The London School of Economics and Political Science

On optimal hedging and redistribution of catastrophe risk in insurance

Oksana Yurievna Savina

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Declaration

I certify that the thesis I have presented for examination for the MPhil/PhD degree of the London School of Economics and Political Science is solely my own work other than where I have clearly indicated that it is the work of others (in which case the extent of any work carried out jointly by me and any other person is clearly identified in it).

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Abstract

The purpose of the thesis is to analyse the management of various forms of risk that affect entire insurance portfolios and thus cannot be eliminated by increasing the number of policies, like catastrophes, financial market events and fluctuating insurance risk conditions. Three distinct frameworks are employed.

First, we study the optimal design of a catastrophe-related index that an insurance company may use to hedge against catastrophe losses in the incomplete market. The optimality is understood in terms of minimising the remaining risk as proposed by Föllmer and Schweizer. We compare seven hypothetical indices for an insurance industry comprising several companies and obtain a number of qualitative and formula-based results in a doubly stochastic Poisson model with the intensity of the shot-noise type.

Second, with a view to the emergence of mortality bonds in life insurance and longevity bonds in pensions, the design of a mortality-related derivative is discussed in a Markov chain environment. We consider longevity in a scenario where specific causes of death are eliminated at random times due to advances in medical science. It is shown that bonds with payoff related to the individual causes of death are superior to bonds based on broad mortality indices, and in the presence of only one cause-specific derivative its design does not affect the hedging error. For one particular mortality bond linked to two causes of death, we calculate the hedging error and study its dependence on the design of the bond.

Finally, we study Pareto-optimal risk exchanges between a group of insurance companies. The existing one-period theory is extended to the multiperiod and continuous cases. The main result is that every multiperiod or continuous Pareto-optimal risk exchange can be reduced to

the one-period case, and can be constructed by pre-setting the ratios of the marginal utilities between the group members.

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Summary

More than a decade has passed since the infamous Hurricane Andrew hit Florida in August 1992 and caused about $18 billion in insured losses. It served as a wake up call for the whole insurance industry and initiated the integration of the financial and insurance markets. One of the primary goals in the insurance industry since then was finding better ways of catastrophe risk securitization. However, the consequences of Hurricane Katrina, which hit the United States in August 2005 and was responsible for over $80 billion in damages, provided evidence that insurance companies remain vulnerable to catastrophe risk. The losses were so severe that, subsequently, Standard & Poor's placed 10 insurance companies on CreditWatch negative. The list included such insurance giants as ACE, Allstate Corp., Swiss Re, and losses ranged from $25 billion to $60 billion (CreditSights (2005)).

This state of affairs provided motivation for our research on optimal ways of redistribution of insurance risk. We consider two major types of insurable risk. The first one is the "high-severity, low-frequency" catastrophe risk, associated with jumps in human and property losses. On the other side of the scale is longevity risk which became relatively recently recognized as a serious threat to the life insurance industry. Overestimation of future mortality rates in the last two decades has caused serious problems to many insurance companies and pension funds.

We consider two strategies that insurance company may use for securitization. The first one involves transferring part of the insurance risk to the financial market and suggests that the company trades a catastrophe-based index of a specified form. Mortality bonds in insurance and longevity bonds in pensions provide an example of transferring mortality risk to the capital market. Unit-linked insurance contracts with

guarantee (e.g., guaranteed annuity options) provide an example of the contract that combines actuarial and financial risks. The second strategy is endogenous in the sense that it does not attract the resources of the financial market and implies co-operating with other insurance companies to form a risk-exchange group and undertake an attempt to lighten the burden of liabilities by exchanging parts of the members' risks. The risk-sharing rules are determined by the companies' utility functions that reflect their attitudes towards risk.

In Chapter 1 of the thesis, we present an overview of the evolution of the insurance-linked derivatives, followed by a survey of the recent developments in the area of optimization of the design of the securities.

In Chapter 2, we focus on the problem of finding the optimal loss-based index using the criterion of risk-minimization as proposed by Föllmer and Sondermann (1986). We consider an insurance industry of n companies working in the same line of insurance business. We suppose that a loss-linked tradeable index is launched, and selling this index will help an insurance company to offset part of their potential catastrophic losses. Seven hypothetical indices are examined, some depending on the total losses within the industry and others based on the number of claims. Also, an insurance-independent severity index, reflecting the severity of the catastrophe, is considered. We assume that the companies' losses are triggered by the same catastrophic events, and also depend on the size of the company and on the specific exposure for each company and each catastrophic event. The loss distribution in consideration is a doubly stochastic Poisson distribution with shot-noise intensity. The value for the hedging error in each case is obtained. Next we compare the results for the seven indices in question and find the optimal one. At the end of that section, we point out how our results can be applied to the case of extreme rainfall and earthquake models.

Chapter 3 uses the same idea of minimizing the quadratic hedging error for a different type of risk, but in a new framework. We examine several ways of hedging mortality risk in a Markov chain economic-demographic environment adopting an approach proposed by Norberg (2007). First, various causes of death are considered; jumps in mortality are linked to elimination or emergence of such causes. The mortality rates can be modelled by a Markov chain, and one can hedge mortality risk by trading suitable mortality-linked securities. In our first example, we compare the hedging error associated with trading two cause-specific bonds to hedging with a bond whose payment process depends on existence of either of the two death causes. The optimal design of this bond is obtained with the help of a computer program for some specific values of the parameters. The interest rate in this model is taken as constant. The risk-minimizing values of the parameters are obtained, and sensitivity to variations in the internal and external parameters is studied.

The rest of the thesis (Chapter 4) deals with an alternative approach to redistribution of risk. It involves the concept of Pareto-optimal risk exchanges (POREX) and the utility theory. As in Chapter 2, the market consists of n insurance companies. Their risk preferences are described by their utility functions. In the well-studied one-period model, the Pareto-optimal risk exchange is the treaty which cannot be replaced by another risk exchange agreement such that the expected utility of at least one agent will increase and the expected utilities of the rest of the group will not be diminished. One of the main results provides necessary and sufficient conditions for such risk exchanges. We extended the theory to the multiperiod and continuous set-ups and have proved that, with some limitations, the multiperiod Pareto-optimal risk exchanges should satisfy necessary and sufficient conditions similar to those in one-period case. In the continuous-time setting the main result states that the Pareto-optimal risk exchange can be reduced to the one-period case, and can be constructed by pre-setting the ratios of the marginal utilities between the group members.

Contents

Chapter 1. Introduction to the insurance of catastrophe risk

In this chapter, we will trace the development and the recent achievements in the market for insurance risk. In Section 1.1, the milestones of this process are surveyed, and in Section 1.2 we review the research literature on optimal design of a catastrophe index.

1.1 Background

Catastrophes are now defined by Property Claim Services (PCS), the insurance industry's statistical agent, as events that cause $25 billion or more in direct insured losses to property and that affect a significant number of policyholders and insurers. Catastrophe losses caused by natural disasters, such as earthquakes, hurricanes, wind- and hailstorms, floods and other perils, have been traditionally insured by the property and casualty insurance industry. Until the early nineties, direct insurers used to buy catastrophe protection from reinsurers. With better capitalization and better-diversified portfolios, the reinsurers are expected to absorb more risk more efficiently. However, it became evident after Hurricane Andrew, which hit Florida in August 1992, caused about $18 billion in insured losses and left at least 11 primary insurers insolvent, that traditional reinsurance is no longer capable of coping with potential mega-catastrophes.

In the wake of this disaster, reinsurers tried to re-evaluate their risks, and, consequently, reinsurance prices rose significantly (more than doubled in some areas), preventing primary insurers from buying adequate reinsurance protection at affordable prices. After 1994, the global reinsurance rates began to fall, almost reaching their 1992-level by 1999

(Pollner (2001)). Nevertheless, major global reinsurers admitted that catastrophe coverage was inadequate in many parts of the world, and that a resolution to this problem should come from the financial market. Pollner (2001) argued that "the capital market capacity stands at approximately $42 trillion or close to 50 times the capital available from the insurance industry, estimated at $850 billion". Inventing mutually advantageous ways of channelling insurance risk to the investors became an issue of vital importance. The solution came in the form of insurance-linked securities (ILS), which can be traded on a stock exchange and whose payout depends on the occurrence of a specified insurance event. These securities were intended to fill the gap between the insurance and the financial market.

Swiss Re (1999) describes three factors that might catalyze the integration of the insurance and finance. Apart from the well-tested first one, which is the occurrence of a new major catastrophe or a series of catastrophes, a downfall in the financial market itself might be beneficial. As Swiss Re (1999) puts it, "investors, whose expectations have been coloured by an extraordinary 15-year bull market in US equities, would grow disillusioned, reduce their equity holdings, and look elsewhere for the new investment opportunities". To some extent, this is exactly what has happened, although in the fixed income market, as bond yields have come down to historically low levels in the recent years. The third factor to accelerate the development of the ILS market would be an increased participation of the credit rating agencies and favourable regulatory treatment. Rating agencies' involvement will lead to the standardisation of deals and increased transparency, enabling investors to compare the risks between ILS as well as those between ILS and purely financial securities. On the regulation side, investors would welcome a more relaxed tax regime. We would also add that availablility of good pricing models and research in the area will make the investors more confident when taking this sort of risk.

However, there are several good features of the ILS that should, sooner or later, and despite the initial suspicion of the financial market players, make them an attractive investment. First, it may offer investors a good diversification instrument: zero-beta assets with a possibility of high returns. Empirical analysis provides evidence that the occurrence of catastrophic events is uncorrelated with price movements of stocks and bonds. For example, Canter et al (1997) studied the relationship between the yearly percentage changes in the S&P500 Index and in the PCS national index using data from 1949 to 1996. The correlation coefficient they obtained was insignificantly different from zero. However, as the catastrophes become more frequent and more severe in the third millennium, and affect greater regions (as, for example, the flood in Europe in August 2002), the question of uncorrelatedness between the occurrence of a natural catastrophe and the general movement of the stock prices may deserve further investigation. One cannot neglect a possible situation when a new catastrophe or a sequence affects several countries and paralyses their economies for a while, thus tying together catastrophe and financial market losses.

The second incentive is ILS' good risk/return characteristics. According to Froot (2001), actual spread at issue over LIBOR of the USAA/Residential Re 1997 transaction was 576 basis points, well above the estimated expected loss of only 63 basis points. Part of the reason for such mispricing is perhaps a relatively low level of competition on the demand side, due to the lack of skills and experience (unlike those for dealing with corporate defaults) as well as a possibility of losing all the capital without any recovery value. Also interesting is the fact that insurance companies seem willing to pay those rates. Froot (2001) analyses several possible explanations, looking mainly at supply-side issues such as lack of reinsurance capital and various imperfections of such a market. In any event, these circumstances seem likely to change only slowly, keeping such instruments attractive. The recent global liquidity contraction will be an interesting test of insurance companies' appetite for CAT bond issuance given higher risk premiums elsewhere. Another way of looking at the risk/return profile of such instruments is their Sharpe ratio. Indeed, according to Swiss Re (2003), Sharpe ratios for catastrophe (CAT) bonds were well above that for, for example, BB-rated corporate bonds. When one considers optimal portfolio construction, allocating a small percentage of the portfolio to such assets with a high Sharpe ratio and low correlation with the rest of the portfolio allows to increase expected portfolio return while reducing, or keeping the same, return volatility.

So while the fundamental reasons for launching such securities are sound, the reality was not that straightforward. It took time to put all the pieces in place, and, back in the early nineties, the conditions were far from perfect. In December 1992, the Chicago Board of Trade (CBOT) began to trade CAT futures and options on futures. Both types of securities were based on the loss ratio index of Insurance Service Office (ISO), settled on loss data from a number of selected companies. The ISO index was calculated as the total loss ratio of these companies. This first attempt was unsuccessful. "The reasons were that the index was announced only once before the settlement date, there was information asymmetry between insurers and investors, and there was a lack of realistic models", explain Christensen and Schmidli (2000).

Next followed PCS options and call spreads in 1995. The underlying index for the options was estimated and published daily. The CBOT's call spreads settled on indices of industry-wide catastrophic property losses in various regions of the USA. The loss indices were compiled by Property Claim Services. There were nine indices: a national index, five regional indices, and three state indices (for California, Florida, and Texas). The indices were based on PCS estimates of catastrophic property losses in the specified geographic areas during quarterly or annual exposure periods. But, according to Sheehan (2003), "the volume of PCS index option contracts peaked at only 15,706 contracts in 1997 and declined to 561 contracts in 1999. The CBOT has since delisted these options because of a lack of interest". Swiss Re (2001) certified that in a survey on the use of CAT options, only 5% of managers polled reported actually using them, with the main reason for remaining cautious being "the lack of market liquidity and the view that derivatives were risky and might lead to increased regulatory oversight, ... there was uncertainty regarding the design and use of these derivatives, ... lack of qualified personnel, the need to educate management." In 1998, the annual volume of CAT risk securitisation reached $1.4 billion and then fell to $1 billion in 1999 (Swiss Re (2001)).

However, a new wave of growth in the market has already started, beginning in Europe this time, with the emergence of the first CAT bond.

As far as the mechanics of the transaction goes, capital raised by issuing CAT bonds is invested in safe securities such as Treasury bonds which are held by a single-purpose reinsurer to insulate investors from the credit risk of the bond issuer. The bond issuer holds a call option on the funds in the single-purpose reinsurer. If the defined event occurs, the bond issuer can withdraw funds from the reinsurer to pay claims, and part or all of the interest and/or principal payments are forgiven. If the defined catastrophic event does not occur, the investors receive their principal plus interest equal to the risk-free rate plus a premium.

The first CAT bond, the Winterthur convertible bond, was issued in 1997 in Switzerland. It was a three-year bond with a possibility of conversion into five Winterthur Insurance registered shares at maturity. The face value was CHF 4,700, and coupons were at risk. The trigger event was the occurrence of a hailstorm resulting in 6,000 or more claims (from Schmock (1999)). Four other CAT bonds were launched in the same year, and, in 2003 the total amount of outstanding ILS reached $4.3 billion (Swiss Re (2004)). In 2006, the total volume of ILS outstanding almost reached $23 billion. According to Swiss Re (2006a), "of this, two-thirds or $15 billion are life bonds, and the remaining $8 billion non-life, ... insurers and investors increasingly benefit from these new opportunities. Since total bonds outstanding are still just a small fraction of the potential market, issuance is likely to see strong growth going forward". In the total, according to Guy Carpenter (2007), in the past decade, "97 transactions have been completed, representing $15.35 billion in catastrophe bond issue". Thirty eight of them took place in years 2005-2006, reflecting an increase of activity in the market. The risk profile of the market has expanded as well, now including both unrated issues and those rated from B to AA (Guy Carpenter (2007)).

Even though more than half of the total CAT bond issuance since 1997 covers hurricane and earthquake risk in the US, the geographic scope of the CAT bond coverage continues to grow. According to Guy Carpenter (2007), about 10% of all amount in the last 10 years covers typhoon and earthquake risk in Japan, 9% are linked to hail and windstorm in Europe, and the proportion of multiple intercontinental risk is also growing and has reached 26% in years 2005-2006. New regions have launched CAT bonds recently:

Taiwan earthquake bond in 2003, Australia multiperil bond and Mexico earthquake bond in 2006. Even some of the abandoned ILS securities have now been revived. In March 2007, NYMEX resumed trading PCS risk futures and options contracts, based on PCS data compiled for one of three regions - a Nationwide index, Texas-Maine index or Florida property insurance claims.

CAT-risk securities are a particularly interesting example of a new type of derivatives where the underlying is not a traded asset or commodity and therefore has no directly observable price which could determine the price of the derivative. In this sense, CAT securities are analogous to other new derivatives with "exotic underlying", such as weather derivatives.

According to the Swiss Re (2003) report, CAT bonds can use a number of trigger mechanisms. The simplest are indemnity-based transactions which are linked to the company's own losses. In order to improve transparency, either index or modelled loss triggers can be used. The most transparent one relies on parametric or physical triggers, i.e., a measure of the severity of a catastrophe. The most popular one, accounting to about 60% of all outstanding notional, is the parametric index that adjusts the parametric trigger for the particular company, thus combining transparency with a closer match to the company's business and coverage mix. A recent trend gaining popularity is a hybrid trigger that uses two or more trigger types in a single transaction.

In addition to the (relatively) standardised marketable products such as CAT bonds, a large number of transactions is done in the principal-to-principal format. This allows for more precise tailoring of the terms to suit both parties. Because the parties and their preferences are known, utility curves can be used to quantify their return requirements and risk tolerance. These can then be combined into an optimal transaction design using the Pareto optimality concept. Optimality, according to the latter, is the state when neither party's utility can be increased without reducing the utility of the other party.

Catastrophes affect not only property insurance but the life insurance sector as well. In 2005, the worst catastrophe in terms of victims was the earthquake (magnitude 7.6) in Kashmir, taking the lives of some 73,300 people (Swiss Re (2006b)). One can think of catastrophic events solely within the life insurance sector, without association with natural disasters, like an outbreak of cholera or a pandemic flu. The first security of this type was launched by Swiss Re in December 2003. According to Cox and Lin (2006), it was a three-year catastrophe bond with principal exposed to mortality risk, defined in terms of an index based on the weighted average annual mortality rates in the US, UK, Italy, Switzerland and France. The principal was reduced proportionally to the change in index as compared to its initial value and was abandoned completely if this ratio had reached 1.5. Total coverage was $400 million. Since then, such "death bonds" have become increasingly popular. Goldstein (2007) reports that "in 2005 about $10 billion worth were transacted, ... and this number rose to $15 billion in 2006, and could double this year. Over the next few decades ... the face value of life settlement deals will top $160 billion a year in today's dollars".

Another relatively new risk associated with the mortality rates is the longevity risk. It results from higher-than-expected payout ratios stemming from a decrease in mortality rates among pensioners and other policyholders with survival benefits. The policies most exposed to longevity risk are those with a guarantee. This tendency was first spotted in 1990s, and soon the need for a mechanism of hedging against such risk became apparent. The solution came in the form of a mortality-linked derivative, and the pioneering product was the long-term longevity bond, issued by EIB/BNP in November 2004. This bond was only partially subscribed and later called back due to a variety of problems with its design (see, for example, Cox and Lin (2006), Cairns et al (2006b), Antolin and Blommenstien (2007)). But Cairns et al (2006b) believed that "these implementation difficulties are essentially teething problems which will be resolved over time, and so leave the way open to the development of flourishing markets in a brand new class of securities".

1.2 In search of optimality

Quite a few attempts have been undertaken to design optimal, under some criteria, CAT derivatives. To date, most of the analyses have been based on historical data. For example, Cummins, Lalonde, Phillips (2000) compared indices with different geographic scope. They conducted a simulation analysis of hedging effectiveness for 255 insurers in Florida for each of five indices: one statewide and four intra-state regional indices. Three hedging objectives have been investigated: reduction in loss variance, value-at-risk (VAR) and the expected loss conditional on losses exceeding a specified loss threshold. The authors found that the firms with large enough market shares can hedge almost as effectively using intra-state index contracts as they can by using perfect-hedge contracts, such as when the company's own losses are used as the index. Hedging with the statewide contracts, on the other hand, is effective only for insurers with the largest state market shares and insurers that are highly diversified throughout the state.

J.Major (1999) indicated that one of the major shortcomings in hedges based on indices is the low correlation of a particular insurer's losses with the index (called *basis risk*). He considered a smaller scale (as compared to Cummins, Lalonde and Phillips), compared zip-based hedge of one-event hurricane risk to the statewide hedge in nine USA states using Monte-Carlo simulation, and explored the effect of insurer market penetration on basis risk. He studied two cases: conditional on event and the unconditional one. He has found out that the correlation between the statewide index and the insurer's losses was 0.661, and that between the zip-based index and the losses was 0.996. Major derived that, in the conditional case, the statewide index rarely achieved more than 25% reduction in conditional volatility, whereas the zip-based hedge typically attained over a 70% reduction. In the unconditional case, when variation in the events is also taken into account, both hedges significantly reduced volatility, the numbers being 50-75% for the statewide index and 90-99% for the zip-based index. Thus, Major concluded, the statewide index is "afflicted by substantial basis risk caused by the variation in market penetration of insured portfolio".

Harrington and Niehaus (1999) have also studied basis risk associated with index-based hedging. The authors compared two indices. The first one was an industry aggregate catastrophe loss ratio for the twenty states included in the CBOT's Eastern PCS contract. The second index was an insurer-specific catastrophe loss ratio of a special form. Harrington and Niehaus employed linear regression analysis based on the historical data of the insurer's catastrophe losses and the values of the loss index. The effectiveness of the hedge was measured by the coefficient of determination (R-squared) between the variable to be hedged and the underlying index of the derivative contract. Their regression analysis suggested that PCS derivatives would have provided a more effective hedge as compared to the one based on state-specific industry loss ratios.

However, even though the results based on historical data analysis do provide some insight into the benefits of hedging catastrophe losses and suggest better ways of doing it, such methods have certain disadvantages. For example, catastrophe loss distributions are highly skewed and have few observations from the tail of the catastrophe loss distribution during a given period. An alternative approach, which we present in the next Chapter, uses the theoretical risk-minimization framework to address the problem of finding an optimal design of a catastrophe index. We are going to compare the remaining risk resulting from hedging with each of five hypothetical indices for three models. The first one is a general model with the shot-noise intensity process for catastrophe losses, the second one involves some assumptions about the correlation among the companies' losses and the third one considers shot-noise process with exponential marks for the losses.

Chapter 2. Optimal design of catastrophe index

This chapter is devoted to finding the optimal design of a catastrophe index; the design that will minimize the basis risk for an insurance company. We consider an insurance industry comprising n companies in the same line of business and subject to more or less the same catastrophe risk. A tradeable index can be launched in the market, linked to the catastrophe-induced losses or some catastrophe parameters; such index can be used by an insurance company for hedging its own catastrophe risk. We introduce five hypothetical indices: two based on the total losses for the companies, two on the number of claims and one - the so-called severity index - based on the severity of the catastrophes within the specified time interval, and calculate the risk associated with hedging each of these indices and the corresponding risk-minimizing strategy. All indices are cumulative, adding data resulting from each catastrophe that occurred within the time period $(0, T]$.

The research is conducted in the risk-minimization framework set forth in the mid-eighties by Föllmer and Schweizer, and is the first attempt in applying this theory for choosing the optimal design of the catastrophe index. We start with the description of the risk-minimization theory in Subsection 2.1a and derive formulae for calculating basis risk in case of a tradeable catastrophe-linked index in Subsection 2.1b.

Then we proceed to more detailed models for the intensity of the claims arrival process in Section 2.2, and further specify the correlation among the losses suffered by the companies in the index. Subsection 2.2a contains formulae for the general shot-noise model for the intensity of the claims arrival process; Subsection 2.2b treats the case when these intensities are proportional to the company's size and the catastrophe exposure which varies with catastrophes. A more thorough comparison of the risk-reducing properties of the indices becomes possible after an additional assumption of proportionality of catastrophe exposure of each company to the absolute catastrophe severity. This case is worked out in Subsection 2.2c, and we have found that, in such case, the best index is the one based on the total losses in the whole insurance industry, next best being the index based on the total number of claims for all n companies, followed by the severity index, which is equal to the sum of catastrophes' severities occurred during time $(0, T]$. The two indices which exclude the hedging company should be placed in reverse order: the index based on the number of claims has a smaller unhedgeable risk part as compared to the index based on the total loss amounts. The last model treated involves the popular shot-noise intensities with exponential mark functions (Subsection 2.2d).

Subsection 2.2e has a more theoretical character. By using the martingale representation theorems for the point-process martingales, we consider indices based on the number of claims and then based on the total losses and establish the general form of the remaining risk for such indices using the coefficients of their martingale representations. In Subsection 2.2f we illustrate how the formulae from 2.1b can be applied for the commonly used Neyman-Scott model for the earthquake occurrence.

2.1 Risk-minimization theory

The concept of risk-minimization was introduced in 1986 by Föllmer and Sondermann who proposed to measure the risk associated with a hedging strategy by a quadratic risk criterion. They have solved the problem of finding the risk-minimizing strategy for the case where the asset price process was a martingale. The general semimartingale problem was worked out soon after by Schweizer (1988). If the market is incomplete, the risk-minimizing strategy in the martingale case is found by projection on the space of the square-integrable martingales given by Galtchouk-Kunita-Watanabe decomposition (see Kunita, Watanabe (1967)). The analogue of this decomposition in the semimartingale case is called

Föllmer-Schweizer decomposition.

These methods are extensively used in the markets with restricted or incomplete information, theoretical foundation being provided in Schweizer (1994) and Föllmer, Schweizer (1990); application to insurance payment streams is presented in Møller (2000,2001).

Our research represents the first application of the risk-minimization theory for the purpose of finding a catastrophe index with best hedging opportunities for an insurance company with a catastrophe risk exposure.

2.1a Basic ideas

Here we are going to briefly introduce the reader to the basic concepts of financial hedging under the criterion of risk-minimization. For a full account of the theory, see Föllmer, Sondermann (1986) or Föllmer, Schweizer (1990); Schweizer (1999) is a good survey reading.

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ be the filtered probability space where T is a fixed finite time horizon given by the maturity of the catastrophe derivative. The filtration $\mathbb{F}$ satisfies the usual assumptions of completeness and right-continuity, $\mathcal{F}_T = \mathcal{F}$ and $\mathcal{F}_0$ is trivial. Let $\mathcal{L}^2(\Omega, \mathcal{F}_t, \mathbb{P})$ denote the space of square-integrable random variables on $(\Omega, \mathcal{F}_t, \mathbb{P})$. To make $\mathcal{L}^2$ a Hilbert space we take as equivalent random variables those that are equal almost surely and define the inner product $<<.,.>>$ and the corresponding norm $\|\bullet\|$ for arbitrary elements $X, Y \in \mathcal{L}^2$ by

$$\|X_t\|^2 = \mathbb{E}[X_t^2],$$
$$<< X_t, Y_t >>= \mathbb{E}[X_t Y_t].$$

Now we introduce the financial market. There are two assets available for trading in the market, a bond and a stock (or any other tradeable asset), and their prices are random processes defined on $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ and denoted as $B = (B_t)_{0\leqslant t\leqslant T}$ and $S = (S_t)_{0\leqslant t\leqslant T}$, respectively. We assume that B_t is deterministic and S_t is real-valued with right-continuous paths with limits from the left. We introduce the discounted stock price $X_t = S_t/B_t$; the discounted value deposited on savings account is, obviously, $B_t/B_t \equiv 1$.

In what follows, we will consider a problem of hedging a contingent claim H_T, or H for simplicity, by means of dynamic $B,S-$ strategies. Our hedging strategy will be represented by a pair $\varphi = (\xi_t, \eta_t)_{0 \leqslant t \leqslant T}$, where ξ_t and η_t denote the number of stocks and bonds in the portfolio, respectively. The process ξ_t is predictable, that is, if we view it as a function of $(t,\omega) \in [0,T] \times \Omega$, it is measurable with respect to the σ-algebra $\mathcal{P}$ generated by the adapted processes whose all paths are left-continuous. This means that ξ_t is fixed just before time t, and given F_{t-}, the amount ξ_t is determined. The process η_t is adapted, or $\mathcal{F}_t-$ measurable. A strategy $\varphi = (\xi_t, \eta_t)_{0 \leqslant t \leqslant T}$ with the components satisfying the conditions above is called *admissible.*

We call a contingent claim H, due as a lump sum at time T, *attainable* if it can be replicated by means of a dynamic self-financing strategy based on the existing assets B and S. Such claims are sometimes called redundant as their payoff can be constructed by some combination of the assets that are already present in the market, and in this sense such claims do not bring new opportunities to the market. The financial market is *complete* if every contingent claim is attainable by means of some $B,S-$ strategy (see also Harrison, Pliska (1981)). It means that the risk associated with hedging can be eliminated completely by choosing a suitable dynamic strategy.

Now, consider a contingent claim at time T given by a random variable $H \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P})$. As before, our hedging strategy is based on the stock S and the savings account B. The discounted *value process* of the resulting portfolio is

$$V_t(\varphi) \equiv \xi_t X_t + \eta_t, \qquad (0 \leqslant t \leqslant T), \tag{2.1}$$

and the *cost* accumulated up to time t is defined as

$$C_t(\varphi) = V_t(\varphi) - \int_0^t \xi_u dX_u, \qquad (0 \leqslant t \leqslant T). \tag{2.2}$$

For the integral in (2.2) to exist, the amount ξ_u invested in stocks should satisfy certain integrability conditions, which we will give on page 25.

DEFINITION 2.1. A strategy is called *self-financing* if its cumulative cost process is constant over time, i.e., if

$$C_t(\varphi) \equiv C_0(\varphi), \qquad \mathbb{P}-a.s., \qquad 0 \leqslant t \leqslant T.$$

Self-financing means that after the initial moment $t = 0$, there is no capital injection to or withdrawal from the portfolio.

DEFINITION 2.2. The market is said to present an *arbitrage opportunity* at time T_0 if there exists a self-financing *(arbitrage) strategy* Y such that with zero initial capital

$$Y_0 = 0$$

the capital at time T is non-negative,

$$Y_T \geqslant 0 \qquad \mathbb{P}-a.s.$$

and is positive with positive probability, that is,

$$\mathbb{P}(Y_T > 0) > 0.$$

If the set of arbitrage strategies is empty, the market is called *arbitrage-free.* In the B,S-market that we consider this condition is equivalent to the existence of an equivalent martingale measure $\mathbb{P}^*$, such that X_t is a square-integrable martingale under $\mathbb{P}^*$; i.e.,

$$\mathbb{E}^*[X_T^2] < \infty$$

and

$$X_t = \mathbb{E}^*[X_T|\mathcal{F}_t], \qquad 0 \leqslant t \leqslant T,$$

where $\mathbb{E}^*[X_T|\mathcal{F}_t]$ denotes the conditional expectation under $\mathbb{P}^*$ with respect to σ-algebra $\mathcal{F}_t$ (for a precise definition, see Harrison, Kreps (1979)). Such a martingale measure is unique if and only if the market is complete. When the market is incomplete, there are infinitely many martingale measures, so one may impose further constraints on the martingale measure and select the one of his preference.

Even though we will be considering an incomplete arbitrage-free market in this part, it is not the aim of our research to go into the issue of the choice of a martingale measure. We just assume that the financial market is arbitrage-free and we have already chosen an equivalent martingale measure $\mathbb{P}^*$ so that X belongs to the space $\mathcal{M}^2$ of square-integrable martingales under $\mathbb{P}^*$.

Later in this section, we will need the following

DEFINITION 2.3. Let $\mathcal{M}^2$ be the subspace of $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ consisting of the square-integrable martingales under $\mathbb{P}$ with respect to the filtration $\mathbb{F}$, and X_t, Y_t are two elements from $\mathcal{M}^2$. Then the *predictable quadratic covariation process* $\langle X, Y\rangle_t$ is the unique predictable process with $\langle X, Y\rangle_0 = 0$ and right-continuous increasing paths such that $X_t Y_t - \langle X, Y\rangle_t$ is a martingale. One can show that

$$\langle X, Y\rangle_t = \int_0^t \mathbb{E}[dX_s dY_s | \mathcal{F}_{s-}],$$

which explains its name. The process $\langle X\rangle_t = \langle X, X\rangle_t$ is called the *predictable quadratic variation process*. The processes X_t, Y_t are orthogonal, written $X_t \perp Y_t$, if $\langle X, Y\rangle_t = 0$.

Now we can specify the integrability conditions on ξ_u. Following the notation as in Föllmer, Sondermann (1986), we denote by $\mathbb{P}_X^*$ the finite measure on $(\Omega \times [0, T], \mathcal{P})$ described by

$$\mathbb{P}_X^*[A] = \mathbb{E}^*\left[\int_0^T I_A(t, \omega) d < X >_t (\omega)\right],$$

and use the notation $\mathcal{L}^2(\mathbb{P}_X^*)$ for the class of predictable processes which, viewed as $\mathcal{P}$-measurable (as defined on page 23) functions on $\Omega \times [0, T]$, are square-integrable with respect to $\mathbb{P}_X^*$. We will only admit strategies such that ξ_u is a predictable process from $\mathcal{L}^2(\mathbb{P}_X^*)$ and the processes $V = (V_t(\varphi))_{0 \leqslant t \leqslant T}$ and $C = (C_t(\varphi))_{0 \leqslant t \leqslant T}$ are square-integrable, have right-continuous paths and satisfy the boundary constraint $V_T(\varphi) = H$ $\mathbb{P}^* - a.s.$

Suppose that the claim H admits the Itô representation

$$H = H_0 + \int_0^T \xi_u^H dX_u \qquad P - a.s. \qquad (0 \leqslant t \leqslant T). \tag{2.3}$$

Then we can use the following replicating strategy:

$$\xi_t := \xi_t^H, \qquad \eta_t := V_t(\varphi) - \xi_t X_t, \tag{2.4}$$

$$V_t(\varphi) := H_0 + \int_0^t \xi_s^H dX_s, \qquad 0 \leqslant t \leqslant T. \tag{2.5}$$

We can also define $V_t(\varphi)$ as a right-continuous version of the martingale

$$V_t^* = \mathbb{E}^*[H|\mathcal{F}_t], \qquad 0 \leqslant t \leqslant T. \tag{2.6}$$

When the market is incomplete, a claim cannot, in general, be replicated by a suitable self–financing strategy based on S. The claims that cannot be replicated by means of a self-financed hedging (or the *non-redundant* contingent claims) are said to carry associated *intrinsic risk.* In such case, one can relax the requirement of achieving the payoff exactly equal to H at time T and find a strategy that is "close", in some suitable sense, to H. Another approach, and the one we are going to follow in this paper, is to stick to the final payoff H but somewhat relax the self-financing constraint. For this purpose, a broader concept of a *mean-self-financing strategy* will be introduced.

DEFINITION 2.4. A strategy is called *mean-self-financing* if the corresponding cost process $C(\varphi) = (C_t(\varphi))_{0 \leqslant t \leqslant T}$ is a martingale.

It means that once we have determined the initial value $V_0(\varphi) = C_0(\varphi)$, the additional cost $C_t(\varphi) - C_0(\varphi)$ is a random variable with expectation $\mathbb{E}^*[C_t(\varphi) - C_0(\varphi)] = 0$ (see Föllmer, Schweizer (1988)), so that additional cost is zero "on average". One can see that any self-financing strategy is mean-self-financing.

We can aim to reduce actual risk associated with hedging non-redundant claims to its intrinsic component and construct an appropriate optimal mean-self-financing strategy. Here we have to define what we mean by "optimal". Using the notation from Föllmer, Sondermann (1986), we measure the remaining risk associated with strategy φ by

$$R_t(\varphi) := \mathbb{E}^*[(C_T(\varphi) - C_t(\varphi))^2|\mathcal{F}_t]. \tag{2.7}$$

A *risk-minimizing strategy* is an admissible strategy $\hat{\varphi}$ which minimizes, at each given time t, the remaining risk $R_t(\varphi)$ over the space of all admissible strategies. A risk-minimizing strategy is necessarily mean-self-financing (for proof we refer the reader to Schweizer (1999)). Föllmer and Sondermann (1986) have proved that if an equivalent martingale measure $\mathbb{P}^*$ exists, then for any claim a unique risk-minimizing strategy can be obtained by using the Galtchouk-Kunita-Watanabe projection of an element H from $\mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}^*)$ on the space $\mathcal{M}^2$ of square-integrable martingales. Then the Galtchouk-Kunita-Watanabe decomposition of the claim H is

$$H = H_0 + \int_0^T \xi_u^H dX_u + L_T^H, \tag{2.8}$$

with $H_0 \in \mathcal{L}^2(\Omega, \mathcal{F}_0, \mathbb{P}^*)$, $\xi \in \mathcal{L}^2(\mathbb{P}_X^*)$, and the unhedgeable part $L^H = (L_t^H)_{0 \leqslant t \leqslant T} \in \mathcal{L}^2(\Omega, \mathcal{F}, \mathbb{P}^*)$ is a zero-mean $\mathbb{P}^*-$ martingale which is orthogonal to the space of stochastic integrals with respect to the process X_t. The risk-minimizing strategy $\hat{\varphi} = (\hat{\xi}, \hat{\eta})$ is then given by

$$\hat{\xi} := \xi^H, \qquad \hat{\eta} := V(\hat{\varphi}) - \hat{\xi} X,$$

with

$$V_t(\hat{\varphi}) = H_0 + \int_0^t \xi_u^H dX_u + L_t^H. \tag{2.9}$$

A claim H is attainable if and only if the associated unhedgeable part L_T^H can be eliminated completely. Since X is a martingale under $\mathbb{P}^*$, the value process $V_t(\hat{\varphi})$ can be calculated directly as a right-continuous version of the martingale $\mathbb{E}^*[H|\mathcal{F}_t]$, $0 \leqslant t \leqslant T$, and the problem of finding the risk-minimizing strategy amounts to projecting this martingale on the martingale X (see Föllmer, Schweizer (1990)). Such projection means decomposing $V_t(\hat{\varphi})$ into a sum of two components: one belonging to the space of stochastic integrals with respect to the process X_t and the other being orthogonal to that space as described in Definition 2.3.

In the following sections, this theory will be applied to the problem of hedging of the catastrophe risk by means of a strategy based on some tradeable catastrophe-linked index D, with D replacing stock S in the theory presented above. We focus on finding the best, in the sense of reducing intrinsic catastrophe-generated risk, index and the corresponding optimal strategy in the market where such index is traded. Even though an index can be constructed as a complicated function of losses or the number of claims, in this part of the thesis we restrict our study to the linear combination of losses, linear combinations of the number of claims and some indices of the knock-out type.

2.1b Application to calculating basis risk associated with hedging with a catastrophe index

First, we are going to state in brief the basic assumptions about our model and the key stochastic processes involved. At the beginning we introduce the catastrophe occurrence process K_t. This process very much depends on the underlying natural phenomena and can be different even for the same peril but happening in various parts of the globe. The occurrence of earthquakes is well described by the Neyman-Scott clustering process (see Vere-Jones (1970)); for the rainfalls, the Bartlett-Lewis model has a better fit (see Rodriguez-Iturbe et al (1987)), and for the windstorms, Poisson process with constant parameter (suggested by Schmock (1999)) can be a good candidate. In Subsections $2.2a-e$ we assume that K_t is a Poisson process with constant intensity μ, which can be easily replaced by a deterministic function μ_t, and in Subsection 2.2f we consider a clustering process for the catastrophe occurrence. Throughout Chapter 2, the process K_t is defined on the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ with finite time horizon $[0,T]$. The filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ describes the amount of information that is available to the hedging company up to and including time t. We consider the natural filtration $(\mathcal{F}_t)_{t\in[0,T]}$, $\mathcal{F}_0$ is trivial.

The market consists of n insurance companies subject to the same catastrophic events. We assume that there is a risk-free bank account in the market, and $X_T^{(1)}, X_T^{(2)}, \ldots, X_T^{(n)}$ are the random variables defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and representing the accumulated catastrophe losses suffered by companies $1, 2, \ldots, n$ at time T. Suppose that an index D, linked to the companies' losses, is compiled by an independent agency. A derivative contract specifying the payment $g(D_T)$ at maturity T, can be traded in the market. For simplicity, we will be considering only the trivial form of such contract, when $g(D_T) = D_T$, and say that the index itself can be traded. The discounted price process of the index is a stochastic process adapted to $\mathbb{F}$, denoted by D_t. As in the previous section, we assume that the market is arbitrage-free, so that we can choose an equivalent martingale measure $\mathbb{P}^*$ such that the process D_t is a martingale under $\mathbb{P}^*$ of the form

$$D_t = \mathbb{E}^*[D_T|\mathcal{F}_t]. \tag{2.10}$$

A specified company, say, number 1, wants to protect itself to some extent against losses associated with potential catastrophes by using a hedging strategy involving the index D. Construction of the optimal hedging strategy for company 1 involves the intrinsic value process $V_t^{(1)} = \mathbb{E}^*\left[X_T^{(1)}|\mathcal{F}_t\right]$. The *intrinsic value* of a business is the value that is determined by the cash inflows and outflows – discounted at an appropriate interest rate – that can be expected to occur during the remaining life of the business.

We aim to find the Galtchouk-Kunita-Watanabe decomposition as in (2.9)

$$V_t^{(1)} = V_0^{(1)} + \int_0^t \xi_s dD_s + L_t^{(1)}, \tag{2.11}$$

where $L_t^{(1)}$ is a zero-mean martingale orthogonal to the space of stochastic integrals of the type $\int_0^t \xi_s dD_s$. In other words, we have to solve with respect to ξ_s the equation

$$V_t^{(1)} = V_0^{(i)} + \int_0^t \xi_s dD_s + L_t^{(1)}, \qquad \text{or} \qquad dV_t^{(1)} = \xi_t dD_t + dL_t^{(1)} \tag{2.12}$$

under the condition $dL_t^{(1)} \perp dD_t$. Since the difference

$$V_t^{(1)} - V_0^{(i)} + \int_0^t \xi_s dD_s = L_t^{(1)}$$

is the cost process corresponding to the hedging strategy based on D_t (see (2.1), (2.2)), the remaining risk (2.7) can be calculated as

$$R_t := \mathbb{E}^*[(L_T^{(1)} - L_t^{(1)})^2|\mathcal{F}_t].$$

After multiplying both sides of (2.12) by dD_t and finding expectation conditional on $\mathcal{F}_{t-}$, we arrive at

$$\begin{aligned}\mathbb{E}^*[dV_t^{(1)}dD_t|\mathcal{F}_{t-}] &= \mathbb{E}^*[\xi_t(dD_t)^2|\mathcal{F}_{t-}] + \mathbb{E}^*[dL_t^{(1)}dD_t|\mathcal{F}_{t-}] = \\ &= \xi_t\mathbb{E}^*[dD_t\, dD_t|\mathcal{F}_{t-}] = \xi_t d < D >_t ,\end{aligned}$$

as ξ_t is predictable and $dL_t^{(1)}$ is orthogonal to dD_t. The optimal amount of the index to be held in portfolio is

$$\xi_t = \frac{\mathbb{E}^*[dV_t^{(1)}\times dD_t|\mathcal{F}_{t-}]}{\mathbb{E}^*[dD_t\times dD_t|\mathcal{F}_{t-}]} = \frac{d<V^{(1)},D>_t}{d<D>_t}. \tag{2.13}$$

The incremental unhedgeable part $dL_t^{(1)}$ of the Galtchouk-Kunita-Watanabe decomposition (2.11) is

$$dL_t^{(1)} = dV_t^{(1)} - \xi_t dD_t = dV_t^{(1)} - \frac{d<V^{(1)},D>_t}{d<D>_t}dD_t. \tag{2.14}$$

Finally, we calculate the remaining risk R_t using the fact that $dV_t^{(1)} - \xi_t dD_t$ is orthogonal to dD_t :

$$\begin{aligned}R_t &= \mathbb{E}^*\left[\int_t^T d < L^{(1)} >_s \;\Big|\; \mathcal{F}_t\right] = \\ &= \mathbb{E}^*\left[\int_t^T \mathbb{E}\left\{\left(dV_s^{(1)} - \frac{d<V^{(1)},\widetilde{D}>_s}{d<\widetilde{D}>_s}dD_s\right)^2 \;\Big|\; \mathcal{F}_{s-}\right\} \;\Big|\; \mathcal{F}_t\right] = \\ &= \mathbb{E}^*\left[\int_t^T d < V^{(1)} >_s - \frac{(d<V^{(1)},D>_s)^2}{d<D>_s} \;\Big|\; \mathcal{F}_t\right] = \\ &= \mathbb{E}^*\left[\int_t^T d < V^{(1)} >_s \;\Big|\; \mathcal{F}_t\right] - \mathbb{E}^*\left[\int_t^T \frac{(d<V^{(1)},D>_s)^2}{d<D>_s} \;\Big|\; \mathcal{F}_t\right].\end{aligned} \tag{2.15}$$

We conclude that R_t depends on the index D only through the last component of the sum (2.15), which is minus

$$\mathbb{E}^*\left[\int_t^T \frac{(d<V^{(1)},D>_s)^2}{d<D>_s} \;\Big|\; \mathcal{F}_t\right]. \tag{2.16}$$

In the sections below, we will call (2.16) the "index-specific risk part". Also, we will assume that the measure we are working with is a martingale measure so that D_t is a martingale under it, and omit star symbol "$*$" when taking conditional and unconditional expectations.

In Section 2.2, we are going to consider several variations of the shot-noise intensity model for catastrophe losses. Within each model, we calculate the remaining risk for several hypothetical indices and find out which index or indices have best risk-reducing properties.

2.2 Application to shot - noise intensity model for the catastrophe losses

In non-catastrophic insurance modelling, a Poisson process with deterministic intensity is usually adopted to describe the claims' arrival process. However, due to the unpredictable nature of catastrophic events, such a process would no longer adequately describe the arrival of catastrophe-related claims. A Poisson process with stochastic intensity is a good alternative. In this Chapter, we will consider the so-called shot-noise intensity model, in which the intensity of the Poisson process has jumps at random times and continuously and monotonously decreases at exponential rate between jumps. The intensity in this model mimics the behaviour of the number of claims associated with catastrophic events: a sudden increase directly after a catastrophe is followed by a gradual decline. Of course, this model is not perfect. Firstly, one can always argue that the jump does not actually happen immediately after the catastrophe because some time is needed for the insured to estimate the risk they have suffered. Secondly, there is a specified period (usually 9 months) within which the claims related to the particular catastrophe can be accepted, and the exponential decrease does not provide for such a stop.

But, despite all these drawbacks, the shot-noise model still provides a fairly good description of the process, it is relatively easy to study and understand, has good analytical properties, and it allows for further modifications and improvements. In our thesis, we consider a general shot-noise process in Subsection 2.2a, with some variations in Subsections 2.2b and 2.2c, and in Subsection 2.2d we examine the shot-noise process with exponential mark functions.

In Subsection 2.2e, we apply the martingale representation theorem for point processes to express the remaining risk via coefficients of the martingale representation of the index' price process. As an example of the application of the formulae from Subsection 2.2a to an existing earthquake model, the Neyman-Scott cluster model, is considered in Section 2.2f.

2.2a General shot-noise intensity process

Let $(\Omega, \mathbb{F}, P)$ be a probability space with the information structure given by $\mathbb{F} = \{\mathcal{F}_t, t \in [0,T]\}$ which includes all information available about the market up to and including time t. As we are going to study losses resulting from catastrophic events, we will begin with the catastrophe arrival process K_t.

i) *Catastrophe arrival process.* Let us denote by K_t the number of catastrophes in the interval $[0,t]$. We assume that $(K_t)_{t\geqslant 0}$ is a Poisson process with intensity μ. Even though in this research the intensity of catastrophe arrival process is assumed to be constant, the transition to a more sophisticated and more realistic (possibly random) function will not pose significant difficulties as long as we assume that both K_t and μ_t are independent of the loss processes $X_t^{(1)}, X_t^{(2)}, \ldots, X_t^{(n)}$, the number of claims $N_t^{(1)}, N_t^{(2)}, \ldots, N_t^{(n)}$ and their intensities $\lambda_t^{(1)}, \lambda_t^{(2)}, \ldots, \lambda_t^{(n)}$. We suppose that catastrophes arrive at random times s_k, $k = 1, 2, \ldots$, and are characterized by marks y_k reflecting the absolute severity of the disaster. Insurers may obtain the values y_k from meteorologists or seismologists. We suppose that for every $k = 1, 2, \ldots$, the marks y_k do not depend on $(K_t)_{t\geqslant 0}$ and are independent and identically distributed random variables with distribution function $G(y)$. We denote the first two moments of this distribution by m_1 and m_2, correspondingly.

ii) *Claims arrival processes* $N_t^{(i)}$ *and their intensities* $\lambda_t^{(i)}$. Losses associated with the catastrophes will result in claims being submitted to the insurance companies. In what follows, we describe the number of claims submitted to the company number i by a doubly stochastic Poisson process $N_t^{(i)}$ driven by the stochastic process $\lambda_t^{(i)}$ which acts as its intensity.

As a more formal definition of the doubly stochastic Poisson process we offer the one adopted by Cox, Isham (1981):

DEFINITION 2.5. The process N_t is called a *doubly stochastic Poisson process* if there exists a real-valued non-negative stochastic process λ_t, such that if F_t^λ denotes a whole realization of λ, then the intensity of the process N_t conditional on F_t^λ is $\Lambda(t)$, where $\Lambda(t)$ is the realized value of λ_t.

Such a definition means that, given the history of the process $\lambda_t^{(i)}$, the increments $N_{t_2}^{(i)} - N_{t_1}^{(i)}$ for $t_1 \leqslant t_2$ are Poisson-distributed with parameter $\int_{t_1}^{t_2} \lambda_s ds$, and

$$Pr\{N_{t_2} - N_{t_1} = k | \lambda_s, t_1 \leqslant s \leqslant t_2\} = \frac{\exp\left(-\int_{t_1}^{t_2} \lambda_s ds\right)\left(\int_{t_1}^{t_2} \lambda_s ds\right)^k}{k!}. \tag{2.17}$$

The stochastic intensity here is assumed to be an observable shot-noise process as introduced by Cox, Isham (1981):

$$\lambda_t^{(i)} = \sum_{0 \leqslant s_k \leqslant t} M_k^{(i)}(t - s_k), \tag{2.18}$$

where the functions $M_k^{(i)}(t - s_k)$ are the marks attributed to catastrophe number k and specific for each company. It will be natural to suggest that they are proportional to the company's size, denoted as c_i, and are linked to the magnitude y_k of the catastrophe number k. In addition, we assume that the catastrophe arrival times s_k do not depend on y_k. In this section, we do not specify the mark functions.

iii) *Claim size.* The losses suffered by the company i up to and including time t are denoted by $X_t^{(i)}$ and equal

$$X_t^{(i)} = \sum_{j=1}^{N_t^{(i)}} Z_j^{(i)}, \tag{2.19}$$

where $Z_j^{(i)}$ is the size of the claim number j submitted to the company number i. We assume that all claims are independent and have the same distribution function $H(z)$. In what follows below, we assume that the intensity of the total losses increase is of the form

$$\lambda_t^{(i)}(dz) = \lambda_t^{(i)} H(dz),$$

where $\lambda_t^{(i)}$ is a nonnegative $\mathcal{F}_t$-predictable process and $H(dz)$ is the probability distribution of the claim sizes. We suggest that $X_0^{(i)} = 0$ for $i = 1,2,\dots,n$. The expression (2.19) can be written in the integral form as

$$X_t^{(i)} = \int_{s=0}^{t} \int_{z=0}^{\infty} z dN_s^{(i)}(dz) \tag{2.20}$$

So the filtration we will be working with is the stream of σ-algebras generated by the processes $\lambda_s^{(i)}, K_s$, $M_k^{(i)}$, $N_s^{(i)}$ and $X_s^{(i)}$; $\mathcal{F}_t = \sigma\{\lambda_s^{(i)}, K_s, M_k^{(i)}, N_s^{(i)}, X_s^{(i)}, \ i = 1,2,\dots,n, \ 0 \leqslant s \leqslant t, \ k \leqslant K_s\}$. One can say that $\mathcal{F}_t$ contains full information about all the involved processes in real time. The filtration $\mathcal{F}_0$ is trivial.

We denote by $V_t^{(i)}$ the intrinsic value process for the company i. It equals

$$V_t^{(i)} = \mathbb{E}\left[X_T^{(i)} | \mathcal{F}_t\right] = \mathbb{E}\left[\sum_{j=0}^{N_T^{(i)}} Z_j^{(i)} | \mathcal{F}_t\right] = \mathbb{E}\left[\sum_{j=0}^{N_t^{(i)}} Z_j^{(i)} | \mathcal{F}_t\right] + \mathbb{E}\left[\sum_{j=N_t+1}^{N_T^{(i)}} Z_j^{(i)} \mid \mathcal{F}_t\right] =$$

$$= X_t^{(i)} + \mathbb{E}[Z]\mathbb{E}\left[N_T^{(i)} - N_t^{(i)} \mid \mathcal{F}_t\right] = X_t^{(i)} + \mathbb{E}[Z] \int_{s=t}^{T} \mathbb{E}(\lambda_s^{(i)} \mid \mathcal{F}_t) ds. \tag{2.21}$$

The second component of the sum in the last expression in (2.21) is computed below:

$$\int\limits_{s=t}^{T} \mathbb{E}(\lambda_s^{(i)} \mid \mathcal{F}_t)ds = \int\limits_{s=t}^{T} \mathbb{E}\left(\sum_{k=0}^{K_s} M_k^{(i)}(s - s_k) \mid \mathcal{F}_t \right) ds =$$

$$= \int\limits_{s=t}^{T} \mathbb{E}\left(\sum_{k=0}^{K_t} M_k^{(i)}(s - s_k) + \sum_{k=K_t+1}^{K_s} M_k^{(i)}(s - s_k) \mid \mathcal{F}_t \right) ds =$$

$$= \sum_{k=0}^{K_t} \int_{s=t}^{T} M_k^{(i)}(s - s_k)ds + \int_{s=t}^{T} \mathbb{E}\left(\sum_{k=K_t+1}^{K_s} M_k^{(i)}(s - s_k) \right) ds. \tag{2.22}$$

So we can rewrite (2.21) as

$$V_t^{(i)} = X_t^{(i)} + \mathbb{E}[Z]\left(\sum_{k=0}^{K_t} \int_{s=t}^{T} M_k^{(i)}(s - s_k)ds + \int_{s=t}^{T} \mathbb{E}\left(\sum_{k=K_t+1}^{K_s} M_k^{(i)}(s - s_k) \right) ds \right). \tag{2.23}$$

The sum $\sum_{k=0}^{K_t} \int_{s=t}^{T} M_k^{(i)}(s - s_k)ds$ above is a random variable observable at time t, while $\int_{s=t}^{T} \mathbb{E}\left(\sum_{k=K_t+1}^{K_s} M_k^{(i)}(s - s_k) \right) ds$ is a deterministic function which we denote by $f^{(i)}(t)$:

$$f^{(i)}(t) = \int_{s=t}^{T} \mathbb{E}\left(\sum_{k=K_t+1}^{K_s} M_k^{(i)}(s - s_k) \right) ds.$$

The dynamics of $V_t^{(i)}$ is given below:

$$dV_t^{(i)} = dX_t^{(i)} + \mathbb{E}[Z]d\left(\sum_{k=0}^{K_t} \int_{s=t}^{T} M_k^{(i)}(s - s_k)ds + f^{(i)}(t) \right) =$$

$$= dX_t^{(i)} + \mathbb{E}[Z]d\left(\sum_{k=0}^{K_t} \int_{s=t}^{T} M_k^{(i)}(s - s_k)ds \right) + \mathbb{E}[Z]f^{(i)\prime}(t)dt. \tag{2.24}$$

The differential of the expression in the round brackets can be split into two parts, the first one representing a new mark function resulting from a possible catastrophe at time t, and the second one standing for the changes in the mark functions $M_k^{(i)}$ related to the catastrophes which have occurred before time t:

$$d\left(\sum_{0 \leqslant s_k \leqslant t} \int_{s=t}^{T} M_k^{(i)}(s - s_k)ds \right) = \left(\int_{s=t}^{T} M_{K_t}^{(i)}(s - s_{K_t})ds \right) dK_t - \sum_{0 \leqslant s_k < t} M_k^{(i)}(t - s_k)dt. \tag{2.25}$$

Since s_{K_t} equals t if $dK_t = 1$ (that is, a catastrophe does occur at time t), we can now write (2.24) as

$$dV_t^{(i)} = dX_t^{(i)} + \mathbb{E}[Z]\left(\int\limits_{s=t}^{T} M_{K_t}^{(i)}(s-t)ds\right)dK_t + \mathbb{E}[Z]\left(f^{(i)\prime}(t) - \sum_{0\leqslant s_k<t} M_k^{(i)}(t-s_k)\right)dt =$$
$$= \int\limits_{z=0}^{\infty} z dN_t^{(i)}(dz) + \mathbb{E}[Z]\left(\int\limits_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)dK_t + \mathbb{E}[Z]\left(f^{(i)\prime}(t) - \sum_{0\leqslant s_k<t} M_k^{(i)}(t-s_k)\right)dt. \tag{2.26}$$

We keep in mind that, as, due to our assumption, catastrophes and claims do not arrive at the same time, at least one of the increments $dX_t^{(i)}$, dK_t should equal zero. Also, the terms with $(dt)^2$ are cancelled out as they are infinitesimals of order 2. Then the quadratic variance of $V_t^{(i)}$ equals

$$d\langle V\rangle_t^{(i)} =$$
$$= \mathbb{E}\left[\left(dX_t^{(i)} + \mathbb{E}[Z]\left(\int\limits_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)dK_t + \mathbb{E}[Z]\left(f^{(i)\prime}(t) - \sum_{0\leqslant s_k<t} M_k^{(i)}(t-s_k)\right)dt\right)^2 \mid \mathcal{F}_{t-}\right] =$$
$$= \mathbb{E}\left[\left(dX_t^{(i)} + \mathbb{E}[Z]\left(\int\limits_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)dK_t\right)^2 \mid \mathcal{F}_{t-}\right] =$$
$$= \mathbb{E}\left[\left(\int\limits_{z=0}^{\infty} z dN_t^{(i)}(dz)\right)^2 \mid \mathcal{F}_{t-}\right] + (\mathbb{E}[Z])^2\mathbb{E}\left[\left(\int\limits_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)^2\right]\mu dt =$$
$$= \left(\mathbb{E}[Z^2]\lambda_t^{(i)} + (\mathbb{E}[Z])^2\mathbb{E}\left[\left(\int\limits_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)^2\right]\mu\right)dt. \tag{2.27}$$

Now we are going to introduce the indices. We specify two groups: indices based on the number of claims filed to the companies, and those based on the total losses incurred. Within each group, one index represents a linear combination of the basic elements (companies' losses or the number of claims) with constant coefficients $(a_1, a_2, \ldots, a_n)$ or $(b_1, b_2, \ldots, b_n)$ (see Table 1 below); to define each combination uniquely, we set the first non-zero coefficient in the combination above equal 1 (that is, $a_1 = b_1 = 1$). After presenting the solution in the general case we look at some special cases. Next we consider indices which exclude information about losses for the first company, that is, when $a_1 = b_1 = 0$. Such indices may be useful if the hedging company is just planning to enter the market or for some reason (small size, for example) cannot be included in the index. Then we consider the so-called "severity" index which depends only on catastrophes' severity. These five indices are described in Table 1 below.

TABLE 1. CATASTROPHE INDICES - DESCRIPTION.

index	formula	description
1	$\sum_{i=1}^{n} a_i X_T^{(i)}$	*weighted sum of losses for all n companies.*
2	$\sum_{i=2}^{n} a_i X_T^{(i)}$	*weighted sum of losses for companies 2,...,n.*
3	$\sum_{i=1}^{n} b_i N_T^{(i)}$	*weighted sum of claims for all n companies.*
4	$\sum_{i=2}^{n} b_i N_T^{(i)}$	*weighted sum of claims for companies 2,...,n.*
5	$\sum_{k \leqslant K_T} y_k$	*sum of catastrophes' severity up to and including T*

We admit that the list above is far from exhaustive. For example, it does not include the indices of the knock-out type, like the Winterthur index. This is because in this case it is hard to derive a closed formula for the basis risk in the case of stochastic intensity for the claims arrival process. However, in Subsection 2.2e we will obtain formulae for the remaining risk via the coefficients of the index' martingale representation. For a special case of a knock-out index with deterministic intensity for the claims arrival process, we will provide a more explicit formula involving only the intensity process and the average size of the claim.

We suppose that all the indices listed above are traded in the market and their no-arbitrage discounted price processes are given by $D_t = \mathbb{E}[D_T|\mathcal{F}_t]$. We shall label these processes by D_t with a superscript indicating the index number. In the rest of this subsection, we are going to calculate the remaining risk associated with optimal hedging of each of the five indices from Table 1. The results will be obtained in the form of the index-specific risk part (2.16) and collected in Table 2.

The price process for the first index is

$$D_t^{(1)} = \mathbb{E}\left[\sum_{i=1}^{n} a_i X_T^{(i)} \mid \mathcal{F}_t\right] = \sum_{i=1}^{n} a_i X_t^{(i)} + \mathbb{E}[Z] \sum_{i=1}^{n} \int_{s=t}^{T} a_i \mathrm{E}\left[\lambda_s^{(i)} \mid \mathcal{F}_t\right] ds.$$

Using (2.27), we can write

$$d\langle D^{(1)}\rangle_t = \mathbb{E}\left[\left(\sum_{i=1}^{n} a_i dX_t^{(i)} + \mathbb{E}[Z] \sum_{i=1}^{n} a_i \left(\int_{s=0}^{T-t} M_{K_t}^{(i)}(s) ds\right) dK_t\right)^2 \mid \mathcal{F}_{t-}\right] =$$

$$= \left(\mathbb{E}[Z^2] \sum_{i=1}^{n} a_i^2 \lambda_t^{(i)} + (\mathbb{E}[Z])^2 \mathbb{E}\left[\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s) ds\right)^2\right] \mu\right) dt. \qquad (2.28)$$

The predictable quadratic covariation process associated with $D_t^{(1)}$ and $V_t^{(1)}$ equals

$$d\langle V^{(1)}, D^{(1)}\rangle_t =$$

$$= \mathbb{E}\left[\left(dX_t^{(1)} + \mathbb{E}[Z]\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s) ds\right) dK_t\right)\right.$$

$$\left.\left(\sum_{i=1}^{n} a_i dX_t^{(i)} + \mathbb{E}[Z] \sum_{i=1}^{n} a_i \left(\int_{s=0}^{T-t} M_{K_t}^{(i)}(s) ds\right) dK_t\right) \mid \mathcal{F}_{t-}\right] =$$

$$= \left(\mathbb{E}[Z^2] \lambda_t^{(1)} + (\mathbb{E}Z)^2 \mathbb{E}\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s) ds\right)\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s) ds\right) \mu\right) dt. \qquad (2.29)$$

Now the value (2.16) is

$$\frac{(d\langle V^{(1)}, D^{(1)}\rangle_t)^2}{d\langle D^{(1)}\rangle_t} = \frac{\left(\mathbb{E}[Z^2]\lambda_t^{(1)} + (\mathbb{E}Z)^2 \mathbb{E}\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s) ds\right)\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s) ds\right)\mu\right)^2 dt^2}{\left(\mathbb{E}[Z^2] \sum_{i=1}^{n} a_i^2 \lambda_t^{(i)} + (\mathbb{E}[Z])^2 \mathbb{E}\left[\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s) ds\right)^2\right]\mu\right) dt} =$$

$$= \frac{\left(\mathbb{E}[Z^2]\lambda_t^{(1)} + (\mathbb{E}Z)^2 \mathbb{E}\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s) ds\right)\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s) ds\right)\mu\right)^2}{\mathbb{E}[Z^2] \sum_{i=1}^{n} a_i^2 \lambda_t^{(i)} + (\mathbb{E}[Z])^2 \mathbb{E}\left[\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s) ds\right)^2\right]\mu} dt. \qquad (2.30)$$

If we consider a similar index but without the hedging company, that is,

$D_T^{(2)} = \sum_{i=2}^{n} a_i X_T^{(i)}$, the resulting index-specific risk part will be

$$\frac{(d\langle V^{(1)}, D^{(2)}\rangle_t)^2}{d\langle D^{(2)}\rangle_t} = \frac{\left((\mathbb{E}Z)^2 \mathbb{E}\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\left(\sum_{i=2}^{n}\int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s)ds\right)\mu\right)^2}{\mathbb{E}[Z^2]\sum_{i=2}^{n} a_i^2\lambda_t^{(i)} + (\mathbb{E}Z)^2\mathbb{E}\left(\sum_{i=2}^{n}\int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s)ds\right)^2\mu.} dt. \tag{2.31}$$

The third index is the weighted sum of companies' numbers of claims.

$$D_T^{(3)} = \sum_{i=1}^{n} b_i N_T^{(i)}.$$

$$D_t^{(3)} = \mathbb{E}\left[\sum_{i=1}^{n} N_T^{(i)} \mid \mathcal{F}_t\right] = \sum_{i=1}^{n} N_t^{(i)} + \sum_{i=1}^{n}\int_{s=t}^{T} \mathbb{E}(\lambda_s^{(i)}|\mathcal{F}_t)ds. \tag{2.32}$$

$$\begin{aligned} d\langle D^{(3)}\rangle_t &= \mathbb{E}\left[\left(\sum_{i=1}^{n} b_i dN_t^{(i)} + dK_t \sum_{i=1}^{n}\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)^2 \mid \mathcal{F}_{t-}\right] = \\ &= \sum_{i=1}^{n} b_i^2\lambda_t^{(i)}dt + \mathbb{E}\left[\left(\sum_{i=1}^{n}\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)^2 dK_t \mid \mathcal{F}_{t-}\right] = \\ &= \left(\sum_{i=1}^{n} b_i^2\lambda_t^{(i)} + \mathbb{E}\left[\left(\sum_{i=1}^{n}\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)^2\right]\mu\right)dt. \end{aligned} \tag{2.33}$$

The predictable quadratic covariation process associated with $V_t^{(1)}$ and $D_t^{(3)}$ equals

$$\begin{aligned} &d\langle V^{(1)}, D^{(3)}\rangle_t = \\ &= \mathbb{E}\Big[\left(dX_t^{(1)} + \mathbb{E}[Z]\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds dK_t\right) \\ &\qquad \left(\sum_{i=1}^{n} b_i dN_t^{(i)} + \sum_{i=1}^{n}\left(\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)dK_t\right) \mid \mathcal{F}_{t-}\Big] = \\ &= \mathbb{E}[Z]\left(\lambda_t^{(1)} + \mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\left(\sum_{i=1}^{n}\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)\right]\mu\right)dt, \end{aligned} \tag{2.34}$$

and the index-specific risk part is

$$\frac{(d<M^{(1)},D^{(3)}>_t)^2}{d<D^{(3)}>_t} = \frac{(\mathbb{E}[Z])^2\left(\lambda_t^{(1)}+\mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\left(\sum_{i=1}^{n}\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)\right]\mu\right)^2}{\sum_{i=1}^{n} b_i^2\lambda_t^{(i)}+\mathbb{E}\left[\left(\sum_{i=1}^{n}\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)^2\right]\mu}dt. \tag{2.35}$$

For the fourth index $D_T^{(4)} = \sum_{i=2}^{n} N_T^{(i)}$, which excludes the hedging company from the pool, we have

$$\frac{(d<V^{(1)},D^{(4)}>_t)^2}{d<D^{(4)}>_t} = \frac{(\mathbb{E}[Z])^2\left(\mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\left(\sum_{i=2}^{n}\int_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)\right]\mu\right)^2}{\sum_{i=2}^{n} b_i^2\lambda_t^{(i)}+\mathbb{E}\left[\left(\sum_{i=2}^{n}\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)^2\right]\mu}dt. \tag{2.36}$$

Another interesting candidate for an index is what we would call the severity index. Namely, it is the sum of catastrophes' magnitudes up to and including T:

$$D_T^{(5)} = \sum_{k\leqslant K_T} y_k.$$

Its price process is

$$D_t^{(5)} = \mathbb{E}\left[\sum_{k\leqslant K_T} y_k \mid \mathcal{F}_t\right] = \sum_{k\leqslant K_t} y_k + \mathbb{E}\left[\sum_{K_t<k\leqslant K_T} y_k\right] = \sum_{k\leqslant K_t} y_k + m_1\mu(T-t), \tag{2.37}$$

$$dD_t^{(5)} = y_{K_t}dK_t - m_1\mu dt. \tag{2.38}$$

$$d\langle D^{(5)}\rangle_t = \mathbb{E}[y_{K_t}^2 dK_t|\mathcal{F}_{t-}] = m_2\mu dt. \tag{2.39}$$

$$\begin{aligned} d < V^{(1)},D^{(5)} >_t &= \\ &= \mathbb{E}\left[\left(dX_t^{(1)} + \mathbb{E}[Z]\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)dK_t\right)(y_{K_t}dK_t - m_1\mu dt) \mid \mathcal{F}_{t-}\right] = \\ &= \mathbb{E}\left[\mathbb{E}[Z]\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)dK_t y_{K_t}dK_t \mid \mathcal{F}_{t-}\right] = \mathbb{E}[Z]\mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\right]m_1\mu dt. \end{aligned}$$

Then the index-specific risk part equals

$$\frac{(d<V^{(1)},D^{(5)}>_t)^2}{d<D^{(5)}>_t} = \frac{\left(\mathbb{E}[Z]\mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\right]m_1\mu\right)^2}{m_2\mu}dt =$$

$$= (\mathbb{E}[Z])^2 \frac{m_1^2}{m_2} \left(\mathbb{E}\left[\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds \right] \right)^2 \mu dt. \tag{2.40}$$

The index-specific remaining risk parts for indices $D^{(1)} - D^{(5)}$ are collected in the Table 2 below.

TABLE 2. INDEX-SPECIFIC REMAINING RISK PARTS.

index	formula	remaining risk part
1	$\sum_{i=1}^{n} a_i X_T^{(i)}$	$\dfrac{\left(\mathbb{E}[Z^2]\lambda_t^{(1)} + (\mathbb{E}[Z])^2 \mathbb{E}\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds \right)\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s)ds \right)\mu \right)^2}{\mathbb{E}[Z^2]\sum_{i=1}^{n} a_i^2 \lambda_t^{(i)} + (\mathbb{E}[Z])^2 \mathbb{E}\left[\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s)ds \right)^2 \right]\mu} dt$
2	$\sum_{i=2}^{n} a_i X_T^{(i)}$	$\dfrac{\left((\mathbb{E}Z)^2 \mathbb{E}\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds \right)\left(\sum_{i=2}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s)ds \right)\mu \right)^2}{\mathbb{E}[Z^2]\sum_{i=2}^{n} a_i^2 \lambda_t^{(i)} + (\mathbb{E}Z)^2 \mathbb{E}\left(\sum_{i=2}^{n} \int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s)ds \right)^2 \mu} dt$
3	$\sum_{i=1}^{n} b_i N_T^{(i)}$	$\dfrac{(\mathbb{E}[Z])^2 \left(\lambda_t^{(1)} + \mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds \right)\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds \right) \right]\mu \right)^2}{\sum_{i=1}^{n} b_i^2 \lambda_t^{(i)} + \mathbb{E}\left[\left(\sum_{i=1}^{n} \int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds \right)^2 \right]\mu} dt$
4	$\sum_{i=2}^{n} b_i N_T^{(i)}$	$\dfrac{(\mathbb{E}[Z])^2 \left(\mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds \right)\left(\sum_{i=2}^{n} \int_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds \right) \right]\mu \right)^2}{\sum_{i=2}^{n} b_i^2 \lambda_t^{(i)} + \mathbb{E}\left[\left(\sum_{i=2}^{n} \int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds \right)^2 \right]\mu} dt$
5	$\sum_{k \leqslant K_T} y_k$	$(\mathbb{E}[Z])^2 \frac{m_1^2}{m_2} \left(\mathbb{E}\left[\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds \right] \right)^2 \mu dt$

Since the model is very general, it is difficult to compare all the indices without further assumptions about the intensity functions. Still, even at this stage we can compare indices 2 and 4 for the same weights $a_i = b_i$, $i = 1, \ldots, n$. If the numerator and the denominator of the ratio for the fourth index were multiplied by $(\mathbb{E}[Z])^2$, the difference between the two indices will be only in the denominators:

$$\mathbb{E}[Z^2]\sum_{i=2}^{n} a_i^2\lambda_t^{(i)} + (\mathbb{E}Z)^2\mathbb{E}\left(\sum_{i=2}^{n}\int_{s=0}^{T-t} a_i M_{K_t}^{(i)}(s)ds\right)^2\mu$$

for the second index and

$$(\mathbb{E}[Z])^2\sum_{i=2}^{n} b_i^2\lambda_t^{(i)} + (\mathbb{E}[Z])^2\mathbb{E}\left[\left(\sum_{i=2}^{n}\int_{s=0}^{T-t} b_i M_{K_t}^{(i)}(s)ds\right)^2\right]\mu$$

for the fourth one.

Since $\mathbb{E}\xi^2 \geqslant (\mathbb{E}\xi)^2$ for every ξ, and $a_i = b_i$, we conclude that the fourth index is better than the second one.

2.2b Intensities proportional to exposure and size

In this Subsection, we are going to consider more specific functions $M_k^{(i)}(t - s_k)$ by stipulating the form of dependence among the companies' losses. We assume that they are proportional to the company size c_i, and take the size of the company 1 as unity. Also, we have mentioned that the marks should be linked to the catastrophe magnitudes. We assume that, as a result of the catastrophe number k with absolute severity y_k, the company number i is hit by an "impact factor" $y_k^{(i)}$. The "impact factors" $y_k^{(i)}$ are identically distributed random variables, with the same distribution function $G(y)$ of the catastrophe magnitude, independent for different catastrophes but correlated for the same catastrophe. We assume that there is dependence among impact factors, which is the same for each catastrophe and varies only from one pair of companies to another. This dependence is expressed in relationships $\mathrm{E}y_k^{(i_1)}y_k^{(i_2)} = m_{i_1,i_2}$, for $i_1 \neq i_2$, $\mathrm{E}y_k y_k^{(i_1)} = m_{0,i_1}$, and is stipulated by the proximity of the objects insured by each pair of companies. This essentially means that the companies, which insure closely situated objects, are likely to experience similar impact factors. As in Subsection 2.2a, m_1 and m_2 stand for the first and second moments of $G(y)$, correspondingly. Then the mark functions and the intensities will be of the form

$$M_k^{(i)}(t-s_k) = c_i y_k^{(i)} g(t-s_k), \tag{2.41}$$
$$\lambda_t^{(i)} = \sum_{0 \leqslant s_k \leqslant t} c_i y_k^{(i)} g(t-s_k), \tag{2.42}$$

where $g(t)$ is a deterministic component of the mark function common to all companies. Then we can write the integral from (2.27) as

$$\int_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds = c_i y_{K_t}^{(i)} \int_{s=0}^{T-t} g(s)ds, \tag{2.43}$$
$$\sum_{i=1}^{n} \int_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds = \sum_{i=1}^{n} \left(c_i y_{K_t}^{(i)}\right) \int_{s=0}^{T-t} g(s-s_{K_t})ds, \tag{2.44}$$

Therefore, the index-specific risk part (2.16) for the index $D^{(1)}$ is

$$\begin{aligned}
\frac{\left(d\langle V^{(1)},D^{(1)}\rangle_t\right)^2}{d\langle D^{(1)}\rangle_t} &= \frac{\left(\mathbb{E}[Z^2]\lambda_t^{(1)}+(\mathbb{E}[Z])^2\cdot\mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\left(\sum_{i=1}^{n}\int_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)\right]\mu\right)^2}{\mathbb{E}[Z^2]\sum_{i=1}^{n}\lambda_t^{(i)}+(\mathbb{E}[Z])^2\cdot\mathbb{E}\left[\left(\sum_{i=1}^{n}\int_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)^2\right]\mu} dt = \\
&= \frac{\left(\mathbb{E}[Z^2]\lambda_t^{(1)}dt+(\mathbb{E}[Z])^2\left(\int_{s=0}^{T-t} g(s)ds\right)^2 \mathbb{E}\left(\sum_{i=1}^{n} c_1 y_{K_t}^{(1)} c_i y_{K_t}^{(i)}\right)\mu\right)^2}{\mathbb{E}[Z^2]\sum_{i=1}^{n}\lambda_t^{(i)}+(\mathbb{E}[Z])^2\left(\int_{s=0}^{T-t} g(s)ds\right)^2 \mathbb{E}\left(\sum_{j=1}^{n}\sum_{i=1}^{n} c_j y_{K_t}^{(j)} c_i y_{K_t}^{(i)}\right)\mu} dt = \\
&= \frac{\left(\mathbb{E}[Z^2]\lambda_t^{(1)}+(\mathbb{E}[Z])^2\left(\int_{s=0}^{T-t} g(s)ds\right)^2\left[c_1 m_2+\sum_{i=2}^{n} c_1 c_i m_{0,t}\right]\mu\right)^2}{\mathbb{E}[Z^2]\sum_{i=1}^{n}\lambda_t^{(i)}+(\mathbb{E}[Z])^2\left(\int_{s=0}^{T-t} g(s)ds\right)^2\left(\sum_{j=1}^{n}\sum_{i=1}^{n} c_j c_i m_{j,i}\right)\mu} dt,
\end{aligned} \tag{2.45}$$

To make the formulae look simpler, we denote $\int_{s=0}^{T-t} g(s)ds$ as $\hat{g}_t$ and $\left(\int_{s=0}^{T-t} g(s)ds\right)^2$ as $\hat{g}_t^2$, so that we can write

$$\frac{\left(d\langle V^{(1)},D^{(1)}\rangle_t\right)^2}{d\langle D^{(1)}\rangle_t} = \frac{\left(\mathbb{E}Z^2\lambda_t^{(1)}+(\mathbb{E}Z)^2\cdot\hat{g}_t^2\left[c_1 m_2+\sum_{i=2}^{n} c_1 c_i m_{1,i}\right]\mu\right)^2}{\mathbb{E}Z^2\sum_{i=1}^{n}\lambda_t^{(i)}+(\mathbb{E}Z)^2\cdot\hat{g}_t^2\left(\sum_{j=1}^{n}\sum_{i=1}^{n} c_j c_i m_{j,i}\right)\mu} dt, \tag{2.46}$$

and for the second index

$$\frac{\left(d\langle V^{(1)},D^{(2)}\rangle_t\right)^2}{d\langle D^{(2)}\rangle_t} = \frac{\left((\mathbb{E}Z)^2 \cdot \hat{g}_t^2 \left[\sum_{i=2}^{n} c_1 c_i m_{1,i}\right] \mu\right)^2}{\mathbb{E}[Z^2] \sum_{i=2}^{n} \lambda_t^{(i)} + (\mathbb{E}Z)^2 \cdot \hat{g}_t^2 \left(\sum_{j=2}^{n}\sum_{i=2}^{n} c_j c_i m_{j,i}\right) \mu} dt. \tag{2.47}$$

For $D_t^{(3)}$ we have

$$\frac{\left(d\langle V^{(1)},D^{(3)}\rangle_t\right)^2}{d\langle D^{(3)}\rangle_t} = \frac{(\mathbb{E}[Z])^2 \left(\lambda_t^{(1)} + \hat{g}_t^2 \left[c_1 m_2 + \sum_{i=2}^{n} c_1 c_i m_{1,i}\right] \mu\right)^2}{\sum_{i=1}^{n} \lambda_t^{(i)} + \hat{g}_t^2 \left(\sum_{j=1}^{n}\sum_{i=1}^{n} c_j c_i m_{j,i}\right) \mu} dt, \tag{2.48}$$

and for $D_t^{(4)}$

$$\frac{\left(d\langle V^{(1)},D^{(4)}\rangle_t\right)^2}{d\langle D^{(4)}\rangle_t} = \frac{(\mathbb{E}[Z])^2 \left(\hat{g}_t^2 \left[\sum_{i=2}^{n} c_1 c_i m_{1,i}\right] \mu\right)^2}{\sum_{i=2}^{n} \lambda_t^{(i)} + \hat{g}_t^2 \left(\sum_{j=2}^{n}\sum_{i=2}^{n} c_j c_i m_{j,i}\right) \mu} dt. \tag{2.49}$$

The expression for $D^{(5)}$ is as follows:

$$\begin{aligned}\frac{\left(d\langle V^{(1)},D^{(5)}\rangle_t\right)^2}{d\langle D^{(5)}\rangle_t} &= (\mathbb{E}[Z])^2 \frac{m_1^2}{m_2} \left(\mathrm{E}\left[\int_{s=0}^{T-t} M_{K_t}^{(1)}(s) ds\right]\right)^2 \mu dt = c_1^2 (\mathbb{E}[Z])^2 \frac{m_1^4}{m_2} \hat{g}_t^2 \mu dt = \\ &= (\mathbb{E}[Z])^2 \frac{m_1^4}{m_2} \hat{g}_t^2 \mu dt. \end{aligned} \tag{2.50}$$

These are the general formulae that can be used by insurance companies which have available the necessary estimators for c_i, $m_{j,i}$ and $m_{0,i}$. In the following section, we are going to make further assumptions about the correlation coefficients ρ_{i_1,i_2}, which will enable us to carry out a more detailed comparison of the indices $D^{(1)} - D^{(5)}$.

2.2c Proportional exposure

The correlation between $y_j^{(i_1)}$, $y_j^{(i_2)}$ in terms of m_{i_1,i_2} and m_1, m_2 is $\rho_{i_1,i_2} = \frac{m_{i_1,i_2} - m_1^2}{m_2 - m_1^2}$. Depending on the value of the correlation coefficient, various modifications of the formulae $(2.52-2.56)$ can be obtained. Here, we shall consider the case when $\rho_{i_1,i_2} \equiv 1$ for all $i_1, i_2 = 1, \dots. n$. It means that the impact factors are linearly dependent. However, taking into account the nature of the processes we study, it is more suitable to suggest that we simply have a proportional relationship among impact factors $y_j^{(i_1)}$. These proportionality coefficients may be dictated by the geographic location of the companies and their proximity to the potential sources of catastrophes. For example, if the companies insure against flood, those who have more policyholders along seashore or a riverbank are likely to be at a greater risk.

So let us assume that $\rho_{i_1,i_2} \equiv 1$ and $y_j^{(i_1)} = \alpha_{i_1} y_j$. This assumption implies that $m_{i_1,i_2} = \mathrm{E} y_j^{(i_1)} y_j^{(i_2)} = \alpha_{i_1}\alpha_{i_2} m_2$ for $i_1, i_2 = 1, \dots n$, and the intensities

$$\lambda_t^{(i)} = \sum_{0 \leqslant s_k \leqslant t} c_i \alpha_i y_k g(t - s_k) = c_i \alpha_i \lambda_t \tag{2.51}$$

differ only in the coefficient $c_i\alpha_i$. Here λ_t denotes $\sum\limits_{0 \leqslant s_k \leqslant t} y_k g(t - s_k)$.

For the first index we have:

$$\frac{(d\langle V^{(1)}, D^{(1)}\rangle_t)^2}{d\langle D^{(1)}\rangle_t} = \frac{\left(\mathbb{E}[Z^2]c_1\alpha_1\lambda_t + (\mathbb{E}[Z])^2 \hat{g}_t^2 c_1\alpha_1 m_2\left(\sum\limits_{i=1}^n c_i\alpha_i\right)\mu\right)^2}{\mathbb{E}[Z^2]\left(\sum\limits_{i=1}^n c_i\alpha_i\right)\lambda_t + (\mathbb{E}[Z])^2\hat{g}_t^2 m_2\left(\sum\limits_{i=1}^n c_i\alpha_i\right)^2\mu} dt =$$

$$= \alpha_1^2 \frac{\mathbb{E}[Z^2]\lambda_t + (\mathbb{E}[Z])^2\hat{g}_t^2 m_2\left(\sum\limits_{i=1}^n c_i\alpha_i\right)\mu}{\sum\limits_{i=1}^n c_i\alpha_i} dt. \tag{2.52}$$

Then for the second one:

$$\frac{(d\langle V^{(1)},D^{(2)}\rangle_t)^2}{d\langle D^{(2)}\rangle_t} = \frac{(\mathbb{E}[Z])^4\left(\sum_{i=2}^{n} c_i\alpha_i\right)(\hat{g}_t^2 c_1\alpha_1 m_2\mu)^2}{\mathbb{E}[Z^2]\lambda_t+(\mathbb{E}[Z])^2\hat{g}_t^2 m_2\left(\sum_{i=2}^{n} c_i\alpha_i\right)\mu.}dt. \tag{2.53}$$

For the third index we obtain:

$$\frac{(d\langle V^{(1)},D^{(3)}\rangle_t)^2}{d\langle D^{(3)}\rangle_t} = \frac{(\mathbb{E}[Z])^2\left(\alpha_1\lambda_t+\hat{g}_t^2\alpha_1 m_2\left(\sum_{i=1}^{n} c_i\alpha_i\right)\mu\right)^2}{\lambda_t\sum_{i=1}^{n} c_i\alpha_i+\hat{g}_t^2 m_2\left(\sum_{i=1}^{n} c_i\alpha_i\right)^2\mu}dt = \alpha_1^2\frac{(\mathbb{E}[Z])^2\left(\lambda_t+\hat{g}_t^2 m_2\mu\sum_{i=1}^{n} c_i\alpha_i\right)}{\sum_{i=1}^{n} c_i\alpha_i}dt. \tag{2.54}$$

The formula for the fourth index is

$$\frac{(d\langle V^{(1)},D^{(4)}\rangle_t)^2}{d\langle D^{(4)}\rangle_t} = \alpha_1^2\frac{(\mathbb{E}[Z])^2(g_t^2 m_2\mu)^2\left(\sum_{i=2}^{n} c_i\alpha_i\right)}{\alpha_1\lambda_t+g_t^2 m_2\mu\left(\sum_{i=2}^{n} c_i\alpha_i\right)}dt. \tag{2.55}$$

And for the fifth one it will be

$$\frac{(d\langle V^{(1)},D^{(5)}\rangle_t)^2}{d\langle D^{(5)}\rangle_t} = (\mathbb{E}[Z])^2\frac{m_1^4}{m_2}\hat{g}_t^2\alpha_1^2\mu dt. \tag{2.56}$$

We summarize the results in the Table 3:

TABLE 3. INDEX-SPECIFIC REMAINING RISK PARTS FOR PROPORTIONAL EXPOSURE.

index	formula	remaining risk part
1	$\sum_{i=1}^{n} X_T^{(i)}$	$\alpha_1^2 \dfrac{\mathbb{E}[Z^2]\lambda_t + (\mathbb{E}[Z])^2 \hat{g}_t^2 m_2 \left(\sum_{i=1}^{n} c_i \alpha_i \right) \mu}{\sum_{i=1}^{n} c_i \alpha_i} dt$
2	$\sum_{i=2}^{n} X_T^{(i)}$	$\dfrac{(\mathbb{E}[Z])^4 \left(\sum_{i=2}^{n} c_i \alpha_i \right) (\hat{g}_t^2 c_1 \alpha_1 m_2 \mu)^2}{\mathbb{E}[Z^2]\lambda_t + (\mathbb{E}[Z])^2 \hat{g}_t^2 m_2 \left(\sum_{i=2}^{n} c_i \alpha_i \right) \mu.} dt$
3	$\sum_{i=1}^{n} N_T^{(i)}$	$\alpha_1^2 (\mathbb{E}[Z])^2 \dfrac{\left(\lambda_t + \hat{g}_t^2 m_2 \mu \sum_{i=1}^{n} c_i \alpha_i \right)}{\sum_{i=1}^{n} c_i \alpha_i} dt$
4	$\sum_{i=2}^{n} N_T^{(i)}$	$\alpha_1^2 (\mathbb{E}[Z])^2 \dfrac{(g_t^2 m_2 \mu)^2 \left(\sum_{i=2}^{n} c_i \alpha_i \right)}{\alpha_1 \lambda_t + g_t^2 m_2 \mu \left(\sum_{i=2}^{n} c_i \alpha_i \right)} dt$
5	$\sum_{k \leqslant K_T} y_k$	$\alpha_1^2 (\mathbb{E}[Z])^2 \frac{m_1^4}{m_2} \hat{g}_t^2 \mu dt$

We already know from Subsection 2.2a that the fourth index is better than the second one. Now we can add that, for the same reason, since $\mathbb{E}[\xi^2] \geqslant (\mathbb{E}[\xi])^2$ for every ξ, the third index is worse than the first one. By looking at the difference between the third and the fourth ratio, we can see that it is strictly positive:

$$\alpha_1^2 \left[\frac{(\mathbb{E}[Z])^2 \left(\lambda_t + \hat{g}_t^2 m_2 \mu \sum_{i=1}^{n} c_i \alpha_i \right)}{\sum_{i=1}^{n} c_i \alpha_i} - \frac{(\mathbb{E}[Z])^2 \left(\sum_{i=2}^{n} c_i \alpha_i \right) (\hat{g}_t^2 m_2 \alpha_1 \mu)^2}{\alpha_1 \lambda_t + \hat{g}_t^2 m_2 \mu \left(\sum_{i=2}^{n} c_i \alpha_i \right)} \right] dt =$$

$$= (\mathbb{E}[Z])^2 \frac{\left(\lambda_t + \hat{g}_t^2 m_2 \mu \sum_{i=1}^{n} c_i \right) \left(\lambda_t + \hat{g}_t^2 m_2 \mu \left(\sum_{i=2}^{n} c_i \right) \right) - \sum_{i=1}^{n} c_i \left(\sum_{i=2}^{n} c_i \right) (\hat{g}_t^2 m_2 \mu)^2}{\left(\lambda_t + \hat{g}_t^2 m_2 \mu \left(\sum_{i=2}^{n} c_i \right) \right) \sum_{i=1}^{n} c_i} dt =$$

$$= (\mathbb{E}[Z])^2 \frac{\lambda_t^2+\hat{g}_t^2 m_2\mu\left(\sum_{i=1}^{n} c_i+\sum_{i=2}^{n} c_i\right)\lambda_{ti}}{\left(\lambda_t+\hat{g}_t^2 m_2\mu\left(\sum_{i=2}^{n} c_i\right)\right)\sum_{i=1}^{n} c_i} dt. \tag{2.57}$$

Thus, we may now conclude that the third index is better than the fourth one. The severity index can be placed between the third and the fourth. For the proof, we estimate the difference between the third ratio and the fifth one ((2.54) minus (2.56)):

$$\frac{(\mathbb{E}[Z])^2\left(\lambda_t+\hat{g}_t^2 m_2\mu\sum_{i=1}^{n} c_i\right)dt}{\sum_{i=1}^{n} c_i} - (\mathbb{E}[Z])^2 \frac{m_1^4}{m_2}\hat{g}_t^2\mu dt =$$

$$= (\mathbb{E}[Z])^2\left(\left(\frac{\lambda_t}{\sum_{i=1}^{n} c_i} + \hat{g}_t^2 m_2\mu\right) - \frac{m_1^4}{m_2^2} m_2\hat{g}_t^2\mu\right)dt \geqslant$$

$$\geqslant (\mathbb{E}[Z])^2\mu\hat{g}_t^2(m_2 - \tfrac{m_1^4}{m_2})dt \geqslant (\mathbb{E}[Z])^2\mu\hat{g}_t^2 m_2\left(1 - \left(\tfrac{m_1^2}{m_2}\right)^2\right)dt \geqslant 0. \tag{2.58}$$

It means that the fifth index is ranked lower than the third one. The following computation shows that, in our hierarchy, the fifth index goes straight after the third index and before the fourth one. We divide the expressions (2.55) for the fourth index and (2.56) for the fifth index by the common factor $(\mathbb{E}[Z])^2\mu dt$ and by doing so we only have to compare

$$\frac{\left(\sum_{i=2}^{n} c_i\right)(\hat{g}_t^2 m_2)^2\mu}{\lambda_t+\hat{g}_t^2 m_2\mu\left(\sum_{i=2}^{n} c_i\right)} \tag{2.59}$$

and

$$\frac{m_1^4}{m_2}\hat{g}_t^2. \tag{2.60}$$

We will look at their reciprocal values:

$$\frac{\lambda_t+\hat{g}_t^2 m_2\mu\left(\sum_{i=2}^{n} c_i\right)}{\left(\sum_{i=2}^{n} c_i\right)(\hat{g}_t^2 m_2)^2\mu} = \frac{\lambda_t}{\left(\sum_{i=2}^{n} c_i\right)(\hat{g}_t^2 m_2)^2\mu} + \frac{1}{\hat{g}_t^2 m_2}, \tag{2.61}$$

and for the expression from fourth index

$$\frac{m_2}{\hat{g}_l^2 m_1^4} \leqslant \frac{m_2}{\hat{g}_l^2 m_2^2} = \frac{1}{\hat{g}_l^2 m_2} \leqslant \frac{\lambda_l}{\left(\sum_{i=2}^{n} c_i\right)(\hat{g}_l^2 m_2)^2 \mu} + \frac{1}{\hat{g}_l^2 m_2}, \tag{2.62}$$

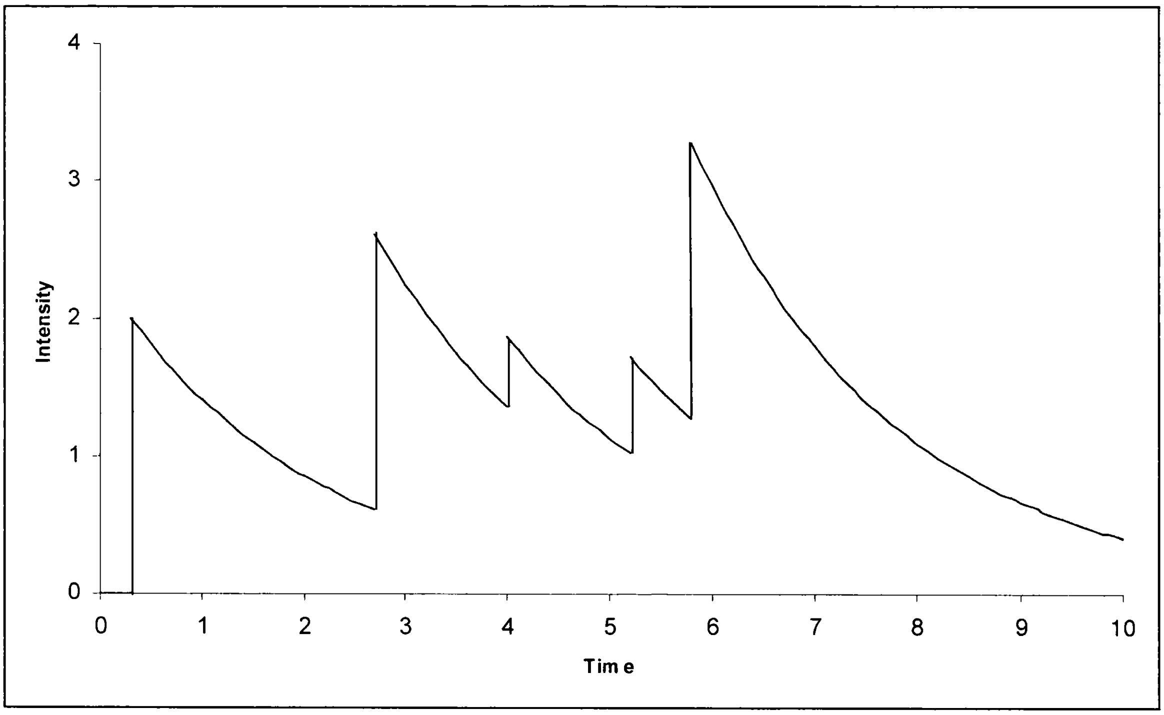
the left-hand side of (2.62) being the inverse of (2.60), which corresponds to the fifth index and right-hand side being the reciprocal of (2.59) which is related to the fourth index. So we conclude that the reciprocal of the expression for the fifth index is smaller than that for the fourth one, and thus the fifth index is better than the fourth one.

We can now order indices from the best to the worst: $D^{(1)}$ - $D^{(3)}$ - $D^{(5)}$ - $D^{(4)}$ - $D^{(2)}$. The first order $D^{(1)}$ - $D^{(3)}$ is no surprise, as with the transfer from the total loss amount to the total number of claims essential information about the claim size is lost. The superiority of index $D^{(4)}$ over index $D^{(2)}$ is less obvious. It can be explained if the information about other companies excluding the hedging company is viewed by it as some noise. In this sense, the less noise the better, and the index based on the number of claims for companies $2,\dots,n$ reduces remaining risk better than the index which settles on the total accumulated losses suffered by the same companies. The severity index is essentially on the borderline: it still contains some information related to company 1, but in a minimal amount, and it still exceeds both indices which do not depend on the data for company 1. We may expect other possible indices to be ranked higher above index $D^{(5)}$ if they involve data on the hedging company and have more information, and lower than index $D^{(5)}$ if they have more information but do not include company 1. Theoretical grounds for this phenomenon based on the martingale representation theorem for point processes are given in Subsection 2.2e.

2.2d Shot - noise with exponential mark functions

In this section we are going to derive the formulae for remaining risk and the hedging strategy for the case of the exponential mark functions. These functions are a good representation of the nature of the claim arrival process. As a catastrophe occurs, there is a sudden jump in the number of submitted claims and then the process declines at the rate of decay σ. These processes have been studied by Dassios, Jang (2003) for the purpose of obtaining the gross premium for stop-loss reinsurance contract and arbitrage-free prices for insurance derivatives. A detailed account of the theory of the point processes with exponential marks can be found in Bremaud (1981) and Daley and Vere-Jones (2002). On the graph below, the trajectory of the intensity of a shot-noise process with exponential mark functions is demonstrated.

GRAPH 4. TRAJECTORY OF INTENSITY OF A SHOT-NOISE PROCESS WITH EXPONENTIAL MARKS.



We are going to find the optimal strategy (2.13) and the remaining risk (2.15) for the three best indices, namely $D^{(1)}$, $D^{(3)}$ and $D^{(5)}$ by using the formulae (2.26), (2.52), (2.54) and (2.56). As before, the terms with $(dt)^2$, which are infinitesimal of order 2, are neglected.

First we will write explicitly what are the values $\hat{g}_t$ and the random processes λ_t, $\lambda_t^{(i)}$ in the case of exponential (decaying at the rate σ) mark functions:

$$\hat{g}_t = \int_{s=0}^{T-t} e^{-\sigma s} ds = \int_{s=0}^{T-t} e^{-\sigma s} ds = \frac{1-e^{-\sigma(T-t)}}{\sigma} = a_t \tag{2.63}$$

$$\lambda_t = \sum_{0 \leqslant s_k \leqslant t} y_k e^{-\sigma(t-s_k)},$$

$$\lambda_t^{(i)} = \sum_{0 \leqslant s_k \leqslant t} c_i \alpha_i y_k e^{-\sigma(t-s_k)}. \tag{2.64}$$

Then the strategy and the remaining risk for the first index are

$$\xi_t^{(1)} = \frac{d<V^{(1)},D^{(1)}>_t}{d<D^{(1)}>_t} = \frac{\alpha_1}{\sum\limits_{i=1}^{n} c_i \alpha_i}, \tag{2.65}$$

$$R_t^{(1)} = \mathbb{E}\left[\int_t^T d < V^{(1)} >_s \mid \mathcal{F}_t)\right] - \mathbb{E}\left[\int_t^T \frac{(d<V^{(1)},D^{(1)}>_s)^2}{d<D>_s} \mid \mathcal{F}_t\right] =$$

$$= \mathbb{E}\left[\int_t^T \left(dX_u^{(1)} + \mathbb{E}[Z]\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)dK_u\right)^2 \mid \mathcal{F}_t\right] -$$

$$- \mathbb{E}\left[\int_t^T \alpha_1^2 \frac{\mathbb{E}[Z^2]\lambda_u + (\mathbb{E}Z)^2\left(\frac{1-e^{-\sigma(T-u)}}{\sigma}\right)^2 m_2\left(\sum\limits_{i=1}^{n} c_i\alpha_i\right)\mu}{\sum\limits_{i=1}^{n} c_i \alpha_i} du \mid \mathcal{F}_t\right] =$$

$$= \alpha_1 \mathbb{E}[Z^2] \int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du + \alpha_1^2(\mathbb{E}[Z])^2 m_2 \mu \int_t^T \left(\frac{1-e^{-\sigma(T-u)}}{\sigma}\right)^2 du -$$

$$- \frac{\alpha_1^2}{\sum\limits_{i=1}^{n} c_i\alpha_i} \mathbb{E}[Z^2] \int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du - \alpha_1^2(\mathbb{E}[Z])^2 m_2 \mu \int_t^T \left(\frac{1-e^{-\sigma(T-u)}}{\sigma}\right)^2 du =$$

$$= \alpha_1 \left(1 - \frac{\alpha_1}{\sum\limits_{i=1}^{n} c_i \alpha_i}\right) \mathbb{E}[Z^2] \int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du. \tag{2.66}$$

We only have to find $\int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du$. The subintegral expression equals

$$\mathbb{E}[\lambda_u|\mathcal{F}_t] = \mathbb{E}\left[\sum_{0 \leqslant s_k \leqslant u} y_k e^{-\sigma(u-s_k)} \mid \mathcal{F}_t\right] = e^{-\sigma(u-t)}\lambda_t + \mathbb{E}\left[\sum_{t < s_k \leqslant u} y_k e^{-\sigma(u-s_k)}\right].$$

The first component in the sum above shows the contribution to intensity due to the catastrophes occurred no later that time t, while the second one shows the input of catastrophes which happened after time t. This second part is computed below:

$$\mathbb{E}\left[\sum_{t<s_k\leqslant u} y_k e^{-\sigma(u-s_k)}\right] = \int_t^u \mathbb{E}y\mu e^{-\sigma(u-s)}ds = e^{-\sigma u}\int_t^u m_1\mu e^{\sigma s}ds = m_1\mu\frac{1-e^{-\sigma(u-t)}}{\sigma}.$$

Now we can write that for $t \leqslant u \leqslant T$

$$\mathbb{E}[\lambda_u|\mathcal{F}_t] = e^{-\sigma(u-t)}\lambda_t + m_1\mu\frac{1-e^{-\sigma(u-t)}}{\sigma}.$$

This, in particular, shows that

$$\mathbb{E}[\lambda_u] = m_1\mu\frac{1-e^{-\sigma u}}{\sigma}.$$

Now we go back to (2.66) and use the notation from (2.63):

$$\begin{aligned}\int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du &= \int_t^T \left(e^{-\sigma(u-t)}\lambda_t + m_1\mu\frac{1-e^{-\sigma(u-t)}}{\sigma}\right)du = \frac{1-e^{-\sigma(T-t)}}{\sigma}\lambda_t + m_1\mu\frac{1-\frac{1-e^{-\sigma(T-t)}}{\sigma}}{\sigma} = \\ &= \frac{1-e^{-\sigma(T-t)}}{\sigma}\lambda_t + m_1\mu\frac{\sigma-1-e^{-\sigma(T-t)}}{\sigma^2} = a_t\lambda_t + m_1\mu\frac{1-a_t}{\sigma},\end{aligned}$$

so the remaining risk is

$$\begin{aligned}R_t^{(1)} &= \alpha_1\left(1-\frac{\alpha_1}{\sum_{i=1}^n c_i\alpha_i}\right)\mathbb{E}[Z^2]\int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du = \\ &= \alpha_1\left(1-\frac{\alpha_1}{\sum_{i=1}^n c_i\alpha_i}\right)\mathbb{E}[Z^2]\left(a_t\lambda_t + m_1\mu\frac{1-a_t}{\sigma}\right).\end{aligned}$$

If the distribution of the catastrophe severity is exponential with parameter γ, then the remaining risk is

$$R_t^{(1)} = \alpha_1\left(1-\frac{\alpha_1}{\sum_{i=1}^n c_i\alpha_i}\right)\mathbb{E}[Z^2]\left(a_t\lambda_t + \gamma\mu\frac{1-a_t}{\sigma}\right). \tag{2.67}$$

Now we calculate the optimal strategy and the remaining risk for the third index:

$$\xi_t^{(3)} = \frac{d\langle V^{(1)}, D^{(3)}\rangle_t}{d\langle D^{(3)}\rangle_t} = \frac{(\mathrm{E}Z)^2\left(\alpha_1\lambda_t + g_2\alpha_1 m_2\left(\sum_{i=1}^{n} c_i\alpha_i\right)\mu\right)}{\lambda_t\sum_{i=1}^{n} c_i\alpha_i + g_2 m_2\left(\sum_{i=1}^{n} c_i\alpha_i\right)^2\mu}dt = \frac{(\mathrm{E}Z)^2\alpha_1}{\sum_{i=1}^{n} c_i\alpha_i}, \tag{2.68}$$

$$\begin{aligned}
R_t^{(3)} &= \alpha_1\mathbb{E}[Z^2]\int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du + \alpha_1^2(\mathrm{E}Z)^2 m_2\mu\int_t^T \left(\tfrac{1-e^{-\sigma(T-u)}}{\sigma}\right)^2 du - \\
&\quad - \mathbb{E}\left[\int_t^T \alpha_1^2\frac{(\mathrm{E}Z)^2\left(\lambda_u + g_t^2 m_2\mu\sum_{i=1}^{n} c_i\alpha_i\right)}{\sum_{i=1}^{n} c_i\alpha_i}du \;\Big|\; \mathcal{F}_t\right] = \\
&= \alpha_1\mathbb{E}[Z^2]\int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du + \alpha_1^2(\mathrm{E}Z)^2 m_2\mu\int_t^T \left(\tfrac{1-e^{-\sigma(T-u)}}{\sigma}\right)^2 du - \\
&\quad - \frac{(\mathrm{E}Z)^2\alpha_1^2}{\sum_{i=1}^{n} c_i\alpha_i}\int_t^T \mathbb{E}\left[\lambda_u + \left(\tfrac{1-e^{-\sigma(T-t)}}{\sigma}\right)^2 m_2\mu\sum_{i=1}^{n} c_i\alpha_i \;\Big|\; \mathcal{F}_t\right]du = \\
&= \left(\alpha_1\mathbb{E}[Z^2] - \frac{(\mathrm{E}Z)^2\alpha_1}{\sum_{i=1}^{n} c_i\alpha_i}\right)\int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du = \\
&= \alpha_1\left(\mathbb{E}[Z^2] - \frac{(\mathbb{E}[Z])^2}{\sum_{i=1}^{n} c_i\alpha_i}\right)\left(a_t\lambda_t + m_1\mu\tfrac{1-a_t}{\sigma}\right).
\end{aligned} \tag{2.69}$$

The optimal strategy and the remaining risk for the severity index $D^{(5)}$ are

$$\xi_t^{(5)} = \frac{\mathrm{E}[Z]\mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\right]m_1\mu}{m_2\mu} = \mathbb{E}[Z]\mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\right]\frac{m_1}{m_2} = (\mathbb{E}[Z])^2\frac{m_1^2}{m_2}a_t\alpha_1^2\mu, \tag{2.70}$$

$$\begin{aligned}
R_t^{(5)} &= \\
&= \alpha_1\mathbb{E}[Z^2]\int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du + \alpha_1^2(\mathrm{E}Z)^2 m_2\mu\int_t^T \left(\tfrac{1-e^{-\sigma(T-u)}}{\sigma}\right)^2 du - \\
&\quad - \mathbb{E}\left[\int_t^T (\mathbb{E}[Z])^2\frac{m_1^4}{m_2}g_u^2\alpha_1^2\mu du \;\Big|\; \mathcal{F}_t\right] = \\
&= \alpha_1\mathbb{E}[Z^2]\int_t^T \mathbb{E}[\lambda_u|\mathcal{F}_t]du + \alpha_1^2(\mathbb{E}[Z])^2\left(m_2 - \frac{m_1^4}{m_2}\right)\mu\int_t^T \left(\tfrac{1-e^{-\sigma(T-u)}}{\sigma}\right)^2 du =
\end{aligned}$$

$$= \alpha_1 \mathbb{E}[Z^2]\left(a_t\lambda_t + m_1\mu\tfrac{1-a_t}{\sigma}\right) + \alpha_1^2(\mathbb{E}[Z])^2\left(m_2 - \tfrac{m_1^4}{m_2}\right)\mu\left(\tfrac{T-t}{\sigma} - \tfrac{a_t}{\sigma}\left[\tfrac{1}{\sigma} + \tfrac{a_t}{2}\right]\right) =$$
$$= \alpha_1 \mathbb{E}[Z^2]a_t\lambda_t + \alpha_1^2(\mathbb{E}[Z])^2\left(m_2 - \tfrac{m_1^4}{m_2}\right)\mu\left(\tfrac{T-t}{\sigma} - \tfrac{a_t}{\sigma}\left[\tfrac{1}{\sigma} + \tfrac{a_t}{2}\right]\right) + \alpha_1\mathbb{E}[Z^2]m_1\mu\tfrac{1-a_t}{\sigma}. \tag{2.71}$$

The formulae (2.65), (2.68) and (2.70) give the optimal hedging strategy in the market with indices $D^{(1)}$, $D^{(3)}$ and $D^{(5)}$. Interestingly, the strategy for the first and the third index is constant over time, while for the fifth index it depends on t.

2.2e General representation of the basis risk for the point-process martingales

In this subsection, we will consider two types of indices in the most general form. The first type includes any index D' that ignores the sizes of the claims and the catastrophe severities and only counts them; that is, the index which is based on the claims arrival processes $N_t^{(1)},\ldots,N_t^{(n)}$ and the catastrophe arrival process K_t. To obtain the formulae for the remaining risk for such index, we will need the following theorem (see Bremaud (1981)):

THEOREM 2.1. (Integral Representation of Point Process Martingales). Let $(N_t^{(1)},\ldots,N_t^{(m)})$ be an $m-$ variate point process on $(\Omega,\mathcal{F},\mathbb{P})$, $\mathbb{P}-$ non-explosive, and let $\mathcal{G}_t$ be its internal history. Suppose that for each i $(1 \leqslant i \leqslant M)$, $N_t^{(i)}$ admits the $(\mathbb{P},\mathcal{G}_t)$-predictable intensity $\lambda_t^{(i)}$. Let now M_t be a right-continuous $(\mathbb{P},\mathcal{G}_t)$ –martingale of the form $M_t = E[M_T|\mathcal{G}_t]$, where M_T is some $\mathbb{P}$ –integrable random variable. Then for each $t \geqslant 0$

$$M_t = M_0 + \sum_{i=1}^{m} \int_0^t G_s^{(i)}(dN_s(i) - \lambda_s(i)ds) \qquad P-a.s.,$$

where for each i $(1 \leqslant i \leqslant M)$, $G_t^{(i)}$ is a $\mathcal{G}_t$ –predictable process satisfying

$$\int_0^t \left|G_s^{(i)}\right|\lambda_s^{(i)}ds < \infty \qquad P-a.s., \qquad t \geqslant 0.$$

We can also write it in the infinitesimal form, which we will mostly be using:

$$dM_t = \sum_{i=1}^{m} G_t^{(i)} \left(dN_t^{(i)} - \lambda_t^{(i)} dt \right) \qquad P - a.s., \tag{2.72}$$

In our setting, a catastrophe index can involve several point processes: the claims arrival processes $N_t^{(1)}, \ldots, N_t^{(n)}$ and the catastrophe arrival process K_t. Since any of the claims arrival processes are linked to the process K_t, the catastrophe arrival process is the only one which must be present in the martingale representation of an index, that is, $G_t^{(i)} \neq 0$. Thus, the dynamics of the index will be of the form

$$dD'_t = G_t^{(0)}(dK_t - \mu dt) + \sum_{i=1}^{n} G_t^{(i)} \left(dN_t^{(i)} - \lambda_t^{(i)} dt \right), \tag{2.73}$$

where some or all of the processes $G_t^{(i)}$, $i = 1, \ldots, n$ can be zero. Then the minimal hedging error associated with trading such index is based on the predictable quadratic variance of the index price process

$$\begin{aligned} d\langle D' \rangle_t &= \mathbb{E}\left[\left(G_t^{(0)}(dK_t - \mu dt) + \sum_{i=1}^{n} G_t^{(i)} \left(dN_t^{(i)} - \lambda_t^{(i)} dt \right) \right)^2 \mid \mathcal{F}_{t-} \right] = \\ &= \left(G_t^{(0)} \right)^2 \mu dt + \sum_{i=1}^{n} \left(G_t^{(i)} \right)^2 \lambda_t^{(i)} dt, \end{aligned}$$

and the predictable quadratic covariance process

$$\begin{aligned} & d < V^{(1)}, D' >_t = \\ & = \mathbb{E}\Bigg[\left(dX_t^{(1)} + \mathbb{E}[Z] \int_{s=0}^{T-t} M_{K_t}^{(1)}(s) ds \, dK_t \right) \\ & \qquad \left(G_t^{(0)}(dK_t - \mu dt) + \sum_{i=1}^{n} G_t^{(i)} \left(dN_t^{(i)} - \lambda_t^{(i)} dt \right) \right) \mid \mathcal{F}_{t-} \Bigg] = \\ & = \mathbb{E}[Z] G_t^{(0)} \mathbb{E}\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s) ds \right) \mu dt + G_t^{(1)} \mathbb{E}[Z] \lambda_t^{(1)} dt. \end{aligned}$$

The increment of the remaining risk part (2.16) for the index D' equals

$$\frac{\left(d\langle V^{(1)},D'\rangle_t\right)^2}{d\langle D'\rangle_t} = \frac{\left(G_t^{(0)}\mathbb{E}\left(\int\limits_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\mu + G_t^{(1)}\lambda_t^{(1)}\right)^2}{\left(G_t^{(0)}\right)^2\mu + \sum\limits_{i=1}^{n}\left(G_t^{(i)}\right)^2\lambda_t^{(i)}}(\mathbb{E}[Z])^2 dt. \tag{2.74}$$

The components $G_t^{(0)}\mathbb{E}\left(\int\limits_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\mu$ in the numerator and $\left(G_t^{(0)}\right)^2\mu$ in the denominator of the expression above are present for any non-trivial index. One can observe that if the hedging company's claims arrival process is excluded from the index, and therefore $G_t^{(1)} \equiv 0$, then the value (2.74) can be maximized by minimizing the denominator to its catastrophe arrival component $\left(G_t^{(0)}\right)^2\mu$ by setting $G_t^{(i)} \equiv 0,\ i = 2,\dots,n$. Then (2.74) reduces to its maximal possible value

$$\frac{\left(d\langle V^{(1)},D'\rangle_t\right)^2}{d\langle D'\rangle_t} = \left[\mathbb{E}\left(\int\limits_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\right]^2 \mu(\mathbb{E}[Z])^2 dt.$$

We conclude that, if the hedging company is not in the index for some reasons, then the best hedge of its catastrophe risk will be provided by the index which depends only on the catastrophe's arrival process K_t. Introduction of the companies' claims arrival processes to the index will have an adverse effect on the quality of the hedge; the smaller is the sum $\sum\limits_{i=1}^{n}\left(G_t^{(i)}\right)^2\lambda_t^{(i)}$, the better.

The second group comprises indices D'' which depend on the catastrophe arrival process K_t, catastrophe severities y_k, claims arrival processes $N_t^{(1)},\dots,N_t^{(n)}$ and claim sizes. In such case, we will be dealing with marked point processes, where y_k and claim sizes $X_j^{(i)}$ will act as marks. The formal definition of a marked point process is given below:

DEFINITION 2.6. Let there be defined on some measurable space $(\Omega,\mathcal{F},\mathbb{P})$:

i) a point process T_n (or N_t),

ii) a sequence $(Z_n, n \geqslant 1)$ of E-valued random variables, where $(E,\mathcal{E})$ is a measurable space.

Then the double sequence $(T_n, Z_n, n \geqslant 1)$ is called an *E-marked point* *process.* The measurable space $(E, \mathcal{E})$ on which the sequence $(Z_n, n \geqslant 1)$ takes its values is the *mark space.*

The internal history of $(T_n, Z_n, n \geqslant 1)$ is defined by $\mathcal{F}_t^p$.

An extension of the Theorem 2.1 to marked point processes is needed for evaluating the remaining risk for the index D'' (the complete account of the relevant theory can be found in Bremaud (1981)). But before we introduce it, we need to make further assumptions about the intensity functions of the claim arrival processes $\lambda_t^{(i)}$ and introduce the following

DEFINITION 2.7. Let $p(dt \times dz)$ be an $E-$ marked point process with $(\mathbb{P}, \mathcal{F}_t)-$ intensity kernel $\lambda_t(dz)$ of the form

$$\lambda_t(dz) = \lambda_t \Phi_t(dz),$$

where λ_t is a nonnegative $\mathcal{F}_t$-predictable process and $\Phi_t(\omega, dz)$ is a probability transition kernel from $(\Omega \times [0, \infty), \mathcal{F} \otimes \mathcal{B}_+)$ into $(E, \mathcal{E})$. The pair $(\lambda_t, \Phi_t(dz))$ is called the $(\mathbb{P}, \mathcal{F}_t)$-local characteristics of $p(dt \times dz)$.

THEOREM 2.2. (Integral Representation of Point Process Martingales). Let $p(dt \times dz)$ be an $E-$ marked point process with the $(\mathbb{P}, \mathcal{F}_t)$-local characteristics $(\lambda_t, \Phi_t(dz))$, and the internal history $\mathcal{F}_t$ has the special form $\mathcal{F}_t = \mathcal{F}_0 \vee \mathcal{F}_t^p$. Then any $(\mathbb{P}, \mathcal{F}_t)$-martingale M_t admits the stochastic integral representation

$$M_t = M_0 + \int_0^t \int_E G(s,z) q(ds \times dz) \qquad P-a.s., \tag{2.75}$$

where

$$q(dt \times dz) = p(dt \times dz) - \lambda_t(dz)dt$$

and $G(t,z)$ is an E-indexed $\mathcal{F}_t$-predictable process such that

$$\int_0^t \int_E |G(s,z)| \lambda_s(dz) ds < \infty \qquad P-a.s., \qquad t \geqslant 0.$$

The infinitesimal form of (2.44) is

$$dM_t = \int_E G(t,z)[p(dt \times dz) - \lambda_t(dz)dt] \qquad P - a.s.,$$

The discounted price process of the index D'',

$$D''_t = \mathbb{E}\left[D''_T \mid \mathcal{F}_t\right].$$

is a right-continuous martingale, and according to the Theorem 2.2 there exists the following infinitesimal representation:

$$dD''_t = G^{(0)}(t,z)y_{K_t}(dK_t - \mu dt) + \sum_{i=1}^{n}\int_E G^{(i)}(t,z)\left(dX_t^{(i)} - \lambda_t^{(i)}(dz)dt\right), \tag{2.76}$$

where y_{K_t} equals the severity of the catastrophe number K_t and $G^{(0)}(t,z)$ is non-zero. Other processes $G^{(i)}(t,z)$, however, can equal zero. Then we can calculate the remaining risk part by computing

$$\begin{aligned}
d\langle D''\rangle_t &= \mathbb{E}\left[\left(G^{(0)}(t,z)y_{K_t}(dK_t - \mu dt) + \sum_{i=1}^{n}\int_E G^{(i)}(t,z)\left(dX_t^{(i)} - \lambda_t^{(i)}(dz)dt\right)\right)^2 \mid \mathcal{F}_{t-}\right] = \\
&= \left(G_t^{(0)}(t,z)\right)^2 \mu dt + \sum_{i=1}^{n}\left(G_t^{(i)}(t,z)\right)^2 \mathbb{E}[Z^2]\lambda_t^{(i)}dt,
\end{aligned}$$

$$\begin{aligned}
&d < V^{(1)}, D'' >_t = \\
&= \mathbb{E}\Bigg[\left(dX_t^{(1)} + \mathbb{E}[Z]\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds dK_t\right) \\
&\qquad \left(G^{(0)}(t,z)y_{K_t}(dK_t - \mu dt) + \sum_{i=1}^{n}\int_E G^{(i)}(t,z)\left(dX_t^{(i)} - \lambda_t^{(i)}(dz)dt\right)\right) \mid \mathcal{F}_{t-}\Bigg] = \\
&= G^{(1)}(t,z)(\mathbb{E}[Z])^2\lambda_t^{(1)}dt;
\end{aligned}$$

and then finding the ratio

$$\frac{\left(d<V^{(1)},D''>_t\right)^2}{d<D''>_t} = \frac{\left(G^{(1)}(t,z)\lambda_t^{(1)}\right)^2}{\sum_{i=1}^{m} a_i^2\left\{\left[G^{(i)}(t,z)\right]^2\lambda_t^{(i)}\right\}}\frac{(\mathbb{E}[Z])^4}{\mathbb{E}[Z^2]}dt. \tag{2.77}$$

The comparison of the risk associated with hedging based on the index of D' –type and the one based on the index of D'' –type requires identifying the processes $G_t^{(i)}$ and $G^{(i)}(t,z)$, which is not trivial. However, we can evaluate the remaining risk for some indices of D' –type in the case when the intensities $\lambda_t^{(i)}$ are deterministic. Let us consider an index D''' that indicates whether the weighted number of claims submitted to n insurance companies exceeds some specified amount B. This will be an index of the knock-out type similar to the Winterthur index (see Section 1.1). We may denote $\sum_{i=0}^{n} b_i N_t^{(i)}$ by N_t, its intensity $\sum_{i=1}^{n} b_i \lambda_t^{(i)}$ by Λ_t, and write the price process for the index D''' as

$$D_t''' = \mathbb{E}\Big[\mathbb{I}_{\{N_T \leqslant B\}} \mid \mathcal{F}_t\Big] = \mathbb{P}[N_T \leqslant B | N_t] = \mathbb{P}[N_T - N_t \leqslant B - N_t | N_t] =$$
$$= \sum_{j=0}^{B-N_t} \frac{(\int_t^T \Lambda_s ds)^j}{j!} e^{-\int_t^T \Lambda_s ds} \mathbb{I}_{\{N_t \leqslant B\}} = h(t, N_t).$$

So D_t''' is a martingale which can be expressed as some function h of t and N_t. Then we can write

$$dD_t''' = (h(t, N_{t-} + 1) - h(t, N_{t-}))(dN_t - \Lambda_t dt) =$$
$$= -\frac{\left(\int_t^T \Lambda_s ds\right)^{B-N_{t-}}}{(n-N_{t-})!} e^{-\int_t^T \Lambda_s ds} \mathbb{I}_{\{N_{t-} \leqslant B\}}(dN_t - \Lambda_t dt) = p(t)\mathbb{I}_{\{N_{t-} \leqslant B\}}(dN_t - \Lambda_t dt),$$

where $p(t)$ is $\mathcal{F}_{t-}$ – measurable.

$$d < D''' >_t \; = \mathbb{E}\big[(p(t)\mathbb{I}_{\{N_{t-} \leqslant B\}}(dN_t - \Lambda_t dt))^2 \mid \mathcal{F}_{t-}\big] = p(t)^2 \mathbb{I}_{\{N_{t-} \leqslant B\}} \Lambda_t dt.$$

$$d < V^{(1)}, D''' >_t = \mathbb{E}\left[\left(dX_t^{(1)} + \mathbb{E}[Z] \int_{s=0}^{T-t} M_{K_t}^{(1)}(s) ds dK_t\right) p(t)\mathbb{I}_{\{N_{t-} \leqslant B\}}(dN_t - \Lambda_t dt) \;\Big|\; \mathcal{F}_{t-}\right] =$$
$$= p(t)\mathbb{I}_{\{N_{t-} \leqslant B\}} \mathbb{E}[Z] \lambda_t^{(1)} dt,$$

$$\frac{(d < V^{(1)}, D''' >_t)^2}{d < D''' >_t} = \frac{\left(p(t)\mathbb{I}_{\{N_{t-} \leqslant B\}} \mathbb{E}[Z]\lambda_t^{(1)}\right)^2}{p(t)^2 \mathbb{I}_{\{N_{t-} \leqslant B\}} \Lambda_t} dt = \frac{\left(\mathbb{E}[Z]\lambda_t^{(1)}\right)^2}{\Lambda_t} dt = \frac{\left(\lambda_t^{(1)}\right)^2}{\sum_{i=1}^{n} b_i \lambda_t^{(i)}} (\mathbb{E}[Z])^2 dt.$$

However, the cases with stochastic intensity are much more difficult, and we leave them for future research.

2.2f Example: application to earthquake model

In the sections above, we assumed that the catastrophe occurrence process is a Poisson process with constant intensity. Of course, this assumption is somewhat idealised, and the real-life models of catastrophic events are more diverse. Empirical studies show that not only do the models for different natural disasters follow different patterns, but even catastrophes of the same type but in remote geographic locations fit into different models. A large number of models use the so-called cluster processes, where the occurrence of one event is accompanied by a cluster of secondary points. Examples are the Neyman-Scott model and the Bartlett-Lewis model, habitually used for modeling earthquakes and heavy rainfalls (empirical analysis and a comparison of these models and their extensions can be found, for example, in Vere-Jones (1970), Ogata, Vere-Jones (1984), Rodriguez-Iturbe et al (1987), Ogata (1988)). As Rodriguez-Iturbe et al (1987) puts it, "the difference between the two is minor, ... it is very unlikely that the empirical analysis of data can be used to choose between them." Both models are based on the Poisson process of cluster centres; the only difference is in the distribution of the secondary points around the centre. In the Neyman-Scott model, these points are independent and identically distributed, while in the Bartlett-Lewis model the inter-arrival times between successive points are independent and identically distributed. Below, we will show how the results of Subsection 2.2a can be incorporated in the Neyman-Scott model.

In the Neyman-Scott model (see Cox, Isham (1980)) Poisson process $\hat{K}_t$ with constant rate $\hat{\rho}$ describes the process of cluster centres T_i. For each cluster centre, there is an associated with it random number O_i of secondary points with locations T_{ij}, $j = 1,2,\ldots O_i$. It is assumed that

- the offsets $T_{ij} - T_i$ are independent and identically distributed around the cluster centre
- O_i and $T_{ij} - T_i$ are independent of one another, of $\hat{K}_t$ and of the O_k and T_{kl} for $k \neq i$.

For the rainfalls, the assumptions are more specific, that is, O_i has Poisson distribution with mean a, and the "offsets" $T_{ij} - T_i$ are exponentially distributed with mean $1/b$. The cluster centre is not observed, but the cluster points are. Under these conditions, one can show (see Karr (1991)) that this Neyman-Scott cluster process is a doubly stochastic Poisson process $\mathring{K}_t$ with directing intensity $\hat{\lambda}_t = ab \int_0^t e^{-b(t-u)} d\hat{K}_u$. For a deeper account of the theory we refer the reader to Vere-Jones (1970) and Ogata (1988).

The assumptions about the claims arrival processes and the claim size distributions remain as before, and the catastrophe arrival processes are independent of claims arrival processes and claim sizes. Finally, the claims and the catastrophes do not arrive simultaneously. Below we shall see that in this case the remaining risk is calculated in almost the same way as in Subsection 2.2a, Table 2 (see also (2.26)). We only have to replace the deterministic intensity μ of the process K_t with the directing intensity $\hat{\lambda}_t = ab \int_0^t e^{-b(t-u)} d\hat{K}_u$ for the process $\mathring{K}_t$:

The index-specific components of the remaining risk (2.16) for the indices $D^{(1)} - D^{(5)}$ will be similar to those calculated in Subsection 2.2a:

$$\frac{\left(d\langle V^{(1)}, D^{(1)}\rangle_t\right)^2}{d\langle D^{(1)}\rangle_t} = \frac{\left(\mathbb{E}[Z^2]\lambda_t^{(1)} + (\mathbb{E}[Z])^2 \mathbb{E}\left(\int_{s=0}^{T-t} M_{\mathring{K}_t}^{(1)}(s)ds\right)\left(\sum_{i=1}^n \int_{s=0}^{T-t} M_{\mathring{K}_t}^{(i)}(s)ds\right) ab\int_0^t e^{-b(t-u)} d\hat{K}_u\right)^2}{\mathbb{E}[Z^2]\sum_{i=1}^n \lambda_t^{(i)} + (\mathbb{E}[Z])^2 \mathbb{E}\left[\left(\sum_{i=1}^n \int_{s=0}^{T-t} M_{\mathring{K}_t}^{(i)}(s)ds\right)^2\right] ab\int_0^t e^{-b(t-u)} d\hat{K}_u} dt. \tag{2.78}$$

$$\frac{\left(d\langle V^{(1)}, \widetilde{D}^{(2)}\rangle_t\right)^2}{d\langle \widetilde{D}^{(2)}\rangle_t} = \frac{\left((\mathbb{E}Z)^2 \mathbb{E}\left(\int_{s=0}^{T-t} M_{\mathring{K}_t}^{(1)}(s)ds\right)\left(\sum_{i=2}^n \int_{s=0}^{T-t} M_{\mathring{K}_t}^{(i)}(s)ds\right) ab\int_0^t e^{-b(t-u)} d\hat{K}_u\right)^2}{\mathbb{E}[Z^2]\sum_{i=2}^n \lambda_t^{(i)} + (\mathbb{E}Z)^2 \mathbb{E}\left(\sum_{i=2}^n \int_{s=0}^{T-t} M_{\mathring{K}_t}^{(i)}(s)ds\right)^2 ab\int_0^t e^{-b(t-u)} d\hat{K}_u} dt. \tag{2.79}$$

$$\frac{\left(d\langle V^{(1)}, D^{(3)}\rangle_t\right)^2}{d\langle D^{(3)}\rangle_t} = \frac{(\mathbb{E}[Z])^2\left(\lambda_t^{(1)} + \mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{\mathring{K}_t}^{(1)}(s)ds\right)\left(\sum_{i=1}^n \int_{s=0}^{T-t} M_{\mathring{K}_t}^{(i)}(s)ds\right)\right] ab\int_0^t e^{-b(t-u)} d\hat{K}_u\right)^2}{\sum_{i=1}^n \lambda_t^{(i)} + \mathbb{E}\left[\left(\sum_{i=1}^n \int_{s=0}^{T-t} M_{\mathring{K}_t}^{(i)}(s)ds\right)^2\right] ab\int_0^t e^{-b(t-u)} d\hat{K}_u} dt. \tag{2.80}$$

$$\frac{\left(d\langle V^{(1)}, D^{(4)}\rangle_t\right)^2}{d\langle D^{(4)}\rangle_t} = \frac{(\mathbb{E}[Z])^2\left(\lambda_t^{(1)} + \mathbb{E}\left[\left(\int_{s=0}^{T-t} M_{K_t}^{(1)}(s)ds\right)\left(\sum_{i=1}^n \int_{s=0}^{T-t} M_{K_t}^{(i)}(s)ds\right)\right] ab\int_0^t e^{-b(t-u)} d\hat{K}_u\right)^2}{\sum_{i=1}^n \lambda_t^{(i)} + \mathbb{E}\left[\left(\sum_{i=1}^n \int_{s=0}^{T-t} M_{\mathring{K}_t}^{(i)}(s)ds\right)^2\right] ab\int_0^t e^{-b(t-u)} d\hat{K}_u} dt. \tag{2.81}$$

$$\frac{\left(d\langle V'^{(1)},D^{(5)}\rangle_t\right)^2}{d\langle D^{(5)}\rangle_t} = (\mathbb{E}[Z])^2 \frac{m_1^2}{m_2}\left(\mathbb{E}\left[\int_{s=0}^{T-t} M_{\mathring{K}_t}^{(1)}(s)ds\right]\right)^2 ab\int_0^t e^{-b(t-u)}d\hat{K}_u dt. \tag{2.82}$$

The further improvements of the model may go in the direction of allowing for the change in the magnitudes of the cluster points: for example, in earthquake modelling the energy released with the subsequent shocks decreases gradually, following roughly an exponential distribution (Vere-Jones (1970)).

Summary

In Chapter 2, we considered the problem of hedging the catastrophe risk in the insurance market consisting of n companies. We introduced five hypothetical tradeable indices: two based on weighted total losses for the industry, with and without the hedging company, two based on weighted total number of claims with and without the hedging company, and the severity index based on the sum of the severities of the catastrophes within the specified time period. Four shot-noise intensity models for the companies' losses, in the descending order of generality, have been presented, and for each of them, we have obtained the closed form formulae for the intrinsic risk associated with hedging the indices named above.

A detailed comparison of the hedging error was possible for the model with proportional catastrophe exposure among the companies. In that case, the best index was the one based on the total losses. The index based on the total number of claims was second, and the severity index was third. Indices which excluded the hedging company were the worst, and we explained the reason for such order by employing the martingale representation theorem for point process martingales. Also, the formulae for the intrinsic risk via the coefficients of the index' martingale representation allow to apply the results for the real catastrophe indices.

The results of the research can be used for the existing catastrophe models.

This is illustrated in the example for the Neyman-Scott earthquake model at the end of Chapter 2. Additionally, the results are in the form where they can be tested for real indices. Indeed, some of the indices used in this Chapter are similar to the kind currently calculated by the PCS. The accumulating amount of data on catastrophes, together with the increasing interest in catastrophe derivatives, would make this testing easier.

Chapter 3. Mortality bonds in the Markov chain environment

3.1 Mortality risk market - history and recent development

The issue of longevity and mortality risk has been around for centuries; according to Tett, Chung (2007), "The very first time that the British state issued a bond - back in the 17th century to fund a war against France... The scheme was devised by Lorenzo Tonti, a Neapolitan economist, and inherited his name "tontine". The government raised money by selling a bond, and then paid bondholders a lump sum each year, divided among the investor pool. Tontines had to be held by a single, named investor, and these instruments expired when that person died. So bond payments were divided each year among the remaining tontine holders, ceasing when the last tontine holder died."

The first tontine was issued in 1693, and soon became very popular in Europe. But, as time passed, tontines became more and more questionable. According to the same source, "the problem was that the government kept getting its estimates of longevity wrong. When it sold the first issue of tontines in 1693, it apparently expected tontine holders to live just a few decades. That seemed a reasonable bet at the time, and ... the early tontine holders included men and women of all ages. But by the middle of the 18th century, investors had become more canny, with the record showing most tontines being bought in the name of girls, usually around five years old. That was because girls lived longer than boys, and because there was a high level of infant mortality until about age four. This produced great results for the tontine holders, some of whom kept collecting money until their nineties". By the end of the eighteenth century, the government decided to abandon the whole scheme as it was devastating for the state financial system.

But the market for the mortality risk did not die, and in the middle of the nineteenth century a new instrument called "penny policies" was launched. Also, the insurance market itself continued to develop, and in the course of the nineteenth century many mutual-insurance societies were founded, providing its members with low-cost insurance. In the early-20th century, as Tett, Chung (2007) state, "new schemes such as pensions, saving plans and annuities appeared. By the 1960s, the sales of these products had become a multibillion-pound business".

The annuities market was relatively stable until the 1990s, when the old strategy of looking back at historical trends and projecting them forward ended up seriously underestimating future mortality rates. As Blake et al (2006b) report, the uncertainty of longevity projections is illustrated by the fact that life expectancy for men aged 60 is more than 5 years longer in 2005 than it was anticipated to be in mortality projections made in the 1980s. This mistake was disastrous for some insurance companies, like Equitable Life Assurance Society, whose guaranteed annuity options became very valuable in the 1990s because of a combination of falling interest rates and improvements in mortality, leading to a downfall of this organization (Cairns et al (2006a)).

However, the correct prediction of the future mortality rates is important not only to the life insurance companies and pensions providers with annuity portfolios but also to any company with a defined-benefit pension scheme. That is why, two centuries later, longevity securities are back and conquering the market, although, of course, their design has become more sophisticated, as have the underlying reasons for mortality rates fluctuations. David Blake, professor of pension economics at City University, forecasts that the new market would eventually outstrip credit derivatives, which have ballooned to $26,000 billion. "The potential is enormous and it will start to happen very soon" (Wighton, Tett (2006)). At the moment, the pace is promising. Lane, Beckwith (2005) state that the total dollar amount in the market before 1999 was $887 million, and in 2005 it more than doubled, reaching $1,803 million. The room for growth is plentiful, as, according to Blake et al (2006b), "the state and private sector exposure to longevity risk in the United Kingdom amounted to £2,520 billion (or $4,424 billion) at the end of 2003 – that is, nearly £40,000 (or $70,000) for every man, woman, and child in the United Kingdom".

Blake et al (2006a) suggest a number of ways of hedging longevity risk that amount to:

- diversification by creating a balanced portfolio of term assurance and annuity business;
- designing the life insurance policies so that the risk is reduced;
- securitization of the line of business exposed to longevity risk; and
- trading mortality-linked securities.

A number of studies have been done on the latter topic; some real instruments have also been launched. For a large part, they use the structures developed elsewhere in the financial theory. Mortality-linked securities, both existing and hypothetical, are studied in great detail in Cairns et al (2006a), and here we list just some examples:

- short-term catastrophe bonds (Swiss Re, Dec.2003);
- long-term longevity bonds (EIB/BNP, Nov.2004);
- survivor swaps (swapping fixed for floating mortality-linked cash flows; some OTC trades have been done);
- annuity futures (traded contract; the underlying instrument is market annuity rate, many exercise dates).

But whichever way of hedging longevity risk an investor chooses to pursue, he will rely on some model for future mortality rates. Until recently, actuaries treated mortality rates at different ages as deterministic. But the latest unanticipated increase in longevity shows that the forces driving the mortality curve down are more complicated than actuaries used to assume. According to the Office of National Statistics (2006a), life expectancy for men aged 65 has increased from 13.2 years in 1983-1985 to 16.6 years, and for women the figures are 17.2 and 19.4 years, respectively. The mortality rate for women in the 55-64 age group in the past 30 years has decreased by approximately 44%. Of that amount, a decrease of only 8% was observed during the first decade, a further 21% in the second one, and an additional 25% in the last ten years.

The reader can find a detailed survey of the models for mortality rates currently in use/discussion in Cairns et al (2007). The authors acknowledge that there is no universal "best" model; they show that England and Wales mortality data are in good agreement with one model while the US mortality data are better explained by another. However, there are some trends in all modern frameworks. For example, stochastic component in the mortality rate modelling has proved to be useful in many applications, especially for the policies involving certain types of a guarantee. In this case, the value of the mortality derivative depends on the level of interest rate at retirement and upon the mortality table being used by the life office at that time (Office for National Statistics (2006a)). The concept was first introduced by Milevsky, Promislow (2001), who employed a continuous-time diffusion process for the hazard rate. It had a Gompertz form $h_t = h_0 e^{gt + \sigma Y_t}$, Y_t being a mean-reverting diffusion process. Later, Ballotta and Haberman (2006) created a framework for pricing of the guaranteed annuity options using a single-factor Heath-Jarrow-Morton framework for the term structure of interest rates and enriching a popular reduction factor model for projecting mortality rates with a stochastic component driven by an Ornstein-Uhlenbeck process. Recently, Lin and Cox (2006) provided some empirical evidence for the necessity of inclusion of jump component in the mortality dynamics modelling and for this purpose proposed a Brownian motion multiplied by a jump process.

Obviously, the level of sophistication in the mortality risk analysis increases, and particular attention is now drawn to the specific causes of death. As Thomas Boardman, a policy director for Prudential, puts it, "In the past five years we have started to look more at the medical profession, and the trends behind why people die" (Tett, Chung (2007)). In line with this trend, Norberg (2007) proposed a new framework for evaluating and pricing mortality risk. The idea stems from the fact that jumps in mortality can be associated with advances in medical technology leading to eradication of some causes of death. Such events can have a strong impact on mortality rates: for example, an (imaginary) discovery of a universal treatment for cancer in 2004 would have prevented 46% of all deaths in the United Kingdom in the age group 55 – 64 and 40% of all deaths in the age group 65 – 74 in the year to follow (data compiled from Heart Statistics (2005)). As all deaths can be classified by cause, and there is a limited number of principal death factors, one can consider the employment of the Markov chain modelling for the economic-demographic environment. The states of this environment are characterised by the causes of death still active at a given time. Transition from one state to another occurs with the elimination of an existing cause or the emergence of a new death factor. This approach provides a framework for analysing mortality securities linked to the existence of a particular death factor. When these securities are launched in the financial market, they would give various players in the pension market an opportunity to hedge against the undesirable components of the mortality risk.

The attractiveness of this model lies in its ability to explain and model jumps in mortality rates and compatibility with the other branches of life insurance which heavily use the Markov-chain models. A good reference is also Norberg (2003), where a financial market driven by a continuous time homogeneous Markov chain is considered together with some aspects of hedging in such markets.

In Section 3.2 of this chapter, we provide the description of the Markov chain environment applied to the mortality risk modelling and state the basic assumptions and features of hedging in this environment as developed in Norberg (2007). In Section 3.3, we look at an economic-demographic environment with two causes of death with the potential of disappearing and four combinations of related mortality bonds. We calculate the hedging error and the amount of bonds that should optimally be held. In Section 3.4, we are once again concerned with finding the optimal design of a derivative. Using numerical data and approximations for mortality intensity, we apply the results of Section 3.3 to the case of a digital longevity bond with principal at risk to construct the optimal, in terms of minimising the hedging error, design of the mortality-linked bonds in this case. The analytical solution can not be obtained explicitly, so we run a computer program to obtain the numerical results for some particular values of the parameters. Sensitivity analysis is presented at the end of Section 3.4, where we study the relationship between the hedging error and the external and internal model parameters.

3.2 Model description and basic results

In this section we will be studying the possibilities of quadratic hedging of mortality derivatives in a Markov chain environment as proposed by Norberg (2007). The model suggests that the economic-demographic environment at time t can be characterised by the causes of death still active at that time. According to a recent chart produced by the Office for National Statistics (2005), more than 86% of the death toll in UK citizens aged $65 - 74$ in 2005 is attributed to diseases of the circulatory system, cancer, diabetes and respiratory diseases. We suppose that these four diseases (and only they) can become curable with time, and group all other causes together and call this group "other causes", so that there are be 5 causes of death, and the state space would consist of 16 possible states corresponding to various combinations of activeness and inactiveness of the first four causes. Transition from one state to another takes place if one cause of death is eliminated because of advances in medical research or a dramatic improvement in people's diet and lifestyle.

Below we provide the formal description of the corresponding Markov chain market, followed by the theoretical facts and formulae that will be used later. Here we closely follow Norberg (2007).

The process $(Y(t))_{t\in[0,T]}$ describes the evolution of the economic-demographic environment. The state space is finite, $\mathcal{Y} = (0, 1, \ldots, J^Y)$, the starting point is $Y(0) = 0$. The process $Y(t)$ is adapted to some filtration $\mathrm{G} = (\mathcal{G}_t)_{t\in[0,T]}$ representing the development of the environment. We assume that $Y(t)$ is a Markov chain with intensities $\lambda_{ef}(t)$, $e,f \in \mathcal{Y}$. By convention, a diagonal element is minus the total intensity of transition out of the current state,

$\lambda_{ee}(t) = -\sum_{f;f\neq e} \lambda_{ef}(t)$. We assume that there exists a market for environmental risk, dictating an equivalent martingale measure $\widetilde{\mathrm{P}}$ under which Y is a Markov chain with intensities $\widetilde{\lambda}_{ef}(t)$. We denote by $\mathcal{Y}_e(t) = \{f; \widetilde{\lambda}_{ef} > 0\}$ the set of states that are directly accessible from state e at time t. The cardinality of this set is denoted by $n_e(t)$.

A life insurance policy terminating at time T is issued at time 0. The state of the policy is a stochastic process $(Z(t))_{t\in[0,T]}$ with the finite state space $\mathcal{Z} = \{0, 1, \ldots, J^Z\}$ starting from $(Z(t)) = 0$. The filtration generated by $Z(t)$ is denoted by $\mathrm{H} = (\mathcal{H}_t)_{t\in[0,T]}$. Next we assume that, conditional on $\mathcal{G}_t$, Z is a Markov process with intensities $\mu_{Y(t),jk}(t)$, $j,k \in \mathcal{Z}$. Then, under $\widetilde{\mathrm{P}}$, the process $X = (Y,Z)$ is a Markov chain with state space $\mathcal{X} = \mathcal{Y} \times \mathcal{Z}$ and intensities

$$\kappa_{ej,fk}(t) = \begin{cases} \widetilde{\lambda}_{ef}(t), & e \neq f, j = k, \\ \mu_{e,jk}(t), & e = f, j \neq k, \\ 0, & e \neq f, j \neq k. \end{cases}$$

Next we introduce the indicator of the event that Y is in state e at time t,

$I(Y(t) = e) = I_e^Y(t)$.

and assume that there is a money market account with state-dependent interest rate

$r(t) = r_{Y(t)} = \sum_e I_e^Y(t) r_e$.

The insurance policy is of the standard type, with deterministic state-wise annuity payment functions B_j and sums assured b_{jk}. The state-wise reserves are denoted by $V_{ej}(t)$. They are the conditional expected present value at time t of benefits less premiums in $(t,T]$, given that $(Y(t),Z(t)) = (e,j)$. The state-wise reserves are the solutions to the backward Thiele differential equations

$$dV_{ej}(t) = V_{ej}(t) r_e dt - dB_j(t) - \sum_{k:k\neq j} R_{ej,ek}(t)\mu_{e,jk}(t)dt - \sum_{f;f\neq e} R_{ej,fj}(t)\widetilde{\lambda}_{e,jk}(t)dt, \tag{3.1}$$

subject to the terminal conditions

$$V_{ej}(T-) \ = \triangle B_j(T) = b. \tag{3.2}$$

Here

$$R_{ej.ek}(t) = b_{jk}(t) + V_{ek}(t) - V_{ej}(t) \tag{3.3}$$

is the sum at risk associated with a transition of the policy from state j to state k at time t when the environment is in state e, and

$$R_{ej,fj}(t) = V_{fj}(t) - V_{ej}(t) \tag{3.4}$$

is the sum at risk associated with the transition of the environment from state e to state f at time t when the policy is in state j (no lump sum is paid upon such a transition). We introduce the counting processes

$$N^Y_{ef}(t) = \ | \{\tau; 0 < \tau \leqslant t, Y(\tau -)= e, Y(\tau -)= f\} \ |,$$

the number of direct transitions of Y from state e to state f in the time interval $(0,t]$, and

$$N^Z_{jk}(t) = \ | \{\tau; 0 < \tau \leqslant t, Z(\tau -)= j, Z(\tau -)= k\} \ |,$$

the number of direct transitions of Z from state j to state k in the time interval $(0,t]$. The martingale

$$\widetilde{M}(t) = \widetilde{\mathbb{E}}\left[\int_{0-}^{T} e^{-\int_0^\tau r(s)ds} dB(\tau) \ | \ \mathcal{G}_t \vee \mathcal{H}_t \right] \tag{3.5}$$

has dynamics

$$d\widetilde{M}(t) = \sum_{e \neq f} d\tilde{M}^Y_{ef}(t) \sum_{j} I^Z_j(t)\tilde{R}_{ej,fj}(t) + \sum_{j \neq k} dM^Z_{jk}(t) \sum_{e} I^Y_e(t)\tilde{R}_{ej,ek}(t), \tag{3.6}$$

where the $\tilde{M}^Y_{ef}$ are the compensated counting processes of the environment with the dynamics given by

$$d\tilde{M}^Y_{ef}(t) = dN^Y_{ef}(t) - I^Y_e(t)\tilde{\lambda}_{ef}dt, \tag{3.7}$$

the M^Z_{jk} are the compensated counting processes of the policy, and their dynamics is

$$dM^Z_{jk}(t) = dN^Z_{jk}(t) - I^Z_j(t) \sum_e I^Y_e(t)\mu_{e,jk}(t)dt. \tag{3.8}$$

The values $\tilde{R}_{ej,fk}$ are the discounted sums at risk,

$$\tilde{R}_{ej,fk}(t) = e^{-\int_0^t r(s)ds} R_{ej,fk}(t). \tag{3.9}$$

Suppose that the market has m mortality-related derivatives with discounted price processes $(\tilde{S}_i(t))$, $i = 1,\ldots,m$. In an arbitrage-free market these are martingales adapted to G under $\tilde{\mathrm{P}}$ with the dynamics of the form

$$d\tilde{S}^{(i)}(t) = \sum_{e \neq f} \xi^{(i)}_{ef}(t) d\tilde{M}^Y_{ef}(t). \tag{3.10}$$

Consider a self-financing portfolio consisting of $\theta^{(i)}(t)$ units of asset number $i = 1,\ldots,m$ at time t, asset number 0 being the money market account. The portfolio in the risky mortality derivatives, $\boldsymbol{\theta}(t) = (\theta^{(1)}(t),\ldots,\theta^{(m)}(t))'$, $t \in [0,T]$, is a predictable stochastic process. The portfolio vector can be written as the sum of its state-wise values,

$$\boldsymbol{\theta}(t) = \sum_e I^Y_e(t)\boldsymbol{\theta}_e(t), \qquad \boldsymbol{\theta}_e(t) = (\theta^{(1)}_e(t),\ldots,\theta^{(m)}_e(t))'.$$

The discounted value process $\tilde{V}^\theta(t)$ is a martingale with dynamics

$$d\tilde{V}^\theta(t) = \sum_{i=1}^m \theta^{(i)}(t) d\tilde{S}^{(i)}(t) = \sum_{e \neq f} \left(\sum_{i=1}^m I^Y_e(t)\theta^{(i)}_e(t)\xi^{(i)}_{ef}(t) \right) d\tilde{M}^Y_{ef}(t). \tag{3.11}$$

We aim to minimize the hedging error or risk, defined as the expected squared difference between the total discounted contractual payments and the discounted terminal value of the portfolio,

$$\rho^\theta = \tilde{\mathbb{E}}(\tilde{M}(T) - \tilde{V}^\theta(T))^2. \tag{3.12}$$

The risk is the squared distance between the random variable $\tilde{M}(T)$ and the linear space of T-values of self-financed portfolios in the Hilbert space of random variables that are square integrable w.r.t. $\tilde{\mathrm{P}}$, the inner product being $<< X, Y >>= \tilde{\mathbb{E}}[X, Y]$. Inserting

$$\tilde{M}(T) = \tilde{\mathbb{E}}\tilde{M}(T) + \int_0^T d\tilde{M}(t) \tag{3.13}$$

and

$$\tilde{V}^\theta(T) = \tilde{V}^\theta(0) + \int_0^T d\tilde{V}^\theta(t) \tag{3.14}$$

and using (3.6) and (3.11), we get

$$\begin{aligned}
\rho^\theta = \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\tilde{M}(T) - \tilde{V}^\theta(0) + \int_0^T \Big(& \sum_{e \neq f} d\tilde{M}^Y_{ef}(t) \sum_j I^Z_j(t)\tilde{R}_{ej,fj}(t) + \\
& + \sum_{j \neq k} dM^Z_{jk}(t) \sum_e I^Y_e(t)\tilde{R}_{ej,ek}(t) - \\
& - \sum_{e \neq f}\sum_{i=1}^m I^Y_e(t)\theta^{(i)}_e(t)\xi^{(i)}_{ef}(t) d\tilde{M}^Y_{ef}(t)\Big)\Big]^2 = \\
= \tilde{\mathbb{E}}\Big[\tilde{\mathbb{E}}\tilde{M}(T) - \tilde{V}^\theta(0) + \int_0^T \Big(& \sum_{j \neq k}\sum_e I^Y_e(t)\tilde{R}_{ej,ek}(t) dM^Z_{jk}(t) + \\
& + \sum_{e \neq f} \Big(\sum_j I^Z_j(t)\tilde{R}_{ej,fj}(t) - \sum_{i=1}^m I^Y_e(t)\theta^{(i)}_e(t)\xi^{(i)}_{ef}(t)\Big) d\tilde{M}^Y_{ef}(t)\Big)\Big]^2 .
\end{aligned} \tag{3.15}$$

The risk above decomposes into three components

$$\rho^\theta = \rho^\theta_0 + \rho^\theta_I + \rho^\theta_E, \tag{3.16}$$

where

$$\rho^\theta_0 = \left[\tilde{\mathbb{E}}\tilde{M}(T) - \tilde{V}^\theta(0)\right]^2 \tag{3.17}$$

is the *basis risk*,

$$\begin{aligned}
\rho^\theta_I &= \tilde{\mathbb{E}}\left[\int_0^T \sum_{j \neq k} \left(\sum_e I^Y_e(t)\tilde{R}_{ej,ek}(t) \right) dM^Z_{jk}(t) \right]^2 = \\
&= \tilde{\mathbb{E}}\left[\int_0^T \sum_e I^Y_e(t) \sum_{j \neq k} I^Z_j(t)\mu^Z_{e;jk}(t)\tilde{R}^2_{ej,ek}(t) dt \right]^2
\end{aligned} \tag{3.18}$$

is the non-systematic *individual risk*, and

$$\rho^\theta_E = \tilde{\mathbb{E}}\left[\int_0^T \sum_{e \neq f} I^Y_e(t) \left(\sum_j I^Z_j(t)\tilde{R}_{ej,fj}(t) - \sum_{i=1}^m \theta^{(i)}_e(t)\xi^{(i)}_{ef}(t) \right) d\tilde{M}^Y_{ef}(t) \right]^2 \tag{3.19}$$

is the systematic *environmental risk* or *hedging error.*

In order to avoid arbitrage, the number of risky assets with linearly independent dynamics in each state should be no greater than the number of sources of randomness $n_e(t)$ (or the number of driving martingales) in this state. In the case of equality the market is complete, and the environmental risk can be eliminated completely. If $n_e(t)$ exceeds the number of active securities in state e, the market is incomplete and there will be a positive hedging error in that state.

The basis risk is minimized by setting

$$\tilde{V}^{\theta}(0) = \tilde{\mathbb{E}}\tilde{M}(T). \tag{3.20}$$

The individual risk ρ_I^{θ} does not depend on the portfolio strategy. Therefore, by choosing the optimal strategy, we only need to minimize the environmental risk (3.19).

However, in the incomplete market one will not be able to get rid of the hedging error entirely. To establish the optimal portfolio and calculate the hedging error in this case, we introduce the following notation:

the $n_e(t)$ –vector

$$\eta_e(t) = (\eta_{ef}(t))_{f\in\mathcal{Y}_e(t)} \tag{3.21}$$

with the elements

$$\eta_{ef}(t) = \sum_j I_j^Z(t)\tilde{R}_{ej,fj}(t), \tag{3.22}$$

the $n_e(t) \times n_e(t)$ diagonal matrix

$$\Lambda_e(t) = Diag_{f\in\mathcal{Y}_e(t)}(\tilde{\lambda}_{ef}(t)), \tag{3.23}$$

and the $n_e(t) \times m$ matrix of price coefficients

$$\Xi_e(t) = (\xi_{ef}^{(i)}(t))_{f\in\mathcal{Y}_e(t)}^{i=1,\dots,m}. \tag{3.24}$$

We will assume that $m < n_e(t)$ and $\Xi_e(t)$ has full rank m. Then the optimal portfolio in state e is

$$\tilde{\theta}_e(t) = \left(\Xi_e'(t)\tilde{\Lambda}_e(t)\Xi_e(t)\right)^{-1}\Xi_e'(t)\tilde{\Lambda}_e(t)\eta_e(t). \tag{3.25}$$

The environmental risk (3.19) in state e can be written as

$$\rho_E^\theta = \tilde{\mathbb{E}}\left[\int_0^T \sum_e I_e^Y(t)\left(\eta_e(t) - \Xi_e(t)\tilde{\theta}_e(t)\right)'\tilde{\Lambda}_e(t)\left(\eta_e(t) - \Xi_e(t)\tilde{\theta}_e(t)\right)dt\right]^2. \tag{3.26}$$

If $m = n_e$, both $\Xi_e(t)$ and $\tilde{\Lambda}_e(t)$ have dimensionality $n_e \times n_e$. Then (3.25) becomes

$$\tilde{\theta}_e(t) = \Xi_e^{-1}(t)\tilde{\Lambda}_e^{-1}(\Xi_e'(t))^{-1}\Xi_e'(t)\tilde{\Lambda}_e(t)\eta_e(t) = \Xi_e^{-1}(t)\eta_e(t), \tag{3.27}$$

and the hedging error, as pointed out above, is zero.

Armed with the main definitions, assumptions and formulae of this theory, we are now ready to proceed to its applications.

3.3 Two-cause model: general results

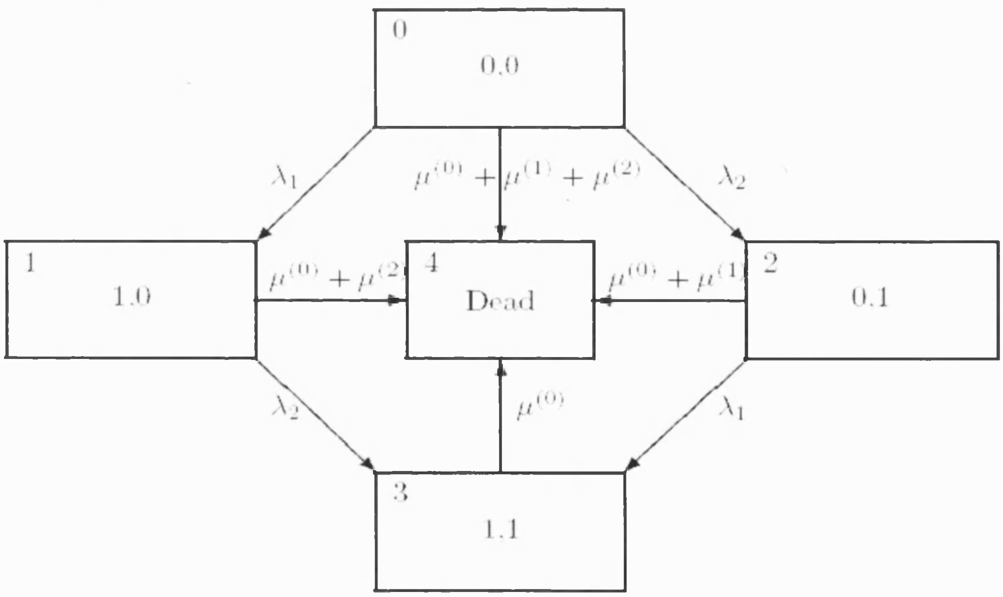
In this section we specify the demographic market $\mathcal{Y}$, the insurance policy and the mortality derivatives. We assume that there are three causes of death: cancer, coronary heart disease (or, simply, heart disease) and "any other" cause. "Any other" cause is static, while the first two causes are "active" at time 0, that is, cancer and heart disease are incurable at $t = 0$. When the scientists find a cure for one of the two diseases, making it curable, we will say that the corresponding cause of death becomes "inactive". Thus, the demographic market can be in one of the four states: $\mathcal{Y} = \{active, active;\ inactive, active;\ active, inactive;\ inactive, inactive\} = \{0, 1, 2, 3\}$. The first word in each pair defines the state of the cancer as a cause of death, and the second word stands for the state of the heart disease cause. For example, in state $1 = (inactive, active)$, the treatment for cancer is already available but for the heart disease is not. We assume that the Markov process Y_t, $t \in [0, T]$, which describes the state of the market, is time homogeneous, and the intensity matrix is

TABLE 5. INTENSITY MATRIX $\widetilde{\Lambda}_{ef}$

$e \backslash f$	0	1	2	3
0	$-(\tilde{\lambda}_1 + \tilde{\lambda}_2)$	$\tilde{\lambda}_1$	$\tilde{\lambda}_2$	0
1	0	$-\tilde{\lambda}_2$	0	$\tilde{\lambda}_2$
2	0	0	$-\tilde{\lambda}_1$	$\tilde{\lambda}_1$
3	0	0	0	0

Remark: in what follows, $\tilde{\lambda}_1$ and $\tilde{\lambda}_2$ can be made time-dependent; all the computations will carry on just the same. Below we provide a diagram of the environment used in our model.

DIAGRAM 6. MARKOV CHAIN ENVIRONMENT, TWO-CAUSE MODEL.



Consider a life endowment of b with term T against level premium c purchased by an x-year old at time 0. There are two states of the policy: $\mathcal{Z} = \{0, 1\} = \{alive, dead\}$. We consider a policyholder who was x years old at time 0, and the stochastic mortality rate at age y is of the form

$$\mu_{Y(t)}(y) = \mu^{(0)}(y) + (I_0^Y(t) + I_2^Y(t))\mu^{(1)}(y) + (I_0^Y(t) + I_1^Y(t))\mu^{(2)}(y). \tag{3.28}$$

The component $\mu^{(1)}$ is associated with the mortality from the first cause, cancer; $\mu^{(2)}$ is attributed to the mortality from the second cause, the heart disease. The first element of the sum (3.28), $\mu^{(0)}(y)$, comprises mortality from all other causes, and we assume here that "all other causes" cannot be eliminated.

Next we consider three zero-coupon digital bonds. The first two are disease-specific: their principal repayment amounts depend on whether the cure for cancer or, correspondingly, heart disease is found (in which case it pays out less than the original principal) or not. We label the first bond and its attributes with superscript C (for cancer) and call it the C-bond. The second bond (H-bond) is labelled with superscript H (for heart). The third bond (CH-bond) depends on the existence of the cure for cancer and the heart disease alike.

The pay-off function of the C-bond does not depend on whether the heart disease is curable or not, so it is not affected by the transition of the environment from state 0 to state 2 or from state 1 to state 3, which are associated with discovering treatment for the heart disease. Similarly, the H-bond is not affected by transitions from state 0 to state 1 and from state 2 to state 3. Therefore the payoff functions for the bonds are

$$d\tilde{S}^C(t) = \sum_{e \neq f} \xi_{ef}^C(t) d\tilde{M}_{ef}^Y(t) = \xi_{01}^C(t) d\tilde{M}_{01}^Y(t) + \xi_{23}^C(t) d\tilde{M}_{23}^Y(t), \tag{3.29}$$

$$d\tilde{S}^H(t) = \sum_{e \neq f} \xi_{ef}^H(t) d\tilde{M}_{ef}^Y(t) = \xi_{02}^H(t) d\tilde{M}_{02}^Y(t) + \xi_{13}^H(t) d\tilde{M}_{13}^Y(t), \tag{3.30}$$

$$\begin{aligned} d\tilde{S}^{CH}(t) &= \sum_{e \neq f} \xi_{ef}^{CH}(t) d\tilde{M}_{ef}^Y(t) = \\ &= \xi_{01}^{CH}(t) d\tilde{M}_{01}^Y(t) + \xi_{02}^{CH}(t) d\tilde{M}_{02}^Y(t) + \xi_{13}^{CH}(t) d\tilde{M}_{01}^Y(t) + \xi_{23}^{CH}(t) d\tilde{M}_{23}^Y(t). \end{aligned} \tag{3.31}$$

We do not specify at the moment the design of the bonds above, since some conclusions and formulae can be obtained in the general case. We will work out in detail the design of the three bonds above in Section 3.4 obtain the expressions for the functions $\xi(t)$ for each of these bonds.

Now we are going to write explicitly the hedging error in these three cases. The C-bond and the H-bond markets are essentially the special cases of the CH-bond market, and we will first calculate the portfolio and the hedging error for the market with the CH-bond.

CH-BOND. First we find $\tilde{\Lambda}_e(t)$, $\Xi_e(t)$ and $\eta_e(t)$.

$\tilde{\Lambda}_e$: from Table 5, we can write the matrices

$$\tilde{\Lambda}_0 = \begin{pmatrix} \tilde{\lambda}_1 & 0 \\ 0 & \tilde{\lambda}_2 \end{pmatrix}; \qquad \tilde{\Lambda}_1 = \tilde{\lambda}_2; \qquad \tilde{\Lambda}_2 = \tilde{\lambda}_1. \tag{3.32}$$

Vectors $\Xi_e(t)$ and $\eta_e(t)$ are:

$$\Xi_0(t) = [\xi_{01}^{CH}(t), \xi_{02}^{CH}(t)]'; \qquad \Xi_1(t) = \xi_{13}^{CH}(t); \qquad \Xi_2(t) = \xi_{23}^{CH}(t); \tag{3.33}$$

$\eta_e(t)$:

$$\eta_0(t) = \begin{pmatrix} I_0^Z(t)\tilde{R}_{00,10}(t) + I_1^Z(t)\tilde{R}_{01,11}(t) \\ I_0^Z(t)\tilde{R}_{00,20}(t) + I_1^Z(t)\tilde{R}_{01,21}(t) \end{pmatrix}; \tag{3.34}$$

$$\eta_1(t) = (I_0^Z(t)\tilde{R}_{10,30}(t) + I_1^Z(t)\tilde{R}_{11,31}(t)); \tag{3.35}$$

$$\eta_2(t) = (I_0^Z(t)\tilde{R}_{20,30}(t) + I_1^Z(t)\tilde{R}_{21,31}(t)). \tag{3.36}$$

When the policy is in state 1 (the policyholder is dead), no assurance shall be paid, and the corresponding $\tilde{R}_{e1,f1}$ equal zero. Then the formulae (3.34) – (3.36) can be simplified:

$$\eta_0(t) = \begin{pmatrix} I_0^Z(t)\tilde{R}_{00,10}(t) \\ I_0^Z(t)\tilde{R}_{00,20}(t) \end{pmatrix}; \tag{3.37}$$

$$\eta_1(t) = I_0^Z(t)\tilde{R}_{10,30}(t); \tag{3.38}$$

$$\eta_2(t) = I_0^Z(t)\tilde{R}_{20,30}(t). \tag{3.39}$$

Now the application of the formula (3.25) yields

$$\tilde{\theta}_0^{CH}(t) = I_0^Z(t)\frac{\xi_{01}^{CH}(t)\tilde{\lambda}_1\tilde{R}_{00,10}(t)+\xi_{02}^{CH}(t)\tilde{\lambda}_2\tilde{R}_{00,20}(t)}{(\xi_{01}^{CH}(t))^2\tilde{\lambda}_1+(\xi_{02}^{CH}(t))^2\tilde{\lambda}_2}; \tag{3.40}$$

$$\tilde{\theta}_1^{CH}(t) = \frac{\xi_{13}^{CH}(t)\tilde{\lambda}_2 I_0^Z(t)\tilde{R}_{10,30}(t)}{(\xi_{13}^{CH}(t))^2\tilde{\lambda}_2} = I_0^Z(t)\frac{\tilde{R}_{10,30}(t)}{\xi_{13}^{CH}(t)}; \tag{3.41}$$

$$\tilde{\theta}_2^{CH}(t) = \frac{\xi_{23}^{CH}(t)\tilde{\lambda}_1 I_0^Z(t)\tilde{R}_{20,30}(t)}{(\xi_{23}^{CH}(t))^2\tilde{\lambda}_1} = I_0^Z(t)\frac{\tilde{R}_{20,30}(t)}{\xi_{23}^{CH}(t)}. \tag{3.42}$$

The hedging error is different from zero only in state 0. It equals (see (3.19))

$$
\begin{aligned}
\tilde{\rho}^{CH}_{Y(t)=0} &= \\
&= \tilde{\mathbb{E}}\Big[\int_0^T I_0^Y(t)\Big(I_0^Z(t)\tilde{R}_{00,10}(t) - \tfrac{\tilde{\xi}_{01}^{CH}(t)\tilde{\lambda}_1 I_0^Z(t)\tilde{R}_{00,10}(t)+\tilde{\xi}_{02}^{CH}(t)\tilde{\lambda}_2 I_0^Z(t)\tilde{R}_{00,20}(t)}{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1+(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2}\tilde{\xi}_{01}^{CH}(t)\Big)d\tilde{M}_{01}^Y(t)\Big]^2 + \\
&\quad +\Big[\int_0^T I_0^Y(t)\Big(I_0^Z(t)\tilde{R}_{00,20}(t) - \tfrac{\tilde{\xi}_{01}^{CH}(t)\tilde{\lambda}_1 I_0^Z(t)\tilde{R}_{00,10}(t)+\tilde{\xi}_{02}^{CH}(t)\tilde{\lambda}_2 I_0^Z(t)\tilde{R}_{00,20}(t)}{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1+(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2}\tilde{\xi}_{02}^{CH}(t)\Big)d\tilde{M}_{02}^Y(t)\Big]^2 = \\
&= \tilde{\mathbb{E}}\Big[\int_0^T I_0^Y(t)I_0^Z(t)\tfrac{(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2\tilde{R}_{00,10}(t)-\tilde{\xi}_{01}^{CH}(t)\tilde{\xi}_{02}^{CH}(t)\tilde{\lambda}_2\tilde{R}_{00,20}(t)}{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1+(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2}d\tilde{M}_{01}^Y(t)\Big]^2 + \\
&\quad + \tilde{\mathbb{E}}\Big[\int_0^T I_0^Y(t)I_0^Z(t)\tfrac{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1\tilde{R}_{00,20}(t)-\tilde{\xi}_{01}^{CH}(t)\tilde{\xi}_{02}^{CH}(t)\tilde{\lambda}_1\tilde{R}_{00,10}(t)}{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1+(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2}d\tilde{M}_{02}^Y(t)\Big]^2.
\end{aligned}
$$

Using formula (3.7) and observing that if $I_0^Y(t) = 1$ then $dN_{01}^Y(t) = dN_{02}^Y(t) = 0$, we can rewrite the last formula as

$$
\begin{aligned}
\tilde{\rho}^{CH}_{Y(t)=0} &= \tilde{\mathbb{E}}\int_0^T I_0^Z(t)I_0^Y(t)\tilde{\lambda}_1\Big(\tfrac{(\tilde{\xi}_{02}^{CH})^2(t)\tilde{\lambda}_2\tilde{R}_{00,10}(t)-\tilde{\xi}_{01}^{CH}(t)\tilde{\xi}_{02}^{CH}(t)\tilde{\lambda}_2\tilde{R}_{00,20}(t)}{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1+(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2}\Big)^2 dt + \\
&\quad + \tilde{\mathbb{E}}\int_0^T I_0^Z(t)I_0^Y(t)\tilde{\lambda}_2\Big(\tfrac{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1\tilde{R}_{00,20}(t)-\tilde{\xi}_{01}^{CH}(t)\tilde{\xi}_{02}^{CH}(t)\tilde{\lambda}_1\tilde{R}_{00,10}(t)}{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1+(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2}\Big)^2 dt = \\
&= \tilde{\mathbb{E}}\int_0^T I_0^Z(t)I_0^Y(t)\tilde{\lambda}_1\tilde{\lambda}_2\Big(\tfrac{\tilde{\xi}_{02}^{CH}(t)\tilde{R}_{00,10}(t)-\tilde{\xi}_{01}^{CH}(t)\tilde{R}_{00,20}(t)}{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1+(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2}\Big)^2\Big[(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1 + (\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2\Big]dt = \\
&= \tilde{\mathbb{E}}\int_0^T I_0^Z(t)I_0^Y(t)\tilde{\lambda}_1\tilde{\lambda}_2\tfrac{(\tilde{\xi}_{02}^{CH}(t)\tilde{R}_{00,10}(t)-\tilde{\xi}_{01}^{CH}(t)\tilde{R}_{00,20}(t))^2}{(\tilde{\xi}_{01}^{CH}(t))^2\tilde{\lambda}_1+(\tilde{\xi}_{02}^{CH}(t))^2\tilde{\lambda}_2}dt. \qquad (3.43)
\end{aligned}
$$

C-BOND. Since this is a cancer-specific bond, $\tilde{\xi}_{02}^C(t) = \tilde{\xi}_{13}^C(t) = 0$. From (3.25) and (3.26), the amount of bond to be held in state 0 is

$$\tilde{\theta}_0^C(t) = \frac{I_0^Z(t)\tilde{R}_{00,10}(t)}{\tilde{\xi}_{01}^C(t)}. \qquad (3.44)$$

The hedging error in this state is determined by the possibility of the environment transition from state 0 to state 2:

$$\rho^C_{Y(t)=0} = \tilde{\mathbb{E}}\Big[\int_0^T I_0^Y(t)I_0^Z(t)\tilde{R}_{00,20}(t)d\tilde{M}_{02}^Y(t)\Big]^2 = \tilde{\mathbb{E}}\int_0^T I_0^Y(t)I_0^Z(t)\tilde{R}_{00,20}^2(t)\tilde{\lambda}_2 dt. \qquad (3.45)$$

The environmental risk in state 2 can be completely eliminated if

$$\tilde{\theta}_2^C(t) = \frac{I_0^Z(t)\tilde{R}_{20,30}(t)}{\tilde{\xi}_{23}^C(t)}, \qquad (3.46)$$

and the hedging error in state 1 will be maximal,

$$\rho^C_{Y(t)=1} = \tilde{\mathbb{E}}\Big[\int_0^T I_1^Y(t) I_0^Z(t) \tilde{R}_{10,30}(t) d\tilde{M}_{13}^Y(t)\Big]^2 = \tilde{\mathbb{E}}\int_0^T I_1^Y(t) I_0^Z(t) \tilde{R}^2_{10,30}(t) \tilde{\lambda}_2 dt. \tag{3.47}$$

Total error then stems from the presence of errors in states 0 and 1 and equals

$$\begin{aligned}\rho^C_E &= \rho^C_{Y(t)=0} + \rho^C_{Y(t)=1} = \\ &= \tilde{\mathbb{E}}\int_0^T I_0^Y(t) I_0^Z(t) \tilde{R}^2_{00,20}(t) \tilde{\lambda}_2 dt + \tilde{\mathbb{E}}\int_0^T I_1^Y(t) I_0^Z(t) \tilde{R}^2_{10,30}(t) \tilde{\lambda}_2 dt. \end{aligned}\tag{3.48}$$

H-BOND. Similarly, in this single-cause mortality derivative market, $\tilde{\xi}^H_{01}(t) = \tilde{\xi}^H_{23}(t) = 0$. Total hedging error will be minimized if

$$\tilde{\theta}_0^H(t) = \frac{I_0^Z(t)\tilde{R}_{00,20}(t)}{\tilde{\xi}^H_{02}(t)}; \tag{3.49}$$

$$\tilde{\theta}_1^H(t) = \frac{I_0^Z(t)\tilde{R}_{10,30}(t)}{\tilde{\xi}^H_{13}(t)}; \tag{3.50}$$

The total hedging error is equal to the sum of the errors in states 0 and 2, which is

$$\begin{aligned}\rho^H_E &= \rho^H_{Y(t)=0} + \rho^H_{Y(t)=2} = \\ &= \tilde{\mathbb{E}}\int_0^T I_0^Y(t) I_0^Z(t) \tilde{R}^2_{00,10}(t) \tilde{\lambda}_1 dt + \tilde{\mathbb{E}}\int_0^T I_2^Y(t) I_0^Z(t) \tilde{R}^2_{20,30}(t) \tilde{\lambda}_1 dt. \end{aligned}\tag{3.51}$$

Thus, as formulae (3.48) and (3.51) indicate, in the case of one cause-specific bond the hedging error does not depend on the design of this disease-specific derivative.

Formulae (3.43), (3.48) and (3.51) represent the hedging error for the CH-bond and the two disease-specific bonds. Next we need to find the survival probabilities $\tilde{\mathbb{E}} I_1^Y(t) I_0^Z(t)$ for the C-bond case, $\tilde{\mathbb{E}} I_2^Y(t) I_0^Z(t)$ for the H-bond and $\tilde{\mathbb{E}} I_0^Y(t) I_0^Z(t)$ for all three cases: C-bond, H-bond and CH-bond. We assume that causes of death go inactive by independent mechanisms, and for the C-bond we find, by conditioning on the time of transition of the environment Y to state 1 from state 0 :

$$\tilde{\mathbb{E}}I_1^Y(t)I_0^Z(t) = \mathbb{P}\{Z(t) = 0 \cap Y(t) = 1\} =$$
$$= \int_{\tau\in[0,t]} \mathbb{P}\{Z(t) = 0 \cap Y(\tau) = 0 \cap Y(\tau + d\tau) = Y(t) = 1\} =$$
$$= \int_0^t e^{-\int_0^\tau \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} \tilde{\lambda}_1 e^{-\int_\tau^t \mu_0(x+s)+\mu_2(x+s)ds} e^{-\tilde{\lambda}_1\tau} e^{-\tilde{\lambda}_2 t} d\tau =$$
$$= \int_0^t e^{-\int_0^\tau \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} \tilde{\lambda}_1 e^{-\int_\tau^t \mu_0(x+s)+\mu_2(x+s)ds} e^{-\tilde{\lambda}_1\tau} e^{-\tilde{\lambda}_2 t} d\tau =$$
$$= e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t-\int_0^t \mu_0(x+s)+\mu_2(x+s)ds} \int_0^t e^{-\int_0^\tau \mu_1(x+s)ds} \tilde{\lambda}_1 e^{\tilde{\lambda}_1(t-\tau)} d\tau.$$

For the H-bond the required value is obtained by conditioning on the time of transition of the environment Y to state 2 from state 0:

$$\tilde{\mathbb{E}}I_2^Y(t)I_0^Z(t) = P\{Z(t) = 0 \cap Y(t) = 2\} =$$
$$= \int_{\tau\in[0,t]} P\{Z(t) = 0 \cap Y(\tau) = 0 \cap Y(\tau + d\tau) = Y(t) = 2\} =$$
$$= e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t-\int_0^t \mu_0(x+s)+\mu_1(x+s)ds} \int_0^t e^{-\int_0^\tau \mu_2(x+s)ds} \tilde{\lambda}_2 e^{\tilde{\lambda}_2(t-\tau)} d\tau,$$

and in the CH-case we have

$$\tilde{\mathbb{E}}I_0^Y(t)I_0^Z(t) = P\{Z(t) = 0 \cap Y(t) = 0\} = e^{-\int_0^t \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t}.$$

Now we rewrite the formulae (3.43), (3.48) and (3.51):

$$\tilde{\rho}_E^{CH} = \int_0^T e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t} e^{-\int_0^t \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} \tilde{\lambda}_1\tilde{\lambda}_2 \frac{\left(\xi_{02}^{CH}(t)\tilde{R}_{00,10}(t)-\xi_{01}^{CH}(t)\tilde{R}_{00,20}(t)\right)^2}{\left(\xi_{01}^{CH}(t)\right)^2\tilde{\lambda}_1+\left(\xi_{02}^{CH}(t)\right)^2\tilde{\lambda}_2} dt, \tag{3.52}$$

$$\rho_E^C = \int_0^T e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t} e^{-\int_0^t \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} \tilde{R}_{00,20}^2(t)\tilde{\lambda}_2 dt +$$
$$+ \int_0^T e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t-\int_0^t \mu_0(x+s)+\mu_2(x+s)ds} \tilde{R}_{10,30}^2(t)\tilde{\lambda}_1\tilde{\lambda}_2 \int_0^t e^{-\int_0^\tau \mu_1(x+s)ds} e^{\tilde{\lambda}_1(t-\tau)} d\tau dt, \tag{3.53}$$

$$\rho_E^H = \int_0^T e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t} e^{-\int_0^t \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} \tilde{R}_{00,10}^2(t)\tilde{\lambda}_1 dt +$$
$$+ \int_0^T e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t-\int_0^t \mu_0(x+s)+\mu_1(x+s)ds} \tilde{R}_{20,30}^2(t)\tilde{\lambda}_1\tilde{\lambda}_2 \int_0^t e^{-\int_0^\tau \mu_2(x+s)ds} e^{\tilde{\lambda}_2(t-\tau)} d\tau dt, \tag{3.54}$$

We have shown so far in the formulae (3.52) and (3.53) that the hedging error in the market with two causes of death and with one cause-specific bond does not depend on the design of this bond, as long as the cause is the same. If both cancer and heart disease bonds are placed in the market at the same time, the hedging error will be eliminated completely. A less trivial situation occurs if the number of derivatives is smaller than the number of driving sources of randomness, as in the market with the CH-bond, when the hedging error cannot be eliminated completely. To evaluate the risk associated with hedging based on the mortality derivative in this case, one has to specify the design of the bond and calculate the coefficients ξ of the martingale representation of the payoff functions of the bonds and solve Thiele differential equations to obtain the values for the reserves $\tilde{R}_{ej,fk}(t)$.

In the next section we are going to specify the design of the C-bond and the H-bond and obtain numerical values for the amount of these assets to be held in the portfolio to eliminate the environmental risk; and for a specific design of the CH-bond, involving two parameters r_1 and r_2, we will derive the formulae for the corresponding ξ-functions and the optimal combination of these parameters which provides the smallest hedging error (3.51). We will also perform sensitivity analysis with respect to the model parameters.

3.4 Two-cause model: hedging error and sensitivity analysis

As in Section 3.3, we consider a life endowment contract of b with term T against level premium c purchased by an x-year old at time 0. There are two states of the policy: $\mathcal{Z} = \{0,1\} = \{alive, dead\}$. The state of the policy is a stochastic process Z_t, $t \in [0,T]$. The stochastic mortality rate at age y and calendar time t is of the form

$$\mu_{Y(t)}(y) = \mu^{(0)}(y) + (I_0^Y(t) + I_2^Y(t))\mu^{(1)}(y) + (I_0^Y(t) + I_1^Y(t))\mu^{(2)}(y) \tag{3.55}$$

For simplicity, the interest rate is taken to be fixed and equal r. First, we will study a mortality-linked C-bond with price function $\tilde{S}^C$ which pays the full principal if no cure for cancer is discovered by the maturity time U, and pays a reduced principal $(1 - r_1)$ otherwise. The price function for such bond is

$$
\begin{aligned}
\tilde{S}^C(t) &= \tilde{\mathbb{E}}[e^{-rU}(1 - r_1[I_1^Y(U) + I_3^Y(U)])|\mathcal{G}_t] = \\
&= e^{-rU} - r_1 e^{-rU} \sum_e I_e^Y(t)\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U))|Y(t) = e] = \\
&= e^{-rU} - r_1 e^{-rU} I_0^Y(t)\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U))|Y(t) = 0] - \\
&\quad - r_1 e^{-rU} I_1^Y(t)\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U))|Y(t) = 1] - \\
&\quad - r_1 e^{-rU} I_2^Y(t)\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U))|Y(t) = 2] - \\
&\quad - r_1 e^{-rU} I_3^Y(t)\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U))|Y(t) = 3] = \\
&= e^{-rU} - r_1 e^{-rU}(1 - e^{-\tilde{\lambda}_1(U-t)})I_0^Y(t) - r_1 e^{-rU} I_1^Y(t) - r_1 e^{-rU}(1 - e^{-\tilde{\lambda}_1(U-t)})I_2^Y(t) - r_1 e^{-rU} I_3^Y(t) = \\
&= e^{-rU} - r_1 e^{-rU}\left[(1 - e^{-\tilde{\lambda}_1(U-t)})I_0^Y(t) + I_1^Y(t) + (1 - e^{-\tilde{\lambda}_1(U-t)})I_2^Y(t) + I_3^Y(t)\right].
\end{aligned}
$$

$$
\begin{aligned}
d\tilde{S}^C(t) &= r_1 e^{-rU}\left[\tilde{\lambda}_1 e^{-\tilde{\lambda}_1(U-t)} I_0^Y(t) + \tilde{\lambda}_1 e^{-\tilde{\lambda}_1(U-t)} I_2^Y(t)\right] + \\
&\quad + r_1 e^{-rU}\left[-e^{-\tilde{\lambda}_1(U-t)} dN_{01}^Y(t) - e^{-\tilde{\lambda}_1(U-t)} dN_{23}^Y(t)\right] = \\
&= r_1 e^{-rU}\left[\tilde{\lambda}_1 e^{-\tilde{\lambda}_1(U-t)} I_0^Y(t) + \tilde{\lambda}_1 e^{-\tilde{\lambda}_1(U-t)} I_2^Y(t)\right] + \\
&\quad + r_1 e^{-rU}\left[-e^{-\tilde{\lambda}_1(U-t)}[dM_{01}^Y(t) + I_0^Y(t)\lambda_{01}dt] - e^{-\tilde{\lambda}_1(U-t)}[dM_{23}^Y(t) + I_2^Y(t)\lambda_{01}dt\right] = \\
&= -r_1 e^{-rU} e^{-\tilde{\lambda}_1(U-t)} dM_{01}^Y(t) - r_1 e^{-rU} e^{-\tilde{\lambda}_1(U-t)} dM_{23}^Y(t).
\end{aligned}
\tag{3.56}
$$

So in the case of the cancer-specific bond we have

$$\tilde{\xi}_{01}^C = \tilde{\xi}_{23}^C = -r_1 e^{-rU+\tilde{\lambda}_1(t-U)}. \tag{3.57}$$

Due to symmetry, if we launch a mortality bond S^H which pays full principal (or 1) at time U if no remedy for the heart disease is found and a smaller amount $(1 - r_2)$ if such a cure becomes available, its coefficients $\tilde{\xi}^H$ will be

$$\tilde{\xi}_{02}^H = \tilde{\xi}_{13}^H = -r_2 e^{-rU+\tilde{\lambda}_1(t-U)}. \tag{3.58}$$

Looking back at the formulae (3.44), (3.46) for the C-bond and (3.49), (3.50) for the H-bond, the amount $\tilde{\theta}_e^C$ of the C-bond and $\tilde{\theta}_e^H$ of the H-bond to be held in state e are

$$\tilde{\theta}_0^C(t) = -I_0^Z(t)\frac{\tilde{R}_{00,10}(t)}{r_1 e^{-rU+\lambda_{01}(t-U)}}, \tag{3.59}$$

$$\tilde{\theta}_2^C(t) = -I_0^Z(t)\frac{\tilde{R}_{20,310}(t)}{r_1 e^{-rU+\lambda_{01}(t-U)}}, \tag{3.60}$$

$$\tilde{\theta}_0^H(t) = -I_0^Z(t)\frac{\tilde{R}_{00,20}(t)}{r_2 e^{-rU+\lambda_{02}(t-U)}}, \tag{3.61}$$

$$\tilde{\theta}_1^H(t) = -I_0^Z(t)\frac{\tilde{R}_{10,30}(t)}{r_2 e^{-rU+\lambda_{02}(t-U)}}. \tag{3.62}$$

Next we consider a digital mortality CH–bond with price function $\tilde{S}^{CH}$ which pays the full principal at maturity time U if both cancer and heart disease remain incurable, $1-r_1$ (or $1-r_2$) if the treatment for cancer (or heart disease) becomes available and $1-(r_1+r_2)$ if both diseases become curable by time U. The natural restriction is that $0 \leqslant r_1 + r_2 \leqslant 1$.

The price function of such bond is

$$\begin{aligned}
\tilde{S}^{CH}(t) &= \tilde{\mathbb{E}}[e^{-rU}(1 - r_1[I_1^Y(U) + I_3^Y(U)] - r_2[I_2^Y(U) + I_3^Y(U)])|\mathcal{G}_t] = \\
&= e^{-rU} - e^{-rU}\sum_e \left\{r_1\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U)|Y(t) = e] + r_2\tilde{\mathbb{E}}[I_2^Y(U) + I_3^Y(U)|Y(t) = e]\right\} = \\
&= e^{-rU} - e^{-rU}\left\{r_1\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U)|Y(t) = 0] + r_2\tilde{\mathbb{E}}[I_2^Y(U) + I_3^Y(U)|Y(t) = 0]\right\}I_0^Y(t) - \\
&\quad - e^{-rU}\left\{r_1\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U))Y(t) = 1] + r_2\tilde{\mathbb{E}}[I_2^Y(U) + I_3^Y(U)|Y(t) = 1]\right\}I_1^Y(t) - \\
&\quad - e^{-rU}\left\{r_1\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U))|Y(t) = 2] + r_2\tilde{\mathbb{E}}[I_2^Y(U) + I_3^Y(U)|Y(t) = 2]\right\}I_2^Y(t) - \\
&\quad - e^{-rU}\left\{r_1\tilde{\mathbb{E}}[I_1^Y(U) + I_3^Y(U))|Y(t) = 3] + r_2\tilde{\mathbb{E}}[I_2^Y(U) + I_3^Y(U)|Y(t) = 3]\right\}I_3^Y(t) = \\
&= e^{-rU} - e^{-rU}\left\{r_1(1 - e^{-\tilde{\lambda}_1(U-t)}) + r_2(1 - e^{-\tilde{\lambda}_2(U-t)})\right\}I_0^Y(t) - \\
&\quad - e^{-rU}\left\{r_1 + r_2(1 - e^{-\tilde{\lambda}_2(U-t)})\right\}I_1^Y(t) - \\
&\quad - e^{-rU}\left\{r_1(1 - e^{-\tilde{\lambda}_1(U-t)}) + r_2\right\}I_2^Y(t) - \\
&\quad - e^{-rU}\{r_1 + r_2\}I_3^Y(t).
\end{aligned} \tag{3.63}$$

The dynamics is of the form

$$\begin{aligned}
d\tilde{S}^{CH}(t) &= e^{-rU}\left(\tilde{\lambda}_1 r_1 e^{-\tilde{\lambda}_1(U-t)} + \tilde{\lambda}_2 r_2 e^{-\tilde{\lambda}_2(U-t)}\right)I_0^Y(t)dt + \\
&\quad + e^{-rU}\tilde{\lambda}_2 r_2 e^{-\tilde{\lambda}_2(U-t)}I_1^Y(t)dt + e^{-rU}\tilde{\lambda}_1 r_1 e^{-\tilde{\lambda}_1(U-t)}I_2^Y(t)dt -
\end{aligned}$$

$$-e^{-rU}r_1 e^{-\tilde{\lambda}_1(U-t)}\left(d\tilde{M}_{01}^Y(t) + I_0^Y(t)\tilde{\lambda}_1 dt\right) - e^{-rU}r_2 e^{-\tilde{\lambda}_2(U-t)}\left(d\tilde{M}_{02}^Y(t) + I_0^Y(t)\tilde{\lambda}_2 dt\right) -$$

$$\begin{aligned}
&-e^{-rU}r_2 e^{-\tilde{\lambda}_2(U-t)}\left(d\tilde{M}_{13}^Y(t) + I_1^Y(t)\tilde{\lambda}_2 dt\right) - e^{-rU}r_1 e^{-\tilde{\lambda}_1(U-t)}\left(d\tilde{M}_{23}^Y(t) + I_2^Y(t)\tilde{\lambda}_1 dt\right) = \\
&\quad = -e^{-rU}r_1 e^{-\tilde{\lambda}_1(U-t)}d\tilde{M}_{01}^Y(t) - e^{-rU}r_2 e^{-\tilde{\lambda}_2(U-t)}d\tilde{M}_{02}^Y(t) - \\
&\quad\quad - e^{-rU}r_2 e^{-\tilde{\lambda}_2(U-t)}d\tilde{M}_{13}^Y(t) - e^{-rU}r_1 e^{-\tilde{\lambda}_1(U-t)}d\tilde{M}_{23}^Y(t).
\end{aligned} \tag{3.64}$$

So the functions $\tilde{\xi}^{CH}$ are

$$\tilde{\xi}^{CH}_{01} = \tilde{\xi}^{CH}_{23} = -r_1 e^{-rU+\tilde{\lambda}_1(t-U)},$$
$$\tilde{\xi}^{CH}_{02} = \tilde{\xi}^{CH}_{13} = -r_2 e^{-rU+\tilde{\lambda}_2(t-U)}.$$

Then the hedging error (3.52) is

$$\begin{aligned}\tilde{\rho}^{CH}_E &= \int_0^T e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t} e^{-\int_0^t \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} \tilde{\lambda}_1\tilde{\lambda}_2 \frac{\left(\tilde{\xi}^{CH}_{02}(t)\tilde{R}_{00,10}(t)-\tilde{\xi}^{CH}_{01}(t)\tilde{R}_{00,20}(t)\right)^2}{\left(\tilde{\xi}^{CH}_{01}(t)\right)^2\tilde{\lambda}_1+\left(\tilde{\xi}^{CH}_{02}(t)\right)^2\tilde{\lambda}_2} dt = \\ &= \int_0^T e^{-(\tilde{\lambda}_1+\tilde{\lambda}_2)t} e^{-\int_0^t \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} \tilde{\lambda}_1\tilde{\lambda}_2 \frac{\left(r_2 e^{\tilde{\lambda}_2(t-U)}\tilde{R}_{00,10}(t)-r_1 e^{\tilde{\lambda}_1(t-U)}\tilde{R}_{00,20}(t)\right)^2}{\left(r_1 e^{\tilde{\lambda}_1(t-U)}\right)^2\tilde{\lambda}_1+\left(r_2 e^{\tilde{\lambda}_2(t-U)}\right)^2\tilde{\lambda}_2} dt.\end{aligned} \tag{3.65}$$

The state-wise sums at risk $\tilde{R}_{e0,f0}(t)$ are found as the discounted values

$$\tilde{R}_{e0,f0}(t) = e^{-rt}(V_{f0}(t) - V_{e0}(t)),$$

and V_{e0} are solutions to the Thiele differential equations

$$\begin{aligned} V'_{00}(t) &= V_{00}(t)r + c + (\mu^{(0)}(x+t) + \mu^{(1)}(x+t) + \mu^{(2)}(x+t))V_{00}(t) - \\ &\quad - \tilde{\lambda}_1(V_{10}(t) - V_{00}(t)) - \tilde{\lambda}_2(V_{20}(t) - V_{00}(t)), && (3.66) \\ V'_{10}(t) &= V_{10}(t)r + c + (\mu^{(0)}(x+t) + \mu^{(2)}(x+t))V_{10}(t) - \tilde{\lambda}_2(V_{30}(t) - V_{10}(t)), && (3.67) \\ V'_{20}(t) &= V_{20}(t)r + c + (\mu^{(0)}(x+t) + \mu^{(1)}(x+t))V_{20}(t) - \tilde{\lambda}_1(V_{30}(t) - V_{20}(t)), && (3.68) \\ V'_{30}(t) &= V_{30}(t)r + c + \mu^{(0)}(x+t)V_{30}(t). && (3.69) \end{aligned}$$

with side conditions

$$V_{e0}(T-) = b.$$

To find the optimal values of r_1 and r_2, one has to compare the hedging error (3.65) for various combinations of r_1 and r_2 such that their sum is between 0 and 1. It seems rather difficult to carry out analytical comparison, so we have written a computer program that solves the Thiele equations using some specific values of the parameters. For the sensitivity analysis, some parameters were fixed while others, one at a time, were allowed to assume a range of values.

For our basic model, the values of the parameters used in the program were fixed as follows:

$\mu^{(0)} = 0.0002 + 0.00005e^{0.038 \ln 10}$ (intensity of mortality due to all other factors);

$\mu^{(1)} = 0.00003 + 0.00003e^{0.038 \ln 10}$ (mortality intensity due to cancer);

$\mu^{(2)} = 0.00002e^{0.038 \ln 10}$ (mortality intensity due to the heart disease);

$b = 100$ (pension endowment);

$x = 40$ (age at purchase of the policy);

$r = 0.05$ (interest rate);

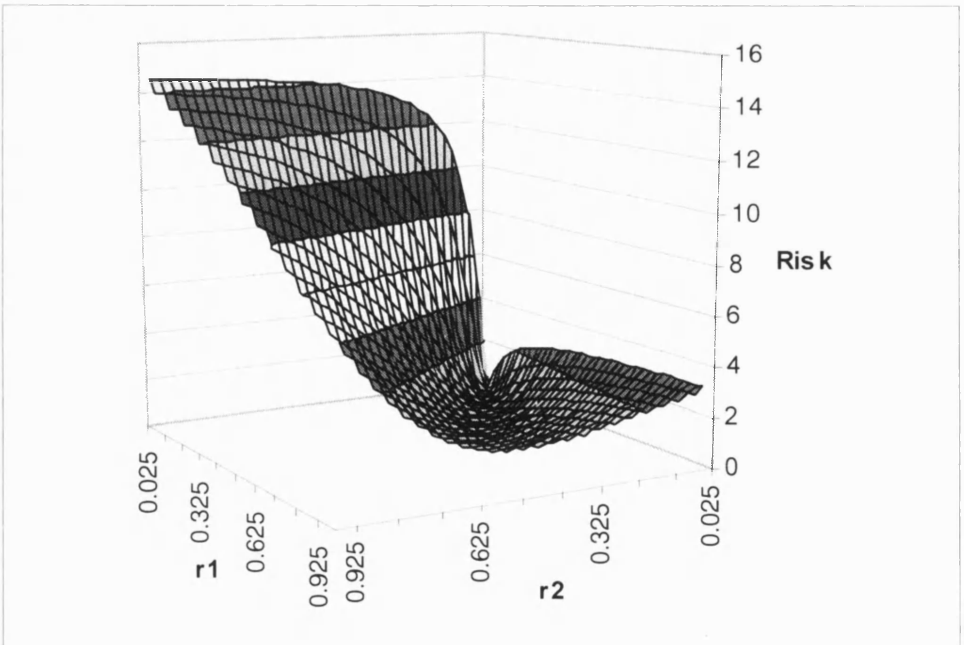
$\tilde{\lambda}_1 = 0.04$ (intensity of finding cure for cancer);

$\tilde{\lambda}_2 = 0.06$ (intensity of finding cure for the heart disease).

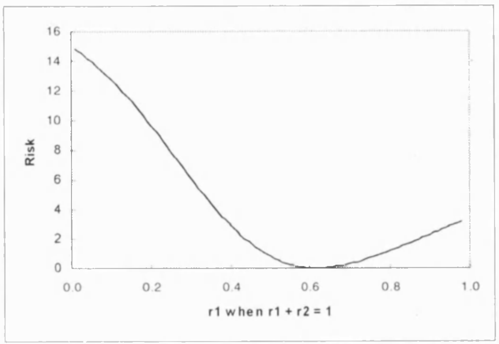
Mortality intensity is of the form $\mu = \theta_1 + \theta_2 e^{\theta_3 \ln 10}$, and we will refer to θ_1 as the first coefficient and to θ_2 as the second coefficient of the intensity representation.

First, we studied dependence between various combinations of values of the parameters r_1 and r_2 and the hedging error. The resulting output is provided in Graphs 7 and 8.

GRAPH 7. HEDGING ERROR AS A FUNCTION OF r_1 AND r_2.



GRAPH 8. HEDGING ERROR AS A FUNCTION OF r_1 WHEN $r_1 + r_2 = 1$



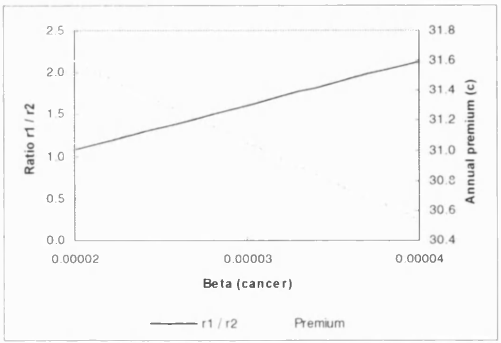
Graph 7 shows that the risk can be reduced virtually to zero along a line in the $r_1 - r_2$ plane. If we dissect the hedging error surface with the vertical plane containing the line $r_1 + r_2 = 1$ (or any other line of the type $r_1 + r_2 = const$), we will see that the ratio $\frac{r_1}{r_2}$ at which risk is minimal is the same for each constant. This is due to the fact that the hedging error (3.65) can be written as a function of $\frac{r_1}{r_2}$:

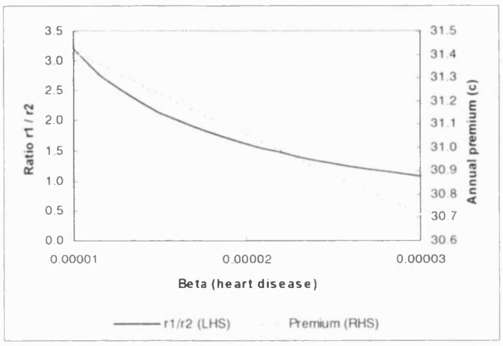
$$\widetilde{\rho}_E^{CH} = \int_0^T e^{-(\widetilde{\lambda}_1+\widetilde{\lambda}_2)t} e^{-\int_0^t \mu_0(x+s)+\mu_1(x+s)+\mu_2(x+s)ds} \widetilde{\lambda}_1 \widetilde{\lambda}_2 \frac{\left(e^{\widetilde{\lambda}_2(t-T)} \widetilde{R}_{00,10}(t) - \frac{r_1}{r_2} e^{\widetilde{\lambda}_1(t-T)} \widetilde{R}_{00,20}(t)\right)^2}{\left(\frac{r_1}{r_2} e^{\widetilde{\lambda}_1(t-T)}\right)^2 \widetilde{\lambda}_1 + \left(e^{\widetilde{\lambda}_2(t-T)}\right)^2 \widetilde{\lambda}_2} dt.$$

For our particular model specification, the optimal ratio approximately equals 1.61; that the ratio is greater than one can be explained by mortality associated with cancer being higher than that associated with heart disease.

Next, using the parameters as above, we study how the ratio depends on the parameters of the model. Additionally, we look at the annual premium, that is, the amount to be paid by the insured after the purchase of the contract until pension age. It depends, among other factors, on the intensities of finding cure for cancer and heart disease and on the age at purchase. Graphs 9 and 10 depict the dependence between the annual premium c (along the right-hand vertical axis) and the second coefficient in the formula for $\mu^{(1)}$ and $\mu^{(2)}$ correspondingly. Our analysis has shown that the annual premium c is almost insensitive to the changes in the first coefficient for $\mu^{(1)}$ or $\mu^{(2)}$ within a reasonable range. Along the left-hand side vertical axis we depict the ratio $\frac{r_1}{r_2}$ which minimizes the hedging error.

GRAPHS 9, 10. RATIO $\frac{r_1}{r_2}$ AND ANNUAL PREMIUM c AS THE FUNCTIONS OF SECOND COEFFICIENT OF $\mu^{(1)}$ and SECOND COEFFICIENT OF $\mu^{(2)}$

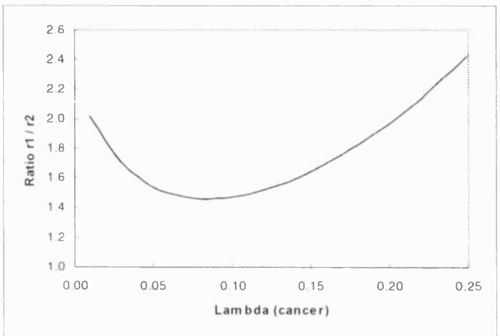


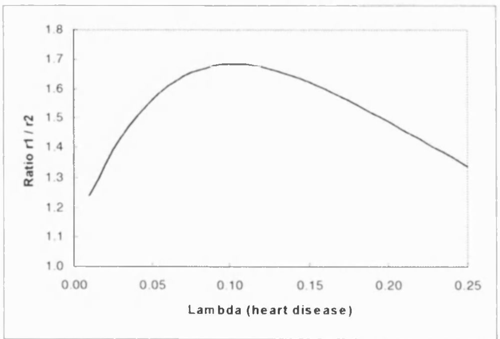


An increase in the second coefficient means a more significant contribution of the particular disease to the total mortality. Correspondingly, finding a cure for this disease will have a larger impact on mortality. We can see, therefore, the expected relationship in the graphs above: the larger contribution of the total mortality is attributed to a disease, the more hedge is required for the event that the cure is found.

Shown on graphs 11 and 12 is the relationship between the risk-minimising ratio $\frac{r_1}{r_2}$ and the intensity of finding cure for cancer and the heart disease. In graph 11, $\tilde{\lambda}_1$ varies within the range $(0, 0.25)$ and $\tilde{\lambda}_2$ equals 0.06, while in graph 12, $\tilde{\lambda}_2$ changes between 0 and 0.25 and $\tilde{\lambda}_1$ equals 0.04.

GRAPHS 11, 12. RATIO $\frac{r_1}{r_2}$ AS A FUNCTION OF THE INTENSITY OF FINDING CURE FOR CANCER $(\tilde{\lambda}_1)$ and CURE FOR HEART DISEASE $(\tilde{\lambda}_2)$



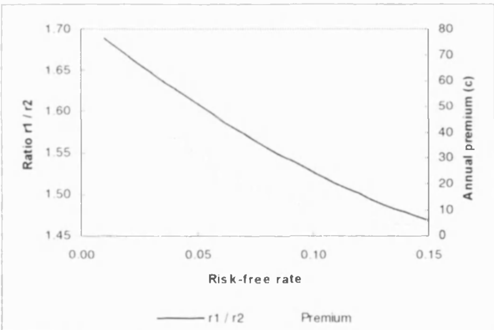


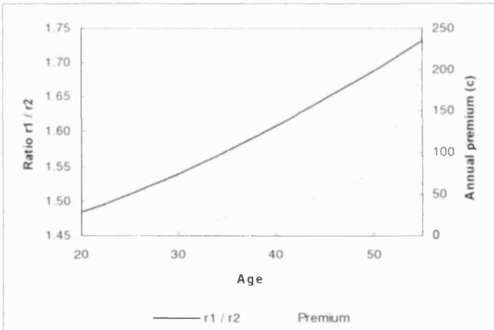
Minimum of the ratio $\frac{r_1}{r_2}$ is attained for $\tilde{\lambda}_1 = 0.08$, $\tilde{\lambda}_2$ being fixed, and maximum is achieved if $\tilde{\lambda}_2 = 0.1$, $\tilde{\lambda}_1$ being fixed.

Finally, we look at the sensitivity of the ratio to the external parameters, namely, risk-free rate and the age at the time of purchase of the policy. Graphs 13 and 14 illustrate how the ratio $\frac{r_1}{r_2}$ and the annual premium depend on the risk-free rate and the starting age of the policy, correspondingly. We can also see that the graphs, with proper scaling, are nearly the mirror images of one another due to the fact that the high interest rate has essentially the same impact on the annual premium and $\frac{r_1}{r_2}$ as the early purchase of the policy.

GRAPHS 13, 14. RATIO $\frac{r_1}{r_2}$ AND ANNUAL PREMIUM c AS FUNCTIONS OF THE

RISK-FREE RATE r and AGE x AT PURCHASE OF THE POLICY





We have shown so far that the design of the CH-bond can be optimized, in the sense of minimizing environmental risk for the insurance company, by a suitably chosen combination of the parameters r_1 and r_2.

Summary

The aim of this chapter has been to address the problem of risk minimization using the tools and methods of the Markov chain theory in economic-demographic environment. For the purpose of hedging mortality risk, a four-state demographic environment was considered, with the states denoting activeness or inactiveness of each of the two potential death causes, cancer and heart disease. Decreases in mortality rate were attributed to discovering the cure for one or both of these diseases. The risk for an insurance company evolves as the policyholders may live longer than estimated if one or both causes of death are eradicated. Partial protection against this sort of risk may come in the form of mortality derivative(s).

We considered three bonds: two disease-specific and one "joint", depending on curability of each of these diseases. We have shown that in the case of disease-specific security, the hedging error does not depend on the design of the security so long as it is based on the same disease. In the case of the "joint" bond, the design does matter, as the number of sources of randomness exceeds the number of mortality derivatives in the market. We have restricted our analysis to the specific design of the bond, which pays $1 - r_1$ or $1 - r_2$ at terminal time U if the cure for cancer or heart disease, correspondingly becomes available, and $1 - (r_1 + r_2)$ if both diseases become curable.

Since analytical results are unattainable in this case, a computer program helps to find optimal, in terms of reducing risk for the insurance company, values for r_1 and r_2. Sensitivity analysis is provided at the end of Chapter 3. The methods described and used in this Chapter can be applied to any mortality derivative. One problem left open is the interests of potential buyers of such a bond. Taking these interests into account might lead to further interesting analysis/conclusions. Another interesting extension of the model discussed would be a further complication of the model by introduction of extra causes of death and derivatives linked to the existence of these causes. A more sound estimation of some of the model parameters, like the intensity of the elimination of death causes would also improve the analysis of the environmental risk.

Chapter 4. Multiperiod and continuous Pareto-optimal risk exchanges

In this chapter, we will be considering the Pareto-optimal redistribution of risk in insurance. The main goal is to extend the results of the existing one-period Pareto-optimality theory to the multiperiod and continuous cases. This would be helpful for insurance companies which may wish to hedge their risks by forming a pool and reallocating risks among themselves.

The one-period theory is already well developed. It was initiated by Borch (1960), (1962), who was "the first to take the ... general approach of deriving the optimal insurance policy form endogenously", as stated by Raviv (1979). In the next two decades, the theory was deeply and extensively developed by Arrow (1964), Borch (1974, 1986), Bühlmann and Gerber (1978), Gerber (1978), Bühlmann and Jewell (1979), Raviv (1979), Bühlmann (1980), (1984). A good unification of their work was created by Taylor (1992) and Aase (2002). An extensive list of references can be found in Kaluszhka (2004).

Recent research in Pareto-optimal risk redistribution has gone in various directions. In some papers the general theory is adjusted for a case with limitations of some sort. For example, Golubin (2005) considers optimal risk exchanges when treaties are allowed only between the insurer and each insured separately, not among the insured themselves. Other researchers seek to extend the existing theory by relaxing constraints. Ermoliev and Flåm (2001) find Pareto-optimal insurance contracts when the underlying loss distribution is unknown. A number of works are concerned with financial applications of the Pareto-optimality theory.

Since the Pareto-optimality concept is so popular, it seems beneficial to extend it beyond the one-period range. An attempt of this kind was recently undertaken by Barrieu, Scandolo (2007), who have studied the Pareto-optimal risk exchanges between two agents in the expected utility framework. The form of the risk exchange in their work is obtained as a solution of a sup-convolution problem between the agents' modified preference functionals. The authors have proved that for the case of bounded risks and concave utility functionals in the real vector space the n -period Pareto-optimal risk exchange is characterised by the same ratio of marginal utilities in each direction.

In this chapter, we start with reporting the basic results from the one-period Pareto-optimality theory in Section 4.1. In Section 4.2 we discuss the potential applications of the multiperiod model and, based on these considerations, we propose two multiperiod utility functionals, one based on the accumulated losses occurred by the end of each period and the other based on the incremental losses, and introduce two corresponding definitions of multiperiod Pareto-optimal risk exchanges (POREX). Then in Section 4.3 we investigate the properties of the multiperiod POREX defined as above. By using a technique different from Barrieu, Scandolo (2007), we achieve the main result for each of the functionals: under some constraints, the ratios of marginal utilities of each pair of agents is almost surely constant and is the same for each year of the multiperiod Pareto-optimal treaty. This essentially means that a multiperiod POREX can be arranged as a sequence of one-period POREX's which maximize the same linear combination as in the Theorem 4.7 below.

In Section 4.4, we develop the Pareto-optimality theory for the continuous case. We partially borrow the technique employed by Barrieu, Scandolo (2007) to prove that the continuous POREX, as the multiperiod one, can be reduced to the one-period case: it can be organized by specifying in advance the ratio of marginal utilities between each pair of agents and keeping it constant in the course of the treaty.

4.1. One-period Pareto-optimal risk exchanges: basic facts

Here we are going to describe the framework and the basic properties of the one-period Pareto-optimal risk exchanges. The insurance industry in consideration includes n companies, or agents, $A_1,\ldots,A_n$, which provide insurance against closely related hazards. Each agent faces a random risk and is willing to exchange parts of their risk through a mutual agreement. Agent A_i's attitude towards risk is reflected by a utility function $u_i(\omega)$, $i = 1,\ldots,n$. These functions are assumed to be twice continuously differentiable, with $u_i(\omega) > 0$ and $u_i(\omega) \leqslant 0$ for all i. Each agent A_i, $i = 1,\ldots,n$, is endowed with a random payoff $X_i(\omega)$, $i = 1,\ldots,n$, which is his initial risk share, including the effect of any direct premium and claims. The random variables $X_i(\omega)$ are defined on the same probability space and are square-integrable. We will also need the total initial risk within the group, $X(\omega) = X_i(\omega)$. The sigma-algebra

$$\mathcal{F} = \sigma\{X_i(\omega),\ i = 1,\ldots,n\}$$

represents the information about the initial risk allocation. The agents can enter into risk exchanges (REX) resulting in a new set of random variables $\boldsymbol{Y} = \{Y_i(\omega)\}$, $i = 1,\ldots,n$, which equal the agent's possible final payout. A REX must satisfy two conditions:

CONDITION 4.1. (*information condition*) The risk shares under the treaty can only depend on the amount of information available at that time, $Y_i(\omega)$ is adapted to $\mathcal{F}$, $i = 1,\ldots,n$.

CONDITION 4.2. (*clearing condition*) No inflow or outflow of capital takes place (the group is closed), $\sum_{i=1}^{n} Y_i(\omega) = \sum_{i=1}^{n} X_i(\omega) = X(\omega)$.

The risk exchange which satisfies both conditions above is called *admissible*. From now on, we will often omit the argument after risk exchange function and write just X_i or Y_i instead of $X_i(\omega)$ or $X_i(\omega)$, always bearing in mind their random character.

To be able to compare different risk exchanges, we employ the standard expected utility as the agents' preference functional, so the agent A_i's satisfaction associated with its risk share Y_i is evaluated as the expected utility $\mathbb{E}u(Y_i)$. It is, of course, impossible to find a risk exchange that is optimal from the isolated point of view of each individual agent. Since the total income (or losses) is to be shared, any increase of one agent's share must be compensated by decreasing the shares of some of the others. This being so, a natural first step is to identify those treaties that are worth negotiating and rule out those which can easily be improved by increasing expected utilities of some agents while the expected utilities of the others are not impaired. This property is summarized in the following

DEFINITION 4.3. An admissible REX $\boldsymbol{Y} = \{Y_i\}$, $i = 1,\ldots,n$, is called *Pareto-optimal risk exchange* (POREX) if there is no admissible allocation $\boldsymbol{Z} = \{Z_i\}$, $i = 1,\ldots,n$, with $\mathbb{E}u_i(Z_i) \geqslant \mathbb{E}u_i(Y_i)$ for all i and with $\mathbb{E}u_j(Z_j) > \mathbb{E}u_j(Y_j)$ for at least one agent A_j.

Pareto-optimal risk exchanges form a subset in the set of all admissible risk exchanges. Similar thing can be said about the filtrations to which they are adapted. The minimal filtration to which Pareto-optimal risk exchanges are adapted is a sub-σ-algebra of $\mathcal{F}$ generated by X, $\mathcal{F}^X \subseteq \mathcal{F}$. This result will be proved in the Theorem 4.5, but first we have to introduce a

DEFINITION 4.4. A REX $\boldsymbol{Y} = \{Y_i\}$, $i = 1,\ldots,n$, is called *global* if the risk shares Y_i, $i = 1,\ldots,n$ are functions of the total income(loss) X.

PROPOSITION 4.5. A risk exchange $\boldsymbol{Y} = \{Y_i\}$, $i = 1,\ldots,n$, is Pareto-optimal only if it is a global REX.

Proof. □Suppose $\boldsymbol{Y} = (Y_1, Y_2, \ldots Y_n)$ is an arbitrary risk exchange, and $\sum_{i=1}^{n} Y_i = X$. We define a new risk exchange $\tilde{\boldsymbol{Y}} = (\tilde{Y}_1, \tilde{Y}_2, \ldots \tilde{Y}_n)$ so that $\tilde{\boldsymbol{Y}}_i = \mathbb{E}[Y_i|\mathcal{F}^X]$. Then $\tilde{\boldsymbol{Y}}$ is admissible, since

$$\sum_{i=1}^{n} \tilde{Y}_i = \sum_{i=1}^{n} \mathbb{E}[Y_i|\mathcal{F}^X] = \mathbb{E}[\sum_{i=1}^{n} Y_i|\mathcal{F}^X] = X.$$

Also, the expected utilities of the agents under $\tilde{\boldsymbol{Y}}$ are not smaller than under $\boldsymbol{Y}$, because by Jensen's inequality for every agent i, $i = 1,\ldots,n$ and the concavity of the utility functions

$$\mathbb{E}[u_i(Y_i)] = \mathbb{E}[\mathbb{E}[u_i(Y_i)|\mathcal{F}^X]] \leqslant \mathbb{E}[u_i(\mathbb{E}[Y_i|\mathcal{F}^X])] = \mathbb{E}[u_i(\tilde{Y}_i)]. \tag{4.1}$$

The equality for all i, $i = 1,\ldots,n$ is achieved if and only if the initial REX was already global. If the inequality (4.1) is not strict for at least one i, for agent A_i the expected utility $\mathbb{E}[u_i(\tilde{Y}_i)]$ will be strictly greater than $\mathbb{E}[u_i(Y_i)]$, which means that the treaty $\tilde{Y}$ improves Y in the sense of Pareto-optimization, and the pool would prefer REX $\tilde{\boldsymbol{Y}}$ to the initial REX $\boldsymbol{Y}$.■

Theorems 4.6 and 4.7 provide further necessary conditions for POREX, and we formulate them without proof. The reader can find it, for instance, in Aase (2002).

THEOREM 4.6. A risk exchange $\boldsymbol{Y} = \{Y_i\}$, $i = 1,\ldots,n$, is Pareto-optimal if and only if the ratios between the marginal utilities $u_i'(Y_i)$ are almost surely constant, that is, there exist constants θ_i, $i = 1,\ldots,n$ such that

$$u_i'(Y_i) = \theta_i u_1'(Y_1), \qquad a.s., \qquad i = 1,\ldots,n \tag{4.2}$$

Necessarily, θ_i must be strictly positive, and $\theta_1 = 1$.

THEOREM 4.7. A risk exchange $\boldsymbol{Y} = \{Y_i\}$, $i = 1,\ldots,n$, is Pareto-optimal if and only if it is a solution to the following optimization problem with some fixed strictly positive constants a_i, $i = 1,\ldots,n$:

$$\max\sum_{i=1}^{n} a_i\mathbb{E}[u(Y_i)] \text{ over all admissible REX.}$$

Later (see Lemma 4.14) we will show that the constants a_i and $\frac{1}{\theta_i}$ are essentially the same for any POREX, and for the multiperiod and continuous framework we will use the two formulations interchangeably.

Having stated the assumptions and results for the one-period theory, we move on to formulating an analogous multiperiod theory and investigating the properties of the multiperiod POREX.

4.2 Potential applications of the multiperiod model

Once the one-period Pareto-optimal risk exchanges have been introduced and studied, the natural further step is to generalize the concept of Pareto-optimality for the multiperiod setting. How can we define the structure of Pareto-optimal REX in this case, and what are the potential applications of it? In this respect, it is instructive to look at the various real-life examples of optimization and the implied models.

First, let us consider a hedge fund. A fund is marked to market at least daily. The goal is to consistently make money, and ability to do so at present matters more than whether one was successful in the past. While a short-term losing streak may be tolerated, risk management systems will often ensure that risky positions are taken off when losses reach a certain limit (a more sophisticated version of the stop-loss selling rule). A longer period of losing money is likely to result in either dismissal of the fund manager responsible or in withdrawal of client money. The latter has essentially the same impact of forcing the fund to liquidate positions. As a rule of thumb, a fund manager's strategy will be seriously questioned after two months of underperformance, and two quarters of negative returns may see the fund fold.

From the above, the following qualitative observations about a hedge fund's utility function can be made:

- tolerance for losses is not high (risk controls and stop-loss rules)
- time horizon for evaluating returns is shorter
- incremental returns rather than cumulative returns matter (dismissal for short-term underperformance).

A somewhat different set of observations can be made about the so-called "real money", such as mutual fund, investment management organizations. The time horizon for positions is typically longer and is measured in months rather than days. While funds are also marked to market on a daily basis, a fund manager is unlikely to be fired if a position has lost money over two months. In addition, while clients look at short-term performance, they also tend to pay attention (rightly or wrongly) to the past track record.

In terms of the utilities in the short and long run, this can be summarized as follows:

- time horizon for evaluating returns is longer
- tolerance for risk at such horizon is higher (and is certainly higher than that of a hedge fund on a shorter horizon)
- both incremental and cumulative returns matter.

Finally, a typical buy-and-hold investor, such as an insurance company, provides still another example. The portfolio of securities is usually designated as held to maturity (rather than trading for hedge funds and mutual funds). This means that such securities (usually bonds) are not required to be marked to market (even though most institutions do so for internal purposes). Instead, provisions are taken against possible losses based on certain rules. For example, securities may have to be written down if their prices fall below a threshold and such impairment is likely to be permanent. Losses, therefore, can come either from defaults or some such permanent impairment. The threshold is usually set as a percentage of par, allowing for a degree of volatility without triggering the write-down. Such an approach results in a greater tolerance for losses in the medium term (quarters, as opposed to days and months). Because managers of insurance companies tend to have longer time horizons (as compared to fund managers), they are more likely to look at cumulative returns rather than short-term incremental returns.

The following, therefore, can be said about utility curves of such buy-and-hold investors:

- time horizon for evaluating returns is long (the shorter of maturity, default, or price falling below the write-down threshold)

- within the limits of the write-down threshold, tolerance for losses is high (but losses are typically smaller, on a relative basis, than in hedge funds!)

- cumulative returns matter perhaps more than incremental returns.

Another potentially interesting application of this theory is the well-known agency problem that arises between company managers and shareholders. Should the management take decisions that they believe will create value for the company in the long term even if that means sacrificing some short-term earnings? Or should they pay attention mainly to making sure quarterly earnings please investors, perhaps to the detriment of somewhat more ambitious long-term plans?

Having in mind the considerations above, in the next subsection we propose a definition of a multiperiod risk-exchange together with the conditions it must satisfy. Then, to be able to compare risk exchanges, we need to introduce some measure of attractiveness of a risk exchange to an agent. In Subsection 4.2b we offer three satisfaction criteria together with the definitions of the corresponding Pareto-optimal risk exchanges and study their properties and the connections among them in Section 4.3.

4.2a Multiperiod risk exchanges - definition

In the multiperiod model, as in the one-period setting, we will be considering n agents $A_1, A_2, \dots, A_n$, all operating in more or less the same line of risky business and seeking possibilities of reducing individual risk through co-operative arrangements. They would typically look to share losses, but may aim to redistribute profits as well. At the outset each agent A_i has a random net income (for example, premiums less paid claims) ∇X_{it} in year $t = 1, \dots, T$. The sum of all agents' risks in year t is denoted by ∇X_t. We assume that the risks are bounded. The sigma-algebra

$$\mathcal{F}_t = \sigma\{\nabla X_{i\tau}\}; \qquad i = 1, 2, \dots, n; \qquad \tau = 1, \dots, t,$$

comprises the information available at time t, and the filtration $\mathbb{F}= \{\mathcal{F}_t\}_{t=1,2,\dots,T}$ stands for the flow of information over time. In order to reduce their individual risks, the agents enter into negotiations of treaties for mutual exchange of risk among them for the T years.

DEFINITION 4.8. A *T-period risk exchange* ($REX_{(1,\dots,T)}$) is a set of random variables $\boldsymbol{Y} = \{\nabla Y_{it};\ i = 1,2,\dots,n;\ t = 1,\dots,T\}$, where ∇Y_{it} is the net income of agent A_i in year t under the treaty.

A REX must satisfy two conditions. Firstly, the *information condition* which states that the amounts exchanged in any year t, $\nabla \boldsymbol{Y}_t = \{\nabla Y_{it};\ i = 1,\dots,n\}$, can only depend on the information available at that time;

$$\nabla \boldsymbol{Y}_t \in \mathcal{F}_t, \qquad t = 1,\dots,T. \tag{4.3}$$

(In other words, $\{\nabla \boldsymbol{Y}_t\}_{t=1,\dots,T}$ is adapted to $\mathbb{F}$). Secondly, since the group of n agents is closed (makes no money transactions with third parties), a REX must also satisfy the *clearing condition* (or *budget constraint*)

$$\sum_{i=1}^{n} \nabla Y_{it} = \sum_{i=1}^{n} \nabla X_{it} = \nabla X_t. \tag{4.4}$$

Let us introduce the accumulated net incomes Y_{it} by period t,

$$Y_{it} = \sum_{\tau=1}^{t} \nabla Y_{i\tau}, \qquad \nabla Y_{it} = Y_{it} - Y_{i,t-1}, \qquad i = 1,\dots,n, \quad t = 1,\dots,T, \tag{4.5}$$

defining $Y_{i0} = 0$. In particular, these definitions apply to the trivial exchange (or rather 'no exchange') $\boldsymbol{X} = \{\nabla X_{it};\ i = 1,2,\dots,n;\ t = 1,\dots,T\}$. Due to (4.5), the information condition (4.3) implies that

$$\boldsymbol{Y}_t \in \mathcal{F}_t, \qquad t = 1,\dots,T, \tag{4.6}$$

and the clearing condition (4.4) can equivalently be cast as

$$\sum_{i=1}^{n} Y_{it} = \sum_{i=1}^{n} X_{it} = X_t, \qquad t = 1,\ldots,T. \tag{4.7}$$

where X_t stands for the total accumulated net income by period t within the group.

4.2b Multiperiod Pareto-optimality

We suppose that, in the multiperiod setup, every agent's attitudes towards risk in year t are described by his/her individual utility function u_{it} as in the one-period model. The usual assumptions are that, for $i = 1,\ldots,n$ and $t = 1,\ldots,T$, these functions are concave, increasing and continuously differentiable on the closure of the support of probability measure. In general, utility functions u_{it} may change over time as the attitudes of the agents are not necessarily constant.

At the beginning of Section 4.2, we described several considerations of the factors that may influence the form of the satisfaction criterion of an agent. Taking these considerations into account, we propose three utility functionals which agent A_i, $i = 1,\ldots,n$ can use to value a given REX $\boldsymbol{Y}$. These satisfaction functionals are labelled by superscripts in the round brackets, and the symbol $V_i^{(*)}$ will be used if the exact form of the functional is not important:

$$V_i^{(1)}(\boldsymbol{Y}) = \sum_{t=1}^{T} \mathbb{E}[u_{it}(\nabla Y_{it})], \tag{4.8}$$

$$V_i^{(2)}(\boldsymbol{Y}) = \sum_{t=1}^{T} \mathbb{E}[u_{it}(Y_{it})], \tag{4.9}$$

$$V_i^{(3)}(\boldsymbol{Y}) = \mathbb{E}[u_{iT}(Y_{iT})]. \tag{4.10}$$

The first functional $V_i^{(1)}(\boldsymbol{Y})$ will typically be employed by businesses like hedge funds, for which it is mostly incremental income that matters. Businesses that are interested in the cumulative changes of income are more likely to measure their utility with the satisfaction functional $V_i^{(2)}(\boldsymbol{Y})$, which sums total income at the end of each year $t \leqslant T$. The final version may reflect the preferences of the long-term investors interested mostly in the final utility. In fact, $V_i^{(3)}(\boldsymbol{Y})$ is a one-period expected utility for the time period of length T, beginning in year 0 and ending in year T.

It is also possible that the market players choose to employ more than one criterion. For example, they may combine the first and the third one or the third and the second criteria, as the utility of the final position in year T may be of particular importance. However, it will be shown that, with appropriate constraints, the first and the second criterion are equivalent and entail the fulfilment of the third one.

It is, of course, impossible to find a risk exchange that is optimal from the isolated point of view of every individual agent. Under the clearing constraint (4.4) any increase in one agent's share must be compensated by a decrease in the shares of the others. This being so, a natural first step is to identify the treaties that are worth negotiating. Thus we introduce

DEFINITION 4.9. *A multiperiod Pareto-optimal risk exchange* $REX_{(1,\dots,T)}\ \boldsymbol{Y}$ under criterion $(*)$, ($POREX^{(*)}_{(1,\dots,T)}\ \boldsymbol{Y}$) is an exchange which cannot be replaced by another $REX_{(1,\dots,T)}\ \tilde{\boldsymbol{Y}}$ such that under $\tilde{\boldsymbol{Y}}$, satisfaction for at least one agent $V_i^{(*)}$ increases while for the rest of the group it does not decrease:

$$\nexists \tilde{\boldsymbol{Y}} : V_i^{(*)}(\boldsymbol{Y}) < V_i^{(*)}(\tilde{\boldsymbol{Y}}) \tag{4.11}$$

for some i, $1 \leqslant i \leqslant n$, and $V_j^{(*)}(\boldsymbol{Y}) \leqslant V_j^{(*)}(\tilde{\boldsymbol{Y}})$, $j = 1,\dots,n$. Another equivalent way of putting it is to say that if a $REX_{(1,\dots,T)}\ \boldsymbol{Y}$ is Pareto-optimal and $V_i^{(*)}(\boldsymbol{Y}) < V_i^{(*)}(\tilde{\boldsymbol{Y}})$ for some i, then we must also have $V_j^{(*)}(\boldsymbol{Y}) > V_j^{(*)}(\tilde{\boldsymbol{Y}})$ for some other j.

Since no agent is forced to enter a risk exchange, a $POREX^{(*)}_{(0,\dots,T)}\ \boldsymbol{Y}$ must satisfy the *feasibility condition*

$$V_i^{(*)}(\boldsymbol{Y}) \geqslant V_i^{(*)}(\boldsymbol{X}), \forall i. \tag{4.12}$$

Suppose that there exists a feasible POREX. Then a risk exchange that is not a POREX can be excluded since it can be replaced by some other REX that improves the position of at least one agent without impairing the position of any other agent. Thus, the negotiable treaties are precisely the feasible Pareto-optimal ones.

4.3 Properties of the multiperiod Pareto-optimal risk exchanges

As in the one-period case, we want to establish the necessary and sufficient conditions of the treaties that are Pareto-optimal under one of the satisfaction criteria given by (4.8), (4.9). In Subsection 4.3*a*, we start with setting forth the necessary conditions under the second satisfaction criterion (4.9) and then proving similar statements is case of the first criterion (4.8). Then we move to the sufficient conditions in Subsection 4.3*b*. These conditions will also establish the connection between the one-period and the multiperiod Pareto-optimal risk exchanges.

4.3a Necessary conditions for a multiperiod POREX

Let us denote by $\mathcal{F}_t^{X_t} = \sigma\{X_t\}$ the sigma-algebra generated by the total accumulated income of the group in year t. The $\mathcal{F}_t^{X_t}$, $t = 1,\ldots,T$, do not constitute a filtration. It is obvious that for every Pareto-optimal risk exchange $\boldsymbol{Y}$, $Y_t \in \mathcal{F}_t^{X_t} \subseteq \mathcal{F}_t^{\min}$ must hold for $t = 1,\ldots,T$ to satisfy the clearing condition. Let $\widetilde{\mathcal{F}}_t$ be a sigma-algebra such that $\mathcal{F}_t^{X_t} \subseteq \widetilde{\mathcal{F}}_t \subseteq \mathcal{F}_t$, $t = 1,\ldots,T$. Thus, the $\widetilde{\mathcal{F}}_t$ represents a flow of information that may be summary and with an imperfect memory but retains the information about the total accumulated group incomes. Let $\boldsymbol{Y}= \{Y_{it}\}$ be some $REX_{(0,\ldots,T)}$ and define $\tilde{\boldsymbol{Y}} = \{\widetilde{Y}_{it} = \mathbb{E}[Y_{it}|\widetilde{\mathcal{F}}_t]\}$. Then $\tilde{\boldsymbol{Y}}_t$ is a $REX_{(1,\ldots,T)}$; it trivially satisfies the information condition (4.6), and it also satisfies the clearing condition since

$$\sum_{i=1}^{n} \tilde{Y}_{it} = \sum_{i=1}^{n} \mathbb{E}[Y_{it}|\tilde{\mathcal{F}}_t] = \mathbb{E}[\sum_{i=1}^{n} Y_{it}|\tilde{\mathcal{F}}_t] = \mathbb{E}[\sum_{i=1}^{n} X_{it}|\tilde{\mathcal{F}}_t] = \sum_{i=1}^{n} X_{it} = X_t. \tag{4.13}$$

Next, we show that under the satisfaction criteria given by (4.9), every agent will enjoy an increase in the level of satisfaction, or at least his level of satisfaction will remain the same, under $REX_{(0,\dots,T)}$ $\tilde{\boldsymbol{Y}}$ compared to $REX_{(0,\dots,T)}$ $\boldsymbol{Y}$. By Jensen's inequality and from the concavity of the utility functions u_{it}, $i = 1,\dots,n$, $t = 1,\dots,T$

$$\begin{aligned} V_{iT}^{(2)}(\boldsymbol{Y}) &= \sum_{t=1}^{T} \mathbb{E}[u_{it}(Y_{it})] = \sum_{t=1}^{T} \mathbb{E}\big[\mathbb{E}\big[u_{it}(Y_{it})|\tilde{\mathcal{F}}_t\big]\big] \leqslant \\ &\leqslant \sum_{t=1}^{T} \mathbb{E}\big[u_{it}\big(\mathbb{E}\big[Y_{it}|\tilde{\mathcal{F}}_t\big]\big)\big] = \sum_{t=1}^{T} \mathbb{E}\big[u_{it}\big(\tilde{Y}_{it}\big)\big] = V_{iT}^{(2)}(\tilde{\boldsymbol{Y}}). \end{aligned} \tag{4.14}$$

Therefore, all agents would prefer $REX_{(1,\dots,T)}$ $\tilde{\boldsymbol{Y}}$ to $REX_{(1,\dots,T)}$ $\boldsymbol{Y}$. The argument also says that $\tilde{\mathcal{F}}_t$ should be chosen as small as possible, which means that $\tilde{\mathcal{F}}_t = \mathcal{F}_t^X$; otherwise one could improve on the position of all agents by conditioning anew on $\mathcal{F}_t^X$. We conclude:

PROPOSITION 4.10. *A necessary condition for a* $REX_{(1,\dots,T)}$ $\boldsymbol{Y}$ *to be a* $POREX_{(1,\dots,T)}^{(2)}$ *is that*

$$Y_{it} \in \mathcal{F}_t^X, \qquad t = 1,\dots,T \tag{4.15}$$

In other words, the risk shares Y_{it} must be functions of X_t only.

A sufficient condition is not easy to find. The reason is the special form of the utility criterion (4.9), which adds the expected utilities for all years. However, a further necessary condition can be derived from this additive structure. Let $\boldsymbol{Y}$ be a $REX_{(1,\dots,T)}$ that is Pareto-optimal in the sense of definition (4.9), but for some year τ, $1 \leqslant \tau \leqslant T$, $\{Y_{it}\}$ is not a *POREX* in the one-period sense for year τ problem of exchanging the accumulated individual incomes $X_{i\tau}$, $i = 1,\dots,n$, when each agent A_i seeks to maximize expected utility in year τ only. Then one can find another year τ *REX* $\{\tilde{Y}_{i\tau}\}$ that satisfies the clearing condition in year τ and is such that $\mathbb{E}(u_{i\tau}(Y_{i\tau})) \leqslant \mathbb{E}(u_{i\tau}(\tilde{Y}_{i\tau}))$ for all i, with strict inequality for some j from $\{1,2,\dots,N\}$. Taking $\tilde{Y}_{it} = Y_{it}$ for $t \neq \tau$, we then find that $\boldsymbol{Y}$ is not $POREX_{(0,\dots,T)}^{(2)}$

by the definition (see (4.11)). We conclude:

PROPOSITION 4.11. *A necessary condition for a $REX_{(1,\ldots,T)}$ $\boldsymbol{Y}$ to be a $POREX^{(2)}_{(0,\ldots,T)}$ is that, for each year t, $\{Y_{it}\}$ is a POREX for the local year t problem of exchanging the accumulated incomes X_t when each agent A_i seeks to maximize his year t expected utility $\mathbb{E}(u_{it}(Y_{it}))$.*

This Proposition also implies that any $POREX^{(2)}_{(0,\ldots,T)}$ $\boldsymbol{Y}$ is $POREX^{(2)}_{(0,\ldots,T_0)}$ $\boldsymbol{Y}$ for any $0 < T_0 < T$, and is also Pareto-optimal for any $0 < T_0 \leqslant T$ under the third satisfaction criterion $V^{(3)}_T(\boldsymbol{Y})$, under which every agent seeks to maximize his final utility $\mathbb{E}[u_{iT}(Y_{iT})]$.

It follows that for $\boldsymbol{Y}$ to be a *POREX*, it must satisfy the Borch conditions for the utility derivatives

$$u'_{it}(Y_{it}) = \theta_{it} u'_{1t}(Y_{1t}) \qquad a.s. \tag{4.16}$$

for some positive constants θ_{it}, $i = 2,3,\ldots,n$, $t = 1,2,\ldots,T$. Actually, Proposition 4.11 already implies Proposition 4.10 due to the well known result that a POREX in the one-period set-up must be a pool. An equivalent necessary condition in the multiperiod setting is that for each t $\{Y_{it}\}$ must maximize

$$\sum_{i=1}^{n} \alpha_{it} \mathbb{E}[u_{it}(Y_{it})] \tag{4.17}$$

over the set of all admissible REX for some strictly positive constants α_{it}. To make the combination $\{\alpha_{it}\}$, $i = 1,\ldots,n$, $t = 1,\ldots,T$, uniquely defined, we set $\alpha_{1t} \equiv 1$ for all t. In fact, these constants are equal to the reciprocal values of θ_{it} used in (4.16). If we want to maximize (4.17) for some linear combination $\{1,\alpha_{2t},\ldots\ldots,\alpha_{nt}\}$ under the clearing constraint, we have to differentiate $\sum_{i=1}^{n} \alpha_{it}\mathbb{E}(u_{it}(Y_{it}))$ with respect to Y_{it}, $i = 2,\ldots n$ (as $Y_{1t} = X_t - (Y_{2t}+\ldots+Y_{nt})$) and equate these derivatives to zero:

$$\frac{\partial\left(\sum_{i=1}^{n}\alpha_{it}\mathbb{E}[u_{it}(Y_{it})]\right)}{\partial Y_{it}} = -\alpha_{1t}\mathbb{E}[u'_{1t}(Y_{1t})] + \alpha_{it}\mathbb{E}[u'_{it}(Y_{it})] = 0 \Rightarrow$$
$$\Rightarrow \mathbb{E}[u'_{1t}(Y_{1t})] = \alpha_{it}\mathbb{E}[u'_{it}(Y_{it})].$$

Since we have at the same time that, according to (4.16), $u'_{it}(Y_{it})) = \theta_{it}u'_{1t}(Y_{1t})$, *a.s.*, we take expectations and see that $\mathbb{E}[u'_{it}(Y_{it})] = \theta_{it}\mathbb{E}[u'_{1t}(Y_{1t})]$ so θ_{it} must equal $\frac{1}{\alpha_{it}}$, $i = 2,\ldots,n$. We have proved so far

LEMMA 4.12. *Each one-period POREX $\{Y_i\}$ maximizes some linear combination $\sum_{i=1}^{n}\alpha_{it}\mathbb{E}[u_{it}(Y_{it})]$, and*

$$u'_{it}(Y_{it})) = \frac{1}{\alpha_{it}}u'_{1t}(Y_{1t}) \qquad a.s. \tag{4.18}$$

Statements similar to those in Propositions 4.10 and 4.11 can be formulated for the first satisfaction criterion given by (4.8). If the goodness of a multiperiod risk exchange is measured as in (4.8), we might reasonably expect that a Pareto-optimal REX during each time period t should depend on the total change of income during that period. Let $\mathcal{F}_t^{\Delta X} = \sigma\{\Delta X_t\}$ be the sigma-algebra generated by the income change during year t, and $\widetilde{\mathcal{F}}_t$ - some sigma-algebra which includes $\mathcal{F}_t^{\Delta X}$: $\mathcal{F}_t^{\Delta X} \subseteq \widetilde{\mathcal{F}}_t \subseteq \mathcal{F}_t$, $t = 1,\ldots,T$. Then for each REX $\boldsymbol{Y}$= $\{\Delta Y_{it}\}$ we define a new risk-exchange $\tilde{\boldsymbol{Y}} = \{\Delta\widetilde{Y}_{it} = \mathbb{E}[\Delta Y_{it}|\widetilde{\mathcal{F}}_t]\}$. Then $\tilde{\boldsymbol{Y}}$ is a $REX_{(0,\ldots,T)}$; it satisfies the information requirement (4.3), and it also satisfies the clearing condition since

$$\sum_{i=1}^{n}\Delta\widetilde{Y}_{it} = \sum_{i=1}^{n}\mathbb{E}[\Delta Y_{it}|\widetilde{\mathcal{F}_t^{\Delta}}] = \mathbb{E}[\sum_{i=1}^{n}\Delta Y_{it}|\widetilde{\mathcal{F}_t^{\Delta}}] = \mathbb{E}[\sum_{i=1}^{n}\Delta X_{it}|\widetilde{\mathcal{F}_t^{\Delta}}] = \Delta X_t. \tag{4.19}$$

The satisfaction of each agent under risk-exchange $\tilde{\boldsymbol{Y}}$ will not be less than that under $\boldsymbol{Y}$, because by Jensen's inequality and from the concavity of the utility functions

$$V_{iT}^{(1)}(\boldsymbol{Y}) = \sum_{t=1}^{T}\mathbb{E}[u_{it}(\Delta Y_{it})] = \sum_{t=1}^{T}\mathbb{E}\left[\mathbb{E}\left[u_{it}(\Delta Y_{it})|\widetilde{\mathcal{F}}_t\right]\right] \leqslant$$
$$\leqslant\sum_{t=1}^{T}\mathbb{E}\left[u_{it}\left(\mathbb{E}\left[\Delta Y_{it}|\widetilde{\mathcal{F}}_t\right]\right)\right] = \sum_{t=1}^{T}\mathbb{E}\left[u_{it}\left(\Delta\widetilde{Y}_{it}\right)\right] = V_{iT}^{(1)}(\tilde{\boldsymbol{Y}}). \tag{4.20}$$

Thus, all agents would agree to switch to *REX* $\tilde{\boldsymbol{Y}}$. As for the $POREX^{(2)}_{(1,\dots,T)}$ at the beginning of Subsection 4.2*a*, $\tilde{\mathcal{F}}_t$ should be chosen as small as possible, which means that $\tilde{\mathcal{F}}_t = \mathcal{F}_t^{\triangle X}$; otherwise one could improve on the position of all agents by conditioning anew on $\mathcal{F}_t^{\triangle X}$. To put it formally, we state

PROPOSITION 4.13. *A necessary condition for a* $REX_{(1,\dots,T)}$ $\boldsymbol{Y}$ *to be a* $POREX^{(1)}_{(1,\dots,T)}$ *is that*

$$\triangle Y_{it} \in \mathcal{F}_t^{\triangle X}, \qquad t = 1,\dots,T \tag{4.21}$$

It means that $\triangle Y_{it}$ must be functions of $\triangle X_t$ only.

Now we are going to examine the structure of the multiperiod $POREX^{(1)}_{(1,\dots,T)}$. It is formulated in the following

LEMMA 4.14. *A necessary condition for a* $REX_{(1,\dots,T)}$ $\boldsymbol{Y}=\{Y_{it}\}$, $i = 1,\dots,n$, $t = t_1,\dots,t_2$, *to be a* $POREX^{(1)}_{(1,\dots,T)}$ *is that for every* t_1,t_2 *such that* $0 < t_1 \leqslant t_2 \leqslant T$, $\{Y_{it}\}$, $i = 1,\dots,n$, $t = t_1,\dots,t_2$, *is a* $POREX^{(1)}_{(t_1,\dots,t_2)}$ *for the local subperiod* $(t_1,\dots,t_2)$ *problem of exchanging incomes* $\triangle X_t$ *when each agent* A_i *seeks to maximize* $\sum_{t=t_1}^{t_2} \mathbb{E}[u_{it}(\triangle Y_{it})]$. *In particular, it is a one-period POREX for every year* t, $0 < t \leqslant T$ *(in this case* $t_2 = t_1 = t$*).*

□Proof. Suppose there exists a $REX_{(1,\dots,T)}$ $\boldsymbol{Y}$ which is a $POREX^{(1)}_{(1,\dots,T)}$, but is not Pareto-optimal for some subperiod $(t_1,\dots,t_2)$, $0 < t_1 \leqslant t_2 \leqslant T$. Then for this subperiod there exists an alternative $REX_{(t_1,\dots,t_2)}$ $\boldsymbol{Z}$ which increases at least one agent's satisfaction function

$$\sum_{t=t_1}^{t_2} \mathbb{E}[u_{jt}(\triangle Z_{jt})] > \sum_{t=t_1}^{t_2} \mathbb{E}[u_{jt}(\triangle Y_{jt})] \quad \text{for some } j, \tag{4.22}$$

while other agents' satisfaction is not impaired:

$$\sum_{t=t_1}^{t_2} \mathbb{E}[u_{jt}(\triangle Z_{jt})] \geqslant \sum_{t=t_1}^{t_2} \mathbb{E}[u_{jt}(\triangle Y_{jt})], \qquad i \neq j. \tag{4.23}$$

Then we can construct a new $REX_{(1,\dots,T)}$ $\tilde{\boldsymbol{Y}}$ such that $\tilde{Y}_{it} = Z_{it}$ for $t \in [t_1,\dots,,t_2]$ and $\tilde{Y}_{it} = Y_{it}$ for $t \notin [t_1,\dots,,t_2]$. Since the risk exchanges $\boldsymbol{Y}$ and $\boldsymbol{Z}$ satisfy the information and the clearing conditions, new $REX_{(0,\dots,T)}$ $\tilde{\boldsymbol{Y}}$ agrees with them as well. From (4.22) and from the construction of $REX_{(0,\dots,T)}$ $\tilde{\boldsymbol{Y}}$ it is evident that for agent j,

$$V_{jT}^{(1)}(\tilde{\boldsymbol{Y}}) = \sum_{t=1}^{T} \mathbb{E}\left[u_{jt}\left(\triangle\tilde{Y}_{jt}\right)\right] = \sum_{t=1}^{t_1-1} \mathbb{E}[u_{jt}(\triangle Y_{jt})] + \sum_{t=t_1}^{t_2} \mathbb{E}[u_{jt}(\triangle Z_{jt})] + \sum_{t=t_2+1}^{T} \mathbb{E}[u_{jt}(\triangle Y_{jt})] >$$
$$> \sum_{t=1}^{T} \mathbb{E}[u_{jt}(\triangle Y_{jt})] = V_{jT}^{(1)}(\boldsymbol{Y}),$$

and for every other agent i, from (4.23) we get

$$V_{iT}^{(1)}(\tilde{\boldsymbol{Y}}) = \sum_{t=1}^{T} \mathbb{E}\left[u_{it}\left(\triangle\tilde{Y}_{it}\right)\right] = \sum_{t=1}^{t_1-1} \mathbb{E}[u_{it}(\triangle Y_{it})] + \sum_{t=t_1}^{t_2} \mathbb{E}[u_{it}(\triangle Z_{it})] + \sum_{t=t_2+1}^{T} \mathbb{E}[u_{it}(\triangle Y_{it})] \geqslant$$
$$\geqslant \sum_{t=1}^{T} \mathbb{E}[u_{it}(\triangle Y_{it})] = V_{iT}^{(1)}(\boldsymbol{Y}).$$

This contradicts the assumption that $\boldsymbol{Y}$ is a $POREX_{(0,\dots,T)}^{(1)}$. So we have proved that $\boldsymbol{Y}$ must be Pareto-optimal for any subperiod $[t_1,\dots,,t_2]$.■

Further necessary conditions can be formulated with some additional assumptions. We consider a special case of time-independent utility functions, $u_{it} = u_i$, $i = 1,\dots,n$, $t = 1,\dots,T$ and non-decreasing risk shares for every agent and year. We will show that in this case for the $POREX_{(0,\dots,T)}^{(2)}$ and the $POREX_{(0,\dots,T)}^{(1)}$ the ratio of the marginal utilities is the same for each year. We start with a simple case of just two agents and the time horizon of two years and formulate the statement for the $POREX_{(0,1,2)}^{(2)}$:

THEOREM 4.15. For $n = 2$ and $T = 2$ under $POREX_{(1,2)}^{(2)}$ the ratio of the derivatives of the utility functions of the agents is the same in each year.

PROOF. □ Suppose that we have a $POREX_{(0,1,2)}^{(2)}$ $\boldsymbol{Y}$, and the total losses in years 1 and 2 are X_1 and X_2 correspondingly. We assume that the risk shares do not decrease, $Y_{11} \leqslant Y_{12}$ and $Y_{21} \leqslant Y_{22}$. Then, according to

Proposition 4.11, $\{Y_{11}, Y_{21}\}$ and $\{Y_{12}, Y_{22}\}$ each form one-period *POREX*. This means that

$$u_1'(Y_{11})) = \theta_1 u_2'(Y_{21}) \quad a.s. \tag{4.24}$$
$$u_1'(Y_{12})) = \theta_2 u_2'(Y_{22}) \quad a.s. \tag{4.25}$$

Suppose that both u_1' and u_2' are continuous and, contrary to the statement of the Theorem, that $\theta_1 \neq \theta_2$. Then there exists a $\delta > 0$ such that for each Y_{11}' from the δ-neighbourhood of Y_{11} and Y_{21}' from the δ-neighbourhood of Y_{21}, $\left|\frac{u_1'(Y_{11}')}{u_2'(Y_{21}')} - \theta_1\right| \leqslant \frac{|\theta_1 - \theta_2|}{2}$ which implies that $\left|\frac{u_1'(Y_{11}')}{u_2'(Y_{21}')} - \theta_2\right| \geqslant \frac{|\theta_1 - \theta_2|}{2}$. Then if $X_2 - X_1 < \delta$, the risk shares Y_{12} and Y_{22} will belong to the δ-neighbourhood of Y_{11} and Y_{21} correspondingly, and $\left|\frac{u_1'(Y_{12})}{u_2'(Y_{22})} - \theta_2\right| \geqslant \frac{|\theta_1 - \theta_2|}{2}$. Since $\mathbb{P}[X_2 - X_1 < \delta] > 0$, the condition (4.25) is not satisfied, the ratio $\frac{u_1'(Y_{12})}{u_2'(Y_{22})}$ is different from θ_2 on a set with positive measure, which contradicts the assumption of Pareto-optimality with $u_1'(Y_{12})) = \theta_2 u_2'(Y_{22})$ *a.s.* in year 2. Thus, the assumption $\theta_1 \neq \theta_2$ is false and for both one-period *POREX*'s the ratio of the derivatives of utility functions must be the same. One can also see that, if $X_2 = 0$ with positive probability, then θ_1 and θ_2 must be the same. ■

Transition to n agents is straightforward, and from $T = 2$ to an arbitrary T it can be performed by induction on the number of years. The corresponding sufficient condition will be proved in Theorem 4.17, Subsection 4.3*b*.

The similar statement for the $POREX^{(1)}_{(1,\dots,T)}$ can not be proved in the same manner. However, we will prove in Theorem 4.16 that under any $POREX^{(1)}_{(1,\dots,T)}$ the ratio of the agents' marginal utilities is the same in each year with additional assumptions about the model. Namely, if we consider exponential utility functions which are widely used in applications, and if total losses are identically distributed for every year, under any $POREX^{(1)}_{(1,\dots,T)}$ the ratio of the marginal utilities should be the same for each year. The sufficient part of this statement follows from the Theorem 4.17 in Subsection 4.3*b*.

THEOREM 4.16. If the agents' utility functions are exponential (have the form $u(y) = -e^{-ay}$) and time-independent, under any $POREX^{(1)}_{(1,\ldots,T)}$ the ratio of the agents' marginal utilities is the same in each year.

PROOF. □ First we will prove the statement of the theorem for $n = 2$ and $T = 2$. Suppose that we have a $POREX^{(1)}_{(1,2)}$ $\mathbb{Y}$, and the incremental losses in years 1 and 2 are $\triangle X_1$ and $\triangle X_2$ correspondingly, and they are independent and identically distributed. Let us say that the utility function of the agent $A_i, i = 1,2$ is $u_i(y) = -e^{-a_i y}$. First we suppose that the statement of the theorem is not true; that is, there exists a $POREX^{(1)}_{(0,1,2)}$ $\mathbb{Y}$ such that in year 1 the ratio of utilities is θ_1 and in year 2 it is θ_2. We will show that then one can find θ such that a risk exchange with the ratio of derivatives of θ in every year will increase both agents' satisfaction under the first satisfaction criterion (4.8).

It is a well-known fact that for exponential utility functions, the Pareto-optimal risk shares are linear functions of total losses. So the shares of the first agent in years 1 and 2 are

$$\triangle Y_{11} = \tfrac{a_1}{a_1+a_2}\triangle X_1 + \tfrac{1}{a_1+a_2}\ln\tfrac{a_2\theta_1}{a_1}$$

and

$$\triangle Y_{12} = \tfrac{a_1}{a_1+a_2}\triangle X_2 + \tfrac{1}{a_1+a_2}\ln\tfrac{a_2\theta_2}{a_1},$$

correspondingly, and for the second agent they are

$$\triangle Y_{21} = \tfrac{a_2}{a_1+a_2}\triangle X_1 - \tfrac{1}{a_1+a_2}\ln\tfrac{a_2\theta_1}{a_1}$$

and

$$\triangle Y_{22} = \tfrac{a_2}{a_1+a_2}\triangle X_2 - \tfrac{1}{a_1+a_2}\ln\tfrac{a_2\theta_2}{a_1}.$$

Then the satisfaction functionals are (we use the fact that $\triangle X_1$ and $\triangle X_2$ have identical distribution):

$$\begin{aligned}
V_1 &= -\mathbb{E}[e^{-a_1\triangle Y_{11}}] - \mathbb{E}[e^{-a_1\triangle Y_{12}}] = \\
&= -\mathbb{E}\left[e^{-a_1\left(\frac{a_1}{a_1+a_2}\triangle X_1 + \frac{1}{a_1+a_2}\ln\frac{a_2\theta_1}{a_1}\right)}\right] - \mathbb{E}\left[e^{-a_1\left(\frac{a_1}{a_1+a_2}\triangle X_2 + \frac{1}{a_1+a_2}\ln\frac{a_2\theta_2}{a_1}\right)}\right] = \\
&= -\left\{e^{-\frac{a_1}{a_1+a_2}\ln\frac{a_2\theta_1}{a_1}} + e^{-\frac{a_1}{a_1+a_2}\ln\frac{a_2\theta_2}{a_1}}\right\}\mathbb{E}\left[e^{-\frac{a_1^2}{a_1+a_2}\triangle X_1}\right] =
\end{aligned}$$

$$= -\left\{\left(\tfrac{a_2\theta_1}{a_1}\right)^{-\frac{a_1}{a_1+a_2}} + \left(\tfrac{a_2\theta_2}{a_1}\right)^{-\frac{a_1}{a_1+a_2}}\right\}\mathbb{E}\left[e^{-\frac{a_1^2}{a_1+a_2}\Delta X_1}\right] =$$

$$= -\left(\tfrac{a_2}{a_1}\right)^{-\frac{a_1}{a_1+a_2}}\left\{\theta_1^{-\frac{a_1}{a_1+a_2}} + \theta_2^{-\frac{a_1}{a_1+a_2}}\right\}\mathbb{E}\left[e^{-\frac{a_1^2}{a_1+a_2}\Delta X_1}\right],$$

$$V_2 = -\left(\tfrac{a_1}{a_2}\right)^{\frac{a_2}{a_1+a_2}}\left\{\theta_1^{\frac{a_2}{a_1+a_2}} + \theta_2^{\frac{a_2}{a_1+a_2}}\right\}\mathbb{E}\left[e^{-\frac{a_2^2}{a_1+a_2}\Delta X_1}\right].$$

We can see that the only part that depends on θ_1, θ_2 is

$\left\{\theta_1^{-\frac{a_1}{a_1+a_2}} + \theta_2^{-\frac{a_1}{a_1+a_2}}\right\}$ for agent 1, and

$\left\{\theta_1^{\frac{a_2}{a_1+a_2}} + \theta_2^{\frac{a_2}{a_1+a_2}}\right\}$ for agent 2.

We replace $\frac{a_1}{a_1+a_2}$ with b_1 and $\frac{a_2}{a_1+a_2}$ with b_2. It is evident that both b_1 and b_2 lie between 0 and 1 and their sum is 1. We will show that one can find a θ such that

$$\theta^{-b_1} + \theta^{-b_1} < \theta_1^{-b_1} + \theta_2^{-b_1} \tag{4.26}$$

and

$$\theta^{b_2} + \theta^{b_2} < \theta_1^{b_2} + \theta_2^{b_2}. \tag{4.27}$$

Next we show that the satisfaction of both agents will increase if we replace the existing $REX_{(0,1,2)}$ with the one for which the ratio of derivatives is θ at every step.

From (4.26) and (4.27) we derive

$$\theta^{-b_1} < \frac{\theta_1^{-b_1} + \theta_2^{-b_1}}{2}$$

and

$$\theta^{b_2} < \frac{\theta_1^{b_2} + \theta_2^{b_2}}{2},$$

equivalent to

$$\theta > \left[\frac{\theta_1^{-b_1} + \theta_2^{-b_1}}{2}\right]^{-\frac{1}{b_1}} \tag{4.28}$$

and

$$\theta < \left[\frac{\theta_1^{b_2} + \theta_2^{b_2}}{2} \right]^{\frac{1}{b_2}}. \tag{4.29}$$

We just have to show that

$$\left[\frac{\theta_1^{b_2} + \theta_2^{b_2}}{2} \right]^{\frac{1}{b_2}} > \left[\frac{\theta_1^{-b_1} + \theta_2^{-b_1}}{2} \right]^{-\frac{1}{b_1}}. \tag{4.30}$$

The mean of degree s of two distinct positive numbers θ_1, θ_2 $\left[\frac{\theta_1^s + \theta_2^s}{2} \right]^{\frac{1}{s}}$ is an increasing function of s on $\mathbb{R}$ (because $\frac{\ln(1+a^s)}{s}$ has positive derivative on $\mathbb{R}$); in the interval $[0, 1]$ it increases from the geometric mean of θ_1, θ_2 to their arithmetic mean. So (4.30) does hold, and we have found a $REX_{(1,2)}$ which improves satisfaction of both agents and the ratio of utilities is the same for both years.

Now proceed to the general case of arbitrary n and T. Suppose there exists a $POREX^{(1)}_{(1,\ldots,T)}$ $\boldsymbol{Y}$ such that there exist two agents (say, i and j) and two years t_1 and t_2 such that the ratios of their marginal utilities are different almost surely in these years. Then, as shown above, their shares in years t_1 and t_2 can be changed in such a way that the sum of their losses in each of these years remains unchanged (so the rest of the REX $\boldsymbol{Y}$ can be left unchanged and still conform with the clearing conditions), the ratio of the marginal utilities is the same and the sums of utilities of agents i and j in years t_1 and t_2 increase. Then the utility functional $V^{(1)}$ will increase for agents i and j and remain the same for all other agents. This contradicts the assumption that $\boldsymbol{Y}$ is $POREX^{(1)}_{(1,\ldots,T)}$.■

We conclude that in case of the time-independent exponential utility functions and identically distributed total losses X_t any $POREX^{(1)}_{(1,\ldots,T)}$ is necessarily characterized by the same ratios of the marginal utilities for each year $t = 1, \ldots, T$ and each pair of agents $i, j \in \{1, \ldots, n\}$. We can rephrase it and say that the same linear combination of expected utilities should be maximized in each year.

4.3b Sufficient conditions for a multiperiod POREX

In the Theorems 4.15 and 4.16 we have proved that, with appropriate constraints, the ratio of the marginal utilities in each year should be almost surely constant for all agents every year. The sufficient condition can be proved in both cases. Below we present the proof for the Pareto-optimality in terms of accumulated risks, as in the satisfaction criterion (4.9), but exactly the same reasoning can be applied to a $POREX^{(1)}_{(0,\dots,T)}$ as well.

THEOREM 4.17. If $REX_{(0,\dots,T)}$ $\boldsymbol{Y}$ is such that for every $i = 1,\dots,n$ and $t = 1,\dots,T$ the ratios of the derivatives of the utility functions are almost surely constant and do not change over time,

$$\frac{u'_{it}(Y_{it})}{u'_{1t}(Y_{1t})} = \theta_i \qquad \text{a.s.,} \tag{4.31}$$

then $\boldsymbol{Y}$ is a $POREX^{(2)}_{(1,\dots,T)}$. Ratios θ_i are strictly positive for every $i = 1,\dots n$.

Proof. □ Suppose, on the contrary, that $\boldsymbol{Y}$ is not a $POREX^{(2)}_{(1,\dots,T)}$. Then there exists another $REX_{(1,\dots,T)}$ $\tilde{\boldsymbol{Y}}$ such that for at least one agent j

$$V_j^{(2)}(\boldsymbol{Y}) = \sum_{t=1}^{T} \mathbb{E}[u_{jt}(Y_{jt})] < V_j^{(2)}(\tilde{\boldsymbol{Y}}) = \sum_{t=1}^{T} \mathbb{E}[u_{jt}(\tilde{Y}_{jt})], \tag{4.32}$$

and for all other agents i,

$$V_i^{(2)}(\boldsymbol{Y}) = \sum_{t=1}^{T} \mathbb{E}[u_{it}(Y_{it})] \leqslant V_i^{(2)}(\tilde{\boldsymbol{Y}}) = \sum_{t=1}^{T} \mathbb{E}[u_{it}(\tilde{Y}_{it})]. \tag{4.33}$$

But from monotonous increase of the utility functions and their concavity, which implies that $u_i' > 0$ and $u_i'' < 0$ for every i, we can write that for each t,

$$u_i(\tilde{Y}_{it}) < u_i(Y_{it}) + u_i'(Y_{it})(\tilde{Y}_{it} - Y_{it}) \qquad \text{a.s.} \tag{4.34}$$

or

$$\frac{u_i(\tilde{Y}_{it}) - u_i(Y_{it})}{u_i'(Y_{it})} < (\tilde{Y}_{it} - Y_{it}) \qquad \text{a.s.}$$

From (4.31) $u'_{it}(Y_{it}) = \theta_i u'_{1t}(Y_{1t})$, so we can write

$$\sum_{i=1}^{n} \frac{u_i(\tilde{Y}_{it})-u_i(Y_{it})}{\theta_i} < u'_{1t}(Y_{1t}) \sum_{i=1}^{n} (\tilde{Y}_{it} - Y_{it}) \qquad a.s. \tag{4.35}$$

Because of the clearing condition the right-hand side equals 0,

$$\sum_{i=1}^{n} \frac{u_i(\tilde{Y}_{it})-u_i(Y_{it})}{\theta_i} < 0 \qquad \text{a.s.} \tag{4.36}$$

Since (4.36) holds for each year t, we can write

$$\sum_{t=1}^{T}\sum_{i=1}^{n} \mathbb{E}\frac{u_i(\tilde{Y}_{it})-u_i(Y_{it})}{\theta_i} < 0,$$

or

$$\sum_{i=1}^{n}\sum_{t=1}^{T} \frac{\mathbb{E}[u_i(\tilde{Y}_{it})]-\mathbb{E}[u_i(Y_{it})]}{\theta_i} = \sum_{i=1}^{n} \frac{V_i^{(2)}(\tilde{\mathbf{Y}})-V_i^{(2)}(\mathbf{Y})}{\theta_i} < 0. \tag{4.37}$$

In accordance with our assumption that $\boldsymbol{Y}$ can be replaced with $\tilde{\boldsymbol{Y}}$ thus improving some agents satisfaction and not impairing others' utility functionals, all the numerators $V_i^{(2)}(\tilde{\boldsymbol{Y}}) - V_i^{(2)}(\boldsymbol{Y})$ must be non-negative. But since the constants θ_i are all positive, (4.37) cannot hold. This contradiction shows that our assumption about $\mathbb{Y}$ was wrong and any multiperiod risk exchange with the same ratio of the derivatives of the utility functions is necessarily Pareto-optimal.■

The main results proved in Subsections 4.3*a* and 4.3*b* states that for both utility functionals (4.8) and (4.9), global risk exchanges are better than non-global ones, and for time-independent utility functions, multiperiod POREX can be represented as a one-period POREX under which the same linear combination $\sum_{i=1}^{n} \alpha_{it}\mathbb{E}[u_{it}(Y_{it})]$ is maximized at each step (additional assumption for the utility functional (4.8) is that utility functions should be exponential). Sufficient condition also takes place: if the ratios of the derivatives of the utility functions are almost surely constant and do not change over time, $u'_{it}(Y_{it}) = \theta_i u'_{1t}(Y_{1t})$ then $\boldsymbol{Y}$ is POREX.

4.4 Continuous Pareto-optimal risk exchanges

Assume now that the agents observe - and assess - their financial position continuously and not just at a finite set of time points. Denote the total income of agent A_i, $i = 1,\dots,n$ in the time interval $(0,t]$ by X_{it}, $t \in [0,T]$. Then $\boldsymbol{X} = \{X_{it};\ i = 1,\dots,n,\ t \in (0,T]\}$ is the initial risk allocation within the group of n agents. The individual income processes X_{it} generate the filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t\in[0,T]}$. The previous definitions and constraints of Section 4 on the multiperiod Pareto-optimal risk exchanges carry over to the continuous time set-up: admissible treaties are those satisfying clearing and information conditions:

DEFINITION 4.18. *Continuous risk exchange in the time interval (0,T]* ($REX^c_{[0,T]}$) $\boldsymbol{Y}$, is a set of random variables $\{Y_{it}\}$, $i = 1,\dots,n$, $t \in (0,T]$ satisfying the following conditions:

CONDITION 4.19 (*information condition*). The amounts exchanged at any moment t, $\boldsymbol{Y} = \{Y_{it}; i = 1,\dots,n\}$ can only depend on the information available at that time,

$$Y_{it} \in \mathcal{F}_t, \qquad t \in (0,T], \tag{4.38}$$

that is, $\{Y_{it}\}_{t=1,\dots,T}$ is adapted to $\mathbb{F}$.

CONDITION 4.20 (*clearing condition*). The group involved in the continuous risk exchange is closed, which means that no money transactions with third parties are allowed,

$$\sum_{i=1}^{n} Y_{it} = \sum_{i=1}^{n} X_{it} = X_t, \qquad t \in (0,T]. \tag{4.39}$$

DEFINITION 4.21. A risk exchange $\mathbb{Y} = \{Y_t^{(1)},\dots,Y_t^{(n)}\}$ is called *admissible* if

$$\sum_{i=1}^{n} Y_{it} = X_t \; \forall t \in [0,T].$$

The set of all admissible risk exchanges is denoted as $\mathcal{A}(\boldsymbol{X})$.

The continuous time analogue to the utility functional as in (4.9) is

$$V_i^c(\boldsymbol{Y}) = \mathbb{E} \int_{t=0}^{T} u_{it}(Y_{it})dt, \tag{4.40}$$

with the superscript c for "continuous", and $u_{it}(\omega)$ is the utility function of the agent A_i at time t. The Riemann integral (4.40) exists if the trajectories of the risk share processes Y_{it} are bounded and continuous almost everywhere, so we make this additional assumption about the risk exchanges we are going to consider. This also imposes the constraint of boundedness and continuity almost everywhere on the total loss process X_t.

We will show that the functional $V_i^c(\boldsymbol{Y})$ is concave. Indeed, for any $\overline{\boldsymbol{Z}}, \widetilde{\boldsymbol{Z}} \in \mathcal{A}(\boldsymbol{X})$, $\alpha \in [0,1]$, $\alpha\overline{\boldsymbol{Z}} + (1-\alpha)\widetilde{\boldsymbol{Z}}$ represents an admissible risk exchange, since it obviously satisfies the information and clearing conditions, so $\alpha\overline{\boldsymbol{Z}} + (1-\alpha)\widetilde{\boldsymbol{Z}} \in \mathcal{A}(\boldsymbol{X})$. Because of the concavity of the utility functions u_{it} we can write that its utility functional

$$\begin{aligned} V_i^c(\alpha\overline{\boldsymbol{Z}} + (1-\alpha)\widetilde{\boldsymbol{Z}}) &= \\ &= \mathbb{E} \int_0^T u_{it}(\alpha\overline{\boldsymbol{Z}}_t^{(i)} + (1-\alpha)\widetilde{\boldsymbol{Z}}_t^{(i)})dt \geqslant \mathbb{E} \int_0^T \alpha u_{it}(\overline{\boldsymbol{Z}}_t^{(i)}) + (1-\alpha)u_{it}\widetilde{\boldsymbol{Z}}_t^{(i)})dt = \\ &= \mathbb{E} \int_0^T \alpha u_{it}(\overline{\boldsymbol{Z}}_t^{(i)})dt + \mathbb{E} \int_0^T (1-\alpha)u_{it}\widetilde{\boldsymbol{Z}}_t^{(i)})dt = \alpha V_i^c(\overline{\boldsymbol{Z}}) + (1-\alpha)V_i^c\widetilde{\boldsymbol{Z}}. \end{aligned} \tag{4.41}$$

It is not possible to construct a continuous analogue to the first satisfaction criterion (4.8), the reason being that an object like $\int_{t=0}^{T} u_{it}(dY_{it})$ is not well-defined. Therefore, we will only use the definition (4.40). Also, we still assume that the utility functions are continuously differentiable. One example of such time-dependent utility function is $u_t(\omega) = ce^{-\alpha\omega-\rho t}$(see Aase

(1999)), α being the coefficient of absolute risk aversion and ρ the time impatience rate.

The Pareto-optimality condition can be defined the following way:

DEFINITION 4.22. *Continuous Pareto-optimal* $REX^c_{(0,T]}$ $\mathbf{Y}$, or $POREX^c_{(0,T]}$ $\mathbf{Y}$ for the time period $(0,T]$ is a risk exchange which cannot increase some agents' satisfaction functional $V_i^c(\mathbf{Y})$ defined by (5.3) without impairing other agents' satisfaction:

$$\forall \tilde{\mathbf{Y}} : V_i^c(\mathbf{Y}) \leqslant V_i^c(\tilde{\mathbf{Y}}), \quad \forall i \Rightarrow V_i^c(\mathbf{Y}) = V_i^c(\tilde{\mathbf{Y}}). \tag{4.42}$$

The following properties can be formulated straight away:

PROPOSITION 4.23. Let $\mathbf{Y}$ be a $POREX^c_{(0,T]}$. Then $\mathbf{Y}$ must be global, that is, at each time instant t, Y_{it} should depend on the total losses X_t only. Proof is identical to that of Proposition 4.10.

LEMMA 4.24. Let $\mathbf{Y}$ be a $POREX^c_{(0,T]}$. Then for any subinterval $(t_1,t_2] \subseteq (0,T]$ $\mathbf{Y}$ must be a $POREX^c_{(t_1,t_2]}$.

Proof. □ The proof rests on the idea that, in case $\mathbf{Y}$ is not a $POREX^c_{(t_1,t_2]}$ for some subinterval $(t_1,t_2]$, then there exists another treaty $\tilde{\mathbf{Y}}$ which improves the satisfaction of at least one agent A_i on $(t_1,t_2]$ without impairing other agents' satisfaction. This risk exchange can then replace $\mathbf{Y}$ on (t_1,t_2). The resulting new treaty $\bar{\mathbf{Y}}$, which is defined on $(0,T]$ and coincides with $\tilde{\mathbf{Y}}$ on (t_1,t_2) and with $\mathbf{Y}$ on $[0,t_1]$ and on $[t_2,T]$, will improve A_i's satisfaction without diminishing the satisfaction of the other agents, and will, consequently, contradict our assumption about Pareto-optimality of $\mathbf{Y}$.■

Next we state the main result of this section: the necessary and sufficient conditions for the continuous *POREX* for a group of n agents.

THEOREM 4.25. Suppose a $REX^c_{(0,T]}$ $\mathbf{Y}=\{Y_t^{(1)},\ldots,Y_t^{(n)}\}$ is Pareto-optimal. Then for almost all moments $t \in (0,T]$, the ratio of the marginal utilities between the agents should be constant, and these constants must be the same for the whole period $(0,T]$.

PROOF. □To prove this statement, we will need the well known

THEOREM 4.26. (Separability of the convex sets in the real linear space, from Aubin (2000)) Let M and N be two convex sets in the real linear space $\mathcal{L}$, and the interior of at least one of them, say M, is not empty and does not intersect with N. Then there exists a non-trivial linear functional on $\mathcal{L}$ which separates M and N.

To apply the theorem above, we need to specify the sets and the space we are working in. For every $REX^c_{[0,T]}\boldsymbol{Z} \in \mathcal{A}(\boldsymbol{X})$, the agents' utility functionals have values $V^c_1(\boldsymbol{Z}),\ldots,V^c_n(\boldsymbol{Z})$ correspondingly. We may view these values as a point in the n-dimensional space $\mathbb{R}^n$. By the same procedure, we may correspond a point from $\mathbb{R}^n$ to every admissible risk exchange. The image of the set $\mathcal{A}(\boldsymbol{X})$ after such mapping will be an $(n-1)$–dimensional surface in $\mathbb{R}^n$. We will denote this set by $\Theta(\boldsymbol{X})$. Now consider a subset of $\mathbb{R}^n$

$$\Pi = \{(\pi^{(1)},\ldots,\pi^{(n)}) : 0 \leqslant \pi^{(i)} \leqslant V^c_i(\boldsymbol{Z}),\, \boldsymbol{Z} \in \mathcal{A}(\boldsymbol{X}),\ i = 1,\ldots,n\}.$$

It is the area between the coordinate hyperplanes in $\mathbb{R}^n$ and $\Theta(\boldsymbol{X})$. The set Π is convex: if we take two points $\overline{\pi} = (\overline{\pi}^{(1)},\ldots,\overline{\pi}^{(n)})$ and $\widetilde{\pi} = (\widetilde{\pi}^{(1)},\ldots,\widetilde{\pi}^{(n)})$ from Π, the area between these points belongs to Π. Indeed, if $0 \leqslant \overline{\pi}^{(i)} \leqslant U^{(i)}(\overline{\boldsymbol{Z}})$ and $0 \leqslant \widetilde{\pi}^{(i)} \leqslant U^{(i)}(\widetilde{\boldsymbol{Z}})$, $\overline{\boldsymbol{Z}},\widetilde{\boldsymbol{Z}} \in \Theta(\boldsymbol{X})$, then

$$\alpha\overline{\pi}^{(i)} + (1-\alpha)\widetilde{\pi}^{(i)} \leqslant \alpha V^c_i(\overline{\boldsymbol{Z}}) + (1-\alpha)V^c_i(\widetilde{\boldsymbol{Z}}) \leqslant V^c_i(\alpha\overline{\boldsymbol{Z}}+(1-\alpha)\widetilde{\boldsymbol{Z}}).$$

Since $\alpha\overline{\boldsymbol{Z}}+(1-\alpha)\widetilde{\boldsymbol{Z}} \in \mathcal{A}(\boldsymbol{X})$, we have shown that $\alpha\overline{\pi} + (1-\alpha)\widetilde{\pi} \in \Pi$.

The second set required for the Theorem 4.26 is

$$\Sigma = \{(\sigma^{(1)},\ldots,\sigma^{(n)}) : \sigma^{(i)} \geqslant V^c_i(\boldsymbol{Y}),\, i = 1,\ldots,n\}\backslash\{(V^c_1(\boldsymbol{Y}),\ldots,V^c_n(\boldsymbol{Y}))\}.$$

This set is obviously convex, and it does not intersect with Π. Also, the interiors of both sets are not empty: the set Σ contains, for example, an interior point $(V^c_1(\boldsymbol{Y})+1,\ldots,V^c_n(\boldsymbol{Y})+1)$, and the set Π contains an interior point $(V_1^c(\boldsymbol{Y})/2,\ldots,V_n^c(\boldsymbol{Y})/2)$.

Now Theorem 4.26 can be applied to the sets Π, Σ in $\mathbb{R}^n$. The theorem states that there exists a set of the coefficients $\{c_1,c_2,\ldots,c_n\}$ such that the linear functional $f = f(x_1,x_2,\ldots,x_n) = \sum_{i=1}^{n} c_i x_i$ in $\mathbb{R}^n$ separates Π and Σ :

$\forall(\pi^{(1)},\ldots,\pi^{(n)}) \in \Pi$ and $\forall(\sigma^{(1)},\ldots,\sigma^{(n)}) \in \Sigma$ the following inequality holds:

$$\sum_{i=1}^{n} c_i \pi^{(i)} \leqslant \sum_{i=1}^{n} c_i \sigma^{(i)}. \tag{4.43}$$

Since $(V_1^c(\boldsymbol{Y}),\ldots,V_n^c(\boldsymbol{Y}))$ belongs to Π and the points $(V_1^c(\boldsymbol{Y})+1, V_2^c(\boldsymbol{Y}),\ldots,V_n^c(\boldsymbol{Y}))$, $(V_1^c(\boldsymbol{Y}), V_2^c(\boldsymbol{Y})+1,\ldots,V_n^c(\boldsymbol{Y}))$, ..., $(V_1^c(\boldsymbol{Y}), V_2^c(\boldsymbol{Y}),\ldots,V_n^c(\boldsymbol{Y})+1)$ belong to Σ, it follows from (4.6) that $c_i \geqslant 0$, $i = 1,2,\ldots,n$. As the functional is non-trivial, at least one $c_i > 0$. We may assume without loss of generality that $c_1 > 0$. As for any $\varepsilon > 0$ $(V_1^c(\boldsymbol{Y})+\varepsilon, V_2^c(\boldsymbol{Y}),\ldots,V_n^c(\boldsymbol{Y})) \in \Sigma$ and therefore

$$\sum_{i=1}^{n} c_i V_i^c(\boldsymbol{Z}) \leqslant c_1\varepsilon + \sum_{i=1}^{n} c_i V_i^c(\boldsymbol{Y}) \qquad \forall \boldsymbol{Z} \in \Theta(\mathbb{X}),$$

and by letting $\varepsilon \searrow 0$ we obtain

$$\sum_{i=1}^{n} c_i V_i^c(\boldsymbol{Z}) \leqslant \sum_{i=1}^{n} c_i V_i^c(\boldsymbol{Y}) \qquad \forall \boldsymbol{Z} \in \Theta(\mathbb{X}).$$

It shows that $(V_1^c(\boldsymbol{Y}),\ldots,V_n^c(\boldsymbol{Y}))$ maximizes the linear combination

$\sum_{i=1}^{n} c_i V_i^c(\boldsymbol{Z})$ over the set $\mathcal{A}(\boldsymbol{X})$.

Finally, we will show that the same linear combination must be maximized almost everywhere on $(0,T]$. Since for every agent A_i, $i = 1,\ldots,n$ the trajectories $u_{it}(Y_{it})$ are continuous almost everywhere, almost everywhere on $(0,T]$ all n trajectories are continuous simultaneously. We denote by Υ the set of points at which every trajectory $u_{it}(Y_{it})$, $i = 1,\ldots,n$ is continuous.

Now suppose that in some point $t \in \Upsilon$ the risk shares $\mathbb{E}u_{it}(Y_{it})$, $i = 1,\ldots,n$ fail to maximize the linear combination $\{c_i\}$. Then, because of the continuity of each realization, it fails to do so in some δ-neighbourhood of t, (t_1,t_2), which contradicts LEMMA 4.24.

We have proved so far that continuous Pareto-optimal risk exchanges necessarily maximize the same linear combination at almost every time instant on $(0,T]$. We conclude with the sufficient part. Suppose some REX^c $\boldsymbol{Z}$ maximizes $\sum_{i=1}^{n} b_i\mathbb{E}u_{it}(Y_{it})$ for almost all $t \in (0,T]$. Then the integration from 0 to T preserves this property, and the utility functionals $V_i(\boldsymbol{Z})$ maximize $\sum_{i=1}^{n} b_i V_i(\boldsymbol{Z})$, which means that $\boldsymbol{Z}$ is Pareto-optimal.■

Summary

In Chapter 4, we have undertaken an attempt to extend the Pareto-optimality concept to the multiperiod and continuous setting. The two utility functionals proposed for the multiperiod model were $V_i^{(1)}(\boldsymbol{Y}) = \sum_{t=1}^{T} \mathbb{E}[u_{it}(\nabla Y_{it})]$ and $V_i^{(2)}(\boldsymbol{Y}) = \sum_{t=1}^{T} \mathbb{E}[u_{it}(Y_{it})]$. The first one is based on incremental risk shares while the second one involves cumulative risk parts. Other utility functionals can be studied as well, the weighted versions of $V_i^{(1)}(\boldsymbol{Y})$ and $V_i^{(2)}(\boldsymbol{Y})$ being the straightforward generalisation of the two functionals above. In this case, the theorems and proofs of Section 4.3 can be carried on just the same.

We have shown that multiperiod POREX's enjoy the following properties. First, they are global, that is, the individual risk shares depend on the total losses for that year only. Second, any multiperiod POREX must be Pareto-optimal for any subperiod of the initial time interval. Third, with some constraints, we have proved that multiperiod Pareto-optimal risk exchanges are those and only those which are Pareto-optimal in the one-period sense and maximise the same linear combination $\sum_{t=1}^{T} a_i \mathbb{E}[u_{it}(Y_{it})]$ at each time period.

The proposed continuous time utility functional is $\mathbb{E} \int_{t=0}^{T} u_{it}(Y_{it})dt$. We have proved in Section 4.4 that continuous Pareto-optimal risk exchanges on $(0,T]$ are global, Pareto-optimal in the continuous sense on any subperiod of $(0,T]$ and necessarily maximize the same linear combination $\sum_{t=1}^{T} a_i \mathbb{E}[u_{it}(Y_{it})]$ at almost every time instant on $(0,T]$.

We conclude that the construction of the multiperiod and continuous POREX studied in Chapter 4 can actually be reduced to the one-period case. The agents must decide on the coefficients $\{a_i\}$ of the linear combination $\sum_{t=1}^{T} a_i \mathbb{E}[u_{it}(Y_{it})]$ beforehand and allocate the risk shares which maximize this combination at each time instant.

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