Spectral Measures and Walk-Generating Functions for Certain Classes of Groups

Spectral Properties of graphs derived from groups

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January 1995

Dissertation for the Degree of
Doctor of Philosophy
University of London

Abstract

This thesis is primarily about spectral measures and walk-generating functions of lattices. Formally a lattice is obtained from a finitely-generated abelian group G, a finite set Υ, and a finite subset L of $G \times \Upsilon \times \Upsilon$, by constructing a graph with vertex set $G \times \Upsilon$, and joining (g_1, v_1) and (g_2, v_2) if $(g_2 - g_1, v_1, v_2) \in L$, though of course we need L to obey extra conditions if we want this graph to be undirected without loops. Informally a lattice is likely to be some structure in n-dimensional space such as a hexagonal or cubic lattice. Spectral measures and walk-generating functions determine each other, and are relevant to Markov Chains and networks of resistances.

Formulae for spectral measures and walk generating functions of lattices are found, and generalised to sum-difference graphs and graphs obtained from groups with large abelian subgroups.

Formulae are also found for walk generating functions for modified lattices. Lattices may be modified by a finite set of changes to edges or vertices, but also by an infinite but periodic set of modifications (such as a row of points being removed). For example, this makes it possible to find exact formulae for Markov Chains where two interacting particles move around a lattice. However only one infinite periodic set of modifications can be so handled; we show that with directed lattices with two infinite periodic sets of modifications, even finding if two points are connected can be equivalent to the Halting Problem.

New methods are found for discovering what a spectral measure looks like. We develop techniques in the theory of complex functions of several variables to provide criteria which make it possible to show that a spectral measure is well-behaved at some point (in the sense that its density function is analytic) if local properties of certain analytic functions are satisfied globally.

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Acknowledgements

I wish to express my gratitude to the following people: Professor Norman Biggs (for supervising me); Sarah Goodall (for helping me to print out my thesis); Dr Jonathan Partington of Leeds University and Dr Gregory Sankaran of St. Catharine's College, Cambridge (for freely giving me helpful advice). The research in this thesis was partially funded by a grant from the Science and Engineering Research Council.

Statement of Originality

The concept of a lattice in Chapter 1 has been around for a long time, but the formal definition is my own. Chapter 2 is standard, except possibly Theorem 2.1:7, which I have never seen stated or proved, though it seems likely enough that it has been done before. Chapter 3 is entirely standard except for the applications of Theorem 2.1:7 and the definitions in Definition 3.1:1. All of Chapter 4 is standard. Section 5.1 is not original and may be found in [MOHAR& WOESS,1989] and [GODSIL&MOHAR,1988].

A finite set of modifications to a Markov Chain based on a lattice was dealt with in [MONTROLL,1969]. Apart from this Subsection 5.1.4 is original. Section 5.2 is already known in that similar results may be found in [MOHAR& WOESS,1989].

There are two different approaches given in Chapter 6 for finding the spectral measures of lattices. The first, given in sections Section 6.1 to Section 6.4, is based on finding the spectral measures of finite lattices, and then using a notion of convergence of lattices to find the spectral measures of infinite lattices. Hints of this technique occur in various places in the literature. In [MONTROLL, 1956], a similar technique is used for solving random walks on certain lattices. In [GODSIL&MOHAR,1988], the technique is applied to the special case of a 1-dimensional hexagonal chain. However, I know of nowhere where the method for general lattices has been completely explained, still less justified. Furthermore, I have given a definition of convergence which is useful for weighted lattices; the existing definitions of convergence that I have seen only make sense for graphs with bounded degrees.

The second approach I have given, in Section 6.6, is original. It has the advantage of providing the walk-generating functions for directed weighted lattices, where the spectral measures may not exist at all.

In Section 6.7, I give some applications of these methods which I have not heard of before. It is original that the methods used for lattices can be applied to the Dihedral Groups and Sum-Difference Lattices.

Chapter 7 is completely original.

Chapter 8 is original. However, the functions ϕ involved in simple local threadings were inspired by similar functions used in [MILNOR,1963], Chapter 1, Theorem 3.1, to compare portions of manifolds. Here they are used in a different way for different purposes. Also the limiting argument used to show

Lemma 8.9.2:9 is similar to one given in [RELLICH,1969] to show that eigenvalues of a hermitian matrix function in a single real parameter can be written as power series in that parameter.

Chapter 1

Lattices

Let G be a finitely generated abelian group and Υ a non-empty finite set. A *Lattice* on (G, Υ) is a triple (G, Υ, L) where L is a finite subset of $G \times \Upsilon \times \Upsilon$ such that:

(1) $\forall g \in G, \forall \upsilon_1, \upsilon_2 \in \Upsilon\, (g, \upsilon_1, \upsilon_2) \in L \iff (-g, \upsilon_2, \upsilon_1) \in L$.

(2) $\forall \upsilon \in \Upsilon\, (0, \upsilon, \upsilon) \notin L$.

In fact I shall usually be lazy, and write the lattice as L, without distinguishing the subset of $G \times \Upsilon \times \Upsilon$ and the triple (G, Υ, L).

We associate each lattice with a graph. The vertex set of the graph will be $G \times \Upsilon$. Vertex (g_1, υ_1) will be adjacent to (g_2, υ_2) if and only $(g_2 - g_1, \upsilon_1, \upsilon_2) \in L$. (1) ensures that this edge condition is symmetric, and from (2) we can see that the graph has no loops; hence it is simple. Furthermore, each vertex has degree at most $|L|$, so the graph has bounded vertex degrees.

For example, suppose Υ to be a singleton set $\{\upsilon\}$, and G to be the group of addition on $\mathbb{Z}^d$, for some positive integer d. We may as well define $L' = \{x \mid (x, \upsilon, \upsilon) \in L\}$. Then the associated graph has one vertex $v_x = (x, \upsilon, \upsilon)$ for each $x \in \mathbb{Z}^d$, and v_x is joined to v_y if and only if $y - x \in L'$. This means that the graph is a Cayley graph on the group $\mathbb{Z}^d$ with edges given by L'. If $d = 2$ and $L' = \{(0,1), (0,-1), (1,0), (-1,0)\}$ then we obtain the familiar two-dimensional grid, as shown in Figure 1. In this and Figure 2 the dashed box will indicate the vertices corresponding to a single group element; hence if the segment inside the dashed box is repeated indefinitely in all directions the complete lattice is obtained.

Now we consider a more complicated example. Take $\Upsilon = \{\upsilon_1, \upsilon_2\}$ and $G = \mathbb{Z}^2$. Let $L = \{((0,0), \upsilon_1, \upsilon_2), ((0,1), \upsilon_1, \upsilon_2), ((1,0), \upsilon_1, \upsilon_2), ((0,0), \upsilon_2, \upsilon_1), ((0,-1), \upsilon_2, \upsilon_1), ((-1,0), \upsilon_2, \upsilon_1)\}$. In this case we obtain the hexagonal lattice, as shown in Figure 2.

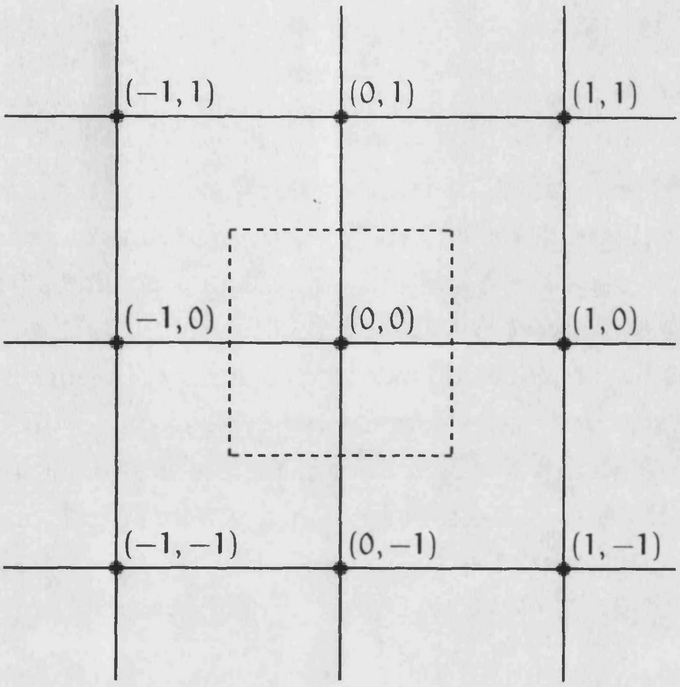

Figure 1. Part of the Two-Dimensional Grid

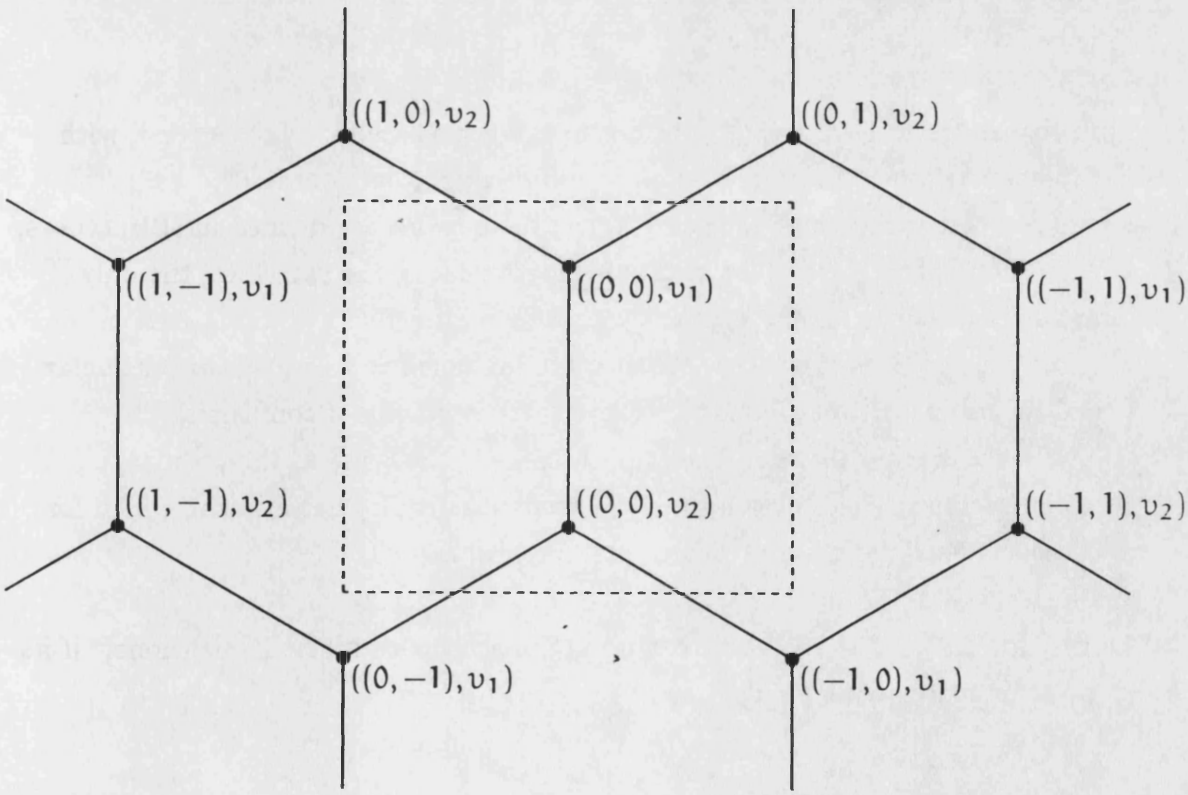

Figure 2. Part of the Hexagonal Lattice

Chapter 2

Banach Spaces

This is a well-studied area, and it is unnecessary for me to try to provide a full introduction to it. Instead, I shall sketch what seem to me to be the important points. This can at most make it clear how my approach differs from others, or remind the reader of what he has already studied.

For a general introduction to functional analysis, the reader is referred to [BOLLOBÁS,1990]. However, this does not cover, for example, the functional calculus; for that and other advanced topics, see [DUNFORD&SCHWARZ,1963].

Let Λ be a set. A Λ-sequence is a function y from Λ to $\mathbb{C}$, mapping λ to y_λ. Let x be such a Λ-sequence. Let $1 \le p \le \infty$.

Definition 2:1. *Define* $\|x\|_p$ *to be*

$$\begin{cases} \left(\sum_{\lambda\in\Lambda} |x_\lambda|^p\right)^{1/p}, & \textit{for } p < \infty; \\ \sup\limits_{\lambda\in\Lambda} |x_\lambda|, & \textit{for } p = \infty. \end{cases}$$

Clearly $0 \le \|x\|_p \le \infty$. The sum in the definition is always defined (though it may be ∞) because $|x_\lambda|$ will always be non-negative.

Definition 2:2. *Define* $\ell_p(\Lambda)$ *to be the* Banach Space $\{x \mid \|x\|_p < \infty\}$, *with pointwise multiplication by scalars and addition, and* Norm $\| \; \|_p$.

I shall not define *Banach Space* and *Norm*; these terms are defined in [BOLLOBÁS, 1990]. Though there are other kinds of Banach spaces, the $\ell_p(\Lambda)$ are the only ones I shall consider for the moment.

For general Banach spaces, I shall write the norm of x as $\|x\|$; in particular, I shall do this sometimes for $\ell_p(\Lambda)$ where there is no risk of confusion.

When Λ has finite order, the Banach space is the same as the ordinary complex vector space of dimension $|\Lambda|$, except that it also has a norm, which for $p = 2$ is the Euclidean norm used to measure distances.

Definition 2:3. *I shall call a (complex) Banach Space* Finite-Dimensional *if its vector space is isomorphic to* $\mathbb{C}^n$ *for some finite* n.

Definition 2:4. *Suppose* $\{z_\theta \mid \theta \in \Theta\}$ *to be a set of elements of a normed vector space (for example, a Banach Space), indexed by* Θ. *Then the* Sum $\sum_{\theta\in\Theta} z_\theta$ *is defined to equal* z *if for any* $\epsilon > 0$, *there is a finite set* $\Phi \subseteq \Theta$ *such that for all finite* Ψ *with* $\Phi \subseteq \Psi \subseteq \Theta$, $\|z - \sum_{\psi\in\Psi} z_\psi\| < \epsilon$

It will be seen that for finite sets Θ, this definition corresponds with that of finite sums, and for $\Theta = \mathbb{N}$, this definition is consistent with the conventional one, but only is defined when there is absolute convergence.

Definition 2:5. *For* $\lambda \in \Lambda$, e^λ *is the function on* Λ *mapping* λ *to* 1 *and all other elements to* 0. *This can be considered as an element of the Banach space* $\ell_p(\Lambda)$; *it has norm* 1. *For* $p < \infty$, *the* Canonical Basis *of* $\ell_p(\Lambda)$ *is the set* $\{e^\lambda \mid \lambda \in \Lambda\}$.

For $p < \infty$, any element x of $\ell_p(\Lambda)$ can be written uniquely as an infinite sum $\sum_\Lambda y_\lambda e^\lambda$, for y a function from Λ to $\mathbb{C}$. Namely, with $y = x$.

§**2.1.** Operators on Banach Spaces

Let X be a Banach Space.

Definition 2.1:1. *Let* $\mathcal{L}(X)$ *be the set of partial linear maps from* X *to* X; *and* $\mathcal{B}(X)$ *the set of total continuous linear maps from* X *to* X. *The elements of* $\mathcal{B}(X)$ *are called* Operators

Clearly $\mathcal{B}(X) \subseteq \mathcal{L}(X)$. If X is finite-dimensional, all total maps in $\mathcal{L}(X)$ are in $\mathcal{B}(X)$, since linear maps on finite dimensional spaces are automatically continuous.

Definition 2.1:2. *The* Identity Operator, I, *is defined to map every element of* X *to itself.*

Clearly I is continuous.

We can compose operators as we compose functions, so that if T_1 and T_2 are two operators, then T_1T_2 is defined as the operator mapping x to $T_1(T_2(x))$. For example, for any operator T, we have $TI = IT = T$.

Definition 2.1:3. *For* $T \in \mathcal{L}(X)$*, the* Operator Norm *of* T, $\|T\|$, *is defined as* $\inf\{M \mid \forall x\, M\|x\| \geq \|Tx\| \text{ where } Tx \text{ is defined}\}$.

For example, we see that $\|I\| = 1$, provided that X contains some non-zero elements.

Definition 2.1:4. *For* $T \in \mathcal{L}(X)$ *we say that* T *is* Bounded *if* $\|T\|$ *is finite.*

The following is proved in [BOLLOBÁS,1990] (Chapter 2, Theorem 2), though I have made some trivial changes.

Lemma 2.1:5. *Let* $T \in \mathcal{L}(X)$ *be defined on all of* X*. Then the following are equivalent:*

(1) $T \in \mathcal{B}(X)$ *(that is to say, it is continuous);*

(2) T *is continuous at some point* $x_0 \in X$*;*

(3) T *is bounded.*

It is also easy to see the following:

Lemma 2.1:6. *Let* $T \in \mathcal{L}(X)$ *be bounded, with norm* $\|T\|$*, and defined on a dense subset of* X*. Then* T *can be extended uniquely to* $\tilde{T} \in \mathcal{B}(X)$*, with* $\|\tilde{T}\| = \|T\|$

I shall not bother to use the notation $\tilde{T}$; in future I shall just write T.

For $X = \ell_p(\Lambda)$, and $p < \infty$, a particularly useful dense subset is the set of finite linear combinations of elements of the canonical basis. For the rest of this section, let F be this set.

I shall now use this to show that a particular class of linear functions is in $\mathcal{B}(\ell_p(\Lambda))$.

Theorem 2.1:7. *Let* $a_{\lambda_1\lambda_2}$ *be a function from* $\Lambda \times \Lambda$ *to* $\mathbb{C}$ *satisfying*

(1) If $p < \infty$ *then for all* λ_1, $\sum_{\lambda_2} |a_{\lambda_1\lambda_2}| \leq M$.

(2) If $p > 1$ *then for all* λ_2, $\sum_{\lambda_1} |a_{\lambda_1\lambda_2}| \leq M$.

where M *a finite constant.*

Then there is an operator $Z \in \mathcal{B}(\ell_p(\Lambda))$ *such that for all* λ_1,

$$Ze^{\lambda_1} = \sum_{\lambda_2} a_{\lambda_1\lambda_2} e^{\lambda_2}; \qquad (1)$$

this satisfies $\|Z\| \le M$. Z *is unique for* $p < \infty$.

Proof. First we prove the result for when $p = \infty$, since this is entirely separate from the rest of the proof.

Lemma 2.1:8. Theorem 2.1:7 *is true for* $p = \infty$

Proof. Let $x \in \ell_\infty(\Lambda)$. I claim that the following definition of $(Zx)_{\lambda_2}$ defines Z satisfactorily:

$$(Zx)_{\lambda_2} \stackrel{\text{def}}{=} \sum_{\lambda_1} x_{\lambda_1} a_{\lambda_1\lambda_2} \tag{2}$$

It is trivial that this satisfies Display (1). I need to show that $\|Zx\| \le M\|x\|$ according to this definition. From the definition of $\| \|_\infty$, we see that:

$$\begin{aligned} \|Zx\| &= \sup_{\lambda_2} \left| \sum_{\lambda_1} x_{\lambda_1} a_{\lambda_1\lambda_2} \right| \\ &\le \sup_{\lambda_2} \sum_{\lambda_1} \|x\| \, |a_{\lambda_1\lambda_2}| \\ &= \|x\| \sup_{\lambda_2} \sum_{\lambda_1} |a_{\lambda_1\lambda_2}| \\ &\le \|x\| M. \end{aligned}$$

which proves the lemma. These inequalities also prove absolute convergence of the sum in Display (2).

Unlike the case where $p < \infty$, it is impossible to ensure uniqueness. For example, consider $\ell_\infty(\mathbb{N})$. Let $Z_1, Z_2 \in \mathcal{B}(\ell_\infty(\mathbb{N}))$ be defined so that Z_1 is the function mapping everything to 0, and $Z_2(x)$ is $L(x)e$, where L is the *Banach Limit*, as defined in [Bollobás,1990] (Chapter 3, Exercise 19), and e_λ is always 1. Like Z_1, Z_2 maps every e^λ to 0, yet $Z_2 \ne Z_1$; for example $Z_2(e) = e$.

It remains to deal with $p < \infty$. It is easy to see that for any $\lambda_1 \in \Lambda$, the sum on the right hand side of Display (1) converges, and in fact Zx can be defined as a linear operator for all $x \in F$ with

$$Zx = \sum_{\lambda_1} \sum_{\lambda_2} x_{\lambda_1} a_{\lambda_1\lambda_2} e^{\lambda_2}. \tag{3}$$

So Z can at least be defined as an element of $\mathcal{L}(\ell_p(\Lambda))$, with domain a dense subset of $\ell_p(\Lambda)$. By Lemma 2.1:6, we see that it suffices to show that this Z has norm at most M. Thus the theorem follows from

Lemma 2.1:9. *Under the conditions of* Theorem 2.1:7, *for* $x \in F$ *and* Zx *defined by* Display (3), $\|Zx\| \leq M\|x\|$.

My strategy will be, first to show Lemma 2.1:9 for $p = 1$, then to deduce it for all $p \in (1, \infty)$.

Lemma 2.1:10. Lemma 2.1:9 *is true for* $p = 1$.

Proof. Let $x \in F$; we need to show that $\|Zx\| \leq M\|x\|$. However from Display (3), and the definition of $\| \; \|_p$, we see that

$$\begin{aligned} \|Zx\| &= \Big\| \sum_{\lambda_1} \sum_{\lambda_2} x_{\lambda_1} a_{\lambda_1 \lambda_2} e^{\lambda_2} \Big\| \\ &= \sum_{\lambda_2} \left| \sum_{\lambda_1} x_{\lambda_1} a_{\lambda_1 \lambda_2} \right| \\ &\leq \sum_{\lambda_1} |x_{\lambda_1}| \sum_{\lambda_2} |a_{\lambda_1 \lambda_2}| \\ &\leq \left(\sum_{\lambda_1} |x_{\lambda_1}| \right) \times M \\ &= M\|x\| \end{aligned}$$

which completes the proof of the lemma.

Now we show the theorem for $p \in (1, \infty)$. To begin with, I shall adapt a standard inequality.

Lemma 2.1:11. *For* $n \geq 1$, *let* $\mu_1, \ldots, \mu_n$ *be positive with* $\sum \mu_i = 1$. *Let* $y_1, \ldots, y_n$ *be non-negative. Let* r *be non-zero. Then*

$$K_r(y) \stackrel{\text{def}}{=} \left(\sum_{i=1}^{n} \mu_i {y_i}^r \right)^{1/r}$$

is monotonic and increasing with r, *that is,* $r_1 \leq r_2$ *implies* $K_{r_1}(y) \leq K_{r_2}(y)$.

Proof. This lemma is part of Theorem 5 in Chapter 1 of [Bollobás,1990] (though I have changed the notation).

Lemma 2.1:12. *Let μ and y be Λ-sequences, with μ_λ non-negative such that $\sum_\lambda \mu_\lambda \le 1$, and y_λ non-negative. Then*

$$K_r(y) \stackrel{\text{def}}{=} \left(\sum_{\lambda\in\Lambda} \mu_\lambda {y_\lambda}^r\right)^{1/r}$$

is monotonic and increasing with r, as before.

Proof. We relax the conditions of Lemma 2.1:11 one by one to obtain the result.

(1) It is easy to see that it doesn't matter if for some λ, $\mu_\lambda = 0$.

(2) Let $\mu = \sum \mu_\lambda$. We have $0 \le \mu \le 1$, and in fact the lemma is trivial if $\mu = 0$. Otherwise let ${\mu_i}' = \frac{\mu_i}{\mu}$, and let ${K_r}'(y)$ be the K_r we obtain using μ' instead of μ. In fact ${K_r}'(y) = \frac{K_r(y)}{\mu^{1/r}}$. Lemma 2.1:11 applied to ${K_r}'$, together with $\mu \in (0,1]$, ensures that $K_r(y)$ is still monotonic and increasing with r.

(3) It remains to show that we can change the sum from $\sum_{i=1}^{n}$ to $\sum_{\lambda\in\Lambda}$. Since everything is non-negative, Definition 2:4 is equivalent to defining the sum over Λ as the supremum of the sum over finite subsets of Λ. Such a finite subset will correspond to $\{1,\ldots,n\}$, where n is its order; and with (2), we know that the sum for any particular finite subset will monotonically increase. As the sum over Λ is a supremum of these sums over finite subsets, this proves the lemma.

We now proceed to complete the proof of the theorem. As in Lemma 2.1:10, let $x \in F$; it may be assumed that $x \ne 0$. I have to show that $\|Zx\|_p \le M\|x\|_p$. By expanding this inequality, we find that it may be assumed that $M = 1$ (by dividing $a_{\lambda_1\lambda_2}$ by M), that $\|x\|_p = 1$ (by replacing x by $\frac{x}{\|x\|}$), and that for any λ_1 λ_1 and λ_2, x_{λ_1} and $a_{\lambda_1\lambda_2}$ are real and non-negative (by replacing each by their absolute values). Define the Λ-sequence y to have $y_\lambda \stackrel{\text{def}}{=} {x_\lambda}^p$, then $\|y\|_1 = 1$. It remains to show that $\|Zx\|_p \le 1$; that is to say:

$$\begin{aligned} 1 &\ge \|\sum_{\lambda_1}\sum_{\lambda_2} x_{\lambda_1} a_{\lambda_1\lambda_2} e^{\lambda_2}\| \\ &= \|\sum_{\lambda_2} e^{\lambda_2} \sum_{\lambda_1} x_{\lambda_1} a_{\lambda_1\lambda_2}\| \\ &= \left(\sum_{\lambda_2}\left|\sum_{\lambda_1} x_{\lambda_1} a_{\lambda_1\lambda_2}\right|^p\right)^{1/p} \end{aligned}$$

which is equivalent to:

$$
\begin{aligned}
1 &\geq \sum_{\lambda_2} \left| \sum_{\lambda_1} x_{\lambda_1} a_{\lambda_1 \lambda_2} \right|^p \\
&= \sum_{\lambda_2} \left(\sum_{\lambda_1} x_{\lambda_1} a_{\lambda_1 \lambda_2} \right)^p \\
&= \sum_{\lambda_2} \left(\sum_{\lambda_1} a_{\lambda_1 \lambda_2} y_{\lambda_1}^{1/p} \right)^p .
\end{aligned}
$$

By Lemma 2.1:12, comparing $r = 1$ and $r = 1/p$, this is

$$
\leq \sum_{\lambda_2} \left(\sum_{\lambda_1} a_{\lambda_1 \lambda_2} y_{\lambda_1} \right) \quad \text{since } \forall \lambda_2 \textstyle\sum_{\lambda_1} a_{\lambda_1 \lambda_2} \leq M = 1.
$$

The result, and hence the whole theorem, now follows from the case $p = 1$, where we bound Zy rather than Zx. □

Both condition (1) & condition (2) are required in the statement of this theorem. Let $\Lambda = \mathbb{N}$ and $M = 1$, then without (1), $a_{\lambda_1 \lambda_2} = \delta_{\lambda_1 1}$ would be a counter-example with $x = e^1$ for any $p < \infty$; and without (2), $a_{\lambda_1 \lambda_2} = \delta_{\lambda_2 1}$ would be a counter-example with $x = \sum_n \frac{1}{n} e^n$ for any $p > 1$. I leave the details of this to the reader.

Chapter 3

Hilbert Space on the Lattice

For any set V, define H(V) to be the Hilbert Space of all functions $x : v \mapsto x_v$ from V to $\mathbb{C}$ satisfying $\sum_v |x_v|^2 < \infty$, with inner product defined to be $\langle x, y \rangle \stackrel{\text{def}}{=} \sum_v x_v \overline{y_v}$. For any $v \in V$, define $e^v \in H(V)$ to map v to 1 and everything else to 0. Let H be any Hilbert Space. For a general introduction to Hilbert Spaces see [BOLLOBÁS,1990].

Definition 3:1.
(1) *For* $x \in H$, *the* Norm *of* x, $\|x\|$, *is* $\sqrt{\langle x, x \rangle}$.
(2) *For* T *an operator on* H, *the* Norm *of* T, $\|T\|$, *is* $\sup\limits_{x \in H} \frac{\|Tx\|}{\|x\|}$.
We let I denote the operator mapping x to itself.

It will be seen that H(V) contains the same elements as $\ell_2(V)$ and has the same norm for elements and operators.

Let K be a function from $V \times V$ to $\mathbb{C}$.

Definition 3:2.
(1) $M_1(K) \stackrel{\text{def}}{=} \sup_{u \in V} \sum_v |K(u,v)|$.
(2) $M_2(K) \stackrel{\text{def}}{=} \sup_{v \in V} \sum_u |K(u,v)|$.
(3) $M(K) \stackrel{\text{def}}{=} \max(M_1(K), M_2(K))$.

Theorem 3:3. *Suppose* $M(K) < \infty$. *Then there is a unique bounded operator* T *on* H(V) *such that for all* u *and* v *in* V, $\langle Te^u, e^v \rangle = K(u,v)$. *Furthermore* $\|T\|$ *is at most* M(K).

Proof. This follows from Theorem 2.1:7.

Corollary 3:4. T *as constructed in the previous theorem is Hermitian if and only if for all* (u,v), $K(v,u) = \overline{K(u,v)}$.

Proof. *only if* is true as $K(u,v) = \langle Te^u, e^v \rangle$. *if* is true as $\langle T^* e^u, e^v \rangle = \overline{\langle e^v, T^* e^u \rangle} = \overline{\langle Te^v, e^u \rangle} = \overline{K(v,u)} = K(u,v)$; thus $T = T^*$ by the uniqueness of T.

We shall also need the following theorem.

Theorem 3:5. *If* $\|T\| < 1$ *then there is an operator* T' *with* $T'(I - T) = (I - T)T' = I$, *and* $\|T'\| \leq (1 - \|T\|)^{-1}$.
This is essentially [BOLLOBÁS,1990], Chapter 12, Theorem 1, in that the existence of T' is in the statement of the theorem and the bound on $\|T'\|$ follows from the proof. □

We now consider Hilbert Spaces on lattices. Specifically, $H(L)$ is defined to be $H(V)$ where V is the set of vertices of the associated graph; namely $G \times \Upsilon$.

Definition 3:6. *For a lattice* L, *the* Adjacency Operator A *is the operator on* $H(L)$ *defined so that* $\langle Ae^u, e^v \rangle$ *is* 1 *if* u *and* v *are adjacent in the graph, and* 0 *otherwise.*
Because the graph has bounded vertex degrees, by Theorem 3:3 A is a well-defined operator with norm bounded above by the maximum degree of the graph. Furthermore, A is Hermitian by Corollary 3:4.

§3.1. Weighted Lattices and Directed Weighted Lattices

Definition 3.1:1.

(1) *For* G *and* Υ *as in* Chapter 1, *a* Directed Lattice Weighting *on* $G(\Upsilon)$ *is a function* f *mapping* $G \times \Upsilon \times \Upsilon$ *to* $\mathbb{C}$, *such that* $\sum_{g,\upsilon_1,\upsilon_2} |f(g,\upsilon_1,\upsilon_2)|$ *is finite. A* Lattice Weighting *on* (G,Υ) *is a directed lattice weighting which also satisfies* $\forall g \in G, \forall \upsilon_1,\upsilon_2 \in \Upsilon\, f(g,\upsilon_1,\upsilon_2) = \overline{f(-g,\upsilon_2,\upsilon_1)}$.

(2) *A* Weighted Lattice *(respectively* Directed Weighted Lattice*) is a triple* (G,Υ,f), *where* f *is a lattice weighting (directed lattice weighting) on* (G,Υ). *The associated Hilbert Space* $H(G,\Upsilon;f)$ *is* $H(G \times \Upsilon)$, *as for lattices.*

(3) *The* Weighted Adjacency Function *of a weighted lattice or a directed weighted lattice* (G,Υ,f) *is the function* B *mapping* $(u = (g_1,\upsilon_1), v = (g_2,\upsilon_2))$ *to* $B_{v\leftarrow u} \stackrel{\text{def}}{=} f(g_2 - g_1,\upsilon_1,\upsilon_2)$. *The* Weighted Adjacency Operator, *also called* B, *is the operator on the associated Hilbert Space defined by* $\langle Be^u, e^v \rangle \stackrel{\text{def}}{=} B_{v\leftarrow u}$.
The weighted adjacency operator B *is well-defined by* Theorem 3:3, *because* $M(B) < \infty$. *Also, by* Corollary 3:4, *the weighted adjacency operator of a weighted lattice is Hermitian.*

We can identify a lattice with a weighted lattice, with the same adjacency operator. For if we construct A from L as we did originally, we obtain the same operator as if we had defined f to map everything in L to 1 and everything else to 0, and then constructed B from f.

Thus weighted lattices and directed weighted lattices generalise lattices. We need this generalisation for Chapter 7, which obtains results about lattices (and in fact directed weighted lattices) with periodic modifications by constructing directed weighted lattices from them.

Chapter 4

Spectral Measures

Throughout this chapter, let T be a Normal operator in a Hilbert Space H. Let Spec(T) denote the *Spectrum* of T, that is to say, the set of $\lambda \in \mathbb{C}$ such that there is no bounded operator on the Hilbert Space inverse to $\lambda I - T$. For finite-dimensional Hilbert spaces, the spectrum is the set of eigenvalues; however for infinite-dimensional Hilbert spaces the situation is considerably more complicated. Thus, if λ is in the spectrum of T, it is not necessarily true that there is a non-zero x in H such that $Tx = \lambda x$. For a discussion of the various possibilities, see [BOLLOBÁS,1990], Chapter 12.

Theorem 4:1.
(1) $\sup\{\lambda \in \text{Spec}(T)\} = \|T\|$.
(2) *If* T *is Hermitian,* $\text{Spec}(T) \subset \mathbb{R}$.

(1) is proved in [BOLLOBÁS,1990], Chapter 12, Theorem 11. (2) is a standard result. To prove it we may proceed as follows. Spec(T) is bounded; let λ be on its boundary. Then by [BOLLOBÁS,1990], Chapter 12, Theorem 7, λ is an *Approximate Eigenvalue* of T. That is to say, there is a sequence $(x^n)_{n=1}^{\infty}$ with $x^n \in H(V)$ and $\|x^n\| = 1$ for all n, such that $(\lambda I - T)x^n \to 0$ as $n \to \infty$. Since $\langle x^n, x^n \rangle = 1$, we have $\lambda - \langle Tx^n, x^n \rangle \to 0$. However $\langle Tx^n, x^n \rangle$ is real because $\overline{\langle Tx^n, x^n \rangle} = \langle x^n, Tx^n \rangle = \langle Tx^n, x^n \rangle$. Thus λ is real, and so the whole boundary of Spec(B) is real; this concludes the proof.

Definition 4:2. *Let* f *be a function mapping any set to* $\mathbb{C}$. *Then*
$$\|f\|_\infty \stackrel{\text{def}}{=} \sup_x |f(x)|.$$
This extends Definition 2:1.

Definition 4:3. $L_\infty(\text{Spec}(T))$ *is the set of measurable functions* f *from* Spec(T) *to* $\mathbb{C}$ *with* $\|f\|_\infty < \infty$.

We regard $L_\infty(\text{Spec}(T))$ as an algebra over the complex numbers by defining $(f+g)(\lambda) \stackrel{\text{def}}{=} f(\lambda) + g(\lambda)$, $(xf)(\lambda) \stackrel{\text{def}}{=} x \times f(\lambda)$, and $(fg)(\lambda) \stackrel{\text{def}}{=} f(\lambda) \times g(\lambda)$; for all f and g in $L_\infty(\text{Spec}(T))$, all $\lambda \in \text{Spec}(T)$, and all complex x.

We also regard the set of bounded operators on H as an algebra, by defining $(V_1 + V_2)(y) \stackrel{\text{def}}{=} V_1(y) + V_2(y)$, $(xV_1)(y) \stackrel{\text{def}}{=} x \times V_1(y)$, and $(V_1V_2)(y) \stackrel{\text{def}}{=} (V_1 \circ V_2)(y)$; for all bounded operators V_1 and V_2, all complex x, and all $y \in H$.

Theorem 4:4. *There is a unique function Φ_T from $L_\infty(\text{Spec}(T))$ into the set of operators on H satisfying the following conditions:*

(1) *Φ_T is an algebra homomorphism.*
(2) *$\|\Phi_T(f)\| \leq \|f\|_\infty$, for all f.*
(3) *$\Phi_T([1]) = I$, where $[1]$ is the constant function mapping everything to 1.*
(4) *$\Phi_T([\lambda]) = T$, where $[\lambda]$ is the function mapping each $\lambda \in \text{Spec}(T)$ to itself.*

Furthermore, the following are also true:

(5) *For any elements x and y of H, there is a unique bounded complex measure $\mu_{x,y}$ on the Borel sets in $\text{Spec}(T)$ such that for any f, $\langle \Phi_T(f)x, y\rangle = \int f \, d\mu_{x,y}$.*
(6) *If f is continuous, $\|\Phi_T(f)\| = \|f\|_\infty$.*

Definition 4:5. *The Φ_T satisfying the conditions in* Theorem 4:4 *will be called the* Functional Calculus *on T.*

These are all standard results, and may be found in [PLESNER,1969] in Section 8.6 and Section 8.7 together with Theorem 9.5.3.

Definition 4:6. *If f and T are as above, then we write $f(T)$ for $\Phi_T(f)$.*

Definition 4:7. *The* Spectral Measures *of T are the measures $\mu_{x,y}$ given in* (5).

Definition 4:8. *Suppose $H = H(V)$. For $u, v \in V$, define $\mu_{u,v}$ to be μ_{e^u,e^v}.*

Lemma 4:9. *If* $H = H(V)$ *then*

$$\mu_{x,y} = \sum_{u,v} x_u \overline{y_v} \mu_{u,v}.$$

Proof. Let f be any function in $L_\infty(\mathrm{Spec}(T))$. It suffices to show that $\int f\, d\mu_{x,y} = \sum_{u,v} x_u \overline{y_v} \int f\, \mu_{u,v}$. Indeed, the left hand side equals $\langle f(T)x, y\rangle$ while the right hand side equals $\sum_{u,v} x_u \overline{y_v} \langle f(T)e^u, e^v\rangle$, which converges since $|\langle f(T)e^u, e^v\rangle| \leq \|f\|_\infty \|e^u\| \|e^v\| = \|f\|_\infty$, so $\sum_{u,v} |x_u \overline{y_v} \langle f(T)e^u, e^v\rangle| \leq \|f\|_\infty \sum_{u,v} |x_u \overline{y_v}|$ which is bounded by the Cauchy-Schwartz inequality. The result then follows from $x = \sum_u x_u e^u$ and $y = \sum_v y_v e^v$.

§**4.1.** Spectral Measures on Finite Dimensional Hilbert Spaces

Consider what happens if H is finite dimensional, with dimension n. H is a vector space. For any basis of H, we can write the elements of H as vectors and the operators on H as matrices. If the basis is orthonormal, and if two elements x and y of H are represented with this basis by vectors containing only real entries, then in fact $\langle x, y\rangle$ is just the conventional dot product of the vectors.

Since T is normal we can choose an orthonormal basis $\{e'^1, \ldots, e'^n\}$ over which it diagonalises. So we can write

$$T = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}.$$

where the off-diagonal elements are all zero. Since H is finite-dimensional, $\mathrm{Spec}(T)$ is equal to the set of eigenvalues of T, which is $\{\lambda_1, \ldots, \lambda_n\}$. e'^i is an eigenvector corresponding to λ_i. Suppose $f \in L_\infty(\mathrm{Spec}(T))$; over the same basis as before it is in fact true that

$$f(T) = \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix}$$

and it may be easily verified that the conditions in Theorem 4:4 hold. In particular, we look at the spectral measures of T.

Definition 4.1:1. *The* Point Measure at λ, δ_λ, *is the measure with* $\delta_\lambda(S)$ *equal to* 1 *if* $\lambda \in S$ *and to* 0 *otherwise.*

Since the spectral measures on T are on the finite set $\{\lambda_1, \ldots, \lambda_n\}$, they must be linear combinations of point measures.

I now give a formula for the spectral measures of T. Let x and y be two vectors, and suppose we want to know $\mu_{x,y}$. Write $x = \sum x_i e'^i$ and $y = \sum y_i e'^i$, where $x_i = \langle x, e'^i \rangle$ and $y_i = \langle y, e'^i \rangle$. We have $\langle f(T)x, y \rangle = \sum_i f(\lambda_i) x_i \overline{y_i}$; hence we have

$$\mu_{x,y} = \sum_i x_i \overline{y_i} \delta_{\lambda_i}. \tag{1}$$

It is easy to check that these spectral measures satisfy the given conditions, which means that our formula is correct.

§**4.2.** Traces and *The* Spectral Measure

Let us for the moment continue to assume that H is finite-dimensional. Let $\{x^1, \ldots, x^n\}$ be any orthonormal basis of H, and let U be any operator acting on H. Then since H is finite dimensional, we can define $\mathrm{Tr}(U)$ and it is equal to $\sum_i \langle Ux^i, x^i \rangle$. Furthermore the trace is of course independent of basis, so $\sum_i \langle Ux^i, x^i \rangle$ will be the same for any orthonormal basis $\{x^1, \ldots, x^n\}$.

Recall that T is a normal operator, with $\mu_{x,y}$ the corresponding spectral measures. Let $f \in L_\infty(\mathrm{Spec}(T))$; we can now write down $\mathrm{Tr}(f(T))$ in terms of the spectral measures. In fact

$$\begin{aligned} \mathrm{Tr}(f(T)) &= \sum_i \langle f(T)x^i, x^i \rangle \\ &= \sum_i \int f \, d\mu_{x^i,x^i} \\ &= \int f \, d(\sum \mu_{x^i,x^i}). \end{aligned}$$

So we have expressed $\mathrm{Tr}(f(T))$ as an integral of f with respect to a certain measure. However in this thesis I do in fact want to do similar things for infinite-dimensional Hilbert Spaces. Unfortunately it may well be true that for any orthonormal basis $\sum \mu_{x^i,x^i}$ is undefined. There is no satisfactory way of dealing with this, but we proceed as follows.

Definition 4.2:1. *Suppose* H *is any Hilbert Space, and that* X *is some orthonormalised basis of* H. *Suppose* M *is a set of functions* $m: X \to \mathbb{C}$, *and suppose we have defined an averaging function mapping* $m \in M$ *to its average in* $\mathbb{C}$. *Then the* Normalised Trace *of an operator* U, NTr(U), *is the average of the function mapping* $x \in X$ *to* $\langle Ux, x\rangle$.

For H finite-dimensional with dimension n, and hence X of size n, we have no problem defining an averaging function; we simply define the average of m to be $\sum_{x \in X} m(x)/n$. Then we have $\mathrm{NTr}(U) = \frac{1}{n}\mathrm{Tr}(U)$, where U is any operator. In fact, I won't use any averaging functions much more complicated than this one, so it won't be necessary to go into much detail.

Definition 4.2:2. *Suppose we have defined an averaging function. Then* The *Spectral Measure of* U *is the measure* μ *defined (if it can be defined) so that* $\mu(S)$ *is the average of* $\mu_{x,x}(S)$ *over* $x \in X$, *where* S *is a measurable set.*

We now return to the finite-dimensional case, with $f \in L_\infty(\mathrm{Spec}(T))$. In the finite-dimensional case, recall that we had $\mathrm{Tr}(f(T)) = \int f \, d(\sum \mu_{x^i,x^i})$. $\mathrm{NTr}(f(T)) = \frac{1}{n}\mathrm{Tr}(f(T))$ and $\mu = \frac{1}{n}\sum \mu_{x^i,x^i}$. So in fact we have $\mathrm{NTr}(f(T)) = \int f \, d\mu$. This will also be trivially true for the averaging functions we shall consider.

Now in fact there is a simple expression for the normalised trace of T when H is finite-dimensional. For recall that since T is normal, we can choose a basis over which it can be written as a diagonal matrix with diagonal entries $\lambda_1, \ldots, \lambda_n$, where these are the eigenvalues of T (counting multiplicities). Then we have

$$f(T) = \begin{pmatrix} f(\lambda_1) & & \\ & \ddots & \\ & & f(\lambda_n) \end{pmatrix}.$$

So $\mathrm{Tr}(f(T) = \sum f(\lambda_i)$ and $\mathrm{NTr}(f(T)) = \frac{1}{n}\sum f(\lambda_i)$. Let μ be the spectral measure, then

$$\mu = \frac{1}{n}\sum_i \delta_{\lambda_i}. \tag{2}$$

This means that we can regard the Spectral Measure as encoding the eigenvalues of T, counting multiplicities.

Chapter 5

Uses of the Spectral Measures: Walk Generating Functions and Markov Chains

Let V be any set. Let B be some function $V \times V \to \mathbb{C}$, mapping (u, v) to $B_{v \leftarrow u}$. Suppose $M(B) < \infty$. Then we can construct an operator (which we shall also call B) on the Hilbert Space H(V) defined by $\langle Be^u, e^v \rangle = B_{v \leftarrow u}$. B is a well-defined operator with operator norm at most M by Theorem 3:3. However we will not in this chapter make any other assumptions on B, except where stated. In particular, we cannot assume that B is Hermitian, nor even that there are spectral measures for it.

§5.1. Walk Generating Functions

Definition 5.1:1.

(1) *A* Walk w *on* V *is a sequence* $(w_0, \ldots, w_l)$ *of elements of* V*; it is said to* Start *at* w_0, End *at* w_l, *or go* From w_0 To w_l, *and the* length len(w) *is defined to be* l. *Walks may have zero length; a zero length walk is a sequence containing just one element.*

(2) *The* Power *of* w, $p(w)$, *is* $\prod_i B_{w_i \leftarrow w_{i-1}}$.

(3) *Given a family of walks* W, *the power of* W *is the sum of the powers of the elements of* W.

(4) *Given walks* w^1 *and* w^2, *such that* w^1 *ends at the vertex* w^2 *starts at, then their product* w^1w^2 *is the sequence* $(w^1_0, \ldots, w^1_{\mathrm{len}(w^1)} = w^2_0, \ldots, w^2_{\mathrm{len}(w^2)})$.

(5) *Given families of walks* W^1 *and* W^2 *such that all walks in* W^1 *end at and all the walks in* W^2 *start at the same vertex, their product* W^1W^2 *is the family of all walks* w^1w^2 *with* $w^1 \in W^1$ *and* $w^2 \in W^2$ *(so each* w *occurs as many times in* W^1W^2 *as there are pairs* $(w^1, w^2) \in W^1 \times W^2$ *with* $w = w^1w^2$*).*

(6) *Given families of walks* W^1 *and* W^2, *their sum* $W^1 + W^2$ *is their union, taken as a union of families, so that the number of times a walk occurs in* $W^1 + W^2$ *is the sum of the number of times it occurs in each of* W^1 *and* W^2.

Lemma 5.1:2. *For two walks w^1 and w^2, and two families of walks W^1 and W^2:*

(1) $\mathrm{len}(w^1w^2) = \mathrm{len}(w^1) + \mathrm{len}(w^2)$, *provided the product is defined.*

(2) $p(w^1w^2) = p(w^1)p(w^2)$, *provided the product is defined.*

(3) $p(W^1W^2) = p(W^1)p(W^2)$, *provided the product is defined.*

(4) $p(W^1 + W^2) = p(W^1) + p(W^2)$.

Proof. These all follow immediately from the definitions.

Definition 5.1:3. *For i a non-negative integer, and $u, v \in V$:*

(1) $W_{v \leftarrow u}$ *is the set of all walks from u to v.*

(2) $W^i_{v \leftarrow u}$ *is the set of all walks of length i from u to v.*

(3) $B^i_{v \leftarrow u}$ *is* $p(W^i_{v \leftarrow u})$.

Lemma 5.1:4.

(1)

  (1.1) *For all u,* $\sum_v |p(W^i_{v \leftarrow u})| \leq M_1(B)^i$.

  (1.2) *For all v,* $\sum_u |p(W^i_{v \leftarrow u})| \leq M_2(B)^i$.

(2) $B^i_{v \leftarrow u}$ *is well-defined by the above definition and,*

  (2.1) *For all u,* $\sum_v |B^i_{v \leftarrow u}| \leq M_1(B)^i$.

  (2.2) *For all v,* $\sum_u |B^i_{v \leftarrow u}| \leq M_2(B)^i$.

Proof. (1) is trivial by induction on i; (2) then follows immediately.

Lemma 5.1:5. $B^i_{v \leftarrow u} = \langle B^i e^u, e^v \rangle$, *where B^i is the i^{th} power of the weighted adjacency operator B.*

Proof. By induction on i.

Definition 5.1:6. *For $z \in \mathbb{C}$:*

(1) $B(z) \stackrel{\text{def}}{=} \sum_{i=0}^{\infty} z^i B^i$, *where the sum is absolutely convergent (as defined by the operator norm).*

(2) *For $u, v \in V$,* $B_{v \leftarrow u}(z) \stackrel{\text{def}}{=} \langle B(z) e^u, e^v \rangle$.

(3) *The* Walk Generating Functions *of B are the functions taking z to $B_{v \leftarrow u}(z)$ as u and v vary in V.*

Lemma 5.1:7. *$B(z)$ is defined for $|z| < \frac{1}{\|B\|}$, and equals $(I - zB)^{-1}$. $B(z)$ is undefined (so the power series isn't absolutely convergent) for all z where $(I - zB)^{-1}$ is undefined.*

Proof. $\|zB\| < 1$ and so the first sentence follows from Theorem 3:5. If the power series is absolutely convergent (as defined by the operator norm) we can define $(I - zB)^{-1}$ to be its sum.

Lemma 5.1:8. $B_{v \leftarrow u}(z) = \sum_i B^i_{v \leftarrow u} z^i$.

Proof. Trivial from the definitions and Lemma 5.1:5.

Definition 5.1:9. *For W a family of walks, we define $W(z) = \sum_{w \in W} z^{\mathrm{len}(w)} p(w)$.*

Lemma 5.1:10. *In fact we have $B_{v \leftarrow u}(z) = W_{v \leftarrow u}(z)$.*

Lemma 5.1:11. *For W_1, W_2 two families of walks*

(1) $(W_1 \cup W_2)(z) = W_1(z) + W_2(z)$;

(2) $(W_1 W_2)(z) = W_1(z) W_2(z)$, *provided the product is defined.*

Proof. This follows immediately from the definitions, as in Lemma 5.1:2.

Definition 5.1:12.

(1) *For any p and q we define $[p = q]$ to be 1 if $p = q$, 0 otherwise.*

(2) *We define $B^*_{v \leftarrow u}(z) = B_{v \leftarrow u}(z) - [u = v]$. So $B^*_{v \leftarrow u}$ is like $B_{v \leftarrow u}$ except that we ignore the zero length walk (u) when $u = v$.*

Lemma 5.1:13.

(1) *For all u and all z with $|z| < \frac{1}{M_1(B)}$,*

$$\sum_v |B^*_{v \leftarrow u}(z)| \leq \frac{|z| M_1(B)}{1 - |z| M_1(B)}.$$

(2) *For all v and all z with $|z| < \frac{1}{M_2(B)}$,*

$$\sum_u |B^*_{v \leftarrow u}(z)| \leq \frac{|z| M_2(B)}{1 - |z| M_2(B)}.$$

Proof. This follows from Lemma 5.1:4 (1). I shall only prove (1) as the proof for (2) is similar.

$$\begin{aligned}\sum_{\nu} |B^*_{\nu\leftarrow u}(z)| &\leq \sum_{\nu}\sum_{i=1} |B^i_{\nu\leftarrow u} z^i| \\ &\leq \sum_{i=1} \left|M_1(B)^i z^i\right| \quad \text{(by Lemma 5.1:4 (2.1))} \\ &= \sum_{i=1} M_1(B)^i |z|^i \\ &= \frac{|z| M_1(B)}{1 - |z| M_1(B)}.\end{aligned}$$

§§**5.1.1.** *Walk Generating Functions in Terms of Spectral Measures*

Suppose that there is a functional calculus on B. Then we can write down a formula for $B(z)$ in terms of its spectral measures, using Lemma 5.1:7. In fact we have $B(z) = (I - zB)^{-1}$. Define $f_z(t)$ to be $1/(1 - zt)$; then it is easy to see that $(I - zB)f_z(B) = I$; therefore (as inverses are unique) $B(z) = f_z(B)$. In terms of the spectral measures we have

$$B_{\nu\leftarrow u}(z) = \int \frac{1}{1 - zt}\, d\mu_{u,\nu}(t). \tag{1}$$

However there is another way of finding $B(z)$, which might work even if B has no functional calculus. We have $B(z) = (I - zB)^{-1}$, so this is what we want to find. Write $C(z) = I - zB$; we want a formula for the inverse of $C(z)$. Write $D(z) = C(z)C(z)^*$. Then $D(z)$ is certainly Hermitian, as $(D(z)^* = (C(z)C(z)^*)^* = C(z)^*(C(z)^*)^* = C(z)C(z)^* = D(z)$. Furthermore we can write $B(z) = C(z)^*(D(z))^{-1}$, provided $D(z)$ has a bounded inverse operator.

Lemma 5.1.1:1. $D(z)$ *has a bounded inverse operator whenever* $B(z)$ *is defined.*

Proof. If $C(z)$ and $C(z)^*$ have bounded inverses, then so has $D(z)$, namely their product. If $C(z)$ does not have a bounded inverse, $B(z)$ isn't defined anyway, by Lemma 5.1:7, so we are done. If $C(z)^* = I - \bar{z}B^*$ does not have a bounded inverse, then $\overline{1/z} \in \mathrm{Spec}(B^*)$. By [BOLLOBÁS,1990], Chapter 12, Theorem 11, $1/z \in \mathrm{Spec}(B)$, so $C(z)$ does not have a bounded inverse, and we are done as before.

$D(z)$ is Hermitian, and therefore has a functional calculus. If $D(z)$ has a bounded inverse then $0 \notin \mathrm{Spec}(D(z))$. Define $f(t) = 1/t$ on $\mathrm{Spec}(D(z))$, then $f(D(z))$ is the inverse of $D(z)$, and is bounded (as $\mathrm{Spec}(D(z))$ is closed). Suppose that we know the spectral measures of $D(z)$, then we know (or at least have a formula for) $f(D(z))$, and hence for $B(z) = C(z)^* f(D(z))$.

§§**5.1.2.** *Spectral Measures in Terms of Walk Generating Functions*
Suppose we know $B(z)$, and that B is Hermitian. Then there is in fact a formula for the spectral measures, and hence the functional calculus. To find this we start from the following, taken from [WIDDER,1946] (I have corrected an obvious misprint):

Theorem 5.1.2:1. *If the integral*

$$f(s) = \int_0^\infty \frac{d\alpha(t)}{s+t} \tag{2}$$

converges then for any particular number ξ,

$$\begin{aligned}\lim_{\eta\to 0+} \frac{1}{2\pi i}\int_0^\xi [f(-\sigma - i\eta) - f(-\sigma + i\eta)]\, d\sigma \\ = \frac{\alpha(\xi+)+\alpha(\xi-)}{2} - \frac{\alpha(0+)+\alpha(0-)}{2}.\end{aligned} \tag{3}$$

The integral $\int d\alpha(t)$ is the *Stieljes Integral* with respect to α; Display (2) means that f is the *Stieljes Transform* of α; and Display (3) is the *Stieljes Inversion Formula* giving α in terms of f. $\alpha(\xi+)$ is defined as $\lim_{t\to\xi+}(\alpha(t))$ and $\alpha(\xi-)$ as $\lim_{t\to\xi-}(\alpha(t))$.

To use this, suppose we know $B_{v\leftarrow u}(z)$ and want to find $\mu_{u,v}$. Let $M > \|B\|$; then in fact $\mathrm{Spec}(B) \subset (-M, M)$ by Theorem 4:1. Define

$$\alpha_0(t) = \mu_{u,v}(-\infty, t) + \frac{1}{2}\mu_{u,v}\{t\}. \tag{4}$$

Lemma 5.1.2:2. Display (4) *determines a unique Borel measure* $\mu_{u,v}$ *which is linear in the function* α_0.

Proof. Suppose α_0 and $\mu_{u,v}$ satisfy Display (4); we show that $\mu_{u,v}$ is determined by and linear in α_0.

(1) $\mu_{u,v}(-\infty, t_1) = \lim_{\epsilon\to 0+} \alpha_0(t_1 - \epsilon)$ and $\mu_{u,v}(-\infty, t_2] = \lim_{\epsilon\to 0+} \alpha_0(t_2 + \epsilon)$.

(2) Hence the measure with respect to $\mu_{u,v}$ of all intervals is determined by α_0; for example $\mu_{u,v}[t_1, t_2] = \mu_{u,v}(-\infty, t_2] - \mu_{u,v}(-\infty, t_1)$.

(3) The proof that $\mu_{u,v}$ is determined on all Borel sets follows by the standard measure-theoretic argument of showing that the set of all sets whose measure with respect to $\mu_{u,v}$ is determined by α_0 is a σ-field, and hence $\mu_{u,v}$ is determined on all Borel sets of $\mathbb{R}$, and hence all Borel sets of $\mathrm{Spec}(B) \subset \mathbb{R}$. Similarly, if $\alpha_0 = \lambda_1\alpha_0^1 + \lambda_2\alpha_0^2$ and $\alpha_0, \alpha_0^1, \alpha_0^2$ determine corresponding measures μ, μ^1, μ^2, then we can prove that $\mu = \lambda_1\mu^1 + \lambda_2\mu^2$ by showing that the set of sets S for which $\mu(S) = \lambda_1\mu^1(S) + \lambda_2\mu^2(S)$ is a σ-field containing all intervals.

Now define $\alpha_M(t) = \alpha_0(t - M)$; let $s = -M - \frac{1}{z}$. Starting with $B_{v\leftarrow u}(z) = \int \frac{1}{1-zt}\, d\mu_{u,v}(t)$, as in Display (1), we find by elementary algebraic manipulation that $f(s) = -zB_{v\leftarrow u}(z)$ and $\alpha = \alpha_M$ satisfy Display (2). Note that $(\alpha_M(\xi+) + \alpha_M(\xi-))/2 = \alpha_M(\xi)$, and $\alpha_M(0) = 0$. We have $z = -\frac{1}{s+M}$. Therefore, from Theorem 5.1.2:1,

$$\begin{aligned}
\alpha_M(\xi) &= \lim_{\eta\to 0+} \frac{1}{2\pi i}\int_0^{\xi} f(-\sigma - i\eta) - f(-\sigma + i\eta)\, d\sigma \\
&= \lim_{\eta\to 0+} \frac{1}{2\pi i}\int_0^{\xi} \frac{B_{v\leftarrow u}\left(\frac{-1}{-(\sigma-M)-i\eta}\right)}{-(\sigma - M) - i\eta} - \frac{B_{v\leftarrow u}\left(\frac{-1}{-(\sigma-M)+i\eta}\right)}{-(\sigma - M) + i\eta}\, d\sigma.
\end{aligned}$$

So

$$\alpha_0(\xi) = \lim_{\eta\to 0+} \frac{1}{2\pi i}\int_{-M}^{\eta} \frac{B_{v\leftarrow u}\left(\frac{1}{\sigma+i\eta}\right)}{-\sigma - i\eta} - \frac{B_{v\leftarrow u}\left(\frac{1}{\sigma-i\eta}\right)}{-\sigma + i\eta}\, d\sigma$$

which by taking $M \to \infty$ equals

$$= \lim_{\eta\to 0+} \frac{1}{2\pi i}\int_{-\infty}^{\eta} \frac{B_{v\leftarrow u}\left(\frac{1}{\sigma+i\eta}\right)}{-\sigma - i\eta} - \frac{B_{v\leftarrow u}\left(\frac{1}{\sigma-i\eta}\right)}{-\sigma + i\eta}\, d\sigma.$$

Thus we deduce

Theorem 5.1.2:3. *For any u, v, the spectral measure $\mu_{u,v}$ of a Hermitian*

Operator B *is determined by and linear in the function* $B_{v\leftarrow u}(z)$.
Using this and Lemma 4:9 we find that

Theorem 5.1.2:4. *All the spectral measures of* B *are determined by and linear in the functions* $B_{v\leftarrow u}(z)$.

§§**5.1.3.** *Equitable Partitions*

Definition 5.1.3:1. *An* Equitable Partition *on* (V, B) *is an equivalence relation* $\sim$ *on* V *such that if* $u, v \in V$ *then*

$$\sum_{v_1 \sim v} B_{v_1 \leftarrow u}$$

remains the same if u *is replaced by any other element of its equivalence class.*

Suppose throughout this subsection that $\sim$ is indeed an equitable partition on (V, B), where $M(B) < \infty$. Let V' be the set of its equivalence classes, and write $[v]$ for the equivalence class of v.

Definition 5.1.3:2.

$$B_{[v]\leftarrow[u]} \stackrel{\text{def}}{=} \sum_{v_1 \sim v} B_{v_1 \leftarrow u}.$$

Thus we may consider B as a function on $V' \times V'$.

Lemma 5.1.3:3. *Write* M' *for* $M(B)$ *when* B *is considered as a function on* $V' \times V'$*, and write* M *for* $M(B)$ *when* B *is considered as a function on* $V \times V$. *Then* $M' \leq M$.

Proof. Consider B as a function on $V' \times V'$; then

$$\begin{aligned} M_1(B) &= \sup_{[u]\in V'} \sum_{[v]\in V'} \left| \sum_{v_1 \sim v} B_{v_1 \leftarrow u} \right| \\ &\leq \sup_{u \in V} \sum_{[v]\in V'} \sum_{v_1 \sim v} |B_{v_1 \leftarrow u}| \end{aligned}$$

$$\begin{aligned} &= \sup_{u \in V} \sum_{v \in V} |B_{v \leftarrow u}| \\ &\leq M. \end{aligned}$$

Similarly for $M_2(B)$. Therefore $M' \leq M$.

This justifies us in considering B as an operator on $H(V')$.

Lemma 5.1.3:4. *For all* $k \geq 0$

$$\left(B_{[v] \leftarrow [u]}\right)^k = \sum_{v_1 \sim v} (B^k)_{v_1 \leftarrow u}.$$

Proof. We use induction on k. The result is trivial for $k = 0$, so suppose $k = n + 1$ where the lemma is true for n. Then

$$\begin{aligned} \left(B_{[v] \leftarrow [u]}\right)^k &= \sum_{[w] \in V'} B_{[w] \leftarrow [u]} \left(B_{[v] \leftarrow [w]}\right)^n \\ &= \sum_{[w] \in V'} \left(\sum_{w_1 \sim w} B_{w_1 \leftarrow u} \right) \left(B_{[v] \leftarrow [w]}\right)^n \\ &= \sum_{[w] \in V'} \sum_{w_1 \sim w} B_{w_1 \leftarrow u} \sum_{v_1 \sim v} B^n_{v_1 \leftarrow w_1} \quad \text{(by induction)} \\ &= \sum_{v_1 \sim v} \sum_{[w] \in V'} \sum_{w_1 \sim w} B_{w_1 \leftarrow u} B^n_{v_1 \leftarrow w_1} \\ &= \sum_{v_1 \sim v} \sum_{w_1 \in V'} B_{w_1 \leftarrow u} B^n_{v_1 \leftarrow w_1} \\ &= \sum_{v_1 \sim v} (B^k)_{v_1 \leftarrow u}. \end{aligned}$$

Definition 5.1.3:5. *An equitable partition* $\sim$ *on* (V, B) *is* Finite *if for any* $v \in V$, $\{v_1 \mid v_1 \sim v\}$ *is finite.*

For the rest of this subsection, I shall assume that $\sim$ is in fact a finite equitable partition.

Theorem 5.1.3:6. *For* $u, v \in V$,

$$B_{[v]\leftarrow[u]}(z) = \sum_{v_1 \in [v]} B_{v_1 \leftarrow u}(z).$$

Proof. This follows from Lemma 5.1.3:4 and Lemma 5.1:8.

□

Theorem 5.1.3:7. *For* B *Hermitian, and for* $u, v \in V$,

$$\mu_{[u],[v]} = \sum_{v_1 \sim v} \mu_{u,v_1}.$$

Proof. This follows from Theorem 5.1.3:6 and Theorem 5.1.2:3.

□

§§5.1.4. *Traps and Other Ways of Adding Operators*

Let $V^{(1)}$ and $V^{(2)}$ be two sets with union V^+ and intersection V'. For i equal to 1 and 2, let $B^{(i)}$ be some function mapping $V^{(i)} \times V^{(i)} \to \mathbb{C}$, mapping (u, v) to ${B^{(i)}}_{v\leftarrow u}$. Suppose $B^{(1)}$ and $B^{(2)}$ correspond to operators on the Hilbert Spaces $H(V^{(1)})$ and $H(V^{(2)})$. Define $B^{(1)}(z)$ and $B^{(2)}(z)$ from $B^{(1)}$ and $B^{(2)}$ as we defined $B(z)$ from B at the beginning of this section. Define B^+ so that $B^+_{v\leftarrow u} = B^{(1)}_{v\leftarrow u} + B^{(2)}_{v\leftarrow u}$, where we take ${B^{(i)}}_{v\leftarrow u}$ to be 0 if either u or v is not in $V^{(i)}$. We can regard $B^{(1)}$ and $B^{(2)}$ as acting on $H(V^+)$, by defining

$$(B^{(i)}(x))_v = \begin{cases} (B^{(i)}(x|_{V^{(i)}}))_v & \text{if } v \in V^{(i)}; \\ 0 & \text{otherwise.} \end{cases}$$

Then we can regard B^+ as the operator equal to $B^{(1)} + B^{(2)}$. Again, define ${B^+}_{v\leftarrow u}(z)$ and other notation from B^+ as we defined notation from B at the start of this section.

For example, suppose $V^{(1)} = \mathbb{Z} \times \mathbb{Z} \times \{\upsilon\}$ and take $B^{(1)}: V^{(1)} \times V^{(1)} \to \mathbb{C}$ to be the adjacency function and operator of the two-dimensional grid described in Chapter 1. Take $m \notin V^{(1)}$ and define $V^{(2)} = \{m, ((0,0),\upsilon), ((0,1),\upsilon)\}$ and $B^{(2)}: V^{(2)} \times V^{(2)} \to \mathbb{C}$ so that $B^{(2)}_{v \leftarrow u}$ is 1 if exactly one of u and v is m; -1 if $\{u,v\} = \{((0,0),\upsilon), ((0,1),\upsilon)\}$; and 0 otherwise. Then $B^{+}_{v \leftarrow u}$ is 1 if u and v are joined by an edge in the graph obtained from the two-dimensional grid by creating a new vertex m in the middle of the edge $(((0,0),\upsilon), ((0,1),\upsilon))$, and 0 otherwise.

I shall show that we can reduce the problem of finding the walk-generating functions ${B^{+}}_{v \leftarrow u}(z)$ to the problems:

(1) Finding the walk-generating functions of $B^{(1)}$ and $B^{(2)}$.

(2) Finding the walk generating functions $B^{\circ}_{v \leftarrow u}(z)$ where B° will be defined on $V^{\circ} \times V^{\circ}$, and $V^{\circ} = V' \times \{1,2\} \cup \{u,v\}$.

So we can find the walk-generating functions of the two-dimensional grid modified as in our example if we know them for the walk-generating functions for the two-dimensional grid, the three-element set $V^{(2)}$ and the (at most) six element set $V' \times \{1,2\} \cup \{u,v\}$. In Chapter 6 I will show how to find walk-generating functions for lattices, and we can find walk-generating functions for finite sets from Lemma 5.1:7. So we can find walk-generating functions on the modified two-dimensional grid. This can be generalised to allow a finite set of local modifications (such as adding or deleting edges or vertices).

Traps are examples of such local modifications. Consider a Markov chain in the form of a particle wandering the vertices of a directed weighted lattice with transition probabilities given by the adjacency function and operator (as described in Section 5.2), except that if the particle reaches a given vertex it stays there. This vertex is a typical trap. More generally we can allow traps which absorb a particle with probability less than 1, and more than one trap. As is shown in Section 5.2, random walk probabilities and expected times are given by walk-generating functions, so the methods of this section make it possible to find formulae for random walk probabilities in lattices with a finite number of traps. This has already been explained [MONTROLL,1969]; however in Chapter 7 we will find procedures for deriving walk-generating functions for lattices with an infinite number of modifications made in a periodic way.

Definition 5.1.4:1.

(1) *A* Modified Walk w *on* V^{+} *is a sequence* $(w_0, b_1, \ldots, b_l, w_l)$ *where each* w_i *is in* V^{+} *and each* b_i *is in* $\{1,2\}$*; it is said to* Start *at* w_0, End *at* w_l*, or go* From w_0 To w_l*, and the length* $\mathrm{len}(w)$ *is defined to be* l*. Modified Walks may have zero length like walks.*

(2) *The Power of* w, $p(w)$, *is* $\prod_i B^{(b_i)}_{w_i \leftarrow w_{i-1}}$.
(3) *Given a family of modified walks* W, *the power of* W *is defined as for walks.*
(4) *Given modified walks* w^1 *and* w^2, *such that* w^1 *ends at the vertex* w^2 *starts at, then their product* w^1w^2 *is the sequence* $(w^1_0, b^1_1 \ldots, b^1_{\text{len}(w^1)}, w^1_{\text{len}(w^1)} = w^2_0, b^2_1, \ldots, b^2_{\text{len}(w^2)}, w^2_{\text{len}(w^2)})$.
(5) *Given families of modified walks* W^1 *and* W^2 *such that all modified walks in* W^1 *end at and all the modified walks in* W^2 *start at the same vertex, their product* W^1W^2 *is defined in terms of the product of modified walks in the same way as the product of families of walks was defined in terms of the product of walks.*
(6) *Given families of modified walks* W^1 *and* W^2, *their sum* $W^1 + W^2$ *is their union, as for walks.*

Lemma 5.1.4:2. *The identities in* Lemma 5.1:2 *are also true for modified walks.*

Proof. This is equally trivial.

Definition 5.1.4:3.
(1) $W^M_{v \leftarrow u}$ *is the set of modified walks from* u *to* v.
(2) *For* W *a family of modified walks,* $W(z) \stackrel{\text{def}}{=} \sum_{w \in W} z^{\text{len}(w)} p(w)$, *as for walks (see* Definition 5.1:9*).*

Definition 5.1.4:4.
(1) *Given a modified walk* $w = (w_0, b_1, \ldots, b_l, w_l)$, *the* Restriction *of* w, $R(w)$, *is the walk* $(w_0, \ldots, w_l)$.
(2) *Given a modified walk* $w = (w_0, b_1, \ldots, b_l, w_l)$, *the* Change Sequence *of* w *is* $(k_1, \ldots, k_m)$ *such that* $0 < k_1 < \cdots < k_m < l$, *and* $i \in \{k_1, \ldots, k_m\}$ *if and only if* $b_i \neq b_{i+1}$.
(3) *Given a modified walk* $w = (w_0, b_1, \ldots, b_l, w_l)$ *the* Summary *of* w *is the sequence* $(w_0, (w_{k_1}, b_{k_1}), \ldots, (w_{k_m}, b_{k_m}), w_l)$ *where* $k_1, \ldots, k_m$ *is the change sequence of* w. *The summary of a modified walk of length* 0 *is just itself.*

For example, if $l = 7$ and $(b_1, b_2, b_3, b_4, b_5, b_6, b_7) = (2, 2, 1, 2, 1, 2, 2)$ then the change sequence is $(2, 3, 4, 5)$. The summary is $(w_0, (w_2, b_2), (w_3, b_3), (w_4, b_4), (w_5, b_5), w_7)$.

Lemma 5.1.4:5. *Let* w *be a walk in* V^+. *Then the power of* w *is the sum of the powers of all modified walks* w' *such that* $R(w') = w$.

Proof. This is easy by induction on the length of w.

Lemma 5.1.4:6. *If w has non-zero power, and the summary of w is $(w_0, (x_1, c_1), \ldots, (x_m, c_m), w_l)$, then each x_i is in V'.*

Proof. If x_i were not in $V^{(1)}$ then both $B^{(1)}_{u \leftarrow x_i}$ and $B^{(1)}_{x_i \leftarrow u}$ are zero for any u, but one or the other occurs as a factor of $p(w)$, so that would have to be zero too. Similarly for $V^{(2)}$.

Now suppose that we want to evaluate the walk generating function $B_{v_0 \leftarrow u_0}(z)$ at some z, for some $u_0, v_0 \in V^+$.

Definition 5.1.4:7.

(1) $V^\circ \stackrel{\text{def}}{=} V' \times \{1,2\} \cup \{u_0, v_0\}$.

(2) *The map B° from $V^\circ \times V^\circ$ to $\mathbb{C}$ is defined so that $B^\circ_{v \leftarrow u}$ is zero except in the following cases, where $w, x \in V'$ and $i \in \{1,2\}$:*

(2.1) $B^\circ_{(x,i) \leftarrow (w,3-i)} = B^{(i)*}_{x \leftarrow w}(z)$.

(2.2) $B^\circ_{(x,i) \leftarrow u_0} = B^{(i)*}_{x \leftarrow u_0}(z)$.

(2.2) $B^\circ_{v_0 \leftarrow (w,3-i)} = B^{(i)*}_{v_0 \leftarrow w}(z)$.

(2.2) $B^\circ_{v_0 \leftarrow u_0} = B^{(1)*}_{v_0 \leftarrow u_0}(z) + B^{(2)*}_{v_0 \leftarrow u_0}(z)$.

Theorem 5.1.4:8. $B^+_{v_0 \leftarrow u_0}(z) = B^\circ_{v_0 \leftarrow u_0}(1)$.

Proof. Starting with Lemma 5.1:8 we find that

$$\begin{aligned}
B^+_{v_0 \leftarrow u_0}(z) &= \sum_i {B^{+i}}_{v_0 \leftarrow u_0} z^i \\
&= \sum_{w \in {W^+}_{v_0 \leftarrow u_0}} p(w) z^{\mathrm{len}(w)} \\
&= \sum_{R(w) \in {W^+}_{v_0 \leftarrow u_0}} p(w) z^{\mathrm{len}(w)} \\
&= \sum_{w \in W^M_{v_0 \leftarrow u_0}} p(w) z^{\mathrm{len}(w)}.
\end{aligned}$$

Therefore we have to show that $W^M_{v_0 \leftarrow u_0}(z)$ (the sum over all modified walks w in V from u_0 to v_0 of $p(w)z^{\mathrm{len}(w)}$) is equal to $B^\circ_{v_0 \leftarrow u_0}(1)$ (the sum of the powers of all walks from u_0 to v_0 in V°).

To do this we shall associate each walk w from u_0 to v_0 in V° with non-zero power with a set W^M of modified walks in $W^M_{v_0 \leftarrow u_0}$ in such a way that each modified walk with non-zero power is associated with exactly one walk, and $p(w) = W^M(z)$. This will prove the theorem.

Therefore let w be a walk from u_0 to v_0 in V° with non-zero power; I shall now define the association, showing in each case that $p(w) = W^M(z)$.

Firstly, if $\mathrm{len}(w) = 0$, then I associate w with the set containing the modified walk $(u_0 = v_0)$, which has length 0. $p(w) = W^M(z) = 1$.

Secondly, if $\mathrm{len}(w) = 1$, then $w = (u_0, v_0)$. I associate w with the set W^M of all modified walks with non-zero length which go from u_0 to v_0 in which the b_i's are constant. $p(w) = B^{\circ}_{v_0 \leftarrow u_0} = B^{(1)*}_{v_0 \leftarrow u_0}(z) + B^{(2)*}_{v_0 \leftarrow u_0}(z)$. The elements of W^M either have all b_i's 1 or all b_i's 2. The elements with all b_i's 1 have the same power as the the restriction has when the power is evaluated with respect to $B^{(1)}$; similarly if all b_i's are 2. Thus $W^M(z)$ is also equal to $B^{(1)*}_{v_0 \leftarrow u_0}(z) + B^{(2)*}_{v_0 \leftarrow u_0}(z)$. So in this case we have $p(w) = W^M(z)$.

Lastly, suppose $\mathrm{len}(w) > 1$. We may assume that $p(w) > 0$, so $w = (u_0, (x_1, b_1), \ldots, (x_m, b_m), v_0)$, where the b_i alternate between 1 and 2. We associate w with the set W^M of all modified walks w^M which have summary w. Define $x_0 = u_0$, $x_{m+1} = v_0$, and $b_{m+1} = 3 - b_m$. Thus w^M is in W^M if and only if there exist $k_1, \ldots, k_m$, with $w^M = (x_0 = u_0, b_1, w_1, \ldots, b_1, x_1 = w_{k_1}, b_2, w_{k_1+1}, \ldots, w_{k_{m-1}}, b_m, w_{k_{m-1}+1}, \ldots, b_m, x_m = w_{k_m}, b_{m+1}, \ldots, x_{m+1} = v_0)$. For $1 \le i \le m+1$, define W^i to be the set of modified walks with non-zero length from x_{i-1} to x_i with all b's equal to b_i. Clearly we have $W^M(z) = W^1(z) \ldots W^{m+1}(z)$. So

$$
\begin{aligned}
W^M(z) &= \prod W^i(z) \\
&= \prod B^{(b_i)*}_{x_i \leftarrow x_{i-1}}(z) \\
&= B^{\circ}_{(x_1, b_1) \leftarrow u_0} \left(\prod_{1 < i \le m} B^{\circ}_{x_i, b_i \leftarrow (x_{i-1}, 3 - b_i)} \right) B^{\circ}_{v_0 \leftarrow (x_m, b_m)} \\
&= p(w).
\end{aligned}
$$

as required.

It should also be clear why each modified walk with non-zero power is associated with exactly one walk; the reason is that each modified walk is associated with its summary; the vertices all lie in V° because of Lemma 5.1.4:6. Thus the theorem is proved. □

Theorem 5.1.4:9. *Suppose* $M(B^{(1)}) < \infty$ *and* $M(B^{(2)}) < \infty$. *Then for* $|z| < \min(\frac{1}{M(B^{(1)})}, \frac{1}{M(B^{(2)})})$, $M(B^{\circ})$ *is finite and so* B° *can be regarded as an operator on the Hilbert Space* $H(V^{\circ})$.

Proof. We need to show that for sufficiently small $|z|$, $M_1(B^{\circ}) < \infty$ and $M_2(B^{\circ}) < \infty$, and we will deduce this from Lemma 5.1:13. But the situation is symmetrical, so I shall only give the proof of $M_1(B^{\circ}) < \infty$.

If $u = u_0$, then for $|z| < \frac{1}{M_1(B^{(1)})}, \frac{1}{M_2(B^{(2)})}$,

$$
\sum_v |B^{\circ}_{v \leftarrow u}| \le \sum_{i, x \in V'} |B^{(i)*}_{x \leftarrow u}(z)| + |B^{(1)*}_{v_0 \leftarrow u}(z)| + |B^{(2)*}_{v_0 \leftarrow u}(z)|
$$

$$\leq 2\left(\frac{|z|M_1(B^{(1)})}{1-|z|M_1(B^{(1)})}+\frac{|z|M_1(B^{(2)})}{1-|z|M_1(B^{(2)})}\right). \quad \text{(Lemma 5.1:13)}$$

If $u = (w, 3-i)$ then for $|z| < M_1(B^{(i)})$,

$$\sum_v |B^\circ_{v\leftarrow u}| \leq \sum_{x\in V'} |B^{(i)*}_{x\leftarrow w}(z)| + |B^{(i)*}_{v_0\leftarrow w}(z)|$$
$$\leq 2\frac{|z|M_1(B^{(i)})}{1-|z|M_1(B^{(i)})}.$$

Therefore $M_1(B^\circ) < \infty$ for all z with $|z| < \min(\frac{1}{M_1(B^{(1)})}, \frac{1}{M_1(B^{(2)})})$. Similarly with M_2 replacing M_1. So the theorem is proved.

§5.2. Markov Chains

Suppose we have a Markov Chain with state set V, with the transition probability from u to v given by $B_{v\leftarrow u}$. For the standard Markov Chain $\sum_v B_{v\leftarrow u} = 1$ for all u; however in this section I shall generalise this slightly and allow $\sum_v B_{v\leftarrow u} < 1$. If the chain reaches such a state u, there is a non-zero probability that at the next step it will not go to any vertex v. If this happens I shall say that the Markov Chain *Terminates* at u.

I shall also make the assumption that $M_2(B) < \infty$. As $M_1(B) = \sup_u \sum_v B_{v\leftarrow u} \leq 1$, we can define the weighted adjacency operator by Theorem 3:3, and then proceed as in Section 5.1, to define the Walk Generating Functions.

In this section we find how various properties of the Markov Chains can be obtained from the Walk Generating Functions, which can in turn be obtained in terms of Spectral Measures, as in Subsection 5.1.1.

First, take $u, v \in V$; suppose the Markov Chain begins at u; let ρ be the probability that the Markov Chain at some time is in state v, and let τ be the expected time before the Markov Chain is first in state v (so if with non-zero probability the Markov Chain never visits v then $\tau = \infty$). We shall find formulae for ρ and τ in terms of the walk generating functions.

Define $W_{v\Leftarrow u}$ to be the set of walks which begin at u and end at v, but do not contain v except at the end.

Lemma 5.2:1.

(1) $\rho = W_{v \Leftarrow u}(1)$.

(2) *If* $\rho = 1$ *then* $\tau = \frac{dW_{v\Leftarrow u}(z)}{dz}\Big|_{z=1}$.

Proof. We shall associate each member $(w_0 = u, \ldots, w_l = v)$ of $W_{v\Leftarrow u}$ with the event that the Markov Chain goes through the states $(w_0, \ldots, w_l)$ from when it starts to when it first reaches v. There is no way the Markov Chain can go to state v without one of these events happening; also these events are all disjoint. The probability of the event associated with a member w of $W_{v\Leftarrow u}$ is $p(w)$. Therefore

(1) $\rho = \sum_{w \in W_{v\Leftarrow u}} p(w) = W_{v\Leftarrow u}(1)$.

(2) If $\rho = 1$ then $\tau = \sum_{w\in W_{v\Leftarrow u}} \text{len}(w)p(w) = W'_{v\Leftarrow u}(1)$, by differentiation of power series.

ρ is certainly defined, and because probabilities are always non-negative real numbers (1) demonstrates that in fact $\sum_{w\in W_{v\Leftarrow u}} p(w)$ is absolutely convergent, and so the $W_{v\Leftarrow u}$ are all (since u and v were arbitrary) well-defined at 1. Since the $W_{v\Leftarrow u}$ are all given as power series this means they are defined at all complex z with $|z| < 1$. However it may well be that the radius of convergence of the $W_{v\Leftarrow u}$ will be exactly 1, in which case it may not be true that they are differentiable there; when they are not τ may be infinite.

Therefore it suffices to find the functions $W_{v\Leftarrow u}(z)$.

Theorem 5.2:2.

$$W_{v\Leftarrow u}(z) = \frac{W_{v\leftarrow u}(z)}{W_{v\leftarrow v}(z)}$$

where $W_{v\leftarrow u}(z)$ *is defined.*

We can decompose any walk in $W_{v\leftarrow u}(z)$ uniquely as a product of the section of the walk up to the first occurrence of v, followed by the remainder which is a walk from v to v. In other words any $w \in W_{v\leftarrow u}(z)$ can be written uniquely as w_1w_2 with $w_1 \in W_{v\Leftarrow u}(z)$ and $w_2 \in W_{v\leftarrow v}(z)$; furthermore $w_1w_2 \in W_{v\leftarrow u}(z)$ for any such w_1 and w_2. Therefore we have, by Lemma 5.1:11, $W_{v\leftarrow u}(z) = W_{v\Leftarrow u}(z)W_{v\leftarrow v}(z)$, from which the result follows.

□

This may also be shown using traps. For define $B^+_{v'\leftarrow u'}$ to be equal to $B_{v'\leftarrow u'}$ except if $v = u'$ when it is 0. The effect of this is to leave the power of any walk unchanged unless it passes out of v, in which case it becomes zero. Clearly $W^{+}_{v\leftarrow u}(z)$ will be equal to $W_{v\Leftarrow u}(z)$, and we can find this using the method in Subsection 5.1.4. However this is fairly mechanical, involving little more than elementary algebraic manipulation, but involves quite a lot of complication, so I shall not do it this way.

Of course, the formula in Theorem 5.2:2 will only be valid when $W_{v\leftarrow u}(z)$ is defined; that is, within its radius of convergence. However to make use of Lemma 5.2:1 we need to know the value or derivative of $W_{v\Leftarrow u}(z)$ when $z = 1$. $M_1(B) \leq 1$, and so by Lemma 5.1:4, (2.1), all the coefficients of $W_{v\leftarrow u}(z)$ have absolute value at most 1, so the radius of convergence of $W_{v\leftarrow u}(z)$ is at least 1. So the only problem here is when the radius of convergence is exactly 1. In this case, we can replace the equations in Lemma 5.2:1 by $\rho = \lim_{r\to 1-} W_{v\Leftarrow u}(r)$ and $\tau = \lim_{r\to 1-} \frac{dW_{v\Leftarrow u}(z)}{dz}\Big|_{z=r}$, and then replace $W_{v\Leftarrow u}$ by the formula in Theorem 5.2:2. It is clear that the first limit will always exist, and the second will fail to exist if and only if τ is infinite.

A more serious problem may arise if we have found a formula for $W_{v\leftarrow u}(z)$ which is only known to be valid for $|z| < R$ where R is some constant in $(0, 1)$. The theoretical solution is obviously to analytically continue this formula inside the whole of $W_{v\leftarrow u}(z)$'s radius of convergence. In practice this may mean proving that the formula can in fact be extended to an analytic function throughout $W_{v\leftarrow u}(z)$'s radius of convergence, in which case this must equal $W_{v\leftarrow u}(z)$.

Chapter 6

Finding Spectral Measures of Lattices

Let L be a weighted lattice on (G, Υ), with lattice weighting f; let V be $G \times \Upsilon$; and let B be L's weighted adjacency operator on the Hilbert Space $H \stackrel{\text{def}}{=} H(L)$. Let $M = M(B)$. When we consider lattices in this chapter, we consider them to be weighted using the construction given in Section 3.1.

I will give two proofs of the formula for spectral measures (which is Theorem 6.4:27). The first one is a development of the method given for the one-dimensional hexagonal chain in [GODSIL&MOHAR,1988], and uses a weighted generalisation of graph convergence. The second, in Section 6.6, does not use convergence, and is probably simpler. However the weighted generalisation of convergence makes sense for non-abelian groups, so it is conceivable that one day it will be useful for finding spectral measures of operators associated with non-abelian groups.

§6.1. Finitely-Generated Abelian Groups

Recall that G is a finitely-generated abelian group. In this section let G be a general finitely-generated abelian group.

Definition 6.1:1. *Suppose* $\{g_1, \ldots, g_k\} \subseteq G$ *is a finite set of generators for* G. *Then it is a* Basis *of* G *if any* $g \in G$ *can be written uniquely as* $\sum_j l_j g_j$ *where* l_j *is an integer modulo the order of* g_j *if* g_j *has finite order, and any integer if* g_j *has infinite order.*

Theorem 6.1:2. *There are non-negative integers* r *and* s, *a basis* $\{g_1, \ldots, g_{r+s}\}$ *of* G, *and positive integers* $k_1, \ldots, k_r$ *such that: for* $1 \leq j \leq r$, g_j *has order* k_j *and for* $r + 1 \leq j \leq r + s$, g_j *has infinite order.*

This follows because G is a module over $\mathbb{Z}$ and from the fundamental theorem on finitely generated modules, which is proved (for Euclidean rings; in particular $\mathbb{Z}$) in [HERSTEIN,1975], Theorem 4.5.1.

For the whole of this chapter, I shall use the notation given in this theorem – namely (r, s, g_j, k_j) – without further comment.

§**6.2.** Finite Lattices

Here we have $s = 0$, so G has a basis $\{g_1, \ldots, g_r\}$ where g_j has order k_j. Let $N = |G| = \prod_j k_j$.

For $1 \leq j \leq r$, define $\omega_j = e^{2\pi i/k_j}$. Define G' to be the set of all r-tuples $\gamma = (\gamma_1, \ldots, \gamma_r)$ where each $\gamma_j^{k_j} = 1$. G' has N elements. For $g \in G$, write it as $\sum l_j g_j$ where for each j, $\frac{-k_j}{2} < l_j \leq \frac{k_j}{2}$. For $\gamma \in (\mathbb{C} \setminus \{0\})^s$ define $\gamma(g) \in \mathbb{C}$; $\chi^\gamma, e^\gamma \in H(G)$ by

$$\begin{aligned}
\gamma\left(\sum l_j g_j\right) &\stackrel{\text{def}}{=} \prod \gamma_j^{l_j};\\
\chi_g^\gamma &\stackrel{\text{def}}{=} \gamma(g);\\
e^\gamma &\stackrel{\text{def}}{=} \frac{1}{\sqrt{N}}\chi^\gamma.
\end{aligned}$$

Lemma 6.2:1. *If* $\gamma^1, \gamma^2 \in G'$, *then* $\langle e^{\gamma^1}, e^{\gamma^2}\rangle = [\gamma^1 = \gamma^2]$; $\{e^\gamma \mid \gamma \in G'\}$ *is an orthonormal basis of* $H(G)$.

Proof. The result is trivially true if $\gamma^1 = \gamma^2$. So we must show it for $\gamma^1 \neq \gamma^2$.

First note that if $\omega = e^{2\pi i/k}$ and a^1 and a^2 are different integers modulo k then $\sum_j \omega^{j(a^1 - a^2)} = 0$, which follows from the geometric progression formula. This incidentally proves the result for $r = 1$.

Since $\gamma^1 \neq \gamma^2$, we may assume without loss of generality that $\gamma_1^1 \neq \gamma_1^2$. Since for $i = 1, 2$, $(\gamma_1^i)^{k_1} = 1$, define integers a^i by $\gamma_1^i = \omega_1^{a^i}$. a^1 and a^2 are different modulo k_1. Then

$$\begin{aligned}
\langle e^{\gamma^1}, e^{\gamma^2}\rangle &= \frac{1}{N}\sum_{\substack{0 \leq l_1 < k_1\\ \cdots\\ 0 \leq l_r < k_r}} \prod (\gamma_j^1)^{l_j} \overline{\prod (\gamma_j^2)^{l_j}}\\
&= \frac{1}{N}\sum_{l_1, \ldots, l_r} \prod \left(\gamma_j^1 \overline{\gamma_j^2}\right)^{l_j}\\
&= \frac{1}{N}\sum_{l_1} \omega_1^{l_1(a^1 - a^2)} \sum_{l_2, \ldots, l_r} \prod \left(\gamma_j^1 \overline{\gamma_j^2}\right)^{l_j}.
\end{aligned}$$

However $\sum_{l_1} \omega_1^{l_1(a^1 - a^2)} = 0$ and $|G| = |G'|$ so we deduce the lemma.

Definition 6.2:2. *For* $g \in G$, *the operator* E_g *on* $H(G)$ *is the unique bounded* *operator such that for any* $h_1, h_2 \in G$,

$$\langle E_g e^{h_1}, e^{h_2} \rangle = \begin{cases} 1 & \text{if } h_2 = h_1 + g; \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3:3, E_g is defined.

We return to the assumption that G is abelian.

Lemma 6.2:3. *For* $\gamma \in G'$ *and* $g_1, g_2 \in G$ *we have* $\gamma(g_1)\gamma(g_2) = \gamma(g_1 + g_2)$.
Proof. Trivial.

Lemma 6.2:4. *For* $g \in G$, *and* $\gamma \in G'$ *we have* $E_g e^\gamma = \overline{\gamma(g)} e^\gamma$.
Proof.

$$\langle E_g e^\gamma, e^h \rangle = \langle e^\gamma, e^{h-g} \rangle = \overline{\gamma(g)} \langle e^\gamma, e^h \rangle$$

by Lemma 6.2:3.

§§**6.2.1.** *Finite Weighted Lattices with* $\Upsilon = \{\upsilon\}$

Now consider a weighted lattice (G, Υ, f) where G is finite and $\Upsilon = \{\upsilon\}$.

We associate $g \in G$ with (g, υ), and e^g with $e^{(g,\upsilon)}$. Then $\{e^\gamma \mid \gamma \in G'\}$ is an orthonormal basis of H. I claim that in fact the e^γ are all eigenvectors of B, and so B is diagonalised by this basis.

Lemma 6.2.1:1.

$$B = \sum_g f(g, \upsilon, \upsilon) E_g. \tag{1}$$

Proof. Since the operator B is defined as the unique operator with $\langle B e^{g_1}, e^{g_2} \rangle = B_{g_2 \leftarrow g_1} = f(g_2 - g_1, \upsilon, \upsilon)$, it is enough to show that the expression on the right hand side of Display (1) satisfies this. In fact

$$\begin{aligned} \langle \sum_g f(g, \upsilon, \upsilon) E_g e^{g_1}, e^{g_2} \rangle &= \langle \sum_g f(g, \upsilon, \upsilon) e^{g+g_1}, e^{g_2} \rangle \\ &= f(g_2 - g_1, \upsilon, \upsilon) \end{aligned}$$

as required.

Theorem 6.2.1:2. $\{e^\gamma \mid \gamma \in G'\}$ *is a complete orthonormal set of eigenvectors of* B*; the eigenvalue corresponding to* e^γ *is* $\sum_g f(g,\upsilon,\upsilon)\overline{\gamma(g)}$.

Proof.

$$\begin{aligned} Be^\gamma &= \sum_g f(g,\upsilon,\upsilon)E_g e^\gamma \quad \text{(Lemma 6.2.1:1)} \\ &= \sum_g f(g,\upsilon,\upsilon)\overline{\gamma(g)}e^\gamma. \quad \text{(Lemma 6.2:4)} \end{aligned}$$

So each e^γ is an eigenvector with the eigenvalue claimed. As the e^γ form an orthonormal basis of H, this shows that they are a complete set of eigenvectors.

In particular, this allows us to write down formulae for the spectral measures of B using the technique in Section 4.1, and so find formulae for walk generating functions and solve random walks as in Chapter 5. These formulae will typically involve sums over the whole of G′, which has the same size as G, so they may be expensive to compute if G is large.

§§6.2.2. *All Finite Weighted Lattices*

Using a more general version of this technique, we can find all the eigenvectors and associated eigenvalues when Υ contains more than one element (though, from the definition, we may assume it is finite). However more complexity is introduced; in particular it is necessary to find all the eigenvalues and eigenvectors of $|\Upsilon| \times |\Upsilon|$ matrices.

The approach I shall use will be, as in Subsection 6.2.1, to produce a complete orthonormal set of eigenvectors.

Definition 6.2.2:1. *For* $\gamma \in \mathbb{C}^r$ *and* $\upsilon \in \Upsilon$, *define*

$$\begin{aligned} \chi^{(\gamma,\upsilon)}_{(g,\upsilon')} &= [\upsilon = \upsilon']\gamma(g); \\ e^{(\gamma,\upsilon)} &= \frac{1}{\sqrt{|G|}}\chi^{(\gamma,\upsilon)}. \end{aligned}$$

We consider $e^{(\gamma,\upsilon)}$ to be an element of H. We need to produce $|G| \times |\Upsilon|$ orthonormal eigenvectors. We will do this by choosing $|\Upsilon|$ orthonormal eigenvectors corresponding to each $\gamma \in G'$. Each eigenvector corresponding to γ will be of the form $\sum_{\upsilon} x_{\upsilon} e^{(\gamma,\upsilon)}$, and is given by x, a function from Υ to $\mathbb{C}$.

We now repeat some of the definitions and lemmas we used to deal with the cases where Υ has just one element, generalised a bit.

Definition 6.2.2:2. *For $g \in G$, $\upsilon_1, \upsilon_2 \in \Upsilon$, the operator $E_{g;\upsilon_2 \leftarrow \upsilon_1}$ on $H(G \times \Upsilon)$ is the unique bounded operator such that for any $h_1, h_2 \in G$ and $\upsilon_1', \upsilon_2' \in \Upsilon$,*

$$\begin{aligned}&\langle E_{g;\upsilon_2 \leftarrow \upsilon_1} e^{(h_1,\upsilon_1')}, e^{(h_2,\upsilon_2')}\rangle \\ &\quad = \begin{cases} 1 & \text{if } h_2 = h_1 + g \text{ and } \upsilon_1 = \upsilon_1' \text{ and } \upsilon_2 = \upsilon_2'; \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

By Theorem 3:3, $E_{g;\upsilon_2 \leftarrow \upsilon_1}$ is defined.

Lemma 6.2.2:3. *For (G, Υ, f) any weighted lattice or directed weighted lattice and B its adjacency operator*

$$B = \sum_{g,\upsilon_1,\upsilon_2} f(g,\upsilon_1,\upsilon_2) E_{g;\upsilon_2 \leftarrow \upsilon_1}. \tag{2}$$

Since the operator B is defined as the unique operator with $\langle Be^{(g_1,\upsilon_1')}, e^{(g_2,\upsilon_2')}\rangle = B_{(g_2,\upsilon_2') \leftarrow (g_1,\upsilon_1')} = f(g_2 - g_1, \upsilon_1', \upsilon_2')$, it is enough to show that the expression on the right hand side of Display (2) satisfies this. In fact

$$\begin{aligned}&\langle \sum_{g,\upsilon_1,\upsilon_2} f(g,\upsilon_1,\upsilon_2) E_{g;\upsilon_2 \leftarrow \upsilon_1} e^{(g_1,\upsilon_1')}, e^{(g_2,\upsilon_2')}\rangle \\ &\quad = \langle \sum_{g,\upsilon_2} f(g,\upsilon_1',\upsilon_2) e^{(g+g_1,\upsilon_2)}, e^{(g_2,\upsilon_2')}\rangle \\ &\quad = f(g_2 - g_1, \upsilon_1', \upsilon_2').\end{aligned}$$

Therefore the result is proved.

We return to the assumption that G is finite and abelian.

Lemma 6.2.2:4. *For* $g \in G$, *and* $\gamma \in G'$ *we have*

$$E_{g;\upsilon_2 \leftarrow \upsilon_1} e^{(\gamma,\upsilon)} = \begin{cases} \overline{\gamma(g)} e^{(\gamma,\upsilon_2)} & \text{if } \upsilon = \upsilon_1; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. This is obvious for $\upsilon \neq \upsilon_1$. Otherwise, we can prove this in exactly the same way as we proved Lemma 6.2:4.

We can now demonstrate the existence of eigenvectors corresponding to each γ. Consider the action of B on $e^{(\gamma,\upsilon)}$.

$$\begin{aligned} Be^{(\gamma,\upsilon)} &= \sum_{g,\upsilon_1,\upsilon_2} f(g,\upsilon_1,\upsilon_2) E_{g;\upsilon_2 \leftarrow \upsilon_1} e^{(\gamma,\upsilon)} \\ &= \sum_{\upsilon_2} e^{(\gamma,\upsilon_2)} \sum_{g} f(g,\upsilon,\upsilon_2)\overline{\gamma(g)}. \quad \text{(Lemma 6.2.2:4)} \end{aligned} \tag{3}$$

So for all fixed γ, B leaves the subspace of H generated by the $e^{(\gamma,\upsilon)}$ invariant. For $\gamma \in G'$, let B^γ be the matrix of B restricted to this subspace, with basis the $e^{(\gamma,\upsilon)}$. Display (3) tells us what the entries of this matrix are (where $B^\gamma_{\upsilon_2\upsilon_1}$ is the entry in row υ_2, column υ_1):

$$B^\gamma_{\upsilon_2\upsilon_1} = \sum_{g} f(g,\upsilon_1,\upsilon_2)\overline{\gamma(g)}. \tag{4}$$

Lemma 6.2.2:5. B^γ *is Hermitian.*

Proof.

$$\begin{aligned} (B^\gamma)^*{}_{\upsilon_2\upsilon_1} &= \overline{B^\gamma_{\upsilon_1\upsilon_2}} \\ &= \overline{\sum_{g} f(g,\upsilon_2,\upsilon_1)\overline{\gamma(g)}} \\ &= \sum_{g} f(-g,\upsilon_1,\upsilon_2)\gamma(g) \\ &= \sum_{g} f(g,\upsilon_1,\upsilon_2)\overline{\gamma(g)} \quad \text{(substituting } -g \text{ for } g) \\ &= B^\gamma_{\upsilon_2\upsilon_1}. \end{aligned}$$

Hence B^γ is Hermitian. Therefore it has a complete set of orthonormal eigenvectors. For $e^{(\gamma,j)}$ one of these eigenvectors, with $(e^{(\gamma,j)})_\upsilon$ the entry of

$e^{(\gamma,j)}$ corresponding to the basis element $e^{(\gamma,\upsilon)}$, we define $\overline{e}^{(\gamma,j)}$ to be the element of H equal to $\sum (e^{(\gamma,j)})_\upsilon e^{(\gamma,\upsilon)}$. Thus these eigenvectors of B^γ, when considered as elements of H, correspond to eigenvectors of B, with the same eigenvalues.

Theorem 6.2.2:6. *As we vary* γ *through* G' *these eigenvectors form a complete orthonormal basis of* H.

Proof. First note that the $e^{(\gamma,\upsilon)}$ form an orthonormal basis for H. For by Lemma 6.2:1, the $e^{(\gamma,\upsilon)}$ are orthonormal for fixed υ. However if $\upsilon_1 \neq \upsilon_2$ then $e^{(\gamma_1,\upsilon_1)}$ and $e^{(\gamma_2,\upsilon_2)}$ have no non-zero entries in common; therefore all the $e^{(\gamma,\upsilon)}$ are orthonormal. There are $|G| \times |\Upsilon|$ of them, the dimension of H, so they form a complete orthonormal basis.

We now need to show that the basis of eigenvectors is orthonormal. We obtained the eigenvectors corresponding to γ by finding the orthonormal eigenvectors of the matrix B^γ and mapping them into H using the basis $e^{(\gamma,\upsilon)}$; as this basis is orthonormal, the eigenvectors in H obtained from a single γ are orthonormal.

It remains to show that if we have $\gamma^1 \neq \gamma^2$, the eigenvectors corresponding to γ^1 are all orthogonal to the eigenvectors corresponding to γ^2. However this is true as the eigenvectors corresponding to γ^i are all obtained as linear combinations of $e^{(\gamma^i,\upsilon)}$, and these are orthogonal as $\gamma^1 \neq \gamma^2$. Thus, as we have $|G| \times |\Upsilon|$ orthonormal eigenvectors altogether, the theorem is proved.

§6.3. Convergence of $(V^{(i)}, B^{(i)})$ and of Measures

In this section, let V be a set and $(V^{(i)})_{i \in \mathbb{N}}$ a sequence of sets. Let $B : V \times V \to \mathbb{C}$, and $B^{(i)} : V^{(i)} \times V^{(i)} \to \mathbb{C}$. Suppose $M(B)$ and each $M(B^{(i)})$ are less than some finite constant M, so B and $B^{(i)}$ can be regarded as operators on $H(V)$ and $H(V^{(i)})$ respectively, with norm always bounded by M. Define this and other notation as in Section 4.1 and Section 5.1. Suppose $\nu^{(i)} : V \to V^{(i)}$ is a surjective map for each i.

Definition 6.3:1. $(V^{(i)}, B^{(i)}) \xrightarrow{\nu} (V, B)$ *if*

(1) *For every* $u \in V$ *and every* $\epsilon > 0$, *there is an* N *such that for all* $i \geq N$ *and all* $v \in V$, $|B^{(i)}{}_{\nu^{(i)}(v) \leftarrow \nu^{(i)}(u)} - B_{v \leftarrow u}| \leq \epsilon$.

(2) *For every* $u, v \in V$, *there is an* N *such that for all* $i \geq N$, $\nu^{(i)}(u) \neq \nu^{(i)}(v)$.

In this section I shall show that if $(V^{(i)}, B^{(i)}) \xrightarrow{\nu} (V, B)$, the spectral measures of the $B^{(i)}$ converge to those of B, in a way to be defined. This will allow us to find the spectral measures of infinite lattices.

Definition 6.3:2. *Define* B^k *and* $(B^{(i)})^k$ *to be the* k^{th} *powers of* B *and* $B^{(i)}$ *considered as operators, and consider them as functions mapping* (u, v) *to* $\langle B^k e^u, e^v \rangle$ *and* $\langle (B^{(i)})^k e^u, e^v \rangle$.

Lemma 6.3:3. *If* $(V^{(i)}, B^{(i)}) \xrightarrow{\nu} (V, B)$, *then for any non-negative integer* k, $(V^{(i)}, (B^{(i)})^k) \xrightarrow{\nu} (V, B^k)$.

Proof. We have two parts of Definition 6.3:1 to establish. (2) is trivial. So we have to prove (1) for each k. We use induction on k. $k = 0$ is trivial. Suppose $k = n + 1$ and that the lemma is true for n. Let us suppose that u and ϵ are given. Then for any v we have

$$B^k_{v \leftarrow u} = \sum_{w \in V} B^n_{w \leftarrow u} B_{v \leftarrow w}$$

$$B^{(i)k}{}_{\nu^{(i)}(v) \leftarrow \nu^{(i)}(u)} = \sum_{w \in V^{(i)}} (B^{(i)})^n_{w \leftarrow \nu^{(i)}(u)} B^{(i)}{}_{\nu^{(i)}(v) \leftarrow w}.$$

Let $\delta > 0$. By Lemma 5.1:4 we can choose a finite nonempty set $S \subseteq V$ such that for any $w \notin S$, $|B^n_{w \leftarrow u}| < \delta$, because the sum over all w of this is at most M^n, which is finite. Now, using the lemma for n, choose N such that for all $i \geq N$ the following conditions are satisfied:

(1) For any $w_1, w_2 \in S$, $w_1 \neq w_2$ implies that $\nu^{(i)}(w_1) \neq \nu^{(i)}(w_2)$.

(2) For any $w \in \{u\} \cup S$, and for any $v \in V$, $|B^n_{v \leftarrow w} - (B^{(i)})^n_{\nu^{(i)}(v) \leftarrow \nu^{(i)}(w)}| < \delta$.

(3) For any $w \in \{u\} \cup S$, and for any $v \in V$, $|B_{v \leftarrow w} - B^{(i)}{}_{\nu^{(i)}(v) \leftarrow \nu^{(i)}(w)}| < \frac{\delta}{|S|}$.

Take $i \geq N$. If $x \in V^{(i)} \setminus \nu^{(i)}(S)$, there must be a $x' \in V \setminus S$ with $\nu^{(i)}(x') = x$, by the surjectivity of $\nu^{(i)}$. Thus by (2),

$$|(B^{(i)})^n_{x \leftarrow \nu^{(i)}(u)}| \leq \delta + |B^n_{x' \leftarrow u}| \leq 2\delta. \tag{5}$$

It is now possible to bound $B^k_{v\leftarrow u} - B^{(i)k}{}_{\nu^{(i)}(v)\leftarrow\nu^{(i)}(u)}$ without reference to v. For we have

$$\left|B^k_{v\leftarrow u} - B^{(i)k}{}_{\nu^{(i)}(v)\leftarrow\nu^{(i)}(u)}\right| =$$

$$\left|\sum_{w\in V} B^n_{w\leftarrow u}B_{v\leftarrow w} - \sum_{w\in V^{(i)}} (B^{(i)})^n_{w\leftarrow\nu^{(i)}(u)}B^{(i)}{}_{\nu^{(i)}(v)\leftarrow w}\right|$$

$$\leq M\delta + M(2\delta) + \left|\sum_{w\in S} B^n_{w\leftarrow u}B_{v\leftarrow w} - \sum_{w\in\nu^{(i)}(S)} (B^{(i)})^n_{w\leftarrow\nu^{(i)}(u)}B^{(i)}{}_{\nu^{(i)}(v)\leftarrow w}\right|$$

(by Display (5))

$$= 3\delta M + \left|\sum_{w\in S} B^n_{w\leftarrow u}B_{v\leftarrow w} - \sum_{w\in S} (B^{(i)})^n_{\nu^{(i)}(w)\leftarrow\nu^{(i)}(u)}B^{(i)}{}_{\nu^{(i)}(v)\leftarrow\nu^{(i)}(w)}\right|$$

(by (1))

$$= 3\delta M + \left|\sum_{w\in S} B^n_{w\leftarrow u}B_{v\leftarrow w} - (B^{(i)})^n_{\nu^{(i)}(w)\leftarrow\nu^{(i)}(u)}B^{(i)}{}_{\nu^{(i)}(v)\leftarrow\nu^{(i)}(w)}\right|$$

$$\leq 3\delta M + \sum_{w\in S} |B^n_{w\leftarrow u}(B_{v\leftarrow w} - B^{(i)}{}_{\nu^{(i)}(v)\leftarrow\nu^{(i)}(w)})| + |(B^n_{w\leftarrow u} - (B^{(i)})^n_{\nu^{(i)}(w)\leftarrow\nu^{(i)}(u)})B_{v\leftarrow w}| + |(B^n_{w\leftarrow u} - (B^{(i)})^n_{\nu^{(i)}(w)\leftarrow\nu^{(i)}(u)})(B_{v\leftarrow w} - B^{(i)}{}_{\nu^{(i)}(v)\leftarrow\nu^{(i)}(w)})|$$

$$\leq 3\delta M + M^n\frac{\delta}{|S|} + \delta M + |S|\delta\frac{\delta}{|S|}$$

$$\leq \delta(4M + M^n + \delta).$$

As M is constant, and δ is arbitrarily small, we can prove the lemma by choosing δ so that $\delta(4M + M^n + \delta) \leq \epsilon$.

For the rest of this section, I shall assume that the operators B and $B^{(i)}$ are Hermitian, and so have spectral measures and a real spectrum.

Definition 6.3:4. *Let Σ be a compact subset of $\mathbb{R}$, and let μ and $\mu^{(i)}$ for $i \in \mathbb{N}$ be (possibly signed or complex) measures on Σ. Then $\mu^{(i)}$* Converges *to μ, $\mu_i \to \mu$, if for every continuous $f : \Sigma \to \mathbb{R}$, $\int f\, d\mu^{(i)} \to \int f\, d\mu$ as $i \to \infty$.*

Note that by Lemma 5.1.2:2, if $\mu^{(i)} \to \mu$, the measure μ is determined by the $\mu^{(i)}$, since

$$\alpha_0(t_0) = \lim_{\epsilon \to 0} \lim_{i \to \infty} \int f_{\epsilon,t_0}(t)\, d\mu^{(i)}$$

where $f_{\epsilon,t_0}(t)$ is 1 if $t \in (-\infty, t_0 - \epsilon)$, 0 if $t \in (t_0 + \epsilon, \infty)$, and linear inbetween these two intervals. We shall in fact choose $\Sigma = [-M, M]$, which contains the spectra of B and of all the $B^{(i)}$.

Definition 6.3:5.

(1) *For $x \in \ell_1(V)$, define $\gamma^{(i)}(x) = \sum_{v \in V} x_v e^{\gamma^{(i)}(v)}$.*

(2) *For any i, define the spectral measures ${\mu_{x,y}}^{(i)}$ from $B^{(i)}$ as the spectral measures $\mu_{x,y}$ were defined from B.*

Theorem 6.3:6. *If $x, y \in \ell_1(V)$ and $B^{(i)} \xrightarrow{\gamma} B$ then ${\mu_{\gamma^{(i)}(x),\gamma^{(i)}(y)}}^{(i)} \to \mu_{x,y}$.*

Proof. By Lemma 4:9, it suffices to establish this if $x = e^u$ and $y = e^v$, for $u, v \in V$.

Let a continuous function $f : [-M, M] \to \mathbb{C}$ and an $\epsilon > 0$ be given; suppose we want to find an N such that for all $i \geq N$,

$$\left| \int f\, d\mu_{\gamma^{(i)}(u),\gamma^{(i)}(v)} - \int f\, d\mu_{u,v} \right| \leq \epsilon.$$

Let $\delta > 0$ be arbitrary. By the Weierstrass Approximation Theorem ([Bollobás, 1990],Chapter 6,Corollary 11), there is a polynomial p such that, for any $t \in [-M, M]$, $|f(t) - p(t)| \leq \delta$. Write $p(t) = \sum_{k=0}^{d} p_k t^k$. Using Lemma 6.3:3, choose N such that for any $i \geq N$, and for any k between 0 and k,

$$\left| {B^{(i)}}^k_{\gamma^{(i)}(v) \leftarrow \gamma^{(i)}(u)} - B^k_{v \leftarrow u} \right| \leq \frac{\delta}{\sum_k |p_k| + 1}.$$

We now show that this N works, for a suitable value of δ. By the definition of the spectral measures, we have $\int p\, d\mu_{u,v} = (\sum_k p_k B^k)_{v \leftarrow u}$. Also, from (2) of

Theorem 4:4, we know that $\|f(B) - p(B)\| \leq \delta$ and so $\int f - p \, d\mu_{x,y} \leq \delta$. And similarly for $\mu_{v^{(i)}(u),v^{(i)}(v)}{}^{(i)}$. So

$$
\begin{aligned}
\left| \int f \, d\mu_{v^{(i)}(u),v^{(i)}(v)} - \int f \, d\mu_{u,v} \right| &\leq 2\delta + \left| \int p \, d\mu_{v^{(i)}(u),v^{(i)}(v)} - \int p \, d\mu_{u,v} \right| \\
&\leq 2\delta + \sum_k |p_k| \left| B^{(i)k}{}_{v^{(i)}(v) \leftarrow v^{(i)}(u)} - B^k_{v \leftarrow u} \right| \\
&\leq 2\delta + \frac{\sum_k |p_k| \delta}{\sum_k |p_k| + 1} \\
&\leq 3\delta.
\end{aligned}
$$

So by taking $\delta = \frac{\epsilon}{3}$, the theorem is proved. □

§**6.4.** Infinite Lattices

In this section I shall use Theorem 6.3:6 to find the spectral measures of the lattice when G may be infinite, by choosing finite weighted lattices for which the weighted adjacency operator converges to the weighted adjacency operator of the lattice.

Definition 6.4:1.

(1) Π *is the set* $\{z \in \mathbb{C} \mid |z| = 1\}$.

(2) G' *is the set of* $(\gamma_1, \ldots, \gamma_{r+s})$ *in* $\mathbb{C}^{r+s}$ *such that for* $1 \leq j \leq r$, γ_j *is a* k_j^{th} *root of unity and for* $r+1 \leq j \leq r+s$, $\gamma_j \in \Pi$.

(3) *For* $\gamma \in G'$ *and* $g = \sum l_j g_j \in G$, *we define*

$$\gamma(g) = \prod \gamma_j^{l_j}. \tag{6}$$

Definition 6.4:2. *For* i *an integer:*

(1) *For* $1 \leq j \leq r+s$ *define* r,

$$k_j^{(i)} = \begin{cases} k_j & \textit{if } j \leq r; \\ i & \textit{otherwise.} \end{cases}$$

(2) $G^{(i)}$ *is the abelian group with a basis* $\{g_1^{(i)}, \ldots, g_{r+s}^{(i)}\}$ *where* $g_j^{(i)}$ *has order* $k_j^{(i)}$.

(3) $\gamma^{(i)} : G \to G^{(i)}$ *is the group homomorphism mapping* g_j *to* $g_j^{(i)}$.
(4) $V^{(i)}$ *is* $G^{(i)} \times \Upsilon$.
(5) $\nu^{(i)} : G \times \Upsilon \to G^{(i)} \times \Upsilon$ *is defined by* $\nu^{(i)}(g,\upsilon) \stackrel{\text{def}}{=} (\gamma^{(i)}(g),\upsilon)$.
(6) *The lattice weighting* $f^{(i)}$ *on* $(G^{(i)},\Upsilon)$*, is defined by*

$$f^{(i)}(g^{(i)},\upsilon_1,\upsilon_2) \stackrel{\text{def}}{=} \sum_{\gamma^{(i)}(g)=g^{(i)}} f(g,\upsilon_1,\upsilon_2). \tag{7}$$

(7) *The weighted lattice* $L^{(i)}$ *is* $(G^{(i)},\Upsilon,f^{(i)})$.
(8) $B^{(i)}$ *is the weighted adjacency function and operator for* $L^{(i)}$.
(9) *The Hilbert Space* $H^{(i)}$ *is* $H(G^{(i)} \times \Upsilon)$.

The weighted lattices $L^{(i)}$ are finite, and we have given a method to find the eigenvalues and eigenvectors of each $B^{(i)}$. To obtain the spectral measures of B, our first task is to show that $B^{(i)} \xrightarrow{\gamma} B$.

Lemma 6.4:3. $B^{(i)} \xrightarrow{\gamma} B$.

In Definition 6.3:1, (2) is trivial. To show (1), suppose $u \in V$ and $\epsilon > 0$ are given. We have to find an N such that for all $i \geq N$ and all $\nu \in V$, $|B^{(i)}{}_{\nu^{(i)}(\nu)\leftarrow\nu^{(i)}(u)} - B_{\nu\leftarrow u}| \leq \epsilon$. Write $u = (g_1,\upsilon_1)$; then for all $(g_2,\upsilon_2) \in G \times \Upsilon$ we want

$$|f^{(i)}(\gamma^{(i)}(g_2) - \gamma^{(i)}(g_1),\upsilon_1,\upsilon_2) - f(g_2 - g_1,\upsilon_1,\upsilon_2)| \leq \epsilon. \tag{8}$$

Recall that $\sum_{g,\upsilon_1,\upsilon_2} |f(g,\upsilon_1,\upsilon_2)| < \infty$. Let $G \setminus n \stackrel{\text{def}}{=} \{\sum l_j g_j \mid \exists j > r, |l_j| \geq n\}$. $\bigcap G \setminus n = \emptyset$, so we can choose N such that for all $n \geq \lceil \frac{N}{2} \rceil$, $\sum_{g \in G\setminus n,\upsilon_1,\upsilon_2} |f(g - g_1,\upsilon_1,\upsilon_2)| \leq \epsilon$. By Display (7), Display (8) is true, so the lemma is true.

So by Theorem 6.3:6 we have a construction of the spectral measures of B as a limit of the spectral measures of the $B^{(i)}$. We now consider how to construct these measures in practice.

Recall that the eigenvectors of the finite lattices $G^{(i)}$ were parameterised by $\gamma_1,\ldots,\gamma_{r+s}$ where each γ_j is a $k_j^{(i)\text{th}}$ root of 1.

Definition 6.4:4. *Let* d *be a non-negative integer, and* S *an arbitrary subset of* $\mathbb{C}^d$.

(1) *A function* $F : S \to \mathbb{C}$ *is* Continuous *on* S *if for every* $\epsilon > 0$, *every* $s \in S$ *has a neighbourhood* $U \subseteq \mathbb{C}^d$ *such that for* $s' \in U \cap S$, $|F(s') - F(s)| \leq \epsilon$.

(2) *A function* $F : S \to \mathbb{C}$ *is* Analytic *on* S *if every* $s \in S$ *has a neighbourhood* $U \subseteq \mathbb{C}^d$ *such that on* $U \cap S$, F *is given by a multivariate power-series.*

(3) *For* e *a non-negative integer,* $F : S \to \mathbb{C}^e$ *is* Continuous *(respectively* Analytic*) on* S *if* $F(s) = (f_1(s), \dots, f_e(s))$, *where each* f_j *is continuous (analytic) on* S. *Similarly a matrix function mapping* S *to matrices (all with the same dimension) is continuous (analytic) if all the entries of the matrices are continuous (analytic) on* S.

(4) *A function with domain* $\mathbb{C}^d$ *is* Continuous *(respectively* Analytic*) if it is continuous (analytic) on* $\mathbb{C}^d$.

I shall not explicitly state and refer to proofs of every elementary fact about analytic functions. For general information, the reader is referred to [HERVÉ, 1987].

Definition 6.4:5.

(1) *Let* G\` *be* $\{(\gamma_1, \dots, \gamma_{r+s}) \mid \nexists j\, \gamma_j = 0\}$.

(2) *For* $\gamma \in$ G\` *and* $g = \sum l_j g_j$ *define* $\gamma(g)$ *as in* Display (6).

G\` is open in $\mathbb{C}^{r+s}$ and contains $G'$. For all g, $\gamma(g)$ is analytic and non-zero on G\`.

Lemma 6.4:6. *If* F *is complex-valued and analytic, as in* Definition 6.4:21, *then* F *is continuous.*

Proof. F has a derivative.

Lemma 6.4:7. *Suppose* $m : G \to \mathbb{C}$ *satisfies* $\sum_g |m(g)| < \infty$. *Then we can define a continuous function* $M : G' \to \mathbb{C}$ *by* $M(\gamma) \stackrel{\text{def}}{=} \sum m(g)\gamma(g)$.

Proof. $M(\gamma)$ is defined because $\sum |m(g)\gamma(g)| = \sum |m(g)| < \infty$ so the sums are absolutely convergent. Let $G \cap n \stackrel{\text{def}}{=} \{\sum l_i g_i \mid \forall i |l_i| \leq n\}$. $G \cap n$ is finite. Define $M_n(\gamma)$ to be $\sum_{g \in G \cap n} m(g)\gamma(g)$. M_n is analytic and so continuous.

$\bigcup_n G \cap n = G$. So given $\epsilon > 0$, we can choose N such that for $n = N$, and hence all $n \geq N$, $\sum_{g \notin G \cap n} |m(g)| \leq \epsilon$. Then for all γ,

$$\begin{aligned} |M(\gamma) - M_n(\gamma)| &= \left| \sum_{g \notin G \cap n} m(g)\gamma(g) \right| \\ &\leq \sum_{g \notin G \cap n} |m(g)\gamma(g)| \\ &= \sum_{g \notin G \cap n} |m(g)| \\ &\leq \epsilon. \end{aligned}$$

So the M_n converge uniformly to M on G'. As each M_n is continuous, M is continuous.

Definition 6.4:8. *For all $\gamma \in G'$, $g \in G$, $\upsilon_1, \upsilon_2 \in \Upsilon$ define the $\Upsilon \times \Upsilon$ matrix $E^{\gamma}_{g;\upsilon_2 \leftarrow \upsilon_1}$ by defining, for all $\upsilon'_1, \upsilon'_2 \in \Upsilon$,*

$$\left(E^{\gamma}_{g;\upsilon_2 \leftarrow \upsilon_1}\right)_{\upsilon'_2 \upsilon'_1} = \frac{[\upsilon'_1 = \upsilon_1][\upsilon'_2 = \upsilon_2]}{\gamma(g)}.$$

For all g, $E^{\gamma}_{g;\upsilon_2 \leftarrow \upsilon_1}$ is analytic in G'.

Definition 6.4:9. *For a finite matrix C we shall use $C_{\upsilon_2 \upsilon_1}$ and $C_{\upsilon_2 \leftarrow \upsilon_1}$ interchangably for the element on row υ_2, column υ_1.*

Definition 6.4:10.

(1)

(1.1) *Extend $\nu^{(i)}$ to map $\ell_1(V)$ to $H(V^{(i)})$ by defining $\nu^{(i)}(x) = \sum_v x_v e^{\nu^{(i)}(v)}$.*

(1.2) *For $g \in G$ with $g = \sum l_j g_j$ and $\gamma \in G'$, define $\chi^{(\gamma,\upsilon)}_{(g,\upsilon')} = [\upsilon = \upsilon'] \prod \gamma^{l_j}$.*

(2) *For $x \in \ell_1(V)$ and $\gamma \in G'$, define $\langle x, \chi^{(\gamma,\upsilon)} \rangle = \sum x_v \overline{\chi_v^{(\gamma,\upsilon)}}$.*

(3) *For x in $\ell_1(V)$ or $\ell_1(V^{(i)})$ and $\gamma \in G'$, define x^{γ} to be the vector with entries indexed by Υ where $(x^{\gamma})_{\upsilon} = \langle x, \chi^{(\gamma,\upsilon)} \rangle$.*

Note that these definitions accord with those for finite G.

Lemma 6.4:11.

(1) *By* Lemma 6.4:7, *the function taking* γ *to* $\langle x, \chi^{(\gamma,\upsilon)}\rangle$ *is defined and continuous.*

(2) *Hence the map taking* γ *to* x^γ *is continuous on* G'.

Lemma 6.4:12. *For* x *in* $\ell_1(V)$ *and* $\gamma \in G'$ $\|x^\gamma\|_2 \le \sqrt{|\Upsilon|}\|x\|_1$ *for all* γ.

Proof.

$$\begin{aligned}(x^\gamma)_\upsilon &= \langle x, \chi^{(\gamma,\upsilon)}\rangle \\ &\le \sum_g |x_{(g,\upsilon)}\overline{\gamma(g)}| \\ &= \sum_g |x_{(g,\upsilon)}| \\ &\le \|x\|_1.\end{aligned}$$

Hence $\|x^\gamma\|_2 \le \sqrt{|\Upsilon|}\|x\|_1$ as required.

Lemma 6.4:13. *For* $\gamma \in G'$ *and* $x \in \ell_1(V)$,

$$(E_{g;\upsilon_2\leftarrow\upsilon_1}x)^\gamma = E^\gamma_{g;\upsilon_2\leftarrow\upsilon_1}x^\gamma.$$

Proof. For $\upsilon \in \Upsilon$:

$$\begin{aligned}((E_{g;\upsilon_2\leftarrow\upsilon_1}x)^\gamma)_\upsilon &= \langle E_{g;\upsilon_2\leftarrow\upsilon_1}x, \chi^{(\gamma,\upsilon)}\rangle \\ &= \sum_h \overline{\gamma(h)}(E_{g;\upsilon_2\leftarrow\upsilon_1}x)_{(h,\upsilon)} \\ &= [\upsilon = \upsilon_2]\sum_h \overline{\gamma(h)}x_{(h-g,\upsilon_1)} \\ &= [\upsilon = \upsilon_2]\sum_{h'} \overline{\gamma(g)\gamma(h')}x_{(h',\upsilon_1)} \quad (h' = h-g) \\ &= [\upsilon = \upsilon_2]\overline{\gamma(g)}\langle x, \chi^{(\gamma,\upsilon_1)}\rangle \\ &= [\upsilon = \upsilon_2]\overline{\gamma(g)}(x^\gamma)_{\upsilon_1} \\ &= \left(E^\gamma_{g;\upsilon_2\leftarrow\upsilon_1}x^\gamma\right)_\upsilon \quad (\text{as } \gamma \in G')\end{aligned}$$

□

Definition 6.4:14. *For all* $\gamma \in G'$, *define*

$$B^{\gamma} = \sum_{g,\upsilon_1,\upsilon_2} f(g,\upsilon_1,\upsilon_2) E^{\gamma}_{g;\upsilon_2 \leftarrow \upsilon_1} \qquad (9)$$

where the sum is convergent.

Note that by Display (4) this is consistent with the previous definition of B^{γ} for finite lattices and $\gamma \in G'$.

Lemma 6.4:15. *Where* B^{γ} *is defined,*

$$(B^{\gamma})_{\upsilon_2 \upsilon_1} = \sum_{g} f(-g,\upsilon_1,\upsilon_2)\gamma(g). \qquad (10)$$

Proof. This follows by Display (9) and Definition 6.4:8.

Lemma 6.4:16. B^{γ}, *considered as a function in* γ, *is defined and continuous in* G'.

Proof. This follows from Display (10) and Lemma 6.4:7.

Lemma 6.4:17. *For* $\gamma \in G'$, B^{γ} *is Hermitian.*

Proof. This follows similarly to Lemma 6.2.2:5.

Lemma 6.4:18. *For* $x \in \ell_1(V)$,

$$(Bx)^{\gamma} = B^{\gamma}(x^{\gamma}).$$

Proof. From Lemma 6.2.2:3 we have $B = \sum_{g,\upsilon_1,\upsilon_2} f(g,\upsilon_1,\upsilon_2) E_{g;\upsilon_2 \leftarrow \upsilon_1}$. The result follows from Lemma 6.4:13.

Lemma 6.4:19. *For* $i \in \mathbb{N}$ *and* $\gamma \in G^{(i)\prime}$*:*
(1) *For* $x \in \ell_1(V)$, $\nu^{(i)}(x)^{\gamma} = x^{\gamma}$.
(2) $(B^{(i)})^{\gamma} = B^{\gamma}$.
(3) *For all* $x \in \ell_1(V)$, $(B^{(i)}x)^{\gamma} = B^{\gamma}(x^{\gamma})$.
Proof. (1) and (2) follow immediately from the definitions and Lemma 6.4:15. (3) is true because $(B^{(i)}x)^{\gamma} = (B^{(i)})^{\gamma}x^{\gamma} = B^{\gamma}x^{\gamma}$.

Lemma 6.4:20. *For all* $\gamma \in G'$, $M(B^{\gamma}) \leq M$.
Proof. In fact $M_1(B^{\gamma}) \leq M_1(B)$ and $M_2(B^{\gamma}) \leq M_2(B)$; the proofs are similar so I only prove the former.

$$M_1(B) = \sup_{\upsilon_1} \sum_{g,\upsilon_2} |f(g,\upsilon_1,\upsilon_2)|$$

and

$$M_1(B^{\gamma}) = \sup_{\upsilon_1} \sum_{\upsilon_2} \left| \sum_{g} \overline{\gamma(g)} f(g,\upsilon_1,\upsilon_2) \right|.$$

As $\gamma \in G'$, $|\gamma(g)| = 1$ for all $g \in G$. The result follows.

As for any operator T, $\mathrm{Spec}(T) \subseteq \{z \mid |z| \leq \|T\|\}$, and $\|T\| \leq M(T)$, it follows that $\mathrm{Spec}(B^{\gamma}) \subseteq \{z \mid |z| \leq M(B)\}$. As B^{γ} is Hermitian, $\mathrm{Spec}(B^{\gamma}) \subseteq [-M, M]$.

Let $x, y \in \ell_1(V)$. We want to find $\mu_{x,y}$. We know that $\mu_{\nu^{(i)}(x),\nu^{(i)}(y)}{}^{(i)} \to \mu_{x,y}$. Let the orthonormalised eigenvectors of B^{γ} be $e^{(\gamma,1)}, \ldots, e^{(\gamma,|\Upsilon|)}$ with corresponding eigenvalues $\lambda^{(\gamma,1)}, \ldots, \lambda^{(\gamma,|\Upsilon|)}$. For x' and y' in $H(\Upsilon)$ define $\mu_{x',y'}{}^{\gamma}$ to be the spectral measure of B^{γ} such that for all functions $g \in L_{\infty}([-M,M])$, $\langle g(B^{\gamma})x', y'\rangle = \int g\, \mu_{x',y'}{}^{\gamma}$. Then by Display (4.1) we have

$$\mu_{x',y'}{}^{\gamma} = \sum_{i} \langle x', e^{(\gamma,i)}\rangle \overline{\langle y', e^{(\gamma,i)}\rangle} \delta_{\lambda^{(\gamma,i)}}. \tag{11}$$

Now let $\bar{e}^{(\gamma,j)}$ be the eigenvector of $B^{(i)}$ corresponding to $e^{(\gamma,j)}$ using the construction in Theorem 6.2.2:6. In other words

$$\bar{e}^{(\gamma,j)} = \sum_{\upsilon} \left(e^{(\gamma,j)}\right)_{\upsilon} e^{(\gamma,\upsilon)}$$

where here we take $e^{(\gamma,\upsilon)} \in \mathsf{H}(V^{(i)})$. Recall that the $e^{(\gamma,j)}$ are orthonormal. Hence

$$\langle \nu^{(i)}(x), \bar{e}^{(\gamma,j)} \rangle = \sum_{\gamma',\upsilon} \langle \nu^{(i)}(x), e^{(\gamma',\upsilon)} \rangle \langle e^{(\gamma',\upsilon)}, \bar{e}^{(\gamma',j)} \rangle$$

(since the $e^{(\gamma,\upsilon)}$ are orthonormal)

$$= \sum_{\upsilon} \langle \nu^{(i)}(x), e^{(\gamma,\upsilon)} \rangle \overline{\left(e^{(\gamma,j)}\right)}_{\upsilon}$$
$$= \frac{1}{\sqrt{|G^{(i)}|}} \sum_{\upsilon} \left((\nu^{(i)}(x))^{\gamma} \right)_{\upsilon} \overline{e^{(\gamma,j)}}_{\upsilon}$$

(since $e^{(\gamma,\upsilon)} = \frac{1}{\sqrt{|G^{(i)}|}} \chi^{(\gamma,\upsilon)}$)

$$= \frac{1}{\sqrt{|G^{(i)}|}} \sum_{\upsilon} (x^{\gamma})_{\upsilon} \overline{\left(e^{(\gamma,j)}\right)}_{\upsilon}$$
$$= \frac{1}{\sqrt{|G^{(i)}|}} \langle x^{\gamma}, e^{(\gamma,j)} \rangle.$$

Therefore, by Display (4.1),

$$\mu_{\nu^{(i)}(x),\nu^{(i)}(y)}{}^{(i)} = \sum_{(\gamma,j)} \langle \nu^{(i)}(x), \bar{e}^{(\gamma,j)} \rangle \overline{\langle \nu^{(i)}(y), \bar{e}^{(\gamma,j)} \rangle} \delta_{\lambda(\gamma,j)}$$
$$= \sum_{(\gamma,j)} \frac{1}{|G^{(i)}|} \langle x^{\gamma}, e^{(\gamma,j)} \rangle \overline{\langle x^{\gamma}, e^{(\gamma,j)} \rangle} \delta_{\lambda(\gamma,j)}$$
$$= \frac{1}{|G^{(i)}|} \sum_{\gamma} \mu_{x^{\gamma},y^{\gamma}}{}^{\gamma} \quad \text{(by Display (11))} \qquad (12)$$

where the sums over γ are taken over all $\gamma \in G^{(i)\prime}$.

Note that this is an average over all possible γ's for $G^{(i)}$. We can now take this to infinity, by replacing averages over γ_j's with $j > r$ by integrals.

Definition 6.4:21. *Let F be a bounded measurable function mapping G' to $\mathbb{C}$. Then we define*

$$\int F(\gamma)\,d\gamma = \frac{1}{k_1 k_2 \cdots k_r} \sum_{\alpha_1=0}^{k_1-1} \cdots \sum_{\alpha_r=0}^{k_r-1} \frac{1}{(2\pi)^s} \times \int_{\theta_{r+1}=0}^{2\pi} \cdots \int_{\theta_{r+s}=0}^{2\pi} F(e^{2\pi i \alpha_1/k_1}, \ldots, e^{2\pi i \alpha_r/k_r}, e^{i\theta_{r+1}}, \ldots, e^{i\theta_{r+s}})\, d\theta_{r+1} \cdots d\theta_{r+s}.$$

Definition 6.4:22. *If F is a bounded measurable function as in* Definition 6.4:21, *we define*

$$F^{(i)} = \frac{1}{k_1 k_2 \cdots k_r i^s} \sum_{\alpha_1=0}^{k_1-1} \cdots \sum_{\alpha_r=0}^{k_r-1} \sum_{\alpha_{r+1}=0}^{i-1} \cdots \sum_{\alpha_{r+s}=0}^{i-1} F(e^{2\pi j \alpha_1/k_1}, \ldots, e^{2\pi j \alpha_r/k_r}, e^{2\pi j \alpha_{r+1}/i}, \ldots, e^{2\pi j \alpha_{r+s}/i})$$

(I have used j here as a synonym for i, a square root of -1, to avoid confusion).

Definition 6.4:23. *For any $s = (s_1, \ldots, s_d) \in \mathbb{C}^d$,* $|s| \stackrel{\text{def}}{=} \max_j |s_j|$.

Lemma 6.4:24. *If F is a continuous function on G' then given $\epsilon > 0$ we can choose δ such that for all $\gamma, \gamma' \in G'$, $|\gamma - \gamma'| \leq \delta$ implies $|F(\gamma) - F(\gamma')| \leq \epsilon$.*

Proof. Define X, a subset of $(G')^2$, by $X = \{(\gamma, \gamma') \mid \gamma, \gamma' \in G' \,\&\, |F(\gamma) - F(\gamma')| \geq \epsilon\}$. By the continuity of F, X is closed, and hence compact. Define $U_\delta = \{(\gamma, \gamma') \in X \mid |\gamma - \gamma'| > \delta\}$. U_δ is open in X; also $\bigcup_{\delta>0} U_\delta = X$. So (because the U_δ form a chain) we can choose δ with $U_\delta = X$, from which the lemma follows.

Lemma 6.4:25. *If* F *is continuous,* $F^{(i)} \to \int F\,d\gamma$.

Proof. Given ϵ, choose δ as in Lemma 6.4:24. Then for all $i \geq \frac{1}{\delta}$, we have $|F^{(i)} - \int F\,d\gamma| \leq \epsilon$, by using the sums in $F^{(i)}$ to estimate the integral in the usual way.

The following theorem is useful now, but we shall also use it later on in this thesis.

Theorem 6.4:26. *Let* K *be a compact subset of a topological space homeomorphic to* $\mathbb{R}^n$ *for some non-negative integer* n*. Then there is a bijection from complex-valued Borel measures* μ *on* K *to functionals* ϕ *(that is, continuous linear maps* $\phi : C(K) \to \mathbb{C}$*, where* $C(K)$ *is the normed space of continuous functions from* K *to* $\mathbb{C}$ *normed by* $\|\,\|_\infty$*), given by*

$$\phi(f) \stackrel{\text{def}}{=} \int f\,d\mu.$$

Proof. The hard part of this is done by [Halmos,1950] in Chapter 10, Section 56, Theorem D and Theorem E, which proves it for positive functionals and positive measures, and Exercise 5, where it is extended to real functionals and signed measures. To extend to complex functionals and measures, we proceed as follows. Split complex measures μ into a real part μ^R defined by $\mu^R(S) = \mathrm{Re}(\mu(S))$, and an imaginary part μ^I defined by $\mu^I(S) = \mathrm{Im}(\mu(S))$ (here we're adopting the convention that $z = \mathrm{Re}(z) + i\,\mathrm{Im}(z)$). Split continuous functions $f : K \to \mathbb{C}$ into f^R and f^I by defining $f^R(x) = \mathrm{Re}(f(x))$ and $f^I(x) = \mathrm{Im}(f(x))$. Split functionals ϕ into ϕ^R and ϕ^I by, for continuous $f^R : K \to \mathbb{R}$, defining $\phi^R(f^R)(x) = \mathrm{Re}(\phi(f^R)(x))$ and $\phi^I(f^R)(x) = \mathrm{Im}(\phi(f^R)(x))$. ϕ^R and ϕ^I are then linear maps from the real continuous functions to $\mathbb{R}$, and we can reconstruct ϕ from them since

$$\begin{aligned}
\phi(f) &= \phi(f^R + if^I) \\
&= \phi(f^R) + i\phi(f^I) \\
&= \phi^R(f^R) + i\phi^I(f^R) + i(\phi^R(f^I) + i\phi^I(f^I)) \\
&= \phi^R(f^R) - \phi^I(f^I) + i(\phi^R(f^I) + \phi^I(f^R)).
\end{aligned}$$

It is then easy to verify that the result is true by applying the results in [Halmos,1950] to ϕ^R, μ^R and ϕ^I, μ^I. □

This is known by various names such as the Reisz Representation Theorem for Measures, and the Reisz-Markov-Kakutani Theorem.

Theorem 6.4:27. *For* $x, y \in \ell_1(V)$, $\mu_{x,y}$ *is the unique (complex) measure such that for all continuous functions* $h : [-M, M] \to \mathbb{C}$,

$$\int h \, d\mu_{x,y} = \int\int h \, d\mu_{x^\gamma, y^\gamma}{}^\gamma \, d\gamma. \tag{13}$$

Uniqueness follows immediately from Theorem 6.4:26. We know that $\mu_{v^{(i)}(x), v^{(i)}(y)}{}^{(i)} \to \mu_{x,y}$. Using the Weierstrass Approximation Theorem as in Theorem 6.3:6, it suffices to show that Display (13) is true for h a polynomial, and in fact for $h(t) = t^k$. So we need to show that as $i \to \infty$

$$\int t^k \, d\mu_{v^{(i)}(x), v^{(i)}(y)}{}^{(i)} \to \int\int t^k \, d\mu_{x^\gamma, y^\gamma}{}^\gamma(t) \, d\gamma. \tag{14}$$

By Display (12), the left-hand side of Display (14) is equal to

$$\frac{1}{|G^{(i)}|} \sum_{\gamma \in G^{(i)\prime}} \int t^k \, d\mu_{x^\gamma, y^\gamma}{}^\gamma. \tag{15}$$

Let $F(\gamma) = \langle (B^\gamma)^k x^\gamma, y^\gamma \rangle$. By the definition of the spectral measures, Display (15) is equal to $F^{(i)}$. The right-hand side of Display (14) is equal to $\int F(\gamma) \, d\gamma$. Thus the theorem is true provided F is continuous on G'. However this follows as B^γ and x^γ are continuous on G' and because conjugates and products of continuous functions are continuous. Therefore the theorem follows. □

§6.5. *The* Spectral Measure for Lattices

Let m be a function from V to $\mathbb{C}$ such that $m(g, v)$ is independent of g. Then I define the *Average* of m to be $\frac{1}{|Y|} \sum_v m(g, v)$. From this we can define the normalised trace and *the* spectral measure as in Section 4.2.

Let g be an arbitrary element of G.

Lemma 6.5:1. *$(e^{(g,\upsilon)})^{\gamma}$ is an orthonormal basis of $H(\Upsilon)$ as υ ranges through Υ, if $\gamma \in G'$.*

Proof. $((e^{(g,\upsilon)})^{\gamma})_{\upsilon'} = \langle e^{(g,\upsilon)}, \chi^{(\gamma,\upsilon')}\rangle = [\upsilon = \upsilon']\overline{\gamma(g)}$. So $(e^{(g,\upsilon)})^{\gamma} = \overline{\gamma(g)}e^{\upsilon}$. As $\gamma \in G'$, $|\gamma(g)| = 1$, so the result follows.

Let μ be the spectral measure of B, and μ^{γ} the spectral measure of B^{γ}.

Theorem 6.5:2. *Let f be a continuous function on $[-M, M]$. Then $\int f\,d\mu = \int\int f\,d\mu^{\gamma}\,d\gamma$.*

By definition $\mu = \frac{1}{|\Upsilon|}\sum_{\upsilon}\mu_{(g,\upsilon),(g,\upsilon)}$. By Theorem 6.4:27, we have

$$\begin{aligned}\int f\,d\mu &= \frac{1}{|\Upsilon|}\sum_{\upsilon}\int\int f\,d\mu_{e^{(g,\upsilon)\gamma},e^{(g,\upsilon)\gamma}}{}^{\gamma}\,d\gamma \\ &= \int\frac{1}{|\Upsilon|}\sum_{\upsilon}\langle f(B^{\gamma})(e^{(g,\upsilon)})^{\gamma}, (e^{(g,\upsilon)})^{\gamma}\rangle\,d\gamma \\ &= \int \mathrm{NTr}(f(B^{\gamma}))\,d\gamma \quad \text{(as the } (e^{(g,\upsilon)})^{\gamma} \text{ are an orthonormal basis)} \\ &= \int\int f\,d\mu^{\gamma}\,d\gamma.\end{aligned}$$

□

Note that by Display (4.2), $\mu^{\gamma} = \frac{1}{|\Upsilon|}\sum_{i}\delta_{\lambda_i}$, where the eigenvalues of B^{γ} are (counting multiplicities) $\lambda_1, \ldots, \lambda_{|\Upsilon|}$.

§6.6. Directed Weighted Lattices

Now let L be a directed weighted lattice on (G, Υ) with directed lattice weighting f and weighted adjacency operator B. B may not be hermitian, or even normal, so there is no reason to suppose it has spectral measures. However we shall find formulae for the walk-generating functions.

Define terms such as $E_{g;\upsilon_2\leftarrow\upsilon_1}$, $E^{\gamma}_{g;\upsilon_2\leftarrow\upsilon_1}$ and x^{γ} as before.

Lemma 6.6:1. *For any* $g \in G$,

$$\int \gamma(g)\, d\gamma = [g = 0].$$

Proof. Write $g = \sum l_j g_j$. Then

$$\begin{aligned}\int \gamma(g)\, d\gamma &= \int \prod \gamma_j^{l_j}\, d\gamma \\ &= \prod_{j=1}^{r} \frac{1}{k_j} \left(\sum_{a_j=0}^{k_j-1} \gamma_j^{l_j} \right) \prod_{j=r+1}^{r+s} \frac{1}{2\pi} \left(\int_0^{2\pi} e^{2\pi i l_j \theta_j}\, d\theta_j \right).\end{aligned}$$

For each j, the term for j in these products is $[l_j = 0]$, from which the result follows.

Lemma 6.6:2. *For any* $g \in G$, $\upsilon_1, \upsilon_2 \in \Upsilon$, *and* $x^1, x^2 \in \ell_1(V)$,

$$\langle E_{g;\upsilon_2 \leftarrow \upsilon_1} x^1, x^2 \rangle = \int \langle E^{\gamma}_{g;\upsilon_2 \leftarrow \upsilon_1} (x^1)^{\gamma}, (x^2)^{\gamma} \rangle\, d\gamma.$$

Proof. $\langle E_{g;\upsilon_2 \leftarrow \upsilon_1} x^1, x^2 \rangle = \sum_{u,v \in V} x^1_u \overline{x^2_v} \langle E_{g;\upsilon_2 \leftarrow \upsilon_1} e^u, e^v \rangle$; and similarly for $\langle E^{\gamma}_{g;\upsilon_2 \leftarrow \upsilon_1} (x^1)^{\gamma}, (x^2)^{\gamma} \rangle$, since the map taking x to x^{γ} is linear from the definition of x^{γ}. Because $\sum |x^1_u|$ and $\sum |x^2_v|$ are finite, it suffices to prove the lemma for $x^i = e^{(g_i, \upsilon'_i)}$.

$$(E^{\gamma}_{g;\upsilon_2 \leftarrow \upsilon_1})_{\upsilon'_2 \upsilon'_1} = \overline{\gamma(g)} [\upsilon'_1 = \upsilon_1][\upsilon'_2 = \upsilon_2].$$

Therefore

$$\begin{aligned}\int \langle E^{\gamma}_{g;\upsilon_2 \leftarrow \upsilon_1} (x^1)^{\gamma}, (x^2)^{\gamma} \rangle\, d\gamma &= [\upsilon'_1 = \upsilon_1][\upsilon'_2 = \upsilon_2] \int \overline{\gamma(g)\gamma(g_1)}\gamma(g_2)\, d\gamma \\ &= [\upsilon'_1 = \upsilon_1][\upsilon'_2 = \upsilon_2] \int \gamma(g_2 - g - g_1)\, d\gamma \\ &= [\upsilon'_1 = \upsilon_1][\upsilon'_2 = \upsilon_2][g + g_1 = g_2] \\ &\qquad \text{(Lemma 6.6:1)} \\ &= \langle E_{g;\upsilon_2 \leftarrow \upsilon_1} x^1, x^2 \rangle.\end{aligned}$$

Definition 6.6:3. *Let* h *be any directed lattice weighting on* (G, Υ). *Then we define*

(1)

$$h(E) \stackrel{\text{def}}{=} \sum_{g,\upsilon_1,\upsilon_2} h(g,\upsilon_1,\upsilon_2) E_{g;\upsilon_2 \leftarrow \upsilon_1}.$$

(2)

$$h(E^\gamma) \stackrel{\text{def}}{=} \sum_{g,\upsilon_1,\upsilon_2} h(g,\upsilon_1,\upsilon_2) E^\gamma_{g;\upsilon_2 \leftarrow \upsilon_1}.$$

In particular $f(E) = B$ by Lemma 6.2.2:3.

Definition 6.6:4. $B^\gamma \stackrel{\text{def}}{=} f(E^\gamma)$.
This definition is identical (except for differences in notation) with Definition 6.4:14.

Lemma 6.6:5. *For all* $\gamma \in G'$, $M(f(E^\gamma)) \leq M(f(E))$.
Proof. This lemma is similar to Lemma 6.4:20. As there, prove $M_1(f(E^\gamma)) \leq M_1(f(E))$; this follows as

$$M_1(f(E)) = \sup_{\upsilon_1} \sum_{g,\upsilon_2} |f(g,\upsilon_1,\upsilon_2)|$$

and

$$M_1(f(E^\gamma)) = \sup_{\upsilon_1} \sum_{\upsilon_2} \left| \sum_g \overline{\gamma(g)} f(g,\upsilon_1,\upsilon_2) \right|.$$

As in Lemma 6.4:20 we deduce that $M_1(f(E^\gamma)) \leq M_1(f(E))$; similarly $M_2(f(E^\gamma)) \leq M_2(f(E))$; hence the lemma follows.

Definition 6.6:6. *Let* h_1, h_2 *be two directed lattice weightings on* (G, Υ). *Let* $\lambda \in \mathbb{C}$. *Then we define the directed lattice weightings* λh_1, $h_1 + h_2$ *and* $h_1 h_2$ *by*

(1) $(\lambda h_1)(g, v_1, v_2) = \lambda(h_1(g, v_1, v_2))$.

(2) $(h_1 + h_2)(g, v_1, v_2) = h_1(g, v_1, v_2) + h_2(g, v_1, v_2)$.

(3) $(h_1 h_2)(g, v_1, v_2) = \sum_{\substack{g_1+g_2=g \\ v \in \Upsilon}} h_1(g_1, v_1, v) h_2(g_2, v, v_2)$.

λh_1 and $h_1 + h_2$ are obviously directed lattice weightings.

$$\sum_{g, v_1, v_2} |(h_1 h_2)(g, v_1, v_2)| \le \sum_{g, v_1, v} |h_1(g, v_1, v)| \sum_{g, v, v_2} |h_1(g, v, v_2)| < \infty.$$

so $h_1 h_2$ is too.

Lemma 6.6:7. *For* h_1, h_2 *directed lattice weightings:*

(1) $(\lambda h_1)(E) = \lambda h_1(E)$ *and* $(\lambda h_1)(E^\gamma) = \lambda h_1(E^\gamma)$.

(2) $(h_1 + h_2)(E) = h_1(E) + h_2(E)$ *and* $(h_1 + h_2)(E^\gamma) = h_1(E^\gamma) + h_2(E^\gamma)$.

(3) $(h_1 h_2)(E) = h_1(E) h_2(E)$ *and* $(h_1 h_2)(E^\gamma) = h_1(E^\gamma) h_2(E^\gamma)$.

Proof. By the definitions of $h_i(E)$ and $h_i(E^\gamma)$, we can assume for each i that each h_i is 1 for some (g, v_1, v_2) and 0 elsewhere; then $h_i(E)$ is $E_{g; v_2 \leftarrow v_1}$ and $h_i(E^\gamma)$ is the corresponding $E^\gamma_{g; v_2 \leftarrow v_1}$; the result is then trivial.

Lemma 6.6:8. *Let* h *be any directed lattice weighting on* (G, Υ), *and* $x, y \in \ell_1(V)$. *Then*

$$\langle h(E)x, y \rangle = \int \langle h(E^\gamma) x^\gamma, y^\gamma \rangle \, d\gamma.$$

Proof. This follows immediately from the definitions of $h(E)$ and $h(E^\gamma)$ together with Lemma 6.6:2 and Lemma 6.6:7.

We can now prove the formula for walk generating functions of directed weighted lattices. Recall from Lemma 5.1:7 that $B_{v \leftarrow u}(z) = \langle (I - zB)^{-1} e^u, e^v \rangle$.

Theorem 6.6:9. *For* $z < \frac{1}{M(B)}$ *and* $\mathbf{x}, \mathbf{y} \in \ell_1(V)$

$$\langle (I - zB)^{-1}\mathbf{x}, \mathbf{y}\rangle = \int \langle (I - zB^{\gamma})^{-1}x^{\gamma}, y^{\gamma}\rangle \, d\gamma.$$

Proof. Let f_k be the directed lattice weighting $\sum_{i=0}^{k} z^i f^i$.

Define $H_k = f_k(E)$. Then $H_k = \sum_{i=0}^{k} z^i B^i$, and so

$$\begin{aligned}
\|(I - zB)^{-1} - H_k\| &\leq \|\sum_{i=k+1}^{\infty} z^i B^i\| \\
&\leq \sum_{i=k+1} |z|^i \|B\|^i \\
&\leq \frac{(|z|\|B\|)^{k+1}}{1 - |z|\|B\|} \\
&\leq \frac{(|z|M(B))^{k+1}}{1 - |z|M(B)} \\
&\to 0 \text{ as } k \to \infty.
\end{aligned}$$

Similarly, if we define $H_k^{\gamma} = f_k(E^{\gamma})$ we find that

$$\begin{aligned}
\|(I - zB^{\gamma})^{-1} - H_k^{\gamma}\| &\leq \frac{(|z|\|B^{\gamma}\|)^{k+1}}{1 - |z|\|B^{\gamma}\|} \\
&\leq \frac{(|z|M(B))^{k+1}}{1 - |z|M(B)} \\
&\to 0 \text{ uniformly as } k \to \infty.
\end{aligned}$$

Therefore $\langle H_k\mathbf{x}, \mathbf{y}\rangle \to \langle (I - zB)^{-1}\mathbf{x}, \mathbf{y}\rangle$ and $\int \langle H_k^{\gamma} x^{\gamma}, y^{\gamma}\rangle \, d\gamma \to \int \langle (I - zB)^{-1} x^{\gamma}, y^{\gamma}\rangle \, d\gamma$.

However by Lemma 6.6:8, for any k, $\langle H_k\mathbf{x}, \mathbf{y}\rangle = \int \langle H_k^{\gamma} x^{\gamma}, y^{\gamma}\rangle \, d\gamma$. Therefore the result follows. □

Theorem 6.6:10. *For* $z < \frac{1}{M(B)}$ *and* $u, v \in V$

(1)

$$B_{v \leftarrow u}(z) = \int \langle (I - zB^{\gamma})^{-1} (e^{u})^{\gamma}, (e^{v})^{\gamma} \rangle \, d\gamma.$$

(2)

$$B^{*}_{v \leftarrow u}(z) = \int \langle zB^{\gamma} (I - zB^{\gamma})^{-1} (e^{u})^{\gamma}, (e^{v})^{\gamma} \rangle \, d\gamma.$$

Proof. (1) follows immediately from Lemma 5.1:7 and Theorem 6.6:9. (2) is true because $B^{*}_{v \leftarrow u}(z) = B_{v \leftarrow u}(z) - [u = v]$ and therefore $B^{*}_{v \leftarrow u}(z) = B_{v \leftarrow u}(z) - B_{v \leftarrow u}(0)$. (2) is thus true because

$$\begin{aligned} (I - zB^{\gamma})^{-1} - (I - 0B^{\gamma})^{-1} &= (I - zB^{\gamma})^{-1} (I - (I - zB^{\gamma})) \\ &= zB^{\gamma} (I - zB^{\gamma})^{-1}. \end{aligned}$$

□

Now suppose again that we are dealing with a weighted lattice, not a directed weighted lattice, so the adjacency operator B is Hermitian. We can now provide the alternative proof of Theorem 6.4:27.

Theorem 6.4:27. *For* $x, y \in \ell_1(V)$, $\mu_{x,y}$ *is the unique (signed) measure such that for all continuous functions* $h : [-M, M] \to \mathbb{C}$,

$$\int h \, d\mu_{x,y} = \int \int h \, d\mu_{x^{\gamma}, y^{\gamma}}{}^{\gamma} \, d\gamma. \tag{16}$$

Proof. Uniqueness follows by Theorem 6.4:26 (so as in the original proof). So it remains to prove Display (16) for all continuous functions $h : [-M, M] \to \mathbb{C}$. Let $\epsilon > 0$ be given. By the Weierstrass Approximation Theorem (as before), let p be a polynomial such that $\sup_t |h(t) - p(t)| \leq \epsilon$. By Theorem 4:4, (2), $\|(h - p)(B)\| \leq \epsilon$. Therefore

$$\int h \, d\mu_{x,y} - \int p \, d\mu_{x,y} \leq \epsilon \|x\| \|y\|$$

and similarly

$$\int h \, d\mu_{x^\gamma, y^\gamma}{}^\gamma - \int p \, d\mu_{x^\gamma, y^\gamma}{}^\gamma \leq \epsilon \|x^\gamma\| \|y^\gamma\|.$$

By Lemma 6.4:12, for constant x and y, $\|x^\gamma\|$ and $\|y^\gamma\|$ are bounded, so the theorem is proved if we can show it for $h = p$.

However for $h = p$, $h(B)$ is a polynomial in B. Therefore $h(B) = (h(f))(E)$ and $h(B^\gamma) = (h(f))(E^\gamma)$, where $h(f)$ is the directed lattice weighting obtained by applying h to f. The theorem then follows by Lemma 6.6:8.

§6.7. Some Examples

In this section I shall illustrate the method of finding spectral measures of lattices by some examples.

§§6.7.1. *The Dihedral Groups (and a generalisation)*

The group $D_{2n} = \langle a, b \mid a^n = 1, b^2 = 1, ab = ba^{-1} \rangle$ is the *Dihedral Group of Order* n. We allow $n = \infty$ by defining $D_{2\infty} = \langle a, b \mid b^2 = 1, ab = ba^{-1} \rangle$. Its elements are $\{1, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$ if $n < \infty$, and $\{a^i, a^i b \mid i \in \mathbb{Z}\}$ if $n = \infty$. Suppose a function $d : D_{2n} \to \mathbb{C}$ is given, satisfying $\sum_{g \in D_{2n}} |d(g)| < \infty$. Let V be the set of elements of D_{2n} and define a function $B : V \times V \to \mathbb{C}$ by $B_{g_2 \leftarrow g_1} \stackrel{\text{def}}{=} d(g_1{}^{-1} g_2)$. Also let B denote the corresponding adjacency operator in $H(V)$. Then I claim we can obtain B as the adjacency operator of a directed weighted lattice over $(\frac{\mathbb{Z}}{n}, \{1, b\})$, where $\frac{\mathbb{Z}}{n}$ is the additive group of integers modulo n. In fact, we shall identify $a^i \in V$ with $(i, 1)$, and $a^i b \in V$ with (i, b). The lattice weighting f will be defined by

$$f(i, j_1, j_2) = d(j_1 a^i j_2),$$

for $i \in \frac{\mathbb{Z}}{n}$, and $j_1, j_2 \in \{1, b\}$. To check that this works, suppose $g_r = a^{i_r} j_r$ for $r = 1, 2$. Then

$$\begin{aligned} B_{g_2 \leftarrow g_1} &\stackrel{\text{def}}{=} d(g_1{}^{-1} g_2) \\ &= d((a^{i_1} j_1)^{-1} a^{i_2} j_2) \\ &= d(j_1 a^{-i_1 + i_2} j_2) \\ &= f(i_2 - i_1, j_1, j_2) \end{aligned}$$

as required. Also $\sum_{i,j_1,j_2} |f(i,j_1,j_2)| \leq 2\sum_{g\in D_{2n}} |d(g)|$ (we divide the summands into those with $j_i = 1$ and $j_i = b$). Therefore B is the adjacency operator of the directed weighted lattice $(\frac{\mathbb{Z}}{n}, \{1,b\}, f)$.

Suppose n is finite. We now find the eigenvalues of B (we could also find all the eigenvectors with a bit more trouble). By Theorem 6.5:2 we know that for any continuous function f, the integral of f over the spectral measure of B is the average of the integrals over the spectral measures of B^γ as γ ranges through $(\frac{\mathbb{Z}}{n})'$. By Display (4.2) this means that the complete family of eigenvalues of B is the union of the families of eigenvalues of the B^γ. This is obvious anyway because by choosing the basis of vectors $e^{(\gamma,\nu)}$, B can be block-diagonalised with the B^γ represented by the matrices on the diagonal, but we don't do it this way because that is not part of the general method.

For example, suppose

$$d(g) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } g = b, a, a^{-1} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

Then a little calculation shows that $f(i,j_1,j_2) = 0$ except when $(i,j_1,j_2) \in \{(1,1,1),(1,b,b),(-1,1,1),(-1,b,b),(0,1,b),(0,b,1)\}$, in which case $f(i,j_1,j_2) = 1$.

$(\frac{\mathbb{Z}}{n})'$ is the set of n^{th} roots of unity (to be pedantic, it is in fact the set of sequences of length one of roots of unity). Let γ be such a root. Then we have $B^\gamma_{\nu_2\leftarrow\nu_1} = \sum_g f(g,\nu_1,\nu_2)/\gamma(g)$. So

$$B^\gamma = \begin{pmatrix} \gamma+\overline{\gamma} & 1 \\ 1 & \gamma+\overline{\gamma} \end{pmatrix},$$

with eigenvalues $\pm 1 + \gamma + \overline{\gamma}$. Therefore the eigenvalues of B are $\pm 1 + \gamma + \overline{\gamma}$ as γ ranges through the n^{th} roots of unity.

Similarly, if n is infinite, things are much the same except that spectral measures replace eigenvalues and eigenvectors. For d as in Display (17) we find that B^γ is the same as for finite n, but γ can be any point in Π. The spectral measure μ of B is then defined

$$\begin{aligned}\int f\,d\mu &= \int\int f\,d\mu^\gamma\,d\gamma \\ &= \int\int f\,d(\frac{1}{2}(\delta_{-1+\overline{\gamma}+\gamma} + \delta_{1+\overline{\gamma}+\gamma})\,d\gamma\end{aligned}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} f(-1+2\cos(\theta)) + f(1+2\cos(\theta))\,d\theta$$

The eigenvalues for finite n of the Dihedral Groups have already been found in [BABAI,1979] using the characters of the dihedral groups.

Definition 6.7.1:1. *A set* S *is* Small *if it is finite and we can find eigenvalues and eigenvectors of* $S \times S$ *matrices.*

Now suppose that H^1 is a small group and that H^2 is an abelian group. Suppose G is a group containing H^1 and H^2 as subgroups, such that for $g \in G$ there is a unique $(h^1, h^2) \in H^1 \times H^2$ with $g = h^2h^1$. Suppose $V = G$, $d : G \to \mathbb{C}$ with $\sum_g |d(g)| < \infty$, and $B_{g_2 \leftarrow g_1} = d(g_1{}^{-1}g_2)$; we want to find the spectral measures of B considered as an operator in $H(V)$. We can consider B as the weighted adjacency operator of the weighted lattice (H^2, H^1, f) where $f(h^2, h^1_1, h^1_2) = d((h^1_1)^{-1}h^2h^1_2)$. For suppose $g_i = h^2_ih^1_i$ where $h^2_i \in H^2$ and $h^1_i \in H^1$, for $i = 1, 2$. Then

$$\begin{aligned} B_{g_2 \leftarrow g_1} &= d(g_1{}^{-1}g_2) \\ &= d((h^1_1)^{-1}(h^2_1)^{-1}h^2_2h^1_2) \\ &= f(h^2_1(h^2_2)^{-1}, h^1_1, h^1_2) \end{aligned}$$

as required.

In particular this means we can find spectral measures when G is the semidirect product of an abelian group with a small group of automorphisms. For example, let a be an element of small order modulo an integer N (and so coprime with it); then we can take G to be the set of functions from $\frac{\mathbb{Z}}{N}$ to $\frac{\mathbb{Z}}{N}$ of the form $x \mapsto a^ix + b$ with H_1 the subgroup of maps $x \mapsto a^ix$, and H_2 the subgroup of maps $x \mapsto x + b$.

§§6.7.2. *Sum-Difference Lattices and a generalisation*

We can also handle sum-difference lattices, and some others, with this technique. Let G be an abelian group, and suppose that two functions $f_+ : G \to \mathbb{C}$ and $f_- : G \to \mathbb{C}$ are defined, satisfying $\sum_g |f_+(g)| < \infty$ and $\sum_g |f_-(g)| < \infty$. Suppose $V = G$ and that $B : V \times V \to \mathbb{C}$ is defined by

$$B_{g_2 \leftarrow g_1} \stackrel{\text{def}}{=} f_-(g_2 - g_1) + f_+(g_2 + g_1).$$

Clearly $M(B)$ is at most $\sum_g |f_+(g)| + \sum_g |f_-(g)| < \infty$ and B corresponds to an operator on $H(V)$ with norm at most $M(B)$. How can we find walk generating functions for B and (if B is Hermitian) spectral measures? It is possible to do this using a lattice with an equitable partition.

Let $\Upsilon = \{+,-\}$. Define the directed lattice weighting f on (G,Υ) by

$$f(g,\upsilon_1,\upsilon_2) = \begin{cases} f_-(g) & \text{if } (\upsilon_1,\upsilon_2) = (+,+). \\ f_-(-g) & \text{if } (\upsilon_1,\upsilon_2) = (-,-). \\ f_+(g) & \text{if } (\upsilon_1,\upsilon_2) = (-,+). \\ f_+(-g) & \text{if } (\upsilon_1,\upsilon_2) = (+,-). \end{cases}$$

Then we consider the directed weighted lattice (G,Υ,f). Let $\bar{V} = G \times \Upsilon$ and $\bar{B}$ be the weighted adjacency function and operator for the lattice. Since this is a directed weighted lattice we can find its walk-generating functions and (if f is a lattice weighting) spectral measures.

We now define a finite equitable partition $\sim$ for which V', the set of equivalence classes, can be identified with G in such a way that $\bar{B}$, considered as an operator on $H(V')$, is equal to B. Hence, by the results in Subsection 5.1.3, we can find the walk-generating functions on B and (if $\bar{B}$ is Hermitian) its spectral measures.

Specifically, define $\sim$ so that $(g_1,\upsilon_1) \sim (g_2,\upsilon_2)$ only when $(g_1,\upsilon_1) = (g_2,\upsilon_2)$, or $g_1 = -g_2$ and $\upsilon_1 \neq \upsilon_2$. This is trivially an equivalence relation with a complete set of equivalence classes given by $[(g,+)]$ as g ranges through G. We will identify V' by V by identifying g with $[(g,+)]$.

Theorem 6.7.2:1. *$\sim$ is an equitable partition of $(\bar{V},\bar{B})$, and*

$$B_{g_2 \leftarrow g_1} = \bar{B}_{[(g_2,+)] \leftarrow [(g_1,+)]}.$$

Proof.

$$\begin{aligned} \sum_{v \sim (g_2,+)} \bar{B}_{v \leftarrow (g_1,+)} &= \bar{B}_{(g_2,+) \leftarrow (g_1,+)} + \bar{B}_{(-g_2,-) \leftarrow (g_1,+)} \\ &= f_-(g_2 - g_1) + f_+(g_2 + g_1). \end{aligned}$$

Similarly

$$\sum_{\nu\sim(g_2,+)} \bar{B}_{\nu\leftarrow(-g_1,-)} = \bar{B}_{(-g_2,-)\leftarrow(-g_1,-)} + \bar{B}_{(g_2,+)\leftarrow(-g_1,-)}$$
$$= f_-(-(-g_2-(-g_1))) + f_+(g_2-(-g_1))$$
$$= f_-(g_2-g_1) + f_+(g_2+g_1)$$

The fact that both are equal to $f_-(g_2-g_1)+f_+(g_2+g_1)$ establishes both parts of the theorem. □

Thus we can find walk-generating functions of sum-difference lattices. If they are Hermitian we can find formulae for the spectral measures using the methods of Subsection 5.1.2. [CHUNG,1989] found eigenvalues and eigenvectors of finite sum graphs (where $f_-=0$).

More generally, let G be an abelian group, and H a small group of automorphisms of G. Suppose for each $h \in H$ there is a function $f_h : G \to \mathbb{C}$ satisfying $\sum_g |f_h(g)| < \infty$. Let $V = G$, and $B : V \times V \to \mathbb{C}$ the function defined by

$$B_{g_2\leftarrow g_1} \stackrel{\text{def}}{=} \sum_{h\in H} f_h(g_2 - h(g_1)).$$

Clearly $M(B) \leq \sum_h \sum_g |f_h(g)|$, so by Theorem 3:3, B can be considered as an operator. We shall now show how to find the walk-generating functions of B by looking at a directed weighted lattice on (G,H) with a suitable equitable partition.

Specifically, define the directed lattice weighting f by

$$f(g,h_1,h_2) = f_{h_2^{-1}h_1}(h_2^{-1}(g)).$$

Let $\bar{V}$ be $G \times H$ and $\bar{B}$ the adjacency function and operator of (G,H,f).

Define $\sim$ by

$$[(g,h)] = \{(h'(g), h'h) \mid h' \in H\}.$$

Clearly $\sim$ is an equivalence relation, and the set $\{(g,1) \mid g \in G\}$ is a complete set of representatives of the equivalence classes.

Theorem 6.7.2:2. *$\sim$ is an equitable partition and*

$$\bar{B}_{[(g_2,1)]\leftarrow[(g_1,1)]} = B_{g_2\leftarrow g_1}$$

Proof. Let $(h_1(g_1), h_1)$ be a typical element of $[(g,1)]$. We need to show that

$$\sum_{h_2\in H} \bar{B}_{(h_2(g_2),h_2)\leftarrow(h_1(g_1),h_1)} = \sum_{h\in H} f_h(g_2 - h(g_1)).$$

In fact:

$$\begin{aligned}
\sum_{h_2\in H} \bar{B}_{(h_2(g_2),h_2)\leftarrow(h_1(g_1),h_1)} &= \sum_{h_2\in H} f(h_2(g_2) - h_1(g_1), h_1, h_2) \\
&= \sum_{h_2\in H} f_{h_2^{-1}h_1}(h_2^{-1}(h_2(g_2) - h_1(g_1))) \\
&= \sum_{h_2\in H} f_{h_2^{-1}h_1}(g_2 - (h_2^{-1}h_1)(g_1)) \\
&\qquad \text{(as } h_2 \text{ is an automorphism)} \\
&= \sum_{h\in H} f_h(g_2 - h(g_1)).
\end{aligned}$$

□

We can then use the results of Subsection 5.1.3.

For example, if $G = \mathbb{Z}^d$ and H is a small group of permutations on $\{1, \ldots, d\}$, then we can consider H as a group of automorphisms on G by defining $h \in H$ to map $(l_1, \ldots, l_d)$ to $(l_{h(1)}, \ldots, l_{h(d)})$. Or another example would be if G is the additive group of integers modulo some N, a is an element with small order in G, and H is the group of automorphisms of G generated by the map taking $x \mapsto ax$.

Suppose more generally that as well as G and H we have a small set Υ, and for each $h \in H$ we have a function $f_h : G \times \Upsilon \times \Upsilon$ with $\sum_g |f_h(g)| < \infty$. Suppose we define $V = G \times \Upsilon$ and we want to find the walk generating functions of B where

$$B_{(g_2,v_2)\leftarrow(g_1,v_1)} = \sum_h f_h(g_2 - h(g_1), v_1, v_2).$$

It should be possible to do this by considering the directed weighted lattice on $(G, H \times \Upsilon)$ with directed lattice weighting

$$f(g,(h_1,\upsilon_1),(h_2,\upsilon_2)) = f_{h_2^{-1}h_1}(h_2^{-1}(g),\upsilon_1,\upsilon_2).$$

Define the equivalence relation $\sim$ by defining

$$[(g,(h,\upsilon_1)] = \{(h'(g),(h'h,\upsilon_1)) \mid h' \in H\}.$$

Let $\bar{B}$ be the adjacency operator and function of this. Then we should be able to show that $\sim$ is an equitable partition and that $\bar{B}$ on the equivalence classes of $\sim$ is equal to B on $G \times \Upsilon$. However I shall not prove this as the proof is identical to that in Theorem 6.7.2:2 except for the added complication of replacing the h_i that appear as arguments of f by (h_i, υ_i), and adding parameters υ_1, υ_2 to f_h, and therefore has no extra mathematical interest.

§§**6.7.3.** *The Two-Dimensional Grid*

See Figure 1. This was defined to be $(\mathbb{Z}^2, \Upsilon, L)$, where $\Upsilon = \{\upsilon\}$ and $L = \{(x,\upsilon,\upsilon) \mid x \in L'\}$ where $L' = \{(0,1),(0,-1),(1,0),(-1,0)\}$. Thus if we consider the two-dimensional grid as a weighted lattice, it is $(\mathbb{Z}^2, \Upsilon, f)$ where f is 1 on L and 0 elsewhere. Let B be the adjacency function and operator of this weighted lattice. We have $M(B) = 4$.

Let $\gamma = (\gamma_1, \gamma_2) \in G' = \Pi^2$. By Definition 6.4:14 and Definition 6.4:8 we have $(B^\gamma)_{\upsilon_2\upsilon_1} = \sum_g f(g,\upsilon_1,\upsilon_2)/\gamma(g)$. So B^γ is the 1×1 matrix with the single entry $\lambda(\gamma) \stackrel{\text{def}}{=} \gamma_1 + \overline{\gamma_1} + \gamma_2 + \overline{\gamma_2}$.

This B^γ has the single eigenvalue λ and eigenvector 1 (we identify elements of $H(\Upsilon)$ with their single value). Therefore its spectral measure μ^γ is δ_λ, and for any continuous function $h\colon [-4,4] \to \mathbb{C}$,

$$\int h \, d\mu = \int h(\lambda(\gamma)) \, d\gamma.$$

Let $u = (i,j)$ and $v = (i+k, j+l)$; then for $|z| < \frac{1}{4}$:

$$\begin{aligned} B_{v \leftarrow u}(z) &= \langle (I - zB)^{-1} \boldsymbol{e}_u, \boldsymbol{e}_v \rangle \\ &= \int \langle (I - zB^\gamma)^{-1} (e_u)^\gamma, (e_v)^\gamma \rangle \, d\gamma \end{aligned}$$

$$
\begin{aligned}
&= \int \left\langle \frac{1}{1 - z(\gamma_1 + \overline{\gamma_1} + \gamma_2 + \overline{\gamma_2})} \overline{\gamma(i,j)}, \overline{\gamma(i+k, j+l)} \right\rangle d\gamma \\
&= \int \frac{1}{1 - z(\gamma_1 + \overline{\gamma_1} + \gamma_2 + \overline{\gamma_2})} \overline{\gamma_1^i \gamma_2^j} \gamma_1^{i+k} \gamma_2^{j+l} \, d\gamma \\
&= \int \frac{\gamma_1^k \gamma_2^l}{1 - z(\gamma_1 + \overline{\gamma_1} + \gamma_2 + \overline{\gamma_2})} \, d\gamma \\
&= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{e^{ik\theta_1 + il\theta_2}}{1 - 2z(\cos(\theta_1) + \cos(\theta_2))} \, d\theta_2 \, d\theta_1 \\
&= \frac{1}{4\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{\cos(k\theta_1 + l\theta_2)}{1 - 2z(\cos(\theta_1) + \cos(\theta_2))} \, d\theta_2 \, d\theta_1
\end{aligned}
$$

as it's real if z is.

§§6.7.4. *The Hexagonal Lattice*

Recall that this is $(\mathbb{Z}^2, \Upsilon, L)$ where $\Upsilon = \{\upsilon_1, \upsilon_2\}$ and $L = \{((0,0), \upsilon_1, \upsilon_2), ((0,1), \upsilon_1, \upsilon_2), ((1,0), \upsilon_1, \upsilon_2), ((0,0), \upsilon_2, \upsilon_1), ((0,-1), \upsilon_2, \upsilon_1), ((-1,0), \upsilon_2, \upsilon_1)\}$. See the picture in Figure 2. The lattice weighting is 1 on elements of L and 0 elsewhere. Let B be the adjacency function and operator. $M(B) = 3$. $G' = \Pi^2$. Let $\gamma = (\gamma_1, \gamma_2)$ be an element of G'. Then we have

$$
B^\gamma = \begin{pmatrix} 0 & 1 + \overline{\gamma_1} + \overline{\gamma_2} \\ 1 + \gamma_1 + \gamma_2 & 0 \end{pmatrix}. \tag{18}
$$

The eigenvalues of B^γ are

$$
\pm\sqrt{(1 + \overline{\gamma_1} + \overline{\gamma_2})(1 + \gamma_1 + \gamma_2)} = \pm|1 + \gamma_1 + \gamma_2|.
$$

So if f is a continuous function on $[-3,3]$ and μ is the spectral measure of B^γ, then

$$
\int f \, d\mu = \int \frac{1}{2}(f(|1 + \gamma_1 + \gamma_2|) + f(-|1 + \gamma_1 + \gamma_2|)) \, d\gamma.
$$

Similarly, by finding the inverse of $I - zB^\gamma$ we can find the walk-generating functions of the hexagonal lattice.

§§**6.7.5.** *Decaying Particles*

We suppose that we have an abelian group G and two finite sets Υ and Σ, together with a function $p : G \times (\Upsilon \times \Sigma) \times (\Upsilon \times \Sigma) \to [0,1]$ such that for any $(\upsilon_1, \sigma_1) \in \Upsilon \times \Sigma$,

$$\sum_{g,\upsilon_2,\sigma_2} p(g,(\upsilon_1,\sigma_1),(\upsilon_2,\sigma_2)) \leq 1.$$

Then $(G, \Upsilon \times \Sigma, p)$ is a directed weighted lattice.

However, we also consider it as a Markov Chain. A particle moves around in $G \times \Upsilon$. In addition to its position (an element of $G \times \Upsilon$), it also has an internal state which is an element of Σ. If it is at (g_1, υ_1) and in internal state σ_1, suppose that with probability $p(g_2 - g_1, (\upsilon_1, \sigma_1), (\upsilon_2, \sigma_2))$ it will move to (g_2, υ_2) and internal state σ_2. So in this situation we can find formulae for the probability of visiting a given vertex and expected time to do so, using the results of Section 5.2.

Thus using directed weighted lattices we can find solutions of Markov Chains with particles which with certain probabilities are transformed into other particles or disappear altogether, provided we can handle formulae involving eigenvalue and eigenvector problems for $(\Upsilon \times \Sigma) \times (\Upsilon \times \Sigma)$ matrices.

Chapter 7

Lattices with Modifications

In Subsection 5.1.4 we have shown how to deal with a finite number of modifications. In fact for lattices we can also deal with one periodic set of modifications (though not more).

I hope I will be forgiven for emphasising how much of an advance this is. The state-of-the-art is perhaps represented by [MONTROLL,1969], [WALSH&KOZAK,1981], [WALSH&KOZAK,1982] and [POLITOWICZ&KOZAK,1983]. [MONTROLL,1969] contains a method for solving random walks on lattices where there are a finite number of traps. [WALSH&KOZAK,1981], [WALSH&KOZAK,1982] and [POLITOWICZ&KOZAK,1983] give methods which speed up the computation of walk-generating functions for finite lattices to which a small number of modifications have been made in a symmetrical way. By virtue of Subsection 5.1.4 and our expressions for the walk-generating functions of lattices, we can already find exact formulae when a finite number of modifications have been made in a lattice. In this section we go further and show how to find formulae for walk-generating functions (and therefore random walk probabilities) when a periodic set of modifications has been made to the lattice. This will allow us to solve, in principle at least, Markov Chains in which two particles move about independently on a lattice except when they get within a certain distance of each other.

However, it is only fair to admit that the formulae which we shall find are quite complex, and a typical formula for a walk-generating function will involve an integral of a matrix expression when the coefficients of the matrices are themselves expressions involving integrals. As I am not a physicist, and have not had time to investigate the possible physical applications of this technique, I cannot say whether these formulae will be useful in practice.

Define (V^+, B^+) as follows.

Suppose that we are given for $i = 1, 2$, a finitely-generated abelian group $G^{(i)}$, a finite set $\Upsilon^{(i)}$, and a directed lattice weighting $f^{(i)}$ on $(G^{(i)}, \Upsilon^{(i)})$; together with functions $\phi^{(i)} : \Upsilon^\circ \to G^{(i)}$ and $\psi^{(i)} : \Upsilon^\circ \to \Upsilon^{(i)}$, where Υ° is a finite set. We require that $G^{(1)}$ and $G^{(2)}$ have intersection a finitely-generated abelian group G°. We require that Υ°, $\Upsilon^{(1)}$ and $\Upsilon^{(2)}$ be disjoint. For $i = 1, 2$, define a function $\theta^{(i)} : G^\circ \times \Upsilon^\circ \to G^{(i)} \times \Upsilon^{(i)}$ by $\theta^{(i)}(g, \upsilon) = (g + \phi^{(i)}(\upsilon), \psi^{(i)}(\upsilon))$. We require that each $\theta^{(i)}$ be injective.

For $i = 1, 2$, define $V^{(i)}$ to be $G^{(i)} \times \Upsilon^{(i)}$ and define $B^{(i)}$ to be the adjacency function and operator of the directed weighted lattice $(G^{(i)}, \Upsilon^{(i)}, f^{(i)})$. For $(g, \upsilon) \in G^\circ \times \Upsilon^\circ$, identify (g, υ) with $\theta^{(1)}(g, \upsilon) \in V^{(1)}$ and $\theta^{(2)}(g, \upsilon) \in V^{(2)}$. Clearly $V^{(1)}$ and $V^{(2)}$ are disjoint except for these identifications. So $V^{(1)} \cap V^{(2)} = G^\circ \times \Upsilon^\circ$. Now define V^+ and B^+ as in Subsection 5.1.4.

Suppose $u_0, v_0 \in V^+$ and $z \in \mathbb{C}$, and that we want to evaluate $B^+_{v_0 \leftarrow u_0}(z)$. We assume, as in Theorem 5.1.4:9, that $|z| < \min(\frac{1}{M(B^{(1)})}, \frac{1}{M(B^{(2)})})$. In Subsection 5.1.4 we showed that $B^+_{v_0 \leftarrow u_0}(z) = B^\circ_{v_0 \leftarrow u_0}(1)$ where $B^\circ : V^\circ \times V^\circ \to \mathbb{C}$ and $V^\circ = V' \times \{1, 2\} \cup \{u_0, v_0\}$ are as defined in Definition 5.1.4:7. To find a formula for $B^\circ_{v_0 \leftarrow u_0}(1)$ we show that restricted to $(V' \times \{1, 2\}) \times (V' \times \{1, 2\})$, B° is the weighted adjacency function and operator of a lattice on $(G^\circ, \Upsilon^\circ \times \{1, 2\})$.

It is clear from the definition of B° that

$$\begin{aligned} B^\circ_{v_0 \leftarrow u_0}(1) = & [u_0 = v_0] + B^{(1)*}_{v_0 \leftarrow u_0}(z) + B^{(2)*}_{v_0 \leftarrow u_0}(z) + \\ & \sum_{i_1, i_2, w_1, w_2} B^{(i_2)*}_{v_0 \leftarrow w_2}(z) B^\circ_{(w_2, 3 - i_2) \leftarrow (w_1, i_1)}(1) B^{(i_1)*}_{w_1 \leftarrow u_0}(z). \end{aligned} \quad (1)$$

Also any walk with non-zero power in (V°, B°) from (w_1, i_1) to $(w_2, 3 - i_2)$ can only go through elements of $V \times \{1, 2\}$, so $B^\circ_{(w_2, 3 - i_2) \leftarrow (w_1, i_1)}(1)$ is unchanged if we restrict B° to $(V' \times \{1, 2\}) \times (V' \times \{1, 2\})$.

To handle the (possibly infinite) sum in this expression, and certain others that will arise, we need to set up some additional machinery for handling characters in G° and $G^{(i)}$, and prove Lemma 7.2:8.

§7.1. G' as the set of Characters of G

Let G be any finitely-generated abelian group.

Definition 7.1:1. *A Character of* G *is a function* $\gamma : G \to \Pi$ *such that for any* $g_1, g_2 \in G$, $\gamma(g_1 + g_2) = \gamma(g_1)\gamma(g_2)$.

By virtue of Definition 6.4:1, (3), we can regard any element of G' as a character. Furthermore, given any character γ and any basis $g_1, \ldots, g_{r+s}$ of G, we can identify γ with the element of G' (with respect to this basis) $(\gamma(g_1), \ldots, \gamma(g_{r+s}))$. This follows as if g_j has finite order k_j, then $1 = \gamma(0) = \gamma(kg_j) = (\gamma(g_j))^k$, so $\gamma(g_j)$ is a k^{th} root of 1. So the correspondence between G' and the characters of G is a bijection.

From now on we shall identify G' and the characters of G according to this correspondence except where stated. Our next task is to define various concepts in a way which is independent of which basis is being used.

Definition 7.1:2. *For each* $g \in G$, *we define* $f_g : G' \to \mathbb{C}$ *by* $f_g(\gamma) = \gamma(g)$.

Definition 7.1:3. *Let the* Character Topology *on the characters of* G *be the* Weak Topology generated by the set of functions $\{f_g \mid g \in G\}$, *where that is as defined in* [BOLLOBÁS,1990], Chapter 8.

Informally the character topology is the weakest possible topology (that is, with the fewest open sets) such that all the functions f_g are continuous (that is, for open $U \subseteq \mathbb{C}$, $f_g^{-1}(U)$ is open).

However we already have a topology for G' given any basis $g_1, \ldots, g_{r+s}$ of G, which is that induced by the ordinary topology in $\mathbb{C}^{r+s}$. In this section we will call this a *Basis Topology*. We now show that this is the same as the topology of Definition 7.1:3.

Lemma 7.1:4. *The Basis Topology is identical to the Character Topology.*

Proof. Every f_g is continuous in the Basis Topology, so every set open in the character topology is open in the basis topology. Hence it is enough to show that for every set U open in the basis topology, and for every $\gamma \in U$, there is a U_γ, $\gamma \in U_\gamma \subseteq U$, such that U_γ is open in the character topology, since then $\bigcup_{\gamma \in U} U_\gamma = U$ is open in the character topology. Suppose $\gamma = (\gamma_1, \ldots, \gamma_{r+s})$; we can find $\epsilon > 0$ such that for all $\gamma' = (\gamma'_1, \ldots, \gamma'_{r+s}) \in G'$ satisfying $\forall j \, |\gamma'_j - \gamma_j| < \epsilon$, $\gamma' \in U$. Then $U_\gamma \stackrel{\text{def}}{=} \bigcap f_{g_j}^{-1}\{z \mid |z - \gamma_j| < \epsilon\} \subseteq U$, contains γ, and must be open in the character topology.

From now on therefore, we shall use only this topology on G', and so can use terms such as 'open', 'converges to' and 'continuous', for G', without ambiguity.

Lemma 7.1:5. G' *is compact.*

Proof. G', as we originally defined it, is a closed subset of Π^{r+s}.

We now consider functions $G' \to \mathbb{C}$.

Definition 7.1:6. *Let* $C(G')$ *be the set of continuous functions on* G'.
Recall that for $f \in C(G')$ we defined $\|f\|_\infty \stackrel{\text{def}}{=} \sup_x |f(x)|$ (in Definition 4:2). This is finite for any $f \in C(G')$ since G' is compact. Clearly $\| \|_\infty$ is a norm on $C(G')$. For $A \subseteq C(G')$, write $\bar{A}$ for the closure of A with respect to this norm.

Lemma 7.1:7. *Let* A *be the linear span of the functions* f_g. *Then* $\bar{A} = C(G')$.
Proof. This is an easy application of the Stone-Weierstrass Theorem for complex functions ([BOLLOBÁS,1990], Chapter 6, Theorem 10). G' is compact. The f_g are closed under pointwise conjugation ($\overline{f_g(\gamma)} = f_{-g}(\gamma)$) and multiplication ($f_g(\gamma) f_h(\gamma) = f_{g+h}(\gamma)$), and so $\bar{A}$ is a closed subalgebra of $C(G)$; it strongly separates the points of G', in the sense of [BOLLOBÁS,1990], because the f_g do, and so the lemma follows.

Lemma 7.1:8. *If* $F \in C(G')$ *and* $\|F\|_\infty \leq \epsilon$ *then for any basis of* G,

$$\left| \int F(\gamma)\, d\gamma \right| \leq \epsilon.$$

Proof.

$$\left| \int F(\gamma)\, d\gamma \right| \leq \int |F(\gamma)|\, d\gamma \leq \int \epsilon\, d\gamma.$$

It is then trivial from Definition 6.4:21 that this is at most ϵ.

The following lemma is not perhaps strictly required, but is good practice for Lemma 7.2:6.

Lemma 7.1:9. *Suppose that for* $i = 1, 2$, $g_1^i, \ldots, g_{r+s}^i$ *is a basis for* G, *and* $\int^i_{G'} d\gamma$ *is defined as in* Definition 6.4:21 *with respect to it. Then for any continuous function* $f \in C(G')$,

$$\int^1_{G'} f(\gamma)\, d\gamma = \int^2_{G'} f(\gamma)\, d\gamma. \tag{2}$$

Proof. First suppose $f = f_g$ for some $g \in G$. Then $f(\gamma) = \gamma(g)$ and by Lemma 6.6:1

$$\int_{G'}^{1} f(\gamma)\, d\gamma = [g = 0] = \int_{G'}^{2} f(\gamma)\, d\gamma.$$

So the lemma is true if $f = f_g$.

Now, as in Lemma 7.1:7, let A be the linear span of the functions f_g, and suppose $f \in A$. Then clearly the lemma is true.

Now suppose $f \in C(G')$, and $\epsilon > 0$ is given. Since $\bar{A} = C(G')$, there must be an $f - F \in A$ where $||F||_\infty \leq \epsilon$. So we find that

$$\int_{G'}^{1} (f - F)(\gamma)\, d\gamma = \int_{G'}^{2} (f - F)(\gamma)\, d\gamma.$$

and by using Lemma 7.1:8 we know that for $i = 1, 2$

$$\left| \int_{G'}^{i} F(\gamma)\, d\gamma \right| \leq \epsilon.$$

So the difference between the left and right sides of Display (2) is at most 2ϵ. As ϵ can be arbitrarily small, the lemma follows.

Hence for any $f \in C(G')$,

$$\int f(\gamma)\, d\gamma$$

can be defined independently of the basis chosen for G.

§**7.2.** Combining Characters of G° and $G^{(i)}$

Let i be equal to 1 or 2. Recall that G° and $G^{(i)}$ are finitely-generated abelian groups, and G° is a subgroup of $G^{(i)}$. Suppose that $\gamma^\circ \in (G^\circ)'$ and $\gamma^{(i)} \in (\frac{G^{(i)}}{G^\circ})'$. We define a character $\gamma^\circ \oplus \gamma^{(i)}$ of $G^{(i)}$, as follows: $(\gamma^\circ \oplus \gamma^{(i)})(g) = \gamma^\circ(g)\gamma^{(i)}(g)$, where γ° and $\gamma^{(i)}$ are extended to become functions on the whole of $G^{(i)}$, and are both characters of $G^{(i)}$. Then clearly $\gamma^\circ \oplus \gamma^{(i)}$ is a character of $G^{(i)}$.

We now have to extend the definitions of γ° and $\gamma^{(i)}$ to the whole of $G^{(i)}$, and show that they are then characters of $G^{(i)}$. $\gamma^{(i)}$ is easier, so we do it first.

Definition 7.2:1. *For* $g \in G^{(i)}$ *and* $\gamma^{(i)} \in (\frac{G^{(i)}}{G^\circ})'$,

$$\gamma^{(i)}(g) \stackrel{\text{def}}{=} \gamma^{(i)}(g + G^\circ).$$

Clearly this turns $\gamma^{(i)}$ into a character of $G^{(i)}$. To turn a character of G° into a character of $G^{(i)}$, we need to make a number of choices and set up some machinery.

Let $h_1 + G^\circ, \ldots, h_t + G^\circ$ be a basis of $\frac{G^{(i)}}{G^\circ}$. We will combine characters using the h_j.

Lemma 7.2:2. *Let* $g \in G^{(i)}$. *Then we can write*

$$g = g^\circ + \sum l^j h_j \tag{3}$$

where $g^\circ \in G^\circ$ *and for all* j, $l^j \in \mathbb{Z}$; *furthermore the cosets* $l^j h_j + G^\circ$ *are unique.*

Proof. $g + G^\circ \in \frac{G^{(i)}}{G^\circ}$. So $g + G^\circ$ can be written as $\sum l^j(h_j + G^\circ)$ where the $l^j(h_j + G^\circ) = l^j h_j + G^\circ$ are unique. Also $g + G^\circ = \sum l^j(h_j + G^\circ)$ if and only if $g - \sum l^j h_j \in G^\circ$, so if and only if there is a $g^\circ \in G^\circ$ such that $g = g^\circ + \sum l^j h_j$. The lemma follows.

As an example, suppose that $G^{(i)}$ is the additive group of $\mathbb{Z}$, and G° is the subgroup of even integers. Then $\frac{G^{(i)}}{G^\circ}$ is the cyclic group of order 2, and is generated by $h + G^\circ$ where h is any odd integer. If we have $g \in G^{(i)}$ and want to write $g = g^\circ + lh$ where $g^\circ \in G^\circ$ and $l \in \mathbb{Z}$, we can choose l to be any integer with the same parity as g, so the representation is a long way from being unique. In particular, $G^{(i)}$ is not a direct product of G° with any other subgroup.

Definition 7.2:3. *Let n be a positive integer and z a non-zero element of $\mathbb{C}$. Then we define*

$$\sqrt[n]{z}$$

to be the complex number w such that $w^n = z$ and

$$0 \leq \arg w < \frac{2\pi}{n}.$$

Definition 7.2:4. *For $1 \leq j \leq t$, let n^j be the order of $h_j + G^\circ$ in $\frac{G^{(i)}}{G^\circ}$.*

Definition 7.2:5. *Suppose γ° to be a character of G°. We extend it to $G^\circ \cup \{h_1, \ldots, h_t\}$ by defining*

$$\gamma^\circ(h_j) = \begin{cases} 1 & \text{if } n^j = \infty; \\ \sqrt[n^j]{\gamma^\circ(n^j h_j)} & \text{if } n^j < \infty. \end{cases}$$

Note that if $n^j < \infty$, then $n^j h_j \in G^\circ$, so $\gamma^\circ(n^j h_j)$ is defined.

By Lemma 7.2:2, this suffices to determine $\gamma^\circ(g)$ for any $g \in G^{(i)}$, if we also have the condition that γ° is to be a character of $G^{(i)}$. Namely, we must have

$$\gamma^\circ(g^\circ + \sum l^j h_j) = \gamma^\circ(g^\circ) \prod_j (\gamma^\circ(h_j))^{l^j} \tag{4}$$

where $g^\circ \in G^\circ$ and the l^j are integers. However, to define γ° this way, we must show that however we write $g = g^\circ + \sum l^j h_j$, we still obtain the same value for $\gamma^\circ(g)$.

Suppose that for $k = 1, 2$, we have $g = g_k^\circ + \sum l_k^j h_j$, where $g_k^\circ \in G^\circ$ and the l_k^j are integers. It will be enough to show that

$$\gamma^\circ(g_1^\circ) \prod_j (\gamma^\circ(h_j))^{l_1^j} = \gamma^\circ(g_2^\circ) \prod_j (\gamma^\circ(h_j))^{l_2^j}. \tag{5}$$

Dividing the left-hand side by the right-hand side, we need to prove that

$$\gamma^\circ(g_1^\circ - g_2^\circ)\prod_j(\gamma^\circ(h_j))^{l_1^j - l_2^j} = 1.$$

Also, because $g_1^\circ + \sum l_1^j h_j = g_2^\circ + \sum l_2^j h_j$, we have

$$(g_1^\circ - g_2^\circ) + \sum (l_1^j - l_2^j)h_j = 0.$$

So, letting $g^\circ = g_1^\circ - g_2^\circ$, and $l^j = l_1^j - l_2^j$, it is enough to show that if $g^\circ + \sum l^j h_j = 0$, then

$$\gamma^\circ(g^\circ)\prod_j(\gamma^\circ(h_j))^{l^j} = 1. \tag{6}$$

We now show this. By Lemma 7.2:2, we must have $l^j h_j \in G^\circ$ for each j. Therefore, for any j, if $n^j = \infty$, $l^j = 0$; while otherwise $n^j \mid l^j$. In the latter case, let $p_j = n^j h_j$ and $m^j = \frac{l^j}{n^j}$. Then $p_j \in G^\circ$ and $l^j h_j = m^j p_j$. So $g^\circ + \sum_{\substack{j \\ n^j<\infty}} m^j p_j = g^\circ + \sum l^j h_j = 0$, and therefore

$$\begin{aligned}\gamma^\circ(g^\circ)\prod_{\substack{j \\ n^j<\infty}}(\gamma^\circ(p_j))^{m^j} &= \gamma^\circ\left(g^\circ + \sum_{\substack{j \\ n^j<\infty}} m^j p_j\right) \\ &= 1.\end{aligned}$$

as γ° is a character on G°. However we have $\gamma^\circ(p_j) = (\gamma^\circ(h_j))^{n^j}$, from the definition of $\gamma^\circ(h_j)$. Display (6) follows, as does our claim that γ° as defined in Definition 7.2:5 can be uniquely extended to a character. This concludes the process necessary to define $\gamma^\circ \oplus \gamma^{(i)}$.

Lemma 7.2:6. *If* $f : (G^{(i)})' \to \mathbb{C}$ *is continuous, then*

$$\int_{(G^{(i)})'} f(\gamma)\, d\gamma = \int_{(G^\circ)'} \int_{(G^{(i)}/G^\circ)'} f(\gamma^\circ \oplus \gamma^{(i)})\, d\gamma^{(i)}\, d\gamma^\circ. \tag{7}$$

Proof. If $F \in C(G^{(i)\prime})$ and $\|F\|_\infty \leq \epsilon$ then it is easily seen that both $\int F(\gamma)\, d\gamma$ and $\int F(\gamma^\circ \oplus \gamma^{(i)})\, d\gamma^{(i)}\, d\gamma^\circ$ have absolute value at most ϵ. Hence, by the same arguments as we used in Lemma 7.1:9, it is enough to show that Display (7) is true for functions f of the form f_g, where $g \in G^{(i)}$. The left hand side of Display (2) is equal to $[g = 0]$; we need to show that the right hand side is too.

As in Lemma 7.2:2, write $g = g^\circ + \sum l^j h_j$. Consider first

$$\begin{aligned}
\int_{(G^{(i)}/G^\circ)'} f_g(\gamma^\circ \oplus \gamma^{(i)})\, d\gamma^{(i)} &= \int_{(G^{(i)}/G^\circ)'} (\gamma^\circ \oplus \gamma^{(i)})(g)\, d\gamma^{(i)} \\
&= \int_{(G^{(i)}/G^\circ)'} \gamma^\circ(g)\gamma^{(i)}(g)\, d\gamma^{(i)} \\
&= \gamma^\circ(g) \int_{(G^{(i)}/G^\circ)'} \gamma^{(i)}(g + G^\circ)\, d\gamma^{(i)} \\
&= \gamma^\circ(g)[g + G^\circ = G^\circ]. \quad \text{(by Lemma 6.6:1)}
\end{aligned}$$

This is 0 unless $g \in G^\circ$, and the same must be true of the right-hand side of Display (7). Suppose otherwise that $g \in G^\circ$. Then $\gamma^{(i)}(g) = \gamma^{(i)}(G^\circ) = 1$. So in this case the right hand side equals

$$\int_{G^\circ} \gamma^\circ(g)\, d\gamma^\circ = [g = 0],$$

using Lemma 6.6:1 again. This completes the proof of the lemma.

□

Lemma 7.2:7. *For* $i = 1, 2$, *if* h *is a continuous function* $(G^{(i)})' \to \mathbb{C}$,

$$\int_{(G^{(i)}/G^\circ)'} h(\gamma_1 \oplus \gamma^{(i)})\, d\gamma^{(i)}$$

is a continuous function of $\gamma_1 \in (G^\circ)'$.

Proof. Once again we use the Stone-Weierstrass theorem, via Lemma 7.1:7.

If for some $g \in G^{(i)}$ we have $h = f_g$, then

$$\begin{aligned}\int_{(G^{(i)}/G^\circ)'} h(\gamma_1 \oplus \gamma^{(i)})\, d\gamma^{(i)} &= \int_{(G^{(i)}/G^\circ)'} (\gamma_1 \oplus \gamma^{(i)})(g)\, d\gamma^{(i)} \\ &= \gamma_1(g) \int_{(G^{(i)}/G^\circ)'} \gamma^{(i)}(g)\, d\gamma^{(i)} \\ &= \gamma_1(g)[g + G^\circ \;=\; G^\circ].\end{aligned} \tag{8}$$

If $g \notin G^\circ$, then $[g + G^\circ \;=\; G^\circ] = 0$, and so certainly the left hand side of Display (8) is continuous. However if $g \in G^\circ$, the left hand side equals $\gamma_1(g)$, and this too is a continuous function of γ_1. Thus the lemma is true whenever h is an f_g, and thus true for any function which is in the linear span of functions f_g.

Now suppose h to be any continuous function $(G^{(i)})' \to \mathbb{C}$. Let $\epsilon > 0$. Then by Lemma 7.1:7, we can find a h' which is in the linear span of the f_g such that $\|h - h'\|_\infty \leq \epsilon$. So

$$\int_{(G^{(i)}/G^\circ)'} h'(\gamma_1 \oplus \gamma^{(i)})\, d\gamma^{(i)}$$

is a continuous function of γ_1, while for any γ_1

$$\left| \int_{(G^{(i)}/G^\circ)'} h(\gamma_1 \oplus \gamma^{(i)})\, d\gamma^{(i)} - \int_{(G^{(i)}/G^\circ)'} h'(\gamma_1 \oplus \gamma^{(i)})\, d\gamma^{(i)} \right| \;\leq\; \epsilon.$$

Thus the function taking γ_1 to

$$\int_{(G^{(i)}/G^\circ)'} h(\gamma_1 \oplus \gamma^{(i)})\, d\gamma^{(i)}$$

is the uniform limit of continuous functions, and so is continuous itself.

Lemma 7.2:8. *For* $i = 1, 2$, *if* h *is a continuous function* $(G^{(i)})' \to \mathbb{C}$, *and*

$$\sum_{g^\circ \in G^\circ} \left| \int_{(G^{(i)})'} \overline{\gamma(g^\circ)} h(\gamma)\, d\gamma \right| < \infty$$

then for all $\gamma_1 \in (G^\circ)'$

$$\sum_{g^\circ \in G^\circ} \gamma_1(g^\circ) \int_{(G^{(i)})'} \overline{\gamma(g^\circ)} h(\gamma)\, d\gamma = \int_{(G^{(i)}/G^\circ)'} h(\gamma_1 \oplus \gamma^{(i)})\, d\gamma^{(i)}.$$

By Lemma 7.2:7, the function taking γ° to $\int_{(G^{(i)}/G^\circ)'} h(\gamma^\circ \oplus \gamma^{(i)})\, d\gamma^{(i)}$ is continuous. Take h to be this function in Lemma 7.2:9; by Lemma 7.2:6 it is enough to show the following lemma.

Lemma 7.2:9. *If* h *is a continuous function* $(G^\circ)' \to \mathbb{C}$ *and*

$$\sum_{g^\circ \in G^\circ} \left| \int_{(G^\circ)'} \overline{\gamma^\circ(g^\circ)} h(\gamma^\circ)\, d\gamma^\circ \right| < \infty \tag{9}$$

then for all $\gamma_1 \in (G^\circ)'$

$$\sum_{g^\circ \in G^\circ} \gamma_1(g^\circ) \int_{(G^\circ)'} \overline{\gamma^\circ(g^\circ)} h(\gamma^\circ)\, d\gamma^\circ = h(\gamma_1). \tag{10}$$

Proof. Because of Display (9) and as $|\gamma_1(g)| = 1$, the sum on the left-hand side of Display (10) is uniformly absolutely convergent to something for all $\gamma_1 \in (G^\circ)'$. Define a function $h' : (G^\circ)' \to \mathbb{C}$ so that $h'(\gamma)$ is the actual value which this sum converges to, so we want to show that for all $\gamma_1 \in (G^\circ)'$, $h'(\gamma_1) = h(\gamma_1)$.

Choose any enumeration $g_1, \ldots$ of G° – it is easy to construct such an enumeration without using the Axiom of Choice given any basis for G°. It is possible that G° is in fact finite, in which case the sum we are trying to show is equal to $h(\gamma_1)$ has finitely many terms and this enumeration will terminate. This case is simpler than that in which G° is infinite and the proof I shall give here will work for this case also provided it is understood that when I refer to the limit of a sequence which is actually finite, the limit is defined to be the last element of this sequence.

For each positive integer $n \le |G^\circ|$ define a function $h_n : (G^\circ)' \to \mathbb{C}$ by

$$h_n(\gamma_1) \stackrel{\text{def}}{=} \sum_{r=1}^{n} \gamma_1(g_r) \int_{(G^\circ)'} \overline{\gamma^\circ(g_r)} h(\gamma^\circ)\, d\gamma^\circ .$$

The h_n converge uniformly to h' (or, if $|G^\circ| < \infty$, $h_{|G^\circ|} = h'$). Also for any $g_r \in G^\circ$, the function taking γ_1 to $\gamma_1(g_r)$ is continuous, and so each h_n is continuous. By a standard argument it follows that h' is continuous. So the lemma follows from

$$\int |h'(\gamma_1) - h(\gamma_1)|^2\, d\gamma_1 = 0.$$

As the h_n converge uniformly to h', it suffices to show that

$$\lim_{n \to \infty} \int |h_n(\gamma_1) - h(\gamma_1)|^2\, d\gamma_1 = 0. \tag{11}$$

In fact I believe this to be a standard result in multi-dimensional Fourier Analysis, and it is in any case a straightforward generalisation of [Bollobás, 1990], Chapter 10, Display (2). So I shall only sketch the proof.

Definition 7.2:10.

(1) *For continuous functions* $f, g : (G^\circ)' \to \mathbb{C}$, *define* $\langle f, g\rangle = \int f(\gamma)\overline{g(\gamma)}\, d\gamma$. *Define* $\|f\| = \sqrt{\langle f, f\rangle}$. *Using the Cauchy-Schwarz inequality, it is easy to show that* $\| \ \|$ *is a norm, and so with it the set of continuous functions* $: (G^\circ)' \to \mathbb{C}$ *becomes a normed vector space.*

(2) *Define* $L_2((G^\circ)')$ *to be Hilbert Space obtained as the completion of this normed vector space with inner product* $\langle\ ,\ \rangle$.

By Lemma 7.1:7 any continuous function on $(G^\circ)'$ can be obtained as a uniform, and so $\| \ \|$, limit of linear combinations of the f_g. Therefore, with respect to $\| \ \|$, the closed linear span of the f_g is $L_2((G^\circ)')$. However it is an immediate consequence of Lemma 6.6:1 that the f_g are in fact orthonormal under $\langle\ ,\ \rangle$. By [Bollobás,1990], Chapter 10, Theorem 8, $h = \sum_{g \in G} \langle h, f_g\rangle f_g$, where this infinite sum converges absolutely in the norm obtained from $\langle\ ,\ \rangle$. So

$$\lim_n \langle h - \sum_{r=1}^{n} \langle h, f_{g_r}\rangle f_{g_r}, h - \sum_{r=1}^{n} \langle h, f_{g_r}\rangle f_{g_r}\rangle = 0.$$

This is exactly the same as the Display (11), after expansion of $\langle\ ,\ \rangle$, f_g, and h_n.

Thus the lemma is proved. □

§7.3. The Method

We now consider the problem of finding a formula for $B^{\circ}_{(w_2,3-i_2)\leftarrow(w_1,i_1)}(1)$. First we find a directed lattice weighting f° on $(G^{\circ}, \Upsilon \times \{1,2\})$ such that B° is the weighted adjacency function and operator of this directed weighted lattice.

Recall (Definition 5.1.4:7, (2.1)) that we defined $B^{\circ}_{(x,i)\leftarrow(w,3-i)} = B^{(i)*}_{x\leftarrow w}(z)$ and $B^{\circ}_{(x,i)\leftarrow(w,i)} = 0$.

Definition 7.3:1.

$$(B^{(i)\gamma})^*(z) \stackrel{\text{def}}{=} zB^{(i)\gamma}(I - zB^{(i)\gamma})^{-1}.$$

By Theorem 6.6:10 (2), for $g_0, g \in G^{(i)}$ and $\upsilon_1, \upsilon_2 \in \Upsilon^{(i)}$ we have

$$B^{(i)*}_{(g_0+g,\upsilon_2)\leftarrow(g_0,\upsilon_1)}(z) = \int \langle (B^{(i)\gamma})^*(z)\left(e^{(g_0,\upsilon_1)}\right)^{\gamma}, \left(e^{(g_0+g,\upsilon_2)}\right)^{\gamma} \rangle \, d\gamma$$

By Definition 6.4:10 (3), $(e^{(g_0,\upsilon_1)})^{\gamma}$ is the vector with υ_1 entry $\overline{\gamma(g_0)}$ and other entries 0; similarly for $(e^{(g_0+g,\upsilon_2)})^{\gamma}$. Hence

$$\begin{aligned} &= \int \overline{\gamma(g_0)}\gamma(g_0+g)\left((B^{(i)\gamma})^*(z)\right)_{\upsilon_2\upsilon_1} d\gamma \\ &= \int \gamma(g)\left((B^{(i)\gamma})^*(z)\right)_{\upsilon_2\upsilon_1} d\gamma. \end{aligned}$$

So if $g_0, g \in G^{\circ}$ and $\upsilon_1, \upsilon_2 \in \Upsilon^{\circ}$ we have

$$\begin{aligned} &B^{(i)*}_{\theta^{(i)}(g_0+g,\upsilon_2)\leftarrow\theta^{(i)}(g_0,\upsilon_1)}(z) \\ &\quad = B^{(i)*}_{(g_0+g+\phi^{(i)}(\upsilon_2),\psi^{(i)}(\upsilon_2))\leftarrow(g_0+\phi^{(i)}(\upsilon_1),\psi^{(i)}(\upsilon_1))} \\ &\quad = \int \gamma(g+\phi^{(i)}(\upsilon_2)-\phi^{(i)}(\upsilon_1))\left((B^{(i)\gamma})^*(z)\right)_{\psi^{(i)}(\upsilon_2)\psi^{(i)}(\upsilon_1)} d\gamma. \end{aligned}$$

Therefore if we define

$$\begin{aligned} &f^{\circ}(g,(\upsilon_1,i'),(\upsilon_2,i)) \\ &\quad = \begin{cases} \int \gamma(g+\phi^{(i)}(\upsilon_2)-\phi^{(i)}(\upsilon_1)) & \\ \qquad \left((B^{(i)\gamma})^*(z)\right)_{\psi^{(i)}(\upsilon_2)\psi^{(i)}(\upsilon_1)} d\gamma & \text{if } i' \neq i; \\ 0 & \text{if } i' = i. \end{cases} \end{aligned}$$

then f° is the required directed lattice weighting provided that $\sum |f^\circ| < \infty$. This follows because $M(B^\circ) < \infty$, by Theorem 5.1.4:9.

We now find a formula for $(B^\circ)^\gamma_{(\upsilon_2,i_2)(\upsilon_1,i_1)}$, where $\gamma \in (G^\circ)'$ and $\upsilon_1, \upsilon_2 \in \Upsilon^\circ$. From Definition 6.6:4 we have

$$(B^\circ)^\gamma_{(\upsilon_2,i_2)(\upsilon_1,i_1)} = \sum_{g \in G^\circ} \overline{\gamma(g)} f^\circ(g, (\upsilon_1, i_1), (\upsilon_2, i_2))$$

This is 0 if $i_1 = i_2$; we assume $i_1 \neq i_2 = i$.

$$\begin{aligned} &= \sum_{g \in G^\circ} \overline{\gamma(g)} \int_{(G^{(i)})'} \gamma'(g + \phi^{(i)}(\upsilon_2) - \phi^{(i)}(\upsilon_1)) \\ &\quad \left((B^{(i)\gamma'})^*(z)\right)_{\psi^{(i)}(\upsilon_2)\psi^{(i)}(\upsilon_1)} d\gamma' \\ &= \sum_{g \in G^\circ} \overline{\gamma(g)} \int_{(G^{(i)})'} \gamma'(g)\gamma'(\phi^{(i)}(\upsilon_2) - \phi^{(i)}(\upsilon_1)) \\ &\quad \left((B^{(i)\gamma'})^*(z)\right)_{\psi^{(i)}(\upsilon_2)\psi^{(i)}(\upsilon_1)} d\gamma' \end{aligned} \tag{12}$$

To evaluate this sum we are going to use Lemma 7.2:8. Define $h : (G^{(i)})' \to \mathbb{C}$ by

$$h(\gamma') \stackrel{\text{def}}{=} \gamma'(\phi^{(i)}(\upsilon_2) - \phi^{(i)}(\upsilon_1)) \left((B^{(i)\gamma'})^*(z)\right)_{\psi^{(i)}(\upsilon_2)\psi^{(i)}(\upsilon_1)}.$$

To apply Lemma 7.2:8, we need to show that h is continuous and that the sum in Display (9) is finite.

We assumed that $|z| < \min(\frac{1}{M(B^{(1)})}, \frac{1}{M(B^{(2)})})$. By Lemma 6.6:5, $|z| < \frac{1}{M(B^{(i)\gamma'})}$ and so by Theorem 3:3, $|z| < \frac{1}{\|(B^{(i)})\|^{\gamma'}}$, and so $\|z(B^{(i)})^{\gamma'}\| < 1$. By Theorem 3:5, $I - z(B^{(i)})^{\gamma'}$ is invertible for all $\gamma' \in (G^{(i)})'$. By Cramer's Rule, we can express each entry of its inverse as a rational function of the complex numbers on the unit circle which make up γ' and their conjugates, and because $I - z(B^{(i)})^{\gamma'}$ is invertible for all $\gamma' \in (G^{(i)})'$, the denominator of this rational function is never 0. So the inverse of $I - z(B^{(i)})^{\gamma'}$ is a continuous function of γ'. And by Definition 7.3:1, so is h.

To show that the sum in Display (9) is finite, we need to show that

$$\sum_{g \in G^\circ} \left| \int_{(G^{(i)})'} \gamma'(g)\gamma'(\phi^{(i)}(\upsilon_2) - \phi^{(i)}(\upsilon_1)) \left((B^{(i)\gamma'})^*(z)\right)_{\psi^{(i)}(\upsilon_2)\psi^{(i)}(\upsilon_1)} d\gamma' \right| < \infty.$$

However (reversing the steps by which we arrived at these integrals) this is equal to $\sum_{g\in G^\circ} |B^\circ_{(g_0+g,v_2,i)\leftarrow(g_0,v_1,3-i)}|$, where g_0 is any element of G°, and so is at most $M(B^\circ)$, which we know to be finite.

From Display (12) we have

$$\begin{aligned}
(B^\circ)^\gamma_{(v_2,i)(v_1,3-i)} &= \sum_{g\in G^\circ} \overline{\gamma(g)} \int_{(G^{(i)})'} \gamma'(g) h(\gamma')\, d\gamma' \\
&= \sum_{g^\circ\in G^\circ} \gamma(g^\circ) \int_{(G^{(i)})'} \overline{\gamma'(g^\circ)} h(\gamma')\, d\gamma \quad (g^\circ = -g) \\
&= \int_{(G^{(i)}/G^\circ)'} h(\gamma \oplus \gamma_2)\, d\gamma_2 \quad \text{(Lemma 7.2:8)} \\
&= \int_{(G^{(i)}/G^\circ)'} (\gamma \oplus \gamma_2)(\phi^{(i)}(v_2) - \phi^{(i)}(v_1)) \\
&\qquad \left((B^{(i)\gamma\oplus\gamma_2})^*(z)\right)_{\psi^{(i)}(v_2)\psi^{(i)}(v_1)} d\gamma_2.
\end{aligned}$$

So we have expressions for the entries of $(B^\circ)^\gamma$ in which each is 0 or an integral over one of the $(\frac{G^{(i)}}{G^\circ})'$.

However we want to find an expression for $B^+_{v_0\leftarrow u_0}(z) = B^\circ_{v_0\leftarrow u_0}(1)$. By Display (1) we have

$$\begin{aligned}
B^\circ_{v_0\leftarrow u_0}(1) =& [u_0 = v_0] + {B^{(1)}}^*_{v_0\leftarrow u_0}(z) + {B^{(2)}}^*_{v_0\leftarrow u_0}(z) + \\
& \sum_{i_1,i_2,w_1,w_2} {B^{(i_2)}}^*_{v_0\leftarrow w_2}(z) B^\circ_{(w_2,3-i_2)\leftarrow(w_1,i_1)}(1) {B^{(i_1)}}^*_{w_1\leftarrow u_0}(z).
\end{aligned}$$

Define $x, y \in H(V' \times \{1,2\})$ by

$$x_{(w_1,i_1)} = \overline{{B^{(i_1)}}^*_{w_1\leftarrow u_0}(z)}$$

and

$$y_{(w_2,3-i_2)} = {B^{(i_2)}}^*_{v_0\leftarrow w_2}(z).$$

So

$$B^\circ_{v_0\leftarrow u_0}(1) = [u_0 = v_0] + {B^{(1)}}^*_{v_0\leftarrow u_0}(z) + {B^{(2)}}^*_{v_0\leftarrow u_0}(z) + \langle B^\circ(1)y, x\rangle.$$

Also, because $M(B^\circ) < \infty$, $x, y \in \ell_1(V' \times \{1,2\})$. So by Theorem 6.6:9,

$$\langle B^\circ(1)y, x\rangle = \int \langle (I - (B^\circ)^\gamma)^{-1} y^\gamma, x^\gamma\rangle \, d\gamma.$$

We already have found expressions for the entries of $(B^\circ)^\gamma$. For the entries of y^γ and x^γ we again use Lemma 7.2:8.

For $\gamma \in (G^\circ)'$ and $\upsilon \in \Upsilon^\circ$ we have $(x^\gamma)_{(\upsilon,i)}$

$$\begin{aligned} &= \langle x, \chi^{(\gamma,(\upsilon,i))}\rangle \\ &= \sum_{g \in G^\circ} \overline{\gamma(g)} x_{(g,\upsilon,i)} \\ &= \sum_{g \in G^\circ} \overline{\gamma(g) B^{(i)*}_{(g+\phi^{(i)}(\upsilon), \psi^{(i)}(\upsilon)) \leftarrow u_0}(z)}. \end{aligned}$$

If $u_0 \notin V^{(i)}$ this is 0. Otherwise suppose $u_0 = (g_u, \upsilon_u) \in V^{(i)}$. Then $(x^\gamma)_{(\upsilon,i)}$

$$\begin{aligned} &= \sum_{g \in G^\circ} \overline{\gamma(g)} \overline{\int_{(G^{(i)})'} \langle ((B^{(i)})^{\gamma'*}(z) e^{(g_u,\upsilon_u)^{\gamma'}}, e^{(g+\phi^{(i)}(\upsilon), \psi^{(i)}(\upsilon))^{\gamma'}}\rangle \, d\gamma'} \\ &= \sum_{g \in G^\circ} \overline{\gamma(g)} \overline{\int_{(G^{(i)})'} ((B^{(i)})^{\gamma'*}(z))_{\psi^{(i)}(\upsilon)\upsilon_u} \overline{\gamma'(g_u)} \gamma'(g + \phi^{(i)}(\upsilon)) \, d\gamma'} \end{aligned}$$

Write $\overline{\gamma}$ for the element of $(G^\circ)'$ which is obtained by conjugating the co-ordinates of γ.

$$= \sum_{g \in G^\circ} \overline{\gamma}(g) \int_{(G^{(i)})'} \overline{\gamma'(g) \left((B^{(i)})^{\gamma'*}(z)\right)_{\psi^{(i)}(\upsilon)\upsilon_u} \gamma'(\phi^{(i)}(\upsilon) - g_u)} \, d\gamma'$$

This satisfies the conditions of Lemma 7.2:8 by a similar argument to that used before. So

$$= \int_{(G^{(i)}/G^\circ)'} \overline{\left((B^{(i)})^{\overline{\gamma}\oplus\gamma_2 *}(z)\right)_{\psi^{(i)}(\upsilon)\upsilon_u} (\overline{\gamma} \oplus \gamma_2)(\phi^{(i)}(\upsilon) - g_u)} \, d\gamma_2.$$

So

$$(x^\gamma)_{(\upsilon,i)} = \begin{cases} \displaystyle\int_{(G^{(i)}/G^\circ)'} \overline{\left((B^{(i)})^{\overline{\gamma}\oplus\gamma_2 *}(z)\right)_{\psi^{(i)}(\upsilon)\upsilon_u}} \\ \qquad \overline{(\overline{\gamma} \oplus \gamma_2)(\phi^{(i)}(\upsilon) - g_u)} \, d\gamma_2 & \text{if } u_0 = (g_u, \upsilon_u) \in V^{(i)}; \\ 0 & \text{if } u_0 \notin V^{(i)}. \end{cases}$$

Similarly we deduce that for $\gamma \in (G^\circ)'$ and $\upsilon \in \Upsilon^\circ$, $(y^\gamma)_{(\upsilon,i)} =$

$$\begin{cases} \displaystyle\int_{(G^{(3-i)}/G^\circ)'} \left((B^{(3-i)})^{\overline{\gamma}\oplus\gamma_2 *}(z)\right)_{\upsilon_\nu \psi^{(3-i)}(\upsilon)} \\ \qquad (\overline{\gamma}\oplus\gamma_2)(g_\nu - \phi^{(3-i)}(\upsilon))\, d\gamma_2 & \text{if } \nu_0 = (g_\nu, \upsilon_\nu) \in V^{(3-i)}; \\ 0 & \text{if } \nu_0 \notin V^{(3-i)}. \end{cases}$$

The other terms in Display (1) are $[u_0 = \nu_0]$, $B^{(1)*}_{\nu_0 \leftarrow u_0}(z)$ and $B^{(2)*}_{\nu_0 \leftarrow u_0}(z)$. $[u_0 = \nu_0]$ is trivial, while the other two terms can be deduced from the formulae for walk-generating functions for the weighted lattices $(G^{(i)}, \Upsilon^{(i)}, f^{(i)})$.

We now give two examples of problems where this method may be applied.

§7.4. Example: The Grid with Every Other edge in a line removed

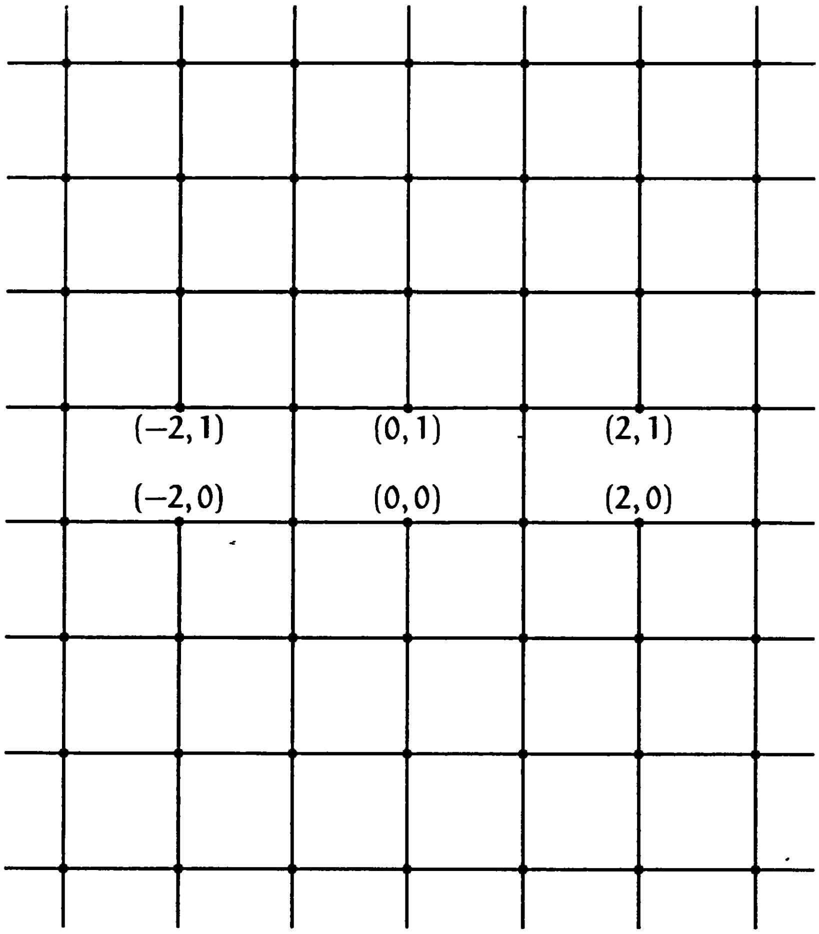

Figure 3. The Grid with every other Edge in a Line deleted

Take the two-dimensional grid, defined at the beginning of this thesis and for which walk-generating functions of the associated adjacency operator B were found in Subsection 6.7.3. Then define a new operator B' by $B'_{v\leftarrow u} = B_{v\leftarrow u}$ unless $\{u,v\} = \{(2k,0),(2k,1)\}$ for some $k \in \mathbb{Z}$, in which case $B'_{v\leftarrow u} = 0$. A picture of this is shown in Figure 3.

We use the construction as given at the start of Chapter 7 as follows.

Define $G^{(1)}$ to be the additive group of $\{(x,y) \mid x,y \in \mathbb{Z}\}$. Define $\Upsilon^{(1)} = \{\upsilon^{(1)}\}$. Define $L' = \{(0,1),(0,-1),(1,0),(-1,0)\}$, define $L = \{(g,\upsilon^{(1)},\upsilon^{(1)}) \mid g \in L'\}$, and define the lattice weighting $f^{(1)}$ on $(G^{(1)},\Upsilon^{(1)})$ as the characteristic function of L. Thus $(G^{(1)},\Upsilon^{(1)},L)$ is the two-dimensional grid, as described at the start of this thesis, and $(G^{(1)},\Upsilon^{(1)},f^{(1)})$ is the corresponding weighted lattice.

Define $G^{(2)}$ to be the additive group of $\{(x,0) \mid 2 \text{ divides } x\}$. Define $\Upsilon^{(2)} = \{\upsilon_1^{(2)},\upsilon_2^{(2)}\}$. Define

$$f^{(2)}((x,0),\upsilon_1,\upsilon_2) = \begin{cases} -1 & \text{if } x=0 \text{ and } \{\upsilon_1,\upsilon_2\} = \{\upsilon_1^{(2)},\upsilon_2^{(2)}\}; \\ 0 & \text{otherwise.} \end{cases}$$

We must have $G^\circ = G^{(1)} \cap G^{(2)} = G^{(2)}$. Define $\Upsilon^\circ = \{\upsilon_1^\circ,\upsilon_2^\circ\}$.

Define $\phi^{(1)}(\upsilon_1^\circ) = (0,0)$. Define $\phi^{(1)}(\upsilon_2^\circ) = (0,1)$. Define $\psi^{(1)}(\upsilon_1^\circ) = \psi^{(1)}(\upsilon_2^\circ) = \upsilon^{(1)}$.

Define $\phi^{(2)}(\upsilon_1^\circ) = \phi^{(2)}(\upsilon_2^\circ) = (0,0)$. Define $\psi^{(2)}(\upsilon_1^\circ) = \upsilon_1^{(2)}$. Define $\psi^{(2)}(\upsilon_2^\circ) = \upsilon_2^{(2)}$.

It can now be shown routinely that the construction at the start of Chapter 7 does produce an operator B^+ equal to B', and so the method there can be applied. I shall not work through the method of Section 7.3 here, as this is a purely mechanical operation; however I will summarise what will happen. We have to find formulae for the matrix $(B^\circ)^\gamma$ and also $(B^\circ)^{\gamma*}(1) = (B^\circ)^\gamma(I-(B^\circ)^\gamma)^{-1}$. $(B^\circ)^\gamma$ will have a row and a column for each pair $(\upsilon \in \Upsilon^\circ, i \in \{1,2\})$, and so the dimension of $(B^\circ)^\gamma$ is $|\Upsilon^\circ| \times 2$, or 4. Also $(B^\circ)^\gamma_{(\upsilon_2,i_2)(\upsilon_1,i_1)}$ is 0 if $i_1 = i_2$, so if we arrange the rows and columns of $(B^\circ)^\gamma$ in order $(\upsilon_1^\circ,1),(\upsilon_2^\circ,1),(\upsilon_1^\circ,2),(\upsilon_2^\circ,2)$, $(B^\circ)^\gamma$ will look like

$$\begin{pmatrix} 0 & A_1 \\ A_2 & 0 \end{pmatrix},$$

where A_1 and A_2 are 2×2 matrices with entries corresponding to values of the walk-generating functions at z for $B^{(1)}$ and $B^{(2)}$. The entries derived from $B^{(2)}$ are easy to compute, as $G^{(2)} = G^{\circ}$, and the walk-generating functions of $B^{(2)}$ can themselves be found by an elementary calculation. However the entries derived from $B^{(1)}$ are integrals over $(\frac{G^{(1)}}{G^{\circ}})' = \Pi$ of expressions themselves involving the walk-generating functions of $B^{(1)}$, so there is no reason why there should be simple expressions for them.

Furthermore, to find formulae for the walk-generating functions of B', we have to find $(B^{\circ})^{\Upsilon *}(1)$, so have to invert $(I - (B^{\circ})^{\Upsilon})$. This involves a further layer of complication.

So even for extremely simple examples like this one, it turns out that finding expressions for the walk-generating functions is quite a complicated operation, and the final expressions are liable to contain integrals within integrals.

§7.5. Example: Markov Chains with Two Interacting Particles

Suppose we have a finitely-generated abelian group G and a finite set Υ. Two particles X and Y move about $G \times \Upsilon$. At time $t \in \mathbb{N}$ the position of X is the random variable X_t, and the position of Y is Y_t; we have $X_t, Y_t \in G \times \Upsilon$. The system is to be a Markov Chain with the state at time t given as (X_t, Y_t). Let $p(x_1, y_1; x_2, y_2)$ be the transition probability from $(X_t, Y_t) = (x_1, y_1)$ to $(X_{t+1}, Y_{t+1}) = (x_2, y_2)$. Define a function B^+ by $B^+_{(x_2,y_2)\leftarrow(x_1,y_1)} = p(x_1, y_1; x_2, y_2)$, and suppose $M(B^+) < \infty$. Using Theorem 3:3, define a corresponding operator B^+ on $H((G \times \Upsilon) \times (G \times \Upsilon))$. In Section 5.2, we found how to derive certain statistics of the Markov Chain from the walk-generating functions $W_{\nu \leftarrow u}(z)$ derived from $(G \times \Upsilon, B^+)$. So we concern ourselves here only with finding the walk-generating functions.

We may summarise the transition probabilities we shall look at as follows. These will be the sum of two components. The first component will be transition probabilities in which the two particles move independently around $G \times \Upsilon$ with their own transition probabilities, each of which are derived from a directed lattice weighting on (G, Υ). The second component is zero except when the difference between the values of G for the positions of the two particles belongs to a certain finite set, and is a function of the differences at the beginning and end of the transition together with the various elements of Υ involved. So, for example, we might have two particles moving about an n-dimensional grid or other lattice, independently except when they come within a certain distance of each other, in which case the transition probabilities are altered depending upon the positions relative to each other of the two particles.

Formally, we suppose f^x and f^y to be two directed lattice weightings on (G, Υ). Let B^x and B^y be corresponding adjacency functions and operators. These correspond to the transition probabilities when the particles are moving around independently. So we let p_1, the first component of the transition probabilities, be defined by

$$p_1(x_1, y_1; x_2, y_2) \stackrel{\text{def}}{=} (B^x_{x_2 \leftarrow x_1})(B^y_{y_2 \leftarrow y_1}).$$

We also suppose Δ to be a finite subset of G and f^c a directed lattice weighting on $(G, \Delta \times \Upsilon \times \Upsilon)$, and define p_2, the second component of the transition probabilities, by

$$p_2((g^x_1, \upsilon^x_1), (g^y_1, \upsilon^y_1); (g^x_2, \upsilon^x_2), (g^y_2, \upsilon^y_2)) \stackrel{\text{def}}{=} \begin{cases} f^c(g^x_2 - g^x_1, (g^y_1 - g^x_1, \upsilon^x_1, \upsilon^y_1), \\ \quad (g^y_2 - g^x_2, \upsilon^x_2, \upsilon^y_2)) & \text{if } g^y_1 - g^x_1, g^y_2 - g^x_2 \in \Delta; \\ 0 & \text{otherwise.} \end{cases}$$

Now define p by

$$p(x_1, y_1; x_2, y_2) = p_1(x_1, y_1; x_2, y_2) + p_2(x_1, y_1; x_2, y_2).$$

This completes the construction of the Markov Chain.

We now show how to apply the method of Section 7.3. $(V^{(1)}, B^{(1)})$ corresponds to the first component, p_1, of the transition probabilities. Let $G^{(1)} = G \times G$ and $\Upsilon^{(1)} = \Upsilon \times \Upsilon$. Define

$$f^{(1)}((g^x, g^y), (\upsilon^x_1, \upsilon^y_1), (\upsilon^x_2, \upsilon^y_2)) \stackrel{\text{def}}{=} f^x(g^x, \upsilon^x_1, \upsilon^x_2) f^y(g^y, \upsilon^y_1, \upsilon^y_2).$$

$(V^{(2)}, B^{(2)})$ corresponds to the second component, p_2, of the transition probabilities. Let $G^{(2)} = \{(g, g) \mid g \in G\}$. Let $\Upsilon^{(2)} = \Delta \times \Upsilon \times \Upsilon$. Define

$$f^{(2)}((g, g), (\delta_1, \upsilon^x_1, \upsilon^y_1), (\delta_2, \upsilon^x_2, \upsilon^y_2)) = f^c(g, (\delta_1, \upsilon^x_1, \upsilon^y_1), (\delta_2, \upsilon^x_2, \upsilon^y_2)).$$

We must have $G^\circ = G^{(1)} \cap G^{(2)} = G^{(2)}$. Define Υ° to be exactly the same as $\Upsilon^{(2)}$, except that the elements are marked in some way so that Υ° and $\Upsilon^{(2)}$ are disjoint. Define $\phi^{(1)}(\delta, \upsilon^x, \upsilon^y) = (0, \delta)$. Define $\psi^{(1)}(\delta, \upsilon^x, \upsilon^y) = (\upsilon^x, \upsilon^y)$. Define $\phi^{(2)}(\delta, \upsilon^x, \upsilon^y) = (0, 0)$. Define $\psi^{(2)} : \Upsilon^\circ \to \Upsilon^{(2)}$ by $\psi^{(2)}(\delta, \upsilon^x, \upsilon^y) = (\delta, \upsilon^x, \upsilon^y)$.

This completes the necessary definitions for the method in Chapter 7 to be applied. I shall not demonstrate that these definitions do indeed define $B^+_{(x_2,y_2)\leftarrow(x_1,y_1)}$ equal to $p(x_1, y_1; x_2, y_2)$, where B^+ is the operator whose walk-generating functions are found by the method, but this is just a routine matter of checking through the various definitions.

§7.6. There is no method for more than one set of Periodic Modifications

Using the method of Section 7.3 it is possible to find expressions (albeit complex ones) for walk-generating functions when there is one set of periodic modifications. Using the method of Subsection 5.1.4, we can then find expressions for walk-generating functions when we make a finite number of changes to individual $B_{v\leftarrow u}$. It is natural to wonder if we can find expressions for walk-generating functions when there are two sets of periodic modifications. In this subsection I shall show that these can in general make the situation sufficiently complex that useful expressions are unlikely to exist. More specifically, I shall explain how to transform a particular instance of the Halting Problem into the problem of determining whether there are any walks with non-zero power between two particular vertices in a two-dimensional lattice in which two sets of periodic modifications have been made. Furthermore, the power of any walk will necessarily be 0 or 1, so this means that the Halting Problem can be transformed into that of determining if the walk-generating function is non-zero when applied to any positive z, and it makes no difference which positive z we choose.

Specifically I shall show that if there is an algorithm which decides if there is a walk with non-zero power between two given vertices in a two-dimensional lattice with two sets of periodic modifications, then we can modify this algorithm into one which solves the Halting Problem for the *Abacus*, as described and discussed in Chapter 6 to Chapter 8 of [BOOLOS&JEFFREYS,1980].

We describe the Abacus now. A *Program* for the Abacus consists of a function from a finite set Σ of *States* with a designated *Initial State* σ_0 into the set of possible *Instructions*. We speak of a *Situation* of the Abacus. A situation is a pairing of a state together with a function $n \in \mathbb{N}_0 \to [n] \in \mathbb{N}_0$, such that for all but finitely many n, $[n] = 0$. We sometimes refer to $[n]$ as the *Contents of Box* n. A program determines a function M with domain the set of situations and codomain the disjoint union of the set of situations and the set $\{\perp\}$. With this function, we *Execute* the program by the following construction; let s_0 be the situation consisting of σ_0 paired with the function mapping everything to 0, and define the (finite or infinite) sequence of situations $s_0, s_1, \ldots$, in that order, so that if $M(s_k) = \perp$ we terminate the sequence at s_k, and otherwise define $s_{k+1} = M(s_k)$ and repeat. If the sequence terminates we say that the execution *Halts*.

Let $s = (\sigma, f)$ be a situation. $M(s)$ is determined by the instruction associated with the state σ by the program. The instructions are of three types.

(1) The *Halt* instruction returns $\perp$ (and therefore terminates the execution).

(2) *Increment* instructions specify

  (2.1) A *Box* b in $\mathbb{N}_0$.

  (2.2) A state σ'.

  When this instruction is obeyed M returns (σ', g) where $g(b) = f(b) + 1$ and for $b' \neq b$, $g(b') = f(b')$.

(3) *Decrement* instructions specify

  (3.1) A *Box* b in $\mathbb{N}_0$.

  (3.2) Two states σ_Z and σ_P.

  Such an instruction is obeyed as follows. If $f(b) = 0$, M returns (σ_Z, f). If $f(b) > 0$, M returns (σ_P, g) where $g(b) = f(b) - 1$ and for $b' \neq b$, $g(b') = f(b')$.

The *Halting Problem* is that of determining if the execution associated with a given program halts. [BOOLOS&JEFFREYS,1980] gives an algorithm which, given a program for a Turing Machine, outputs a program for the Abacus which will halt if and only if the program for the Turing Machine halts; hence there is no algorithm for solving the halting problem for the Abacus.

In fact we shall strengthen this. Because there are only a finite number of states for any program, only a finite number of instructions are used, and hence only a finite number of boxes b occur in the whole program. We say that the *Arity* of the program is the number of boxes which occur. I shall show that there is no algorithm for solving the halting problem even for programs in which the *Arity* is at most 2, because the only boxes which occur are 0 and 1, where the execution of the program only halts when all boxes contain 0, and where there is only one halt instruction. Call a program satisfying these conditions a *Restricted Program*, then this follows immediately from:

Lemma 7.6:1. *There is an algorithm which, given a program* P, *outputs a restricted program* P′ *such that the execution of* P′ *halts if and only if that of* P *halts.*

Proof. We replace each instruction of P by a set of instructions in P′, adding extra states as necessary. We provide an injection k from situations of P into situations of P′ such that when an instruction is obeyed for P to map one situation s_1 to a new one s_2, the corresponding set of instructions in P′ map $k(s_1)$ to $k(s_2)$. Let the primes for the positive integers be $p_0, p_1, \ldots$. We define k as follows: it maps a situation (σ, f), where σ is a state and $f : \mathbb{N}_0 \to \mathbb{N}_0$, to (σ, f'), where

$$f'(0) = \prod_{i=0}^{\infty} p_i^{f(i)} \tag{13}$$

and $f'(b) = 0$ for all $b > 0$. The product in Display (13) is finite as all but finitely many $f(i)$ are 0. k is injective by the Fundamental Theorem of Arithmetic.

When $f(i)$ is incremented, $f'(0)$ is multiplied by p_i. $f(i)$ is non-zero if and only if $p_i \mid f'(0)$, and if so when $f(i)$ is decremented, $f'(0)$ is divided by p_i.

We now construct the transformation required to turn P into P′. We give six *program fragments* which will be connected together to make P′. In each program fragment we will specify an initial state; program fragments can also refer to exit states, which must be set to initial states of other program fragments upon connection. Four of the program fragments may need to be copied more than once; when this is done it is necessary to make their states disjoint. For each program fragment we give the conditions assumed 'on entry', when the instruction associated with the initial state is obeyed and what subsequent behaviour follows from these conditions, together with names for the exit states; we then specify the instructions of the program fragment, preceding each by the name of the states with which it is associated. Two of the fragments are also variable in a number N.

(1) On entry we assume that $[0] = [1] = 0$. There is a single exit state σ; the execution reaches the situation (σ, f) where $f(0) = 1$ and $f(1) = 0$. The

instructions are (just one this time):

I: An instruction to increment box 0, specifying σ.

(2) On entry we assume that $[1] = 0$. The execution halts with $[0] = 0$. The instructions are:

D: A decrement instruction specifying box 0, specifying $\sigma_Z = H$ and $\sigma_P = D$.

H: A halt instruction.

(3) On entry we assume that $[1] = 0$. There is one exit state σ. If on entry $r = [0]$, then the execution reaches (σ, f), where $f(0) = 0$ and $f(1) = r$.

D: An instruction to decrement box 0, specifying $\sigma_Z = \sigma$, and $\sigma_P = I$.

I: An instruction to increment box 1, specifying D.

(4) On entry we assume that $[0] = 0$. There is one exit state σ. If on entry $r = [1]$, then the execution reaches (σ, f), where $f(0) = r$ and $f(1) = 0$. The instructions for this fragment are identical to those of (3) except that "box 0" and "box 1" are interchanged.

(5) There are no assumptions on entry. There is a single exit state σ. If on entry $r = [0]$ and $m = [1]$, the execution reaches (σ, f) where $f(0) = r + N \times m$ and $f(1) = 0$. N must be a non-negative integer.

D: An instruction to decrement box 1, specifying $\sigma_Z = \sigma$ and $\sigma_P = I_1$.

For $1 \leq j \leq N - 1$:

I_j: An instruction to increment box 0, specifying I_{j+1}.

I_N: An instruction to increment box 0, specifying D.

(6) On entry we assume that $[1] = 0$. There are two exit states, P and Z. Let r be the value of [0] on entry. The program reaches (σ, f); here $\sigma = P$ if N divides r, $\sigma = Z$ if N does not divide r; while $f(1) = \lfloor \frac{r}{N} \rfloor$, and $f(0) = r - Nf(1)$. N must be an integer greater than or equal to 2.

D_0 : An instruction to decrement box 0, specifying $\sigma_Z = P$ and $\sigma_P = D_1$.

For $1 \leq j \leq N-2$:

D_j: An instruction to decrement box 0, specifying $\sigma_Z = I_j$ and $\sigma_P = I_{j+1}$.

D_{N-1}: An instruction to decrement box 0, specifying $\sigma_Z = I_{N-1}$ and $\sigma_P = I$.

I: An instruction to increment box 1, specifying D_0.

I_1: An instruction to increment box 0, specifying Z.

For $2 \leq j \leq N-2$:

I_j: An instruction to increment box 0, specifying I_{j-1}.

If $N > 2$:

I_{N-1}: An instruction to increment box 0, specifying I_{N-2}.

We leave the verification that these program fragments work to the reader. We change P into P′ by performing the following transformations, in order, starting with $P' = P$:

(1) We insert (1) into P′, specifying its initial state to be the initial state of P′, and letting the exit state be the initial state of P.

(2) We insert (2) into P′ and delete all halt instructions originally in P, replacing all references to their associated states with references to the initial state of (2).

(3) For each box b, we replace each increment instruction of box b originally in P by separate copies of (3) and (5), with $N = p_b$, replacing all references to the associated state by a reference to the initial state of (3). The exit state of (3) is the initial state of (5); the exit state of (5) is the state specified by the increment instruction.

(4) For each box b, we replace each decrement instruction of box b originally in P by separate copies of (4), (6) and (5), with $N = p_b$, replacing all references to the associated state by a reference to the initial state of (6). The P exit state of (6) is set to the initial state of (4); the exit state of (4) is set to the σ_P state of the original decrement instruction. The Z exit state of (6) is set to the initial state of (5). The exit state of (5) is set to the σ_Z state of the original decrement instruction.

We now show how to transform a restricted program with state set Σ, initial state σ_0, and single halt instruction associated with state σ_H, into a directed weighted lattice over $(\mathbb{Z}^2, \Sigma)$ with two sets of periodic modifications such that any walk with non-zero power from $((0,0), \sigma_0)$ to $((0,0), \sigma_H)$ corresponds to an execution of the program which halts, and vice-versa.

To do this we identify any situation with state σ, box 0 containing [0], and box 1 containing [1], with the vertex of the lattice $(([0],[1]), \sigma)$.

Define $N : ((\mathbb{N}_0 \times \mathbb{N}_0) \times \Sigma) \times ((\mathbb{N}_0 \times \mathbb{N}_0) \times \Sigma) \to \{0,1\}$ so that $N_{((x_0^2,x_1^2),\sigma_2)\leftarrow((x_0^1,x_1^1),\sigma_1)}$ is 1 if and only if the function M associated with the program maps the situation associated with $((x_0^1, x_1^1), \sigma_1)$ to that associated with $((x_0^2, x_1^2), \sigma_2)$, and is 0 otherwise.

We will define B^+, the adjacency function and operator of a directed weighted lattice with two sets of periodic modifications, so that when $x_0^1, x_1^1, x_0^2, x_1^2 \in \mathbb{N}_0$ and $\sigma_1, \sigma_2 \in \Sigma$, $B^+_{((x_0^2,x_1^2),\sigma_2)\leftarrow((x_0^1,x_1^1),\sigma_1)} = N_{((x_0^2,x_1^2),\sigma_2)\leftarrow((x_0^1,x_1^1),\sigma_1)}$. Thus if $u, v \in (\mathbb{N}_0 \times \mathbb{N}_0) \times \Sigma$, $B^+_{v\leftarrow u} \neq 0$ if and only if $M(u) = v$, in which case it is 1. $M(u) = v$ is equivalent to saying that if the situation u occurs in the execution of the program, v will come next. Any execution which halts must necessarily correspond with a walk with non-zero power (and therefore power equal to 1) starting at $((0,0), \sigma_0)$ and ending at $((0,0), \sigma_H)$ (recall that we have arranged that the abacus only halts when all boxes have content 0). Similarly, any such walk corresponds to an execution which halts. So this suffices.

Of course, there is in fact only one possible execution of a given program, so either the walk generating function from $((0,0), \sigma_0)$ to $((0,0), \sigma_H)$ is identically 0 if the execution doesn't halt, or takes z to some z^n where the execution (considered as a sequence of situations) has length $n+1$.

I have not in fact defined what a lattice with two periodic sets of modifications is, but I do not think such formality is necessary, since it should be clear that what I shall describe is one.

The directed weighted lattice $L^{(2)}$ will be $(\mathbb{Z} \times \mathbb{Z}, \Sigma, f^{(2)})$, where $f^{(2)}((\delta_0, \delta_1), \sigma_1, \sigma_2)$ equals 0 except in the following cases in which it equals 1:

(1) The instruction associated with σ_1 is an increment instruction for box i (which must be 0 or 1) specifying σ_2, and $\delta_i = 1$, $\delta_{1-i} = 0$.

(2) The instruction associated with σ_1 is a decrement instruction for box i, specifying σ_2 as σ_P, and $\delta_i = -1$, $\delta_{1-i} = 0$.

Let $B^{(2)}$ be the adjacency function and operator of $L^{(2)}$. Observe that if $x_0^1, x_1^1, x_0^2, x_1^2 \in \mathbb{N}_0$ and $\sigma_1, \sigma_2 \in \Sigma$, $B^{(2)}_{((x_0^2,x_1^2),\sigma_2)\leftarrow((x_0^1,x_1^1),\sigma_1)} = N_{((x_0^2,x_1^2),\sigma_2)\leftarrow((x_0^1,x_1^1),\sigma_1)}$ unless for some $b \in \{0,1\}$, $x_b^1 = 0$, and σ_1 corresponds to a decrement instruction for box b.

We now proceed to deal with this. Define $G^{(0)} = \{(0,x) \mid x \in \mathbb{Z}\}$. Define $\Upsilon^{(0)} = \Sigma \times \{0,-1\}$. Define $f^{(0)}$, a directed lattice weighting on $(G^{(0)}, \Upsilon^{(0)})$, so that $f^{(0)}((0,x),(\sigma_1,i_1),(\sigma_2,i_2))$ is 0 except when $x = 0$, σ_1 corresponds to an instruction to decrement box 0 specifying σ_Z and σ_P, and $i_1 = 0$. In this case we define

$$f^{(0)}((0,0),(\sigma_1,0),(\sigma_2,i_2)) = \begin{cases} 1 & \text{if } \sigma_2 = \sigma_Z \text{ and } i_2 = 0; \\ -1 & \text{if } \sigma_2 = \sigma_P \text{ and } i_2 = -1; \\ 0 & \text{otherwise.} \end{cases}$$

Define $\phi^{(0)}(\sigma,i) = (i,0)$ and $\psi^{(0)}(\sigma,i) = \sigma$. The purpose of this is to arrange that B^+ will be different from $B^{(2)}$ to correct for what happens when box 0 contains 0 and we come to a decrement instruction. We set $f^{(0)} = 1$ to correspond with the situation moved to in this case, and we set $f^{(0)} = -1$ to compensate for the definition of $B^{(2)}_{((-1,x_2),\sigma_2)\leftarrow((0,x_2),\sigma_1)} = 1$.

Similarly we define $G^{(1)} = \{(x,0) \mid x \in \mathbb{Z}\}$, and similarly $\Upsilon^{(1)}, f^{(1)}, \phi^{(1)}, \psi^{(1)}$, reading $(x,0)$ instead of $(0,x)$, box 1 instead of box 0, and $(0,i)$ instead of $(i,0)$.

Define $G^+ = G^{(2)} = \mathbb{Z} \times \mathbb{Z}$, $\Upsilon^+ = \Upsilon^{(2)} = \Sigma$, $\phi(\sigma) = (0,0)$, $\psi(\sigma) = \sigma$.

We now have a similar situation to that at the start of Chapter 7. I shall not formalise the process of generalising the definitions there from combining two lattices to combining three lattices, as this is routine. It should however be clear what will happen, and that this will define B^+ as required.

Chapter 8

Properties of the Spectral Measures of Lattices

Definition 8:1. *Let μ be a (signed) measure on some bounded subset of $\mathbb{R}$.*

(1) *The* Cumulative Distribution Function C *of μ is defined by* $C(\lambda) = \mu(-\infty, \lambda]$.

(2) *μ is* Analytic *at $\lambda \in \mathbb{R}$ if C is given by a power series in a real neighbourhood of λ.*

From Section 6.5 we see that *the* spectral measure of a weighted lattice (G, Υ, f) is an integral over G' of a measure μ^{γ} enumerating the eigenvalues of B^{γ}. The eigenvalues of B^{γ} are the roots of its characteristic function, so (Lemma 8.9.2:5) *the* spectral measure equals a measure μ^{G} giving the distribution of the roots of the characteristic function.

We will develop sufficient conditions for *the* spectral measure to be analytic in Section 8.9. These are based on the results of the earlier sections; the main one (Theorem 8.7:1) gives sufficient conditions the distribution of the roots of a function to be analytic. Section 8.8 gives a converse result.

§8.1. Distributions of Roots of Functions

In this section we show how to define a measure giving the distribution of the real roots in the first variable of an $s+1$-variable function as the other s variables are varied. Here the s variables vary over the torus $\left(\frac{\mathbb{R}}{2\pi}\right)^{s}$. Of course we could use another surface by applying an analytic bijection first; this is what we will do for lattices.

Definition 8.1:1.

(1) *Let $\left(\frac{\mathbb{C}}{2\pi}\right)$ be the set of complex numbers modulo 2π, so that we identify x and y if $x - y = 2\pi n$ for some $n \in \mathbb{Z}$.*

(2) *Let $\left(\frac{\mathbb{R}}{2\pi}\right)$ be the subset of $\left(\frac{\mathbb{C}}{2\pi}\right)$ containing just real numbers modulo 2π.*

(3) *We regard any function on a subset of $\left(\frac{\mathbb{C}}{2\pi}\right)$ as also being a function on $\mathbb{C}$, by forgetting this identification. Similarly, if we have a function f defined on a subset $\mathbb{C}$, we can and do regard it as a function on $\left(\frac{\mathbb{C}}{2\pi}\right)$, provided that if $x, y \in \mathbb{C}$, f is defined for both x and y, and $y - x = 2\pi n$ for some $n \in \mathbb{Z}$, then $f(x) = f(y)$.*

(4) *Throughout this section, s will be a positive integer.*

(5) $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ *may of course be regarded as* $\mathbb{C}^s$ *where* $(x_1, \ldots, x_s)$ *is identified with* $(y_1, \ldots, y_s)$ *if there exist* $(n_1, \ldots, n_s) \in \mathbb{Z}^s$ *such that for any* j, $x_j - y_j = 2\pi n_j$. *We regard functions on* $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ *as also being functions on* $\mathbb{C}^s$ *similarly to* (3), *by ignoring the identification.*

(6) *We define terms as* Analytic *and* Continuous *for functions on subsets of* $\left(\frac{\mathbb{C}}{2\pi}\right)$ *and* $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ *by using the correspondences in* (3) *and* (5).

Of course, there is already a topology on $\left(\frac{\mathbb{C}}{2\pi}\right)$, $\left(\frac{\mathbb{R}}{2\pi}\right)$, $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ and $\left(\frac{\mathbb{R}}{2\pi}\right)^s$, namely quotient topologies and their direct products derived from the standard topologies on $\mathbb{C}$ and $\mathbb{R}$. It may easily be verified that our definition of continuity for $\left(\frac{\mathbb{C}}{2\pi}\right)$ and $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ is identical to the one derived from this topology.

Definition 8.1:2. *In this chapter it will often be necessary to refer to sets as being 'open' or 'neighbourhoods' when they could be either subsets of some* $\mathbb{C}^r \times \left(\frac{\mathbb{C}}{2\pi}\right)^s$ *or of its subset* $\mathbb{R}^r \times \left(\frac{\mathbb{R}}{2\pi}\right)^s$. *We will take them as being open in* $\mathbb{C}^r \times \left(\frac{\mathbb{C}}{2\pi}\right)^s$ *except when* $\mathbb{R}^r \times \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *is implied (as in 'real open set' or a 'real neighbourhood').*

Lemma 8.1:3. $\left(\frac{\mathbb{R}}{2\pi}\right)$ *and* $\left(\frac{\mathbb{R}}{2\pi}\right)^s$ *are compact.*

Proof. $\left(\frac{\mathbb{R}}{2\pi}\right)$ is homeomorphic via tha map $t \to e^{it}$ to Π, the unit circle in the complex plane, which is compact, and so $\left(\frac{\mathbb{R}}{2\pi}\right)$ is. Hence $\left(\frac{\mathbb{R}}{2\pi}\right)^s$ is compact since finite products of compact spaces are compact.

Definition 8.1:4. *Throughout this section, we suppose that* n *is a positive integer,* U *an open subset of* $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ *containing* $\left(\frac{\mathbb{R}}{2\pi}\right)^s$, *and that* G *is a function* $\mathbb{C} \times U \to \mathbb{C}$ *given by*

$$G(\lambda; \theta) = \lambda^n + \sum_{j=0}^{n-1} \lambda^j g^j(\theta), \qquad (1)$$

where each g^j *is an analytic function taking* U *to* $\mathbb{C}$.

In particular, since n has to be positive, $G(\lambda; \theta)$ must be dependent on λ.

Here, as throughout this chapter, a semicolon (';') is used to separate two arguments to G which are to be treated in different ways; this will be done throughout this chapter, not only for G, but for other functions whose arguments can be divided in the same way, or even in expressions such as $(x; y)$ which we take to be formally the same as (x, y), but where the semicolon indicates the different treatment of x and y.

Lemma 8.1:5. *There is a constant* M *such that for any* $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, *all* n *roots of the equation in* λ,

$$G(\lambda; \theta) = 0$$

have absolute value at most M.

Proof. Since $\left(\frac{\mathbb{R}}{2\pi}\right)^s$ is compact, the functions $|g^0|, \ldots, |g^{n-1}|$ are all bounded on it; let N be the supremum of all $|g^j(\theta)|$ for $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$. I claim that $M = N+1$ will do. For suppose $|\lambda| > M$. Let $A = |\lambda|$. Then $|G(\lambda;\theta) - \lambda^n| = |\sum^{n-1} \lambda^j g^j(\theta)| \leq \sum^{n-1} A^j N = N(A^n - 1)/(A-1) \leq NA^n/(A-1) < A^n$, where the last inequality follows easily from $A > M = N+1$. So $|G(\lambda;\theta) - \lambda^n| < A^n$; since $|\lambda^n| = A^n$, we have $G(\lambda;\theta) \neq 0$, and so λ is not a root.

We will take M as in Lemma 8.1:5 throughout this section.

Lemma 8.1:6. *Let* $\theta^0 \in \left(\frac{\mathbb{C}}{2\pi}\right)^s$. *Suppose the distinct roots in* λ *of* $G(\lambda;\theta^0) = 0$ *are* $\lambda^1, \ldots, \lambda^r$, *and that* λ^i *has multiplicity* n^i. *Suppose that for each* i, U^i *is a neighbourhood of* λ^i, *and that the* U^i *are disjoint. Then there is a neighbourhood* V *of* θ^0 *such that if* $\theta \in V$ *then for any* j, $G(\lambda;\theta) = 0$ *has precisely* n^i *roots, counting multiplicities, in* U^i, *and no others.*

Proof. First we show that for any i there is a neighbourhood V^i of θ^0 such that if $\theta \in V^i$ then $G(\lambda;\theta) = 0$ has at least n^i roots in U^i. This is a standard result; we adapt part of the proof of [HERVÉ,1987], Chapter 2, Theorem 1. Specifically, choose some i, and some $\epsilon > 0$ such that $D \stackrel{\text{def}}{=} \{\lambda \mid |\lambda - \lambda^i| \leq \epsilon\} \subseteq U^i$. Let γ be the positively oriented circle which is the boundary of D. So if $\lambda \in \gamma$, $G(\lambda;\theta^0) \neq 0$. As γ is compact, we can choose a connected neighbourhood V^i of θ^0 such that for $\theta \in V^i$ and $\lambda \in \gamma$, $G(\lambda;\theta) \neq 0$. For otherwise there would be sequences $(\theta_j)_{j\in\mathbb{N}}$ and $(\gamma_j)_{j\in\mathbb{N}}$ with $\theta_j \to \theta^0$ and each $\gamma_j \in \gamma$ such that $G(\gamma_j;\theta_j) = 0$; since γ is compact we could replace (θ_j) and (γ_j) by subsequences with $\gamma_j \to \gamma^0$ for some $\gamma^0 \in \gamma$; then $G(\gamma^0;\theta^0) = 0$; a contradiction.

For $\theta \in V^i$, let

$$\sigma(\theta) = \frac{1}{2\pi i}\int_\gamma \frac{\partial G(\lambda;\theta)}{\partial\lambda}\frac{d\lambda}{G(\lambda;\theta)}.$$

Then $\sigma(\theta)$ is the number of roots $\lambda \in D$ of $G(\lambda;\theta) = 0$. However it is integer valued and, because of the compactness of γ, a continuous function of θ. As V^i is connected, this means that $\sigma(\theta)$ is a constant, and thus $\sigma(\theta) = \sigma(\theta^0) = n^i$. Thus for $\theta \in V^i$, the number of roots $\lambda \in D \subseteq U^i$ of $G(\lambda;\theta) = 0$ is exactly n^i, which suffices.

To complete the proof take $V = \bigcap V^i$. If $\theta \in V$, $G(\lambda;\theta) = 0$ has at least n^i roots in U^i. However the equations in λ, $G(\lambda;\theta^0) = 0$ and $G(\lambda;\theta) = 0$, both have degree n and so, counting multiplicities, n roots. Using the disjointness of the U^i, we deduce that each U^i contains exactly n^i roots, and there can be no others, as required.

Definition 8.1:7. *In this section* $\Delta = \{\lambda \mid |\lambda| \leq M\}$.

Definition 8.1:8. *Given* $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, *the measure* $\mu_T^{G;\theta}$ *supported on* Δ *is defined as* $\frac{1}{n}\sum_{j=1}^{n}\delta_{\lambda_j}$ *where* $\lambda_1,\ldots,\lambda_n$ *are the roots in* λ *of* $G(\lambda;\theta) = 0$, *including repeated ones, and* δ_{λ_j} *is the point measure at* λ_j, *as defined in* Definition 4.1:1.

Lemma 8.1:9. *Let* $f : \Delta \to \mathbb{R}$ *be continuous. Then*

$$\int f\, d\mu_T^{G;\theta}$$

is a continuous function of θ, *and never more in absolute value than* $\|f\|_\infty$.

Proof. First we establish continuity. Suppose θ^0 and $\epsilon > 0$ given; we will find a neighbourhood V of θ^0 such that if $\theta \in V$ then

$$\left|\int f\, d\mu_T^{G;\theta} - \int f\, d\mu_T^{G;\theta^0}\right| < \epsilon.$$

By the definition of $\mu_T^{G;\theta}$,

$$\int f \, d\mu_T^{G;\theta} = \frac{1}{n} \sum_j f(\lambda_j) \tag{2}$$

where the roots of $G(\lambda;\theta) = 0$ are $\lambda_1, \ldots, \lambda_n$. Let the distinct roots of $G(\lambda;\theta^0) = 0$ be $\lambda^1, \ldots, \lambda^r$, where λ^i has multiplicity n^i. Clearly we have $\sum n^i = n$. Choose disjoint neighbourhoods U^i of each λ^i such that for $\lambda \in U^i$, $|f(\lambda) - f(\lambda^i)| < \epsilon$. Find V as in Lemma 8.1:6. Suppose $\theta \in V$. Let the roots of $G(\lambda;\theta) = 0$ be $\lambda_1^1, \ldots, \lambda_{n^1}^1, \ldots, \lambda_1^r, \ldots, \lambda_{n^r}^r$, where for $1 \le j \le n^i$, $\lambda_j^i \in U^i$ (using Lemma 8.1:6). Then by Display (2),

$$\begin{aligned}
\left| \int f \, d\mu_T^{G;\theta} - \int f \, d\mu_T^{G;\theta^0} \right| &= \frac{1}{n} \left| \sum_{i=1}^{r} \left(\sum_{j=1}^{n^i} f(\lambda_j^i) - f(\lambda^i) \right) \right| \\
&< \frac{1}{n} \left| \sum_i n^i \epsilon \right| \quad (\text{as } |f(\lambda_j^i) - f(\lambda^i)| < \epsilon) \\
&= \epsilon. \quad (\text{as } \textstyle\sum n^i = n)
\end{aligned}$$

Thus we have proved continuity.

If the roots of $G(\lambda;\theta) = 0$ are $\lambda_1, \ldots, \lambda_n$, then by Display (2) we see that

$$\int f \, d\mu_T^{G;\theta} \le \frac{1}{n} \sum_{j=1}^{n} \|f\|_\infty = \|f\|_\infty.$$

This completes the proof of the lemma.

Definition 8.1:10.

(1) *For* $f : \left(\frac{\mathbb{R}}{2\pi}\right)^s \to \mathbb{C}$ *a bounded measurable function, define*

$$\int f(\theta) \, d\theta = \frac{1}{(2\pi)^s} \int_0^{2\pi} \cdots \int_0^{2\pi} f(\theta_1, \ldots, \theta_s) \, d\theta_1 \ldots d\theta_s.$$

(2) *For* $X \subseteq \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *and* $f : X \to \mathbb{C}$, *define*

$$\int_X f(\theta)\, d\theta \;=\; \int \bar{f}(\theta)\, d\theta$$

where

$$\bar{f}(\theta) \stackrel{\text{def}}{=} \begin{cases} f(\theta) & \text{if } \theta \in X; \\ 0 & \text{otherwise.} \end{cases}$$

The definition of $\int d\theta$ is analogous to our previous definition of $\int d\gamma$ in Definition 6.4:21.

Definition 8.1:11. *The Borel measure* μ_T^G *with support* Δ *is defined so that for all continuous functions* $f : \Delta \to \mathbb{R}$,

$$\int f\, d\mu_T^G \;=\; \int\int f\, d\mu_T^{G;\theta}\, d\theta. \tag{3}$$

We use Theorem 6.4:26 to justify this definition. The integral with respect to θ in Display (3) is defined since, by Lemma 8.1:9, it is of a continuous function bounded by $\|f\|_\infty$. This also means that the integral is at most $\|f\|_\infty$ in absolute value, showing that the functional from continuous functions f to $\int\int f\, d\mu_T^{G;\theta}\, d\theta$ has norm at most 1. Finally, Δ is compact. Thus Theorem 6.4:26 applies, and Display (3) does indeed uniquely define μ_T^G, on all Borel subsets of Δ.

Definition 8.1:12. *For* $\epsilon > 0$ *and* $\lambda \in \mathbb{R}$, *define the function* $t_\epsilon^\lambda : \mathbb{R} \to \mathbb{R}$ *by*

$$t_\epsilon^\lambda(y) \;=\; \begin{cases} 1 - \frac{|y-\lambda|}{\epsilon} & \text{if } |y-\lambda| < \epsilon; \\ 0 & \text{otherwise.} \end{cases}$$

Definition 8.1:13.

(1) *Given* $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$*, the real measure supported on* $[-M, M]$, $\mu^{G;\theta}$*, is defined by* $\mu^{G;\theta}(S) = \mu_T^{G;\theta}(S)$ *where* S *is any subset of* $[-M, M]$.

(2) *The measure* μ^G *with support* $[-M, M]$ *is defined by* $\mu^G(S) = \mu_T^G(S)$ *where* S *is any subset of* $[-M, M]$.

Since μ_T^G is defined on all Borel subsets of Δ, μ^G is defined on all Borel subsets of $[-M, M]$.

Lemma 8.1:14. μ^G *is the unique Borel measure such that for any continuous function* $f : [-M, M] \to \mathbb{R}$,

$$\int f \, d\mu^G = \int\int f \, d\mu^{G;\theta} \, d\theta. \tag{4}$$

Proof. Uniqueness follows from Theorem 6.4:26. To show Display (4), extend f to map Δ by defining it to be 0 outside $[-M, M]$. Define $f_\epsilon : \Delta \to \mathbb{R}$ by

$$f_\epsilon(x + iy) = f(x) t_\epsilon^0(y)$$

for $x, y \in \mathbb{R}$. Then each f_ϵ is a continuous (so measurable) function on Δ, so

$$\int f_\epsilon \, d\mu_T^G = \int\int f_\epsilon \, d\mu_T^{G;\theta} \, d\theta.$$

As $\epsilon \to 0$, the f_ϵ decrease pointwise to f; and for any $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, $\int f_\epsilon \, d\mu_T^{G;\theta}$ decreases to $\int f \, d\mu_T^{G;\theta}$. It follows (for example by [HALMOS,1950], Chapter 5, Section 27, Theorem B) that Display (4) is true.

So $\mu^{G;\theta}$ and μ^G ignore all non-real roots in λ of $G(\lambda; \theta) = 0$. In fact, in the context in which we shall apply this theory for lattices and weighted lattices, it will be the case that for all $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, all roots in λ of $G(\lambda; \theta) = 0$ are real. However we do not need such an assumption for this section, so we do not make it.

Lemma 8.1:15. *The measures* $\mu_T^{G;\theta}, \mu^{G;\theta}, \mu_T^G, \mu^G$ *are all non-negative.*

Proof. This follows immediately from their definitions.

Lemma 8.1:16. *Let* I *be any interval in* $\mathbb{R}$. *Then*

$$\mu^G(I) = \int \mu^{G;\theta}(I)\, d\theta. \tag{5}$$

Proof. Suppose that $I = (-\infty, a]$. For $\epsilon > 0$, define $f_\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$f_\epsilon(x) = \begin{cases} 1 & \text{if } x \le a; \\ 1 - \frac{(x-a)}{\epsilon} & \text{if } a < x \le a + \epsilon; \\ 0 & \text{otherwise.} \end{cases}$$

Then f_ϵ is continuous and for any non-negative measure μ, $\int f_\epsilon\, d\mu$ is a monotonically increasing function of ϵ. Furthermore $\mu(-\infty, a] \le \int f_\epsilon\, d\mu \le \mu(-\infty, a+\epsilon)$, so as ϵ tends to 0, $\int f_\epsilon\, d\mu \to \mu(I)$. We know that

$$\int f_\epsilon\, d\mu^G = \int\int f_\epsilon\, d\mu^{G;\theta}\, d\theta. \tag{6}$$

For n a positive integer let $\epsilon = \frac{1}{n}$. Hence as $n \to \infty$, the left hand side of Display (6) tends to $\mu^G(I)$. By [HALMOS,1950], Chapter 5, Section 27, Theorem B the right hand side tends to $\int \mu^{G;\theta}(I)\, d\theta$. Thus Display (5) follows if $I = (-\infty, a]$.

It also follows if $I = \mathbb{R}$ (by letting $f \equiv 1$ in Lemma 8.1:14) and if $I = (-\infty, b)$, by letting

$$f_\epsilon(x) = \begin{cases} 1 & \text{if } x \le b - \epsilon; \\ \frac{(b-x)}{\epsilon} & \text{if } b - \epsilon < x \le b; \\ 0 & \text{otherwise.} \end{cases}$$

and proceeding in the same way as for $I = (-\infty, a]$, with some changes of sign. Furthermore if Display (5) is true for I equal to I_1 and I_2, and $I_1 \subseteq I_2$, then it is also true for I equal to $I_2 \setminus I_1$. All intervals can be obtained in this way from intervals of the form $\mathbb{R}$, $(-\infty, a]$, $(-\infty, b)$; for example $(a, b) = (-\infty, b) \setminus (-\infty, a]$. So Display (5) is true for all intervals and the lemma follows.

§8.2. Germs

From this section to Section 8.5 we will consider only local properties of G. In this chapter we shall look at factorisations of G; to discover if the spectral measure is analytic at λ^0, we would like to be able to write $G = G_1 \cdots G_r$ so that for any θ^0 and j, the equation $G_j(\lambda; \theta^0) = 0$ does not have a root of multiplicity 2 or more at $\lambda = \lambda^0$. But we will allow local factorisations; this makes our results more general, as it is possible for a function to have a local factorisation which does not correspond to a global factorisation. Consider for example the elliptic curve with equation $y^2 - x^2 + x^3 = 0$. This is irreducible globally. Nevertheless, at the point $(0,0)$ there is a double point where two separate parts of the curve cross each other; at this point $y^2 - x^2 + x^3$ factors locally into two parts, each of which is of the form $y - f(x)$, where f is an analytic function of x.

Germs are a way of looking at functions locally and we shall discuss them in this section. We will go on to establish some results about roots of equations of the form $g = 0$ where g is a germ.

Let X be some space of the form $\mathbb{C}^r \times \left(\frac{\mathbb{C}}{2\pi}\right)^s$, where $r, s \geq 0$.

Definition 8.2:1. *For $x \in X$, a* Germ at x *is a triple (U, x, f), where U is an open subset of X containing x and f is an analytic function mapping U to $\mathbb{C}$, and two germs (U, x, f) and (V, x, g) are defined to be equal if and only if there is a neighbourhood W of x for which $f|_W = g|_W$.*

Germs are defined in a similar way in [HERVÉ,1987], Chapter 1, Section 2, Definition 2, to which the reader is referred for statements and proofs of their elementary properties and definitions.

Definition 8.2:2. *If two germs f and g on x are equal, clearly the associated functions must have the same value on x. We say this is the* Value at x *of the germ.*

If we have an analytic function f defined on any neighbourhood U of x then (U, x, f) is a germ. However because of our definition of equality it doesn't matter what U is; for example we could replace it by any neighbourhood of x contained in U. Also it is rarely necessary to specify x, for we shall not often need to deal with germs which are at different points, so in statements involving more than one germ, we assume that all the germs are at the same x, unless it is stated otherwise. Thus we will normally just refer to the germ as f, if necessary making it clear (as in "the germ f") that we are talking of a germ and not a function. Similarly, we will sometimes treat germs as functions and manipulate them thus in expressions, but of course to do this it is necessary to make sure that the statements we make are true whatever (U, x, f) we use to represent the germ. For example, if g, g_1, g_2 are germs, the equation $g = g_1 + g_2$ is meaningful because of the definition of equality of germs. So we can define $g_1 + g_2$ as a germ, or more generally given germs $g_1, \ldots, g_n$ and an analytic function $F : \mathbb{C}^n \to \mathbb{C}$ we can define a germ $F(g_1, \ldots, g_n)$. Similarly, since the functions corresponding with germs must be analytic, we can define derivative functions of germs which are also germs.

The following is a typical elementary property of germs:

Lemma 8.2:3. *Let* f *and* g *be two germs on* x *and suppose that the value of* g *at* x *is non-zero. Then there is a germ* $\frac{f}{g}$ *at* x *such that* $g(\frac{f}{g}) = f$.

Proof. Suppose $f = (U, x, f')$ and $g = (V, x, g')$. As g' is analytic, and $g'(x) \neq 0$, we can find a neighbourhood $W \subseteq U \cap V$ of x such that g' is non-zero on W. Thus $\frac{f'}{g'}$ is analytic on W and it is easily seen that $\frac{f}{g} = (W, x, \frac{f'}{g'})$ will do.

Definition 8.2:4. *A* Vector Germ *at* x *of* Dimension k *(a non-negative integer) is a list* $f = (f_1, \ldots, f_k)$*, where each* f_j *is a germ at* x*. We call the* f_j *the* Entries *of* f*, and, as here, denote the* j^{th} *entry of* f *by* f_j.

Consider the set of germs at $x \in \mathbb{C}^r \times \left(\frac{\mathbb{C}}{2\pi}\right)^s$. We can drop the identification implicit in $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ and regard x as belonging to $\mathbb{C}^{r+s}$. Suppose $x = (\Lambda; \Theta)$ where $\Theta \in \mathbb{C}^r$ and $\Lambda \in \left(\frac{\mathbb{C}}{2\pi}\right)^s$. Take some positive $R < \pi$ and let B be the open ball in $\mathbb{C}^s$ about Θ of radius R. No distinct elements of B are identified in $\left(\frac{\mathbb{C}}{2\pi}\right)^s$.

Definition 8.2:5. *Given a germ* (U, x, f) *at* $x \in \mathbb{C}^{r+s}$*, we identify it with the germ* $(U \cap (\mathbb{C}^r \times B), x, f)$ *at* $x \in \mathbb{C}^r \times \left(\frac{\mathbb{C}}{2\pi}\right)^s$.

By the definition of equality for germs (in Definition 8.2:1) this identification bijects germs at $x \in \mathbb{C}^{r+s}$ and germs at $x \in \mathbb{C}^r \times \left(\frac{\mathbb{C}}{2\pi}\right)^s$ and commutes with addition, multiplication, composition with other analytic functions, and differentiation. So we can and shall use this identification implicitly.

In this chapter we denote the co-ordinates of $\mathbb{C} \times \left(\frac{\mathbb{C}}{2\pi}\right)^s$ by (in order) the names $\lambda, \theta_1, \ldots, \theta_s$.

Definition 8.2:6. *Let* G *be any germ on* $\mathbb{C} \times \mathbb{C}^s$ *and* g *any germ on* $\mathbb{C}^s$.

(1) G_λ *is the germ* $\frac{\partial G}{\partial \lambda}$.

(2) $G_{\lambda^{(n)}}$ *for* $n \geq 0$ *is the germ* $\frac{\partial^n G}{\partial \lambda^n}$.

(3) *For any* j *between* 1 *and* s, G_{θ_j} *is the germ* $\frac{\partial G}{\partial \theta_j}$ *and* g_{θ_j} *is the germ* $\frac{\partial g}{\partial \theta_j}$.

(4) ∇G *is the vector germ* $(G_{\theta_1}, \ldots, G_{\theta_s})$ *and* ∇g *is the vector germ* $(g_{\theta_1}, \ldots, g_{\theta_s})$.

For example, suppose that the germ G is represented by (U, x, g). Then G_λ is the germ represented by (U, x, g'), where g' is defined so that

$$g'(\lambda; \theta) = \frac{\partial g(t; \theta)}{\partial t}\Big|_{t=\lambda}.$$

Note that ∇G does not involve G_λ. This is because we will treat derivatives with respect to the θ_j separately from derivatives with respect to λ.

Definition 8.2:7. *If* g *is a germ at* x*, then* g *is* Defined on U *if the following are true:*

(1) U *is a convex open set containing* x.

(2) g *can be represented as* (U, x, g') *where* $g' : U \to \mathbb{C}$ *is analytic.*

If $x \in \left(\frac{\mathbb{C}}{2\pi}\right)^u \times \mathbb{C}^v$, we nevertheless regard U as a subset of $\mathbb{C}^{u+v}$ and define convexity in that.

Definition 8.2:8. *If* g *is a germ at* x*, then* g *is* Defined at y *if there is a set* U *containing* y *on which* g *is defined.*

Definition 8.2:9. *If* g *is a germ at* x*, and* Z *is a set,* g *is* Defined Throughout Z *if* g *is defined at every element of* Z.

Lemma 8.2:10. *If* U *is a connected open set,* g *is a germ, and* $y \in U$*, then for any* (U, x, g') *representing* g*,* $g'(y)$ *is the same.*

Proof. If g is represented by (U, x, g_1') and (U, x, g_2'), then g_1' and g_2' must be equal on an open set V contained in U. The lemma follows from the Principle of Analytic Continuation ([HERVÉ,1987], Chapter 1, Section 1, Corollary 2).

Lemma 8.2:11. *If* g *is a germ defined at* y*, then for any* U *containing* y *such that* g *is defined on* U*, and for any* (U, x, g') *representing* g*,* $g'(y)$ *is the same.*

Proof. Suppose g is defined on U_1 and U_2, both containing y. Suppose g is represented by (U_1, x, g_1') and (U_2, x, g_2'). We need to show that $g_1'(y) = g_2'(y)$. Let $U = U_1 \cap U_2$. Both U_1 and U_2 are convex and contain y; therefore U is convex, hence connected, and contains y. g is defined on U, and g can be represented by (U, x, g_1') and (U, x, g_2'). Thus by Lemma 8.2:10, $g_1'(y) = g_2'(y)$ and we are done.

Definition 8.2:12. *In the circumstances of* Lemma 8.2:11, *then* $g(y)$*, the* Value at y*, is equal to* $g'(y)$.

Any germ g at x is defined on some set; for example we can just choose a sufficient small open ball about x. Thus this definition generalises Definition 8.2:2.

Definition 8.2:13. *A vector germ* $f = (f_1, \ldots, f_k)$ *at* x *is* Defined *on* U *if each of its entries is defined on* U. *For* $y \in U$, *its value at* y, $f(y)$, *is* $(f_1(y), \ldots, f_k(y))$.

If we have two germs or vector germs at x, f and g, defined on U and V respectively, let $W = U \cap V$; then f and g are defined on W. By the definition of $f + g$, there is a neighbourhood of x such that $(f + g)(y) = f(y) + g(y)$. Let h be the analytic function on W taking y to $f(y) + g(y)$; then it follows that the germ (W, x, h) equals $f + g$. Hence $h = f + g$ and we deduce that for all $y \in W$, $(f + g)(y) = f(y) + g(y)$. In a similar way we deduce that if f and g are germs at x, then for all $y \in W$, $(f \times g)(y) = f(y)g(y)$.

The operations of addition and multiplication make the set of germs at x a commutative ring, which we denote by $\mathcal{H}_x$. $\mathcal{H}_x$ contains a copy of $\mathbb{C}$, with $z \in \mathbb{C}$ being represented by the germ of the function taking everything to z. We shall identify this germ with z. $\mathcal{H}_x$ has additive and multiplicative identities, which are thus 0 and 1 respectively. By Lemma 8.2:3, any germ g with $g(x) \neq 0$ is invertible in this ring. Conversely, if g is a germ with $g(x) = 0$, then it cannot be invertible for if h was its inverse we would have $f(x)g(x) = 1$.

Definition 8.2:14. $\mathcal{H}'_x \stackrel{\text{def}}{=} \{g \in \mathcal{H}_x \mid g(x) = 0\}$.

Thus $\mathcal{H}'_x$ is the set of all non-invertible germs at x.

Definition 8.2:15.

(1) *Let* U *be a set. Two germs* $g_1, g_2 \in \mathcal{H}_x$ *are said to be* Equivalent on U *if they are defined throughout* U *and there is an analytic function* $h: U \to \mathbb{C}$ *such that for all* $u \in U$, $h(u) \neq 0$ *and* $g_1(u) = h(u)g_2(u)$.

(2) $g_1, g_2 \in \mathcal{H}_x$ *are said to be* Equivalent *if they are equivalent on some open set containing* x.

It is easy to verify that $g_1, g_2 \in \mathcal{H}_x$ are equivalent if and only if there is some germ h with $h(x) \neq 0$ and $g_1 = hg_2$, by expanding definitions and remembering that $h(x) \neq 0$ if and only if there is an open set containing $h(x)$ throughout which h is non-zero.

It is also easy to verify that equivalence is an equivalence relation, and that for fixed U equivalence on U is an equivalence relation. However it is important to keep in mind that if two germs are equivalent, it does not follow that they are equal. For example, if g is a germ and λ is a non-zero complex number then g is equivalent to λg.

Lemma 8.2:16. *If germs at* x*,* g_1 *and* g_2*, are equivalent on a set containing* y *and* $g_1(y) = g_2(y) = 0$*, then at* y*, for any co-ordinate function* x_j*,* $\frac{\partial g_1}{\partial x_j} = 0$ *if and only if* $\frac{\partial g_2}{\partial x_j} = 0$*.*

Proof. Write $g_1 = hg_2$ with $h(y) \neq 0$. Then at y, by the product rule, $\frac{\partial g_1}{\partial x_j} = h(y)\frac{\partial g_2}{\partial x_j}$; hence the lemma follows.

Definition 8.2:17. *If* f *is a non-invertible germ at* x*, it is* Reducible *if there are two non-invertible germs at* x*,* f_1, f_2*, with* $f = f_1 f_2$*. Otherwise,* f *is* Irreducible.

Now in fact the ring of germs at x is a unique factorisation domain, so:

Theorem 8.2:18. *Given a non-invertible germ* f *at* x*, we can find a finite family* F *of irreducible germs at* x *with product* f*; furthermore this family is unique up to equivalence.*

Proof. This is [HERVÉ,1987], Chapter 2, Section 3, Theorem 3.

Definition 8.2:19. *Let* f *be a germ at* $(\lambda;\theta)$*. Then* $I(f)$ *is the set of all germs* h *at* $(\lambda;\theta)$ *such that on some neighbourhood* U *of* $(\lambda;\theta)$ *on which* h *and* f *are defined,* $\{u \in U \mid f(u) = 0\} \subseteq \{u \in U \mid h(u) = 0\}$*.*

Lemma 8.2:20. *Let* f *and* F *be as in* Theorem 8.2:18. *Then* $I(f) = \bigcap_{h \in F} I(h)$*.*
This follows by [HERVÉ,1987], Chapter 2, Section 8, Definition 9, Example 2.

Lemma 8.2:21. *Let* G *be a germ at* $(\lambda;\theta)$ *with* $G_\lambda(\lambda;\theta) \neq 0$*. Then* G *is irreducible.*

Proof. Otherwise, suppose $G = AB$ where $A, B \in \mathcal{H}'_x$; then $A(\lambda;\theta) = B(\lambda;\theta) = 0$, so $G_\lambda = A_\lambda B + AB_\lambda$ is equal to 0 at $(\lambda;\theta)$, a contradiction.

Lemma 8.2:22. *Let* G, G_1 *be germs at* $(\lambda;\theta)$ *with* $(G_1)_\lambda(\lambda;\theta) \neq 0$ *and suppose there is a neighbourhood* U *of* $(\lambda;\theta)$ *such that* $\{a \in U \mid G_1(a) = 0\} \subseteq \{a \in U \mid G(a) = 0\}$*. Then there is a germ* h *at* $(\lambda;\theta)$ *with* $G = hG_1$*.*

Proof. By Lemma 8.2:20 and Lemma 8.2:21, $I(G_1)$ is generated by G_1. However clearly $G \in I(G_1)$. Thus we can find an h with $G = hG_1$ and the lemma follows.

Lemma 8.2:23. *Let* $G, G_1, \ldots, G_m$ *be germs at* $(\lambda^0; \theta^0)$ *such that for each* G_i, $(G_i)_\lambda(\lambda^0; \theta^0) \neq 0$, *and suppose there is a neighbourhood* U *of* $(\lambda^0; \theta^0)$ *such that at any* $(\lambda; \theta) \in U$ *with some* $G_i(\lambda; \theta) = 0$, $G(\lambda; \theta) = 0$, *and the multiplicity of the solution* $\lambda' = \lambda$ *to* $G(\lambda'; \theta) = 0$ *is greater than or equal to the number of* G_i *with* $G_i(\lambda; \theta) = 0$. *Then there is a germ* h *such that* $G = hG_1 \cdots G_m$.

Proof. We prove this by induction on m. Clearly it is trivial if $m = 0$. If $m = k + 1$, apply Lemma 8.2:22 to find a germ h' with $G = h'G_1$ and let U' be a neighbourhood of $(\lambda^0; \theta^0)$ contained in U such that for $(\lambda; \theta) \in U'$, $(G_1)_\lambda(\lambda; \theta) \neq 0$. By elementary properties of multiplicities, the conditions of the lemma hold for $h, G_2, \ldots, G_m$ with U', so inductively we can find h with $h' = hG_2 \cdots G_m$, whence $G = G_1 \cdots G_m$.

Lemma 8.2:24. *Suppose* G_1 *and* G_2 *are analytic functions defined on some open set* U *such that at any* $a \in U$ *with* $G_1(a) = 0$, *there is a germ* h *at* a *with* $G_2 = hG_1$, *where both sides are considered as germs at* a. *Then there is an analytic function* H *defined on* U *with* $G_2 = HG_1$.

Proof. If G_1 is identically 0 throughout U, then so is G_2 and the lemma is trivial. Otherwise, let $S = \{x \in U \mid G_1(x) = 0\}$, and define $H = \frac{G_2}{G_1}$ on $U \setminus S$; H is analytic on $U \setminus S$. We need to show that H can be extended to the whole of U. To show this we will apply [HERVÉ,1987], Chapter 3, Section 1, Theorem 2. By this, we can extend H to the whole of U provided that any $a \in S$ has a neighbourhood on which H is bounded. To show this, let h be a germ at a with $G_2 = hG_1$; then $h = H$ on a neighbourhood of a, but h is continuous at a, so the lemma follows.

Definition 8.2:25.

(1) *Suppose that* g *is a germ at* x *defined on* U *and that* $y \in U$. *Then* g^y *is the germ at* y *given by* (U, y, g).

(2) *Suppose that* g *is a vector germ of dimension* k *at* x *defined on* U. *Then* g^y *is the vector germ at* y *given by* $((U, y, (g_1)^y), \ldots, (U, y, (g_k)^y))$.

In this definition we see that g^y is also defined on U.

Definition 8.2:26. *For* f *a diffeomorphism and* y *in its domain write* $J(f)(y)$ *for the Jacobian (or determinant of the matrix of derivatives) of* f *at* y.

We now state a version of the Implicit Function Theorem.

Theorem 8.2:27. *Let* G *be a germ at* $(\lambda^0;\theta^0)$ *defined on* U, *and suppose* $G_\lambda(\lambda^0;\theta^0) \neq 0$ *and* $G(\lambda^0;\theta^0) = 0$. *Then we can find open neighbourhoods* $V \subseteq U$ *and* W *of* $(\lambda^0;\theta^0)$ *and* θ^0 *respectively, together with an analytic function* $g : W \to \mathbb{C}$ *such that the sets* $\{(\lambda;\theta) \in V \mid G(\lambda;\theta) = 0\}$ *and* $\{(g(\theta);\theta) \mid \theta \in W\}$ *are identical. Furthermore, for* $(\lambda;\theta) \in V$ $G_\lambda(\lambda;\theta) \neq 0$.

Proof. We use the inverse function theorem, as proved in [HERVÉ,1987], Chapter 1, Section 5, Theorem 1. We may assume that U is sufficiently small that it can be regarded as a subset of $\mathbb{C} \times \mathbb{C}^s$, ignoring the identification involved in $\left(\frac{\mathbb{C}}{2\pi}\right)^s$.

Define $f : U \to \mathbb{C}$ by $f(\lambda;\theta) = (G(\lambda;\theta);\theta)$. Then $J(f)(\lambda^0;\theta^0) = G_\lambda(\lambda^0;\theta^0)$, and so is non-zero. Thus by the inverse function theorem, there are open neighbourhoods V of $(\lambda^0;\theta^0)$ and V' of $f(\lambda^0;\theta^0) = (0;\theta^0)$, such that f is a bijection from V to V' with an analytic inverse h. Clearly $h(t;\theta) = (\lambda;\theta)$ where $G(\lambda;\theta) = t$. Define $W = \{\theta \mid (0;\theta) \in V'\}$. Define $g : W \to \mathbb{C}$ so that $h(0;\theta) = (g(\theta);\theta)$. Then it is easily seen that V, W and g satisfy the conditions of the theorem, with the exception of the last sentence. For this, note that since $f : V \to V'$ and $h : V' \to V$ are inverses and both are differentiable $J(f)(\lambda;\theta)J(h)(f(\lambda;\theta)) = 1$. So for any $(\lambda;\theta) \in V$, $J(f)(\lambda;\theta) \neq 0$ and hence $G_\lambda(\lambda;\theta) \neq 0$. □

Corollary 8.2:28.

(1) *For any* $(\lambda';\theta') \in V$, *the equation in* λ, $G(\lambda;\theta') = 0$, *does not have a root of multiplicity 2 or more at* $\lambda = \lambda'$.

(2) *For* $\theta \in W$, $(g(\theta);\theta) \in V$.

(3) $g(\theta^0) = \lambda^0$.

Proof. These all follow immediately from the statement of the theorem.

Theorem 8.2:29. *In* Theorem 8.2:27, *we may assume that* $V = W^1 \times W$ *where* W^1 *is a convex open neighbourhood of* λ^0 *and* W *is an open convex neighbourhood of* θ^0.

Proof. Find V, W and g as in Theorem 8.2:27. Let W^1 be a convex open neighbourhood of λ^0 and X a neighbourhood of θ^0 such that $W^1 \times X \subseteq V$. The set $\{\theta \in X \mid g(\theta) \in W^1\}$ is an open neighbourhood of θ^0. Choose Y to be a convex open neighbourhood of θ^0 contained in this set. Then in the statement of Theorem 8.2:27 replace W by Y and V by $W^1 \times Y$. It is then easy to check that this suffices.

Corollary 8.2:30. *In* Theorem 8.2:29 W^1 *and* W *can be made arbitrarily small, so that if* T^1 *and* T *are any open neighbourhoods of* λ^0 *and* θ^0 *respectively, we can additionally require that* $W^1 \subseteq T^1$ *and* $\overline{W} \subseteq T$, *and that* g *can be analytically extended to* $\overline{W}$.

Proof. Let T' be a neighbourhood of θ^0 such that $\overline{T'} \subseteq T$. In Theorem 8.2:27, and so Theorem 8.2:29, we can replace U by $U \cap T^1 \times T'$; then we must have $V = W^1 \times W \subseteq T^1 \times T'$ and so $W^1 \subseteq T^1$ and $\overline{W} \subseteq T$.

Definition 8.2:31. *We say that* g *is a* Root Function *for* G *on* $W^1 \times W$ *if it satisfies the conditions of* Theorem 8.2:29 *and, with* $V = W^1 \times W$, Theorem 8.2:27.

Thus Theorem 8.2:29 says that we can find W^1, W, and a root function for G on $W^1 \times W$, and Corollary 8.2:30 says that W^1 and W can be made arbitrarily small.

Lemma 8.2:32. *If* G *is a germ at* $(\lambda^0; \theta^0)$ *with root functions for* $i = 1, 2$, g_i *on* $W_i^1 \times W_i$, *then* $g_1 = g_2$ *on* $W_1 \cap W_2$.

Proof. Let $W = \{\theta \in W_1 \cap W_2 \mid g_1(\theta), g_2(\theta) \in W_1^1 \cap W_2^1\}$.
Since W_1, W_2, W_1^1, W_2^2 are open, $\theta^0 \in W^1 \cap W^2$, $\lambda^0 \in W_1^1 \cap W_2^2$, and $g^1(\theta^0) = g^2(\theta^0) = \lambda^0$ (by Corollary 8.2:28 (3)), W is an open neighbourhood of θ^0. So by Lemma 8.2:10 it is enough to show that g_1 and g_2 are equal on W.

But if $\theta \in W$, then for $i = 1, 2$, $g_i(\theta)$ must be the unique element λ of W_i^1 such that $G(\lambda; \theta) = 0$; as $g_i(\theta) \in W_1^1 \cap W_2^1$ we must have $g_1(\theta) = g_2(\theta)$ which concludes the proof of the lemma.

Lemma 8.2:33. *If* G *is a germ at* $(\lambda^0;\theta^0)$, *and* g *is a root function for* G *on* $W^1 \times W$ *(so in particular* $G(\lambda^0;\theta^0) = 0$*) then for any* $(\lambda^1;\theta^1) \in W^1 \times W$ *with* $G(\lambda^1;\theta^1) = 0$, g *is a root function for* $G^{(\lambda^1;\theta^1)}$ *on* $W^1 \times W$.

Proof. This is true as we made no mention of $(\lambda^0;\theta^0)$ in Theorem 8.2:29 except by requiring $(\lambda^0;\theta^0) \in W^1 \times W$ and $G(\lambda^0;\theta^0) = 0$, so we can replace $(\lambda^0;\theta^0)$ by $(\lambda^1;\theta^1)$ without problems.

Theorem 8.2:34. *For* $i = 1,2$ *suppose* g^i *is a root function for* G^i, *a germ at* $(\lambda^0;\theta^0)$, *on* $W^1 \times W$. *Then the following are equivalent:*

(1) G^1 *and* G^2 *are equivalent on* $W^1 \times W$.

(2) G^1 *and* G^2 *are equivalent at* $(\lambda^0;\theta^0)$.

(3) *For* $\theta \in W$, $g^1(\theta) = g^2(\theta)$.

(4) *The germs* $(W,(\lambda^0;\theta^0),g^1)$ *and* $(W,(\lambda^0;\theta^0),g^2)$ *are equal.*

Proof. (4) and (3) are equivalent by Lemma 8.2:10. Now we suppose (2) and deduce (4). Since G^1 and G^2 are equivalent at $(\lambda^0;\theta^0)$, by Definition 8.2:15, there is a convex open set $U \subseteq W^1 \times W$ containing $(\lambda^0;\theta^0)$ and an analytic H non-zero on U with $G^2 = HG^1$. Thus if $(\lambda;\theta) \in U$ then $G^1(\lambda;\theta) = 0$ if and only if $G^2(\lambda;\theta) = 0$. However by the definition of a root function, we know that if $\lambda \in W^1$ and $\theta \in W$ then for $i = 1,2$, $G^i(\lambda;\theta) = 0$ if and only if $\lambda = g^i(\theta)$. The functions taking θ to $(g^i(\theta);\theta)$ is continuous; also by Corollary 8.2:28, (3) each $g^i(\theta^0) = \lambda^0$. So let U' be an open neighbourhood of θ^0 such that for $\theta \in U'$, each $(g^i(\theta);\theta) \in U$. Then we see that for $\theta \in U'$, $g^1(\theta) = g^2(\theta)$ from which (4) follows.

By Definition 8.2:15, (1) implies (2). So we finish the proof of the theorem by showing that (3) implies (1). So suppose (3).

For $i = 1,2$, let $X^i = \{(\lambda;\theta) \in W^1 \times W \mid g^i(\theta) = \lambda\}$. Since on W, $g^1 = g^2$, we deduce that $X^1 = X^2$; let $X = X^1 = X^2$. Let $(\lambda;\theta) \in X$. For $i = 1,2$, consider G^i as a germ at $(\lambda;\theta)$; By Lemma 8.2:21 G^i is irreducible, and so by Lemma 8.2:20 $I(G^i)$ is the ideal generated by G^i. However, since $g^1 = g^2$ on W, the zero sets of G^1 and G^2 are equal on $W \ni (\lambda;\theta)$, and so $G^2 \in I(G^1)$. Thus we can find a germ h at $(\lambda;\theta)$ such that $G^2 = hG^1$. So by Lemma 8.2:24, there is an analytic function $H : W^1 \times W$ with $G^2 = HG^1$. Applying the same arguments with G^1 and G^2 exchanged, we see that there is also an analytic function $H' : W \times W \to \mathbb{C}$ with $G^1 = H'G^2$. Thus for $u \in W^1 \times W \setminus X$, $G^1(u) \neq 0$, so $H'(u)H(u)G^1(u) = G^1(u)$, so $H(u) \neq 0$. For $u \in X$, we have $G^1(u) = 0$ and $G^2_\lambda(u) \neq 0$; however $G^2_\lambda(u) = H_\lambda(u)G^1(u) + H(u)G^1_\lambda(u) = H(u)G^1_\lambda(u)$; so

$H(u) \neq 0$. This completes the proof that G^1 and G^2 are equivalent and the proof of the theorem. □

Lemma 8.2:35. *Suppose* g *is a root function for* G *on* $W^1 \times W$. *Then the function* h *on* $W^1 \times W$ *defined by* $h(\lambda; \theta) = \lambda - g(\theta)$ *is equivalent to* G.

Proof. It is easy to check that g is a root function for h on $W^1 \times W$. Therefore the lemma follows from Theorem 8.2:34.

Theorem 8.2:36. *Let* m *be a positive integer and suppose* f *is a germ at the origin of* $\mathbb{C}^m$, *which we call* $\mathbf{0}$, *mapping to* $\mathbb{C}$, *and that* f *is not identically* 0, *but* $f(\mathbf{0}) = 0$. *Then we can choose a basis of* $\mathbb{C}^m$ *with respect to which there is a germ* ψ *equivalent to* f *of the form*

$$\psi(x_1, \ldots, x_m) = x_m^p + \sum_{k=1}^{p} c_k(x_1, \ldots, x_{m-1}) x_m^{p-k} \qquad (7)$$

where each c_k *is analytic and equal to* 0 *at the origin of* $\mathbb{C}^{m-1}$. *The value of* p *is unique.*

This is part of the Weierstrass Preparation Theorem, which is proved in [HERVÉ, 1987], Chapter 2, Section 1, Theorem 1. From this proof we also observe

Corollary 8.2:37.

(1) *In* Theorem 8.2:36, *we can choose any basis of* $\mathbb{C}^m$ *provided that* f *is defined at its* m^{th} *element,* $\mathbf{a}$, *and* $f(\mathbf{a}) \neq 0$.

(2) *In* Theorem 8.2:36 *we can choose the basis to consist entirely of real vectors.*

Proof. (1) follows immediately from the proof in [HERVÉ,1987]. For (2), observe that we only need to find a real vector $\mathbf{a}$ lying sufficiently close to $\mathbf{0}$ such that $f(\mathbf{a}) \neq 0$. This must exist, for otherwise all the derivatives to any order of f would be 0, and so its Taylor series would be 0; hence f would be identically zero.

Lemma 8.2:38. *Suppose* m *is a non-negative integer,* f *is a germ at some element* x *of* $\mathbb{R}^m$*, and* f *is not identically* 0*. Then there is a neighbourhood* U *of* x *on which* f *is defined such that the set* $\{y \in U \cap \mathbb{R}^m \mid f(y) = 0\}$ *has zero measure in* $\mathbb{R}^m$*.*

Proof. If $m = 0$ the lemma is trivial. Otherwise let ψ and p be as in Corollary 8.2:37, (2) and let U be such that ψ is defined on U and f and ψ are equivalent on U. From Display (7), for any $y' \in \mathbb{R}^{m-1}$, there are at most p values of t such that $\psi(y', t) = 0$. As the measure of $\mathbb{R}^{m-1}$ is 0 in $\mathbb{R}^m$, the lemma follows.

§8.3. Differential Equations

In Section 8.4 we will construct germs which we shall call threadings as solutions of differential equations. To justify this construction, we need some basic results about solutions to differential equations.

Theorem 8.3:1. *Suppose* Z *is a vector germ of dimension* s *at* $(\lambda^0; \theta^0)$ *where* $\lambda^0 \in \mathbb{C}$ *and* $\theta^0 \in \mathbb{C}^s$*. Then there is a unique vector germ of dimension* s*,* ϕ*, at* $(\lambda^0, \lambda^0; \theta^0)$ *(which is in* $\mathbb{C} \times \mathbb{C} \times \mathbb{C}^s$*) such that the following equations are satisfied, where left and right hand sides are compared as germs of functions in* $(\lambda, \mu; \theta)$*.*

(1)

$$\phi(\lambda, \lambda; \theta) = \theta.$$

(2)

$$\frac{\partial \phi(\lambda', \mu; \theta)}{\partial \lambda'}\Big|_{\lambda'=\lambda} = Z(\lambda; \phi(\lambda, \mu; \theta)).$$

There is a standard argument for proving the existence and uniqueness of solutions to differential equations, developed, for example, in [Birkhoff&Rota, 1989], Chapter 6. We will now adapt it to prove Theorem 8.3:1. Let U be an open set containing $(\lambda^0; \theta^0)$ on which Z is defined.

We turn $\mathbb{C}^s$ into a Banach Space by equipping it with the Euclidean norm $\| \, \| \stackrel{\text{def}}{=} \ell_2()$.

Choose compact convex sets $C \subset \mathbb{C}^s$ and $C^1 \in \mathbb{C}$ such that $C^1 \times C$ is a subset of U and $(\lambda^0; \theta^0)$ is in its interior.

For $(\lambda;\theta) \in U$, define the matrix $Z'(\lambda;\theta)$ by

$$(Z'(\lambda;\theta))_{ij} = \frac{\partial(Z(\lambda;\theta)_i)}{\partial\theta^j}.$$

For $x \in \mathbb{C}^s$, $(\lambda;\theta+tx) \in U$,

$$\frac{dZ(\lambda;\theta+tx)}{dt}\Big|_{t=0} = Z'(\lambda;\theta+tx)x. \tag{8}$$

Recall from Theorem 3:3 that the operator norm of Z' is bounded by $M(Z')$. As Z is analytic, $M(Z')$ is continuous. Let M be its supremum on $C^1 \times C$. Let N be the supremum of $\|Z(\lambda;\theta)\|$ for $(\lambda;\theta) \in C^1 \times C$.

Let D be any open convex subset of C containing θ^0 such that there is a $\delta > 0$ satisfying for all $x \in D$ and $y \notin C$, $\|y - x\| > \delta$.

Let $\epsilon > 0$ be less than $\frac{\delta}{N}$ and $\frac{1}{M}$. Let D^1 be an open convex subset of C^1 containing λ^0 of diameter at most ϵ. Therefore $D^1 \times D$ is an convex subset of U. Let $V = D^1 \times D^1 \times D$.

Lemma 8.3:2. *For* $\lambda \in D^1$ *and* $\theta^1, \theta^2 \in C$,

$$\|Z(\lambda;\theta^1) - Z(\lambda;\theta^2)\| \le M\|\theta^1 - \theta^2\|.$$

Proof. This follows from the Mean Value Theorem applied to the function taking t to $Z(\lambda;\theta^1 + t(\theta^2 - \theta^1))$, by Display (8).

For $\psi : V \to C$ an analytic function, define $z(\psi) : V \to \mathbb{C}^s$ by

$$z(\psi)(\lambda',\lambda;\theta) = \theta + \int_{\lambda}^{\lambda'} Z(l,\psi(l,\lambda;\theta))\,dl, \tag{9}$$

where the integral is evaluated along any path in D^1. $Z(l,\psi(l,\lambda;\theta))$ is an analytic function of l, λ, θ, and so as D^1 is simply-connected the integral in Display (9) is the same if evaluated along any path from λ to λ' in D^1. Its derivative with respect to λ' is $Z(\lambda',\psi(\lambda',\lambda;\theta))$, which is analytic; hence it is analytic and so is $z(\psi)$.

Furthermore, since $\psi(l,\lambda;\theta) \in C$, $\|Z(l,\psi(l,\lambda;\theta))\| \le N$, and therefore $\|z(\psi)(\lambda',\lambda;\theta) - \theta\| \le N\epsilon < \delta$. So $z(\psi) : V \to C$, by the definition of δ.

For $\psi_1, \psi_2 : V \to \mathbb{C}^s$ two analytic functions, define

$$d(\psi_1, \psi_2) = \sup_{(\lambda', \lambda; \theta) \in V} \|\psi_1(\lambda', \lambda; \theta) - \psi_2(\lambda', \lambda; \theta)\|.$$

Lemma 8.3:3.

$$d(z(\psi_1), z(\psi_2)) < c d(\psi_1, \psi_2),$$

where c *is a constant less than* 1.

Proof. For $(\lambda', \lambda; \theta) \in V$,

$$\begin{aligned}
&\|z(\psi_1)(\lambda', \lambda; \theta) - z(\psi_2)(\lambda', \lambda; \theta)\| \\
&= \left\| \int_\lambda^{\lambda'} Z(l, \psi_1(l, \lambda; \theta))\, dl - \int_\lambda^{\lambda'} Z(l, \psi_2(l, \lambda; \theta))\, dl \right\| \\
&\leq \int_\lambda^{\lambda'} \|Z(l, \psi_1(l, \lambda; \theta)) - Z(l, \psi_2(l, \lambda; \theta))\|\, dl \\
&\leq M\epsilon d(\psi_1, \psi_2),
\end{aligned}$$

using Lemma 8.3:2 and integrating along the straight line from λ to λ'. However $M\epsilon < 1$, so the lemma follows.

Thus d is a contraction mapping. Furthermore, it is easily seen if an analytic ϕ satisfies $d(\phi) = \phi$, then it satisfies (1) and (2), and vice-versa. This demonstrates the uniqueness of ϕ, since if we had two different solutions ϕ_1 and ϕ_2, then, if necessary choosing C_1 and C sufficiently small that both are defined on an open set containing $C_1 \times C_1 \times C$, we would have $d(\phi_1, \phi_2) < d(\phi_1, \phi_2)$, which is a contradiction. Existence of a solution follows from the contraction mapping theorem.

To apply the contraction mapping theorem we need to show that the set of analytic functions $\psi : V \to C$ is a complete metric space with the respect to the metric d. This follows immediately from Weierstrass's Theorem, which I have copied (changing notation and removing part of it) from [HERVÉ,1987], Chapter 1, Section 4.

Theorem 8.3:4. *If a sequence* (h_n) *of functions, analytic on an open subset* U *of* $\mathbb{C}^s$*, converges uniformly on every compact subset of* U*, then the limit function* h *is analytic in* U.

This completes the proof of Theorem 8.3:1. □

We will take ϕ and Z as in the statement of Theorem 8.3:1.

Definition 8.3:5. *We shall say that* $(D^1; D)$ *is Mapped by* ϕ *if*

(1) D^1 *is an open convex set containing* λ^0.

(2) D *is an open convex set containing* θ^0.

(3) ϕ *is defined and analytic on* $\overline{D^1} \times \overline{D^1} \times \overline{D}$.

(4) Z *is defined on* $D^1 \times D$.

(5) *For* $\lambda, \mu \in D^1$ *and* $\theta \in D$ *we have* $\phi(\lambda, \lambda; \theta) = \theta$ *and* $\frac{\partial \phi(\lambda', \mu; \theta)}{\partial \lambda'}\big|_{\lambda'=\lambda} = Z(\lambda; \phi(\lambda, \mu; \theta))$.

By taking $D^1 \times D^1 \times D$ with closure inside some ball-shaped neighbourhood of $(\lambda^0, \lambda^0; \theta^0)$ in which the two equations in the statement of Theorem 8.3:1 are true considered as functions of $(\lambda, \mu; \theta)$, we can certainly find some $D^1 \times D$ mapped by ϕ. Or we can simply take D^1 and D as we constructed them while proving Theorem 8.3:1, if necessary reducing D^1 and D so that ϕ is defined on $\overline{D^1} \times \overline{D^1} \times \overline{D}$.

Lemma 8.3:6. *If* $(D^1; D)$ *is mapped by* ϕ *and* $(\lambda^1; \theta^1) \in (D^1 \times D)$*, then the germ at* $(\lambda^1, \lambda^1; \theta^1)$*,* $\phi^{(\lambda^1, \lambda^1; \theta^1)}$ *(see* Definition 8.2:25*), is the unique solution obtained by* Theorem 8.3:1 *for the germ at* $(\lambda^1; \theta^1)$*,* $Z^{(\lambda^1; \theta^1)}$*, and* $(D^1; D)$ *is mapped by* $\phi^{(\lambda^1, \lambda^1; \theta^1)}$.

Proof. $\phi^{(\lambda^1, \lambda^1; \theta^1)}$ satisfies the conditions of Theorem 8.3:1, replacing Z by $Z^{(\lambda^1; \theta^1)}$.

We now give some additional assumptions that can be made if necessary about D^1 and D.

We can require D^1 and D to be arbitrarily small.

Lemma 8.3:7. *Let* U *be a neighbourhood of* $(\lambda^0, \lambda^0; \theta^0)$*; then we can choose* D^1 *and* D *with* $D^1 \times D^1 \times D \subseteq U$.

Proof. This is easily seen if U is a ball about $(\lambda^0, \lambda^0; \theta^0)$, and therefore if U is any set containing such a ball; clearly U must contain such a ball since it is open.

Lemma 8.3:8. *Let* V *be a neighbourhood of* $(\lambda^0;\theta^0)$*; then we can choose* D^1 *and* D *such that for all* $\lambda,\mu \in D^1$ *and* $\theta \in D$, $(\lambda;\phi(\lambda,\mu;\theta)) \in V$.

Proof. The function f taking $(\lambda,\mu;\theta)$ to $(\lambda;\phi(\lambda,\mu;\theta))$ is continuous and at $(\lambda^0,\lambda^0;\theta^0)$ it has value $(\lambda^0;\theta^0)$. Define U to be $f^{-1}(V)$; then U is a neighbourhood of $(\lambda^0,\lambda^0;\theta^0)$ and the result follows from Lemma 8.3:7.

In fact the uniqueness condition can be considerably strengthened.

Lemma 8.3:9. *Suppose* $(D^1;D)$ *is mapped by* ϕ*. Let* $\mu \in D^1$ *and* $\theta \in D$.
Suppose that $w : D^1 \to \mathbb{C}^s$ *is a differentiable function satisfying:*

(1) $w(\mu) = \theta$.

(2) *For* $\lambda \in D^1$

$$\frac{\partial w(\lambda)}{\partial \lambda} = Z(\lambda; w(\lambda)). \tag{10}$$

Then for $\lambda \in D^1$, $w(\lambda) = \phi(\lambda,\mu;\theta)$.

Proof. Clearly $w(\lambda) = \phi(\lambda,\mu;\theta)$ satisfies (1) and (2). So it suffices to show that there is at most one w satisfying (1) and (2). This follows from [BIRKHOFF&ROTA,1989], Chapter 6, Section 12, Theorem 11, where we reduce the differential equation to a real-value problem by choosing any $\lambda \in D^1$ and defining $x(t) = w(\mu + t(\lambda-\mu))$, and letting $X(x,t) = Z(\mu + t(\lambda - \mu); x)$.

This strengthens the uniqueness part of Theorem 8.3:1 since D^1 could have been made arbitrarily small without affecting it.

Theorem 8.3:10. *Suppose* $\lambda,\nu,\mu \in D^1$ *and* $\theta, \phi(\mu,\nu;\theta) \in D$*. Then*

$$\phi(\lambda,\nu;\theta) = \phi(\lambda,\mu;\phi(\mu,\nu;\theta)) \tag{11}$$

Proof. Define $w(\lambda') = \phi(\lambda',\nu;\theta)$. Then for $\lambda' \in D^1$, $w(\lambda')$ is defined. Also $w(\mu) = \phi(\mu,\nu;\theta)$ and Lemma 8.3:9, (2) is true. So Display (11) follows from Lemma 8.3:9.

Definition 8.3:11. *For* $x, y \in \mathbb{R}$, *we define* $\mathrm{Re}(x+iy) = x$ *and* $\mathrm{Im}(x+iy) = y$. *For* $\nu = (\nu_1,\ldots,\nu_s)$, *we define* $\mathrm{Re}(\nu) = (\mathrm{Re}(\nu_1),\ldots,\mathrm{Re}(\nu_s))$ *and* $\mathrm{Im}(\nu) = (\mathrm{Im}(\nu_1),\ldots,\mathrm{Im}(\nu_s))$.

Lemma 8.3:12. *If* $X \subseteq \mathbb{R}$ *and* $f : X \to \mathbb{C}$ *or* $f : X \to \mathbb{C}^s$ *is analytic on* X, *then* $\mathrm{Re}(f)$ *and* $\mathrm{Im}(f)$ *are analytic on* X.

Proof. If $f : X \to \mathbb{C}$ then for any $x \in X$, we can convert any power series for f at x to power series for $\mathrm{Re}(f)$ and $\mathrm{Im}(f)$. If $f = (f_1, \ldots, f_s) \in \mathbb{C}^s$ the lemma follows by looking at the f_j.

Lemma 8.3:13. *Suppose* $t_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}^s$, *and* x *and* X *are germs such that*

(1) x *is defined at* t_0 *and* $x(t_0) = x_0$; X *is defined at* $(t_0; x_0)$.

(2) *Considered as germs at* μ *in* t, $\frac{dx(t)}{dt} = X(x(t), t)$.

(3) *If* X *is defined on* $(y, t) \in \mathbb{R}^s \times \mathbb{R}$, $X(y, t)$ *is real.*

Then for all $t \in \mathbb{R}$ *with* $x(t)$ *defined,* $x(t) \in \mathbb{R}^s$.

Proof. By [BIRKHOFF&ROTA,1989], Chapter 6, Section 9, Theorem 8, there is a function x_1 mapping to $\mathbb{R}^s$ defined on a real neighbourhood of t_0 such that $\frac{dx_1(t)}{dt} = X(x_1(t), t)$. Transform the complex differential equation $\frac{dx(t)}{dt} = X(x(t), t)$ into a real one by defining $\bar{x}(t) = (\mathrm{Re}(x(t)), \mathrm{Im}(x(t)))$ and (for $y_1, y_2 \in \mathbb{R}^s$) $\bar{X}((y_1, y_2), t) = (\mathrm{Re}(X(y_1 + iy_2, t)), \mathrm{Im}(X(y_1 + iy_2, t)))$. By Lemma 8.3:12 $\bar{X}$ is analytic. By the uniqueness of solutions to real differential equations ([BIRKHOFF&ROTA,1989], Chapter 6, Section 3, Theorem 1), on any connected real neighbourhood of t_0, $\bar{x}$ is uniquely determined by the two conditions $\bar{x}(t_0) = (x_0, 0)$ and $\frac{d\bar{x}(t)}{dt} = \bar{X}(\bar{x}(t), t)$. However these are also true in a real neighbourhood of t_0 for $\bar{x}(t) = (x_1(t), 0)$, so $\mathrm{Im}(x(t)) = 0$ for all $t \in \mathbb{R}$ in a neighbourhood of t_0. By Lemma 8.3:12 and Lemma 8.2:10, $\mathrm{Im}(x(t)) = 0$ for all $t \in \mathbb{R}$ at which x is defined, as required.

§8.4. Simple Local Threadings

As before, suppose $\lambda^0 \in \mathbb{C}$ and $\theta^0 \in \left(\frac{\mathbb{C}}{2\pi}\right)^s$. If G is simple (Definition 8.4:2) and $G(\lambda^0; \theta^0) = 0$, then there are neighbourhoods V^1 and V of λ^0 and θ^0 respectively such that for any $\theta \in V$ there is exactly one $\lambda \in V^1$ with $G(\lambda; \theta) = 0$; this root has multiplicity 1 in λ.

A simple local threading provides an analytic function $\phi : V^1 \times V^1 \times V \to V$ with the property that for $\lambda, \mu \in V^1$ the map taking θ to $\phi(\lambda, \mu; \theta)$ bijects $\{\theta \mid G(\mu; \theta) = 0\}$ to $\{\theta \mid G(\lambda; \theta) = 0\}$. For fixed μ and θ we can think of the function taking λ to $\phi(\lambda, \mu; \theta)$ as a *Thread* through V; *Simple Local Threadings* are so called as they divide V into such threads.

We will eventually use simple local threadings to show that measures giving distributions of roots are analytic by using them to perturb λ. In this section we state and prove conditions for simple local threadings to exist.

Definition 8.4:1. *Suppose* $x = (x_1, \ldots, x_s)$ *and* $y = (y_1, \ldots, y_s)$ *are two vectors in* $\mathbb{C}^s$. *Then we define* $\bullet$ *by* $x \bullet y = \sum x_j y_j$.

Of course, this is very similar to the definition of $\langle\, , \rangle$ at the start of Chapter 3. However there is no conjugation involved in the definition of $\bullet$, so $x \bullet y$ is actually analytic in the entries of x and y. If these entries are real, we have $x \bullet y = \langle x, y\rangle$.

Let G be a germ at $(\lambda^0; \theta^0)$ such that $G(\lambda^0; \theta^0) = 0$.

Definition 8.4:2. G *is* Simple *if at* $(\lambda^0; \theta^0)$, $G_\lambda \neq 0$ *and* $(\nabla G \bullet \nabla G) \neq 0$.

If the entries of ∇G are real, then the latter inequality is equivalent to $\nabla G \neq 0$.

We will assume that G is simple throughout this section.

Definition 8.4:3. *The vector germ at* $(\lambda^0; \theta^0)$, *Z, is defined by*

$$Z \overset{\text{def}}{=} -\frac{G_\lambda \nabla G}{\nabla G \bullet \nabla G}. \tag{12}$$

Since the germ at $(\lambda^0; \theta^0)$, $\nabla G \bullet \nabla G$, has non-zero value at $(\lambda^0; \theta^0)$, the division in Display (12) is well-defined by Lemma 8.2:3. We now apply Theorem 8.3:1 to this Z.

Definition 8.4:4. *Construct* ϕ, D^1, D *from* Z *as in* Section 8.3.

Thus for $\lambda, \mu \in D^1$ and $\theta \in D$ we have

$$\phi(\lambda, \lambda; \theta) = \theta \tag{13}$$

and

$$\frac{\partial \phi(\lambda', \mu; \theta)}{\partial \lambda'}\Big|_{\lambda'=\lambda} = -\left(\frac{G_\lambda \nabla G}{\nabla G \bullet \nabla G}\right)(\lambda; \phi(\lambda, \mu; \theta)). \tag{14}$$

Definition 8.4:5. *If* ϕ, D^1 *and* D *satisfy* Definition 8.3:5 *with* Z *as in* Display (12), *we say that* ϕ *is a* Simple Local Threading *of* G *at* $(\lambda^0; \theta^0)$, *and, as in* Definition 8.3:5, *say that* $(D^1; D)$ are Mapped *by* ϕ.

Lemma 8.4:6. *Let* $\lambda, \mu \in D^1$ *and* $\theta \in D$. *Then*

$$G(\lambda; \phi(\lambda, \mu; \theta)) = G(\mu; \theta).$$

Proof. Write $f(\lambda) = G(\lambda; \phi(\lambda, \mu; \theta))$; we want to show that $f(\lambda) = G(\mu; \theta)$. Since $\phi(\mu, \mu; \theta) = \theta$, this is true if $\lambda = \mu$. As ϕ and G are analytic, so is f. So it is enough to show that $\frac{df}{d\lambda} = 0$.

However, by the chain rule, we have

$$\begin{aligned} \frac{df}{d\lambda} &= G_\lambda + \nabla G \bullet \frac{\partial \phi}{\partial \lambda} \\ &= G_\lambda + \nabla G \bullet - \left(\frac{G_\lambda \nabla G}{\nabla G \bullet \nabla G} \right) \\ &= 0. \end{aligned}$$

This completes the proof of the lemma.

Lemma 8.4:7. *If* G^1 *and* G^2 *are equivalent and for* $i = 1, 2$, ϕ^i *is a simple local threading of* G^i *at* $(\lambda^0; \theta^0)$, *then there is a neighbourhood* U *of* θ^0 *and a neighbourhood* U^1 *of* λ^0 *such that for any* $\theta \in U$ *and any* $\lambda, \mu \in U^1$ *with* $G^1(\mu; \theta) = 0$, $\phi^1(\lambda, \mu; \theta) = \phi^2(\lambda, \mu; \theta)$.

Proof. We choose U, U^1 and an analytic H such that $(U^1; U)$ is mapped by each ϕ^i; and for any $\lambda, \mu \in U^1$ and $\theta \in U$, at $(\lambda; \phi^1(\lambda, \mu; \theta))$, H is defined and non-zero, and $G^1 = HG^2$. This is possible by Lemma 8.3:8.

Choose $\mu \in U^1$ and $\theta \in U$ with $G^1(\mu; \theta) = 0$. As $G^1 = HG^2$, if $G^1 = 0$ at some point in $U^1 \times U$, then $G^2 = 0$ at that point. In particular, $G^2(\mu; \theta) = 0$.

To prove the lemma we shall use Lemma 8.3:9. Define $w : U^1 \to U$ by $w(\lambda) = \phi^1(\lambda, \mu; \theta)$. By Lemma 8.4:6, $G^1(\lambda; w(\lambda)) = 0$ and so $G^2(\lambda; w(\lambda)) = 0$.

By the definition of ϕ^1,

$$\frac{dw}{d\lambda} = -\left(\frac{G^1_\lambda \nabla G^1}{\nabla G^1 \bullet \nabla G^1} \right) (\lambda; w(\lambda)). \tag{15}$$

Let t be either λ or some θ_j. Then for $\lambda \in U^1$, since $G^2(\lambda; w(\lambda)) = 0$, the following holds at $(\lambda; w(\lambda))$:

$$\frac{\partial G^1}{\partial t} = \frac{\partial (HG^2)}{\partial t} = G^2 \frac{\partial H}{\partial t} + H \frac{\partial G^2}{\partial t} = H \frac{\partial G^2}{\partial t}.$$

Therefore at $(\lambda; w(\lambda))$, $G^1_\lambda = HG^2_\lambda$ and $\nabla G^1 = H\nabla G^2$. So from Display (15) we see that

$$\frac{dw}{d\lambda} = -\left(\frac{G^2_\lambda \nabla G^2}{\nabla G^2 \bullet \nabla G^2}\right)(\lambda; w(\lambda)). \tag{16}$$

Also we have $w(\mu) = \phi^1(\mu, \mu; \theta) = \theta$ and so we deduce from Lemma 8.3:9 applied to ϕ^2 that $\phi^1(\lambda, \mu; \theta) = w(\lambda) = \phi^2(\lambda, \mu; \theta)$. This concludes the proof of the lemma.

Theorem 8.4:8. *We can find $V^1 \subseteq \mathbb{C}$ and $V \subseteq \left(\frac{\mathbb{C}}{2\pi}\right)^s$ and an analytic function $g : V \to \mathbb{C}$ such that*

(1) G *is defined throughout* $V^1 \times V$. *There is a* $(D^1; D)$ *mapped by* ϕ *with* $V^1 \subseteq D^1$ *and* $V \subseteq D$.
(2) $\lambda^0 \in V^1$ *and* $\theta^0 \in V$.
(3) V^1 *is convex and open;* V *is open.*
(4) g *is a root function for* G *on* $V^1 \times V$ *and can be extended analytically to* $V^1 \times \overline{V}$.
(5) *For any* $\lambda, \mu \in V^1$ *the map taking* θ *to* $\phi(\lambda, \mu; \theta)$ *bijects* $\{\theta \in V \mid G(\mu; \theta) = 0\}$ *to* $\{\theta \in V \mid G(\lambda; \theta) = 0\}$, *with inverse taking* θ *to* $\phi(\mu, \lambda; \theta)$.

Proof. Apply Corollary 8.2:30 to find W^1, W and g from G, such that $W^1 \subseteq D^1$ and $\overline{W} \subseteq D$. Apply Lemma 8.3:8 to find $D^1 \subseteq W^1$ and D such that $(D^1; D)$ are mapped by ϕ and for $\lambda, \mu \in D^1$ and $\theta \in D$, $\phi(\lambda, \mu; \theta) \in W$ (hence for $\theta \in D$, $\phi(\lambda^0, \lambda^0; \theta) = \theta \in W$ and so $D \subseteq W$). Apply Lemma 8.3:8 again to find $E^1 \subseteq D^1$ and E such that $(E^1; E)$ are mapped by ϕ and for $\lambda, \mu \in E^1$ and $\theta \in E$, $\phi(\lambda, \mu; \theta) \in D$ (hence $E \subseteq D \subseteq W$). Define $V^1 = E^1$. Define $V' = \{\theta \in E \mid G(\lambda^0; \theta) = 0\}$. Define $V = \{\phi(\lambda, \lambda^0; \theta) \mid \lambda \in E^1 \; \& \; \theta \in V'\}$.

It remains to show that this V^1 and V will do. (1) is trivial since $V^1 = E^1 \subseteq D^1$ and $V \subseteq D \subseteq W$. (2) is trivial since $\lambda^0 \in V^1 = E^1$ and $\theta^0 \in E$ (by construction) and $\theta^0 = \phi(\lambda^0, \lambda^0; \theta^0) \in V$.

Next we show (4). For $\theta \in V$, $\theta = \phi(\lambda, \lambda^0; \theta')$ for some $\lambda \in D^1$ and $\theta' \in V'$. As $\theta' \in V'$, $G(\lambda^0; \theta') = 0$ and so $G(\lambda; \theta) = 0$. So the equation $G(\lambda'; \theta) = 0$ has at least one solution for $\lambda' \in V^1$; namely $\lambda = \lambda'$. Furthermore, any $\lambda' \in V^1 \subseteq W^1$ with $G(\lambda'; \theta) = 0$ must equal $g(\theta)$ by Corollary 8.2:30, so (4) follows. Now we show (5). Take $\lambda, \mu \in V^1$ and $\theta \in V$ with $G(\mu; \theta) = 0$. As $\theta \in V$, there must be a $\mu' \in V^1$ and $\theta' \in V'$ with $\theta = \phi(\mu', \lambda^0; \theta')$. As $\theta' \in V'$, $G(\mu'; \theta) = 0$, and so by (4) $\mu' = \mu = g(\theta)$. So $\theta = \phi(\mu, \lambda^0; \theta')$ with $\theta' \in E \subseteq D$, $\theta \in D$, and as $\lambda, \mu \subseteq V^1 = E^1 \subseteq D^1$, we have, by Theorem 8.3:10,

$$\phi(\lambda, \mu; \theta) = \phi(\lambda, \mu; \phi(\mu, \lambda^0; \theta')) = \phi(\lambda, \lambda^0; \theta').$$

So $\phi(\lambda, \mu; \theta) \in V$, and of course $G(\lambda; \phi(\lambda, \mu; \theta)) = G(\mu; \theta) = 0$. Therefore the map taking θ to $\phi(\lambda, \mu; \theta)$ maps $\{\theta \in V \mid G(\mu; \theta) = 0\}$ to $\{\theta \in V \mid G(\lambda; \theta) = 0\}$. To show that it is a bijection, note that by similar reasoning, if $\theta \in V$ and $G(\lambda; \theta) = 0$, $\phi(\mu, \lambda; \theta) \in \{\theta \in V \mid G(\mu; \theta) = 0\}$, and as by Theorem 8.3:10 again

$$\phi(\mu, \lambda; \phi(\lambda, \mu; \theta)) = \phi(\lambda, \mu; \phi(\mu, \lambda; \theta)) = \theta,$$

the map taking $\theta \in V$ with $G(\lambda; \theta) = 0$ to $\phi(\mu, \lambda; \theta)$ is a two-sided inverse of the map taking $\theta \in V$ with $G(\mu; \theta) = 0$ to $\phi(\lambda, \mu; \theta)$. Hence both are bijections, and (5) is true.

It remains to show (3). V^1 is convex and open by the construction of E^1, since $V^1 = E^1$. Let $f(\theta) = \phi(\lambda^0, g(\theta); \theta)$ and $O = \{\theta \in D \mid g(\theta) \in V^1 \ \&\ f(\theta) \in E\}$. D, E and V^1 are open, and f and g are continuous. So O is open. We will show that in fact $V = O$.

If $\theta \in V$ then $\theta \in D$ and it easily follows that $\theta \in O$. On the other hand, if $\theta \in O$, then $f(\theta) \in E$, and $G(\lambda^0; f(\theta)) = 0$, $f(\theta) \in V'$; also $g(\theta) \in V^1$; hence $\phi(g(\theta), \lambda^0; f(\theta)) \in V$; but by Theorem 8.3:10,

$$\phi(g(\theta), \lambda^0; f(\theta)) = \phi(g(\theta), \lambda^0; \phi(\lambda^0, g(\theta); \theta)) = \theta$$

so $\theta \in V$ and $O \subseteq V$; thus the theorem is proved. □

We now package all the notation used in the previous theorem.

Definition 8.4:9. *A* Detailed Simple Local Threading *of* G *at* $(\lambda^0; \theta^0)$ *is a tuple* (ϕ, V^1, V, g) *where* ϕ *is a simple local threading of* G *at* $(\lambda^0; \theta^0)$ *and* ϕ, V^1, V, g *satisfy the conditions of* Theorem 8.4:8.

Thus in this section we have shown that if G is simple at $(\lambda^0; \theta^0)$, it has a detailed simple local threading.

Lemma 8.4:10. *If* (ϕ, V^1, V, g) *is a detailed simple local threading of* G *at* $(\lambda^0; \theta^0)$, *then it is also a detailed simple local threading of* G *at any* $(\lambda; \theta) \in V^1 \times V$ *such that* $G(\lambda; \theta) = 0$.

Proof. While the definition of detailed simple local threadings is quite complicated, it is routine to check that all the clauses will work equally well with $(\lambda; \theta)$ as with $(\lambda^0; \theta^0)$, with the aid of Lemma 8.3:6 and Lemma 8.2:33.

Lemma 8.4:11. *Suppose* G *has a detailed simple local threading* (ϕ, V^1, V, g). *If* $\lambda, \mu \in V^1$, $\theta \in V$, *and* $G(\mu; \theta) = 0$, *then*

$$\frac{\partial \phi(\lambda', \mu; \theta)}{\partial \lambda'}\Big|_{\lambda'=\lambda} = \left(\frac{\nabla g}{\nabla g \bullet \nabla g}\right) \phi(\lambda, \mu; \theta). \tag{17}$$

Proof. By Lemma 8.2:35, the germ G^2 taking $(\lambda; \theta)$ to $\lambda - g(\theta)$ is equivalent to G. However

$$-\left(\frac{G^2_\lambda \nabla G^2}{\nabla G^2 \bullet \nabla G^2}\right) = \left(\frac{\nabla g}{\nabla g \bullet \nabla g}\right).$$

Thus by Lemma 8.4:7, Display (17) is true for $G(\mu; \theta) = 0$ and $(\lambda, \mu; \theta)$ in some neighbourhood of $(\lambda^0, \lambda^0; \theta^0)$. In particular, it is true for $\mu = g(\theta)$ and $(\lambda; \theta)$ in some neighbourhood of $(\lambda^0; \theta^0)$. If we replace μ in Display (17) by $g(\theta)$, both sides are analytic in $(\lambda; \theta)$, so by Lemma 8.2:10 Display (17) is true for all $\lambda \in V^1$, $\theta \in V$, and $\mu = g(\theta)$. But by the definition of a root function, if $G(\mu; \theta) = 0$ and $\mu \in V^1$, we must have $\mu = g(\theta)$, so the lemma follows.

Definition 8.4:12. *A detailed simple local threading* (ϕ, V^1, V, g) *is said to be* Contained *in an open set* $U \in \mathbb{C} \times \mathbb{C}^s$ *if for* $\lambda, \mu \in V^1$ *and* $\theta \in V$, $(\lambda; \phi(\lambda, \mu; \theta)) \in U$.

Lemma 8.4:13. *Let* U *be an open set containing* $(\lambda^0; \theta^0)$. *Then if* G *is simple at* $(\lambda^0; \theta^0)$, *it has a detailed simple local threading contained in* U.

Proof. This follows by the same construction used in Theorem 8.4:8, using Lemma 8.3:8 and Corollary 8.2:30.

We now prove a theorem which will be crucial when we knit local threadings together to obtain global results.

Theorem 8.4:14. *Suppose that for* $i = 1, 2$, G_i *has a detailed simple local threading at* $(\lambda^i;\theta^i)$, $\Phi_i = (\phi_i, V_i^1, V_i, g_i)$, *and* G_1 *and* G_2 *are equivalent on* $(V_1^1 \cap V_2^1) \times (V_1 \cap V_2)$. *Then for* $\mu \in V_1^1 \cap V_2^1$ *and* $\theta \in V_1 \cap V_2$, $G_1(\mu;\theta) = 0$ *if and only if* $G_2(\mu;\theta) = 0$. *Furthermore if so, and* $\lambda \in V_1^1 \cap V_2^1$, *then* $\phi_1(\lambda,\mu;\theta) = \phi_2(\lambda,\mu;\theta)$.

Proof. If for all $(\lambda^1;\theta^1) \in (V_1^1 \cap V_2^1) \times (V_1 \cap V_2)$, and $i = 1$ or 2, $G_i(\lambda^1;\theta^1) \neq 0$, the theorem is vacuously true. Otherwise suppose without loss of generality that $G_1(\lambda^1;\theta^1) = 0$; then since G_1 and G_2 are equivalent, $G_2(\lambda^1;\theta^1) = 0$. For $i = 1, 2$, by Lemma 8.4:10, Φ_i is a detailed simple local threading of G_i at $(\lambda^1;\theta^1)$.

For $i = 1, 2$, define the function f_i with domain $(V_1^1 \cap V_2^1) \times (V_1 \cap V_2)$ by $f_i(\lambda';\theta') = \phi_i(\lambda', g_i(\theta)';\theta')$. Suppose $\lambda, \mu \in (V_1^1 \cap V_2^1)$ and $\theta \in (V_1 \cap V_2)$, such that $G_1(\mu;\theta) = G_2(\mu;\theta) = 0$. We want to show that $\phi_1(\lambda,\mu;\theta) = \phi_2(\lambda,\mu;\theta)$. Since each g_i is a root function, we have $g_1(\theta) = g_2(\theta) = \mu$. So it is enough to show that $f_1(\lambda;\theta) = f_2(\lambda;\theta)$. We now apply Lemma 8.4:7. As we must have $g_1(\theta^1) = g_2(\theta^1) = \lambda^1$, and $G_1(g_1(\theta');\theta') = G_2(g_2(\theta');\theta') = 0$, $f_1 = f_2$ in a neighbourhood of $(\lambda^1;\theta^1)$. The result follows by Lemma 8.2:10.

□

§8.5. Local Threadings

We now generalise simple local threadings to local threadings. In constructing a local threading of G at a point $(\lambda^0;\theta^0)$, we factorise G, and (if possible) construct a simple local threading for each factor. As we are eventually only going to be interested in real λ, we allow local threadings to ignore a factor H of G provided that there is a real neighbourhood of $(\lambda^0;\theta^0)$ containing no roots of H except possibly $(\lambda^0;\theta^0)$.

Once we have set up this generalisation we shall prove some results showing that if two local threadings (possibly at different points) overlap, they are similar in some ways within this overlap. This is necessary as we will eventually obtain *Global Threadings* by putting together lots of local threadings.

Throughout this section, we suppose that $\lambda^0 \in \mathbb{R}$, $\theta^0 \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, and G is a germ at $(\lambda^0;\theta^0)$ with $G(\lambda^0;\theta^0) = 0$.

Definition 8.5:1. *Let* K *be a simple germ at* $(\lambda^0;\theta^0)$. K *is said to be* Really Simple *if for some root function* g *at* $(\lambda^0;\theta^0)$ *of* K *on some* $D^1 \times D$, *if* $\theta \in D \cap \mathbb{R}^s$, $g(\theta) \in \mathbb{R}$.

Lemma 8.5:2. *If* K *is really simple and* g *is any root function of* K *at* $(\lambda^0;\theta^0)$ *on* $D^1 \times D$, *then for* $\theta \in D \cap \mathbb{R}^s$, $g(\theta) \in \mathbb{R}$.

Proof. There must be some root function satisfying the conditions of Definition 8.5:1, and it is clear from the definition of root functions that this root function agrees with g near θ^0; thus for θ in a neighbourhood of θ^0, if $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, $g(\theta) \in \mathbb{R}$. Consider the function taking $\theta \in D \cap \mathbb{R}^s$ to $\mathrm{Im}(g(\theta))$; by Lemma 8.3:12 this function is analytic on $D \cap \mathbb{R}^s$; it is 0 in a neighbourhood of θ^0, and so as D is convex, the function is everywhere 0. So the lemma follows.

Lemma 8.5:3. *If* K_1, K_2 *are equivalent germs at* $(\lambda^0;\theta^0)$ *then* K_1 *is really simple if and only if* K_2 *is.*

Proof. This follows from Theorem 8.2:34 and Lemma 8.5:2.

Lemma 8.5:4. *If* K *is really simple and* (ϕ, V^1, V, g) *is a detailed simple local threading of* K, *then if* $\lambda, \mu \in V^1 \cap \mathbb{R}$, $\theta \in V \cap \mathbb{R}^s$, *and* $K(\mu;\theta) = 0$, *then* $\phi(\lambda, \mu; \theta) \in \mathbb{R}^s$.

Proof. This follows from Lemma 8.4:11 and Lemma 8.3:13.

Definition 8.5:5. G *is* Locally Threadable *at* $(\lambda^0;\theta^0)$ *if there exist germs* $H, G_1, \ldots, G_r$ *and a neighbourhood* $\mathfrak{U}$ *of* $(\lambda^0;\theta^0)$ *with:*

(1) $G = HG_1 \cdots G_r$.

(2) H *is defined throughout* $\mathfrak{U}$, *but there is no* $(\lambda;\theta) \in \mathfrak{U} \cap (\mathbb{R} \times \mathbb{R}^s)$ *with* $H(\lambda;\theta) = 0$ *except possibly* $(\lambda^0;\theta^0)$.

(3) *Each* G_i *is really simple.*

$(\mathfrak{U}, H, G')$ *where* G' *is the family* $\{G_1, \ldots, G_r\}$ *is said to be a* Factoring *of* G *at* $(\lambda^0;\theta^0)$.

Definition 8.5:6. *A* Local Threading of G at $(\lambda^0;\theta^0)$ *is a family given by* $\{\Phi_1, \ldots, \Phi_r\}$ *where there is a factoring* $(\mathfrak{U}, H, G')$ *of* G *such that we can write* $G' = \{G_1, \ldots, G_r\}$ *where* Φ_j *is a detailed simple local threading of* G_j.

Lemma 8.5:7. *If* G *is locally threadable at* $(\lambda^0;\theta^0)$, *it has a local threading there.*

Proof. This follows from Theorem 8.4:8.

Definition 8.5:8. *The* Range *of a local threading* $T=\{\Phi_1,\ldots,\Phi_r\}$ *of* G *at* $(\lambda^0;\theta^0)$ *with respect to the factoring* (U,H,G') *is* $(\bigcap_{j=1}^r V_j^1; U\cap\bigcap_{i=j}^r V_j)$ *where* $\Phi_j=(\phi_j,V_j^1,V_j,g_j)$

Since each V_j^1 is an open neighbourhood of λ^0 and each V_j is an open neighbourhood of θ^0, the range of a local threading is a pair $(V^1;V)$ where V^1 is an open neighbourhood of λ^0 and V is an open neighbourhood of θ^0. Usually there will only be one factoring of G under consideration; then we will not bother to specify the factoring in question when we refer to the range.

Definition 8.5:9. *Adopt notation as in* Definition 8.5:8. *A* Subrange *of* T *is a pair of open sets* $(W^1;W)$ *with* $\lambda^0\in W^1\subseteq V^1$ *and* $\theta^0\in W\subseteq V$ *such that for* $\lambda,\mu\in W^1$ *and* $\theta\in W$, *and any* ϕ_j, $\phi_j(\lambda,\mu;\theta)\in V$.

Lemma 8.5:10. *Given a local threading* T *at* $(\lambda^0;\theta^0)$ *we can always find a subrange for it.*

Proof. This is immediate from Lemma 8.3:8.

Lemma 8.5:11. *For* $i=1,2$, *let* (U^i,H^i,G'^i) *be two factorings of* G *at* $(\lambda^0;\theta^0)$. *Then up to equivalence at* $(\lambda^0;\theta^0)$ *of germs* H^1 *is the same as* H^2 *and* G'^1 *is the same as* G'^2.

Proof. Using Theorem 8.2:18, write G as a product of irreducible germs $G_1\cdots G_{r_1}H_1\cdots H_{r_2}$, where $G_1,\ldots,G_{r_1}$ are really simple and $H_1,\ldots,H_{r_2}$ are not. Using Lemma 8.5:3, there is only one way in which this can be done, up to equivalence of the families $\{G_1,\ldots,G_{r_1}\}$ and $\{H_1,\ldots,H_{r_2}\}$. We now show that for each i, up to equivalence of germs, the family G'^i equals the family $\{G_1,\ldots,G_{r_1}\}$ and $H^i=\prod H_j$; this will suffice to prove the lemma.

Write $G'^i=(G_1',\ldots,G_r')$. Then each $G_j'\in\mathcal{H}_x'$ and by Lemma 8.2:21, each is irreducible. Since $G=G_1'\ldots G_r'H^i$, by Theorem 8.2:18, the family $\{G_1',\ldots,G_r'\}$ is contained in the family $\{G_1,\ldots,G_{r_1}\}$. Furthermore, if the two families are not equal we must have $H^i=G_jL$ for some j and some germ L, by looking at factorisations of H^i. This is not possible, for if g were a root function for

G_j, $H^i(g(\theta);\theta)$ would equal 0 for θ near θ^0, by Lemma 8.5:2 contradicting the definition of a factoring (in Definition 8.5:5).

Lemma 8.5:12. *Let $F = (U, H, \{G_1, \ldots, G_r\})$ be a factoring of G at $(\lambda^0;\theta^0)$, with a threading $T = \{\Phi_1, \ldots, \Phi_r\}$ and range $(V^1;V)$ where Φ_i is a detailed simple local threading of G_i. Let $(V^1;V)$ be the range of T with respect to F. Suppose $\lambda^1 \in V^1 \cap \mathbb{R}$ and $\theta^1 \in V \cap \left(\frac{\mathbb{R}}{2\pi}\right)^s$ satisfy $G(\lambda^1;\theta^1) = 0$. Let the i such that $G_i(\lambda^1;\theta^1) = 0$ be $i_1, \ldots, i_{r'}$. Then there is a U' and H' such that*

(1) *There is a factoring of G at $(\lambda^1;\theta^1)$ $F' = (U', H', \{G_{i_1}, \ldots, G_{i_{r'}}\})$.*

(2) *$\{\Phi_{i_1}, \ldots, \Phi_{i_{r'}}\}$ is a local threading of G at $(\lambda^1;\theta^1)$ with Φ_{i_j} a detailed simple local threading of G_{i_j}.*

Proof. If $(\lambda^0;\theta^0) = (\lambda^1;\theta^1)$ then let $U' = U$ and $H' = H$; we have nothing to do. Otherwise let U' be an open subset of $U \setminus \{(\lambda^0;\theta^0)\}$ containing $(\lambda^1;\theta^1)$ such that if $i \notin \{i_1, \ldots, i_{r'}\}$, G_i is non-zero on U', and let

$$H' = H \prod_{i \notin \{i_1, \ldots, i_{r'}\}} G_i.$$

Then this lemma follows by checking definitions, with the aid of Lemma 8.4:10.

Theorem 8.5:13. *Suppose that for $i = 1, 2$, G has a factoring $F_i = (U_i, H_i, G_i')$ at $(\lambda^i;\theta^i)$ with a corresponding local threading T_i with range $(V_i^1;V_i)$. Suppose $\lambda' \in V_1^1 \cap V_2^1$ and $\theta' \in V_1 \cap V_2$. Then we find a non-negative integer k and order each G_i' as $G_i = \{G_{i1}, \ldots, G_{ik_i}\}$ and each T_i as $\{\Phi_{i1}, \ldots, \Phi_{ik_i}\}$ so that, letting $\Phi_{ij} = (\phi_{ij}, V_{ij}^1, V_{ij}, g_{ij})$, for all $\lambda \in V_1^1 \cap V_2^1$ and $\theta \in V_1 \cap V_2$:*

(1) *For all i, j, Φ_{ij} is a detailed simple local threading of G_{ij} at $(\lambda^i;\theta^i)$.*

(2) *For $i = 1, 2$, $k \leq k_i$.*

(3) *$G_{ij}(\lambda';\theta') = 0$ if and only if $j \leq k$.*

(4) *For $1 \leq j \leq k$, $g_{1j}(\theta) = g_{2j}(\theta)$.*

(5) *For $1 \leq j \leq k$, G_{1j} is equivalent to G_{2j} on $(V_1^1 \cap V_2^1) \times (V_1 \cap V_2)$.*

(6) *For $1 \leq j \leq k$, $\phi_{1j}(\lambda, \lambda';\theta') = \phi_{2j}(\lambda, \lambda';\theta')$.*

Proof. For $i = 1, 2$, choose k^i and order G_i' as $\{G_{i1}, \ldots, G_{ik_i}\}$ so that $G_{ij}(\lambda';\theta') = 0$ if and only if $j \leq k^i$. Use Lemma 8.5:12 to construct a factoring for G at $(\lambda';\theta')$ of the form $(U_1', H_1', \{G_{i1}, \ldots, G_{ik^i}\})$. Using Lemma 8.5:11, we deduce that $k^1 = k^2$; let $k = k^1 = k^2$; and reorder $\{G_{11}, \ldots, G_{1k}\}$ so that for $1 \leq j \leq k$, G_{1j} is equivalent to G_{2j}. We also reorder the T_i so that Φ_{ij} is a detailed simple local threading of G_{ij} at $(\lambda^i;\theta^i)$. From this we deduce (1), (2), and (3).

Now suppose $j \le k$. By Lemma 8.5:12, we see that Φ_{ij} is a detailed simple local threading of G_{ij} at $(\lambda';\theta')$. Thus in a neighbourhood of θ', $g_{1j} = g_{2j}$. As $(V_1 \cap V_2)$ is convex, and so connected, by Lemma 8.2:10, $g_{1j}(\theta) = g_{2j}(\theta)$ and from this we deduce (4). Write $g_j(\theta) = g_{1j}(\theta) = g_{2j}(\theta)$. It is easily seen that g_j is a root function for each G_{ij} on $(V_1^1 \cap V_2^1) \times (V_1 \cap V_2)$. (5) follows from Theorem 8.2:34. We deduce (6) from Theorem 8.4:14. □

Definition 8.5:14. *Let* $F = (U, H, \{G_1, \ldots, G_r\})$ *be a factoring of* G *with a local threading* $T = \{\Phi_1, \ldots, \Phi_r\}$*, ordered so that* $\Phi_j = (\phi_j, V_j^1, V_j, g_j)$ *is a detailed simple local threading of* G_j *with range* $(V^1;V)$*. Define the* Association Function of T, A_T, *mapping* $V^1 \times V^1 \times \left(\frac{\mathbb{R}}{2\pi}\right)^s \times V$ *to the non-negative integers by*

$$A_T(\lambda,\lambda';\theta,\theta') \\ = \#\{j \mid G_j(\lambda';\theta') = 0 \,\&\, \theta = \phi_j(\lambda,\lambda';\theta')\}. \tag{18}$$

Lemma 8.5:15. *If* $A_T(\lambda,\lambda';\theta,\theta') \neq 0$ *then* $G(\lambda;\theta) = G(\lambda';\theta') = 0$.

Proof. Let notation be as in Definition 8.5:14. If j is some element of the set on the right hand side of Display (18), then $G_j(\lambda';\theta') = 0$ and since $\theta = \phi_j(\lambda,\lambda';\theta')$, $G_j(\lambda;\theta) = G_j(\lambda';\theta') = 0$; hence the lemma follows.

Theorem 8.5:16. *If* T_1 *and* T_2 *are two local threadings with ranges* $(V_1^1;V_1)$ *and* (V_2^1,V_2) *respectively, and* $\lambda,\lambda' \in V_1^1 \cap V_2^1$*, and* $\theta' \in V_1 \cap V_2$ *then*

(1) *If* $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *then*

$$A_{T_1}(\lambda,\lambda';\theta,\theta') = A_{T_2}(\lambda,\lambda';\theta,\theta'). \tag{19}$$

(2) *For* $i = 1,2$ *take* $T_i = (U_i, H_i, G_i')$ *and let* $G^* = \prod_{f \in G_i'} f$. *Then*

$$\sum_{\theta \in (\mathbb{R}/2\pi)^s} A_{T_i}(\lambda,\lambda';\theta,\theta') \tag{20}$$

is 0 *if* $G^*(\lambda';\theta') \neq 0$, *and otherwise is the multiplicity of the solution* $t = \lambda'$ *of the equation* $G^*(t;\theta) = 0$.

Proof. We adopt the notation of Theorem 8.5:13 (apart from θ which has already been defined differently) and without loss of generality order threadings and factorings in Definition 8.5:14 as in that theorem. From (3) we see that

$$G_{1j}(\lambda';\theta') = 0 \iff G_{2j}(\lambda';\theta') = 0 \iff j \leq k.$$

From (6) we see that if $j \leq k$, then

$$\phi_{1j}(\lambda,\lambda';\theta') = \phi_{2j}(\lambda,\lambda';\theta).$$

So the set

$$\{j \mid G_{ij}(\lambda';\theta') = 0 \,\&\, \theta = \phi_{ij}(\lambda,\lambda';\theta')\}$$

is the same for $i = 1$ and for $i = 2$ whence we deduce Display (19) and (1).

To show (2), note that from the foregoing considerations, the sum of Display (20) equals k, while $G_{ij}(\lambda';\theta') = 0$ if and only if $j \leq k$. Furthermore by Corollary 8.2:28, (1), if $j \leq k$ then the equation $G_{ij}(t;\theta') = 0$ has a root of multiplicity exactly 1 at $t = \lambda'$. So $G^*(t;\theta') = \prod_j G_{ij}(t;\theta')$ has a root of multiplicity exactly k at $t = \lambda'$; as this equals Display (20) we are done. □

§8.6. Global Threadings

Now let $\lambda^0 \in \mathbb{R}$ and G be an analytic function, n a positive integer, and U an open subset of $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ containing $\left(\frac{\mathbb{R}}{2\pi}\right)^s$, as in Definition 8.1:4.

Definition 8.6:1. G *is* Globally Threadable *at* λ^0 *if for every* $\theta^0 \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *with* $G(\lambda^0;\theta^0) = 0$, G *is locally threadable at* $(\lambda^0;\theta^0)$.

We will show

Theorem 8.7:1. *If* G *is globally threadable at* λ^0, μ^G *is analytic at* λ^0.

Definition 8.6:2. *A* Global Threading *of* G *at* λ^0 *is a finite list* $T = \{T_1, \ldots, T_r\}$ *and a list* $\{(W_1^1; W_1), \ldots, (W_r^1; W_r)\}$ *such that*

(1) *Each* T_i *is a local threading of* G *at* $(\lambda^0; \theta^i)$ *where* $\theta^i \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *and* $G(\lambda^0; \theta^i) = 0$.

(2) *For each* i, $(W_i^1; W_i)$ *is a subrange of* T_i, *and*

$$\bigcup W_i \supseteq \{\theta \in \left(\tfrac{\mathbb{R}}{2\pi}\right)^s \mid G(\lambda^0; \theta) = 0\}.$$

Suppose the range of each T_i *is* $(V_i^1; V_i)$. *If* $r \geq 1$ *then the* Range *of* T *is* $\bigcap V_i^1$. *If* $r = 0$ *the range is* $\mathbb{C}$. *The* Extent *of* T *is* $\bigcup W_i$.

The range is a open set in $\mathbb{C}$ containing λ^0. The extent is an open set in $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ containing all θ with $G(\lambda^0; \theta) = 0$.

Lemma 8.6:3. *If* G *is globally threadable at* λ^0, *it has a global threading at* λ^0.

Proof. Let $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ be such that $G(\lambda^0; \theta) = 0$. By Lemma 8.5:7 there is a local threading T_θ at $(\lambda^0; \theta)$; let a subrange of this be $(W_\theta^1; W_\theta)$. W_θ is open and contains θ. As $\left(\frac{\mathbb{R}}{2\pi}\right)^s$ is compact, so is $\{\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s \mid G(\lambda^0; \theta) = 0\}$, and we can choose a finite subcover of it by the W_θ; let the T_i be the associated local threadings and we are done.

Lemma 8.6:4. *If* G *has a global threading* T *at* λ^0 *we can find an open interval* R *containing* λ^0 *and such that if* $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *and* $\lambda \in R$ *satisfy* $G(\lambda; \theta) = 0$, *then* θ *is contained in the extent of* T.

Proof. Let E be the extent of T. Since the range of T is a neighbourhood of λ^0, if the lemma is false there must exist sequences $\lambda^i \in \mathbb{R}$ and $\theta^i \in \left(\frac{\mathbb{R}}{2\pi}\right)^s \setminus E$ with each $G(\lambda^i; \theta^i) = 0$ and $\lambda^i \to \lambda^0$. Since $\left(\frac{\mathbb{R}}{2\pi}\right)^s$ is compact, we can, by replacing these sequences with subsequences, assume that in addition the θ^i converge, say to θ. Hence $G(\lambda^0; \theta) = 0$. By the definition of a global threading θ is contained in E. As E is open and $\theta^i \to \theta$ we must have some $\theta^i \in E$, which is a contradiction.

One possibility is that there are no $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ with $G(\lambda^0; \theta) = 0$; then the empty set is a global threading at λ^0. When the empty set is a global threading, we shall call it the *Trivial* global threading. To avoid having to allow for this possibility throughout this section, we prove the following, from which it is obvious from Lemma 8.1:16 that μ^G is analytic (and indeed 0) in a neighbourhood of λ^0.

Lemma 8.6:5. *If* G *has a trivial global threading* T *at* λ^0, *then there is an open interval* R *containing* λ^0 *such that for any* $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, *the equation* $G(\lambda;\theta) = 0$ *has no roots* $\lambda \in R$.

Proof. This is immediate from Lemma 8.6:4, since in this case the extent of T is the empty set.

§8.7. μ^G is analytic at λ^0

In this section we prove the following theorem.

Theorem 8.7:1. *If* G *is globally threadable at* λ^0, *then* μ^G *is analytic at* λ^0.

We will do this by using a partition of unity to rewrite $\mu^G(I)$, for intervals I, as a sum of volume integrals, one for each simple local threading contained in a global threading of G. To show that these integrals are analytic we will use Jacobians with the simple local threadings providing the change of variable.

Using Lemma 8.6:3, we assume that at λ^0, G is globally threadable with global threading $T = \{T_1, \ldots, T_r\}$ and $\{(W_1^1; W_1), \ldots, (W_r^1; W_r)\}$, where the range of T_i is $(V_i^1; V_i)$, and for $1 \leq i \leq r$, $(W_i^1; W_i)$ is a subrange of T_i such that $\bigcup W_i \supseteq \{\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s \mid G(\lambda^0;\theta) = 0\}$ as in Definition 8.6:2, (2). We let R be as in Lemma 8.6:4; by replacing R with $R \bigcap W_i^1$ if necessary we assume that $R \subseteq \bigcap W_i^1$. Let E be the extent of T.

We also define more notation so that for $1 \leq i \leq r$, T_i is the local threading $\{\Phi_{i1}, \ldots, \Phi_{ir_i}\}$ at $(\lambda^i; \theta^i)$ associated with the factoring $(U_i, H_i, \{G_{i1}, \ldots, G_{ir_i}\})$, where each $\Phi_{ij} = (\phi_{ij}, V_{ij}^1, V_{ij}, g_{ij})$ is a detailed simple local threading of G_{ij}.

§§8.7.1. μ^G *in terms of integrals*

Definition 8.7.1:1. *For* $\lambda \in \mathbb{C}$ *and* $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, *let*

$$\mu(\lambda;\theta) = \begin{cases} 0 & \text{if } G(\lambda;\theta) \neq 0; \\ k & \text{if the equation in } \lambda',\ G(\lambda';\theta) = 0, \\ & \quad \text{has a root of multiplicity } k \text{ at } \lambda' = \lambda. \end{cases}$$

Lemma 8.7.1:2. *Let* I *be an interval in* $\mathbb{R}$. *Then*

$$\mu^G(I) = \int_{(\mathbb{R}/2\pi)^s} \sum_{\lambda \in I} \mu(\lambda;\theta)\, d\theta.$$

Proof. In fact this is Lemma 8.1:16, rephrased, since if the roots in λ' of $G(\lambda';\theta) = 0$ are (including repeated ones) $\lambda_1, \ldots, \lambda_n$, then $\mu(\lambda;\theta)$ is the number of j with $\lambda_j = \lambda$, and so $\sum_{\lambda \in I} \mu(\lambda;\theta) = \mu^{G;\theta}(I)$.

Definition 8.7.1:3. *For* $1 \le i \le r$, $\theta \in V_i$ *and* $\lambda \in R$, *let*

$$\mu^*(\lambda;\theta) = \begin{cases} 0 & \text{if } \prod_j G_{ij}(\lambda;\theta) \neq 0; \\ k & \text{if the equation in } \lambda', \prod_j G_{ij}(\lambda';\theta) = 0, \\ & \quad \text{has a root of multiplicity } k \text{ at } \lambda' = \lambda. \end{cases}$$

Lemma 8.7.1:4. *For* $\lambda \in \mathbb{R}$, $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$:

(1) $\mu^*(\lambda;\theta)$ *is well-defined and equals* $\sum_{\theta^0 \in (\mathbb{R}/2\pi)^s} A_{T_i}(\lambda^0, \lambda; \theta^0, \theta)$ *(so, in* Definition 8.7.1:3 *it makes no difference which* V_i *containing* θ *we choose);*

(2) *Unless* $(\lambda;\theta)$ *is equal to some* $(\lambda^i;\theta^i)$, $\mu^*(\lambda;\theta) = \mu(\lambda;\theta)$.

Proof. (1) follows from Theorem 8.5:16. (2) follows from the definition of a factoring, since if $(\lambda;\theta) \in V_i$ but is not equal to $(\lambda^i;\theta^i)$, we must have $H(\lambda;\theta) \neq 0$.

Lemma 8.7.1:5.

$$\mu^G(I) = \int_{(\mathbb{R}/2\pi)^s} \sum_{\lambda \in I} \mu^*(\lambda;\theta)\, d\theta.$$

Proof. This follows from Lemma 8.7.1:4, (2) and Lemma 8.7.1:2, since there are only finitely many $(\lambda^i;\theta^i)$.

§§8.7.2. *Partitions of Unity*

Definition 8.7.2:1. *Let* $\{f_1, \ldots, f_r\}$ *be a set of continuous functions from* $\{\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s \mid G(\lambda^0;\theta) = 0\}$ *to* $[0,1]$ *such that* $\sum_{j=1}^{r} f_i(x) = 1$ *for all* x *and* $f_i(x) = 0$ *for* $x \notin W_i$.

$\{f_1, \ldots, f_r\}$ is a partition of unity subordinate to the cover $\{W_1, \ldots, W_r\}$, and the proof of its existence appears to be part of [BOLLOBÁS,1990], Chapter 6, Exercise 22 (though there is a misprint in the first edition). A more general result is given in [SPIVAK,1965], Chapter 3, Theorem 11.

Lemma 8.7.2:2. *If* $\lambda \in R$ *and* $\theta \in E$ *then*

$$\mu^*(\lambda;\theta) = \sum_{\substack{i,j \\ \theta \in V_{ij} \\ \lambda = g_{ij}(\theta)}} f_i(\phi_{ij}(\lambda^0,\lambda;\theta)). \tag{21}$$

Proof. Since $\theta \in E$ there is an i' with $\theta \in W_{i'}$. Then by Lemma 8.7.1:4, (1),

$$\mu^*(\lambda;\theta) = \sum_{\theta^0} A_{T_{i'}}(\lambda^0,\lambda;\theta^0;\theta)$$

If $A_{T_{i'}}(\lambda^0,\lambda;\theta^0;\theta) \neq 0$ then by Lemma 8.5:15, $G(\lambda^0;\theta^0) = 0$ so $\mu^*(\lambda;\theta)$

$$\begin{aligned} &= \sum_{\theta^0}\left(\sum_i f_i(\theta^0)\right) A_{T_{i'}}(\lambda^0,\lambda;\theta^0,\theta) \\ &= \sum_i \sum_{\theta^0 \in W^i} f_i(\theta^0) A_{T_{i'}}(\lambda^0,\lambda;\theta^0,\theta) \end{aligned}$$

Suppose $\theta^0 \in W^i$ and $A_{T_{i'}}(\lambda^0,\lambda;\theta^0,\theta) \neq 0$. Then for some j' we have $\theta^0 = \phi_{i'j'}(\lambda^0,\lambda;\theta)$ and $G_{i'j'}(\lambda^0;\theta^0) = 0$ (by Definition 8.5:14 and Lemma 8.5:15). $\theta^0 \in V_i$ (since $W_i \subseteq V_i$) and $\theta^0 \in V_{i'}$ (as $\theta \in W_{i'}$ and $\theta^0 = \phi_{i'j'}(\lambda^0,\lambda;\theta)$). We apply Theorem 8.5:13, (6); thus there is a j with $\phi_{ij}(\lambda,\lambda^0;\theta^0) = \phi_{i'j'}(\lambda,\lambda^0;\theta^0)$. By Theorem 8.4:8, (5), $\phi_{i'j'}(\lambda,\lambda^0;\theta^0) = \theta$; hence $\theta \in V_i$. As $\theta \in W_{i'} \subseteq V_{i'}$ we deduce from Theorem 8.5:16, (1) that $A_{T_i}(\lambda^0,\lambda;\theta^0,\theta) = A_{T_{i'}}(\lambda^0,\lambda;\theta^0,\theta)$, and so $\mu^*(\lambda;\theta)$

$$= \sum_i \sum_{\theta^0 \in W^i} f_i(\theta^0) A_{T_i}(\lambda^0, \lambda; \theta^0, \theta)$$
$$= \sum_{\substack{i,j \\ \theta \in V_{ij} \\ \lambda = g_{ij}(\theta)}} f_i(\phi_{ij}(\lambda^0, \lambda; \theta))$$

by Definition 8.5:14 and Lemma 8.5:4, and because if $f_i(\phi_{ij}(\lambda^0, \lambda; \theta)) \neq 0$, $\phi_{ij}(\lambda^0, \lambda; \theta) \in W_i$, so as before $\theta \in V_i \subseteq V_{ij}$. This proves the lemma.

□

Definition 8.7.2:3. *For* $1 \leq i \leq r$, $1 \leq j \leq r_i$, *and* I *an interval contained in* $\mathbb{R}$, *define*

$$C_{ij}(I) = \int_{\substack{\theta \in V_{ij} \\ g_{ij}(\theta) \in I}} f_i(\phi_{ij}(\lambda^0, g_{ij}(\theta); \theta))\, d\theta.$$

Theorem 8.7.2:4.

$$\mu^G(I) = \sum_{i,j} C_{ij}(I).$$

Proof. From Lemma 8.7.1:5 we have

$$\begin{aligned}
\mu^G(I) &= \int_{(\mathbb{R}/2\pi)^s} \sum_{\lambda \in I} \mu^*(\lambda; \theta)\, d\theta \\
&= \int_E \sum_{\lambda \in I} \mu^*(\lambda; \theta)\, d\theta \quad \text{(by Lemma 8.6:4)} \\
&= \int_E \sum_{\lambda \in I} \sum_{\substack{i,j \\ \theta \in V_{ij} \\ \lambda = g_{ij}(\theta)}} f_i(\phi_{ij}(\lambda^0, \lambda; \theta))\, d\theta \quad \text{(by Lemma 8.7.2:2)} \\
&= \sum_{i,j} \int_{\substack{\theta \in V_{ij} \\ g_{ij}(\theta) \in I}} f_i(\phi_{ij}(\lambda^0, g_{ij}(\theta); \theta))\, d\theta.
\end{aligned}$$

This proves the theorem. □

To show Theorem 8.7:1, it will be enough to show that the cumulative distribution function taking λ to $\mu^G(-\infty, \lambda)$ is analytic for $\lambda \in R$. We will in fact show that if $\lambda^1 \in R$ and $R' = \{\lambda \in R \mid \lambda > \lambda^1\}$ then the function taking λ to $\mu^G(\lambda^1,\lambda)$ is analytic for $\lambda \in R'$. This will do, since $\mu^G(-\infty,\lambda) = \mu^G(-\infty,\lambda^1] + \mu^G(\lambda^1,\lambda)$.

By Theorem 8.7.2:4 this follows from the following theorem.

Theorem 8.7.2:5. *$C_{ij}(\lambda^1,\lambda)$ is analytic for $\lambda \in R'$.*
We will prove this in the rest of this section.

Definition 8.7.2:6.
(1) *For $\lambda \in R'$ define $V_\lambda = \{\theta \in V_{ij} \cap \left(\frac{\mathbb{R}}{2\pi}\right)^s \mid g_{ij}(\theta) \in (\lambda^1,\lambda)\}$.*
(2) *For $\lambda, \mu \in V_i^1$, define*

$$k_{\mu\leftarrow\lambda}(t) = \frac{\mu-\lambda^1}{\lambda-\lambda^1}(t-\lambda^1)+\lambda^1.$$

(3) *Define*

$$S = \{(\lambda,\mu;\theta) \mid \lambda,\mu \in \overline{V_i^1}\&\theta \in \overline{V_{ij}}\&k_{\mu\leftarrow\lambda}(g_{ij}(\theta)) \in \overline{V_i^1}\}.$$

(4) *For $(\lambda,\mu;\theta) \in S$ define*

$$h_{\mu\leftarrow\lambda}(\theta) = \phi_{ij}(k_{\mu\leftarrow\lambda}(g_{ij}(\theta)), g_{ij}(\theta);\theta). \tag{22}$$

Note that $k_{\mu\leftarrow\lambda}$ is analytic and bijects (λ^1,λ) to (λ^1,μ).

Lemma 8.7.2:7.
(1) *S is compact.*
(2) *If $\lambda,\mu \in R'$ and $\theta \in V_\mu$ then $(\lambda,\mu;\theta) \in \mathrm{Int}(S)$.*

Proof. (1) follows as S is closed and bounded. (2) follows as $(\lambda,\mu;\theta) \in \{(\lambda,\mu;\theta) \mid \lambda,\mu \in V_i^1\&\theta \in V_{ij}\&k_{\mu\leftarrow\lambda}(g_{ij}(\theta)) \in V_i^1\}$ which is an open subset of S.

Lemma 8.7.2:8. $h_{\mu\leftarrow\lambda}(\theta)$ *is analytic for* $(\lambda,\mu;\theta)\in S$.

Proof. This follows as it was obtained entirely by composition of analytic functions, and from the information that ϕ_{ij} can be extended analytically to $\overline{V^1_{ij}}\times\overline{V^1_{ij}}\times\overline{V_{ij}}$, (Theorem 8.4:8, (1) and Definition 8.3:5, (3)); that g_{ij} can be extended analytically to $\overline{V_{ij}}$ (Theorem 8.4:8, (4)); and that its value on the frontier of V_{ij} must lie within $\overline{V^1_{ij}}$ since it maps V_{ij} into V^1_{ij}.

Lemma 8.7.2:9. $h_{\mu\leftarrow\lambda}$ *is a bijection from* V_λ *to* V_μ*, and has inverse* $h_{\lambda\leftarrow\mu}$.

Proof. Define $V'_\lambda=\{\theta\in V_{ij}\mid g_{ij}(\theta)\in(\lambda^1,\lambda')\}$. It will be sufficient to show the following:

(1) $h_{\mu\leftarrow\lambda}(V_\lambda)\subseteq\left(\frac{\mathbb{R}}{2\pi}\right)^s$. This follows from Lemma 8.5:4, as $G_{ij}(g_{ij}(\theta);\theta)$ is necessarily 0.

(2) $h_{\mu\leftarrow\lambda}(V'_\lambda)\subseteq V'_\mu$ and $h_{\lambda\leftarrow\mu}$ is an inverse to $h_{\mu\leftarrow\lambda}$. This follows from Theorem 8.4:8, (5).
This proves Lemma 8.7.2:9.

Lemma 8.7.2:10. *For* $\lambda,\lambda'\in R'$ *and* $\theta\in V_{\lambda'}$,

$$\phi_{ij}(\lambda^0,g_{ij}(h_{\lambda\leftarrow\lambda'}(\theta));h_{\lambda\leftarrow\lambda'}(\theta)) = \phi_{ij}(\lambda^0,g_{ij}(\theta);\theta).$$

Proof. This follows from Theorem 8.3:10 and as, from Theorem 8.4:8 and Display (22), $g_{ij}(h_{\lambda\leftarrow\lambda'}(\theta))=k_{\lambda\leftarrow\lambda'}(g_{ij}(\theta))$.

Lemma 8.7.2:11. $J(h_{\mu\leftarrow\lambda})(\theta)$ *is positive for* $\theta\in V_\lambda$.

Proof. $J\stackrel{\text{def}}{=}J(h_{\mu\leftarrow\lambda})(\theta)$ is always real since $h_{\mu\leftarrow\lambda}(\theta)$ maps $V_\lambda\subseteq\left(\frac{\mathbb{R}}{2\pi}\right)^s$ to $V_\mu\subseteq\left(\frac{\mathbb{R}}{2\pi}\right)^s$. J is always non-zero since $h_{\mu\leftarrow\lambda}$ has inverse $h_{\lambda\leftarrow\mu}$, so the product of their Jacobians is always 1. J must be a continuous function of λ,μ, since $h_{\mu\leftarrow\lambda}$ is analytic in λ and μ. Hence J always has the same sign, since R' is path-connected, by the intermediate value theorem. $h_{\lambda\leftarrow\lambda}$ is the identity transformation (from Display (13)), and so $J(h_{\lambda\leftarrow\lambda})=1$. Thus $J(h_{\lambda\leftarrow\lambda})$ is always positive, as required.

Lemma 8.7.2:12. $J(h_{\mu\leftarrow\lambda})(\theta)$ *is analytic and bounded for* $(\lambda, \mu; \theta) \in S$.

Proof. J's analyticity follows from Lemma 8.7.2:8; its boundedness follows as S is compact (Lemma 8.7.2:7, (1)).

Lemma 8.7.2:13. *For any fixed* $\lambda' \in R'$ *there is an open set* $O \subseteq \mathbb{C}$ *containing* R' *such that the function taking* $\lambda \in O$ *and* $\theta \in \overline{V_\lambda}$ *to* $J(h_{\lambda\leftarrow\lambda'})(\theta)$ *is analytic in* μ*, continuous in* θ*, and bounded.*

Proof. Apply Lemma 8.7.2:12 and observe that the function taking λ to $(\lambda, \lambda'; \theta)$ is continuous.

We now refer to a theorem from [SPIVAK,1965], (Chapter 3, Theorem 13), justifying the use of Jacobians in changes of variable.

Theorem 8.7.2:14. *Let* $A \subset \mathbb{R}^n$ *be an open set and* $g : A \to \mathbb{R}^n$ *a 1-1, continuously differentiable function such that* $\det g'(x) \neq 0$ *for all* $x \in A$. *If* $f : g(A) \to \mathbb{R}$ *is an integrable function then*

$$\int_{g(A)} f = \int_A (f \circ g) \cdot |\det g'|.$$

Here g' is the matrix of derivatives of g, so $\det g'(\theta) = J(g)(\theta)$. I am not going to define an *Integrable* function, but in particular $f : g(A) \to \mathbb{R}$ is integrable if it is continuous and bounded and $g(A)$ is bounded.

Now fix $\lambda' \in R'$. Let $\lambda \in R'$. We have

$$\begin{aligned} C_{ij}(\lambda^1, \lambda) &= \int_{\substack{\theta \in V_{ij} \\ g_{ij}(\theta) \in (\lambda^1, \lambda)}} f_i(\phi_{ij}(\lambda^0, g_{ij}(\theta); \theta))\, d\theta \\ &= \int_{\theta \in V_\lambda} f_i(\phi_{ij}(\lambda^0, g_{ij}(\theta); \theta))\, d\theta \\ &= \int_{\theta \in V_{\lambda'}} f_i(\phi_{ij}(\lambda^0, g_{ij}(h_{\lambda\leftarrow\lambda'}(\theta)); h_{\lambda\leftarrow\lambda'}(\theta))) J(h_{\lambda\leftarrow\lambda'})(\theta)\, d\theta; \end{aligned}$$

by Theorem 8.7.2:14 and Lemma 8.7.2:11

$$= \int_{\theta \in V_{\lambda'}} f_i(\phi_{ij}(\lambda^0, g_{ij}(\theta); \theta)) J(h_{\lambda\leftarrow\lambda'})(\theta)\, d\theta \qquad (23)$$

by Lemma 8.7.2:10.

To prove that this is an analytic function of λ, we need some lemmas about integrals of analytic functions.

Lemma 8.7.2:15. *Let* O *be an open set in* $\mathbb{C}$ *and* K *a compact subset of* $\left(\frac{\mathbb{R}}{2\pi}\right)^s$. *Suppose* $p : K \to \mathbb{C}$ *is continuous and* $q : O \times K \to \mathbb{C}$ *is analytic. Then the function* $F(\lambda) = \int_K p(\theta)q(\lambda;\theta)\,d\theta$ *is analytic for* $\lambda \in O$.

Proof. This is immediate from part of [HERVÉ,1987], Chapter 1, Section 4, Corollary 2.

Lemma 8.7.2:16. Lemma 8.7.2:15 *remains true if we drop the requirement that* K *is compact and replace it with requirements that* K *be open in* $\left(\frac{\mathbb{R}}{2\pi}\right)^s$ *and that* $|p|$ *and* $|q|$ *are bounded.*

Proof. For $\delta > 0$ let K_δ be the set $\{\theta \in K \mid \forall \theta' \in \left(\frac{\mathbb{R}}{2\pi}\right)^s \setminus K \, \|\theta - \theta'\|_2 \geq \delta\}$. Each K_δ is closed and (as it is a subset of $\left(\frac{\mathbb{R}}{2\pi}\right)^s$) compact; the K_δ form a chain of sets and as K is open $\bigcup_{\delta>0} K_\delta = K$. Define

$$F_\delta(\lambda) = \int_{K_\delta} p(\theta)q(\lambda,\theta)\,d\theta.$$

Then by Lemma 8.7.2:15, F_δ is analytic. Let $N_\delta = \int_{K \setminus K_\delta} 1$; then $N_\delta \to 0$ as $\delta \to 0$. Let P be an upper bound for $|p|$ and Q an upper bound for $|q|$. Then $F(\lambda)$ is defined and $|F_\delta(\lambda) - F(\lambda)| \leq PQN_\delta$ for any λ. So the functions F_δ converge uniformly to F throughout O, and so by Theorem 8.3:4, $F(\lambda)$ is analytic.

We now apply this to Display (23). Let $p(\theta) = \Big| f_i(\phi_{ij}(\lambda^0, g_{ij}(\theta);\theta))$ and $q(\lambda;\theta) = J(h_{\lambda \leftarrow \lambda'})(\theta)$. $p(\theta)$ is bounded and continuous since f_i is, by its definition in Definition 8.7.2:1. We use Lemma 8.7.2:13 to find O. Hence $q(\lambda;\theta)$ is bounded and analytic. Thus we can apply Lemma 8.7.2:16 to show that Display (23) is analytic in λ. This proves Theorem 8.7.2:5 and Theorem 8.7:1. □

§8.8. When $\mu^G(\{\lambda\}) \neq 0$

In this section we examine one reason why μ^G might not be analytic at λ^0, when it contains point masses. We find a criterion for this.

Write μ for Lebesgue measure in $\left(\frac{\mathbb{R}}{2\pi}\right)^s$.

Lemma 8.8:1. *If* $\mu^G(\{\lambda\}) \neq 0$ *and* $X = \{\theta \in (\frac{\mathbb{R}}{2\pi})^s \mid G(\lambda;\theta) = 0\}$ *then* $\mu(X) \neq 0$.
Proof. This is a consequence of Lemma 8.1:16 applied to $I = \{\lambda\} = [\lambda, \lambda]$.

Lemma 8.8:2. *If* $\mu^G(\{\lambda\}) \neq 0$ *then for all* $\theta \in (\frac{\mathbb{R}}{2\pi})^s$, $G(\lambda;\theta) = 0$.
Proof. Let $f(\theta) = G(\lambda;\theta)$. Let $X = \{\theta \mid f(\theta) = 0\}$; by Lemma 8.8:1 X has non-zero measure. Suppose the lemma is false so f is not 0 throughout $(\frac{\mathbb{R}}{2\pi})^s$. Thus by Lemma 8.2:10, f is not identically 0 in any open set in $(\frac{\mathbb{R}}{2\pi})^s$. We apply Lemma 8.2:38. So for every $\theta \in (\frac{\mathbb{R}}{2\pi})^s$ we can find an set U_θ open in $(\frac{\mathbb{R}}{2\pi})^s$ such that $X \cap U_\theta$ has zero measure. Since $(\frac{\mathbb{R}}{2\pi})^s$ is compact, choose a finite open cover $U_{\theta_1}, \ldots$ of $(\frac{\mathbb{R}}{2\pi})^s$; then $\mu(X) \leq \sum \mu(X \cap U_{\theta_i}) = 0$; a contradiction.

Definition 8.8:3. *Let* U' *be the component of* U *containing* $(\frac{\mathbb{R}}{2\pi})^s$.

Lemma 8.8:4. *If* $\mu^G(\{\lambda\}) \neq 0$ *then for all* $\theta \in U'$, $G(\lambda;\theta) = 0$.
Proof. Let f be as in the proof of Lemma 8.8:2. Choose some $\theta^0 \in (\frac{\mathbb{R}}{2\pi})^s$; by Lemma 8.2:10 it suffices to show that that $f = 0$ in a neighbourhood of θ^0. However this follows from Lemma 8.8:2, since all partial derivatives of f at θ^0 to any order must be 0; thus the power series for f at θ^0 must have just zero coefficients.

Now suppose $\mu^G(\{\lambda\}) \neq 0$.

Then we know that for all $\theta \in U'$, $G(\lambda;\theta) = 0$. Note that at any $\theta^0 \in U'$, the function taking $(\lambda';\theta')$ to $\lambda' - \lambda$, considered as a germ at $(\lambda^0;\theta^0)$, is irreducible by Lemma 8.2:21 and divides G by Lemma 8.2:20; so by Lemma 8.2:24, there is an analytic H with $G(\lambda';\theta') = (\lambda' - \lambda)H(\lambda';\theta')$. By dividing Display (1) by $\lambda' - \lambda$ we see that we can write

$$H(\lambda';\theta') = \lambda'^{n-1} + \sum_{j=0}^{n-2} \lambda'^j h^j(\theta') + \frac{h'(\theta')}{(\lambda' - \lambda)}$$

where each h^j and h' is analytic in $\theta' \in U'$. However as H is analytic for all λ', we must have $h' \equiv 0$, and so we can write H in the same way as we wrote G in Definition 8.1:4.

For any θ we have $\mu^{G;\theta} = \delta_\lambda + \mu^{H;\theta}$, and hence $\mu^G = \frac{1}{n}\delta_\lambda + \frac{n-1}{n}\mu^H$. Thus, for example, if H is globally threadable at λ^0; then from Theorem 8.7:1 it will follow that μ^H is analytic at λ^0; and thus that near λ^0, μ^G is the sum of a measure which is analytic at λ^0 and a point mass $\frac{1}{n}\delta_\lambda$.

§8.9. Application to Lattices

Let $L = (G, \Upsilon, f)$ be a weighted lattice. Define notation as in Chapter 6, so for example define B to be the corresponding adjacency function and operator. Let $n = |\Upsilon|$. We will soon assume that $r = 0$, so that G contains no elements of finite order apart from the identity, and that f is a well-behaved lattice weighting; a term which will shortly be defined. These will then be assumed for the rest of the chapter.

§§8.9.1. *If* $r > 0$

We now find out how, if $r > 0$, we can write the spectral measures, and in particular *the* spectral measure, as a spectral measure of a lattice where $r = 0$.

Suppose $r > 0$. Define terms as in Theorem 6.1:2. Write G^{fin} for the subgroup of G generated by $\{g_1, \ldots, g_r\}$, and G^{∞} for the subgroup of G generated by $\{g_{r+1}, \ldots, g_{r+s}\}$. Thus we can write any element of G uniquely as $g^{\mathrm{fin}} + g^{\infty}$ where $g^{\mathrm{fin}} \in G^{f}$ and $g^{\infty} \in G^{\infty}$. Define the lattice weighting f' on $(G^{\infty}, G^{\mathrm{fin}} \times \Upsilon)$ by $f'(g, (g_1, \upsilon_1), (g_2, \upsilon_2)) = f(g + g_2 - g_1, \upsilon_1, \upsilon_2)$. Let B be the weighted adjacency operator obtained from f and B' that obtained from f'. Given $x \in H(G \times \Upsilon)$ define $x' \in H(G^{\infty}, G^{\mathrm{fin}} \times \Upsilon)$ by $(x')_{(g^{\infty},(g^{\mathrm{fin}},\upsilon))} = x_{(g^{\infty}+g^{\mathrm{fin}},\upsilon)}$. Then the map $x \mapsto x'$ is an isomorphism between $H(G \times \Upsilon)$ and $H(G^{\infty} \times (G^{\mathrm{fin}} \times \Upsilon))$ commuting with $\langle\, , \rangle$, and it is easily seen that $\forall x, y \in H(G \times \Upsilon)$, $\langle Bx, y\rangle = \langle B'x', y'\rangle$; thus the spectral measure $\mu_{x',y'}$ (obtained from B') and $\mu_{x,y}$ (obtained from B) are equal.

Thus, from now on in this chapter, we assume that $r = 0$, so that G has a basis $g_1, \ldots, g_s$ of elements of infinite order.

Definition 8.9.1:1. *Well-Behaved Lattice Weightings*

In this chapter it will be necessary to impose a further condition on lattice weightings. Recall that in Definition 3.1:1, (1), where we defined a lattice weighting, one of the conditions for f to be a lattice weighting was that $\sum_{g,\upsilon_1,\upsilon_2} |f(g, \upsilon_1, \upsilon_2)|$ is finite. We now strengthen this condition.

Definition 8.9.1:2.

(1) *A* Well-Behaved Lattice Weighting *on* (G, Υ) *is a lattice weighting* f *on* (G, Υ)*, such that for some open set* $W \subseteq G^{\backslash}$ *containing* G'*, and for all* $\gamma \in W$*,*

$$\sum_{g,\upsilon_1,\upsilon_2} |\gamma(g)f(g,\upsilon_1,\upsilon_2)| < \infty. \tag{24}$$

(2) *A* Well-Behaved Weighted Lattice *on* (G, Υ) *is a weighted lattice* (G, Υ, f)*, where* f *is a well-behaved lattice weighting.*

In particular any lattice weighting on (G, Υ) with finite support is well-behaved. So using the identification in Section 3.1, any lattice is well-behaved.

For the rest of this chapter we assume that L is a well-behaved weighted lattice, and we consider W to be an open set satisfying Definition 8.9.1:2. In fact, I shall use the next lemma to restrict the sort of set W can be, without loss of generality.

Lemma 8.9.1:3. *We can choose* $\delta > 0$ *such that* L *is well-behaved with respect to the set* $(\{z \mid |z| \in (1-\delta, 1+\delta)\})^s$.

Proof. $1^s \in G' \subset W$, so choose $\delta > 0$ so that $(\{z \mid |1-z| < \delta\})^s \subseteq W$. In particular $(1-\delta, 1+\delta)^s \subseteq W$. However it is obvious that only the magnitudes of the co-ordinates of γ are relevant in Display (24), and the lemma follows immediately.

For the rest of this chapter, therefore, I will assume that W is of the form $(\{z \mid |z| \in (r, R)\})^s$, where $0 < r < 1 < R$.

We showed in Lemma 6.4:16 that B^γ is continuous in γ. The main use of the assumption that L is well-behaved is the following:

Theorem 8.9.1:4. B^γ *is analytic for* $\gamma \in W$.

Proof. By Display (6.10) we have

$$(B^\gamma)_{\upsilon_2\upsilon_1} = \sum_g f(-g, \upsilon_1, \upsilon_2)\gamma(g).$$

We are going to use Theorem 8.3:4. Choose any enumeration $g^1, \dots$ of G – for example by choosing a basis of G and bijecting $\mathbb{Z}^s$ to the positive integers in one of the various standard ways available. Define

$$h_n(\gamma) = \sum_{i=1}^{n} f(-g^i, v_1, v_2)\gamma(g^i).$$

For $\gamma \in W$, no co-ordinate of γ is 0. So for each $g \in G$, the function taking γ to $\gamma(g)$ is analytic in W, so each $h_n(\gamma)$ is analytic in W. By Theorem 8.3:4, it will be enough to show that on every compact subset of W, the h_n tend uniformly to $\sum_g f(-g, v_1, v_2)\gamma(g)$. So suppose $\epsilon > 0$ and a compact set V to be given. We will show that we can choose an n such that for any $\gamma \in V$,

$$\sum_{i=n+1}^{\infty} |f(-g^i, v_1, v_2)\gamma(g^i)| \leq \epsilon.$$

First, we show that we may assume that V has a particular form. Write $\gamma = (\gamma_1, \dots, \gamma_s)$. We know that $\gamma \in W$ if and only if for all j, $|\gamma_j| \in (r, R)$. Define $r' = \inf\{|\gamma_j| \mid (\gamma_1, \dots, \gamma_s) \in V \ \& \ 1 \leq j \leq s\}$. Then because V is compact this infimum is attained somewhere inside V which itself is inside W, so we must have $r < r'$. Similarly, define $R' = \sup\{|\gamma_j| \mid (\gamma_1, \dots, \gamma_s) \in V \ \& \ 1 \leq j \leq s\}$; then we have $R' < R$. Clearly $V \subseteq V' \stackrel{\text{def}}{=} (\{z \mid |z| \in [r', R']\})^s$, and V' is also compact, so we may assume $V = V'$.

Let $g_1, \dots, g_s$ be a basis of G. We know that any $g \in G$ can be written uniquely as $\sum l_j g_j$ with all $l_j \in \mathbb{Z}$; write $l_j(g) = l_j$, so for any $g \in G$ we have $g = \sum l_j(g) g_j$. Define

$$\sigma(l) = \begin{cases} +1 & \text{if } l \geq 0; \\ -1 & \text{if } l < 0. \end{cases}$$

and define $\sigma(g) = (\sigma(l_1(g)), \dots, \sigma(l_s(g)))$. There are only 2^s possible values for $\sigma(g)$ as g ranges over G. If τ is any of these values write $G_\tau = \{g \in G \mid \sigma(g) = \tau\}$. It suffices to show that there is an n_τ such that for any $\gamma \in V$.

$$\sum_{\substack{i \geq n_\tau + 1 \\ g^i \in G_\tau}} |f(-g^i, v_1, v_2)\gamma(g^i)| \leq \frac{\epsilon}{2^s}, \tag{25}$$

because then we can choose n to be the largest of the n_τ.

So this is what we shall do. Suppose τ given. γ only occurs in Display (25) in $\gamma(g^i)$, and there only the magnitude matters. Write $\gamma = (\gamma_1, \ldots, \gamma_s)$; then $\gamma(g^i) = \prod \gamma_j^{l_j(g^i)}$. The magnitude of this is maximised for $\gamma \in V$ by $\gamma = (\gamma_1, \ldots, \gamma_s)$ where

$$\gamma_j \stackrel{\text{def}}{=} \begin{cases} R' & \text{if } \sigma(l_j(g^i)) = +1; \\ r' & \text{if } \sigma(l_j(g^i)) = -1. \end{cases}$$

Furthermore, because τ determines each $\sigma(l_j(g^i))$ for $g^i \in G_\tau$, this γ maximises the left-hand side of Display (25). So we only need to show that there is an n_τ such that for this particular γ, Display (25) holds. However, because $r < r'$ and $R' < R$, this γ is in W. Therefore, because L is well-behaved,

$$\sum_{\substack{i=1 \\ g^i \in G_\tau}}^{\infty} |f(-g^i, \upsilon_1, \upsilon_2)\gamma(g^i)| < \infty.$$

Hence, by the convergence of this sum, the result is proved.

□

§§8.9.2. *The Characteristic Function*

Define $F : \mathbb{C} \times W \to \mathbb{C}$ by

$$F(\lambda; \gamma) = \det(\lambda I - B^\gamma).$$

The function taking x to e^{ix} is analytic on $\left(\frac{\mathbb{C}}{2\pi}\right)$, and bijects between $\left(\frac{\mathbb{C}}{2\pi}\right)$ and the set of non-zero complex numbers. In particular it bijects $\left(\frac{\mathbb{R}}{2\pi}\right)$ and Π.

Definition 8.9.2:1. *For $\theta \in \left(\frac{\mathbb{C}}{2\pi}\right)^s$ with $\theta = (\theta_1, \ldots, \theta_s)$, we define $e^{i\theta}$ by*

$$e^{i(\theta_1, \ldots, \theta_s)} \stackrel{\text{def}}{=} (e^{i\theta_1}, \ldots, e^{i\theta_s}).$$

Now let $U = \{\theta \in \left(\frac{\mathbb{C}}{2\pi}\right)^s \mid e^{i\theta} \in W\}$. U is open. Define $G : \mathbb{C} \times U \to \mathbb{C}$ by

$$G(\lambda; \theta) = F(\lambda; e^{i\theta}).$$

Definition 8.9.2:2. *An analytic function* $G : \mathbb{C} \times U \to \mathbb{C}$ *where* U *is an open subset of* $\left(\frac{\mathbb{C}}{2\pi}\right)^s$ *containing* $\left(\frac{\mathbb{R}}{2\pi}\right)^s$ *is a* Real-Root *function of* Degree n *if*

(1) G *and* n *satisfy* Definition 8.1:4.

(2) $G(\lambda;\theta) = 0$ *has a complete set of* n *real roots in* λ, *for any* $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$.

Lemma 8.9.2:3. G *as obtained above from lattices or well-behaved weighted lattices is a real-root function of degree* n.

Proof. $B^{e^{i\theta}}$ is an $n \times n$ matrix. By Theorem 8.9.1:4, its entries are analytic in θ. The lemma follows from the definitions of F and G, because the eigenvalues of $B^{e^{i\theta}}$ are precisely the roots of the characteristic equation of $B^{e^{i\theta}}$, or the roots in λ of $F(\lambda; e^{i\theta}) = 0$, or the roots in λ of $G(\lambda;\theta) = 0$, which are hence all real, since for $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, by Lemma 6.4:17, $B^{e^{i\theta}}$ is Hermitian, while by Theorem 4:1, (1), all its eigenvalues are real.

Definition 8.9.2:4. *Let* μ *be the* spectral measure *of* B. *Let* μ^θ *be the* spectral measure *of* $B^{e^{i\theta}}$.

Lemma 8.9.2:5. $\mu = \mu^G$.

Proof. By Display (4.2), μ^θ is the average of the point measures corresponding to the eigenvalues (including repetitions) of $B^{e^{i\theta}}$. As before all the roots of $G(\lambda;\theta) = 0$ are real. Hence we deduce

$$\mu^\theta = \mu_T^{G;\theta} = \mu^{G;\theta}.$$

Combining Theorem 6.5:2 and Lemma 8.1:14 we see that

$$\mu = \mu^G.$$

We now consider some of the implications of Theorem 8.7:1 in terms of finding when μ^G is analytic at some λ^0 for general real-root functions G, rather than ones obtained just from lattices and weighted lattices. The conditions we shall obtain will depend largely on derivatives of G. Of course, for lattices and weighted lattices these can be obtained by expanding out the whole characteristic polynomial in terms of the θ_j. But in practice this may not be the best method, since if A is a $n \times n$ matrix-valued function, the derivative of the determinant of A is the sum of the determinants $A(i)$, for $1 \le i \le n$, where $A(i)$ is the determinant of the matrix obtained from A by taking the derivative of the i^{th} row and leaving the others unchanged; this rule follows from the one for differentiation of products.

Lemma 8.9.2:6. *If* $G : \mathbb{C} \times U \to \mathbb{C}$ *is real-root then* μ^G *is analytic at* λ^0 *if for every* $\theta^0 \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *with* $G(\lambda^0;\theta^0) = 0$, $G_\lambda(\lambda^0;\theta^0) \neq 0$, *and* $\nabla G \neq 0$.

Proof. In this case take any $\theta^0 \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ such that $G(\lambda^0;\theta^0) = 0$; then since G is a real-root function, it is really simple at $(\lambda^0;\theta^0)$, and so $(U, 1, \{G\})$ is a factoring of G at $(\lambda^0;\theta^0)$. Therefore G is globally threadable; the result follows by Theorem 8.7:1.

More generally, the following is also true.

Theorem 8.9.2:7. *If* G *is real-root then* μ^G *is analytic at* λ^0 *if for every* $\theta^0 \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *with* $G(\lambda^0;\theta^0) = 0$, *there is an open set* $V^1 \subseteq \mathbb{C}$ *with* $\lambda^0 \in V^1$ *and an open set* $V \subseteq \left(\frac{\mathbb{C}}{2\pi}\right)^s$ *with* $\theta^0 \in V$, *a non-negative integer* m, *and analytic functions* $g_1, \ldots, g_m : V \to V^1$ *such that:*

(1) *For* $\theta \in V$, *the roots* $\lambda \in V^1$ *of* $G(\lambda;\theta) = 0$ *are precisely, counting multiplicities,* $g_1(\theta), \ldots, g_m(\theta)$.

(2) *For every* g_i, $g_i(\theta^0) = \lambda^0$ *and* $\nabla g_i(\theta^0) \neq 0$.

Proof. Let $\theta^0 \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ be such that $G(\lambda^0;\theta^0) = 0$, and choose $V^1, V, m, g_1, \ldots, g_m$ as in the statement of the theorem. For any g_i, define the germ G_i at $(\lambda^0;\theta^0)$ to be $G_i(\lambda;\theta) = \lambda - g_i(\theta)$. We show the existence of an open set U' and an analytic function H such that $(U', H, \{G_1, \ldots, G_m\})$ is a factoring of G at $(\lambda^0;\theta^0)$. Certainly, since G is real-root, and $\nabla g_i(\theta^0) \neq 0$, each G_i is really simple at $(\lambda^0;\theta^0)$. So it is enough to find U' and an analytic $H : U' \to \mathbb{C}$ which is non-zero throughout U' such that $G = HG_1 \ldots G_m$. Find a germ h with $G = hG_1 \ldots G_m$ using Lemma 8.2:23. By (1) and (2), the multiplicity of the root $\lambda = \lambda^0$ of $G(\lambda;\theta^0) = 0$ is precisely m, which is the number of G_i with $G_i(\lambda^0;\theta^0) = 0$, so we deduce that $h(\lambda^0;\theta^0) \neq 0$. Choose U to be a neighbourhood of $(\lambda^0;\theta^0)$ contained in $V^1 \times V$ throughout which h is defined and non-zero; then the theorem follows.

It is in fact possible to simplify Lemma 8.9.2:6 by removing the condition that $G_\lambda(\lambda^0;\theta^0)$ should not equal 0 for every $\theta^0 \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ with $G(\lambda^0;\theta^0) = 0$. This is because it turns out that, for any real-root function G, if $G(\lambda^0;\theta^0) = 0$ and $G_\lambda(\lambda^0;\theta^0) = 0$ then $\nabla G(\lambda^0;\theta^0) = 0$. To prove this we proceed as follows.

Lemma 8.9.2:8. *Suppose that $f(l,t)$ is an analytic function defined for (l,t) in a $\mathbb{C}^2$-neighbourhood U of $(0,0)$, satisfying the following conditions:*

(1) *For constant t, $f(l,t)$ is a monic polynomial in l of degree D, where D is independent of t;*

(2) *At $t=0$, the equation $f(l,t)=0$ has a root of multiplicity at least 2 at $l=0$. Then there is a positive integer m and functions $y_1(z), y_2(z)$ analytic on a $\mathbb{C}$-neighbourhood V of 0, with $y_1(0)=y_2(0)=0$, such that for $z \in V$,*

$$f(y_1(z), z^m) = f(y_2(z), z^m) = 0,$$

and if $y_1(z)=y_2(z)=l'$ then $f(l, z^m)=0$ has a root of multiplicity at least 2 at $l=l'$.

Proof. This is a consequence of the existence of Puiseux Series, for which see [BRIESKORN&KNÖRRER,1986]. Specifically write

$$f(l,t) = \prod_{j=1}^{k} f_j(l,t)$$

where each f_j is analytic in a complex neighbourhood of $(0,0)$, such that for constant t, f_j is a monic polynomial in l of degree $D_j > 0$, and such that k is as large as possible subject to these conditions. This is possible, since we must have $\sum D_j = D$, and hence $k \leq D$.

Hence no f_j can be written as a product $g(l,t)h(l,t)$ where g and h are monic non-constant polynomials in l for any constant t. Also, because at $t=0$, we have a root with multiplicity at least 2 of f at $l=0$, for $t=0$ we must have either a root with multiplicity at least 2 of some f_j at $l=0$, or else there must exist $j_1 \neq j_2$ such that f_{j_1} and f_{j_2} both have a root of multiplicity 1 at $l=0$. Hence by [BRIESKORN&KNÖRRER,1986], Section 8.3, Theorem 1 we are done, by taking either the Puiseux series of f_j with m equal to the multiplicity of the root of $f_j(l,0)=0$ at $l=0$, or of f_{j_1} and f_{j_2} with $m=1$.

Lemma 8.9.2:9. *Let* f *satisfy the conditions of* Lemma 8.9.2:8. *Suppose in addition that for any* t *in a real neighbourhood* W *of* 0, *the equation* $f(l,t) = 0$ *has a complete set of real roots for* l. *Then in* Lemma 8.9.2:8 *we may take* $m = 1$.

For $j = 1, 2$, let y_j be as in Lemma 8.9.2:8. For any $z \in \mathbb{C}$ such that $z^m \in W$, we must have each $y_j(z)$ real. Suppose that in a neighbourhood of $z = 0$, $y_j(z) = \sum a_k z^k$. We will show that $a_k = 0$ unless $m \mid k$. Then we can define $y_j'(t) = \sum a_{mk} t^k$ and $m' = 1$; then we can use (y_j', m') rather than (y_j, m).

Let $g(z) = \sum_{m|k} a_k z^k$ and $h(z) = y_j(z) - g(z)$. We want to show that $h(z)$ is 0 in a neighbourhood of 0. Otherwise suppose that k is minimal such that $a_k \neq 0$ and $m \nmid k$. Then $a_k z^k$ is the leading term in the Taylor expansion of $h(z)$. If ω is any m^{th} root of unity then for any real τ we have $(\tau\omega)^m \in \mathbb{R}$ and so for sufficiently small τ we have $h(\tau\omega) \in \mathbb{R}$ (since $g(\tau\omega) \in \mathbb{R}$). But we also have

$$\lim_{\substack{\tau \to 0 \\ \tau \in \mathbb{R}}} \frac{h(\tau\omega)}{(\tau\omega)^k} = a_k \tag{26}$$

since $a_k z^k$ is the leading term of $h(z)$. If we look at the argument of Display (26), we see that $k \arg(\omega)$ is constant. This is impossible, since ω is any m^{th} root of unity and $m \nmid k$. Thus we have a contradiction and the lemma follows.

Theorem 8.9.2:10. *Suppose* $G : \mathbb{C} \times U \to \mathbb{C}$ *is a real-root function, where* U *is an open subset of* $\left(\frac{\mathbb{C}}{2\pi}\right)^s$, *and* $(\lambda^0; \theta^0) \in U \cap \left(\mathbb{R} \times \left(\frac{\mathbb{R}}{2\pi}\right)^s\right)$ *satisfies* $G(\lambda^0; \theta^0) = 0$ *and* $G_\lambda(\lambda^0; \theta^0) = 0$. *Then*

$$\nabla G(\lambda^0; \theta^0) = 0.$$

Proof. Choose j from $\{1, \ldots, s\}$. We will show that $G_{\theta_j}(\lambda^0; \theta^0) = 0$.

Define the analytic function $f(l, t)$ by

$$f(l,t) \stackrel{\text{def}}{=} G(\lambda^0 + l; \theta_1^0, \ldots, \theta_j^0 + t, \ldots, \theta_s^0).$$

Then at $(l,t) = (0,0)$, $f = \frac{\partial f}{\partial l} = 0$. So at $t = 0$, the equation in l, $f(l,t) = 0$, has a root of multiplicity at least 2 at $l = 0$. Furthermore, for any real t, the equation

$f(l,t) = 0$ has a complete set of real roots in l, since G is a real-root function. Therefore the conditions of Lemma 8.9.2:9 are satisfied.

So there exist analytic functions y_1, y_2 with $y_1(0) = y_2(0) = 0$ such that in a neighbourhood of $t = 0$, $f(y_1(t), t) = f(y_2(t), t) = 0$, and if $y_1(t) = y_2(t) = l'$ then the equation in l, $f(l,t) = 0$ has a root root of multiplicity at least 2 at $l = l'$.

We now apply Lemma 8.2:23 to $f, l - y_1(t), l - y_2(t)$, to find an analytic h such that in the neighbourhood of $(0,0)$, $f = h(l - y_1(t))(l - y_2(t))$. Therefore at $(l,t) = (0,0)$, since $y_1(0) = y_2(0) = 0$, we deduce from the ordinary product rule of differentiation that $\frac{\partial f}{\partial t} = 0$. However this is equal to $G_{\theta_j}(\lambda^0; \theta^0)$, and the theorem follows. □

Hence we deduce

Theorem 8.9.2:11. *If* G *is real-root then* μ^G *is analytic at* λ^0 *if for every* $\theta^0 \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$ *with* $G(\lambda^0; \theta^0) = 0$, $\nabla G \neq 0$.

If then some lattice or weighted lattice is given and we wish to find out where *the* spectral measure μ is analytic, we can apply Theorem 8.9.2:11, and if the conditions of that theorem are not met, perhaps the more complex Theorem 8.9.2:7. Note one advantage of Theorem 8.9.2:11: if we want to find pairs $(\lambda^0; \theta^0)$ where it does not hold, we have $s+1$ variables $(\lambda^0, \theta^0_1, \ldots, \theta^0_s)$, and $s+1$ equations ($G = 0$, $G_{\theta_1} = 0$, ..., $G_{\theta_s} = 0$). This thesis presents no converse results which enable us to establish that *the* spectral measure is not analytic at some λ^0, with the exception of those in Section 8.8.

We now consider two lattices, which exemplify these results.

§§8.9.3. *The Hexagonal Lattice*

We considered this in Subsection 6.7.4. Using the same notation as we used there, if $\theta = (\theta_1, \theta_2)$ we have

$$B^{e^{i\theta}} = \begin{pmatrix} 0 & 1 + e^{-i\theta_1} + e^{-i\theta_2} \\ 1 + e^{i\theta_1} + e^{i\theta_2} & 0 \end{pmatrix}.$$

We now define $G(\lambda; \theta) = \det(\lambda I - B^{e^{i\theta}})$ as before. To find where *the* spectral measure $\mu = \mu^G$ of the hexagonal lattice is analytic, we shall use Theorem 8.9.2:11 and look for pairs $(\lambda; \theta) \in \mathbb{R} \times \left(\frac{\mathbb{R}}{2\pi}\right)^s$ where $G(\lambda; \theta) = 0$ and $\nabla G = 0$.

To do this we expand out the characteristic polynomial. Then we find that

$$\begin{aligned} G(\lambda;\theta) &= \lambda^2 - (1 + e^{-i\theta_1} + e^{-i\theta_2})(1 + e^{i\theta_1} + e^{i\theta_2}) \\ &= \lambda^2 - 3 - 2(\cos(\theta_1) + \cos(\theta_2) + \cos(\theta_1 - \theta_2)). \end{aligned}$$

Thus if the spectral measure is not analytic at λ there must be some θ such that this is equal to 0, and also

$$\begin{aligned} G_{\theta_1} &= 2(\sin(\theta_1) + \sin(\theta_1 - \theta_2)) = 0; \\ G_{\theta_2} &= 2(\sin(\theta_2) + \sin(\theta_2 - \theta_1)) = 0. \end{aligned}$$

If we solve these equations we find that they have the following solutions:
(1) $\lambda = \pm 3$ with $(\theta_1, \theta_2) = 0$.
(2) $\lambda = \pm 1$ with $(\theta_1, \theta_2) \in \{(0, \pi), (\pi, 0), (\pi, \pi)\}$.
(3) $\lambda = 0$ with $(\theta_1, \theta_2) \in \{(\frac{2\pi}{3}, \frac{4\pi}{3}), (\frac{4\pi}{3}, \frac{2\pi}{3})\}$.

We can therefore conclude that *the* spectral measure is analytic at all $\lambda \in \mathbb{R}$ except for ± 3, ± 1, and 0. This is consistent with Figure 4, which is the output of a computer program which approximated *the* spectral measure of the hexagonal lattice by taking a large number of random values of $\theta \in \left(\frac{\mathbb{R}}{2\pi}\right)^s$, and finding the corresponding eigenvalues of $B^{e^{i\theta}}$. The histogram shows the distribution of the eigenvalues; we would expect this to approximate a density plot of *the* spectral measure. The Y axis shows the number of computed eigenvalues falling in each range; the X axis corresponds to values of the eigenvalues. As will be observed, the histogram does indeed look relatively smooth except at values on the X-axis of ± 3, ± 1 and 0.

The results in this thesis do not explain the type of bad behaviour that occurs at each of these values. In the case of the hexagonal lattice, there are further techniques which can be applied to each case; these are explained in [BIGGS,1994], which applies the results of these thesis and some others to the hexagonal lattice, and a rather more complex one, the Laves graph.

§§8.9.4. Y3

This lattice was described in [BIGGS&BURFORD,1985], and I will not describe it here. In our notation it is of the form $(\mathbb{Z}^4, \Upsilon, L)$, where Υ has 6 elements, and (as it is a cubic lattice) L has 9. So it is a 4-dimensional cubic lattice, with a repeating pattern containing 6 points. Unfortunately it is quite hard, though theoretically possible, to apply the methods of this thesis to this lattice, since as Υ has 6 elements, the matrices involved are 6×6. In fact they are given by the formula

$$B^{e^{i\theta}} = \begin{pmatrix} 0 & \frac{\gamma_2\gamma_4}{\gamma_1\gamma_3} & \frac{\gamma_1}{\gamma_4} & \frac{\gamma_3}{\gamma_2} & 0 & 0 \\ \frac{\gamma_1\gamma_3}{\gamma_2\gamma_4} & 0 & 0 & 0 & \frac{\gamma_2}{\gamma_1} & \frac{\gamma_4}{\gamma_3} \\ \frac{\gamma_4}{\gamma_1} & 0 & 0 & 0 & \gamma_1 & \frac{1}{\gamma_4} \\ \frac{\gamma_2}{\gamma_3} & 0 & 0 & 0 & \frac{1}{\gamma_2} & \gamma_3 \\ 0 & \frac{\gamma_1}{\gamma_2} & \frac{1}{\gamma_1} & \gamma_2 & 0 & 0 \\ 0 & \frac{\gamma_3}{\gamma_4} & \gamma_4 & \frac{1}{\gamma_3} & 0 & 0 \end{pmatrix}$$

where we take $\gamma_j = e^{i\theta_j}$. Hence the corresponding characteristic functions G are of the form $G(\lambda; \theta_1, \theta_2, \theta_3, \theta_4)$ and have degree 6 in λ. For this reason I have not been able to conduct an exhaustive search for all $(\lambda; \theta_1, \dots, \theta_4) \in \mathbb{R} \times \left(\frac{\mathbb{R}}{2\pi}\right)^s$ for which $G = 0$ and $\nabla G = 0$, though this seems possible with appropriate techniques of numerical analysis. However the obvious alternative to using the methods of this thesis is to solve the simultaneous equations of degree 6 in λ directly for the eigenvalues and find some bare-hands approach for showing that except at certain points, the distributions of the eigenvalues are analytic; this would appear to be a much more forbidding computation.

I have found some solutions to $G = 0$, $\nabla G = 0$; these have λ equal to ± 3, ± 2, and 0. It may be instructive to consider this in relation to Figure 5, which was generated using the same method as Figure 4. Indeed the histogram does not appear to be smooth at these values on the X-axis; however it also does not appear to be smooth at some other points, so it seems likely that my list of solutions is not exhaustive.

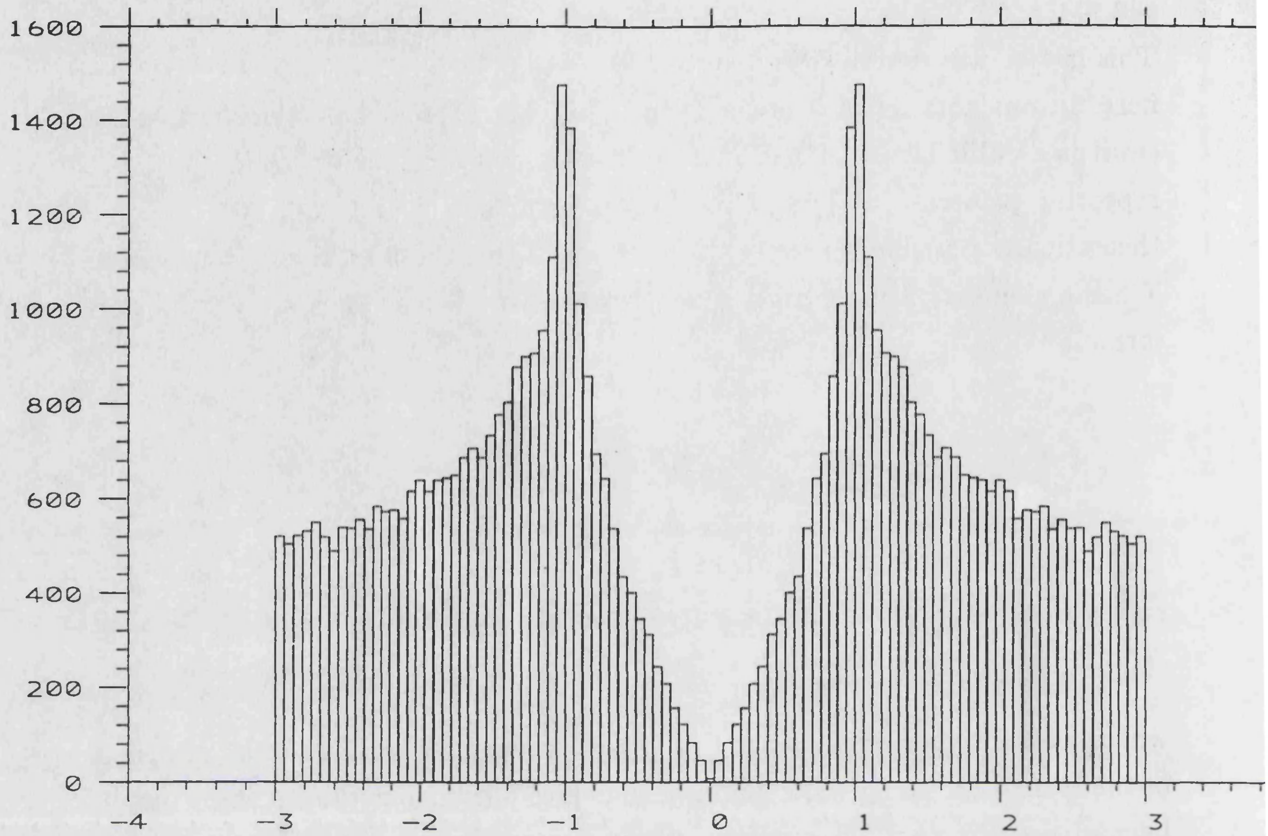

Figure 4. Histogram for spectral measure of the hexagonal lattice

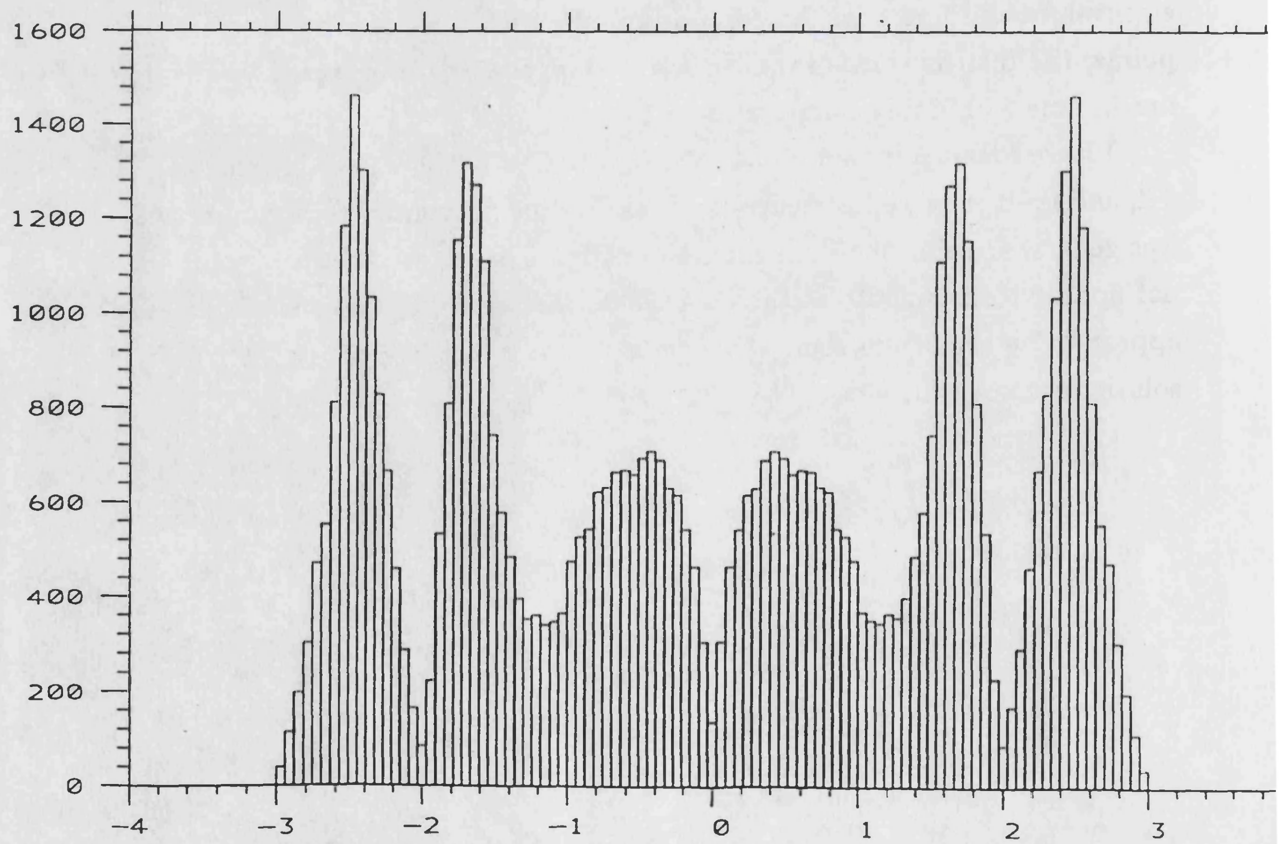

Figure 5. Histogram of spectral measure of Y3

§8.10. Directions for Future Research

It will be seen that the results presented in this chapter are incomplete in that while they are useful for demonstrating that *the* spectral measure is analytic, provided we can find an exhaustive set of solutions to the corresponding simultaneous equations, it has not been shown that *the* spectral measure is not analytic where the characteristic function G does not satisfy the conditions of Theorem 8.9.2:11 or Theorem 8.9.2:7, nor have ways of classifying the different types of non-analytic behaviour been found. However, it seems likely to me that some progress will be made in this area. Indeed, I believe I have already found methods using Morse theory (see [MILNOR,1963]) which classify some circumstances in which the spectral measure μ, or indeed the measure μ^G, is not analytic at particular points, but these are not included here, though it is my intention to publish them soon. Even these methods are quite limited; they classify the different types of behaviour of the spectral measure of the hexagonal lattice for $\lambda = \pm 3$ or ± 1, but cannot be used to predict what will happen at $\lambda = 0$; this requires an *ad hoc* technique of the sort given in [BIGGS,1994], which would be hard to generalise to more complex lattices.

The most promising approach to obtaining more general results would seem to me to involve some sort of surgery, where pairs $(\lambda^0; \theta^0)$ which break the conditions of Theorem 8.7:1 are removed, possibly together with small neighbourhoods surrounding them, some sort of threading is constructed for the remaining $(\lambda; \theta)$, and the total measure is written as a sum of 'nasty' contributions from the points which have been removed, and a 'nice' contribution from what remains.

I have deliberately organised this chapter so that the main result, Theorem 8.7:1, is stated as a fairly general theorem on functions of several variables, rather than just as a theorem about spectral measures of lattices. Indeed, it seems possible these results, and the techniques by which they were obtained, could be used to gain insights into the behaviour of singularities.

References

[BABAI,1979] BABAI, LÁSZLÓ, Spectra of Cayley Graphs, *Journal of Combinatorial Theory, Series B* 27, 180–189, 1979

[BIGGS,1994] BIGGS, NORMAN, How to Compute the Spectral Density of a Lattice and its Quotients, *Mathematics Preprint Series* 74, Mathematics Department, London School of Economics, WC2A 2AE, UK 1994

[BIGGS&BURFORD,1985] BIGGS, NL & BURFORD, SK, Infinite Coverings of Cages, *Europ. J. Combinatorics* 6, 7–12, 1985

[BIRKHOFF&ROTA,1989] BIRKHOFF, GARRETT & ROTA, GIAN-CARLO, Ordinary Differential Equations (4th ed.), John Wiley 1989

[BOLLOBÁS,1990] BOLLOBÁS, BÉLA, Linear Analysis, Cambridge University Press 1990

[BOOLOS&JEFFREYS,1980] BOOLOS, GEORGE & JEFFREYS, RICHARD, Computability and Logic (second edition), Cambridge University Press 1980

[BRIESKORN&KNÖRRER,1986] BRIESKORN, ERNEST & KNÖRRER, HORST (TRANSLATOR JOHN STILLWELL), Plane Algebraic Curves, Birkhäuser Verlag 1986

[CHUNG,1989] CHUNG, FRK, Diameters and Eigenvalues, *Journal of the American Mathematical Society* 2#2, 187–196, 1989

[DUNFORD&SCHWARZ,1963] DUNFORD, N & SCHWARZ, JT, Linear Operators, Interscience, New York 1963

[GODSIL&MOHAR,1988] GODSIL, CD & MOHAR, B, Walk Generating Functions and Spectral Measures of Infinite Graphs, *Linear Algebra and its Applications* 107, 191–206, 1988

[HALMOS,1950] HALMOS, PAUL R, Measure Theory, 1950

[HERSTEIN,1975] HERSTEIN, IN, Topics in Algebra (2nd Edition), 1975

[HERVÉ,1987] HERVÉ, M, Several Complex Variables, Local Theory, Oxford University Press 1987

[MILNOR,1963] MILNOR, J, Morse Theory, Princeton University Press 1963

[MOHAR&WOESS,1989] MOHAR, BOJAN & WOESS, WOLFGANG, A Survey on Spectra of Infinite Graphs, *Bulletin of the London Mathematical Society* 21, 209–234, 1989

[MONTROLL,1956] MONTROLL, EW, Random walks in multi-dimensional spaces, especially on periodic lattices, *Journal of the SIAM* 4, 241–260, 1956

[MONTROLL,1969] MONTROLL, ELLIOT W, Random Walks on Lattices Containing Traps, in *Proceedings of the International Conference on Statistical Mechanics 1968*, **book edited by** THE PHYSICAL SOCIETY OF JAPAN, 6–10, (published as a supplement to volume 26 of the Journal of the Physical Society of Japan) 1969

[PLESNER,1969] PLESNER, ABRAM I (TRANSLATORS NESTELL, MERLYND K AND GIBBS, ALAN G), Spectral Theory of Linear Operators volume II, Frederick Ungar Publishing Co., New York 1969

[POLITOWICZ&KOZAK,1983] POLITOWICZ, PHILIP A & KOZAK, JOHN J, Exact Algorithm for d-Dimensional Walks on Finite and Infinite Lattices with Traps III. Role of Lattice Valency in Influencing the Efficiency of Diffusion-Controlled Reactions, *Physical Review B* 28#10, 5549–5569, 1983

[RELLICH,1969] RELLICH, FRANZ, Perturbation Theory of Eigenvalue Problems, Gordon and Breach 1969

[SPIVAK,1965] SPIVAK, MICHAEL, Calculus on Manifolds, W.A. Benjamin, Inc. 1965

[WALSH&KOZAK,1981] WALSH, CECILIA A & KOZAK, JOHN J, Exact Algorithm for d-Dimensional Walks on Finite and Infinite Lattices with Traps, *Physical Review Letters* 47#21, 1500–1502, 1981

[WALSH&KOZAK,1982] WALSH, CECILIA A & KOZAK, JOHN J, Exact Algorithm for d-Dimensional Walks on Finite and Infinite Lattices with Traps II. General Formulation and Application to Diffusion-Controlled Reactions, *Physical Review B* 26#8, 4166–4189, 1982

[WIDDER,1946] WIDDER, DAVID VERNON, The Laplace Transform, Princeton University Press 1946