\{0,\lambda_{-1}\}) = \Big\{\lambda_3(1-1/\lambda_2)\lambda_{10}^{\lambda_3}\exp\left(\lambda_3(1-1/\lambda_2)\mathsf{x}_1\right) \cdot\left(1-\lambda_{10}\left(\mathbf{E}\exp\left(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\right)\right)^{1/\lambda_3}\right)^{-\lambda_3}, 0, 0\Big\}$$

while

$$\mathbb{F}_\kappa(\mathsf{x},\{0,\lambda_{-1}\},\xi) = \Big\{\mathsf{x}_2, \lambda_5\mathsf{x}_2, \lambda_9(\mathsf{x}_3-\lambda_7)+\lambda_8\xi_3\Big\}.$$

Defining $\iota(\mathsf{x},\lambda_{-1}) := \mathbf{E}\big(G_{\mathsf{x}}(\mathbb{F}(\mathsf{x},\{0,\lambda_{-1}\},\xi),\{0,\lambda_{-1}\})\cdot\mathbb{F}_{\kappa}(\mathsf{x},\{0,\lambda_{-1}\},\xi)\big)\big)$ we can take the scalar product of both these vectors to immediately obtain

$$\begin{aligned}\iota(\mathsf{x},\lambda_{-1}) &= \mathsf{x}_2\lambda_3(1-1/\lambda_2)\lambda_{10}^{\lambda_3}\cdot\exp\big(\lambda_3(1-1/\lambda_2)\mathsf{x}_1\big)\\ &\quad\cdot\Big(1-\lambda_{10}\big(\mathbf{E}\exp\big(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\big)\big)^{1/\lambda_3}\Big)^{-\lambda_3}.\end{aligned}$$

We thus know that $G_{\kappa}(\mathsf{x},\{0,\lambda_{-1}\})$ satisfies

$$\begin{aligned}G_{\kappa}(\mathsf{x},\{0,\lambda_{-1}\}) &= G(\mathsf{x},\{0,\lambda_{-1}\})\cdot\frac{\nu(\lambda_{-1})^{1/\lambda_3-1}}{1+\nu(\lambda_{-1})^{1/\lambda_3}}\\ &\quad\cdot\Big(\iota(\mathsf{x},\lambda_{-1})+\mathbf{E}G_{\kappa}\big(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\{0,\lambda_{-1}\}\big)\Big).\end{aligned}$$

As previously, plugging in $\mathsf{x}=\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\}$, defining $\vartheta(\lambda_{-1}) := \mathbf{E}G_{\kappa}\big(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\{0,\lambda_{-1}\}\big)$ and taking expectations we obtain

$$\begin{aligned}\vartheta(\lambda_{-1}) &= \mathbf{E}G(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\{0,\lambda_{-1}\})\cdot\frac{\nu(\lambda_{-1})^{1/\lambda_3-1}}{1+\nu(\lambda_{-1})^{1/\lambda_3}}\\ &\quad\cdot\Big(\iota(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\lambda_{-1})+\vartheta(\lambda_{-1})\Big)\end{aligned}$$

which again is a linear equation in $\vartheta(\lambda_{-1})$ and so can easily be solved to yield

$$\begin{aligned}\vartheta(\lambda_{-1}) &= \nu(\lambda_{-1})^{1/\lambda_3-1}\cdot\mathbf{E}\Big(G(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\{0,\lambda_{-1}\})\\ &\quad\cdot\iota(\{\lambda_4+\sqrt{\lambda_7}\xi_1,\lambda_6\sqrt{\lambda_7}\xi_2,\lambda_7\},\lambda_{-1})\Big).\end{aligned}\tag{4.102}$$

Accordingly, we have computed all the terms in the expression for the first derivative of the unknown function $G_{\kappa}(\mathsf{x},\{0,\lambda_{-1}\})$

$$G_{\kappa}(\mathsf{x},\{0,\lambda_{-1}\}) \;=\; G(\mathsf{x},\{0,\lambda_{-1}\})\cdot\frac{\nu(\lambda_{-1})^{1/\lambda_3-1}}{1+\nu(\lambda_{-1})^{1/\lambda_3}}\cdot\Big(\iota(\mathsf{x},\lambda_{-1})\;+\;\vartheta(\lambda_{-1})\Big).$$

and can use this expression to write down a formula for the first-order Taylor series approximation to $G(\mathsf{x},\lambda)$

$$\begin{aligned}G(\mathsf{x},\lambda) &\approx G(\mathsf{x},\{0,\lambda_{-1}\})+\kappa\cdot G_{\kappa}(\mathsf{x},\{0,\lambda_{-1}\})\\ &= G(\mathsf{x},\{0,\lambda_{-1}\})\Big(1+\kappa\frac{\nu(\lambda_{-1})^{1/\lambda_3-1}}{1+\nu(\lambda_{-1})^{1/\lambda_3}}\cdot\big(\iota(\mathsf{x},\lambda_{-1})+\vartheta(\lambda_{-1})\big)\Big)\end{aligned}\tag{4.103}$$

and by extension also

$$\begin{aligned}
1+g(\mathsf{x},\lambda) &= \big(G(\mathsf{x},\lambda)^{1/\lambda_3}\big)\cdot\big(\lambda_{10}\big(\exp((1-1/\lambda_2)\mathsf{x}_1)\big)\big)^{-1}\\
&\approx \frac{G(\mathsf{x},\{0,\lambda_{-1}\})^{1/\lambda_3}}{\lambda_{10}\big(\exp((1-1/\lambda_2)\mathsf{x}_1)\big)}\cdot\left(1+\kappa\frac{\nu(\lambda_{-1})^{1/\lambda_3-1}}{1+\nu(\lambda_{-1})^{1/\lambda_3}}\cdot\big(\iota(\mathsf{x},\lambda_{-1})+\vartheta(\lambda_{-1})\big)\right)^{1/\lambda_3}\\
&= \frac{\big(1+\kappa(\nu(\lambda_{-1})^{1/\lambda_3-1})/(1+\nu(\lambda_{-1})^{1/\lambda_3})\cdot\big(\iota(\mathsf{x},\lambda_{-1})+\vartheta(\lambda_{-1})\big)\big)^{1/\lambda_3}}{1-\lambda_{10}\big(\mathbf{E}\exp\big(\lambda_3(1-1/\lambda_2)(\lambda_4+\sqrt{\lambda_7}\xi_1)\big)\big)^{1/\lambda_3}}. \quad\blacksquare
\end{aligned}$$

Conclusions

To conclude, we reiterate the main messages from each of the four essays.

In the first chapter, we used the simplest possible framework to analyze the determinants of risk premium dynamics. We demonstrated that due to changing recession risks, risk premia can be *procyclical* even though the volatility of consumption is *constant* and despite a *countercyclically* varying risk aversion coefficient. We have also documented that persistent habits, shocks or features generating hump shaped consumption responses are all likely to make the premium countercyclical.

Fundamentally, the countercyclicality of the premium in our model, rests on agents' belief that changes in economic conditions are persistent. In other words, after an adverse shock, more risk-averse agents will only require a larger premium on risky assets if they don't expect their future conditions to improve massively. Thus, expressed alternatively, our work explicitly explores the role of countercyclical recession risks – a feature that is implicit in Campbell and Cochrane (1999), and similar in spirit to the mechanism driving the results in Bansal and Yaron (2004). Our results suggest that factors which help match activity data - i.e. allowing for consumption habits and persistent shocks - are also likely to help along the asset pricing dimension.

Changes in premia substantially contribute to asset price volatility and so having a good understanding of factors driving them is crucial for modeling asset prices. Given the increasing frequency with which macroeconomic models are being used to address asset pricing puzzles, it is key to clarify how and why changes in standard modeling assumptions translate into different dynamics of premia. While our study attempted to clarify these issues, further analysis of risk premium dynamics could still be undertaken. For example, in production-economy models the dynamics of term-premia or the inflation risk premium would depend on other aspects of the transmission mechanism. We believe that examining these aspects would be of interest.

Our results in the second essay show that, following persistent adverse shocks, policy-makers might be well advised to steer off predictions of linear models and conduct more accommodative policy. Equally, when demand and supply conditions are improving, taking note of the precautionary saving motives justifies 'leaning against the wind'. Since the size of the precautionary correction is increasing in the degree of volatility, mistakenly ignoring this channel would be most costly during highly turbulent periods. We believe that formally accounting for stochastic volatility and enriching the framework by considering Epstein-Zin preferences would both make for interesting extensions.

In the third chapter I derive closed-form solutions for the equity price-dividend ratio and equity risk-premium in a model in which agents have difference-form external habits. The setup allows for arbitrary shock distributions, correlated consumption growth and nests extensions of the keeping-up and catching-up with the Joneses specifications. I then use the exact solutions to study the ability of alternative estimated models – including one capturing rare events – to simultaneously account for consumption, equity and bond returns in the UK and US.

My results highlight tensions which exist when one tries to simultaneously match the properties of consumption growth, bond returns and equity returns. While the habit specification proposed by Campbell and Cochrane (1999) is capable of cutting the umbilical-cord linking all three, the same cannot be achieved with the more parsimonious model which this papers solves in closed-form. We document the dimensions in which the model fails and argue that more exotic shock distributions are unlikely to solve the underlying problems. However, explicitly accounting for differences between consumption and dividends would provide for an interesting extension and could potentially help tackle some of them.

In the final essay I propose a new method of approximating solutions of models with known endogenous state-variable dynamics. Focusing on asset pricing models, I show how to re-express the Euler equation as a fixed-point problem in Banach spaces. This is key to finding solutions of nested problems, which are then used to approximate the full solution. I show that even low-order approximations are not 'certainty equivalent' and that the zeroth order approximation could allow for time-variation in risk premia. Arguably – by allowing the approximation to be an arbitrary, nonlinear function of the state – the method makes the approximations more parsimonious while retaining high precision. The paper also clarifies how to use the Lipschitz constant to provide upper bounds on the resulting approximation errors. I believe that generalizing the method to frameworks in which the dynamics of state variables is not known as well as a more thorough investigation of its accuracy and asymptotic properties would both make for worthwhile extensions.

Bibliography

ABEL, A. B. (1990): "Asset Prices under Habit Formation and Catching Up with the Joneses," *American Economic Review*, 80, 38–42.

——— (2008): "Equity Premia with Benchmark Levels of Consumption: Closed-Form Results," in *Handbook of the Equity Risk Premium*, ed. by R. Mehra, Elsevier, 117 – 157.

ALESSIE, R. AND A. LUSARDI (1997): "Consumption, saving, and habit formation," *Economics Letters*, 55, 103–108.

AMATO, J. D. AND T. LAUBACH (2004): "Implications of habit formation for optimal monetary policy," *Journal of Monetary Economics*, 51, 305–325.

BANERJEE, R. AND N. BATINI (2003): "UK Consumers' Habits," Discussion Papers 13, Monetary Policy Committee Unit, Bank of England.

BANSAL, R. AND A. YARON (2004): "Risks for the Long Run: A Potential Resolution of Asset Pricing Puzzles," *Journal of Finance*, 59, 1481 – 1509.

BARRO, R. (2006): "Rare Disasters and Asset Markets in the Twentieth Century," *Quarterly Journal of Economics*, 121, 823 – 866.

BARRO, R., E. NAKAMURA, J. STEINSSON, AND J. URSUA (2009): "Crises and Recoveries in an Empirical Model of Consumption Disasters," Harvard University, mimeo.

BARRO, R. AND J. URSUA (2008): "Consumption Disasters in the Twentieth Century," *American Economic Review*, 98, 58–63.

BENIGNO, P. AND M. WOODFORD (2005): "Inflation Stabilization And Welfare: The Case Of A Distorted Steady State," *Journal of the European Economic Association*, 3, 1185–1236.

BIDARKOTA, P. V. AND J. H. MCCULLOCH (2003): "Consumption asset pricing with stable shocks–exploring a solution and its implications for mean equity returns," *Journal of Economic Dynamics and Control*, 27, 399–421.

BOLDRIN, M., L. CHRISTIANO, AND J. FISHER (2001): "Habit persistence, asset returns, and the business cycle," *American Economic Review*, 91, 149–166.

BURNSIDE, C. (1998): "Solving asset pricing models with Gaussian shocks," *Journal of Economic Dynamics and Control*, 22, 329–340.

CALIN, O. L., Y. CHEN, T. F. COSIMANO, AND A. A. HIMONAS (2005): "Solving Asset Pricing Models when the Price-Dividend Function Is Analytic," *Econometrica*, 73, 961–982.

CALVO, G. (1983): "Staggered Prices in a Utility-Maximizing Framework," *Journal of Monetary Economics*, 12, 383–98.

CAMPBELL, J. Y. AND J. H. COCHRANE (1999): "By force of habit: A consumption-based explanation of aggregate stock market behavior," *Journal of Political Economy*, 107, 205–251.

——— (2000): "Explaining the Poor Performance of Consumption-based Asset Pricing Models," *Journal of Finance*, 55, 2863–2878.

CAMPBELL, J. Y., A. W. LO, AND A. C. MACKINLAY (1997): *The Econometrics of Financial Markets*, Princeton University Press.

CAMPBELL, J. Y. AND J. R. SHILLER (1988): "The dividend-price ratio and expectations of future dividends and discount factors," *Review of Financial Studies*, 1, 195–227.

——— (1991): "Yield Spread and Interest Rate Movements: A Bird's Eye View," *The Review of Economic Studies*, 58, 495–514.

CANZONERI, M., R. CUMBY, AND B. DIBA (2007): "The Cost of Nominal Rigidity in NNS Models pp. R. J." *Journal of Money, Credit and Banking*, 39, 1563–1586.

CARROLL, C. (1992): "The Buffer-Stock Theory of Saving: Some Macroeconomic Evidence," *Brookings Papers on Economic Activity*, 2, 61–156.

——— (2000): "Solving Consumption Models with Multiplicative Habits," *Economics Letters*, 68, 67–77.

CARROLL, C., M. SOMMER, AND J. SLACALEK (2008): "International Evidence on Sticky Consumption Growth," John Hopkins University, mimeo.

CARROLL, C. D., J. OVERLAND, AND D. N. WEIL (2000): "Saving and Growth with Habit Formation," *American Economic Review*, 90, 341–355.

CARROLL, C. D. AND A. A. SAMWICK (1998): "How Important Is Precautionary Saving?" *The Review of Economics and Statistics*, 80, 410–419.

CHEN, X. AND S. LUDVIGSON (2009): "Land of Addicts? An Empirical Investigation of Habit-Based Asset Pricing Models," Journal of Applied Econometrics, Forthcoming.

CHRISTIANO, L., M. EICHENBAUM, AND C. EVANS (2005): "Nominal Rigidities and the Dynamic Effects of a Shock to Monetary Policy," *Journal of Political Economy*, 113, 1–45.

COCHRANE, J. AND M. PIAZESSI (2005): "Bond Risk Premium," *American Economic Review*, 95, 138–160.

COLLARD, F., P. FEVE, AND I. GHATTASSI (2006): "Predictability and habit persistence," *Journal of Economic Dynamics and Control*, 30, 2217–2260.

CONSTANTINIDES, G. M. (1990): "Habit formation: A resolution of the equity premium puzzle," *Journal of Political Economy*, 98, 519–543.

DE PAOLI, B. AND J. SONDERGAARD (2009): "A canonical general equilibrium model of the FX risk premium," Bank of England Working Paper, Forthcoming.

DE PAOLI, B., O. WEEKEN, AND A. SCOTT (2007): "Asset pricing implications of a New Keynesian model," Bank of England Working Paper nr. 326.

DE PAOLI, B. AND P. ZABCZYK (2009): "Why do Risk Premia Vary Over Time?" Bank of England WP 361.

DEN HAAN, W. (1995): "The term structure of interest rates in real and monetary economies," *Journal of Economic Dynamics and Control*, 19, 909–940.

DEN HAAN, W. AND J. DE WIND (2009): "How well-behaved are higher order perturbation solutions?" University of Amsterdam, mimeo.

DIEUDONNÉ, J. (1960): *Foundations of Modern Analysis*, Academic Press, International Edition.

DUESENBERRY, J. S. (1949): *Income, Saving and the Theory of Consumer Behavior*, vol. 87 of *Harvard Economic Studies*, Harvard University Press, Cambridge, MA.

DYNAN, K. E. (2000): "Habit Formation in Consumer Preferences: Evidence from Panel Data," *American Economic Review*, 90, 391–406.

EPSTEIN, L. G. AND S. E. ZIN (1989): "Substitution, Risk Aversion and the Temporal Behaviour of Consumption and Asset Returns: A Theoretical Framework," *Econometrica*, 57, 937 – 969.

——— (1991): "Substitution, Risk Aversion and the Temporal Behaviour of Consumption and Asset Returns: An Empirical Analysis," *Journal of Political Economy*, 99, 263 – 286.

FLODEN, M. (2008): "Aggregate Savings When Individual Income Varies," *Review of Economic Dynamics*, 11, 70–82.

FUHRER, J. C. (2000): "Habit Formation in Consumption and Its Implications for Monetary-Policy Models," *The American Economic Review*, 90, 367–390.

GALI, J. (1994): "Keeping Up with the Joneses: Consumption Externalities, Portfolio Choice and Asset Prices," *Journal of Money, Credit and Banking*, 26, 1–8.

GALLMEYER, M. F., B. HOLLIFIELD, F. PALOMINO, AND S. E. ZIN (2009): "Term Premium Dynamics and the Taylor Rule," SSRN eLibrary, Working Paper Series.

GARLEANU, N. AND S. PANAGEAS (2008): "Young, Old, Conservative and Bold: The Implications of Heterogeneity and Finite Lives for Asset Pricing," Mimeo. Chicago GSB.

GUVENEN, F. (2003): "A Parsimonious Macroeconomic Model for Asset Pricing: Habit Formation or Cross-sectional Heterogeneity?" Mimeo, University of Rochester.

HALL, P. (1927): "The Distribution of Means for Samples of Size N Drawn from a Population in which the Variate Takes Values Between 0 and 1, All Such Values Being Equally Probable," *Biometrika*, 19, 240245.

HARVEY, C. R. (1989): "Time-Varying Conditional Covariances in Tests of Asset Pricing Models," *Journal of Financial Economics*, 24, 289–317.

HEATON, J. (1995): "An Empirical Investigation of Asset Pricing with Temporally Dependent Preference Specifications," *Econometrica*, 63, 681–717.

HEIEN, D. AND C. DURHAM (1991): "A Test of the Habit Formation Hypothesis Using Household Data," *The Review of Economics and Statistics*, 73, 189–99.

HORDAHL, P., O. TRISTANI, AND D. VESTIN (2008): "The Yield Curve and Macroeconomic Dynamics," *The Economic Journal*, 118, 1937–1970.

HUGGETT, M. AND S. OSPINA (2001): "Aggregate precautionary savings: When is the third derivative irrelevant," *Journal of Monetary Economics*, 48, 373–396.

IRWIN, J. (1927): "On the Frequency Distribution of the Means of Samples from a Population Having any Law of Frequency with Finite Moments, with Special Reference to Pearson's Type II," *Biometrika*, 19, 225239.

JACCARD, I. (2007): "Asset Pricing, Habit Memory, and the Labor Market," Mimeo, The Wharton School.

JERMANN, U. (1998): "Asset pricing in production economies," *Journal of Monetary Economics*, 41, 257–275.

JUDD, K. L. (1998): *Numerical Methods in Economics*, MIT Press, Cambridge, MA.

JUILLARD, M. AND O. KAMENIK (2005): "Solving SDGE Models: Approximation About The Stochastic Steady State," Computing in Economics and Finance 2005 106, Society for Computational Economics.

JUILLARD, M., P. KARAM, D. LAXTON, AND P. PESENTI (2006): "Welfare-based monetary policy rules in an estimated DSGE model of the US economy," Working Paper Series 613, European Central Bank.

JULLIARD, C. AND A. GHOSH (2009): "Can Rare Events Explain the Equity Premium Puzzle?" Mimeo, London School of Economics.

KANDEL, S. AND R. F. STAMBAUGH (1989): "A Mean-Variance Framework for Tests of Asset Pricing Models," *Review of Financial Studies*, 2, 125–56.

——— (1990): "Expectations and Volatility of Consumption and Asset Returns," *Review of Financial Studies*, 3, 207–32.

——— (1991): "Asset returns and intertemporal preferences," *Journal of Monetary Economics*, 27, 39–71.

KAZAROSIAN, M. (1997): "Precautionary Savings - A Panel Study," *The Review of Economics and Statistics*, 79, 241–247.

KIMBALL, M. S. (1990): "Precautionary Saving in the Small and in the Large," *Econometrica*, 58, 53–73.

KOCHERLAKOTA, N. (1996): "The Equity Premium: It's Still a Puzzle," *Journal of Economic Literature*, 34, 42–71.

LABADIE, P. (1989): "Stochastic inflation and the equity premium," *Journal of Monetary Economics*, 24, 277–298.

LELAND, H. E. (1968): "Saving and Uncertainty: The Precautionary Demand for Saving," *The Quarterly Journal of Economics*, 82, 465–473.

LI, G. (2007): "Time-varying risk aversion and asset prices," *Journal of Banking & Finance*, 31, 243–257.

LI, Y. (2001): "Expected returns and habit persistence," *Review of Financial Studies*, 14, 861–899.

LJUNGQVIST, L. AND H. UHLIG (2000): "Tax Policy and Aggregate Demand Management under Catching Up with the Joneses," *Journal of Political Economy*, 90, 356–366.

LUCAS, R. E. (1978): "Asset Prices in an Exchange Economy," *Econometrica*, 46, 1429–1445.

LUDVIGSON, S. C. AND A. MICHAELIDES (2001): "Does Buffer-Stock Saving Explain the Smoothness and Excess Sensitivity of Consumption?" *American Economic Review*, 91, 631–647.

LUSTIG, H. AND A. VERDELHAN (2007): "The Cross-Section of Foreign Currency Risk Premia and Consumption Growth Risk," *American Economic Review*, 97, 89 – 117.

MARSHALL, A. (1898): *Principles of Economics*, MacMillan, New York, eighth ed.

MEHRA, R. AND E. PRESCOTT (1985): "The Equity Premium: A Puzzle," *Journal of Monetary Economics*, 15, 145–161.

——— (2004): "The Equity Premium in Retrospect," in *Handbook of the Economics of Finance, vol. 1B. Financial Markets and Asset Pricing*, ed. by G. Constantinides, M. Harris, and R. Stulz, Elsevier, North Holland, 887–936.

MUELLBAUER, J. N. (1988): "Habits, Rationality and Myopia in the Life Cycle Consumption Function," *Annales d'Economie et de Statistique*, 9, 47–70.

NASON, J. (1988): "The equity premium and time-varying risk behavior," Finance and Economics Discussion Series 11, Board of Governors of the Federal Reserve System (U.S.).

PIGOU, A. C. (1903): "Some remarks on utility," *Economic Journal*, 13, 58–68.

POLKOVNICHENKO, V. (2007): "Life-Cycle Portfolio Choice with Additive Habit Formation Preferences and Uninsurable Labor Income Risk," *Review of Financial Studies*, 20, 83–124.

RIETZ, T. A. (1988): "The Equity Risk Premium a Solution," *Journal of Monetary Economics*, 22, 117–131.

RUDEBUSCH, G. AND E. SWANSON (2008): "Examining the Bond Premium Puzzle with a DSGE Model," *Journal of Monetary Economics*, 55, S111–S126.

SANDMO, A. (1970): "The Effect of Uncertainty on Saving Decisions," *The Review of Economic Studies*, 37, 353–360.

SANTOS, M. S. (2000): "Accuracy of Numerical Solutions using the Euler Equation Residuals," *Econometrica*, 68, 1377–1402.

SCHMITT-GROHE, S. AND M. URIBE (2004): "Solving dynamic general equilibrium models using a second-order approximation to the policy function," *Journal of Economic Dynamics and Control*, 28, 755–775.

SMETS, F. AND R. WOUTERS (2003): "An Estimated Dynamic Stochastic General Equilibrium Model of the Euro Area," *Journal of the European Economic Association*, 1, 1123–1175.

——— (2007): "Shocks and Frictions in US Business Cycles: A Bayesian DSGE Approach," *American Economic Review*, 97, 586–606.

SMITH, A. (1776): "An Inquiry into the Nature and Causes of the Wealth of Nations : Volumes 1 and 2," in *Glasgow Edition of the Works and Correspondence of Adam Smith*, ed. by R. Campbell, A. S. Skinner, and W. Todd, Liberty Fund Inc., 1 – 1094.

STOKEY, N. L. AND LUCAS, R. E. WITH PRESCOTT, E. C. (1989): *Recursive Methods in Economic Dynamics*, Harvard University Press, Cambridge MA.

TALLARINI, T. D. AND H. H. ZHANG (2005): "External Habit and the Cyclicality of Expected Stock Returns," *Journal of Business*, 78, 1023–1048.

TAUCHEN, G. AND R. HUSSEY (1991): "Quadrature-Based Methods for Obtaining Approximate Solutions to the Integral Equations of Nonlinear Rational Expectations Models," *Econometrica*, 59, 371–396.

TSIONAS, E. (2005): "Likelihood Evidence on the Asset Returns Puzzle," *Review of Economic Studies*, 72, 917–946.

TSIONAS, E. G. (2003): "Exact solution of asset pricing models with arbitrary shock distributions," *Journal of Economic Dynamics and Control*, 27, 843–851.

UHLIG, H. (2004): "Macroeconomics and Asset Markets: Some Mutual Implications," Humboldt University, mimeo.

——— (2007): "Explaining Asset Prices with External Habits and Wage Rigidities in a DSGE Model," *American Economic Review*, 97, 239–243.

VERDELHAN, A. (2009): "A Habit-Based Explanation of the Exchange Rate Risk Premium," Journal of Finance, forthcoming.

WACHTER, J. A. (2006): "A consumption-based model of the term structure," *Journal of Financial Economics*, 79, 365–399.

WOODFORD, M. (2003): *Interest and Prices: Foundations of a Theory of Monetary Policy*, Princeton University Press.