Essays on Non-Fundamental Speculation, Trading Behaviour and Strategic Information Sharing

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Abstract

This thesis is mainly divided into three chapters. Even though the three chapters have different aims, they all concern investment environment and process. With respect to the former aspect, we consider how stocks and bonds are traded in securities markets. The second and third chapter deal with markets where a single risky asset is sequentially traded in batch auctions. In the fourth chapter we consider bond markets where transaction costs create frictions. As for the investment process, we study how market participants should choose their investment, and when should it be made. We analyze trading strategies for market participants in the second and third chapter, while active bond portfolio management is dynamically characterized in the fourth chapter. Chapter 2 develops a dynamic trading game in which fundamental insiders coexist with non-fundamental speculators. We study inclusions in the S&P 500 as an example in which non-fundamental speculation arises due to preannouncing index replacements. Evidence on volume and liquidity is consistent with our theoretical analysis. Chapter 3 deals with asymmetrically informed traders engaging in information sharing about the asset's fundamental value. In the presence of information sharing, trading activity and price volatility both cluster at the end of the trading period, and price informativeness is reduced. Our model predicts a rich variety of patterns for liquidity, volume and return volatility. Chapter 4 focuses on affine term structure models as portfolio management tools. We use returns implied by different models as inputs for an investor's portfolio optimization problem. Each period we determine the optimal investment, and then characterize the financial properties of trading strategies. We show that evaluating term structure models from a financial perspective may yield conflicting results with those arising from a statistical metric.

To Piera, with love

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1

Introduction

> To theorize is to abstract. One builds a *model* –a description of a toy world; one simple enough to be thoroughly understood. In such a world all relationships are clear, and the implications of any possible change can be determined precisely. Such an approach may be used in two ways. A normative model is a guide to action; it indicates the manner in which decisions should be made. A positive model is predictive in nature; it describes the manner in which decisions are made and the relationships among things such as prices, quantities sold, etc. (William F. Sharpe, 1970, *Portfolio Theory and Capital Markets*, New York: McGraw-Hill)

This thesis investigates distinct features of asset markets and their impact on investors' trading behaviour. Starting from the 1980s, information asymmetries in asset markets have emerged as a primary line of research. Market participants are heterogenous in that they hold different opinions about asset markets. The vast majority of contributions to this literature have focused on diverse knowledge about securities' fundamental value, and characterized trading strategies as well as aggregate variables such as prices, volume and volatility. In Chapter 2 we study non-fundamental information as an alternative source of information asymmetries. We consider agents endowed with superior information about future trades, rather than future payoffs. Brokers incarnate the idea of non-fundamental traders: they place orders on their account in addition to the orders submitted on behalf of their customers. Chapter 3 focuses on information sharing in securities markets. Once again, heterogenously informed agents consititute the starting point. In line with most of the literature –and unlike chapter 2– market participants hold different views about future payoffs in chapter 3, and we study whether traders have incentives to communicate their information to one another. While the existing literature focuses on trading as the main vehicle to convey private information to the market at large, we create a role for direct communication as a channel to information revelation in securities markets. The empirical appeal of our analysis hinges on the observation that every day traders exchange opinions and share their views about financial assets, either because they discuss upon meeting at the marketplace or via message posting in financial forums. We move away from the information asymmetries paradigm in chapter 4. Here we consider alternative term structure models developed in the financial literature. The merits of such models are usually assessed via a statistical metric. The main criterion to validate one model is its ability to predict bond prices in line with the ones actually observed. Rather than focusing on such properties, we analyze term structure models as forecasting tools within a multi-period optimal portfolio problem. In fact we believe it is useful to understand what portfolio policy is prescribed by a term structure model in order to evaluate its practical applicability.

The three chapters contribute to the analysis of investment environment and process. With respect to the former aspect, we consider how stocks and bonds are traded in securities markets. The second and third chapter deal with markets where a single risky asset is sequentially traded in batch auctions. In chapter 4 we consider bond markets where transaction costs create frictions. As for the investment process, procedures for selecting investments under uncertainty are the basic ingredients to modern portfolio theory. We study how market participants should choose their investment, and when should it be made. In the second and third chapter we define trading strategies for informed traders and the market maker, while active bond portfolio management is dynamically characterized in the fourth chapter.

The appendices at the end of each chapter contain proofs of propositions, corollaries and lemmas. The outline of the thesis is as follows:

Chapter 2: A Market Microstructure Rationale for the S&P Game. We develop a dynamic trading game in which fundamental insiders coexist with non-fundamental speculators. The latter traders possess superior information about the future noise trades and are able to make sharper inference about the fundamental value with respect to the market maker. We show that non-fundamental speculators decrease market depth as well as the insider's ex-ante gains. We study inclusions in the S&P 500 as an example in which non-fundamental speculation may arise due to the preannouncement practice in index replacements. The evidence on trading activity and the bid-ask spread is consistent with our theoretical analysis.

Chapter 3: Information Sharing and Dynamic Trading. [joint with Antonio Mele] We develop a dynamic model with asymmetrically informed traders. Agents can engage in information sharing about the long-term value of an asset before trading, and the equilibrium outcome is affected by the amount of signals that are privately exchanged in the market. We show that in the presence of information sharing, the equilibrium price process is affected by the amount of information shared; trading activity and price volatility both cluster at the end of the trading period, and price informativeness is reduced. The previous effects are particularly severe, and generate high private incentives to share long-lived information. Our model predicts a rich variety of new patterns of liquidity, volume and return volatility.

Chapter 4: A Portfolio-Based Evaluation of Affine Term Structure Models. [joint with Andrea Beltratti] We use several multi-factor term structure models to produce forecasts for the future values of the state variables. Starting from the conditional moments of the state vector implied by the multi-factor term structure models we employ, we introduce binomial approximations to come up with discrete scenarios for the future state variables. We use returns predicted by these models as inputs for the portfolio optimization problem faced by an investor with a six month horizon, taking into account the possibility to rebalance after one quarter. The sequence of optimal portfolios is then evaluated in terms of financial properties. The results show that a financial based evaluation of term structure models may yield results conflicting with those obtained from a statistical evaluation.

2

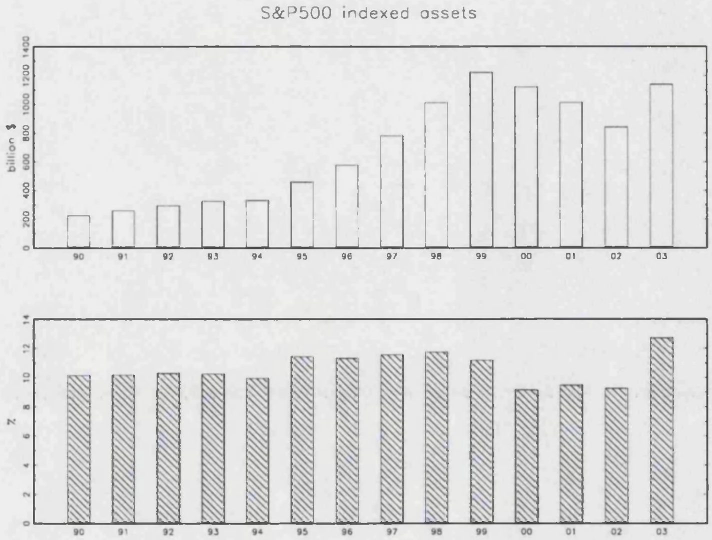
A Market Microstructure Rationale for the S&P Game

2.1 Introduction

The effect on stock prices induced by changes in the composition of broad market indexes has been addressed by many researchers. Most of the empirical work conducted so far focus on the Standard and Poor's 500.[1] There are several reasons behind the attention devoted to the S&P 500. First of all, both investors and institutions can easily trade stocks included in the S&P 500. At the end of 2003 more than $1 trillion were indexed directly or indirectly to the S&P 500, representing roughly 12% of the total index capitalization. As a result, index changes are followed by the financial community at large. Secondly, even though Standard and Poor's sets out several criteria for companies to be included in the index, changes to the S&P 500 roster entail some degree of subjectivity. Thus inclusions in the index are unpredictable and cannot be anticipated by the market as a whole. Finally, changes to the S&P 500 are publicly announced usually five business days before they become effective. Different intervals are occasionally used by S&P.

Each year Standard and Poor's publishes a list of the leading S&P 500 passive fund managers together with their assets under management. From the S&P annual survey of indexed assets (2003) it emerges that more that $1.1 trillion dollars were pegged to the S&P 500 at the end of 2003. This figure is possibly a conservative estimate, since Standard and Poor's claims that it captures approximately 90-95% of the total indexed assets in its survey. According to Blume and Edelen (2004) full replication and stratified sampling are the replicating strategies commonly implemented by S&P 500 indexers.

FIGURE 2.1. Assets indexed to the S&P 500.



End of the year net asset value for passive assets is obtained from the S&P annual survey (2003), while yearly S&P 500 capitalization is obtained from the S&P website http://www.standardandpoors.com. Top panel: S&P 500 indexed assets (billion USD); bottom panel: indexed assets NAV relative to S&P 500 capitalization (percentage).

Full replication requires holdings in all the 500 stocks in the exact proportion to their weights in the index at all times, while sampling strategies hold less that 500 stocks. Index replacements therefore represent a clear rebalancing opportunity for indexers: when a stock is added to the S&P 500, passive funds should buy it. In order to achieve full replication, indexers should replicate the weight it has in the S&P 500. Strategies based on sampling are likely to result in purchasing the included stock as well, even if the portfolio weight might differ from the one in the index. Pruitt and Wei (1989) find changes in institutional investors' holdings to be positively correlated with the abnormal returns experienced by additions to the S&P 500 over the period 1973-1986.

The appeal of passive techniques to investors has increased during the last two decades. In 1976, \$19 billion out of a total market value of \$662 billion were pegged to the S&P 500, which corresponds to 3% of the index capitalization [see Wurgler and Zhuravskaya (2002)]. Figure 2.1 presents the indexed assets over the period 1990-2003 as well as the passive industry weight relative to the whole S&P 500 market capitalization. There is no doubt that passive assets have grown over the last 15 years, and their weight relative to the index capitalization has risen to 12% at the end of 2003. Within the financial literature, there is general agreement that index changes result in a temporary demand shift represented by index funds' trading activity, which causes prices to increase for included stocks (and to a decrease for deleted companies). Shleifer (1986) does not find any significant price impact over a sample consisting of 144 additions during 1966-76, and relates this evidence to the small value of the S&P 500 owned by index funds –less than 0.5% in 1975. A similar explanation is given in Harris and Gurel (1986) for the evidence that prices for stocks added to the S&P 500 are not significantly affected over the period 1973-77. Along the same lines, Beneish and Gardner (1995) do not find any effect on the price and trading volume of newly included firms in the Dow Jones Industrial Average, and they point at the scarcity of funds pegged to the DJIA as the main reason for this. According to figure 2.1, inclusion in the S&P 500 during 2003 implies an additional demand due to indexers for about 12% of the outstanding shares. While the role of this demand shift is generally acknowledged in all the studies on list changes, researchers disagree on the temporary/permanent nature of the price impact as well as on the explanation for it [Chen, Noronha and Singal (2004) and Singal (2003) contain a detailed literature survey].

Until October 1989, Standard and Poor's announced the inclusion of a new stock in the S&P 500 after the close, the change becoming effective by the following open. After October 1989, Standard and Poor's switched to preannouncing changes in the S&P 500 usually five days before the inclusion. The aim of this new practice was to ease post-announcement order imbalances for companies added to the index. As documented in Beneish and Whaley (1996), Lynch and Mendenhall (1997), and more recently Blume and Edelen (2004), prices increase after the announcement but they do not immediately adjust to the level prevailing upon inclusion. This pattern clearly opens the way to profitable opportunities. In fact, Beneish and Whaley (1996) argue that indexers might enhance their returns buying earlier during the announcement period, thus making the entire price adjustment occur after annoucement. However, Blume and Edelen (2004) show that this early-trading strategy dampens passive managers' performance resulting in higher tracking errors. Looking at the volume pattern around index replacements, they conclude that half of the funds pegged to the S&P 500 submit their orders during the effective day of inclusion. Similarly, Beneish and Whaley (1996) find that prices tend to increase from the open to the close on the effective day over their 1989-1994 sample, supporting last day buying pressure by index funds. Moreover they document a temporary upward shift in the average trade size, which the authors relate to pegged funds waiting until the effective day to rebalance. These findings are consistent with daily tracking error being the driving criterion for indexers' performance evaluation.

While the announcement timing of index replacements does not affect passive managers' behaviour, it has relevant effects on other market participants. Under the old announcement practice, indexers would step into the market at the open immediately after the Standard and Poor's public announcement. It follows that there would not be profitable speculation unless the announcement is anticipated by some traders. However inclusions in the S&P 500 do not seem to be predictable due to the above mentioned Standard and Poor's discretionality in selecting stocks for the index, casting doubts on investors anticipating replacements. Singal (2003) provides anecdotal evidence on failures in predicting index changes. On the other hand the new preannouncement practice makes attractive front-running passive assets through the so-called 'S&P game': buy the included stock immediately after the announcement, and sell it at possibly higher prices after the indexers' demand is satisfied. Trading activity dynamics exhibit abnormal average volume following the announcement, which one can attribute to investors –rather than indexers– playing the S&P game. Beneish and Whaley (1996) show how such a strategy yields significant abnormal returns, even accounting for transaction costs. Blume and Edelen (2004) report a 19.2 basis point yearly return associated with the S&P game over their 1995-2000 sample. Early-trading profitability is also documented in Singal (2003) for inclusions between January and July 2002. These findings support the argument that 'an investor who requires that an indexer maintain tracking errors of just a few basis points a year is giving up additional returns. [...] Forgoing these additional returns can be viewed as an agency cost in delegating investment decisions' [Blume and Edelen (2004), p. 3].

Taking the empirical evidence mentioned above as a starting point, this paper contributes to the literature in several ways. In the first place we provide a modelling framework for index replacements: while several studies document returns and trading activity patterns around inclusions, on the theoretical side little work has been done. In Wurgler and Zhuravskaya (2002) demand shifts generate large stock price movements whenever stocks are not perfectly substitutes. Their model is static and, admittedly, cannot be applied to preannounced index changes. Within the market microstructure literature, several authors considered the value of anticipating uninformed trades such as passive funds' demand. Building up on Kyle (1985) some extensions have been proposed addressing this issue. Rochet and Vila (1994) develop a static game in which the insider is aware of both the final liquidation value and the noise traders' demand while submitting his (limit) order. With respect to the static Kyle (1985) equilibrium, they show that an informed investor trades less aggressively on his price signal and in the opposite direction of his volume signal, offsetting half of the uninformed trades. The aggregate order flow and market liquidity decrease, while prices as well as the insider's unconditional profits are unaltered.

The latter result seems to preclude any role for profitable speculation based on knowledge of uninformed trades. However in a dynamic setting this is no longer the case, as shown in Yu (1999). At every batch auction the insider's information set – in addition to the final liquidation value– comprises a noisy signal of the uninformed trades. Comparing the insider's expected profits arising from this model to the sequential auction equilibrium in Kyle (1985), it is shown that both the value of knowing (current) noise trades and market liquidity depend on the signal's precision. Our analysis is closely related to the two-period trading model in Madrigal (1996), where a (non-fundamental) speculator profits from privileged information on past uninformed trades he is endowed with. The author shows that this superior knowledge enables the speculator to make sharper forecasts of the final liquidation value with respect to the market maker. The profitability of strategies based on non-fundamental information is analyzed in Foucault and Lescourret (2003) as well.

In this chapter we explicitly model the preannouncement practice in S&P 500 replacements after October 1989 considering a market in which an insider coexists with a speculator who possesses superior information with respect to *future* uninformed trades, i.e. passive funds' entry at the effective inclusion date. The second contibution of our paper lies in the empirical evidence we provide. We analyze trading volume and bid-ask spreads around index additions between 1989 and 1999. While volume patterns have been extensively documented (and our findings are in line with the existing literature), spread dynamics have received little attention. Edmister, Graham and Pirie (1996) and Erwin and Miller (1998) document improved liquidity, i.e. tighter bid-ask spreads, after inclusion. However both works consider additions before October 1989, thus offering no grounds for studying the S&P game. To our knowledge spreads under the S&P preannouncing policy are analyzed in Beneish and Whaley (1996) only. The authors report a significant spread decrease on the day following the inclusion. On the other hand we find that index additions worsen liquidity. One possible reason for this contrasting evidence is the different sample, since we consider 108 inclusions whereas Beneish and Whaley (1996) deal with 30 companies added to the S&P 500.

The outline of this chapter is as follows. The benchmark model is presented in section 2.2 where the equilibrium in the absence of the speculator is analyzed. Section 2.3 explicitly introduces a role for non-fundamental speculation based on strategies like the S&P game. We show that front-running index funds is indeed profitable, and results in higher volume and lower liquidity. Section 2.4 discusses testable implications arising from the theoretical model, while section 2.5 presents the empirical evidence on S&P 500 inclusions. Finally section 2.6 concludes.

2.2 Passive funds and index replacements

2.2.1 Model setup

2.2.1.1 Asset markets and changes announcements

We develop a two period sequential trading game along the lines of Kyle (1985). Trading takes place at two dates $t = 1, 2$ and the market operates as a batch auction. There are two traded assets: a riskfree asset whose net payoff is normalized to zero, and a risky asset with final liquidation value $f \sim N\left(p_0, \sigma^2_{f,0}\right)$ whose realization occurs after the second trading round. The trading dates capture the timing in index replacements as follows. Before trading takes place at $t = 1$ the authorities announce the change in the index composition. Further to the stock(s) added to/removed from the index, it is announced that the change is effective after the second trading round.

2.2.1.2 Agents

There are three types of agents in the market: an insider, a market-maker and noise (or uninformed) traders. Both the insider and the market maker are risk neutral. At each date the trading process is modeled as a two-stage game: in the first stage the insider and the uninformed traders submit their orders to the market maker; in the second stage the market maker determines the price at which the market is cleared. The insider submits his orders $\{x_t\}_{t=1,2}$ at both dates. The noise in the market comes from two different sources: liquidity traders and passive funds. The main difference between these two groups is that passive funds enter the market at date 2 only, while liquidity traders submit their orders $\{u_t\}_{t=1,2}$ at both dates. More specifically we assume that $u_1 \sim N(0, \sigma^2_{u_1})$ and $u_2 \sim N(0, \sigma^2_{u_2})$, with (f, u_1, u_2) mutually independent. Further to the liquidity traders there are indexers active at the second trading round. Passive trades are denoted by $z_2 \sim N(\bar{z}_0, \sigma^2_{z,0})$ and are orthogonal to the other random variables f, u_1 and u_2.[2] The joint distribution of (f, u_1, u_2, z_2) is common knowledge among market participants before the game starts. The date t aggregate order flow $\{\omega_t\}_{t=1,2}$ is given by $\omega_1 \equiv x_1 + u_1$ and $\omega_2 \equiv x_2 + u_2 + z_2$ respectively.

The noise trading specification slightly departs from the standard assumptions, and the way we model z_2 aims at capturing several aspects in passive managers' behaviour. First of all we allow liquidity trades' variance $\{\sigma^2_{u_t}\}_{t=1,2}$ to vary over time. Later on we compare equilibrium parameters under different market conditions, and use this flexibility in order to keep the overall uninformed variance constant through time. Secondly, replicating strategies are not based on any information related to the asset fundamental value. As mentioned in the Introduction, every time the index composition changes, passive managers should rebalance their portfolios. As such indexers can be regarded as uninformed traders submitting orders due to changes in the benchmark they replicate. In the third place, pegged funds' performance is assessed via tracking error procedures, and in our model the index replacement is effective after date 2. Optimizing the fund's performance (relative to the index) on a daily basis thus leads passive managers to rebalance on the inclusion day rather than immediately after the announcement, i.e. at date 2 rather than at date 1 in our model, consistently with the evidence in Beneish and Whaley (1996) and Blume and Edelen (2004). Eventually, we consider a shift in expected uninformed trades between the two dates via the term $\bar{z}_0$, which reflects passive funds stepping into the market at the second round. In general we relate the magnitude of this shift to the weight pegged funds have relative to other liquidity traders.

2.2.1.3 Information structure

Within our strategic trading setup –as well as in the various extensions to Kyle (1985)– uncertainty among market participants is captured by two random variables: the final liquidation value and uninformed trades. We therefore distinguish the information related to these variables as fundamental and non-fundamental respectively, along the same lines of Madrigal (1996). Let Φ^I_t and Φ^M_t denote the insider's and market maker's information set at time t. At each trading round the market maker observes the ag-

TABLE 2.1. Information structure.

	date 1		date 2	
	info set Φ_1^i	strategy	info set Φ_2^i	strategy
insider (I)	f, u_1	$x_1(f, u_1)$	$\Phi_1^I \cup u_2, \omega_1$	$x_2(f, u_2)$
market maker (M)	ω_1	$p_1(\omega_1)$	$\Phi_1^M \cup \omega_2$	$p_2(\omega_1, \omega_2)$
insider (I)	f, u_1	$x_1(f, u_1, E(s\vert\Phi_1^I))$	$\Phi_1^I \cup u_2, s, v, \omega_1$	$x_2(f, u_2, s, v)$
speculator (S)	v	$y_1(v)$	$\Phi_1^S \cup s, \omega_1$	$y_2(s, v)$
market maker (M)	ω_1	$p_1(\omega_1)$	$\Phi_1^M \cup \omega_2$	$p_2(\omega_1, \omega_2)$

Information sets and strategies for market participants. Top panel: game without non-fundamental speculation (see section 2.2); bottom panel: game with non-fundamental speculation (see section 2.3).

gregate order flow, such that $\Phi_t^M = \{\omega_s, s \leq t\}$. The price in period t is assumed to satisfy the semi-strong efficiency condition:[3]

$$p_t = E\left(f \mid \Phi_t^M\right) \quad , \quad t = 1, 2 \tag{2.1}$$

After each trading round, the price becomes common knowledge among market participants. The insider possesses superior information regarding both the asset's fundamental value and other non-fundamental aspects of the market. The insider is aware of the final liquidation value before the trading game starts.[4] Further to this fundamental information, the insider knows the quantity submitted by liquidity traders –but not passive funds– at date t before filling his order x_t, i.e. $\Phi_1^I = \{f, u_1\}$ and $\Phi_2^I = \Phi_1^I \cup \{p_1, u_2\}$. Thus the insider possesses long-lived fundamental information as well as short-lived non-fundamental information. The information structure is summarized in table 2.1.

Our information structure departs from the existing literature in the following aspects. As in Foster and Viswanathan (1994) and Kyle (1985) the insider is endowed with long-lived information on the final payoff f. Further, in our game the insider is also aware of the contemporaneous liquidity trades, thus making our setup closer to Rochet and Vila (1994) and Yu (1999). In the absence of date 2 pegged trades our trading game reduces to a two-period version of Rochet and Vila (1994) or, equivalently, to the game in Yu (1999) with non-distorted information on uninformed trades. However the entry of passive funds moves the insider away from complete knowledge about noise trades at the second trading round. Therefore our specification resembles a two-period version of Yu (1999) with time-varying quality of the insider's signal about uninformed trades.

2.2.2 Equilibrium construction and description

We focus on linear equilibria for our trading game. For the insider we denote the period t profit by $\{\pi_t^I = \pi_t^I(\omega_s, s \leq t)\}$, i.e. $\pi_1^I = x_1(f - p_1(\omega_1))$ and $\pi_2^I = x_2(f - p_2(\omega_1, \omega_2))$. A Bayes-Nash equilibrium (BNE) is defined by a set of linear functions $\{x_t(\cdot), p_t(\cdot)\}_{t=1,2}$ such that the following conditions hold:

1. *insider's profit maximization*: the insider chooses x_1 to maximize total profits

$$E\left[\pi_1^I(\omega_1) + \pi_2^I(\omega_1, \omega_2) \middle| \Phi_1^I\right], \tag{2.2}$$

given that x_2 maximizes second period profits

$$E\left[\pi_2^I(\omega_1, \omega_2) \middle| \Phi_2^I\right]. \tag{2.3}$$

2. *market efficiency*: the market maker sets prices according to equation (2.1), i.e.

$$p_1 = E\left(f \middle| \Phi_1^M\right) \tag{2.4}$$

$$p_2 = E\left(f \middle| \Phi_2^M\right) \tag{2.5}$$

Proposition 1 *Let the following conditions hold:*

$$a_1 = \frac{2\lambda_2 - \lambda_1}{\lambda_1(4\lambda_2 - \lambda_1)} \quad ; \quad b_1 = a_1\lambda_1 \quad ; \quad \lambda_1 = \frac{a_1\sigma_{f,0}^2}{a_1^2\sigma_{f,0}^2 + (1 - b_1)^2\sigma_{u_1}^2}$$

$$a_2 = \frac{1}{2\lambda_2} \quad ; \quad \lambda_2 = \frac{\sigma_{f,1}}{\left(\sigma_{u_2}^2 + 4\sigma_{z,0}^2\right)^{1/2}}$$

where $\sigma_{f,1}$ is the fundamental value residual variance after the first trading round. Then there exists a linear BNE in which strategies and prices are of the form

$$x_1 = a_1(f - p_0) - b_1u_1 \tag{2.6}$$

$$x_2 = a_2(f - p_1) - u_2/2 \tag{2.7}$$

$$p_1 = p_0 + \lambda_1\omega_1 \tag{2.8}$$

$$p_2 = p_1 + \lambda_2(\omega_2 - \bar{z}_0) \tag{2.9}$$

Furthermore, if the following condition holds

$$\lambda_1 (4\lambda_2 - \lambda_1) > 0 \tag{2.10}$$

the equilibrium is unique.

The equilibrium strategies in Proposition 1 have the following interpretation. Before trading takes place at date 1, the market maker's forecast of the random variables (f, u_1) coincides with the unconditional means $(p_0, 0)$. Thus at time 1 the insider trades on the market maker's misperception of the final liquidation value $(f - p_0)$, and current liquidity trades u_1. The insider places a positive weight $(a_1 > 0)$ and a negative one $(-b_1 < 0)$ respectively on the former and the latter forecast error. After the first trading round, the market maker updates his beliefs about the liquidation value to p_1. Since the first period aggregate order flow does not contain any information about z_2, the market maker doesn't learn anything about passive trades. As a consequence, date 2 passive funds' conditional mean coincides with its unconditional counterpart $\bar{z}_0$. In section 2.3 we discuss how non-fundamental speculation arising from the S&P game modifies the latter feature. Thus at date 2 the insider trades on the market maker's misperception of the final liquidation value $(f - p_1)$, and current liquidity trades u_2. The weights on the market maker's errors are consistent with the ones prevailing during the first trading round: positive on $(f - p_1)$ –since $a_2 > 0$–, and negative on u_2. The insider's trading intensities in eqs. (2.6) and (2.7) are consistent with the previous literature: date t trading aggressiveness on the fundamental information –as captured by a_1 and a_2– is positive like in Kyle (1985). Moreover at every batch auction the insider trades against current uninformed orders like in Yu (1999), given that the intensities on u_t are negative. Finally at date 2 the insider offsets half of the (contemporaneous) liquidity trades as in Rochet and Vila (1994). Equilibrium prices have the usual linear form with λ_t capturing the price response induced by unit-size changes in the aggregate order flow. Equivalently, $1/\lambda_t$ is the date t market depth (or liquidity[5]): large values for λ_t imply that prices are extremely sensitive to changes in the order flow, which occurs in illiquid markets. Finally, we define the fundamental value residual variance as $\sigma^2_{f,t} = var\left(f|\Phi^M_t\right)$, i.e. the final payoff variance after t rounds of trading. $\left\{1/\sigma^2_{f,t}\right\}_{t=1,2}$ therefore gives the speed at which private information about f is revealed to the market, and it can be thought of as measuring market efficiency.

The equilibrium for our trading game is investigated in figures 2.2–2.6. We normalize the initial fundamental volatility setting $\sigma_{f,0}^2 = 1$, and consider uninformed trades' uncertainty at both dates to be equal to the fundamental variance, i.e. $\sigma_{u_1}^2 = 1$ and $\sigma_{u_2}^2 + \sigma_{z,0}^2 = 1$. Further we define $k_I \equiv \sigma_{u_2}^2/\sigma_{z,0}^2$, and refer to k_I as the quality of the insider's non-fundamental information (or equivalently the insider's informational advantage[6]). In fact $\sigma_{u_2}^2 = \sigma_{u_2}^2 \left(\sigma_{u_2}^2 + \sigma_{z,0}^2\right)^{-1}$ can be interpreted as the share of date 2 uninformed orders channeled by the insider to the market maker. Therefore k_I denotes the insider's informational advantage (relative to the market maker) with respect to date 2 noise trades: high values for k_I correspond to small passive funds' volatility, which in turn implies that $\sigma_{u_2}^2$ captures most of the noise trading volatility at date 2. For example if $k_I = 1$, the variance of the uninformed trades observed by the insider is half of the entire noise trading variance faced by the market maker at date 2. We consider several values[7] for k_I and plot the parameters in Proposition 1 in figures 2.2–2.6 (solid line). The dashed line corresponds to a two-period Rochet and Vila (1994) trading game (henceforth RV) in which passive funds are absent at date 2, such that the insider is aware of current liquidity trades at both dates. Clearly, our trading game resembles RV when k_I is large, or equivalently when $\sigma_{z,0}^2$ is negligible relative to $\sigma_{u_2}^2$.[8]

We plot insider's intensities a_1, b_1 and a_2 in figure 2.2, while values for λ_1 and λ_2 are reported in figure 2.3. Since λ_t measures the adverse selection costs faced by the market maker at round t, it is not surprising that λ_2 increases in the insider's advantage k_I (figure 2.3-panel B). Recall from the equilibrium strategy (2.7) that the second period trading intensity on current liquidity orders does not depend on k_I, and is equal to $-1/2$ as in RV and Yu(1999). Therefore at date 2 the insider's advantage $\sigma_{u_2}^2/\sigma_{z,0}^2$ affects the trading aggressiveness a_2 only, which is shown to be decreasing in k_I (figure 2.2-panel C). This is due to the mentioned finding that date 2 liquidity –as measured by $1/\lambda_2$– decreases with k_I.

Turning to date 1 parameters, we note that both the trading intensities a_1 and b_1 increase in k_I (figure 2.2-panel A and B respectively). The bottom line of figure 2.2 is that the insider increases his trading intensity with respect to *both* sources of information together with his informational advantage. This means that the insider incorporates more information on both f and u_1 in his trade x_1 as k_I increases: since the insider anticipates the negative relationship between k_I and date 2 liquidity, he increases his aggressiveness with the information quality during the first trading round.

FIGURE 2.2. Insider intensities.

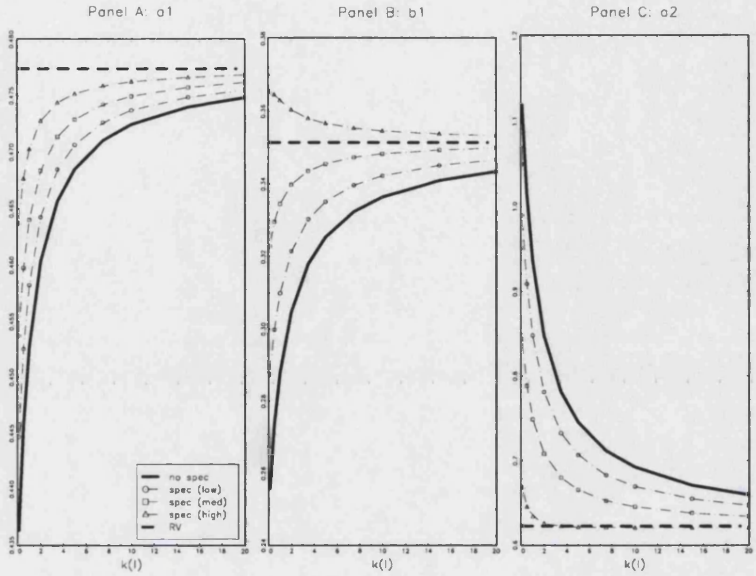
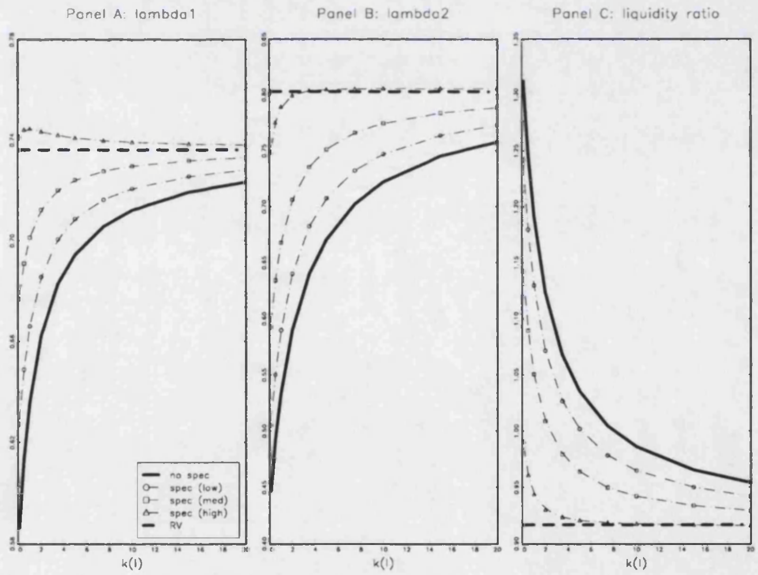


FIGURE 2.3. Market liquidity.



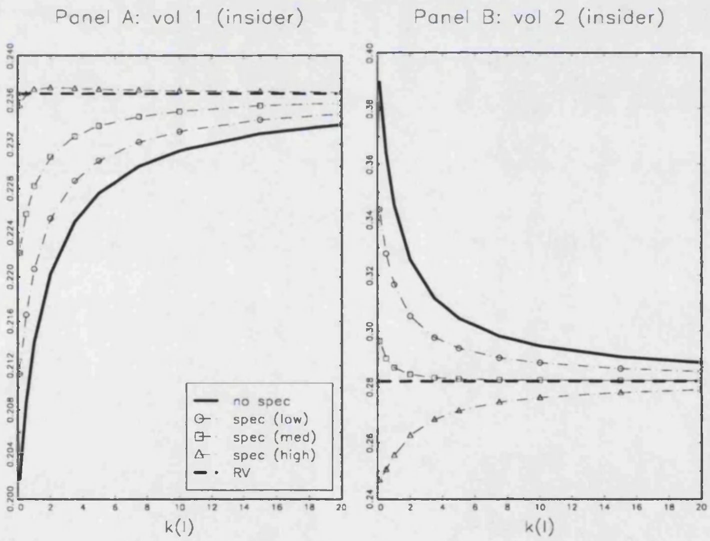
The market maker's reaction is to make date 1 liquidity decreasing with k_I as well (figure 2.3-panel A).

Following Admati and Pfleiderer (1988), we decompose date t trading volume into its components. The contribution of the insider to the expected total volume is therefore given by (see section 2.4 and appendix A for further details[9])

$$V_t^I \equiv \sqrt{\frac{var(x_t)}{2\pi}}, \quad t = 1, 2$$

We plot V_1^I and V_2^I in figure 2.4 (panel A and B respectively). Consistently with the previous analysis for the trading intensities, V_1^I increases (resp. V_2^I decreases) with the informational advantage. The residual variances $\sigma_{f,1}^2$ and $\sigma_{f,2}^2$ are depicted in figure 2.5 (panel A and B respectively), as well as the ratio $\sigma_{f,1}^2/\sigma_{f,2}^2 = \left(\frac{1/\sigma_{f,2}^2}{1/\sigma_{f,1}^2}\right)$ which captures the market efficiency dynamics through time (panel C). As shown in appendix A, $\sigma_{f,t}^2$ is negatively related to the insider intensity a_t and the price sensitivity λ_t. Therefore, date 1 efficiency $1/\sigma_{f,1}^2$ increases in k_I, since both a_1 and λ_1 increase with the informational advantage. Furthermore $\sigma_{f,1}^2/\sigma_{f,2}^2$ does not depend on k_I, which means that the positive relation between k_I and date 2 efficiency in panel B is entirely due to the increase in date 1 market efficiency.

FIGURE 2.4. Insider volume.



The insider's unconditional expected profits are depicted in figure 2.6 (panel A). Unlike other variables, ex-ante gains are non-monotonic in k_I. At a first sight this might seem surprising, as one would expect insider's profits to increase together with the information quality $\sigma_{u2}^2/\sigma_{z,0}^2$. On the other hand figure 2.6 suggests that the insider is (ex-ante) worse off with more precise information whenever k_I is below some threshold value (in figure 2.6 the minimum value is 0.8527 corresponding to $k_I = 1.4$). Yu (1999)

FIGURE 2.5. Price informativeness.

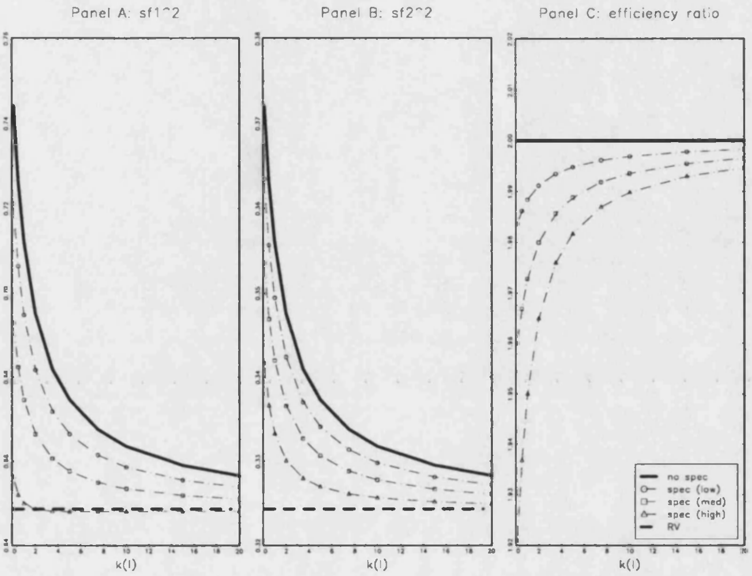
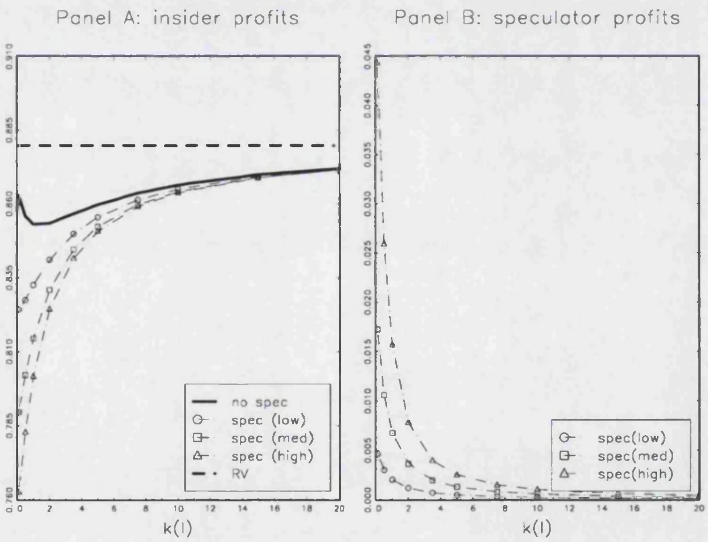


FIGURE 2.6. Profits.



(figure 2, p. 92) documents a similar behaviour and notes that the insider is not necessarily better off with more precise non-fundamental information. Furthermore he shows that a U-shaped curve for ex-ante gains is more likely to emerge when the number of batch auctions is small, as in our model. Therefore the pattern in figure 2.6 is in line with results in Yu (1999). Comparison between figures 2.3 and 2.6 suggests that the insider expects to lose out to a lower date 1 market depth as his information becomes more precise until a threshold level: within this region, liquidity decreases very rapidly with k_I such that profits decrease with the informational advantage.

2.3 Non-fundamental speculation and the S&P game

2.3.1 Model setup

In what follows we explicitly introduce a role for purely non-fundamental speculation (the S&P game) within the setup outlined in section 2.2. As explained in the Introduction, what lies behind this speculative opportunity is privileged information about passive assets, together with index changes preannouncement. For what is not mentioned in this subsection, we maintain the assumptions in subsection 2.2.1.

2.3.1.1 Agents

We introduce another risk-neutral informed trader, the (non-fundamental) speculator. While the insider receives both fundamental and non-fundamental information, the speculator is endowed with non-fundamental information only, as specified later in this subsection. At both trading dates the speculator submits orders $\{y_t\}_{t=1,2}$ to the market maker. The aggregate order flow therefore becomes $\omega_1 \equiv x_1 + y_1 + u_1$ and $\omega_2 \equiv x_2 + y_2 + u_2 + z_2$.

2.3.1.2 Information structure

Let Φ_t^S denote the speculator's information set at time t. The speculator knows the demand submitted by a subset of passive funds before the game starts. As a consequence we decompose the passive industry demand z_2 into two components v and w: the former aggregates the trades known by the speculator, while the latter groups the demand submitted by other passive funds:

$$z_2 = v + w \sim N\left(\bar{v}_0 + \bar{w}, \sigma_{v,0}^2 + \sigma_w^2\right)$$

where $\bar{v}_0 = E(v)$, $\bar{w} = \bar{z}_0 - \bar{v}_0$, $\sigma_{v,0}^2 = var(v)$ and $\sigma_w^2 = \sigma_{z,0}^2 - \sigma_{v,0}^2$. The speculator's information sets are given by $\Phi_1^S = \{v\}$ and $\Phi_2^S = \Phi_1^S \cup \{p_1\}$, implying that the speculator is endowed with (long-lived) non-fundamental information.

The focus on the role of purely non-fundamental speculation [not considered in Kyle (1985), Foster and Viswanathan (1994), RV and Yu (1999)] closely resembles the analysis in Madrigal (1996). However we depart from Madrigal (1996) in several aspects. First of all the speculator is endowed with superior knowledge about a fraction of *future* –rather than past– uninformed trades. It follows that in our model the speculator exploits his advantage trading at both dates, while in Madrigal (1996) he enters the picture at date 2 only. In the second place the speculator acts as a *monopolist* on his privileged information in the first trading round, and competes with the insider at the date 2, whereas Madrigal (1996) focuses on the latter feature only.

Most of the literature on asymmetries in financial markets is concerned with fundamental information, and knowledge about the final payoff is widely accepted as arising from analysts' research activity as well as confidential discussions. On the other hand informational advantages on uninformed orders can be traced to brokers engaging in proprietary –or dual– trading. Brokers both execute trades on behalf of their (liquidity) customers and fill in orders on their own account. As a consequence, brokers can engage in dual-trading based on the ability to observe their clients' orders. In Madrigal (1996) the speculator channels liquidity orders in the first round and then uses this information (together with the price set by the market maker) to forecast the final liquidation value.[10] Similarly in Foucault and Lescourret (2003) the speculator is not endowed with fundamental information, but he observes contemporaneous liquidity trades before submitting his order. This leaves open the question as how our speculator gathers *more precise* information about indexed assets *ahead* of other market participants. As a matter of fact one might object that preannouncing index changes conveys information to the whole market about passive funds' entry at the inclusion. For example one might use publicly available data on pegged funds capitalization [like the S&P survey (2003)] and infer the realization of z_2. However this estimate would be accurate only in case passive funds track the index via full replication, i.e. buy all the stocks in the index and in the same proportion, and if funds do not experience inflows and outflows during the year –which is rather unlikely.[11] Even though in principle full replication allows to track the index very closely, it entails substantial administrative costs due to the number of stocks to be bought/sold and, consequently, the number of dividends to be handled. Given that these costs might dampen passive funds' performance and result in larger tracking errors, indexers can resort to other strategies such as stratified sampling or optimization techniques. Based on the Morningstar database, Blume and Edelen (2004) report that the vast majority of funds indexed to the S&P

500 hold roughly all the stocks included in the index. However, as the authors suggest, this does not necessarily imply that all the funds implement full replication techniques. For instance Blume and Edelen (2004) argue that the increase in the tracking error for the Vanguard 500 Index Fund –one of the largest passive funds– after 1998 is inconsistent with full replication. This example suggests that knowledge about the tracking procedures actually implemented by individual passive managers is inherently difficult to gather, and as a consequence the realization of z_2 cannot be regarded as public information. Thus we consider public data on passive industry capitalization as providing the *expected* passive funds' orders $\bar{z}_0 = \bar{v}_0 + \bar{w}$ to the whole market, and reasonably conceive that some traders are endowed with superior information about z_2. For example, a broker might learn something about the replication technique implemented by a given fund manager because he previously executed his trades. Alternatively, an indexer can direct his order to a broker under the agreement that execution occurs at a specified future date. Both these cases would generate non-fundamental informational advantages consistent with our speculator's information sets. Confidential discussions with passive fund managers would fit into the same specification and result in long-lived information on future uninformed trades as well.

As a consequence of these assumptions, our trading game inherits several interesting features. When trading at date 1 both the insider and the speculator impound their information into orders x_1 and y_1. Time 1 noise trades u_1 keep the aggregate order flow away from fully revealing both the insider's information (f, u_1) as well as the speculator's information v. After observing the aggregate order flow ω_1, the market maker forms an estimate $\bar{v}_1$ of future passive trades:

$$\bar{v}_1 = E\left(v | \Phi_1^M\right) \tag{2.11}$$

Note that our trading game allows the market maker to update his beliefs on (a fraction of) the second period uninformed trades as well as on the final liquidation value – through the price p_1– and to use these updates when setting the market clearing price at date 2. The existing literature concentrates on the market maker's inference on the final payoff only: posteriors on noise trades are not considered, since informed agents are endowed with signals on either current or past uninformed orders. In Yu (1999) the insider receives at each date t (a signal of) contemporaneous noise trades. Nonetheless the independence through time of liquidity-motivated orders prevents the market maker from extracting any signal on time $t+1$ noise trading based on the order flow received at time t. A similar argument holds for both Foucault and Lescourret (2003) and Madrigal (1996).

The information on v, together with the price realization p_1, allows the speculator to form a superior estimate of f relative to the market maker. After the first trading round, the speculator nets out the insider's and liquidity traders' demand out of the aggregate order flow –due to price linearity in ω_1– and extracts a signal s of the fundamental value that is more precise than the market maker's expectation:

$$s = E\left(f \mid \Phi_2^S\right) = E\left(f \mid x_1 + u_1\right) \tag{2.12}$$

Therefore the speculator can profit on the difference $(s - p_1)$ because noise trades in period 2 will prevent the order flow from revealing the speculator's information. The insider reacts to the speculator's presence incorporating an estimate of s when trading in the first round. A similar signal extraction problem and the incentives for the insider to manipulate the first period price are analyzed in Madrigal (1996).

On the other hand the insider infers the speculator's information about v after observing the first period price. Note, however, that the information structure enables the insider to know the realization (of a fraction) of passive funds' trades v, while the speculator extracts only a signal s of the fundamental value f. As such our model displays a hierarchical information structure[12] during the second trading round. Borrowing the terminology in Foster and Viswanathan (1994) the insider is the 'better informed trader' and the speculator is the 'lesser informed trader' at date 2.[13] The information structure is summarized in table 2.1.

2.3.2 Equilibrium construction and description

Let date t speculator's profits be defined along the same lines as in section 2.2, i.e. $\pi_1^S = y_1\left(f - p_1\left(\omega_1\right)\right)$ and $\pi_2^S = y_2\left(f - p_2\left(\omega_1, \omega_2\right)\right)$. A BNE for our trading game is given by a set of linear functions $\left\{x_t\left(\cdot\right), y_t\left(\cdot\right), p_t\left(\cdot\right)\right\}_{t=1,2}$ satisfying the insider's profit maximization [see conditions (2.2, 2.3)], market efficiency [see conditions (2.4, 2.5)] and the following:

speculator's profit maximization: the speculator chooses y_1 to maximize total profits

$$E\left[\pi_1^S\left(\omega_1\right) + \pi_2^S\left(\omega_1, \omega_2\right) \mid \Phi_1^S\right] \ , \tag{2.13}$$

given that y_2 maximizes second period profits

$$E\left[\pi_2^S(\omega_1,\omega_2)\middle|\,\Phi_2^S\right]\ . \tag{2.14}$$

Requirements (2.13, 2.14) amount to look for a pair of linear functions $y_1(\cdot)$ and $y_2(\cdot)$ such that $y_1 = y_1(v)$ and $y_2 = y_2(s,v)$, where s is defined in eq. (2.12). Recall that within our informational structure the insider knows -prior to trading at time 2- the signal s that the speculator extracts from p_1. As a consequence, when trading at date 1, the insider keeps into account the effect of his order on the speculator's estimate of the final liquidation value $E\left(s|\,\Phi_1^I\right) = E\left(E\left(f|\,\Phi_2^S\right)|\,\Phi_1^I\right)$. The insider's trading strategies are therefore given by a pair of linear functions $x_1(\cdot)$ and $x_2(\cdot)$ such that $x_1 = x_1\left(f, u_1, E\left(s|\,\Phi_1^I\right)\right)$ and $x_2 = x_2(f, u_2, s, v)$. At $t=1$ the insider trades on the speculator's (expected) mispricing, i.e. the difference between $E\left(s|\,\Phi_1^I\right)$ and the true liquidation value, as well as on the market maker's mispricing, i.e. the difference between the realization f and p_0. This amounts to conjecture the following form for x_1:

$$x_1 = \alpha(f-p_0) + \beta u_1 + \gamma\left(E\left(s|\,\Phi_1^I\right) - f\right) \tag{2.15}$$

Note that the insider's date 1 trade depends on the (estimate) of the speculator's conjecture of the final liquidation value, which depends itself on the insider's first period trade. Thus one needs to solve for $E\left(s|\,\Phi_1^I\right)$ and then verify the consistency between the resulting expression for x_1 and the speculator's belief s.[14]

Proposition 2 *Let the following conditions hold:*

$$a_1 = d^{-1}\left(1 - \frac{2\lambda_1 + \phi - 2\lambda_2\mu}{6\lambda_2}\right) \quad ; \quad b_1 = 1 - d^{-1}\lambda_1 \quad ; \quad d = 2\lambda_1 - \frac{(2\lambda_1 + \phi - 2\lambda_2\mu)^2}{18\lambda_2}$$

$$C_1 = D^{-1}\left(\frac{\lambda_1 - \lambda_2\mu}{9}\right) \quad ; \quad D = \lambda_1 - \frac{(\lambda_1 - \lambda_2\mu)^2}{9\lambda_2}$$

$$a_2 = \frac{1}{2\lambda_2} \quad ; \quad C_2 = \frac{2a_2}{3}$$

$$\lambda_1 = \frac{a_1\sigma_{f,0}^2}{a_1^2\sigma_{f,0}^2 + (1-b_1)^2\sigma_{u_1}^2 + C_1^2\sigma_{v,0}^2}$$

$$\lambda_2 = \Sigma^{-1}\left(-2\sigma_{fv,1} + \left[4\sigma_{fv,1}^2 + \left(9\sigma_{f,1}^2 - \sigma_{s,1}^2\right)\Sigma\right]^{1/2}\right) \quad ; \quad \Sigma = 9\sigma_{u_2}^2 + 36\sigma_w^2 + 4\sigma_{v,1}^2$$

where $\sigma^2_{f,1}$, $\sigma^2_{s,1}$ and $\sigma_{fv,1}$ are residual variances after the first trading round (defined in appendix B). Then there exists a linear BNE in which trading strategies and prices are of the form

$$\begin{aligned}
x_1 &= a_1(f-p_0) - b_1 u_1 && (2.16)\\
x_2 &= a_2(f-p_1) - u_2/2 - C_2(s-p_1)/2 - (v-\bar{v}_1)/3 && (2.17)\\
y_1 &= C_1(v-\bar{v}_0) && (2.18)\\
y_2 &= C_2(s-p_1) - (v-\bar{v}_1)/3 && (2.19)\\
p_1 &= p_0 + \lambda_1\omega_1 && (2.20)\\
p_2 &= p_1 + \lambda_2(\omega_2 - \bar{z}_1) && (2.21)
\end{aligned}$$

where s is the speculator's belief on f conditional on Φ_2^S, and $\bar{z}_1 = \bar{v}_1 + \bar{w}$ is the market maker's belief on v conditional on Φ_1^M

$$s = p_0 + \phi(x_1 + u_1) \tag{2.22}$$

$$\bar{v}_1 = \bar{v}_0 + \mu(x_1 + y_1 + u_1) \tag{2.23}$$

The updating coefficients in (2.22),(2.23) are given by

$$\phi = \frac{a_1\sigma^2_{f,0}}{a_1^2\sigma^2_{f,0} + (1-b_1)\sigma^2_{u_1}} \tag{2.24}$$

$$\mu = \frac{C_1\sigma^2_{f,0}}{a_1^2\sigma^2_{f,0} + (1-b_1)\sigma^2_{u_1} + C_1\sigma^2_{f,0}} \tag{2.25}$$

Furthermore, if the following condition holds

$$2\lambda_1 - \frac{(2\lambda_1 + \phi - 2\lambda_2\mu)^2}{18\lambda_2} > 0 \tag{2.26}$$

the equilibrium is unique.

The equilibrium strategies in Proposition 2 have the following interpretation. Before trading takes place at date 1, the market maker's forecast of the random variables (f, u_1, z_2) coincides with their unconditional mean $(p_0, 0, \bar{z}_0)$. Thus at time 1 the insider trades on (1) the market maker's misperception of the fundamental value $(f - p_0)$ as well as current liquidity trades u_1, and (2) the speculator's (expected) forecast error $E\left(s|\,\Phi_1^I\right) - f$. Therefore the presence of the speculator results in the insider manipulating his trades (relative to the equilibrium in Proposition 1) as a reaction to the speculator's extraction of the signal s. On the other hand the speculator trades on his informational advantage –with respect to both the market maker and the insider– $(v - \bar{v}_0)$. After observing ω_1, the market maker forms the posterior $\bar{v}_1$ according to (2.11), such that the passive industry conditional mean is given by $\bar{z}_1 = \bar{v}_1 + \bar{w}$. Therefore, after the first trading round, the market maker's update on (f, z_2) is given by $(p_1, \bar{z}_1)$.[15] A similar argument to the one used for first period orders ensures that at date 2 the insider trades on (1) the market maker's misperception of the final liquidation value $(f - p_1)$ as well as current liquidity trades u_2 and (2) the speculator's forecast error $(s - f)$. The linearity assumption further implies that the insider trades on the market maker's misperception of the passive orders $(v - \bar{v}_1)$. Similarly, the speculator trades on his informational advantage (with respect to the market maker only) captured by the terms $(v - \bar{v}_1)$ and $(s - p_1)$.

As mentioned in the Introduction, the S&P game consists in front-running index funds. In our trading framework, the S&P game would translate into the speculator buying at date 1 whenever the market underestimates the realization of v, and subsequently selling at date 2. Conversely, the speculator should sell at date 1 whenever the market overestimates pegged trades, i.e. $v < \bar{v}_0$, and buy back at date 2. Since the speculator trades against $(v - \bar{v}_1)$ [see equation (2.19)] during the second trading round, the occurrence of the S&P game depends on the sign of the coefficient C_1 in (2.18). In fact, we show the following:

Corollary 1 *In equilibrium the speculator plays the S&P game, i.e.* $C_1 > 0$.

The effect of the speculator on equilibrium parameters is analyzed in figures 2.2–2.9. Along the same lines we used in section 2.2 for the insider's non-fundamental advantage k_I, we let $k_S \equiv \sigma_{v,0}^2/\sigma_w^2$ be the precision of the speculator's information. Higher values for k_S imply that the speculator is able to make sharper inference relative to the rest of the market. We set k_S equal to 0.33, 1 and 3, and refer to these three cases as low, medium and high (speculator's) informational advantage respectively. In figures 2.2–2.9 we plot equilibrium values corresponding to $k_S = 0.33$, 1 and 3 with circles, squares and triangles respectively. The other underlying parameters are set accordingly to section 2.2.

FIGURE 2.7. Speculator intensities.

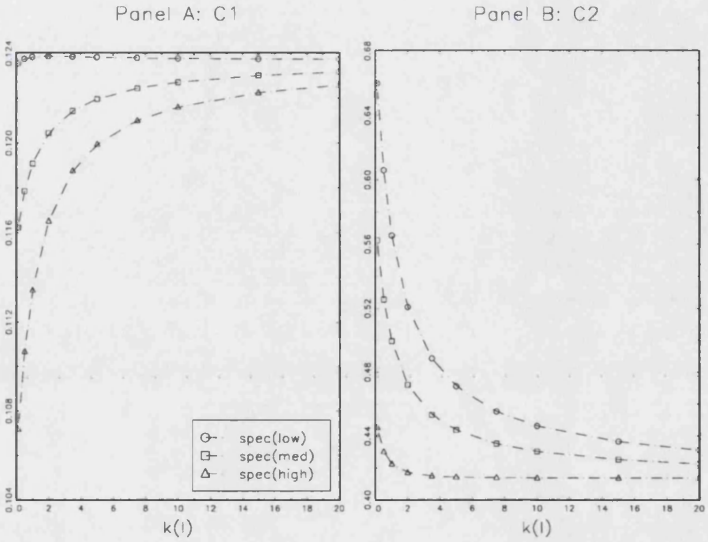
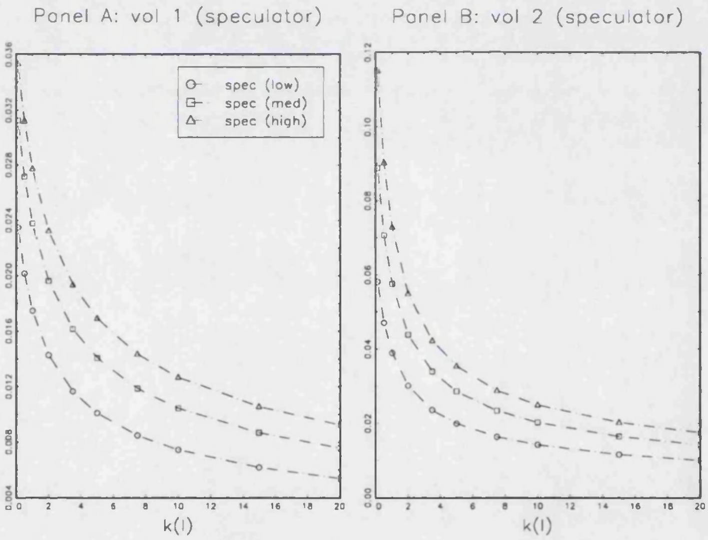


FIGURE 2.8. Speculator volume.



Trade aggressiveness and market liquidity

When trading at date 1, the speculator places a positive weight ($C_1 > 0$ from Corollary 1) on the market maker's initial forecast error ($v - \bar{v}_0$) and then reverses his strategy at date 2 (with intensity equal to $-1/3$). From the expression for y_2 the speculator offsets one third of the (current) passive trades at date 2. This finding is consistent with RV and the trading game in Proposition 1, keeping into account that in Proposition 2 both the speculator *and* the insider trade on the same information

v at the second round, thus offsetting 2/3 of the date 2 indexed trades. As is known, Cournot competition on the indexers' order v between the two informed agents results in higher aggregate intensity on v. Furthermore from eq. (2.17) –as well as from RV and eq. (2.7)–, the insider acts as a monopolist on date 2 liquidity trades, thus offsetting 50% of u_2. Consider the case in which the demand from passive funds known by the speculator is large relative to its unconditional value, i.e. $(v - \bar{v}_0) >> 0$. Other things equal, this would determine an increase in p_2 thanks to the linear pricing rule (2.21) and $\lambda_2 > 0$. Since the speculator observes the realization v one period ahead of the market maker, he forecasts the increase in p_2 induced by unexpectedly large passive trades v. As a reaction the speculator trades positively on $(v - \bar{v}_0)$ at the first date and profits from the price difference between the two trading rounds. Note the difference in the trading intensity on v between the two dates (figure 2.7): the trading aggressiveness on v increases through time, since C_1 is always smaller than 1/3. This arises from the fact that the information about v impounded by the speculator's trades at date 1 allows the market maker[16] to make a sharper inference about the second period indexed assets via the posterior $\bar{v}_1$. Furthermore C_1 decreases with k_S and increases with k_I. The first finding is consistent with the speculator trying to keep his advantage in forecasting the final liquidation value with respect to the market maker, thus incorporating less information whenever k_S is large. The second finding is related to the insider's behaviour at the first trading round. From Proposition 1 we have that the insider trades more aggressively on both $(f - p_0)$ and u_1 the larger is his advantage k_I. Thus the speculator can hide more of his information to the market maker as k_I increases, and as a consequence C_1 increases in the insider's advantage.

The insider reacts to the presence of the speculator increasing his trading intensities at date 1 (figure 2.2-panel A and B) relative to Proposition 1. It is worth noting that while a_1 monotonically increases in k_I –as it happens without the speculator– b_1 decreases with k_I when the speculator's advantage is relatively high ($k_S = 3$), while it increases in k_I in the absence of the speculator as well as for low values of k_S. In order to understand this finding note from figure 2.3 that the speculator's entry decreases liquidity at both dates. The market maker faces more severe information asymmetries than in the absence of the speculator, and market depth is reduced: λ_t increases with both k_I and k_S in all cases but when the speculator's advantage is high (panel A). When this occurs, both λ_1 and b_1 decrease in k_I. Other things equal, the speculator decreases date 1 liquidity. The insider reacts trading less aggressively on u_1 in order to counterbalance the negative effect on liquidity due to the speculator's trades. The net result is that date 1 liquidity improves with k_I due to the insider's reaction when the speculator's advantage is high. Finally, note from equation (2.17) that the insider trades at date 2 in the opposite direction of the signal extracted by the speculator as it occurs in Madrigal (1996).

Trading volume

The insider's contribution to the total trading volume is defined along the same lines as in section 2.2, and is shown in figure 2.4. V_1^I increases due to the speculator's entry. This is due to the externality imposed by the speculator, which results in the insider trading more aggressively in order to exploit his fundamental advantage before the speculator makes his superior inference. As a consequence V_1^I increases with the speculator's advantage k_S. At the second round, the insider's intensity on his fundamental information decreases with k_S as in figure 2.2. Furthermore the insider trades in the opposite direction of the speculator's aggressiveness on his misperception $(s - p_1)$, which is again decreasing in k_S (see figure 2.7). The overall effect on trading volume is that V_2^I decreases with k_S as in figure 2.4 (panel B).

Similarly to V_t^I, the speculator's volume V_t^S is defined as

$$V_t^S \equiv \sqrt{\frac{var\,(y_t)}{2\pi}}, \quad t = 1, 2$$

and it is shown to be increasing in the informational advantage k_S in figure 2.8. At a first glance this finding might seem inconsistent with the pattern for the trading intensities C_1 and C_2 in figure 2.7. However in appendix B we show that V_t^S depends positively on both the trading intensity C_t and the passive funds' variance $\sigma_{v,0}^2$, the latter dependence implying that the quantity traded by the speculator increases in $\sigma_{v,0}^2/\sigma_w^2$. This means that larger uncertainty on the passive trades' volatility offers more opportunities to hide the speculator's informational advantage, thus justifying the pattern in figure 2.8.

Market efficiency

Market efficiency improves due to the speculator's entry: both $1/\sigma_{f,1}^2$ and $1/\sigma_{f,2}^2$ go up with respect to Proposition 1 due to the increased trading aggressiveness of the insider in the presence of the speculator. The insider impounds more information about the final liquidation value in order to anticipate the speculator's signal extraction, and as a result residual variances are lower in the presence of the speculator. From panel C in figure 2.5 it emerges that improvements in market efficiency come mainly from the first trading round. Recall that in the absence of the speculator, efficiency gains $\sigma^2_{f,1}/\sigma^2_{f,2}$ are not affected by k_I, while they depend on both informational advantages in Proposition 2: the higher is the quality of the speculator's information, the lower is the efficiency ratio. In other words the speculator's precision reduces efficiency gains over time.

Ex-ante incentives

As for the insider's expected profits (figure 2.6-panel A) the speculator reduces the insider's motives to trade, like in Madrigal (1996). We note however that in Madrigal (1996) the speculator acts as a free-rider on the insider's information extracting the signal s, which is a better forecast of the final liquidation value than the price set by the market maker. In our game the speculator is able to extract the signal s *at the additional cost* of revealing his privileged information both to the insider (which knows the realization v after the first trading round) and to the market maker (which forms the posterior $\bar{z}_1$ on passive trades after observing the order flow ω_1). Therefore one might expect that the insider makes higher profits in the presence of the speculator due to the additional information about v. However figure 2.6 shows that this is not the case: the negative externality imposed by the speculator on the insider, i.e. the loss due to the speculator's signal gathering activity, dominates the benefit of knowing v in addition to liquidity trades u_2.

As for the speculator's ex-ante incentives, they increase with his own advantage k_S and decrease with k_I (figure 2.6-panel B). This behaviour hinges on the very same trading motives for the insider. Whenever the quality of the insider's information is relatively high, the insider impounds more information on the fundamental value into his orders. As a result p_1 improves its precision as a forecasting tool for the final payoff. The speculator's inferential ability in extracting the signal s reduces relative to the improved market maker's forecast, and speculator's ex-ante gains drop.

In summary the effects of the S&P game are as follows. The presence of the speculator makes the insider trade more aggressively on both the fundamental value and the current liquidity trades at date 1. This arises from the externality imposed by the speculator on the insider via the signal s extracted from $x_1 + u_1$. The insider has an incentive to tilt his trades at date 1 and manipulate p_1 in order to avoid the speculator's inference. Market efficiency improves thanks to the increase in the insider's trading intensity following the speculator's entry. However relative market efficiency is worsened by the speculator. Market depth is lowered by the speculator's entry at both dates. This is due to the higher adverse selection costs faced by the market maker in the presence of the speculator. Finally, speculator's profits increase with the quality of his non-fundamental information, while the insider's ex-ante gains are reduced. The consequences on market volume and liquidity are further investigated in the following section.

2.4 Testable implications

When bringing our model in section 2.3 to the data, we interpret days as rounds. This way the first date coincides with the day following the announcement, while the second date is the inclusion day.

2.4.1 Trading volume

We follow Admati and Pfleiderer (1988) and decompose the expected total volume into the contribution of each group of traders. For the model in section 2.2 one has:

$$V_1 \equiv V_1^I + V_1^L + V_1^M = \frac{1}{2}\left(E_0\left|x_1\right| + E_0\left|u_1\right| + E_0\left|\omega_1\right|\right) \tag{2.27a}$$

$$V_2 \equiv V_2^I + V_2^L + V_2^P + V_2^M = \frac{1}{2}\left(E_1\left|x_2\right| + E_1\left|u_2\right| + E_1\left|z_2\right| + E_1\left|\omega_2\right|\right) \tag{2.27b}$$

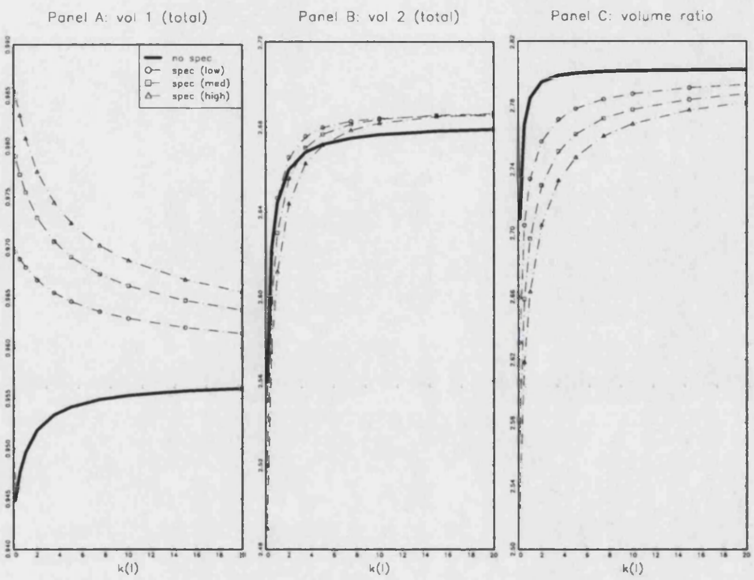
where $E_t(\cdot)$ denotes the expectation conditional on time $t-1$ public information, and superscripts L and P refer respectively to liquidity and passive traders. Since all orders but z_2 and ω_2 are conditionally normal with mean zero, the contributions to date 1 total trading volume follow from Admati and Pfleiderer (1988). On the other hand, in the absence of the speculator one has $E_1(z_2) = E_1(\omega_2) = \bar{z}_0 \neq 0$, which implies that volume at the inclusion depends on the (unconditional) expectation of the passive trades (we derive expressions for V_2^P and V_2^M in appendix A). We plot V_1 and V_2 in figure 2.9 setting $\bar{z}_0 = 2$ as a representative case. Note that we do not consider volume in RV in figure 2.9, since passive trades are absent in this model. During the first trading round V_1 increases in k_I due the insider's contribution V_1^I.[17] The fact that V_2 increases with the insider's informational advantage as well (panel B) might seem in contrast with the analysis for V_2^I, which was shown to be decreasing in k_I. In fact, one can show that V_2^P decreases with k_I as well, since passive volume is proportional to $\sigma_{z,0}^2$. However, liquidity trades V_2^L increase in their own variance σ_{u2}^2, or equivalently in the informational advantage k_I. The latter dependence dominates the other two effects, and as a result V_2 increases in k_I. Moreover we note that an increase in the mean passive trades $\bar{z}_0$ (not reported for reasons of space) would move V_2 further up. Finally, the ratio V_2/V_1 (panel C) is always above unity, as to say that volume is expected to be higher upon inclusion.

For the model developed in section 2.3, the time t market volume is decomposed similarly to $(2.27a - 2.27b)$ as:

$$
\begin{aligned}
V_1 &\equiv V_1^I + V_1^S + V_1^L + V_1^M = \frac{1}{2}\left(E_0\left|x_1\right| + E_0\left|y_1\right| + E_0\left|u_1\right| + E_0\left|\omega_1\right|\right) \\
V_2 &\equiv V_2^I + V_s^S + V_2^L + V_2^P + V_2^M = \frac{1}{2}\left(E_1\left|x_2\right| + E_1\left|y_2\right| + E_1\left|u_2\right| + E_1\left|z_2\right| + E_1\left|\omega_2\right|\right)
\end{aligned}
$$

The presence of the speculator increases volume after the announcement (figure 2.9-panel A). This stems from the volume generated by the speculator (figure 2.8-panel A) as well as from the insider's manipulative incentives (figure 2.4-panel A). Given that both V_1^I and V_1^S increase in the speculator's advantage, it is not surprising that V_1 increases with k_S. Again, while date 1 trades are centered around zero, volume at the inclusion depends on the posterior $\bar{z}_1 = \bar{z}_0 + \bar{v}_1$ in the presence of the speculator. Hence the (conditional) expectation of passive trades after the first round plays a role in determining both V_2^P and V_2^M. Moreover from eq. (2.23) the posterior $\bar{z}_1$ depends on the realization of the first period aggregate order flow (as well as on $\bar{z}_0$). This implies that every realization of the first period trades $\omega_1 = x_1(f, u_1) + y_1(v) + u_1$ generates a different date 2 expected volume. In order to assess the effect of non-fundamental speculation on market volume, we therefore replace $\bar{v}_1$ by its estimate $\hat{v}_1$ using Monte Carlo simulation with 1000 draws for f, v and u_1, and then use $\bar{z}_0 + \hat{v}_1$ instead of $\bar{z}_1$ in the expression for V_2. Volume at the inclusion increases in k_I along the same lines as V_2 in $(2.27b)$ (see figure 2.9-panel B). Note that V_2 (as well as the volume ratio V_2/V_1 in panel C) is inversely related to k_S. In fact, while the speculator generates more volume when his advantage is sharp (figure 2.8-panel B), for the insider the opposite holds true (figure 2.4-panel B). The net result is that the latter effect offsets the former. Finally, an increase in expected passive trades $\bar{z}_0$ (not reported for reasons of space) increases volume at the inclusion, like in the absence of the speculator.

FIGURE 2.9. Market volume.



2.4.2 Market liquidity

The impact of the S&P game on market liquidity pattern is analyzed in figure 2.3-panel C, which plots the ratio λ_1/λ_2. Since date t market liquidity is given by $1/\lambda_t$, the ratio λ_1/λ_2 gives the evolution of market liquidity though time: for example $\lambda_1/\lambda_2 = \left(\frac{1/\lambda_2}{1/\lambda_1}\right) > 1$ means that the market is deeper at date 2 than at date 1. In the absence of the speculator, liquidity decreases over time with the insider's non-fundamental information quality: in fact, large values for k_I imply that the information asymmetry faced by the market maker is relatively severe. Thus the market maker's reaction to large values of k_I is to decrease market liquidity at both dates. Note that it takes a rather precise non-fundamental information ($k_I > 8$) in order to observe more illiquid markets at date 2. This means that depth decreases upon inclusion whenever the (volatility of the) noise coming from the passive industry is extremely small relative to other liquidity traders, i.e. $\sigma^2_{z,0} < \sigma^2_{u_2}/8$. Hence the reduction in spreads before October 1989 [see Beneish and Whaley (1996), Edmister, Graham and Pirie (1996) and Erwin and Miller (1998)] suggests that $k_I < 8$ is in fact a reasonable bound for the insider's advantage.

The speculator's entry causes liquidity to decrease at both date (figure 2.3-panel A and B), since informational asymmetries are now more severe. The stock becomes more illiquid the higher is the speculator's advantage k_S. Moreover market depth reduces at the inclusion relative to the previous day (figure 2.3-panel C), and again this reduction is positively related to the speculator's informational advantage. In particular note that $\lambda_1/\lambda_2 < 1$ when $k_S = 3$ *regardless of the insider's informational advantage.* This means that –irrespective of the passive trades volatility relative to other liquidity-motivated orders– liquidity decreases at date 2 whenever the speculator is aware of at least 3/4 of the indexers' trades.

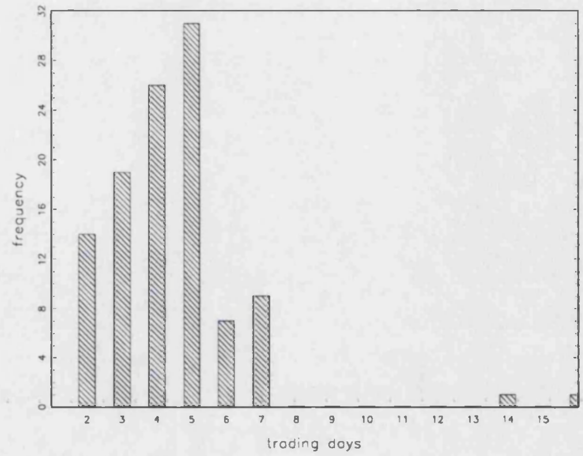
Recall from the Introduction that the main reason for moving to preannouncing index changes hinges on the attempt to reduce trading imbalances after the announcement. The volume ratio seems to confirm this, since non-fundamental speculation reduces V_2/V_1. On the other hand, the S&P game reduces market liquidity at the inclusion. These two opposite effects might allow to cast some doubts on the effectiveness of the S&P change in the announcement practice.

2.5 Empirical study

2.5.1 Data set description

Between October 1989 and December 1999 there have been 248 replacements in the S&P 500.[18] As in the previous literature, we concentrate on market additions due to the fact that stocks deleted from the S&P 500 often do not trade after the list change [see Chen, Noronha and Singal (2004) and the references therein for empirical studies on index deletions]. For notational convenience let AD denote the announcement day (i.e. the day in which after the close the announcement is made) and CD the effective day (i.e. the day in which after the close the change is effective). As previously noted, after October 1989 the replacement is effective at least one day after AD. From the total sample we removed some stocks. First of all we drop companies added and deleted from the index due to name changes (33 stocks) as well as stocks included due to merger (20) or spin-off (17) with another S&P 500 company. In all of these cases we would not observe the demand shock arising from passive traders which is the driving force for non-fundamental speculation in the model developed in section 2.3. In the second place we exclude companies for which we are not certain about the announcement date and/or the effective date (19) as well as stocks for which the inclusion occurs the day after the announcement (30). This latter requirement arises naturally from the time line underlying our theoretical model, since whenever AD+1 coincides with CD one

FIGURE 2.10. Announcement frequencies.



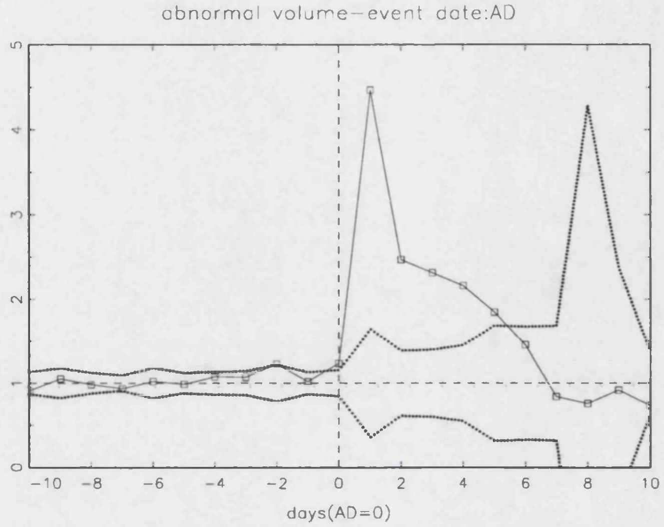
Frequency distribution of the number of trading days between the announcement and the effective day over the period October 1989-December 1999 for S&P 500 (108 inclusions)

cannot disentangle the effect of non-fundamental speculation from indexers' demand. For each company we collect daily data from CRSP on (1) bid price, (2) ask price, (3) volume (number of shares traded) and (4) outstanding shares. Eventually we require stock data availability for a period ranging from 250 days before to 40 days after the announcement, which resulted in dropping 21 companies. The final data set comprises 108 stocks. Figure 2.10 shows the frequency distribution of the number of trading days between AD and CD for the inclusions occurred under the preannouncement practice. The support ranges from one to sixteen trading days and the mode (resp. mean) is five (resp. 4.43), documenting the S&P common practice to preannounce changes five business days beforehand. This evidence is consistent with Beneish and Whaley (1996) for announcements between October 1989 and June 1994.

2.5.2 Trading volume

As explained in the Introduction, the appeal to investors of passive techniques is widely documented by the growth in the net asset value experienced by the major funds pegged to the S&P 500 in the last two decades [see Beneish and Whaley (1996) and Wurgler and Zhuravskaya (2002) among others]. The widespread use of indexed funds can be assessed by looking at the trading volume pattern around AD and CD, since in section 2.4 we have shown that an increase in average passive trades –captured by $\bar{z}_0$– results in higher volume at the inclusion. Given that indexers' performance is assessed by daily tracking error minimization, pegged funds' rebalancing should

FIGURE 2.11. Volume around announcement.



Mean abnormal volume is defined in (2.29). The $MAVR$'s are displayed (bold solid line) for each trading day over the window (AD-10,AD+10) together with the 95% confidence interval for the null hypothesis $MAVR = 1$ (dotted line).

occur at CD. Moreover the presence of risk arbitrageurs (i.e. the speculator in our model in section 2.3) increases volume after the announcement, i.e. over the window AD+1,...,CD-1. Finally, abnormal trading volume on AD may provide evidence that leakage of information regarding index inclusion has occurred.

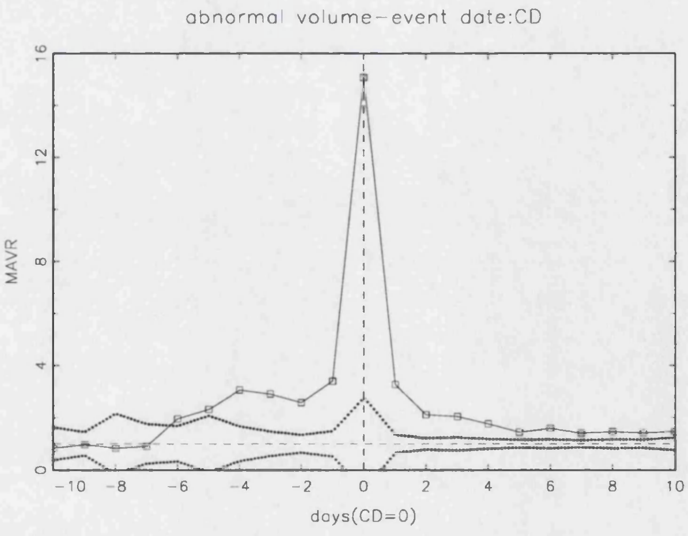
Let $V_{i,t}$ denote the daily turnover for stock i on day t as measured by the ratio between the number of shares traded and the number of outstanding shares for company i during day t. We use daily turnover as a measure of the daily trading volume since it accounts for splits experienced by the stock, thus making turnover[19] preferred to raw volume. Therefore the abnormal trading volume on day t is the ratio between $V_{i,t}$ and the average trading volume in the 40 days[20] preceding the announcement day $\bar{V}_i \equiv \left(\sum_{t=AD-40}^{AD-1} V_{i,t}\right)/40$. Eventually we let $MAVR_t$ denote the cross-section average for the abnormal trading volume over a sample of size N_t:

$$AVR_{i,t} = V_{i,t}/\bar{V}_i \quad ; \quad MAVR_t = \frac{1}{N_t}\sum_{i=1}^{N_t} AVR_{i,t} \tag{2.29}$$

Results from inclusions in the S&P 500 are summarized in table 2.2 and figures 2.11–2.12, taking into account the number of trading days between AD and CD. Under the assumption that individual abnormal volume ratios are (cross-sectionally) independently and identically normally distributed, the resulting statistic for $MAVR_t$ follows a Student-t distribution with $N_t - 1$ degrees of freedom. Moreover, in order to assess the impact of outliers in our analysis, we perform a binomial test for the null hypothesis that the percentage of companies with $AVR_{i,t} > 1$ is different from 50%.[21] Table 2.2 reports sample size[22], mean abnormal volume ratio ($MAVR_t$), cross-sectional t-ratio ($t(MAVR)$) and the percentage of companies for which $AVR_{i,t}$ is greater than one over the window $AD - 10, ..., CD + 10$. Figure 2.11 (resp. figure 2.12) plots $MAVR_t$ and its 95% confidence interval around AD (resp. CD).

On the day after the announcement trading volume is more than 4 times larger than the average daily volume over the 8 weeks base period, and is statistically significant at 5% level. Abnormal volume appears to be persistent in that $MAVR_t$ is greater than one for the whole week after AD+1, even though its magnitude is far from the increase experienced during AD+1, and $MAVR_t$ is not significantly different from one after AD+5. This evidence is consistent with the presence of non-fundamental speculators stepping into the market after the announcement, and diluting their orders over the days preceding the effective change. On AD the estimated mean abnormal volume is roughly 25% above the level in the 40 days preceding the announcement. Further, the t-statistic rejects $MAVR_{AD} = 1$ at 5% significance level. The latter finding is in line with all the above mentioned empirical studies on S&P inclusions after October 1989, and might suggest leakage of information about index replacements before

FIGURE 2.12. Volume around inclusion.



Mean abnormal volume is defined in (2.29). The $MAVR$'s are displayed (bold solid line) for each trading day over the window (CD-10,CD+10) together with the 95% confidence interval for the null hypothesis $MAVR = 1$ (dotted line).

TABLE 2.2. Daily abnormal volume.

day	panel A – event day: AD				panel B – event day: CD			
	N	$MAVR$	$t(MAVR)$	$AVR > 1$	N	$MAVR$	$t(MAVR)$	$AVR > 1$
-10	108	0.912	-1.349	**28.70**	2	0.833	-1.684	0.00
-9	108	1.051	0.576	**34.26**	2	0.966	-0.479	50.00
-8	108	0.984	-0.258	**37.04**	2	0.834	-0.911	50.00
-7	108	0.934	-1.464	**39.81**	2	0.906	-0.785	50.00
-6	108	1.021	0.229	**33.33**	11	1.951	2.536	72.73
-5	108	0.988	-0.191	**37.04**	18	**2.313**	2.139	61.11
-4	108	1.077	1.132	46.30	49	**3.055**	5.196	**77.55**
-3	108	1.071	0.999	41.67	75	**2.897**	6.762	**78.67**
-2	108	**1.229**	2.078	41.67	94	**2.569**	7.639	**86.17**
-1	108	1.018	0.276	**40.74**	108	**3.394**	8.215	**91.67**
0	108	**1.234**	2.990	51.85	108	**15.086**	13.259	**99.07**
1	108	**4.471**	10.652	**96.30**	108	**3.266**	10.945	**94.44**
2	64	**2.468**	7.504	**85.11**	108	**2.112**	8.360	**84.26**
3	75	**2.313**	6.548	**78.67**	108	**2.039**	6.878	**75.93**
4	49	**2.160**	5.172	**71.43**	108	**1.763**	6.953	**72.22**
5	18	**1.846**	2.602	**77.78**	108	**1.445**	4.870	**63.89**
6	11	1.457	1.510	63.64	108	**1.590**	5.696	**70.37**
7	2	0.840	-2.966	0.00	108	**1.408**	4.857	**62.96**
8	2	0.758	-0.937	50.00	108	**1.465**	4.494	**60.19**
9	2	0.918	-0.762	50.00	108	**1.421**	4.546	**63.89**
10	2	0.738	-8.752	0.00	108	**1.471**	3.153	55.56

Abnormal trading volume is defined for each stock in (2.29). In the sample AD precedes CD by at least one day. Since the number of trading days between AD and CD varies across firms, the columns labeled N report the number of companies included in the cross-section for each day. Boldface numbers in columns labeled $MAVR$ denote mean trading volume significantly different from one (5% significance level). Boldface numbers in columns labeled $AVR > 1$ denote percentage of companies with $MAVR_{i,t} > 1$ significantly different from 50% (5% significance level).

announcement. Results from the ten days preceding AD do not detect abnormal trading activity, with the only exception of mean abnormal volume significantly greater than unity documented for AD-2. Comparing the percentage of individual firms whose $AVR_{i,t}$ is greater than one is useful to determine whether the $MAVR$'s are driven by outliers. More than 95% of the cross-section have individual $AVR_{i,AD+1}$ greater than one, this percentage being statistically different from 50% at 5% significance level. Over the window (AD+2,AD+5) more than 70% of the stocks in our sample display abnormal trading volume, which we regard as strengthening the evidence in favour of front-running strategies implemented after the announcement. On the other hand, the percentage of stocks with $AVR_{i,t} > 1$ on both AD-2 and AD is not statistically different from 50%, and we conclude that the abnormal trading volume documented for these two dates is due to a small number of companies.

Trading volume on CD is roughly 15 times higher than the base period and is statistically significant at 5% level. This suggests that passive managers actually wait until the effective day to rebalance their portfolios. It is noteworthy that virtually all of the companies experience an increase in trading volume upon inclusion. Trading activity for the five days before CD is at least twice the average volume during the 8 weeks preceding the announcement, and can be attributed to risk-arbitrageurs' activity. The increase in volume tends to be permanent, in that $MAVR_t$ is significantly different from one in all the days from CD+1 to CD+10, even though it steadily decreases after CD. The percentage of companies with $AVR_{i,t}$ different from one around CD shows that these findings do not appear to be driven by outliers over the fourteen days ranging from CD-4 to CD+9.

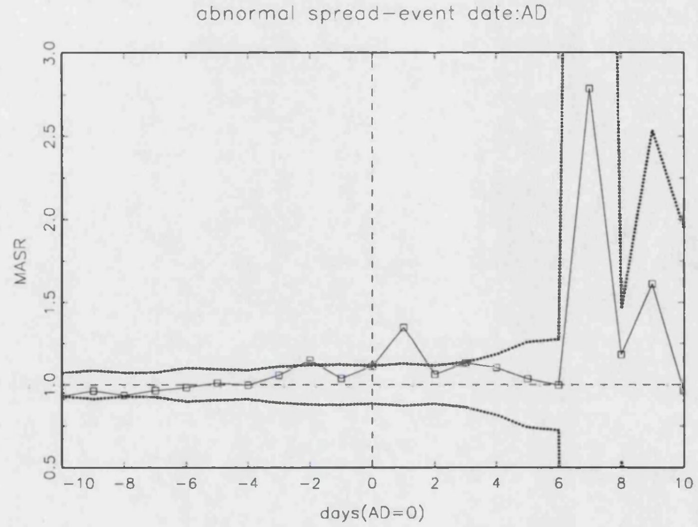
2.5.3 Liquidity

While several authors focused on trading volume around inclusions in the S&P 500, market depth has received little attention. Beneish and Whaley (1996) analyze the bid-ask spread for inclusions: as mentioned in the Introduction they find reductions in the spread after the effective date. Furthermore, the authors report a significant 13% liquidity improvement for stocks included between 1986 and 1989 as well. Erwin and Miller (1998) focus on additions between 1984 and 1988, and document a significant spread decrease over the 30 days following the index change. Similarly, included companies experience liquidity improvements between 1983 and 1989 according to Edmister, Graham and Pirie (1996). In what follows we employ the relative bid-ask spread as a proxy for market liquidity. A measure for abnormal depth can be constructed along the same lines used for the trading volume analysis. Let $A_{i,t} - B_{i,t}$ be stock i's absolute bid-ask spread during day t, and $Q_{i,t}$ the quote midpoint, i.e. $Q_{i,t} = (A_{i,t} + B_{i,t})/2$. The relative bid-ask spread is $S_{i,t} = (A_{i,t} - B_{i,t})/Q_{i,t}$ and $\bar{S}_i \equiv \left(\sum_{t=AD-40}^{AD-1} S_{i,t}\right)/40$ denotes the average relative bid-ask spread over the base period.[23] The abnormal spread ratio $ASR_{i,t}$ and its cross-section counterpart $MASR_t$ are defined as follows:

$$ASR_{i,t} = S_{i,t}/\bar{S}_i \quad , \quad MASR_t = \frac{1}{N_t}\sum_{i=1}^{N_t} ASR_{i,t} \tag{2.30}$$

For the announcement and the inclusion not to affect market depth one should observe $MASR_t$ close to one both around AD and CD. On the other hand, a situation in which $MASR_t$ is less (resp. greater) than one detects a reduction (raise) in the average spread during day t relative to the base period, i.e. an increase (decrease) in market depth on day t relative to the 8 weeks preceding the announcement.

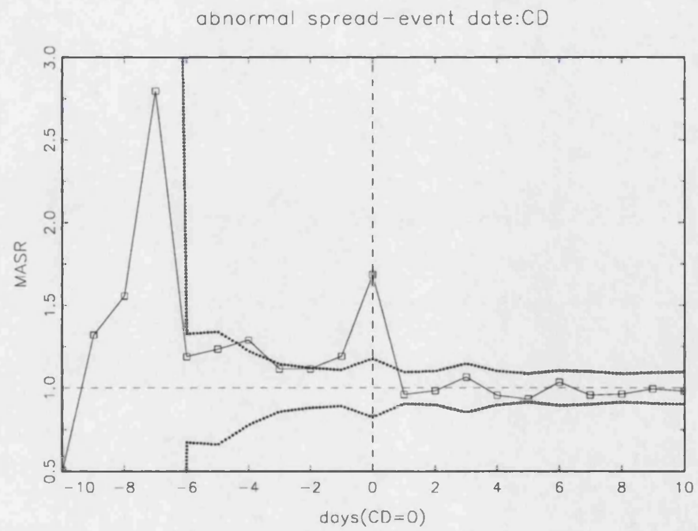
FIGURE 2.13. Bid-ask spread around announcement.



Mean bid-ask spread is defined in (2.30). The $MASR$'s are displayed (bold solid line) for each trading day over the window (AD-10,AD+10) together with the 95% confidence interval for the null hypothesis $MASR = 1$ (dotted line).

Table 2.3 and figures 2.13–2.14 report the mean abnormal spread over the window $AD - 10, ..., CD + 10$. Stocks experience a statistically significant 35% increase in the bid-ask spread the day after the announcement. No clear pattern emerges from our sample for the other days in the event window, with the exception of $MASR_{AD-2}$ being

FIGURE 2.14. Bid-ask spread around inclusion.



Mean bid-ask spread is defined in (2.30). The $MASR$'s are displayed (bold solid line) for each trading day over the window (CD-10,CD+10) together with the 95% confidence interval for the null hypothesis $MASR = 1$ (dotted line).

TABLE 2.3. Daily abnormal bid-ask spread.

day	panel A – event day: AD				panel B – event day: CD			
	N	$MASR$	$t(MASR)$	$ASR > 1$	N	$MASR$	$t(MASR)$	$ASR > 1$
-10	108	0.929	-1.962	**37.96**	2	0.504	-2.170	0.00
-9	108	0.961	-0.877	**38.89**	2	1.320	0.654	50.00
-8	108	0.934	-1.811	**37.96**	2	1.557	1.354	100.00
-7	108	0.965	-0.933	**39.81**	2	2.796	1.713	100.00
-6	108	0.983	-0.323	**37.96**	11	1.188	1.279	72.73
-5	108	1.010	0.214	43.52	18	1.233	1.444	55.56
-4	108	0.997	-0.058	43.52	49	**1.289**	2.611	57.14
-3	108	1.059	1.038	**40.74**	75	1.115	1.596	46.67
-2	108	**1.152**	2.437	53.70	94	1.114	1.885	47.87
-1	108	1.041	0.647	**36.11**	108	**1.190**	3.467	53.70
0	108	1.115	1.943	42.59	108	**1.685**	7.679	**81.48**
1	108	**1.355**	5.409	**65.74**	108	0.961	-0.816	**35.19**
2	64	1.063	1.065	44.68	108	0.983	-0.339	41.67
3	75	1.134	1.919	46.67	108	1.065	0.881	**37.04**
4	49	1.106	1.128	42.86	108	0.955	-0.885	**33.33**
5	18	1.035	0.280	44.44	108	0.933	-1.551	**36.11**
6	11	0.994	-0.051	45.45	108	1.034	0.645	46.30
7	2	2.827	1.795	100.00	108	0.956	-0.893	**35.19**
8	2	1.178	5.427	100.00	108	0.963	-0.851	42.59
9	2	1.617	4.727	100.00	108	0.994	-0.129	44.44
10	2	0.960	-0.508	50.00	108	0.981	-0.382	42.59

Abnormal bid-ask spread is defined for each stock in (2.30). In the sample AD precedes CD by at least one day. Since the number of trading days between AD and CD varies across firms, the columns labeled N report the number of companies included in the cross-section for each day. Boldface numbers in columns labeled $MASR$ denote mean bid-ask spread significantly different from one (5% significance level). Boldface numbers in columns labeled $ASR > 1$ denote percentage of companies with $MASR_{i,t} > 1$ significantly different from 50% (5% significance level).

statistically different from unity. Before AD the market is more liquid over the week AD-10 to AD-6, while liquidity decreases over the week following the announcement. However one cannot reject the null hypothesis that liquidity is statistically different from the base period over both weeks. The percentage of firms experiencing wider spreads on AD+1 is significantly greater than 50%, while this does not occur on AD-2. We conclude that the latter reduction in liquidity is affected by few observations, while the former is not.

Market liquidity significantly decreases around inclusion. There is an average 20% increase in the spread during the day preceding the index change, which rises to almost 70% upon inclusion. The abnormal spread ratio is statistically different from one (5% significance level) on the effective day, as well as on CD-1 and CD-4. Notice, however, that the reduction in market liquidity during CD is not driven by outliers (approximately 80% stocks of the cross-section experience abnormal spreads), while this is not the case for both $MASR_{CD-1}$ and $MASR_{CD-4}$. Our findings are in contrast with

Beneish and Whaley (1996), even though their results might be affected by their small sample size. In fact, their data set comprises 30 index inclusions from October 1989 through June 1994: the authors report a spread decrease during CD and the following days, even though the spread is significantly below normal only for CD+1. Recall from subsection 2.5.2 that stocks experience a significant increase in trading volume upon inclusion, which is driven by passive traders' demand. Beneish and Whaley (1996) argue that the specialist might temporarily charge a lower spread, given that the increase in trading volume would cover the operation costs. On the other hand the trading game in section 2.3 is consistent with an increase in the spread during the day of inclusion: liquidity should decrease over time as a consequence of the higher adverse selection costs faced by the market maker in the presence of (sufficiently accurate) non-fundamental information. Our sample seems to confirm this implication, and we argue that any reduction in operational costs arising from greater volume is more than offset by the asymmetric information costs faced by the market maker in the presence of non-fundamental speculators.

2.5.4 Assessing the importance of the S&P game

The findings in the previous two subsections point at (1) a significant increase in trading volume following the announcement and upon inclusion (2) a significant decrease in liquidity both after the announcement and upon inclusion. The empirical implications arising from our theoretical model(s) stand on the comparison of both volume and liquidity between the day after the announcement and the effective day (respectively the first and second trading round in the models developed in sections 2.2 and 2.3). However for most of the companies in our sample there is more than one day between AD and CD. Therefore, for each of these stocks we define $\overline{AVR_i}$ as the average abnormal volume over the window (AD+1,CD-1), i.e. $\overline{AVR_i} \equiv \sum_{t=AD+1}^{CD-1} AVR_{i,t}$, and then average the $\overline{AVR}$s across stocks to get $M\overline{AVR} = N^{-1} \sum_{i=1}^{N} \overline{AVR_i}$, where N denotes the sample size ($\overline{ASR_i}$ and $M\overline{ASR}$ are defined along the same lines). In order to measure the change in volume we perform a two-sided test for the null $M\overline{AVR} = MAVR_{CD}$ (the change in liquidity is assessed in the same way).

Results on the entire dataset are summarized in the first row of table 2.4. Both volume and bid-ask spread significantly increase upon inclusion relative to the window (AD+1,CD-1). Our findings appear robust, since the binomial test rejects the

TABLE 2.4. Assessing the S&P game: change in volume and liquidity.

sample	N	panel A – volume			panel B – bid-ask spread		
		$DMAVR$	$t(DMAVR)$	$DAVR > 0$	$DMASR$	$t(DMASR)$	$DASR > 0$
all obs	108	**11.515**	10.435	**99.07**	**0.503**	4.95	**72.23**
1989-94	29	**10.102**	4.526	**100.00**	0.268	1.553	58.62
1995-97	40	**10.675**	6.269	**97.50**	**0.591**	3.507	**75.00**
1998-99	39	**13.429**	7.266	**100.00**	**0.587**	3.228	**79.48**

Abnormal volume and bid-ask spread are defined for each stock in (2.29) and (2.30) respectively. In the sample AD precedes CD by at least one day. The test for the equality of mean trading volume (panel A) and bid-ask spread (panel B) around inclusion is presented. Boldface numbers in columns labeled $DMAVR$ denote mean volume on CD significantly different from mean volume over AD+1 to CD-1 (5% significance level). Boldface numbers in columns labeled $DAVR > 0$ denote percentage significantly different from 0.5 (5% significance level). Headers for the bid-ask spread have a similar interpretation.

hypothesis that the percentage of firms experiencing this increase is equal to 50% at 5% significance level. The increase in volume is consistent with both models in sections 2.2 and 2.3, since it may simply reflect the presence of passive funds stepping into the market on the effective day. Similarly –as explained in subsection 2.4.2– the decrease in liquidity might be due to an insider with highly accurate non-fundamental information in the absence of the speculator ($k_I > 8$). Alternatively, a speculator endowed with relatively precise information ($k_S > 3$) might be responsible of the spread increase. We tend however to disregard the first explanation given that all the authors focusing on inclusions before October 1989 document decline in spreads for stocks included in the S&P 500. During this period, the simultaneous occurrence of announcement and inclusion *de facto* rules out the S&P game. As previously noted, taking the model in section 2.2 as a reference, the findings in Edmister, Graham and Pirie (1996) and Erwin and Miller (1998) are consistent with a relatively poor quality of the insider's signal ($k_I < 8$). We therefore attribute the worsening in liquidity documented in table 2.3 to non-fundamental speculators front-running index funds after October 1989. In order to assess the robustness of the S&P game we test for significant changes in volume and liquidity splitting our dataset in three subsamples: 1989-1994, 1995-1997 and 1998-99. Reasonable sample size is one of the criteria we used in choosing these subsamples. Moreover the average net asset value for passive funds over these periods is equal to 287, 606 and 1117.5 billion USD respectively. Expected passive trades thus increase across the subsamples, but are relatively stable within each subsample (see figure 2.1). It is further reasonable to conjecture that the shift towards passive strategies resulted in an increase in the number of indexers, which in turn implies that passive trades' variance increases over time. This is particularly true for the last subsample, given that funds moved away from full replication in recent years [see Blume and Edelen (2004)]. Recall from section 2.4 that, absent the speculator, the combined effect of an increase in both $\bar{z}_0$ and $\sigma^2_{z,0}$ is higher volume and tighter spreads upon inclusion. The former is due to larger orders coming from indexers, while the latter stems from a reduction in the insider's advantage $k_I = \sigma^2_{u_2}/\sigma^2_{z,0}$. On the other hand, an increase in both $\bar{z}_0$ and $\sigma^2_{z,0}$ is compatible with higher volume and lower liquidity in the presence of a speculator with highly accurate information.

From table 2.4 it emerges that volume significantly increases upon inclusion in all subsamples, while liquidity significantly decreases from 1995 onwards. Our findings support the following argument: it took some time for investors to start front-running indexers after the change in the S&P announcement practice. Before 1995, the volume-liquidity pattern points at a statistically significant increase in trading activity, while spreads are unaltered. Starting from 1995 the S&P game has gained appeal, yielding positive profits and significantly worsening liquidity.

2.6 Conclusion

In the last two decades passive funds have gained an increasing consideration among investors as a relatively cheap tool to achieve portfolio diversification. Passive funds aim at mimicking a benchmark index. Portfolio rebalancing, as well as performance evaluation, is carried out by means of tracking error procedures. Index replacements stand as a clear rebalancing opportunity for passive managers. Starting from October 1989, changes in the S&P 500 composition are preannounced by Standard and Poor's usually five days beforehand. Passive funds are not affected by the announcement timing and their portfolio rebalancing occurs during the effective day. On the other hand this preannouncement policy induces non-fundamental speculators to enter the market. Non-fundamental speculators do not possess any information on the asset's fundamental value, rather they buy the included stock ahead of passive funds and sell it a few days later at possibly higher prices. We develop a dynamic model that explicitly keeps into account this preannouncement practice. We show that strategies based on non-fundamental information are profitable and determine a drop in market liquidity, as a direct consequence of the increased adverse selection costs faced by the specialist. Examining S&P 500 inclusions from October 1989 to December 1999 we find evidence consistent with our theoretical analysis.

2.A Appendix A: Equilibrium without speculator

Proof (Proposition 1). Given the pricing function (2.9), the insider chooses his second period trade x_2 in order to maximize the objective function (2.3)

$$E\left[x_2\left(f-p_2\left(\omega_2\right)\right)\middle|\,\Phi_2^I\right]=x_2\left(f-p_1-\lambda_2 x_2-\lambda_2 u_2\right)\ .$$

The first order condition gives $x_2=\dfrac{f-p_1}{2\lambda_2}-\dfrac{u_2}{2}$, such that eq. (2.7) obtains with $a_2=(2\lambda_2)^{-1}$. Further the second order condition is $\lambda_2>0$. Plugging eq. (2.7) in the objective function (2.3) gives

$$E\left(\pi_2^I|\Phi_2^I\right)=\frac{(f-p_1)^2}{4\lambda_2}+\frac{\lambda_2 u_2^2}{4}-\frac{(f-p_1)\,u_2}{2}\ . \tag{2.31}$$

In the first trading round the insider chooses x_1 to maximize (2.2), i.e.

$$E\left[x_1\left(f-p_1\left(\omega_1\right)\right)+\pi_2^I\left(x_1\right)\middle|\,\Phi_1^I\right]\ .$$

By the Law of Iterated Expectations $E\left[\pi_2^I\left(x_1\right)|\Phi_1^I\right]=E\left[E\left(\pi_2^I|\Phi_2^I\right)|\Phi_1^I\right]$, where (2.31) gives $E\left(\pi_2^I|\Phi_2^I\right)$. Therefore when submitting his order x_1 the insider has to keep into account the impact of his trade on the price $p_1\left(x_1\right)$. Assuming that the first period price is set according to eq. (2.8), the first order condition is

$$\left(1-\tfrac{\lambda_1}{2\lambda_2}\right)E\left(f-p_1\left(\omega_1\right)\middle|\,\Phi_1^I\right)-\lambda_1 x_1=0\ ,$$

or equivalently:

$$x_1=\frac{2\lambda_2-\lambda_1}{\lambda_1\left(4\lambda_2-\lambda_1\right)}\left(f-p_0\right)-\frac{2\lambda_2-\lambda_1}{4\lambda_2-\lambda_1}u_1\ ,$$

such that eq. (2.6) obtains with $a_1=\frac{2\lambda_2-\lambda_1}{\lambda_1(4\lambda_2-\lambda_1)}$ and $b_1=\lambda_1 a_1$. Finally the second order condition for (2.2) is $\lambda_1\left(2-\frac{\lambda_1}{2\lambda_2}\right)>0$, and the inequality (2.10) follows since $\lambda_2>0$. We now determine equilibrium prices. Let $\sigma_{\omega,0}^2$ and $\sigma_{f\omega,0}$ denote respectively the unconditional variance of the first period aggregate order flow and the unconditional covariance between ω_1 and f. The unconditional distribution for the random variables (f,u_1) together with the first period trade (2.6) yields $\sigma_{\omega,0}^2=a_1^2\sigma_{f,0}^2+\left(1-b_1\right)^2\sigma_{u_1}^2$ and $\sigma_{f\omega,0}=a_1\sigma_{f,0}^2$.Therefore the efficiency condition (2.4) together with the Projection Theorem gives the price in eq. (2.8), where the regression coefficient $\lambda_1=\sigma_{f\omega,0}/\sigma_{\omega,0}^2$

is defined as

$$\lambda_1 = \frac{a_1 \sigma_{f,0}^2}{a_1^2 \sigma_{f,0}^2 + (1-b_1)^2 \sigma_{u_1}^2} , \tag{2.32}$$

and the fundamental value posterior variance is $\sigma_{f,1}^2 = var\left(f|\omega_1\right) = \left(1 - a_1\lambda_1\right)\sigma_{f,0}^2$.

During the second trading round the market maker observes the order flow

$$\omega_2 = x_2 + u_2 + z_2 = a_2\left(f - p_1\right) + u_2/2 + z_2 .$$

It follows that $\omega_2|\omega_1 \sim N\left(\bar{z}_0, \sigma_{\omega,1}^2\right)$ with $\sigma_{\omega,1}^2 = var\left(\omega_2|\omega_1\right) = a_2^2\sigma_{f,1}^2 + \sigma_{u_2}^2/4 + \sigma_{z,0}^2$, and the second period price is given by eq. (2.9) with

$$\lambda_2 = \frac{\sigma_{f\omega,1}}{\sigma_{\omega,1}^2} = \frac{a_2\sigma_{f,1}^2}{a_2^2\sigma_{f,1}^2 + \sigma_{u_2}^2/4 + \sigma_{z,0}^2} .$$

Substituting for $a_2 = \left(2\lambda_2\right)^{-1}$ in the latter results in a second order equation in λ_2 which admits the unique root (uniqueness follows from the insider's date 2 second order condition)

$$\lambda_2 = \frac{\sigma_{f,1}}{\left(\sigma_{u_2}^2 + 4\sigma_{z,0}^2\right)^{1/2}} . \tag{2.33}$$

The fundamental value residual variance after the second trading round is $\sigma_{f,2}^2 = var\left(f|\omega_2\right) = \left(1 - a_2\lambda_2\right)\sigma_{f,1}^2$ .In order to compute the insider's unconditional profits note that the Law of Iterated Expectations applied to (2.31) gives

$$E\left(\pi_2^I\right) = E\left[E\left(\pi_2^I|\Phi_2^I\right)\right] = \frac{\sigma_{f,1}^2}{4\lambda_2} + \frac{\lambda_2\sigma_{u_2}^2}{4} .$$

First period unconditional profits are obtained substituting the equilibrium trade (2.6) into (2.2), yielding

$$E\left(\pi_1^I\right) = E\left[E\left(\pi_1^I|\Phi_1^I\right)\right] = a_1\left(1 - a_1\lambda_1\right)\sigma_{f,0}^2 - b_1\lambda_1\left(1 - b_1\right)\sigma_{u_1}^2 .$$

Adding up the last two equations gives total unconditional profits as

$$E\left(\pi_1^I + \pi_2^I\right) = a_1\left(1 - a_1\lambda_1\right)\sigma_{f,0}^2 - b_1\lambda_1\left(1 - b_1\right)\sigma_{u_1}^2 + \frac{\sigma_{f,1}^2}{4\lambda_2} + \frac{\lambda_2\sigma_{u_2}^2}{4} . \tag{2.34}$$

$a_2 > 0$ follows from $a_2 = \left(2\lambda_2\right)^{-1}$ and $\lambda_2 > 0$. From the inequality (2.10) one has $4\lambda_1\lambda_2 - \lambda_1^2 > 0$ or equivalently $4\lambda_1\lambda_2 > \lambda_1^2 \geq 0$. Therefore $\lambda_1 > 0$ since $\lambda_2 > 0$.

$a_1 > 0$ follows from $\lambda_1 > 0$ and the expression for λ_1 in eq. (2.32). Finally $b_1 > 0$ since $b_1 = \lambda_1 a_1$. Before deriving the expression for the expected volume, we prove the following:

Lemma 1 *Let $X \sim N(\mu, \sigma^2)$. Then*

$$E|X| = \sqrt{\frac{2}{\pi}}\sigma e^{-\frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2} + \mu\left[1 - 2\Phi\left(-\mu/\sigma\right)\right] \tag{2.35}$$

where $\Phi(\cdot)$ is the cumulative distribution for the standard normal distribution.

Proof. Let $f(x)$ be the normal probability distribution. Then:

$$\begin{aligned}
E|X| &= \int_0^{+\infty} xf(x)\,dx - \int_{-\infty}^{0} xf(x)\,dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\left(\int_{-\mu/\sigma}^{+\infty} ze^{-z^2/2}dz - \int_{-\infty}^{-\mu/\sigma} ze^{-z^2/2}dz\right) \\
&\quad +\mu\left(\int_{-\mu/\sigma}^{+\infty} \frac{e^{-z^2/2}}{\sqrt{2\pi}}dz - \int_{-\infty}^{-\mu/\sigma} \frac{e^{-z^2/2}}{\sqrt{2\pi}}dz\right) \\
&= \frac{\sigma}{\sqrt{2\pi}}\left[2e^{-(\mu/\sigma)^2/2}\right] + \mu\left[1 - 2\Phi\left(-\mu/\sigma\right)\right]
\end{aligned}$$

where the change in variable via $z = (x - \mu)/\sigma$ gives the second line, and the last line follows from straightforward computations. The trading volume in Admati and Pfleiderer (1988) clearly follows from eq. (2.35) setting $\mu = 0$. ■

The contributions of each group of traders to the total expected volume as in $(2.27a, 2.27b)$ are therefore given by

$$\begin{aligned}
V_1^I &= \sqrt{\frac{a_1^2\sigma_{f,0}^2 + b_1^2\sigma_{u_1}^2}{2\pi}} \quad ; \quad V_1^L = \frac{\sigma_{u_1}}{\sqrt{2\pi}} \quad ; \quad V_1^M = \frac{\sigma_{\omega,0}}{\sqrt{2\pi}} \\
V_2^I &= \sqrt{\frac{a_2^2\sigma_{f,1}^2 + \sigma_{u_2}^2/4}{2\pi}} \quad ; \quad V_2^L = \frac{\sigma_{u_2}}{\sqrt{2\pi}} \\
V_2^P &= \frac{\sigma_{z,0}}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{z}_0}{\sigma_{z,0}}\right)^2} + \frac{\bar{z}_0}{2}\left(1 - 2\Phi\left(-\bar{z}_0/\sigma_{z,0}\right)\right) \\
V_2^M &= \frac{\sigma_{\omega,1}}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{\bar{z}_0}{\sigma_{\omega,1}}\right)^2} + \frac{\bar{z}_0}{2}\left(1 - 2\Phi\left(-\bar{z}_0/\sigma_{\omega,1}\right)\right)
\end{aligned} \tag{2.36}$$

■

Remark (parameters in RV). The case of complete knowledge about date 2 noise trades can be obtained from Proposition 1 substituting (2.7) with the following

$$x_2 = a_2 (f - p_0) - (u_2 + z)/2 .$$

The second period expected profits in (2.31) become

$$E\left(\pi_2^I | \Phi_2^I\right) = \frac{(f - p_1)^2}{4\lambda_2} + \frac{\lambda_2 (u_2 + z)^2}{4} - \frac{(f - p_1)(u_2 + z)}{2} ,$$

while date 2 price sensitivity in (2.33) is $\lambda_2 = \sigma_{f,1}\left(\sigma_{u_2}^2 + \sigma_{z,0}^2\right)^{-1/2}$. Finally unconditional profits in (2.34) become

$$E\left(\pi_1^I + \pi_2^I\right) = a_1 (1 - a_1\lambda_1)\sigma_{f,0}^2 - b_1\lambda_1 (1 - b_1)\sigma_{u_1}^2 + \frac{\sigma_{f,1}^2}{4\lambda_2} + \frac{\lambda_2\left(\sigma_{u_2}^2 + \sigma_{z,0}^2\right)}{4} .$$

Other parameters and variables are defined like in Proposition 1. From the above formulas it emerges that parameter values for RV can be obtained setting $z = 0$ and $\sigma_{z,0}^2 = 0$ in Proposition 1.

2.B Appendix B: Equilibrium with speculator

Proof (Proposition 2). The proof is organized in three steps.

Step 1: date 2 trades

Given the speculator's trade (2.19) and the pricing function (2.21), the insider chooses his second period trade x_2 in order to maximize the objective function (2.3), i.e.

$$E\left[x_2 (f - p_2(\omega_2)) | \Phi_2^I\right] = x_2 \left[f - p_1 - \lambda_2 x_2 - C_2\lambda_2 (s - p_1) - 2\lambda_2 (v - \bar{v}_1)/3 - \lambda_2 u_2\right] . \quad (2.37)$$

The optimality conditions for (2.37) are:

$$\begin{aligned} x_2 &= \frac{f - p_1}{2\lambda_2} - \frac{u_2}{2} - \frac{C_2}{2}(s - p_1) - (v - \bar{v}_1)/3 \\ \lambda_2 &> 0 \end{aligned}$$

such that eq. (2.17) obtains with $a_2 = (2\lambda_2)^{-1}$. Similarly, given the insider's order (2.17) and the pricing function (2.21), the speculator chooses y_2 in order to maximize the objective function (2.14):

$$E\left[y_2\left(f - p_2\left(\omega_2\right)\right)\middle|\,\Phi_2^S\right] = y_2\left[-\lambda_2 y_2 - 2\lambda_2\left(v - \bar{v}_1\right)/3 + \left(1 - \left(a_2 - C_2/2\right)\lambda_2\right)\left(s - p_1\right)\right] , \tag{2.38}$$

yielding the optimality conditions:

$$\begin{aligned} y_2 &= -\frac{v - \bar{v}_1}{3} + \frac{1 - \left(a_2 - C_2/2\right)\lambda_2}{2\lambda_2}\left(s - p_1\right) \\ \lambda_2 &> 0 \end{aligned}$$

Equation (2.19) then obtains with $C_2 = \frac{1 - \left(a_2 - C_2/2\right)\lambda_2}{2\lambda_2}$. Solving for the coefficient C_2 yields $C_2 = 2a_2/3$ and the time 2 trades (2.17) and (2.19) become:

$$x_2 = \frac{f - p_1}{2\lambda_2} - \frac{u_2}{2} - \frac{s - p_1}{6\lambda_2} - \frac{v - \bar{v}_1}{3} \quad \text{and} \quad y_2 = \frac{s - p_1}{3\lambda_2} - \frac{v - \bar{v}_1}{3}$$

Note that date 2 trading intensities depend on λ_2 only. Plugging the above expressions for x_2 and y_2 in the objective functions (2.37) and (2.38) gives the conditional profits as

$$\begin{aligned} E\left(\pi_2^I|\Phi_2^I\right) = {} & \frac{\left(f - p_1\right)^2}{4\lambda_2} + \frac{\lambda_2 u_2^2}{4} + \frac{\left(s - p_1\right)^2}{36\lambda_2} + \frac{\lambda_2\left(v - \bar{v}_1\right)^2}{9} \\ & - \frac{\left(f - p_1\right)u_2}{2} - \frac{\left(f - p_1\right)\left(s - p_1\right)}{6\lambda_2} - \frac{\left(f - p_1\right)\left(v - \bar{v}_1\right)}{3} \\ & + \frac{\lambda_2 u_2\left(v - \bar{v}_1\right)}{3} + \frac{u_2\left(s - p_1\right)}{6} + \frac{\left(s - p_1\right)\left(v - \bar{v}_1\right)}{9} \end{aligned}$$

$$E\left(\pi_2^S|\Phi_2^S\right) = \frac{\left(s - p_1\right)^2}{9\lambda_2} + \frac{\lambda_2\left(v - \bar{v}_1\right)^2}{9} - \frac{2\left(s - p_1\right)\left(v - \bar{v}_1\right)}{9} \tag{2.39}$$

Recall that the insider knows the final liquidation value, i.e. $f \in \Phi_2^I$. Using the decomposition $\left(s - p_1\right) = \left(f - p_1\right) + \left(s - f\right)$ and $\left(s - p_1\right)^2 = \left(f - p_1\right)^2 + \left(s - f\right)^2 + 2\left(f - p_1\right)\left(s - f\right)$ the insider's expected profits can be equivalently written as

$$\begin{aligned} E\left(\pi_2^I|\Phi_2^I\right) = {} & \frac{\left(f - p_1\right)^2}{9\lambda_2} + \frac{\lambda_2 u_2^2}{4} + \frac{\left(s - f\right)^2}{36\lambda_2} + \frac{\lambda_2\left(v - \bar{v}_1\right)^2}{9} \\ & - \frac{\left(f - p_1\right)u_2}{3} + \frac{\left(s - f\right)u_2}{6} - \frac{2\left(f - p_1\right)\left(v - \bar{v}_1\right)}{9} \\ & + \frac{\lambda_2 u_2\left(v - \bar{v}_1\right)}{3} - \frac{\left(f - p_1\right)\left(s - f\right)}{9\lambda_2} + \frac{\left(s - f\right)\left(v - \bar{v}_1\right)}{9} . \end{aligned} \tag{2.40}$$

Step 2: date 1 trades

During the first trading round the insider chooses x_1 to maximize (2.2), or equivalently:

$$E\left[x_1\left(f-p_1\left(\omega_1\right)\right)\mid\Phi_1^I\right]+E\left[E\left(\pi_2^I|\Phi_2^I\right)|\Phi_1^I\right] \ , \tag{2.41}$$

where $E\left(\pi_2^I|\Phi_2^I\right)$ is as in eq. (2.40). Therefore when submitting his order x_1 the insider has to keep into account the impact of his trade on the price $p_1\left(x_1\right)$ *and* the speculator's forecast of the final liquidation value $s\left(x_1\right)$. Assume that under the insider's conjecture the first period price follows (2.20) and the speculator updates his beliefs on f according to eq. (2.12). The expression for x_1 depends on the insider's estimate –conditional on Φ_1^I– of the speculator's forecast, $E\left[E\left(f\mid\Phi_2^S\right)|\Phi_1^I\right]$, and the signal s depends on the speculator's conjecture of the form of x_1. The following Lemma gives the insider's estimate of the speculator's forecast s:

Lemma 2 *In equilibrium* $E\left(s-f\mid\Phi_1^I\right)=\chi\left(f-p_0\right)+\psi u_1$

Proof. Assuming that Lemma 2 holds, the insider's trading strategy (2.15) can be rewritten as $x_1 = a_1\left(v-p_0\right)-b_1u_1$, and (2.16) obtains setting $a_1=\alpha+\gamma\chi$ and $b_1=-\left(\beta+\gamma\psi\right)$. Given eq. (2.16), the speculator updates his belief on f after the first round according to $s=p_0+\phi\left(x_1+u_1\right)$ –which is eq. (2.22)– with regression coefficient

$$\phi=\frac{cov\left(f,x_1+u_1\mid\Phi_1^S\right)}{var\left(x_1+u_1\mid\Phi_1^S\right)}=\frac{a_1\sigma_{f,0}^2}{a_1^2\sigma_{f,0}^2+\left(1-b_1\right)^2\sigma_{u_1}^2} \ .$$

Plugging eq. (2.16) into the speculator's forecast (2.22) and taking expectations conditional on Φ_1^I gives $E\left(s-f\mid\Phi_1^I\right) = \chi\left(f-p_0\right)+\psi u_1$, where $\chi = a_1\phi-1$ and $\psi=\left(1-b_1\right)\phi$. Recall that under equation (2.16) the parameters a_1 and b_1 depend on the insider's forecast via χ and ψ, such that

$$a_1=\frac{\alpha-\gamma}{1-\gamma\phi} \quad \text{and} \quad b_1=-\frac{\beta+\gamma\phi}{1-\gamma\phi} \ .$$

For these expressions for a_1 and b_1, and the speculator's update (2.22) the conjecture in Lemma 2 is verified. ∎

Letting $\kappa_I = 2\lambda_1 + \phi - 2\lambda_2\mu$, under conjectures (2.20) and (2.22) the optimality conditions[24] for the objective function (2.41) are:

$$E\left(f - p_1(\omega_1)|\,\Phi_1^I\right)\left(1 - \frac{\kappa_I}{9\lambda_2}\right) - \lambda_1 x_1 + E\left(v - \bar{v}_1(\omega_1)|\,\Phi_1^I\right)\frac{\kappa_I}{9} + E\left(s - f|\,\Phi_1^I\right)\frac{\kappa_I}{18\lambda_2} = 0$$

$$2\lambda_1 - \frac{\kappa_I^2}{18\lambda_2} > 0$$

The insider's first period trade is linear, i.e.

$$x_1 = d^{-1}e\left(f - p_0\right) - \left(1 - d^{-1}\lambda_1\right)u_1\ ,$$

yielding eq. (2.16) with a_1, b_1, and d as given in the main text, and $e = 1 - \kappa_I (6\lambda_2)^{-1}$.

The speculator chooses his first period trade y_1 maximizing (2.13), or equivalently

$$E\left[y_1\left(f - p_1(\omega_1)\right)|\,\Phi_1^S\right] + E\left[E\left(\pi_2^S|\Phi_1^S\right)\right]\ ,$$

where $E\left(\pi_2^S|\Phi_1^S\right)$ is as in eq. (2.39). Plugging the expression for p_1 [see eq. (2.20)], the speculator's update [see eq. (2.22)], and the conjecture for x_1 [see eq. (2.16)] into the latter results in the following optimality conditions[25]:

$$E\left(s - p_1(\omega_1)|\,\Phi_1^S\right)\left(1 - \frac{2\kappa_S}{9\lambda_2}\right) - \lambda_1 y_1 + E\left(v - \bar{v}_1(\omega_1)|\,\Phi_1^S\right)\frac{2\kappa_S}{9} = 0$$

$$2\lambda_1 - \frac{2\kappa_S^2}{9\lambda_2} > 0$$

where $\kappa_S = \lambda_1 - \lambda_2\mu$. The speculator's first period trade is linear in $(v - \bar{v}_0)$, i.e. $y_1 = D^{-1}\frac{2\kappa_S}{9}(v - \bar{v}_0)$, and eq. (2.18) obtains with C_1 and D as in the main text.

Step 3: Prices

We now turn to determine the equilibrium prices. Let $\mathbf{r}$ denote the 3×1 random vector containing the final liquidation value, the passive trades known by the speculator, and the speculator's signal about the final payoff, i.e. $\mathbf{r} \equiv (f, v, s)^\top$. For the i–th component of the vector $\mathbf{r}$ we denote the unconditional variance by $\sigma_{r_i,0}^2 = var(r_i)$; similarly for $i \neq j$ the unconditional covariance is $\sigma_{r_i r_j,0} = cov(r_i, r_j)$. Further the unconditional variance of the first period aggregate order flow is denoted by $\sigma_{\omega,0}^2 = var(\omega_1)$ and the covariance between ω_1 and $r_i \in \mathbf{r}$ is $\sigma_{r_i\omega,0} = cov(r_i, \omega_1)$. The unconditional distribution for the random variables (f, u_1, v) together with the first period trades (2.16) and (2.18) and the speculator's signal s [see eq. (2.22)] yields

$$
\begin{aligned}
\sigma_{s,0}^2 &= \phi^2 \left(a_1^2\sigma_{f,0}^2 + (1-b_1)^2\sigma_{u_1}^2\right) \\
\sigma_{\omega,0}^2 &= a_1^2\sigma_{f,0}^2 + (1-b_1)^2\sigma_{u_1}^2 + C_1^2\sigma_{v,0}^2 \\
\sigma_{fs,0} &= \phi a_1 \sigma_{f,0}^2 \\
\sigma_{f\omega,0} &= a_1\sigma_{f,0}^2 \\
\sigma_{v\omega,0} &= C_1\sigma_{v,0}^2 \\
\sigma_{s\omega,0} &= \phi\left(a_1^2\sigma_{f,0}^2 + (1-b_1)^2\sigma_{u_1}^2\right)
\end{aligned}
\tag{2.42}
$$

Note that plugging the expression for ϕ [see eq. (2.24)] in the above variances and covariances gives $\sigma_{fs,0} = \sigma_{s,0}^2$ and $\sigma_{s\omega,0} = \sigma_{f\omega,0}$. Therefore the market maker's prior joint distribution for $(\mathbf{r}^\top, \omega_1)$ becomes

$$
\begin{bmatrix} \mathbf{r} \\ \omega_1 \end{bmatrix} \sim N\left(\begin{bmatrix} E_{\mathbf{r},0} \\ 0 \end{bmatrix}, \begin{bmatrix} \Sigma_{\mathbf{r},0} & \Sigma_{\mathbf{r}\omega,0} \\ \Sigma_{\mathbf{r}\omega,0}^\top & \sigma_{\omega,0}^2 \end{bmatrix} \right),
$$

where $E_{\mathbf{r},0} = (p_0, \bar{v}_0, p_0)^\top$, and

$$
\begin{aligned}
\Sigma_{\mathbf{r},0} &\equiv E\left[(\mathbf{r} - E_{\mathbf{r},0})(\mathbf{r} - E_{\mathbf{r},0})^\top\right] = \begin{bmatrix} \sigma_{f,0}^2 & 0 & \sigma_{s,0}^2 \\ 0 & \sigma_{v,0}^2 & 0 \\ \sigma_{s,0}^2 & 0 & \sigma_{s,0}^2 \end{bmatrix} \\
\Sigma_{\mathbf{r}\omega,0} &\equiv E\left[(\mathbf{r} - E_{\mathbf{r},0})\,\omega_1\right] = \begin{bmatrix} \sigma_{f\omega,0} & \sigma_{v\omega,0} & \sigma_{f\omega,0} \end{bmatrix}^\top .
\end{aligned}
$$

After observing the first period aggregate order flow the market maker updates his distribution for the random vector $\mathbf{r}$. The Projection Theorem together with eqs. (2.4) and (2.11) give[26] $p_1 = p_0 + \lambda_1\omega_1$ and $\bar{v}_1 = \bar{v}_0 + \mu\omega_1$, with regression coefficients $\lambda_1 = \sigma_{f\omega,0}/\sigma_{\omega,0}^2$ and $\mu = \sigma_{v\omega,0}/\sigma_{\omega,0}^2$ respectively. Using the variance-covariance matrix in (2.42) one has:

$$
\begin{aligned}
\lambda_1 &= \frac{a_1\sigma_{f,0}^2}{a_1^2\sigma_{f,0}^2 + (1-b_1)^2\sigma_{u_1}^2 + C_1^2\sigma_{v,0}^2} \\
\mu &= \frac{C_1\sigma_{v,0}^2}{a_1^2\sigma_{f,0}^2 + (1-b_1)^2\sigma_{u_1}^2 + C_1^2\sigma_{v,0}^2} .
\end{aligned}
$$

We now determine $\Sigma_{\mathbf{r},1}$ (the posterior variance for the vector $\mathbf{r}$). From $\sigma_{fs,0} = \sigma_{s,0}^2$ and $\sigma_{s\omega,0} = \sigma_{f\omega,0}$ it follows that $\sigma_{fs,1} = \sigma_{s,1}^2$ and $\sigma_{vs,1} = \sigma_{fv,1}$. Furthermore using the unconditional variance-covariance matrix in (2.42) together with the expression for ϕ, μ and λ_1 in the main text yields:

$$\begin{aligned} \sigma_{f,1}^2 &= (1 - a_1\lambda_1)\,\sigma_{f,0}^2 \\ \sigma_{v,1}^2 &= (1 - C_1\mu)\,\sigma_{v,0}^2 \\ \sigma_{s,1}^2 &= a_1(\phi - \lambda_1)\,\sigma_{f,0}^2 \\ \sigma_{fv,1} &= -a_1\mu\sigma_{v,0}^2 \end{aligned} \tag{2.43}$$

Note that the conditional variance between the fundamental value and the passive trades v can alternatively be written as $\sigma_{fv,1} = -C_1\lambda_1\sigma_{f,0}^2$.

Therefore $\mathbf{r}|\omega_1 \sim N(E_{\mathbf{r},1}, \Sigma_{\mathbf{r},1})$, where

$$E_{\mathbf{r},1} = E(\mathbf{r}|\omega_1) = (p_1, \bar{v}_1, p_1)^\top \ ,$$

and

$$\Sigma_{\mathbf{r},1} = E\left[(\mathbf{r} - E_{\mathbf{r},1})(\mathbf{r} - E_{\mathbf{r},1})^\top \Big| \omega_1\right] = \begin{bmatrix} \sigma_{f,1}^2 & \sigma_{fv,1} & \sigma_{s,1}^2 \\ \sigma_{fv,1} & \sigma_{v,1}^2 & \sigma_{fv,1} \\ \sigma_{s,1}^2 & \sigma_{fv,1} & \sigma_{s,1}^2 \end{bmatrix}$$

In the second trading round the market maker sets prices according to (2.5). Using the expression for date 2 trades (2.17) and (2.19) one has:

$$\omega_2 = a_2(f - p_1) + u_2/2 + a_2(s - p_1)/3 + v/3 + w + 2\bar{v}_1/3 \ .$$

It follows that $\omega_2|\omega_1 \sim N\left(\bar{z}_1, \sigma_{\omega,1}^2\right)$ with $\bar{z}_1 = \bar{v}_1 + \bar{w}$, and

$$\sigma_{\omega,1}^2 = var(\omega_2|\omega_1) = a_2^2\sigma_{f,1}^2 + \sigma_{u_2}^2/4 + 7a_2^2\sigma_{s,1}^2/9 + \sigma_{v,1}^2/9 + \sigma_w^2 + 8a_2\sigma_{fv,1}/9 \ .$$

Furthermore the conditional covariance between the fundamental value and the date 2 order flow is

$$\sigma_{f\omega,1} = cov(f, \omega_2|\omega_1) = a_2\sigma_{f,1}^2 + a_2\sigma_{s,1}^2/3 + \sigma_{fv,1}/3 \ .$$

Letting $\lambda_2 = \sigma_{f\omega,1}/\sigma_{\omega,1}^2$, date 2 prices follow from the Projection Theorem:

$$p_2 = p_1 + \lambda_2(\omega_2 - \bar{z}_1)$$

which is eq. (2.21). Note that both $\sigma_{f\omega,1}$ and $\sigma^2_{\omega,1}$ depend on the insider's trading aggressiveness a_2, which in turn depends on λ_2 only. Since $a_2 = (2\lambda_2)^{-1}$, the regression coefficient λ_2 solves the quadratic equation

$$\Sigma\lambda_2^2 + 4\lambda_2\sigma_{fv,1} + \left(\sigma^2_{s,1} - 9\sigma^2_{f,1}\right) = 0 \tag{2.44}$$

where Σ is defined in the main text. Lemma 3 addresses the existence of real roots for eq. (2.44) as well as the uniqueness for λ_2.

Lemma 3 *In equilibrium λ_2 is given by:*

$$\lambda_2 = \Sigma^{-1}\left(-2\sigma_{fv,1} + \left[4\left(\sigma_{fv,1}\right)^2 + \Sigma\left(9\sigma^2_{f,1} - \sigma^2_{s,1}\right)\right]^{1/2}\right)$$

Proof. To prove Lemma 3 we proceed in two steps. First we show that both the solutions for λ_2 in eq. (2.44) are real; then we use the second order conditions to pin down the positive root and get λ_2 as in the main text. From eq. (2.44) a sufficient condition for λ_2 to belong to the real line is $\sigma^2_{f,1} > \sigma^2_{s,1}$. Making use of the conditional variances in (2.43) one has $\sigma^2_{f,1} - \sigma^2_{s,1} = (1 - a_1\phi)\,\sigma^2_{f,0}$. Substituting for the expression for ϕ in the latter gives:

$$\sigma^2_{f,1} - \sigma^2_{s,1} = \frac{(1-b_1)^2\,\sigma^2_{u_1}}{a_1^2\sigma^2_{f,0} + (1-b_1)^2\,\sigma^2_{u_1}}\sigma^2_{f,0} > 0$$

such that the roots for (2.44) are real. Recall that in equilibrium $\lambda_2 > 0$. Thus, regardless of the sign of $\sigma_{fv,1}$, the negative root in (2.44) can be discarded and λ_2 in the main text obtains. ■

Eventually the liquidation value residual variance after the second trading round becomes:

$$\sigma^2_{f,2} = \sigma^2_{f,1} - \lambda_2\sigma_{f\omega,1} = \left(3\sigma^2_{f,1} - \sigma^2_{s,1} - 2\lambda_2\sigma_{fv,1}\right)/6\ .$$

Therefore an equilibrium for the trading game is described by solutions for (a_1, b_1, C_1, μ, ϕ, λ_1, λ_2) in Proposition 2 subject to the nonlinear constraint (2.26).

As for the expected trading volume, note that date 1 orders have mean zero. It follows that the expressions for V_1^I, V_1^L and V_1^M are given by the first line in (2.36) [keeping into account that parameters a_1 and b_1 are as in Proposition 2 and $\sigma^2_{\omega,0}$ is given in eq. (2.42)]. Similarly $E_1(u_2) = 0$, such that V_2^L is like in the second line of

(2.36). The other contributions to the trading volume are:

$$\begin{aligned}
V_1^S &= \frac{C_1\sigma_{v,0}}{\sqrt{2\pi}} \\
V_2^I &= \frac{\sigma_{x,1}}{\sqrt{2\pi}} \quad ; \quad V_2^S = \sqrt{\frac{C_2^2\sigma_{s,1}^2 + \sigma_{v,1}^2/9 - 2C_2\sigma_{fv,1}/3}{2\pi}} \\
V_2^P &= \frac{\sigma_{z,1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\bar{z}_1}{\sigma_{z,1}}\right)^2} + \frac{\bar{z}_1}{2}\left(1 - 2\Phi\left(-\bar{z}_1/\sigma_{z,1}\right)\right) \\
V_2^M &= \frac{\sigma_{\omega,1}}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\bar{z}_1}{\sigma_{\omega,1}}\right)^2} + \frac{\bar{z}_1}{2}\left(1 - 2\Phi\left(-\bar{z}_1/\sigma_{\omega,1}\right)\right)
\end{aligned}$$

where $\sigma_{x,1} = \left(a_2^2\sigma_{f,1}^2 + \sigma_{u2}^2/4 - 5a_2^2\sigma_{s,1}^2/9 + \sigma_{v,1}^2/9 - 4a_2\sigma_{fv,1}/9\right)^{1/2}$ and $\sigma_{z,1} = \left(\sigma_{w,0}^2 + \sigma_{v,1}^2\right)$.

Proof (Corollary 1). From the date 2 second order condition $\lambda_2 > 0$. Therefore from the date 1 second order condition $\lambda_1 > 0$, since $\lambda_1 > \kappa_s^2/18\lambda_2 \geq 0$. Now suppose that $C_1 \leq 0$. From the expression for C_1 this implies that $\kappa_s \leq 0$ and from the expression for μ this implies that $\mu \leq 0$. Since κ_S is defined by $\kappa_S = \lambda_1 - \lambda_2\mu$ one has $\kappa_S \leq 0$ if and only if $\lambda_1 \leq \lambda_2\mu$. However $\lambda_2 > 0$ and $\mu \leq 0$, implying $\lambda_1 \leq 0$ which cannot occur in equilibrium. Finally note that the insider's second order condition can be written as $2\lambda_1 - \frac{2\kappa_S^2}{9\lambda_2} - \frac{\phi^2+4\kappa_S\phi}{18\lambda_2} > 0$. Since $\kappa_S > 0$, then $2\lambda_1 - \frac{2\kappa_S^2}{9\lambda_2} - \frac{\phi^2+4\kappa_S\phi}{18\lambda_2} < 2\lambda_1 - \frac{\phi^2+4\kappa_S\phi}{18\lambda_2}$ and (2.26) is sufficient for $2\lambda_1 - \frac{2\kappa_S^2}{9\lambda_2} > 0$. ■

Notes

[1] Investigations on indices other than the S&P 500 can be found in Bos (2000) and Jain (1987) who report evidence for supplementary S&P Indices. Madhavan (2003) focuses on changes to the Russell 2000 and 3000 indices. The Dow Jones Industrial Average is considered in Beneish and Gardner (1995) and Polonchek and Krehbiel (1994), while Deininger, Kaserer and Roos (2000) study the German DAX and MDAX.

[2] We stress the fact that indexers enter the market at the second date only with the subscript 2 in the passive industry demand.

[3] As in Kyle (1985) the efficiency condition arises from price competion à-la-Bertrand in the market making sector. In equilibrium one can consider a single market maker that operates according to a zero expected profits condition.

[4] One can accomodate the insider receiving a noisy signal of the fundamental value rather than the realization f. In this case the insider's informational advantage would be captured by the signal to noise variance, rather than the fundamental value variance only. Qualitatively this does not affect our main conclusions.

[5] We use depth and liquidity as synonyms, even though they capture different aspects of market behaviour.

[6] The insider's informational advantage with respect to the fundamental value is captured by the ratio between fundamental and non-fundamental uncertainty, like in Kyle (1985) and its various extensions. Since we are focusing on the role of non-fundamental information, we refer to k_I as the insider's advantage only.

[7] Parameters at equilibrium are computed using the following values for k_I = 0.01, 0.5, 1, 2, 3.5, 5, 7.5, 10, 15 and 20.

[8] This is not surprising since parameters for RV can be obtained from Proposition 1 setting $\sigma^2_{z,0} = 0$ (appendix A contains further details). For $k_I = 500$ the difference between the parameters in Proposition 1 and their RV counterparts is of the order of 10^{-4}. For reasons of space we consider $k_I \leq 20$ in figures 2.2–2.6.

[9] The following expression follows from the fact that insider's orders are conditionally normal with mean zero, implying that the volume is (a multiple of) the conditional standard deviation. The total volume pattern is discussed in section 2.4.

[10] We note that it might be difficult to justify that both the speculator and the insider observe the *same* fraction of date 1 liquidity orders like in Madrigal (1996).

[11] A related problem is that other fund managers might implement mimicking techniques only for a fraction of their portfolios, and thus would not show up in surveys on completely passive funds like the Standard and Poor's (2003).

[12] Madrigal (1996) imposes a hierarchical information structure as well. As previously noted in his model the insider and the speculator share the knowledge of past noise trades when trading

at date 2. This way the insider knows –on top of the final liquidation value– the speculator's informational advantage relative to the market maker, i.e. the difference $s - p_1$, like in our specification

[13] On the other hand the first period information sets are non-nested. This assumption can be easily modified including v into Φ_1^I. In this case, the speculator would lose his informational advantage (with respect to the insider) and competition between the two informed agents would arise at date 1. However, given the previous considerations on the difficulty in gathering information about passive funds' techniques, we do not regard this situation as particularly interesting.

[14] In equilibrium $E\left(s|\,\Phi_1^I\right) = f + \chi\left(f - p_0\right) + \psi u_1$ and $s = p_0 + \phi\left(x_1 + u_1\right)$ are mutually consistent, and as a consequence the insider's first period trade can be expressed as $x_1 = a_1\left(f - p_0\right) - b_1 u_1$. We derive the expression for coefficients χ, ψ and ϕ in appendix B (see Lemma 2).

[15] The first period aggregate order flow does not contain information about u_2. Therefore conditional on ω_1 the second period liquidity trades have mean zero. Recall that liquidity trades are independent through time and orthogonal to passive funds' orders.

[16] Recall that the speculator has long-lived non-fundamental information, as in Kyle (1985) the insider has long-lived information about the final liquidation value f. Thus it is not surprising that the speculator's behaviour with respect to v closely resembles the insider's aggressiveness on f in Kyle (1985).

[17] V_1^M decreases with k_I, but a slower rate than the increase for V_1^I; moreover V_1^L does not depend on k_I.

[18] We are grateful to Nicholas Barberis and Jeffrey Wurgler for sharing their dataset.

[19] Other authors use market adjusted trading turnover, i.e. $V_{i,t}/V_{M,t}$, where $V_{M,t}$ is the NYSE volume during day t. Market adjustment results in stronger tests by taking into account market variation. Harris and Gurel (1986) report that the qualitative results are not affected by the way one measures trading volume. Accordingly, our results are qualitatively the same when using raw trading volume $V_{i,t}$. Cusick (2002) and Lynch and Mendenhall (1997) use a logarithmic transformation of the market adjusted trading volume.

[20] The evidence reported in this and the following subsections is quite robust to the pre-event window choice. Inclusion of the announcement day in computing average volume and bid-ask spread does not qualitatively change our results.

[21] When computing the $t-$statistic we opt for the cross-sectional dispersion of $MAVR_t$ to estimate its variance. See Lynch and Mendenhall (1997) for an alternative method of computing standard errors. Details on the binomial test are in Hollander and Wolfe (1999).

[22] Since the number of trading days between AD and CD varies across companies (see figure 2.10), the column labeled N_t in each panel in table 2.2 reports the number of stocks included in the sample. For each of the ten days after AD in panel A, only those firms for which CD has

not yet occurred are included. This is why the sample size in the second column decreases over the days after AD in panel A. Similarly, for the ten days preceding CD, only firms for which AD has not yet occurred are included, such that the sample size increases over the ten days before CD in panel B.

23 Using the effective relative spread $2\,|\ln(P_{i,t}/Q_{i,t})|$ does not change the results presented here. For reasons of space we report findings for the relative spread only.

24 Using eqs. (2.20) and (2.22) the terms in the insider's profits (2.40) can be written as functions of x_1 according to $(f - p_1) = (f - p_0) - \lambda_1 [x_1 + C_1 (v - \bar{v}_0) + u_1]$, $(s - f) = -(f - p_0) + \phi(x_1 + u_1)$ and $(v - \bar{v}_1) = (v - \bar{v}_0)(1 - C_1\mu) - \mu(x_1 + u_1)$.

25 Note that (2.20), (2.22) and (2.16) allow to write $f - p_1 = (1 - a_1\lambda_1)(f - p_0) - \lambda_1 [y_1 + (1 - b_1) u_1]$, $s - p_1 = (\phi - \lambda_1)[a_1 (v - p_0) + (1 - b_1) u_1] - \lambda_1 y_1$ and $v - \bar{v}_1 = -\mu [a_1 (f - p_0) + (1 - b_1) u_1] - \mu y_1 + (u_2 - \bar{u}_0)$. Further, since the date 1 speculator's information set does not include f one has $E\left(f - p_1 \middle| \Phi_1^S\right) = E\left(s - p_1 \middle| \Phi_1^S\right) = -\lambda_1 y_1$.

26 From the definition of s one has $E(s|\omega_1) = E[E(f|x_1 + u_1)|\omega_1]$. Since ω_1 is coarser than $x_1 + u_1$, then using the Law of Iterated Expectations $E[E(f|x_1 + u_1)|\omega_1] = E(f|\omega_1) = p_1$.

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3

Information Sharing and Dynamic Trading

3.1 Introduction

The vast majority of the literature on asymmetric information in asset markets have pointed at the role market prices have in aggregating the information dispersed among agents. Starting with the seminal work by Grossman and Stiglitz (1980), countless models have been proposed to investigate how investors incorporate their information into trades, and how market participants extract information from prices and other aggregate variables over time. The way an insider dilutes his information over time to hinder the market maker's inference is best described by Kyle (1985). By trading more aggressively, the insider impounds more information into his order. This in turn reduces the informational asymmetry with respect to the market maker. As a result liquidity improves, and the insider benefits from better terms of trade. As the insider trades more aggressively however, he loses some of his informational advantage, hence foregoing part of his gains. In equilibrium orders are submitted in such a way that the former effect counterbalances the latter. Foster and Viswanathan (1996) generalize Kyle (1985) allowing for multiple insiders. In their model each trader has not only to factor in the market maker's inference [as in Kyle (1985)], but also the information other insiders' extract from the aggregate order flow. These models illustrate in an exemplary way how the literature has emphasized the role of trading as a vehicle to convey private information to the market at large.

There is another channel through which information can be revealed to agents: direct information sharing. Every day traders exchange opinions and share their views about financial assets, either because they discuss upon meetings at the marketplace or via message posting in financial forums. At a first glance these activities might be difficult to conceive. In fact, standard market microstructure theory predicts that asymmetric information is crucial to generate profits from trading. Direct communication of private signals about asset payoffs would therefore seem to reduce traders' informational rent. Yet the empirical literature has documented several instances of peer-to-peer communication and related social interactions phenomena in financial markets. As an example, several studies have analyzed the impact of internet discussion sites and message postings on aggregate market variables such as prices, volume and return volatility [e.g., Tumarkin and Whitelaw (2001) and Antweiler and Frank (2004)]. Furthermore, there is strong evidence that proximity influences investors' portfolio choices. Coval and Moskowitz (1999) show that US fund managers are more prone to invest in locally headquartered firms. They suggest that the preference for geographically proximate investments is driven by asymmetric information between local and nonlocal investors. Hong, Kubik and Stein (2003) document that fund managers quartered in the same city exhibit similar portfolio choices, and show that this finding holds even when controlling for the distance between managers and traded stocks. Thus the authors argue that such correlated portfolio choices arise (1) through peer-to-peer communication, and/or (2) simply because fund managers in a given area commit themselves to investment decisions based upon common sources of information - such as a local newspaper or TV station.

In this chapter, we develop a dynamic model of trading in which asymmetrically informed traders share (portions of) their information endowments. A key feature of our model is that traders engage in, or experience, information sharing through local connections. These local connections give rise to "reference groups" which may include only one's closest neighbours or even the entire market. Indeed, there are no obvious arguments suggesting whether information sharing phenomena should be best thought of as being local or global. In some important cases, however, the size of reference groups endogenously emerges in our model. Our framework encompasses two natural modes of information sharing. In the first one, information sharing arises endogeneously –as a result of traders' optimal choices. In the second one, information sharing occurs because traders have access to common sources of information. We characterize market conditions favouring information sharing among traders. We find that information sharing entails significant gains under a wide range of conditions on initial beliefs heterogeneity and the market structure –as summarized by the number of traders and batch auctions. The explanation for our findings is indeed simple. In our model, traders face a crucial trade-off. On the one hand, information sharing entails a loss in the traders' monopolistic power. On the other hand, information sharing improves the quality of traders' inference about the fundamental asset value. When the initial correlation between individual signals is high enough, the losses generated by the former effect are smaller than the gains associated with the latter effect. One important prediction of the model is that, in the presence of information sharing, traders postpone as long as possible their trades to compensate for their loss in monopolistic power. This in turn generates an intense waiting game, and deeper information asymmetries with respect to the market maker. If again the initial correlation between individual signals is sufficiently high, these information asymmetries also contribute to making information sharing beneficial. Finally, our model predicts that in many circumstances, traders may find it profitable to engage in information sharing with an *optimal* number of peers not necessarily equal to the remaining traders. In other terms, (local) information sharing groups might emerge as a result of optimal choices of traders.

Our analytical framework extends the one in Foster and Viswanathan (1996). In Foster and Viswanathan, every trader is endowed with one signal about the future value of an asset, and the correlation between any two signals is the same for all agents. In our model, information sharing destroys such an homogeneity assumption, and induces patterns of signals correlation varying with traders' geographical proximity. As a result of local information sharing, some traders may thus agree more with some and less with other peers. Furthermore, our model predicts that in some cases, any two traders may not be directly exchanging their initial signals, but still exhibit highly correlated information endowments. This phenomenon occurs when two traders share information with a third trader who is exchanging his signal with each of the initial two traders.

On a strictly theoretical standpoint, our information sharing model is closely related to the literature on markets for information. For instance, Admati and Pfleiderer (1986, 1988a, 1990), Biais and Germain (2002), and Brennan and Chordia (1993) focus on the sale of information by financial intermediaries. Financial intermediaries sell information to their clients either directly, via newsletters and buy/sell recommendations, or indirectly, through a mutual fund. Similarly, we consider traders exchanging information among themselves (rather than selling information against a monetary price). However our model differs from the mentioned literature on the sale of information in many respects. First, our market for information is not monopolistic because all agents are endowed with some information they may subsequently exchange. Second, previous work has focused on markets in which one monopolistic financial intermediary sells information to a pool of *uninformed* investors. Third, no *explicit* compensation scheme for the information seller is needed in our model. Naturally, this does not mean that information cannot be priced. In our model, the value of information is measured by the difference in each agent's ex-ante expected profit with and without information exchange. This in turn simplifies our problem, since we do not have to arbitrarily specify how the information provider is compensated.[1] Finally, agents both exchange information and subsequently trade in securities markets. In Admati and Pfleiderer (1986) and Brennan and Chordia (1993), for example, the financial intermediary does not trade on his account. Finally, information is exchanged only once, while trading takes place in a sequence of batch auctions. Thus, we are ruling out any role for reputation. However, we still consider traders extracting other traders' information from past prices. Such a dynamic inference problem is not considered in Admati and Pfleiderer (1986, 1988a, 1990), Brennan and Chordia (1993) and Biais and Germain (2002), who instead develop static models.

An important assumption of our framework (as well as of all the previously mentioned models) is that traders engage in *truthful* information exchange. Our model thus rules out strategic information transmission [e.g., Benabou and Laroque (1992)] or information-based price manipulation [as in Allen and Gale's (1992) terminology]. In contrast, the model we consider can be thought of one analyzing a market for information in which no trader is a "guru" with respect to others. Information-based price manipulation is illegal in many countries, and cases of information-based manipulation appear to be relatively limited. As an example, there is weak evidence that internet stock messages are posted to manipulate markets. Tumarkin and Whitelaw (2001) do not find any causal link between message board activity and returns (or volume) based on observations from the RacingBull.com discussion forum. Similarly, Antweiler and Frank (2004) find that the impact of Yahoo! internet posting on returns predictability is economically very small. However, understanding the role of strategic communication within our model can be an interesting topic for future research.

The incentives to information exchange activities have been addressed in related fields. Within the industrial organization literature, Novshek and Sonnenschein (1982), Clarke (1983) and Vives (1984) pioneered the role of information sharing in oligopoly, which has been subsequently analyzed thoroughly under different market structures [see Raith (1996) for a survey].[2] This chapter introduces information sharing activities in models with asymmetric information in financial markets, but it has two important distinctive features. First, the demand elasticity faced by oligopolists is exogeneous in the industrial organization literature; here instead we consider a model *à la* Kyle (1985), and the price reaction to the order flow is therefore endogeneous. Second, all models in the industrial organization literature consider global, "marketwide" information sharing. Our model also allows traders to share their information with selected neighbours.

Information sharing motives have been studied in credit markets as well. Pagano and Jappelli (1993) and Padilla and Pagano (1997, 2000) identify conditions under which banks find it profitable to exchange information about their customers' quality [see Jappelli and Pagano (2000) for a recent survey on these theoretical models]. Under uncertainty about the borrower's quality, credit bureaus allow lenders to improve their knowledge about new customers, at the cost of giving up to competitors one's informational rent about existing customers. As we argued, asymmetrically informed traders in financial markets face a similar trade-off in our model. However, our analysis is further complicated because the price reaction to traders' strategies is endogeneously determined. The adverse selection costs faced by the market maker constitute another factor traders must consider when choosing whether to engage in information sharing activities.

The article is organized in the following manner. In the next two sections, we develop the basic information structure of our trading game. In section 3.4, we derive a dynamic equilibrium and in section 3.5, we analyze its properties. Section 3.6 concludes. The appendix contains all technical details omitted in the main text.

3.2 Information structure

3.2.1 Asset market and signals distribution

As in Kyle (1985) and in his subsequent extensions, we consider a market for one risky asset organized in $N \geq 1$ batch auctions. This asset pays a random payoff $f \sim N(0, \sigma_{f,0}^2)$ at the end of the trading period, and M traders receive private signals about f. Let trader i be endowed with signal $s_{i,0}$ and let $(x_{i,n})_{n=1}^N$ be his orders submitted over the trading period. Individual signals are jointly normal with mean zero and $\Psi_0 \equiv E\left[(s_{1,0}, \cdots, s_{M,0})^\top (s_{1,0}, \cdots, s_{M,0})\right]$ variance-covariance matrix. The signal unconditional distribution is symmetric in that: (1) each signal has variance Λ_0, (2) the covariance between any two signals is Ω_0, and (3) the covariance between each signal and the fundamental value is c_0. Let $\mathbf{s}_0$ be the $M \times 1$ vector of individual signals and $\mathbf{1}$ be the $M \times 1$ vector of ones. The joint distribution of the vector $(f, s_{1,0}, \cdots, s_{M,0})^\top$ is given by:

$$\begin{bmatrix} f \\ \mathbf{s}_0 \end{bmatrix} \sim N\left(\begin{bmatrix} 0 \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \sigma_f^2 & c_0\mathbf{1}^\top \\ c_0\mathbf{1} & \Psi_0 \end{bmatrix} \right), \quad \Psi_0 \equiv \begin{bmatrix} \Lambda_0 & \Omega_0 & \cdots & \Omega_0 \\ & \Lambda_0 & & \Omega_0 \\ & & \ddots & \vdots \\ & & & \Lambda_0 \end{bmatrix}. \tag{3.1}$$

Finally, a sector of noise traders submit (perhaps liquidity motivated) orders $(u_n)_{i=1}^N$, where $u_n \sim NID\left(0, \sigma_u^2\right)$ for all n. The aggregate order flow is therefore given by:

$$y_n = \sum_{i=1}^M x_{i,n} + u_n, \quad n = 1, \cdots, N. \tag{3.2}$$

The $(M+1)$-th market participant is a market maker who commits himself to offset the order flow according to the Semi-Strong efficiency rule:

$$p_n = E\left(f \mid y_1, \cdots, y_n\right), \quad n = 1, \cdots, N.$$

The previous asset market and signals distribution structure is essentially the one in Foster and Viswanathan (1996). One of its attractive features lies in the tractability of the resulting model. As an example, the homogeneous correlation structure simplifies each trader's dynamic inference about other traders' signals, and allows to avoid infinite regress problems. While such an homogeneity property is analytically appealing, it is also interesting to analyze situations in which agents may experience some degree of heterogeneity in their signals correlation structure. In the next subsection, we use the previous (homogeneous correlation) information framework as a starting point, and develop novel information structures with heterogenous correlation distributions which are still exempt from infinite regress issues.

3.2.2 Traders' location and information sharing protocols

We view the previous market structure as one in which traders are physically located around a circle. All traders are risk neutral and solve the following optimization problem

$$\text{Max}_{(x_{i,n})_{n=1}^{N}} E\left[\sum_{n=1}^{N} (f - p_n)\, x_{i,n} | F_{i,n}\right]$$

where $F_{i,n} = \{s_{i,0}, (y_{i,t})_{t=1}^{n-1}, (x_{i,t})_{t=1}^{n-1}\}$ denotes trader i information sets at the n-th batch auction. By convention, we assume that traders are ordered clockwise, as to say that trader i has trader $i+1$ to his left and trader $i-1$ to his right (see figure 3.1). For reasons developed below, we assume that M is an odd number. We now describe two possible patterns of signal sharing amongst any two "sufficiently" adjacent traders.

3.2.2.1 Information sharing I: *Endogeneous*

The game begins when traders choose whether to share their information or not. We consider the simple situation in which any information sharing agreement may only take place before (and therefore is not affected by) the very observation of the signals. While such an agreement may be undertaken for a variety of reasons, we shall focus on situations in which information sharing arises because it is simply in the best interest of agents. Furthermore, we assume that agents do not make any strategic use of their private information. Thus, if any two traders choose to share their signals, both of them would communicate the precise content of their signals. This corresponds to the "external agency" assumption in the industrial organization literature, by which firms delegate an external agency to (1) receive their signals, and (2) redistribute the signals to all firms participating to the information sharing agreement [see, for example, Gal-Or (1985)].

As for the remaining institutional details, we assume that trader i can share his information with traders to his right and to his left. We consider "double-sided" information sharing. That is, if trader i decides to share his signal with G traders to his right, then he shares his signal with G traders to his left as well. For example trader i may want to share his signal with traders $i-1$ and $i+1$ (see figure 3.1). In this case he gives $s_{i,0}$ to both traders and receives $s_{i-1,0}$ and $s_{i+1,0}$ in exchange. Since we focus on symmetric equilibria, all traders choose the same G. After exchanging information with other $2G$ traders, the i-th trader information set becomes $\mathrm{s}_{i,0} = (s_{i-G,0}, \cdots, s_{i,0}, \cdots, s_{i+G,0})^{\top}$, $G \in [0, (M-1)/2]$.[3] To easy notation, we let $\hat{G} = 2G+1$ be the reference group size, that is the number of signals in each trader's information set after information sharing has occurred between traders. Thus, if no information sharing occurs, $\hat{G} = 1$, and $\mathrm{s}_{i,0} = s_{i,0}$ for all i. Complete information sharing occurs whenever $\hat{G} = M$, in which case $\mathrm{s}_{i,0} = \mathrm{s}_0$ for all i. Values of $\hat{G}$ between these two polar cases identify intermediate cases. As is clear, our model generates equilibria which are isomorphic to others in which any two traders are credibly *selling* their own signals to each other. This is so because the price of every signal is exactly the same in such symmetric equilibria.

3.2.2.2 Information sharing II: *Exogeneous*

An alternative information sharing protocol may arise at the outset, as a result of the physical location of traders. According to this mechanism, every signal $s_{i,0}$ is made available at trader i's location, and observed by trader i as well as his adjacent peers (i.e. traders $i \mp k$, $k = 1, \cdots, G$). As an example, every signal $s_{i,0}$ can be thought of as being broadcasted to trader's i location through a local newspaper or TV station. Reference groups among traders then arise because different traders have access to the same source of information. If $\hat{G} = 1$, every trader gathers information from a unique local source of financial news, and there is no information sharing amongst agents. At the other extreme, all pieces of information are provided at a marketwide level whenever $\hat{G} = M$. In general, the group size $\hat{G} \in [1, M]$, and thus represents the media coverage of information providers.

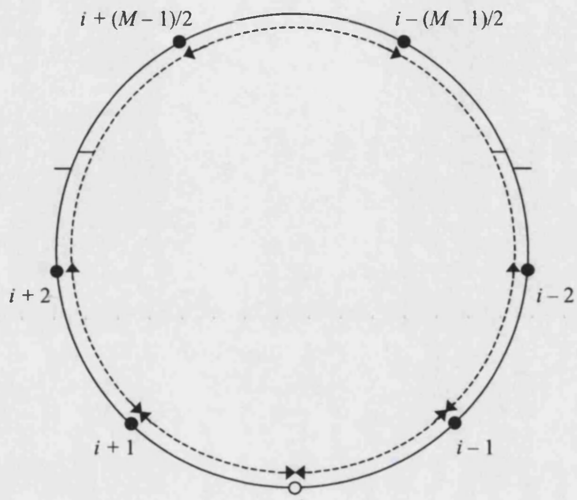
3.3 Heterogeneous signals distribution structures

3.3.1 Average signals

Let $\bar{s}_{i,0}$ denote trader i's average signal:

$$\bar{s}_{i,0} = \hat{G}^{-1} \sum_{k=-G}^{G} s_{i+k,0}. \tag{3.3}$$

FIGURE 3.1. Geographical location of traders.



The i-th trader has $(M-1)/2$ traders to his left and $(M-1)/2$ traders to his right. In this particular example, every trader shares his own signal with $2G = 2$ traders: one trader on his left and one trader on his right.

We refer to the *full information liquidation value* as the expectation of the final value conditional on the information disseminated among traders, i.e. $E(f|\mathbf{s}_0)$. Let $\kappa = c_0(\Lambda_0 + (M-1)\Omega_0)^{-1}$, $\theta = \kappa M$ and $\bar{s} = M^{-1}\sum_{i=1}^{M}\bar{s}_{i,0}$. By eq. (3.1),

$$E(f|\mathbf{s}_0) = \theta\bar{s}. \tag{3.4}$$

Therefore $\bar{s}$, the average of the individual average signals, is a sufficient statistic for the full information liquidation value. Note that θ is well defined whenever the matrix Ψ_0 is invertible. Such an invertibility condition requires the following restriction on the model parameters:

$$\Lambda_0 > -(M-1)\Omega_0. \tag{3.5}$$

The unconditional variance-covariance matrix of the average signals $(\bar{s}_{i,0})_{i=1}^{M}$ is denoted as $\bar{\Psi}_0 \equiv E\left[(\bar{s}_{1,0},\cdots,\bar{s}_{M,0})^\top(\bar{s}_{1,0},\cdots,\bar{s}_{M,0})\right]$. In general we expect the elements of this matrix to depend on the group size $\hat{G}$. Accordingly, we set $\bar{\Psi}_0 \equiv \bar{\Psi}_0(G)$, where

$$\bar{\Psi}_0(G) \equiv \begin{bmatrix} \bar{\Lambda}_0(G) & \bar{\Omega}_0(1,G) & \cdots & \bar{\Omega}_0\left(\frac{M-1}{2},G\right) & \cdots & \bar{\Omega}_0(-1,G) \\ & \bar{\Lambda}_0(G) & & & & \bar{\Omega}_0(-2,G) \\ & & & \ddots & & \vdots \\ & & & & & \bar{\Lambda}_0(G) \end{bmatrix},$$

and the elements

$$\begin{cases} \bar{\Lambda}_0(G) & = \quad var(\bar{s}_{i,0}) \\ \bar{\Omega}_0(k,G) & = \quad cov(\bar{s}_{i+k,0}, \bar{s}_{i,0}), \quad k = \mp 1, \mp 2, \cdots, \mp\frac{M-1}{2} \end{cases}$$

denote, respectively, the unconditional variance of average signals and the unconditional covariance between the average signals of any two traders who are k-positions apart ($k \neq 0$). $\bar{\Lambda}_0(G)$ is constant across traders due to the symmetric unconditional distribution in eq. (3.1) and the fact that G is the same for all traders in equilibrium. Furthermore, due to the symmetric nature of our information sharing protocol, one has $\bar{\Omega}_0(k,G) = \bar{\Omega}_0(-k,G)$. Finally, we define the unconditional covariance between the sum of other traders' average signals with $\bar{s}_{i,0}$ as:

$$\bar{\Gamma}_0(G) = cov\left(\sum_{j\neq i} \bar{s}_{j,0}, \bar{s}_{i,0}\right).$$

Due to the geographical location of agents in this model, $\bar{\Gamma}_0(G)$ is independent of i. Finally, the covariance between the average signal $\bar{s}_{i,0}$ and the fundamental value is simply

$$\bar{c}_0 = cov(f, \bar{s}_{i,0}) = c_0,$$

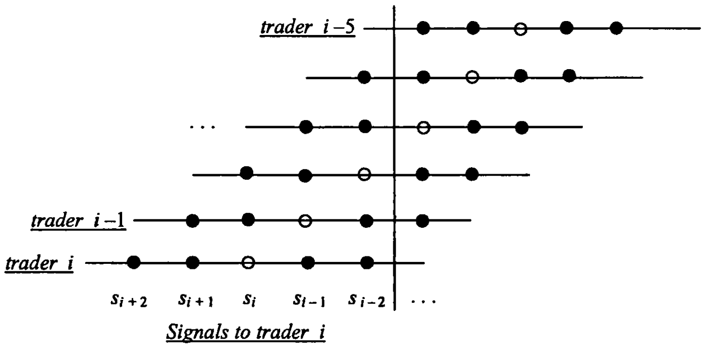
and does not depend on G.

3.3.2 Correlations

By the distributional assumption in (3.1), and the definition of the average signal $\bar{s}_{i,0}$ in (3.3),

$$\bar{\Lambda}_0(G) = \frac{\Lambda_0 + 2G\Omega_0}{\hat{G}}. \tag{3.6}$$

FIGURE 3.2. Overlapping information sets ($G = 2$, $M = 11$).



For any agent, empty circles denote bits of information this particular agent is endowed with. Filled circles are bits of information received by neighbors. Left (to empty) signals are received by left neghbors, and right (to empty) signals are received by right neighbors. Alternatively, any empty circle is transmitted up and down (e.g. $i-1$ gives his signal up and down).

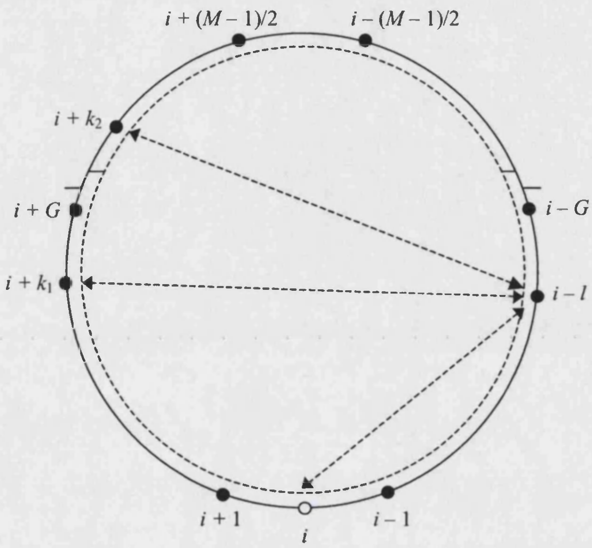
Take any two traders i and $j = i + k$ with $k \neq 0$, and consider the unconditional covariance between average signals $\bar{\Omega}_0(k, G)$. Whenever $\hat{G} < M$, one might expect that the covariance between average signals depends not only on G but on k as well, i.e. the distance between trader i and $i + k$.[4] This is due to the fact that information sharing results in trader i gathering $2G$ additional signals from his neighbours. However the *number* of individual signals each trader shares with other market participants depends on their relative position along the circle. As an example, assume that $2G < (M-1)/2$. In this case trader i shares $2G$ signals with trader $i+1$, $2G-1$ signals with trader $i+2$ and in general $2G+1-k$ signals with trader $i+k$. Eventually, trader i shares no signals with trader $i+2G+1$ and beyond (see figure 3.2). As the simple example in figure 3.2 demonstrates, the covariance between *average* signals does in general depend on k in our model.

To compute the various covariances, we have to distinguish between two cases according to whether $2G$ is less or greater than $(M-1)/2$. If $2G \leq (M-1)/2$, one has $\mathrm{s}_{i+k,0} \cap \mathrm{s}_{i,0} = \{\emptyset\}$ whenever $|k| > 2G$ (as in figure 3.2), which implies $\bar{\Omega}_0(k, G) = \Omega_0$. On the other hand, $s_{i+k,0} \in \mathrm{s}_{i,0}$ for all $|k| \leq 2G$. As we show in the appendix,

$$\text{For } 2G \leq \tfrac{M-1}{2},\ \bar{\Omega}_0(k, G) = \begin{cases} \bar{\Lambda}_0(G) - \hat{G}^{-2}k(\Lambda_0 - \Omega_0), & \text{for } k \in [1, 2G+1] \\ \\ \Omega_0, & \text{for } k \in \left[2G+1, \tfrac{M-1}{2}\right] \end{cases} \tag{3.7a}$$

If (and only if) $2G \geq (M-1)/2$, agents may share additional signals due to a double overlap occurring when traders on the right semicircle within trader i's reach

FIGURE 3.3. Double overlap.



Traders on the left semicircle receive bits of information sent by traders on the right semicircle sharing information with trader # i.

send signals to traders on the left semicircle (see figure 3.3). In figure 3.3 trader i shares his signal with trader $i-\ell$, but no information sharing occurs with trader $i+k_2$. On the other hand, traders $i+k_2$ and $i-\ell$ directly exchange their signals. This implies that trader i knows $s_{i-\ell,0} \in \mathbf{s}_{i+k_2,0}$, and $\mathbf{s}_{i+k_2,0} \cap \mathbf{s}_{i,0} \neq \{\emptyset\}$. In the appendix, we demonstrate that the occurrence of double overlap modifies the correlation structure in (3.7a) as follows:

$$\text{For } 2G \geq \tfrac{M-1}{2},\ \bar{\Omega}_0(k,G) = \begin{cases} \bar{\Lambda}_0(G) - \hat{G}^{-2}k(\Lambda_0 - \Omega_0), & \text{for } k \in \left[1, 2\left(\frac{M-1}{2} - G\right)\right] \\ \\ 2\bar{\Lambda}_0(G) - \hat{G}^{-2}M(\Lambda_0 - \Omega_0) - \Omega_0, & \text{for } k \in \left[2\left(\frac{M-1}{2} - G\right), \frac{M-1}{2}\right] \end{cases} \tag{3.7b}$$

By eqs. (3.6), (3.7a) and (3.7b), the variance-covariance matrix between average signals depends on the information sharing parameter G, as previously mentioned in this section. While the elements on the main diagonal in $\bar{\Psi}_0(G)$ are identical [see eq. (3.6)], the off-diagonal elements decrease with the distance from the main diagonal according to the pattern dictated by eqs. (3.7a)-(3.7b). It is worth noting that the sum of the off-diagonal elements is constant across different rows in $\bar{\Psi}_0(G)$. This sum is precisely what we previously defined as $\bar{\Gamma}_0(G)$, and is therefore identical across traders. This fact will allow us to avoid the infinite regress problem. In particular, in the appendix we show that eqs. (3.7a)–(3.7b) imply that

$$\bar{\Gamma}_0(G) = (M-1)\,\Omega_0 + \frac{2G}{\hat{G}}(\Lambda_0 - \Omega_0), \qquad \text{for all } G \in \left[0, \tfrac{M-1}{2}\right]. \tag{3.8}$$

3.3.3 *Forecasts*

We now turn to consider how trader i forecasts the final liquidation value as well as the sum of other traders' average signals conditionally on his information set $\mathbf{s}_{i,0}$. By eq. (3.1),

$$E\left(f \mid \mathbf{s}_{i,0}\right) = \hat{G}\eta_1 \bar{s}_{i,0}, \tag{3.9}$$

where $\eta_1 \equiv c_0(\Lambda_0 + 2G\Omega_0)^{-1}$. By eqs. (3.7a)–(3.7b), the correlation between average signals changes with any two agents' relative location. Hence, by the Projection Theorem, each trader's expectation of other traders' average signals depends on the relative distance k. However, the expectation of the sum of all other traders' average signals is independent on k and linear in $\bar{s}_{i,0}$:

$$E\left(\textstyle\sum_{j\neq i} \bar{s}_{j,0} | \mathbf{s}_{i,0}\right) = E\left(\textstyle\sum_{j\neq i} \bar{s}_{j,0} | \bar{s}_{i,0}\right) = (M-1)\,\phi_1 \bar{s}_{i,0}, \tag{3.10}$$

where the regression coefficient is

$$\phi_1 \equiv \frac{\bar{\Gamma}_0(G)}{(M-1)\,\bar{\Lambda}_0(G)},$$

and $\bar{\Lambda}_0(G)$ and $\bar{\Gamma}_0(G)$ are given by eqs. (3.6) and (3.8). Clearly, this result follows because $\bar{\Gamma}_0(G)$ does not depend on k as in (3.8).

3.4 Equilibrium characterization

3.4.1 *Market maker's inference*

Let $z_{i,t} = y_t - x_{i,t}$ be the residual order flow as of trader i. We let $F_{i,n} = \{\mathbf{s}_{i,0}, (z_{i,t})_{t=1}^{n-1}, (x_{i,t})_{t=1}^{n-1}\}$ and $F_{M+1,n} = \{(y_t)_{t=1}^{n}\}$ denote trader i and market maker information sets at the n-th batch auction. The market maker sets prices according to the Semi-Strong efficiency condition

$$p_n = E\left(f \mid F_{M+1,n}\right),$$

and updates his estimate of *individual* signals as follows

$$t_{i,n} = E\left(s_{i,0} \middle| F_{M+1,n}\right).$$

Given our symmetric information structure for individual signals, the previous expectation does not depend on agent i, and we set $t_{i,n} = t_n$. A simple but important point is that t_n is also the updated estimate of each *individual average* signal

$$E\left(\bar{s}_{i,0}|F_{M+1,n}\right) = \hat{G}^{-1} E\left[\sum_{k=-G}^{G} s_{i+k,0} \middle| F_{M+1,n}\right] = \hat{G}^{-1} \sum_{k=-G}^{G} t_{i+k,n} = t_n. \tag{3.11}$$

The relationship between p_n (market maker's updated estimate of the asset value) and t_n (market maker's updated estimate of the individual average signal) is given by:

$$p_n = \theta t_n. \tag{3.12}$$

Let $s_{i,n}$ denote the i-th trader residual informational advantage on his own signal (relative to the market maker) after n rounds of trading,

$$s_{i,n} \equiv s_{i,0} - E\left(s_{i,0}|F_{M+1,n}\right) = s_{i,0} - t_n.$$

Trader i informational advantage on his average signal, $\bar{s}_{i,n}$, has a similar interpretation, and due to eq. (3.11) is given by:

$$\bar{s}_{i,n} = \bar{s}_{i,0} - E\left(\bar{s}_{i,0}|F_{M+1,n}\right) = \bar{s}_{i,0} - t_n. \tag{3.13}$$

The market maker's update results in the following residual variances:

$$\begin{aligned}
\sigma^2_{f,n} &= var\left[E\left(f|\mathrm{s}_0\right)|F_{M+1,n}\right] = var\left(\theta\bar{s} - p_n|F_{M+1,n}\right) \\
\Lambda_n &= var\left(s_{i,0}|F_{M+1,n}\right) = var\left(s_{i,n}|F_{M+1,n}\right) \\
\Omega_n &= cov\left(s_{i,0}, s_{j,0}|F_{M+1,n}\right) = cov\left(s_{i,n}, s_{j,n}|F_{M+1,n}\right) \\
\bar{\Lambda}_n(G) &= var\left(\bar{s}_{i,n}|F_{M+1,n}\right) \\
\bar{\Omega}_n(k,G) &= cov\left(\bar{s}_{i,n}, \bar{s}_{i+k,n}|F_{M+1,n}\right)
\end{aligned} \tag{3.14}$$

$\sigma^2_{f,n}$ is the residual variance of the full information fundamental value after n rounds of trading; Λ_n and Ω_n are the individual signal residual variance and covariance respectively; finally, $\bar{\Lambda}_n(G)$ and $\bar{\Omega}_n(k,G)$ are the average signal counterparts. Then we have:

$$\sigma^2_{f,n} = \frac{\theta^2}{M}\left[\Lambda_n + (M-1)\,\Omega_n\right]. \tag{3.15}$$

Furthermore the following recursions hold:

$$\begin{aligned}
\Omega_{n-1} - \Omega_n &= \Lambda_{n-1} - \Lambda_n && && (3.16a)\\
\sigma^2_{f,n-1} - \sigma^2_{f,n} &= \theta^2\left(\Lambda_{n-1} - \Lambda_n\right) && && (3.16b)\\
\bar{\Lambda}_{n-1}(G) - \bar{\Lambda}_n(G) &= \Lambda_{n-1} - \Lambda_n, && \text{all } G && (3.16c)\\
\bar{\Omega}_{n-1}(k,G) - \bar{\Omega}_n(k,G) &= \Lambda_{n-1} - \Lambda_n, && \text{all } k, G && (3.16d)\\
\bar{\Gamma}_{n-1}(G) - \bar{\Gamma}_n(G) &= (M-1)\left(\Lambda_{n-1} - \Lambda_n\right), && \text{all } G && (3.16e)
\end{aligned}$$

Therefore, the off-diagonal elements in $\bar{\Psi}_n(G) \equiv E\left[(\bar{s}_{1,0},\cdots,\bar{s}_{M,0})^\top (\bar{s}_{1,0},\cdots,\bar{s}_{M,0}) \middle| F_{M+1,n}\right]$ depend on G, while the difference $\bar{\Psi}_{n-1}(G) - \bar{\Psi}_n(G)$ does not:

$$\bar{\Psi}_{n-1}(G) - \bar{\Psi}_n(G) = \left(\Lambda_{n-1} - \Lambda_n\right)\mathbf{1}\mathbf{1}^\top.$$

To easy notation, we now supress the dependence of the various coefficients on G.

3.4.2 Dimensionality issues

We focus on equilibria in which each trader's forecasts of the asset value and the forecasts of others are linear in the trader's average signal. In these equilibria, all higher order forecasts of other traders' forecasts are also linear in the same average signals. Consequently, average signals constitute sufficient statistics for both the asset value and the forecasts of others. Furthermore, we focus on equilibria independent from forecasts' history. As it turns out, our information structure makes the strategic gaming in our model comparable to the one introduced by Foster and Viswanathan (1996). Specifically, we assume that in equilibrium traders' demand and the market maker's learning about the asset value take the following form:

$$\begin{aligned}
x_{i,n} &= \hat{G}\beta_n \bar{s}_{i,n-1} && (3.17)\\
p_n &= p_{n-1} + \lambda_n y_n && (3.18)
\end{aligned}$$

Moreover, the market maker learning about individual (and average) signals evolves according to

$$t_n = t_{n-1} + \zeta_n y_n. \tag{3.19}$$

The relationship between the updating parameters ζ_n and λ_n is given by:

$$\lambda_n = \theta \zeta_n. \tag{3.20}$$

At the n-th trading round, trader i forecasts the fundamental value that is not predicted by the market maker after $n-1$ rounds, using his information $F_{i,n}$. By the assumption that trading strategies are linear [see eq. (3.17)], and the market maker's recursive update in eqs. (3.13) and (3.19),

$$x_{i,n} = \hat{G}\beta_n \bar{s}_{i,n-1} = \hat{G}\beta_n \left(\bar{s}_{i,0} - t_{n-1}\right) = \hat{G}\beta_n \left(\bar{s}_{i,0} - \sum_{r=1}^{n-1} \zeta_r y_r\right).$$

Therefore, the residual order flow $(z_{i,t})_{t=1}^{n-1}$ is redundant, and we set $F_{i,n} = \{\mathbf{s}_{i,0}, (y_t)_{t=1}^{n-1}\}$. As in Foster and Viswanathan (1996), trader i can manipulate other traders' beliefs about the asset value only through the aggregate order flow. As a result, every trader forecasts the asset value as follows:

$$E\left(f - p_{n-1} | F_{i,n}\right) = \hat{G}\eta_n \bar{s}_{i,n-1}. \tag{3.21}$$

That is, $\bar{s}_{i,n-1}$ is sufficient for trader i to forecast the fundamental value before submitting his order at time n. Note that eq. (3.21) is the dynamic analog to the projection in eq. (3.9). Similarly, trader i forecasts (the sum of) other traders' forecasts of the fundamental value according to [and analogously to the static case in eq. (3.10)]

$$E\left(\sum_{j \neq i} \bar{s}_{j,n-1} | F_{i,n}\right) = (M-1)\, \phi_n \bar{s}_{i,n-1}. \tag{3.22}$$

As is clear, linear strategies as in eq. (3.17) play a key role in resolving the dimensionality issue, since they allow to conclude that the forecasts of the forecasts of others are linear in each trader's average signal.

3.4.3 Equilibrium and deviation

Our linearity assumptions rule out the problem of increasing state history over time. The argument hinges on the fact that linear strategies in eq. (3.17) are played *in equilibrium.* When moving to consider deviations from the optimal play by trader i, one has to keep into account that $\bar{s}_{i,n-1}$ is no longer a sufficient statistic for predicting the fundamental value as well as other traders' forecast as in eqs. (3.21)–(3.22). In fact $\bar{s}_{i,n-1}$ is sufficient only if trader i played the strategy (3.17) in the first $n-1$ trading rounds. Let us denote deviation from the equilibrium path with a prime ($'$). Trader i deviation from the equilibrium play (3.17) to $(x'_{i,k})_{k=1}^{n-1}$ during the first $n-1$ auctions would generate the aggregate order flow $\{y'_k = y_k - (x_{i,k} - x'_{i,k})\}_{k=1}^{n-1}$. Since the market maker's update on the fundamental value and the average signals are linear in the order flow due to eqs. (3.18)–(3.19), trader i deviation modifies the market maker's learning process as well, resulting in $(p'_k)_{k=1}^{n-1}$ and $(t'_k)_{k=1}^{n-1}$. Given past suboptimal play, it turns out that the residual average signal along the equilibrium path $\bar{s}_{i,n-1}$ and the price deviation $\left(p_{n-1} - p'_{n-1}\right)$ are *jointly* sufficient to forecast the fundamental value as well as the forecasts of other traders. This result allows to conjecture that trader i's value function after n auctions takes the form:

$$W_{i,n} = \alpha_n \bar{s}_{i,n}^2 + \psi_n \bar{s}_{i,n} \left(p_n - p'_n\right) + \mu_n \left(p_n - p'_n\right)^2 + \delta_n. \tag{3.23}$$

Past suboptimal play is captured by the second and third term in the value function in eq. (3.23). Moreover, trader i deviation coincides with the equilibrium strategy in eq. (3.17) plus an additional term reflecting the price deviation induced by suboptimal play in the previous $n-1$ rounds:

$$x'_{i,n} = \hat{G}\beta_n \bar{s}_{i,n-1} + \gamma_n \left(p_{n-1} - p'_{n-1}\right). \tag{3.24}$$

The necessary and sufficient conditions for an equilibrium in our trading game hinge upon the mutual consistency between the conjectured value function in eq. (3.23) and the deviation in eq. (3.24). We have:

Proposition 1 *There exists a symmetric linear recursive Bayesian equilibrium in which trading strategies and prices are as in eqs. (3.17)-(3.18);* λ_n *is the unique real,* *positive solution to:*

$$\frac{\theta\left(M-\hat{G}\right)\left(\Lambda_n-\Omega_n\right)\sigma_u^4}{\hat{G}M^2\sigma_{f,n}^4}\lambda_n^4+\frac{\sigma_u^2\psi_n\bar{\Lambda}_n}{\sigma_{f,n}^2}\lambda_n^3-\frac{\theta\sigma_u^2\left[2\Lambda_n+(M-1)\Omega_n-\frac{2G}{\hat{G}}\left(\Lambda_n-\Omega_n\right)\right]}{M\sigma_{f,n}^2}\lambda_n^2$$
$$-\psi_n\bar{\Lambda}_n\lambda_n+\frac{\theta}{M}\left[\Lambda_n+(M-1)\Omega_n\right]=0 \tag{3.25}$$

and the trading strategy coefficients β_n and γ_n are given by:

$$\beta_n = \frac{\theta\lambda_n\sigma_u^2}{\hat{G}M\sigma_{f,n}^2} \tag{3.26}$$

$$\gamma_n = \frac{\left(1-2\lambda_n\mu_n\right)\left[1-\theta^{-1}\hat{G}\left(M-1\right)\beta_n\lambda_n\right]}{2\lambda_n\left(1-\lambda_n\mu_n\right)} \tag{3.27}$$

The value function coefficients satisfy the recursions:

$$\alpha_{n-1} = \alpha_n\left[1-\theta^{-1}\hat{G}\left(1+(M-1)\phi_n\right)\beta_n\lambda_n\right]^2+\hat{G}^2\beta_n\left[\eta_n-\beta_n\lambda_n\left(1+(M-1)\phi_n\right)\right]$$

$$\begin{aligned}\psi_{n-1} = {} & \psi_n\left[1-\lambda_n\gamma_n-\theta^{-1}\hat{G}\left(M-1\right)\beta_n\lambda_n\right]\left[1-\theta^{-1}\hat{G}\left(1+(M-1)\phi_n\right)\beta_n\lambda_n\right]\\ & +\hat{G}\left\{\gamma_n\left[\eta_n-\beta_n\lambda_n\left(1+(M-1)\phi_n\right)\right]-\beta_n\gamma_n\lambda_n+\beta_n\left[1-\theta^{-1}\hat{G}\left(M-1\right)\beta_n\lambda_n\right]\right\}\end{aligned} \tag{3.28}$$

$$\mu_{n-1} = \mu_n\left[1-\lambda_n\gamma_n-\theta^{-1}\hat{G}\left(M-1\right)\beta_n\lambda_n\right]^2+\gamma_n\left[1-\lambda_n\gamma_n-\theta^{-1}\hat{G}\left(M-1\right)\beta_n\lambda_n\right]$$

$$\delta_{n-1} = \delta_n+\theta^{-2}\alpha_n\lambda_n^2\sigma_u^2+\theta^{-2}\hat{G}^2\alpha_n\lambda_n^2\beta_n^2 var\left(\textstyle\sum_{j\neq i}\bar{s}_{j,n-1}\middle| F_{i,n}\right)$$

where $\alpha_N=\psi_N=\mu_N=\delta_N=0$, and

$$\phi_n = \frac{\bar{\Gamma}_{n-1}}{\left(M-1\right)\bar{\Lambda}_{n-1}} \tag{3.29}$$

$$\eta_n = \frac{\theta\left(\bar{\Gamma}_{n-1}+\bar{\Lambda}_{n-1}\right)}{\hat{G}M\bar{\Lambda}_{n-1}} \tag{3.30}$$

$$var\left(\textstyle\sum_{j\neq i}\bar{s}_{j,n-1}\middle| F_{i,n}\right) = M\left[\Lambda_{n-1}+(M-1)\Omega_{n-1}\right]-\left[1+\phi_n^2\left(M-1\right)^2\right]\bar{\Lambda}_{n-1}-2\bar{\Gamma}_{n-1}.$$

Furthermore the following inequality must hold:

$$\lambda_n\left(1-\lambda_n\mu_n\right)>0,$$

and the following recursion on the full information residual variance must hold:

$$\sigma^2_{f,n} = \left(1 - \theta^{-1}\hat{G}M\beta_n\lambda_n\right)\sigma^2_{f,n-1}.$$

In our model, not only are traders concerned with learning from the information that other traders possess. This learning process is also complicated by every trader's geographical location and the amount of information every trader shares with neighbours. As Proposition 1 reveals, trading strategies and value functions are heavily affected by the heterogeneous correlation structure arising as a result of information sharing –a fact that we will examine in great detail in section 3.5. We now turn to illustrate the computational aspects of the equilibrium.

3.4.4 Computation of the equilibrium

The model parameters are: (1) the number of traders and batch auctions (M, N), (2) the fundamental and non-fundamental uncertainty $(\sigma^2_{f,0}, \sigma^2_u)$, (3) the initial signal distribution (Λ_0, Ω_0), and the covariance between signals and the fundamental value c_0, and (4) the group size $\hat{G}$, or equivalently the amount of information sharing G. Once these parameters are fixed at some value, the regression coefficient θ in eq. (3.4) and the matrix $\bar{\Psi}_0$ are uniquely determined. We then solve for the equilibrium using backward induction. By eq. (3.16a), $\Lambda_n - \Omega_n = \Lambda_0 - \Omega_0$. We fix a terminal value for Λ_N and compute $\Omega_N = \Lambda_N + \Omega_0 - \Lambda_0$ using eq. (3.16a). $\sigma^2_{f,N}$ then follows by eq. (3.15). Since $\alpha_N = \psi_N = \mu_N = \delta_N = 0$, we solve for λ_N in eq. (3.25), which yields β_N and γ_N via eqs. (3.26)-(3.27). To compute the value function coefficients as of at time $N-1$, one needs to express Λ_{N-1} and Ω_{N-1} in terms of variables known at time N. In the appendix, we show that:

$$\Lambda_{n-1} = \frac{\theta\Lambda_n - \hat{G}(M-1)\lambda_n\beta_n(\Lambda_n - \Omega_n)}{\theta - \hat{G}M\lambda_n\beta_n}. \tag{3.31}$$

Then, Λ_{N-1} is obtained by evaluating eq. (3.31) at $n = N$, and Ω_{N-1} is obtained by the equality $\Omega_{N-1} = \Lambda_{N-1} + \Omega_0 - \Lambda_0$. Finally, we retrieve regression coefficients ϕ_N and η_N through eqs. (3.29)-(3.30) and the equality $\bar{\Gamma}_{N-1} = \bar{\Gamma}_N + (M-1)(\Lambda_{N-1} - \Lambda_N)$ [see eq. (3.16e)]. The value function coefficients at time $N-1$ are therefore uniquely determined by eq. (3.28).

The above procedure is then applied at each trading round $n \in [1, N]$ yielding the inital value of Λ_0 implied by the choice of the terminal value of Λ_N. The resulting initial value of Λ_0 is then compared to the one we posited as initial parameter, and the procedure is repeated for different choices of Λ_N until convergence is achieved.

3.5 Market dynamics implications

This section analyzes properties of the equilibrium price formation predicted by our model in a variety of specific cases. We consider an experimental design with three distinct trading periods lengths ($N = 5$, 10 and 40) and three sizes of informed traders ($M = 5$, 7 and 21). In practice, we are spanning a wide range of cases, from one extreme in which $M = 5$ and $N = 5$ to the other extreme in which $M = 21$ and $N = 40$. To explore the asymptotic properties of the model (as N becomes large), we will occasionally study selected properties of the model in the case arising when $M = 7$ and $N = 100$.[5]

We specialize the information structure to one in which the sum of all traders' signals equals the truth, viz $f = \sum_{i=1}^{M} s_i$, and set $\sigma_{f,0}^2 = 1$ and $\sigma_u^2 = N^{-1}$. Therefore, we analyze a situation in which the fundamental uncertainty equals the total non-fundamental uncertainty across all batch auctions. As a result of the choice about the distribution of f, $c_0 = M^{-1}$, $\Lambda_0 = M^{-1} / \left[(M-1)\rho + 1\right]$ and $\Omega_0 \equiv \rho\Lambda_0$. In the remainder, we thus parametrize equilibrium properties by: (1) the initial correlation among the informed traders signals, ρ, (2) the number of trading periods, N, (3) the number of informed traders, M, and (4) the amount of information sharing, G. For a given tuple (ρ, N, M), we define $\mathcal{E} \equiv \mathcal{E}(\rho, N, M)$ as the *initial market structure*, and describe equilibrium properties of the model over $(\mathcal{E}, G)$.

3.5.1 Strategic information sharing

We analyze the pattern of traders' expected profits over $(\mathcal{E}, G)$. Given a fixed tuple (ρ, N, M), we say that information sharing is not optimal [for the corresponding initial market structure $\mathcal{E}(\rho, N, M)$], or does not arise, if expected profits are the highest at $G = 0$. Similarly, we say that given a fixed tuple (ρ, N, M), *partial* (resp. *complete*) *information sharing* is the optimal traders' strategy [for the corresponding initial market structure $\mathcal{E}(\rho, N, M)$] if expected profits are the highest at some $G \in \left[1, \frac{M-1}{2} - 1\right]$ (resp. at $G = \frac{M-1}{2}$).

FIGURE 3.4. Optimal information sharing.

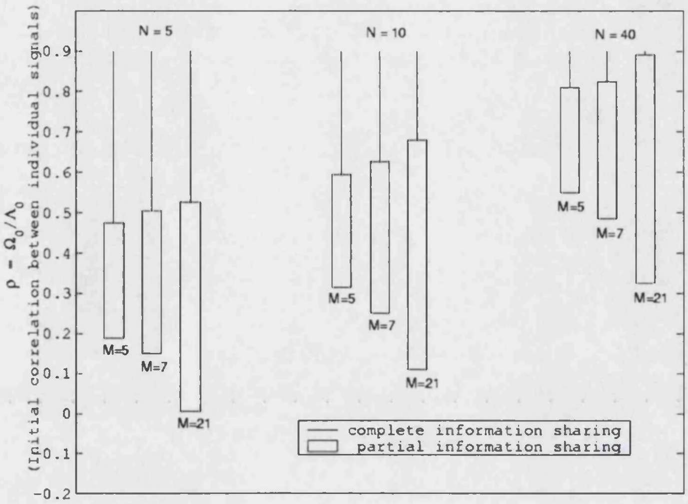


Figure 3.4 shows the ranges of initial correlation between individual signals ρ within which information sharing is optimal for given combinations of (N, M). We summarize our main findings as follows.

Result 1. *In all cases, information sharing (both* partial *and* complete*) always arises in correspondence of some positive range of initial correlations.*

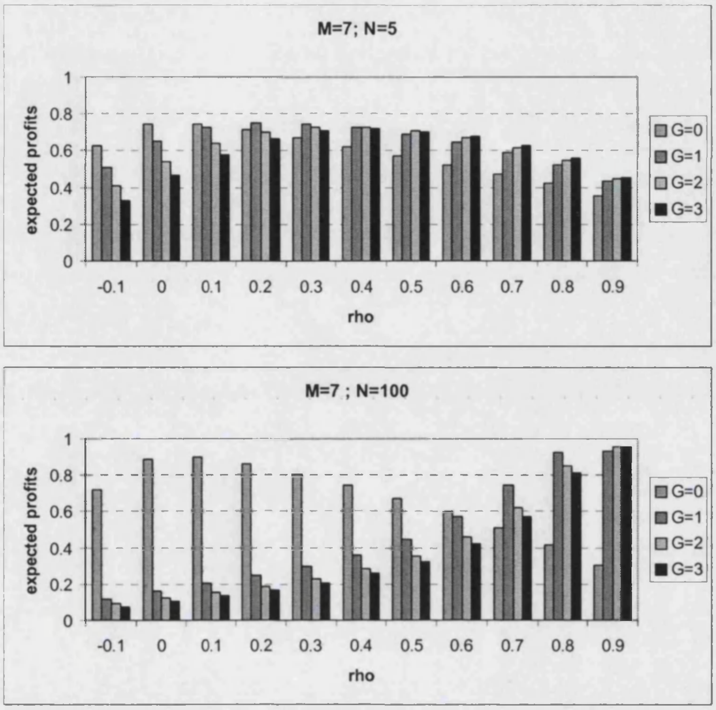
Result 2. *The range of initial correlations within which information sharing is not optimal widens with N.*

Result 3. *For each M, the range of initial correlations within which* complete *information sharing arises shrinks as N increases. For each M, the range of initial correlations within which* partial *information sharing occurs does not shrink as N increases.*

Result 4. *The range of initial correlations within which information sharing is not optimal shrinks as M increases.*

As Result 1 reveals, information sharing is optimal in correspondence of positive values of ρ, and is indeed a robust phenomenon.[6] As N increases, information sharing is optimal for even progressively higher positive values of ρ - consistently with Result 2. A further prediction of our model is that informed agents may only want to share

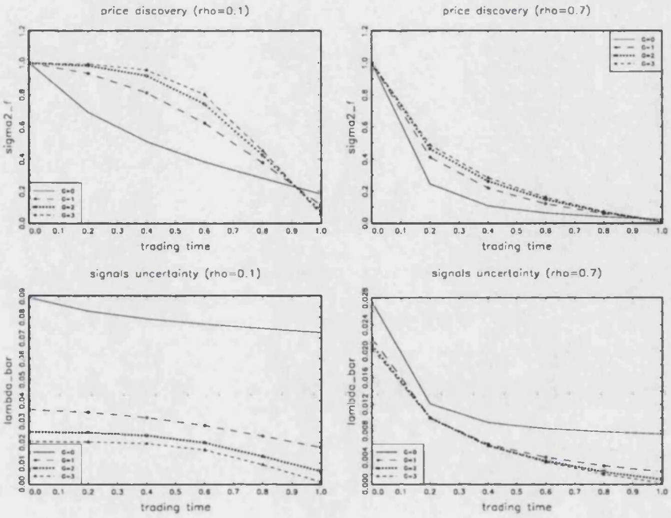
FIGURE 3.5. Expected profits: discrete-time versus continuous-time ($M = 7$).



information with *selected peers*. According to Result 3, this feature of the model becomes more and more pronounced as N increases. We regard this result as particularly important because it reveals that our traders' geographical location structure does not lead to trivial results. In other terms, a simple model in which informed agents only consider *complete* information sharing agreements is not flexible enough to capture the gains from information sharing. Finally, figure 3.4 reveals that the incentives to share long-lived information widen hugely as the number of informed traders increases –as we state in Result 4. As an example, if $M = 21$ and $N = 5$, information sharing is optimal with a correlation as low as $\rho = 0.1$.

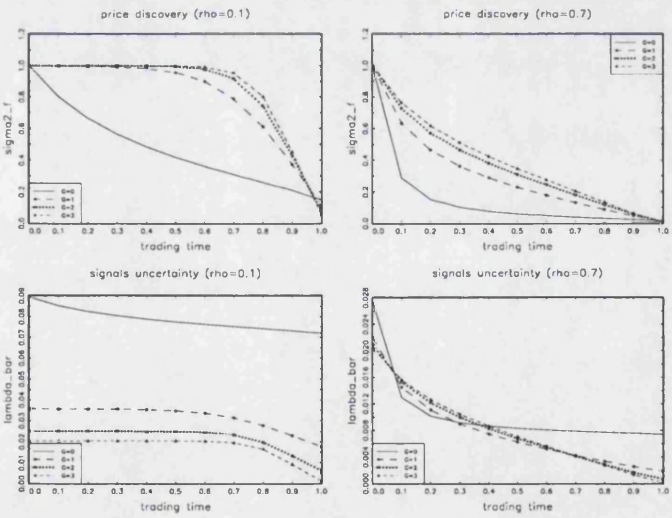
The results in figure 3.4 may seem to imply that the incentives to information sharing die off as $N \rightarrow \infty$. We are not able to provide a formal proof (or refutation) of such a statement. However, we document that the gains from information sharing persist even with a number of batch auctions as high as one hundred. In figure 3.5, we depict the pattern of expected profits when $M = 7$ in two cases: $N = 5$ and $N = 100$. Consistently with Result 2, the range of ρ within which information sharing is optimal with $N = 100$ is smaller than in the previous cases for $N = 5$, 10 and 40. Yet if

FIGURE 3.6. Benefits of information sharing ($M = 7$, $N = 5$).



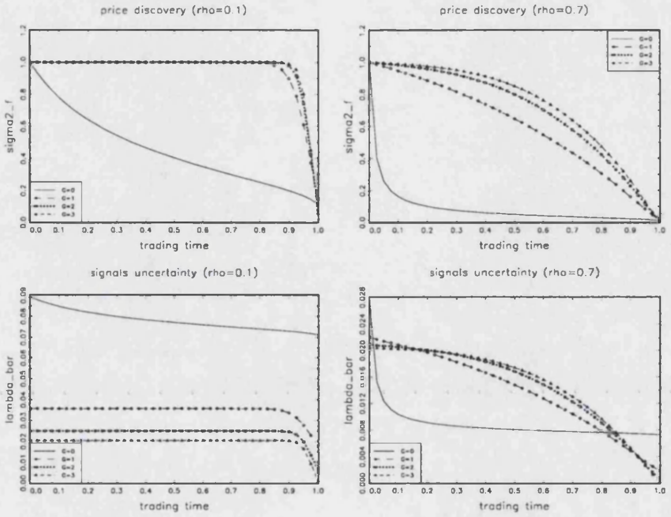
$N = 100$, no information sharing is never optimal for values of ρ larger than some threshold level between 0.6 and 0.7.

FIGURE 3.7. Benefits of information sharing ($M = 7$, $N = 10$).



What are the origins of these results? As is clear, heterogeneity in private information is a source of monopolistic power for traders. Yet such a monopolistic power deteriorates as ρ and/or N increases. This is so because as ρ increases, every trader loses more and more bits of information endowments available only to him. And as N increases, every trader reveals more of his information through his trading activity. As it turns out, information sharing may restore the loss in monopolistic power induced by the market

FIGURE 3.8. Benefits of information sharing ($M = 7$, $N = 40$).

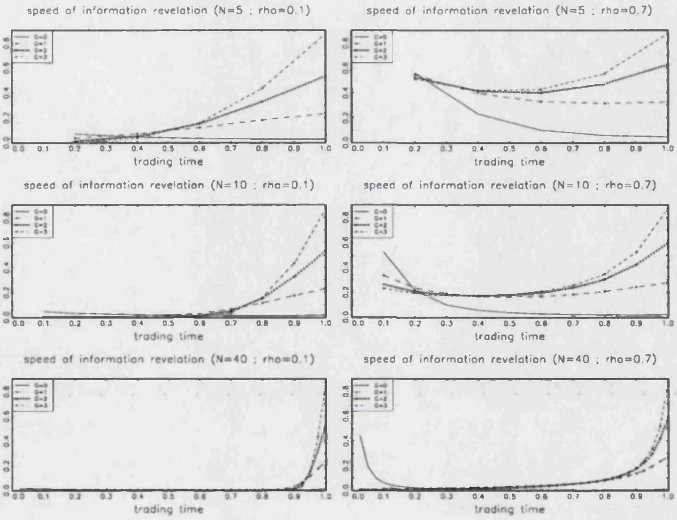


maker's observation of the order flow, and his inference about (1) the full-information asset value, (2) each trader's average signal, and (3) the correlation between each trader's average signal with the average signals of neighbours. We now explain how each of these three factors contributes to make information sharing beneficial to traders.

(1) *Price discovery.* Figures 3.6–3.8 depict the pattern of $\sigma^2_{f,n}$ over time when $\rho = 0.1$ and $\rho = 0.7$. Clearly, the speed of price discovery increases with ρ. Hence, traders are worse off with high values of ρ because increasingly bigger portions of their information endowments are lost as ρ increases. Figures 3.6–3.8 also reveal the important point that for all $n < N$ and G, $\partial\sigma^2_{f,n}/\partial G > 0$. Therefore, as ρ increases informed traders may find it more and more beneficial to share information to hinder the price discovery process enhanced by initially more correlated private signals.

(2) *Signals uncertainty.* Figures 3.6–3.8 also depict the market maker's uncertainty related to traders' average signals ($\bar{\Lambda}_n$). In principle, low values of $\bar{\Lambda}_0$ help imperfectly informed traders to improve their estimates about the asset value. But as more and more batch auctions are added, the quality of each trader's inference is revealed to the market maker. The results in figures 3.6–3.8 show that the level of $\bar{\Lambda}_n$ decreases as ρ increases. Furthermore, the decay rate of $\bar{\Lambda}_n$ increases as ρ increases (similarly as for $\sigma^2_{f,n}$). Finally, $\bar{\Lambda}_n$ decreases significantly with G when ρ is low. That is, if ρ is low, information sharing improves the market maker's

FIGURE 3.9. Speed of information revelation.



Time variation in the information asymmetry, given by the ratio v_n [see eq. (3.32)] for $M = 7$.

inference about private signals. However, such an improvement deteriorates as ρ increases. For high values of N and ρ, information sharing even entails significant losses in the quality of the market maker's inference about private signals. Therefore, given a fixed N, traders may profit from information sharing when ρ is sufficiently high. The impact of the market maker's inference activity on traders' profits may be viewed from an alternative angle. Intuitively, traders' profits are higher (a) the lower is the initial signal uncertainty (i.e. $\bar{\Lambda}_0$), *and* (b) the longer the market maker is kept away from traders' information endowments. A measure of information asymmetry is the ratio between the variance of the market maker's forecast $t_n = E(\bar{s}_{i,0}|F_{M+1,n})$ and the average signal uncertainty $\bar{\Lambda}_n$, viz

$$v_n \equiv \frac{var\left[E\left(\bar{s}_{i,0}|F_{M+1,n}\right)|\, F_{M+1,n-1}\right]}{var\left(\bar{s}_{i,0}|\, F_{M+1,n-1}\right)}.$$

By construction, $v_n \leq 1$, all n. Furthermore, v_n approaches unity as the market maker learns more and more about traders' information sets. As it turns out, the ratio v_n is precisely the decay rate in the average signal uncertainty:

$$v_n = \frac{\bar{\Lambda}_{n-1} - \bar{\Lambda}_n}{\bar{\Lambda}_{n-1}}. \tag{3.32}$$

Figure 3.9 reveals that as ρ and N become higher and higher, v_n is initially lower in the presence of information sharing, and (initially) decreases with G. By eq.

(3.32), this means that information sharing makes the market maker's dynamic inference slow down significantly when ρ and N are sufficiently high. And by the definition of v_n, this also means that information sharing makes information asymmetries persist longer and longer as ρ and N become sufficiently high.

(3) *Signals correlation with neighbours.* Figures 3.10–3.12 show the time variation in the average signal correlation every trader has with his neighbours, or $\bar{\rho}_n(k) \equiv \bar{\Omega}_n(k)/\bar{\Lambda}_n$ for $k = 1, 2, 3$. Clearly, this correlation is inversely related to monopolistic power deriving from the information shared by every trader with neighbours. Interestingly,

$$corr\left(x_{i,n}, x_{i+k,n} | F_{M+1,n-1}\right) = \bar{\rho}_{n-1}(k).$$

Therefore, $\bar{\rho}_{\cdot}(\cdot)$ also measures how information sharing makes neighbours's trades "resemble" one another. In general, traders are worse off as $\bar{\rho}_{\cdot}(\cdot)$ increases. As one might have expected, $\bar{\rho}_n(k)$ is increasing in ρ for all n and k. Furthermore, for all k, $\bar{\rho}_n(k)$ increases with G for almost all of the time. In other terms, information sharing generally entails a loss in traders' monopolistic power. Intuitively, this is so because information sharing induces an increase in the initial correlation $\bar{\rho}_0(\cdot)$ between all traders' average signals. And as it turns out, our model also predicts that the conditional correlation $\bar{\rho}_n(k)$ increases for almost all k and subsequent $n > 1$.

FIGURE 3.10. Correlation heterogeneity ($M = 7$, $N = 5$).

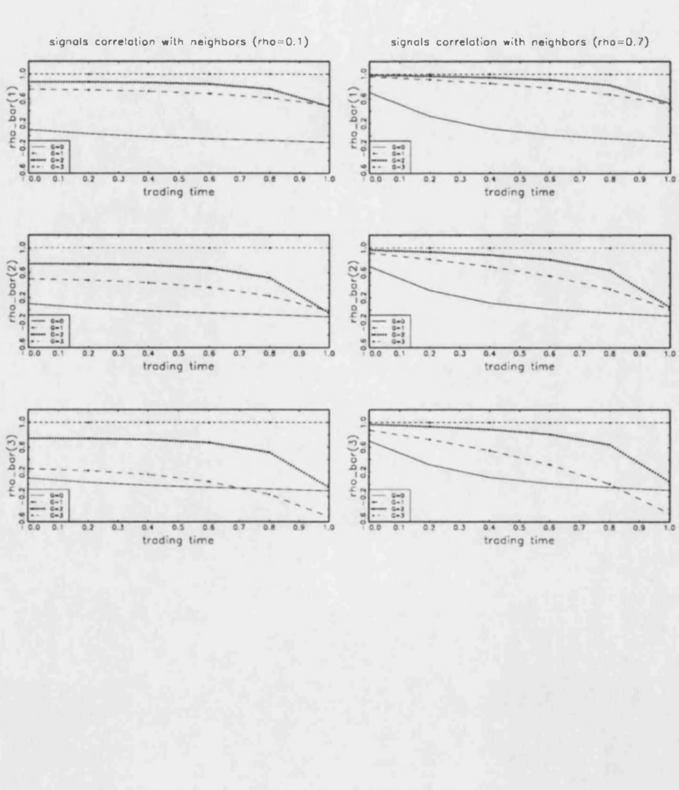
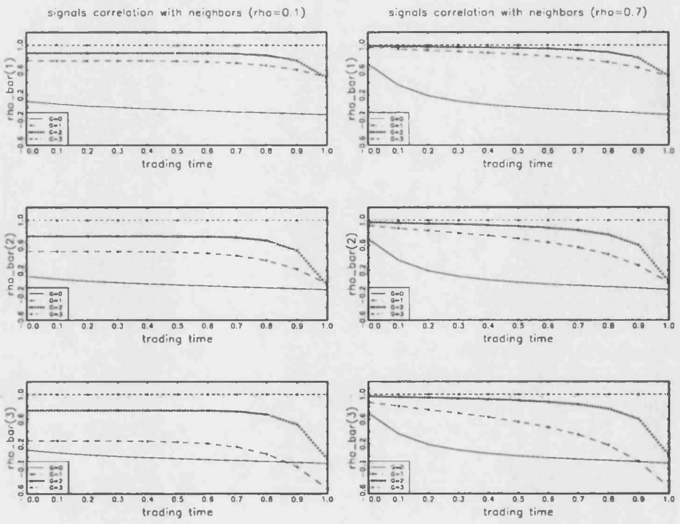


FIGURE 3.11. Correlation heterogeneity ($M = 7$, $N = 10$).



Traders play their entire game so as to keep as much as possible of their monopolistic information power. This power is less and less important as ρ increases. The previous price discovery and signals uncertainty factors (1) and (2) suggest that information sharing may restore part of the traders' power destroyed by high values of ρ. Naturally, information sharing does not come at no cost. Due to the previous signals-correlation-with-neighbours factor (3), information sharing also entails a loss in each trader's monopolistic power. But when ρ is sufficiently high, the first two factors make information sharing beneficial overall, as indicated by Result 1. Figures 3.6–3.8 also suggest that in the absence of any information sharing, both $\sigma^2_{f,n}$ and $\bar{\Lambda}_n$ decrease as N increases.[7] Furthermore, when N is high, such a reduction is more and more important as ρ increases. This explains Results 2 and 3: as N becomes larger and larger, information sharing is beneficial to traders in correspondence of increasing values of ρ. Furthermore, it seems that the benefits of information sharing derive more from the signals uncertainty factor than the price discovery factor. Finally, Result 4 follows because an increase in M corresponds to a reduction in traders' market power. Therefore, as M increases, traders benefit from information sharing with lower levels of the initial correlation ρ.

The previous discussion has been based on the pattern of price discovery and signals uncertainty that are predicted by our model. We now turn to analyze in deeper detail both these patterns and their determinants.

FIGURE 3.12. Correlation heterogeneity ($M = 7$, $N = 40$).

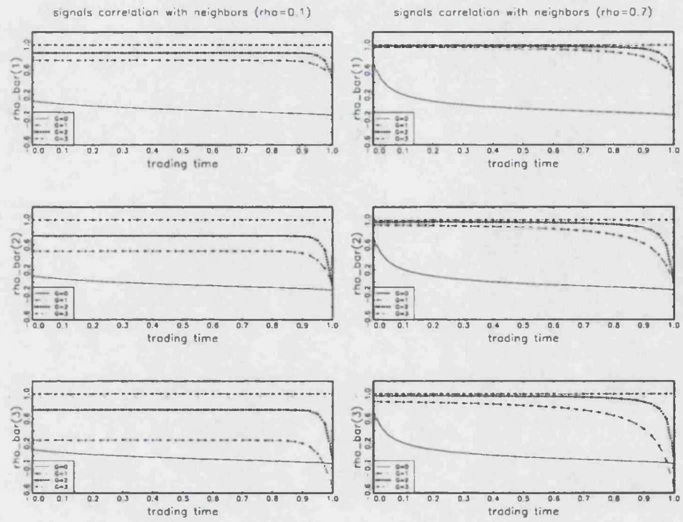
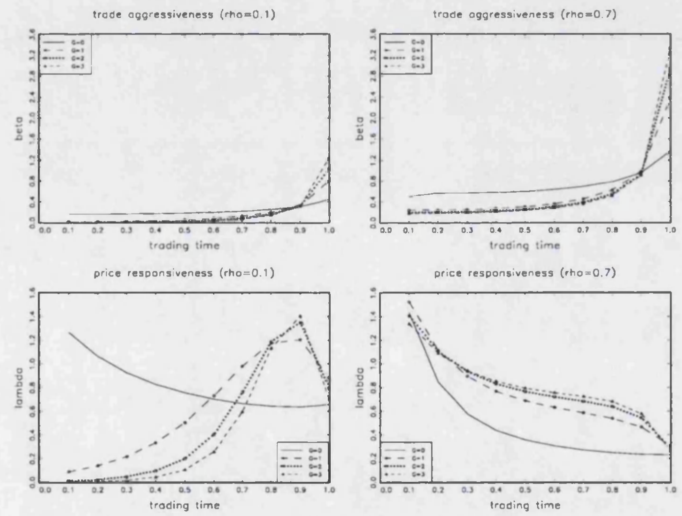


FIGURE 3.13. Trade intensity and market liquidity ($M = 7$, $N = 10$).



Note: beta stands for $\beta_n^G \equiv \hat{G}\beta_n$

3.5.2 *Liquidity and price discovery*

We analyze the link between liquidity, price discovery, trading behavior and information sharing. To simplify the exposition, we report results for the case $M = 7$ only. In figure 3.13 we only illustrate the representative cases corresponding to $\rho = 0.1$ and $\rho = 0.7$ (as for figures 3.6–3.12), and for $N = 10$. The results reported in this section have been obtained through a series of solutions of the model covering a much wider range of cases (see footnote 5).

We begin with the analysis of traders' behavior. Precisely, we measure and track *trade aggressiveness* by $\beta_n^G \equiv \hat{G}\beta_n$, $n = 1, \cdots, N$. We say that a *waiting game* arises whenever β_n^G is increasing and convex in n. Furthermore, we say that the waiting game is more and more intense, or *pronounced*, as *both* the initial value β_1^G decreases *and* the final value β_N^G increases due to some parameters' change. We have:

Result 5. *The* waiting game *becomes more and more* pronounced *as G increases. Its pattern is independent of ρ and N. However, the overall* trade aggressiveness *increases with ρ and decreases with N.*

At the last batch auction, β_N^G is the highest in correspondence of complete information sharing. This result is consistent with a wide number of previous models. As an example, it is well-known that in static Kyle's (1985) type models, insiders endowed with the same signals trade very aggressively. The novel result here is that the waiting game is very intense when traders experience information sharing. In fact, the slow price and signal discovery properties noted in the previous subsection (see figures 3.6–3.8) are the expression of the amplification of the waiting game induced by information sharing. We summarize our findings related to the price discovery process in the following:

Result 6. *Assume that traders experience information sharing (i.e. $G \geq 1$, for strategic or non-strategic reasons). Then the speed of price discovery lowers as both G and N increase. However, the overall (or final) price discovery improves as G increases.*

According to Result 5, trading activity is more and more clustered towards the end of the trading period as traders exchange more and more signals. At the end of the trading period, trading activity is so intense that *overall*, price discovery is higher with than without information sharing (i.e., $\sigma_{f,N}^2$ is decreasing in G). Therefore, price discovery follows the pattern described in Result 6. Such a phenomenon is particularly severe. In all the information sharing cases, the bulk of price discovery takes place only towards the end of the trading process when ρ is low and $N \geq 10$. Even when ρ is high, price discovery arising in the presence of information sharing is much more gradual than in the no-information sharing case. Furthermore, Result 6 predicts that in all information sharing cases, price discovery worsens as N increases. We emphasize that such a result is robust; and that it is in sharp contrast with the predictions in Foster and Viswanathan (1996) based on homogeneous correlation structures. As is well-known, the Foster and Viswanathan (1996) model predicts that the speed of information revelation is higher with more trading rounds. [Our model confirms these findings in the no information share case (i.e. $G = 0$) also in cases not reported by the authors.] Here we find that quite the opposite happens when agents experience information sharing.

We now document our last finding related to the market maker's inference activity. It regards the conditional correlation between private signals, defined as $\rho_n \equiv \Omega_n / \Lambda_n$. As in Foster and Viswanathan (1996), we find that in all information sharing cases, conditional correlation decreases and eventually becomes negative. Furthermore, we have:

Result 7. *For all $G \geq 1$, the conditional correlation ρ_n converges to negative values more and more slowly as G and N increase. However, the terminal conditional correlation ρ_N decreases with G.*

As in Foster and Viswanathan (1996), our model predicts that the market maker learns more about the order flow than about private signals. As a consequence, the conditional correlation ρ_n eventually becomes negative. However, an intense waiting game implies that during the early stages of the trading process, the market maker does not learn from the order flow either. In contrast, his learning about the order flow clusters towards the end of the trading period. This implies Result 7. In fact, a similar explanation holds for the results on the average signals' correlation $\bar{\rho}_n$ depicted in figures 3.10–3.12.

How do these phenomena affect market liquidity? As one may expect, the answer depends on the specific values taken by the initial correlation ρ. Figure 9 also depicts the time variation in λ_n in correspondence of two values of ρ. These values are representative of a variety of situations that we may summarize as follows:

Result 8. *If $G > 0$ and ρ is sufficiently low, liquidity is high at the beginning of the trading period and decreases towards the end of the trading period. Furthermore, towards the end of the trading period liquidity decreases as G increases, and is always* sharing cases, price discovery worsens as N increases. We emphasize that such a result is robust; and that it is in sharp contrast with the predictions in Foster and Viswanathan (1996) based on homogeneous correlation structures. As is well-known, the Foster and Viswanathan (1996) model predicts that the speed of information revelation is higher with more trading rounds. [Our model confirms these findings in the no information share case (i.e. $G = 0$) also in cases not reported by the authors.] Here we find that quite the opposite happens when agents experience information sharing.

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Result 8. *If $G > 0$ and ρ is sufficiently low, liquidity is high at the beginning of the trading period and decreases towards the end of the trading period. Furthermore, towards the end of the trading period liquidity decreases as G increases, and is always* *lower than in the case of no information sharing; this phenomenon is more pronounced as ρ decreases and N increases. Finally, liquidity is increasing over the batch auctions when ρ is sufficiently high; in this case, the increasing pattern in liquidity becomes more and more pronounced as N increases, and the level of liquidity decreases with G for almost all of the trading period.*

Again, the deep waiting game is driving many of the previous results. The market maker knows that for $G \geq 1$, traders concentrate their trading activity at the end of the trading period. Furthermore, trade aggressiveness is overall very low when ρ is low. As a consequence, liquidity costs –as measured by the price sensitivity λ_n– are very low for almost all of the trading period, and increase substantially towards the end –with a decrease at the very end. On the contrary, when ρ is sufficiently high the speed of information revelation is higher, and therefore liquidity costs smoothly decrease over time. However, due to progressively deeper information asymmetries deriving from information sharing, liquidity costs typically become higher and higher as G increases.

As for any other asset pricing model with asymmetric information, the testable implications of our model must be understood in relation to a "reference trading period" in which information (1) arrives at the beginning, and (2) is (at least partially) publicly revealed at the end of the same period. As an example, one may be interested in implications for trading periods preceding earning announcements. Our model predicts that in the anticipation of such information releases, diversely informed traders may find it useful to exchange information if the initial correlation ρ is not too low. The model also predicts that in this case, liquidity costs should increase around public information releases whenever the initial correlation ρ is not too high. Interestingly, adverse selection components of trading costs are well-known to increase significantly around earning announcement days [see, for example, Lee, Mucklow and Ready (1993) and Krinsky and Lee (1996).] Previous dynamic models generate this liquidity pattern only under a set of very restrictive values of the initial correlation ρ and the number of batch auctions N. As an example, the Foster and Viswanathan (1996) model is consistent with liquidity costs increasing over time only when $\rho < 0$ and N is low. Alternatively, the same model predicts that when ρ is low and N is extremely high, liquidity costs are U-shaped.[8] While this prediction is consistent with overwhelming evidence on intraday stock market behavior [see Chan, Chung and Johnson (1995)], a typical trading day for a NYSE stock does not necessarily correspond to the reference trading period hypothesized above. In any event, the Foster and Viswanathan (1996) model is based on the assumption that the correlation of traders' signals is the same for all traders. Our results suggest that as soon as such an homogeneity assumption is relaxed, empirical predictions become dramatically different.

3.5.3 Volume and volatility

Finally, we study how information sharing affects additional variables such as trading volume and asset return volatility. As in Admati and Pfleiderer (1988b), we decompose the (expected) volume at the n-th auction in terms of the contribution of the market maker, the M traders, and the liquidity traders. Precisely, we identify each component with its conditional standard deviation, and set $Vol_n = Vol_n^M + Vol_n^I + Vol_n^U$, where

$$Vol_n^M = \sqrt{\hat{G}^2\beta_n^2 M\left[\Lambda_{n-1} + (M-1)\,\Omega_{n-1}\right] + \sigma_u^2};$$

$$Vol_n^I = \hat{G}\beta_n\sqrt{M\left[\Lambda_{n-1} + (M-1)\,\Omega_{n-1}\right]};$$

and $Vol_n^U = \sigma_u$, for all n. Furthermore, we compute the asset return volatility by also conditioning on the market maker's information set:

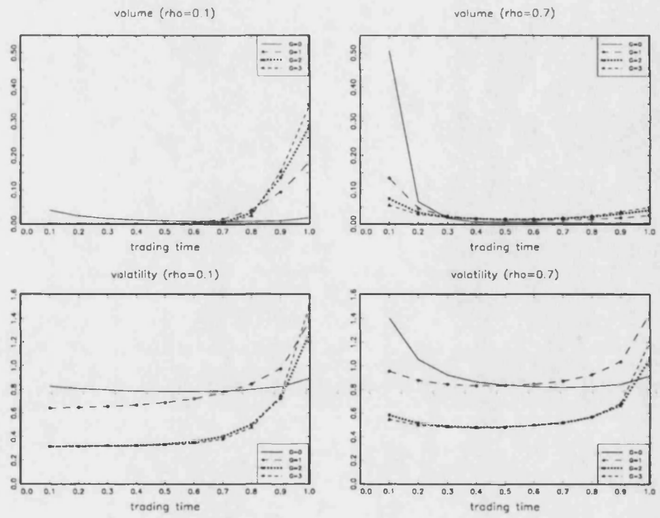
$$var\left(p_n - p_{n-1}\right| F_{M+1,n-1}) = \lambda_{n-1}^2\left\{\left(\hat{G}\beta_{n-1}\right)^2 M\left[\Lambda_{n-1} + (M-1)\,\Omega_{n-1}\right] + \sigma_u^2\right\}, \quad n = 1, \cdots, N.$$

Figure 3.14 depicts some of our results obtained with $N = 10$, in the cases $\rho = 0.1$ and $\rho = 0.7$. We have:[9]

Result 9. *As G increases, volume decreases at the beginning of the trading period and increases towards the end of the trading period. When traders experience information sharing and ρ is sufficiently low, volume at the end of the trading period is always higher than at the beginning of the trading period.*

Empirical research on the behavior of trading volume around earnings announcement days has shown that trading volume is essentially constant, and significantly clusters around announcement days [see, e.g., Lee, Mucklow and Ready (1993)]. Our information sharing model predicts a similar behavior of trading volume when ρ is suf-

FIGURE 3.14. Volume and volatility ($M = 7$, $N = 10$).



ficiently low. The model also predicts that at least partially, returns volatility exhibits a somehow similar behavior:

Result 10. *As G increases, volatility decreases at the beginning of the trading period. When traders experience information sharing, volatility at the end of the trading period is always higher than at the beginning of the trading period.*

As figure 3.14 reveals, asset returns volatility can be decreasing and/or substantially constant over time in the benchmark case of no-information sharing (i.e., when $G = 0$). Therefore, our model makes a sharp prediction about the behaviour of returns volatility around dates of information releases. Understanding returns volatility around these dates seems to us to be an interesting topic of future empirical research.

3.6 Conclusion

We have developed a dynamic trading model of information sharing, and demonstrated that in many important instances information sharing has a positive value. In general, information sharing generates a deep waiting game, and thus reduces price informativeness for almost all of the trading period. We have produced novel predictions on the behaviour of market variables such as trades' correlatedness, volume, return volatility, and liquidity. Incentive problems arising from strategic use of private information certainly constitute an important topic of future research, although the task doesn't seem to be any easy in a dynamic context such as the one we studied here.

3.A Appendix A: Preliminary results

Derivation of eqs. (3.6)–(3.8). To derive eq. (3.6), we use the definition of the average signal in eq. (3.3). A simple computation leaves:

$$var\left(\bar{s}_{i,0}\right) = \frac{\hat{G}\Lambda_0 + 2G\hat{G}\Omega_0}{\hat{G}^2} ,$$

or equivalently (3.6). Next, we derive eqs. (3.7a) and (3.7b). These equations correspond to two cases: *a*) $2G \leq (M-1)/2$ and *b*) $2G \geq (M-1)/2$, which we now study separately.

Case a) ($2G \leq (M-1)/2$). Consider traders i and $j = i+k$, $k \neq 0$. We have: $s_{i+k,0} \notin \mathbf{s}_{i,0}$ for all $|k| > G$. Therefore

$$\bar{\Omega}_0(k,G) \equiv cov\left(\bar{s}_{i,0}, \bar{s}_{i+k,0}\right) = \hat{G}^{-2} \sum_{l=-G}^{G} \sum_{m=-G}^{G} cov\left(s_{i+l,0}, s_{i+k+m,0}\right) = \Omega_0 ,$$

for all $|k| > 2G$, which is the second line in (3.7a). If instead $|k| \leq 2G$, $s_{i+k,0} \in \mathbf{s}_{i,0}$ for all $|k| \leq G$ and $\mathbf{s}_{i+k,0} \cap \mathbf{s}_{i,0} \neq \{\emptyset\}$. In particular, trader i shares $(2G+1-k)$ signals with trader $i+k$. Each of these signals contributes for $(\Lambda_0 + 2G\Omega_0)/(2G+1)^2$ to $\bar{\Omega}_0(k,G)$. Shared signals thus contribute for

$$\frac{\Lambda_0 + 2G\Omega_0}{(2G+1)^2} \cdot (2G+1-k)$$

to $\bar{\Omega}_0(k,G)$. The remaining (not shared) k signals contribute for

$$\frac{(2G+1)\,\Omega_0}{(2G+1)^2} \cdot k$$

to $\bar{\Omega}_0(k,G)$. Therefore,

$$\bar{\Omega}_0(k,G) = \frac{(\Lambda_0 + 2G\Omega_0)\left(\hat{G} - k\right) + \hat{G}\Omega_0 k}{\hat{G}^2} .$$

Grouping terms in the previous expression yields the first line in (3.7a).

Case b) ($2G \geq (M-1)/2$). Due to the double overlap phenomenon discussed in the main text, the number of signals shared by traders i and $i+k$ is:

$$L(k,G) \equiv 2G+1-k+n(k,G), \qquad k=1,\cdots,\tfrac{M-1}{2}\ . \tag{3.33}$$

The term $n(k,G)$ arises because traders on trader i's r.h.s. semicircle might be sending signals to traders lying between $i+1$ and $i+(M-1)/2$ on the l.h.s. semicircle (see figure 3); and obviously the i-th trader receives signals from traders lying between $i-1$ and $i-G$ as well. Double overlap occurs if and only if trader $i+k$ on the l.h.s. semicircle and trader $i-\ell$ with $\ell \in [1, \frac{M-1}{2}]$ on the r.h.s. semicircle are such that ℓ and k satisfy:

$$\begin{cases} \dfrac{M-1}{2}-(\ell-1)+\dfrac{M-1}{2}-k \leq G \\ G \geq \ell \geq 1 \\ \dfrac{M-1}{2} \geq k \geq 1 \end{cases}$$

The first inequality of the previous restrictions requires trader $i-\ell$ to send his signal to trader $i+k$. The second and third constraints restrict trader $i-\ell$ to be on the r.h.s. semicircle and trader $i+k$ to be on the l.h.s. semicircle relative to trader i. Thus, for fixed k ($1 \leq k \leq (M-1)/2$), double overlap occurs if and only if

$$G \geq \ell \geq M-G-k, \qquad k=1,\cdots,\tfrac{M-1}{2},$$

and $\ell \geq 1$. Clearly, $\min_k (M-G-k) = \frac{M-1}{2} - G + 1 \geq 1$. Hence, the constraint that $\ell \geq 1$ is redundant. By the previous inequalities, it immediately follows that:

$$n(k,G) = \max\left[G-(M-G-k)+1, 0\right]\ .$$

By replacing this result into eq. (3.33) leaves:

$$L(k,G) = \begin{cases} 4G+1-(M-1), & \frac{M-1}{2} \geq k \geq 2\left(\frac{M-1}{2}-G\right) \\ \\ 2G+1-k, & 1 \leq k \leq 2\left(\frac{M-1}{2}-G\right) \end{cases}$$

For all $k \in \left[1, 2\left(\frac{M-1}{2}-G\right)\right]$, $\bar{\Omega}_0(k,G)$ is thus exactly as in case $a)$ for $k \in [1, 2G]$, and the first line of eqs. (3.7b) follows. For all $k \in \left[2\left(\frac{M-1}{2}-G\right), \frac{M-1}{2}\right]$, tedious but straightforward computations lead to the second line of eqs. (3.7b).

Finally, we demonstrate that eq. (3.8) holds true. As usual, we consider the two cases in which $2G \gtrless (M-1)/2$. If $0 \leq 2G \leq (M-1)/2$, there are $[M-(4G+1)]$ traders $i+k$ such that $\mathrm{s}_{i+k,0} \cap \mathrm{s}_{i,0} = \{\emptyset\}$. In correspondence of these indexes, $cov(\bar{s}_{i+k,0}, \bar{s}_{i,0}) = \Omega_0$. Therefore,

$$\bar{\Gamma}_0(G) = 2\sum_{k=1}^{2G} \bar{\Omega}_0(k,G) + [M-(4G+1)]\,\Omega_0 .$$

The $2G$ covariances in the summation can be computed through the first line in (3.7a). Eq. (3.8) then follows by using the expression for $\bar{\Lambda}_0(G)$ in eq. (3.6). Next, consider the case $(M-1)/2 \leq 2G \leq M-1$. We have:

$$\bar{\Gamma}_0(G) = 2\left\{\sum_{k=1}^{M-1-2G} \bar{\Omega}_0(k,G) + \left(2G - \frac{M-1}{2}\right)\left[2\bar{\Lambda}_0(G) - \frac{M(\Lambda_0 - \Omega_0)}{\hat{G}^2} - \Omega_0\right]\right\} .$$

By plugging eqs. (3.7b) and (3.6) into the previous equation, we find that the expression of $\bar{\Gamma}_0(G)$ coincides with the one obtained in the case $0 \leq 2G \leq (M-1)/2$, and eq. (3.8) follows. ■

Derivation of eq. (3.10). First, consider the problem of projecting $s_{i+k,0}$ onto $\mathrm{s}_{i,0}$ for $|k| \leq G$. We have $s_{i+k,0} \in \mathrm{s}_{i,0}$ for $k = 0, \mp 1, \cdots, \mp G$. Hence,

$$E(s_{i+k,0} | \mathrm{s}_{i,0}) = s_{i+k,0} .$$

Next, consider a signal outside the i-th individual reach, i.e. take $s_{i+k,0}$ for $k = \mp(G+1), \cdots, \mp\frac{M-1}{2}$. Let $\Psi_{0,G} = E\left(\mathrm{s}_{i,0}\mathrm{s}_{i,0}^{\top}\right)$ be the $\hat{G} \times \hat{G}$ variance-covariance matrix of the vector $\mathrm{s}_{i,0}$. $\Psi_{0,G}$ is a $\hat{G} \times \hat{G}$ submatrix extracted from Ψ_0, and its inverse can be obtained with the same strategy of proof as in Foster and Viswanathan (1996) (p. 1479). Let $K \equiv [(\Lambda_0 - \Omega_0)(\Lambda_0 + 2G\Omega_0)]^{-1}$. We have:

$$(\Psi_{0,G})^{-1} = K \cdot \begin{bmatrix} \Lambda_0 + (2G-1)\Omega_0 & -\Omega_0 & \cdots & -\Omega_0 \\ & \Lambda_0 + (2G-1)\Omega_0 & & -\Omega_0 \\ & & \ddots & \vdots \\ & & & \Lambda_0 + (2G-1)\Omega_0 \end{bmatrix} .$$

Since $cov\left(s_{i+k,0}, \mathbf{s}_{i,0}\right) = \Omega_0 \mathbf{1}$ for $|k| > G$, the Projection Theorem then leaves:

$$\begin{aligned}
E\left(s_{i+k,0}|\mathbf{s}_{i,0}\right) &= \Omega_0 \mathbf{1}^\top \left(\Psi_{0,G}\right)^{-1} \mathbf{s}_{i,0} \\
&= \Omega_0 \frac{\Lambda_0 + (2G-1)\,\Omega_0 - 2G\Omega_0}{\left(\Lambda_0 - \Omega_0\right)\left(\Lambda_0 + 2G\Omega_0\right)} \mathbf{1}^\top \mathbf{s}_{i,0} \\
&= \frac{\hat{G}\Omega_0}{\Lambda_0 + 2G\Omega_0} \bar{s}_{i,0} \ .
\end{aligned}$$

Thus individual signals are projected according to:

$$E\left(s_{i+k,0}|\mathbf{s}_{i,0}\right) = \begin{cases} s_{i+k,0} & \text{for } \ k = 0, \mp 1, \cdots, \mp G \\ \\ \dfrac{\hat{G}\Omega_0}{\Lambda_0 + 2G\Omega_0} \bar{s}_{i,0} & \text{for } \ k = \mp\left(G+1\right), \cdots, \mp\frac{M-1}{2} \end{cases}$$

We now turn to consider how trader i projects the sum of other traders' average signals onto $\mathbf{s}_{i,0}$. Since $\bar{s}_{i,0} \in \mathbf{s}_{i,0}$,

$$E\left(\sum_{j \neq i} \bar{s}_{j,0} \middle| \mathbf{s}_{i,0}\right) = E\left(\sum_{j=1}^{M} \bar{s}_{j,0} \middle| \mathbf{s}_{i,0}\right) - \bar{s}_{i,0} \ .$$

Furthermore,

$$\sum_{j=1}^{M} \bar{s}_{j,0} = \sum_{j=1}^{M} s_{j,0} \ .$$

Using the last two equations,

$$\begin{aligned}
E\left(\sum_{j \neq i} \bar{s}_{j,0} \middle| \mathbf{s}_{i,0}\right) &= E\left(\sum_{j=1}^{M} s_{j,0} \middle| \mathbf{s}_{i,0}\right) - \bar{s}_{i,0} \\
&= \sum_{j=1}^{M} E\left(s_{j,0}|\, \mathbf{s}_{i,0}\right) - \bar{s}_{i,0} \\
&= \mathbf{1}^\top \mathbf{s}_{i,0} + \sum_{|k|>G} E\left(s_{i+k,0}|\, \mathbf{s}_{i,0}\right) - \bar{s}_{i,0} \\
&= (2G+1)\,\bar{s}_{i,0} + \frac{\left(M - (2G+1)\right)\Omega_0}{\Lambda_0 + 2G\Omega_0}\left(2G+1\right)\bar{s}_{i,0} - \bar{s}_{i,0} \\
&= \frac{\Lambda_0 + (M-1)\,\Omega_0}{\Lambda_0 + 2G\Omega_0} \hat{G}\bar{s}_{i,0} - \bar{s}_{i,0} \\
&= \frac{(M-1)(2G+1)\,\Omega_0 + 2G\left(\Lambda_0 - \Omega_0\right)}{\Lambda_0 + 2G\Omega_0} \bar{s}_{i,0} \ .
\end{aligned}$$

Grouping terms in the last equality yields eq. (3.10), where ϕ_1 is as in the main text.

■

3.B Appendix B: One-shot game

In this appendix, we provide an analytical solution (and characterization) of the model in the single auction case, i.e. $N = 1$. We have:

Proposition B1. *There exists a unique symmetric linear Bayesian equilibrium in which traders' strategies and prices are given by:*

$$x_{i,1} = \hat{G}\beta_1 \bar{s}_{i,0} \tag{3.34}$$

$$p_1 = \lambda_1 y_1 \tag{3.35}$$

where β_1 and λ_1 are functions of G given in the appendix, and $\lambda_1 > 0$. Furthermore, for each $M > 3$, there exist constants ρ^* and ρ^{**} depending on M, satisfying $-1/(M-1) \leq \rho^* \leq \rho^{**} \leq 0$, and such that no-information sharing is the optimal strategy for all $\rho \equiv \Omega_0/\Lambda_0 \in (\rho^{**}, 1)$; partial information sharing is the optimal strategy for all $\rho \in (\rho^*, \rho^{**})$; and complete information sharing is the optimal strategy for all $\rho \in (-1/(M-1), \rho^*)$. Finally, let $M = 3$. Then information sharing is the optimal strategy if and only if $\rho \in (-1/2, -1/3)$.

Proof. We organize the proof in two parts. In the first part, we take G as exogenous, and derive the equilibrium price and strategies. In the second part, we derive the strategic information sharing implications of the equilibrium.

Part a) Equilibrium

Suppose that the equilibrium pricing strategy is as in eq. (3.35). By the projection of the fundamental value in eq. (3.9), the problem solved by trader i is:

$$\max_{x_{i,1}} x_{i,1} \left[\eta_1 (2G+1)\bar{s}_{i,0} - \lambda_1 \left(x_{i,1} + E\left(\left.\sum_{j\neq i} x_{j,1}\right| \bar{s}_{i,0}\right)\right)\right] .$$

Now suppose that the trading strategy of all remaining traders is as in eq. (3.34). By eq. (3.10),

$$E\left(\left.\sum_{j\neq i} x_{j,1}\right| \bar{s}_{i,0}\right) = \hat{G}\beta_1 E\left(\left.\sum_{j\neq i} \bar{s}_{j,0}\right| \bar{s}_{i,0}\right) = \hat{G}(M-1)\beta_1\phi_1\bar{s}_{i,0} .$$

Therefore, the optimality conditions for trader i problem are:

$$x_{i,1} = \frac{\hat{G}\left[\eta_1 - \lambda_1\beta_1 (M-1)\phi_1\right]}{2\lambda_1}\bar{s}_{i,0} \ ,$$

and $\lambda_1 > 0$. Solving for the trading intensity β_1 yields:

$$\beta_1 = \frac{\eta_1}{\lambda_1\left[2 + (M-1)\phi_1\right]} \ ,$$

We now solve for the equilibrium price. By the equilibrium strategies conjectured in eq. (3.34),

$$cov\left(f, \sum_{i=1}^M x_{i,1}\right) = \hat{G}\beta_1 cov\left(f, \sum_{i=1}^M \bar{s}_{i,0}\right) = \hat{G}\beta_1 M c_0 \ ,$$

where we have made use of the equality $\sum_{i=1}^M \bar{s}_{i,0} = \sum_{i=1}^M s_{i,0}$. Similarly,

$$var\left(\sum_{i=1}^M x_{i,1}\right) = \hat{G}^2\beta_1^2 var\left(\sum_{i=1}^M \bar{s}_{i,0}\right) = \hat{G}^2\beta_1^2 M\left[\Lambda_0 + (M-1)\Omega_0\right] \ .$$

Hence, the regression coefficient λ_1 in the price function is:

$$\lambda_1 = \frac{\hat{G}\beta_1 M c_0}{\hat{G}^2\beta_1^2 M\left[\Lambda_0 + (M-1)\Omega_0\right] + \sigma_u^2} \ .$$

By substituting for β_1 leaves:

$$\lambda_1 = \frac{c_0\sqrt{M\bar{\Lambda}_0}}{\sigma_u\left[2\bar{\Lambda}_0 + \bar{\Gamma}_0\right]} \ .$$

Part b) Information sharing

The unconditional expected profit is given by

$$E\left[x_{i,1}\cdot(f - \lambda_1 y_1)\right] = \frac{c_0\sigma_u}{\sqrt{M\Lambda_0}}\sqrt{h(G)}, \qquad i = 1, \cdots, M,$$

where

$$h(G) \equiv \frac{\hat{G}(1 + 2G\rho)}{\left[2(G+1) + (\hat{G}M - 1)\rho\right]^2} \ .$$

First, consider the case $M = 3$. By a direct computation, $h(0) - h(1) < 0 \Leftrightarrow 15\rho^2 + 14\rho + 3 < 0 \Leftrightarrow \rho \in (-3/5, -1/3)$. The claim in the proposition now follows by the restriction that $\rho > -1/(M-1)$.

We now consider the case $M > 3$. We have:

$$h'(G) = 2(1-\rho)\frac{\omega(G,\rho)}{\{2(G+1) + [(2G+1)M - 1]\rho\}^3},$$

where

$$\omega(G,\rho) \equiv -2G - [2G(M-2) + M - 1]\rho .$$

Clearly, the denominator of h' is strictly positive for all $\rho \in (-1/(M-1), 1]$. Therefore, the sign of h' is equal to the sign of ω. We now use this fact and prove our claims in four steps. The first three steps deal with optimality of partial information sharing, complete information sharing, and no-information sharing. The last step contains refinements on optimality of partial information sharing.

- *Step 1: Partial information.* For all $\rho \in [0,1]$, $\omega(G,\rho) < 0$; and for all $\rho \in (-1/(M-1), 0)$, $\omega((M-1)/2, \rho) \leq 0$. Furthermore, $\omega(\cdot, \rho)$ has a zero at

$$G_\rho \equiv -\frac{(M-1)\rho}{2[(M-2)\rho + 1]} . \tag{3.36}$$

For all $\rho \in (-1/(M-1), 0)$, $G_\rho > 0$. Complete information sharing is never optimal for all $\rho \in C^N$, where

$$C^N \equiv \left\{x \in \left(-\tfrac{1}{M-1}, 0\right) : G_x < \tfrac{M-1}{2} - 1\right\} = \left\{x \in \left(-\tfrac{1}{M-1}, 0\right) : x > \rho_1\right\},$$

where $\rho_1 \equiv -(M-3)/(M^2 - 4M + 5)$. By simple computations, $\rho_1 \in (-1/(M-1), 0)$. Hence, $C^N = (\rho_1, 0)$. On the other hand, information sharing is optimal for all $\rho \in S$, where

$$S \equiv \left\{x \in \left(-\tfrac{1}{M-1}, 0\right) : G_x \geq 1\right\} = \left(-\tfrac{1}{M-1}, \rho_2\right),$$

and $\rho_2 \equiv -2/(3M-5)$. One may easily verify that $\rho_2 > \rho_1$. Our claim in the proposition follows because partial information sharing is optimal for all $\rho \in C^N \cap S = (\rho_1, \rho_2)$.

- *Step 2*: *Complete information sharing.* Let $\Delta(M,\rho) \equiv h((M-1)/2) - h(-1+(M-1)/2)$. We claim that for each M, complete information sharing is optimal for all $\rho \in C$, where we define

$$C \equiv \overline{C^N} \bigcap \left\{ x \in \left(-\tfrac{1}{M-1}, 0\right) : \Delta(M,x) > 0 \right\} ,$$

and $\overline{C^N} \equiv \{x \in (-1/(M-1), 0) : G_x \geq -1 + (M-1)/2\}$ is the complement of C^N with respect to the set $(-1/(M-1), 0)$. Indeed, fix a $\rho_* \in C$. Then, h is decreasing for $G \in \left(G_{\rho_*}, (M-1)/2\right)$ but $h((M-1)/2) > h(-1+(M-1)/2)$ by construction. Furthermore, h is increasing for all $G \in \left(0, G_{\rho_*}\right)$. Therefore, $h(-1+(M-1)/2) \geq h(G)$ for all $G \in (0, -1+(M-1)/2)$. We now demonstrate that C is not empty. We have

$$\Delta(M,\rho) = \frac{-2(1-\rho)\,\epsilon(M,\rho)}{[M+1+(M^2-1)\rho]^2 \{M-1+[(M-2)M-1]\rho\}^2} ,$$

where $\epsilon(M,\rho) \equiv a_1(M)\rho^2 + a_2(M)\rho + a_3(M)$, $a_1(M) \equiv M^4 - 4M^3 + 4M^2 + 2M - 3$, $a_2(M) \equiv 2M^3 - 6M^2 + M + 4$, and $a_3(M) \equiv M^2 - 2M - 1$. $\epsilon(M,\cdot)$ is a parabola, with $\epsilon(M,\cdot) > 0$; and roots $-1/(M-1)$ and $\rho^* \equiv -\left(M^2 - 2M - 1\right)/\left(M^3 - 3M^2 + M + 3\right)$. It is possible to show that $\rho^* \in (-1/(M-1), \rho_1)$. Furthermore, $\overline{C^N} = (-1/(M-1), \rho_1]$. Hence, $C = (-1/(M-1), \rho^*)$, and the claim in the proposition follows.

- *Step 3*: *No-information sharing.* For all $\rho \in [0,1]$, $G_\rho \leq 0$ [see eq. (3.36)]. Therefore, no-information sharing is the optimal strategy for all $\rho \in [0,1]$. We are left to show that there exists a $\rho^{**} \leq 0$ such that no-information sharing is the optimal strategy for all $\rho \in (\rho^{**}, 0]$ as well. We claim that no-information sharing is the optimal strategy for all $\rho \in S^N$, where

$$S^N \equiv \{x \in (\rho_2, 0] : d(x) > 0\} \bigcap N^I, \quad d(\rho) \equiv h(0) - h(1) ,$$

and

$$N^I \equiv \bigcap_{G=1}^{(M-1)/2} \{x \in (\rho_2, 0] : \omega(G,x) < 0\} .$$

Indeed, we have shown that information sharing is optimal for all $\rho \in (0, \rho_2)$. Furthermore, for all $\rho \in S^N$, $h(0) > h(1)$ and $h(1) \geq h(G)$, $G \in [1, (M-1)/2]$, both by construction. We now verify that S^N is not empty. First, by a direct computation, $\partial\rho^0(G)/\partial G < 0$, where $\rho^0(G) \equiv -2G/(2G(M-2)+M-1)$, and $\omega\left(G,\rho^{0}\left(G\right)\right)=0$. Hence, $N^{I}=\{x\in(\rho_2,0]:\omega(1,x)<0\}=\overline{S}$, where $\overline{S}$ is the complement of S with respect to the set $(-1/(M-1),0]$, or $\overline{S}=(\rho_2,0]$. It follows that

$$S^{N}\equiv\{x\in(\rho_2,0]:d(x)>0\}=\{x\in(\rho_2,0]:\tau(x)>0\}\ ,$$

where $\tau(x)\equiv 3(M-1)^2x^2+2(3M-4)x+2$, and the last equality follows by simple computations. One may readily show that $\tau>0$ on $(\rho^{**},0)$, where $\rho^{**}\equiv\frac{4-3M+\sqrt{3M^2-12M+10}}{3(M-1)^2}<0$. Finally, it is tedious but straight forward to check that $\rho^{**}>\rho_2$.

- *Step 4: Partial information sharing refinements.* By step 3, $h(1)>h(0)$ for all $\rho\in(\rho_2,\rho^{**})$. But by step 1, complete information sharing is not optimal on the same set. Therefore, partial information sharing is the optimal strategy on (ρ_2,ρ^{**}). Similarly, by step 2, complete information sharing is not optimal on (ρ^{*},ρ_1), and by step 1, information sharing is optimal on the same set. Hence, partial information sharing is the optimal strategy on (ρ^{*},ρ_1). ■

3.C Appendix C: Dynamic game

3.C.1 Preliminary results

Derivation of eq. (3.12). By Semi-Strong market efficiency, $p_n=E(f|\,F_{M+1,n})$. By eq. (3.4), $\bar{s}$ is a sufficient statistic for $E(f|\,\mathrm{s}_0)$. Therefore,

$$\begin{aligned}
p_n &= E(f|\,F_{M+1,n})\\
&= E\{E[E(f|\,\mathrm{s}_0)|\,\bar{s},F_{M+1,n}]|\,F_{M+1,n}\}\\
&= E[E(\theta\bar{s}|\,\bar{s},F_{M+1,n})|\,F_{M+1,n}]\\
&= \frac{\theta}{M}E\left(\sum_{i=1}^{M}\bar{s}_{i,0}\middle|\,F_{M+1,n}\right)\\
&= \theta t_n\ ,
\end{aligned}$$

where the last line follows by eq. (3.11). ■

Derivation of eq. (3.15). By the Law of Iterated Expectations:

$$E\left(\theta\bar{s}\right| F_{M+1,n}) = E\left[E\left(f\right| \mathrm{s}_0)\right| F_{M+1,n}] = E\left(f\right| F_{M+1,n}) = p_n \ .$$

Hence,

$$\begin{aligned}
\sigma^2_{f,n} &= E\left[\left(\frac{\theta}{M}\sum_{i=1}^{M}\bar{s}_{i,0} - \frac{\theta}{M}\sum_{i=1}^{M}t_{i,n}\right)^2 \middle| F_{M+1,n}\right] \\
&= \frac{\theta^2}{M^2}E\left[\left(\sum_{i=1}^{M}\bar{s}_{i,n}\right)^2 \middle| F_{M+1,n}\right] \\
&= \frac{\theta^2}{M^2}E\left[\left(\sum_{i=1}^{M}s_{i,n}\right)^2 \middle| F_{M+1,n}\right] \\
&= \frac{\theta^2}{M}\left[\Lambda_n + (M-1)\,\Omega_n\right] \ . \blacksquare
\end{aligned}$$

Derivation of eqs. (3.16a)-(3.16e). Let $c_n \equiv cov\left(s_{i,n-1}, y_n | F_{M+1,n-1}\right)$, which is independent of i, and $\Psi_n \equiv E\left[\left(s_{1,0} - t_n, \cdots, s_{M,0} - t_n\right)^{\top}\left(s_{1,0} - t_n, \cdots, s_{M,0} - t_n\right)\middle| F_{M+1,n}\right]$. By the Projection Theorem,

$$\Psi_n = \Psi_{n-1} - \frac{c_n^2}{var\left(y_n\right| F_{M+1,n-1})}\mathbf{1}\mathbf{1}^{\top} \ ,$$

which gives the recursions:

$$\begin{aligned}
\Lambda_n &= \Lambda_{n-1} - \frac{c_n^2}{var\left(y_n\right| F_{M+1,n-1})} \\
\Omega_n &= \Omega_{n-1} - \frac{c_n^2}{var\left(y_n\right| F_{M+1,n-1})}
\end{aligned}$$

or equivalently (3.16a). Taking one lag in equation (3.15) yields:

$$\sigma^2_{f,n-1} = \frac{\theta^2}{M}\left[\Lambda_{n-1} + (M-1)\,\Omega_{n-1}\right] \ ,$$

giving the recursion:

$$\begin{aligned}
\sigma^2_{f,n-1} - \sigma^2_{f,n} &= \frac{\theta^2}{M}\left[\Lambda_{n-1} - \Lambda_n + (M-1)\left(\Omega_{n-1} - \Omega_n\right)\right] \\
&= \theta^2\left(\Lambda_{n-1} - \Lambda_n\right) \ ,
\end{aligned}$$

where the last equality follows by eq. (3.16a). Now consider the variance of average signals $\bar{\Lambda}_n(G)$. By eq. (3.6), $\bar{\Lambda}_n(G)$ can be expressed in terms of the elements in the individual signals variance-covariance matrix Ψ_n as:

$$\bar{\Lambda}_n(G) = \frac{(\Lambda_n + 2G\Omega_n)}{\hat{G}} ,$$

and eq. (3.16c) follows by eq. (3.16a) and by simple computations. We now provide the update for $\bar{\Omega}_n(k,G)$, thus completing the specification of the variance-covariance matrix $\bar{\Psi}_n(G) \equiv E\left[(\bar{s}_{1,0} - t_n, \cdots, \bar{s}_{M,0} - t_n)^\top (\bar{s}_{1,0} - t_n, \cdots, \bar{s}_{M,0} - t_n) \middle| F_{M+1,n}\right]$. By eq. (3.16a) and the expression of the off-diagonal elements in $\bar{\Psi}_n(G)$ [see eqs. (3.7a) and (3.7b) evaluated at n],

$$\bar{\Omega}_{n-1}(k,G) - \bar{\Omega}_n(k,G) = \Lambda_{n-1} - \Lambda_n, \qquad \text{all } k, G .$$

Note that while the time n covariance term $\bar{\Omega}_n(k,G)$ depends on G as in (3.7a) and (3.7b), the update given in (3.16d) does not.

Finally, from eq. (3.8) one has the recursion:

$$\bar{\Gamma}_{n-1}(G) - \bar{\Gamma}_n(G) = (M-1)(\Lambda_{n-1} - \Lambda_n) \ . \blacksquare$$

Derivation of eq. (3.19). By the definition of t_n and s_n,

$$t_n - t_{n-1} = E(s_{i,0} - t_{n-1} | F_{M+1,n}) = E(s_{i,n-1} | F_{M+1,n}) = \zeta_n y_n ,$$

where ζ_n is the regression coefficient of $s_{i,n-1}$ (or equivalently of $\bar{s}_{i,n-1}$) on y_n, viz

$$\zeta_n = \frac{cov(s_{i,n-1}, y_n | F_{M+1,n-1})}{var(y_n | F_{M+1,n-1})} = \frac{c_n}{var(y_n | F_{M+1,n-1})} \ . \blacksquare$$

Derivation of eq. (3.20). We have:

$$E\left(\frac{1}{M}\sum_{i=1}^{M} \bar{s}_{i,n-1} \middle| F_{M+1,n}\right) = \frac{1}{M} E\left(\sum_{i=1}^{M} s_{i,n-1} \middle| F_{M+1,n}\right) = \zeta_n y_n \ .$$

By eq. (3.12), $p_{n-1} = \theta t_{n-1}$. Hence,

$$\bar{s} - \frac{1}{\theta} p_{n-1} = \bar{s} - t_{n-1} = \frac{1}{M} \sum_{i=1}^{M} (\bar{s}_{i,0} - t_{n-1}) = \frac{1}{M} \sum_{i=1}^{M} \bar{s}_{i,n-1} \ .$$

Therefore,

$$E\left(\theta\bar{s} - p_{n-1} \middle| F_{M+1,n}\right) = \theta E\left(\frac{1}{M} \sum_{i=1}^{M} \bar{s}_{i,n-1} \middle| F_{M+1,n}\right) = \theta \zeta_n y_n \ .$$

But by eq. (3.4), $E\left[E\left(f \mid \mathrm{s}_0\right) \mid F_{M+1,n}\right] = E\left(\theta\bar{s} \mid F_{M+1,n}\right)$. Hence, by the Law of Iterated Expectations,

$$E\left(f - p_{n-1} | F_{M+1,n}\right) = E\left[E\left(f \mid \mathrm{s}_0\right) - p_{n-1} | F_{M+1,n}\right] = E\left(\theta\bar{s} - p_{n-1} \middle| F_{M+1,n}\right) = \theta\zeta_n y_n \ ,$$

and eq. (3.20) follows by eq. (3.18) and $E\left(\theta\bar{s} \mid F_{M+1,n}\right) = p_n$. ■

The following lemma is needed to derive eqs. (3.21)–(3.22).

Lemma 1. *For all $i, j = 1, \cdots, M$ and $n = 2, \cdots, N$,*

$$E\left(s_{j,n-1} \mid F_{i,n}\right) = E\left(s_{j,n-1} \mid s_{i,n-1}\right) \ .$$

Proof. By eqs. (3.17), (3.19) and (3.20), the aggregate order flow can be recursively written as

$$y_n = \left(\frac{\beta_n a_{n-1}}{\beta_{n-1}}\right) y_{n-1} + \varepsilon_n \ ,$$

where $a_{n-1} \equiv 1 - \theta^{-1} \hat{G} M \beta_{n-1} \lambda_{n-1}$ and $\varepsilon_n \equiv u_n - \beta_n \beta_{n-1}^{-1} u_{n-1}$. Solving backward the above recursion gives

$$y_n = b_{n-1} y_1 + \textstyle\sum_{j=1}^{n-2} b_j \varepsilon_{n-j} + \varepsilon_n \ ,$$

where $b_j \equiv \beta_n \beta_{n-j}^{-1} \prod_{h=1}^{j} a_{n-h}$. Hence, $\{y_n\}_{n \geq 1}$ is a Gaussian process. Therefore, it is sufficient to show that $cov[s_{j,n-1}, (y_1, \cdots, y_{n-1})^{\top}] = \mathbf{0}_{(n-1)\times 1}$. For all $k \leq n-1$,

$$\begin{aligned} cov\left(y_k, s_{j,n-1}\right) &= E\left(y_k \cdot s_{j,n-1}\right) \\ &= E\left[E\left(y_k \cdot s_{j,n-1} \mid y_1, \cdots, y_{n-1}\right)\right] \\ &= E\left[y_k \cdot E\left(s_{j,n-1} \mid y_1, \cdots, y_{n-1}\right)\right] \\ &= 0 \ , \end{aligned}$$

where the first line follows because $E\left(y_1,\cdots,y_{n-1}\right)^{\top}=\mathbf{0}_{(n-1)\times 1}$, the second line holds by the Law of Iterated Expectations, and the last line follows by the definition of $s_{j,n-1}$. ■

Derivation of eqs. (3.21)-(3.22). We have:

$$\begin{aligned}
E\left(f-p_{n-1}\middle|\,F_{i,n}\right) &= E\left[E\left(f-p_{n-1}\middle|\,\mathbf{s}_0\right)\middle|\,\mathbf{s}_{i,0},F_{M+1,n-1}\right] \\
&= E\left[\theta\bar{s}-p_{n-1}\middle|\,\mathbf{s}_{i,0},F_{M+1,n-1}\right] \\
&= \theta E\left(\frac{1}{M}\sum_{i=1}^{M}\bar{s}_{i,n-1}\middle|\,\mathbf{s}_{i,n-1},F_{M+1,n-1}\right) \\
&= \theta E\left(\frac{1}{M}\sum_{i=1}^{M}s_{i,n-1}\middle|\,\mathbf{s}_{i,n-1},F_{M+1,n-1}\right) \\
&= \frac{\theta}{M}E\left(s_{i,n-1}+\textstyle\sum_{j\neq i}s_{j,n-1}\middle|\,\bar{s}_{i,n-1}\right) \\
&= \frac{\theta}{M}\left(1+\frac{\bar{\Gamma}_{n-1}}{\bar{\Lambda}_{n-1}}\right)\bar{s}_{i,n-1}\ ,
\end{aligned}$$

by the Law of Iterated Expectations, the fact that $\theta\bar{s}-p_{n-1}=\frac{\theta}{M}\sum_{i=1}^{M}\bar{s}_{i,n-1}$ [see the derivation of eq. (3.20)], and lemma 1. This is eq. (3.21) with $\eta_n=\theta(\hat{G}M)^{-1}(1+\bar{\Lambda}_{n-1}^{-1}\bar{\Gamma}_{n-1})$. By the same arguments,

$$E\left(\textstyle\sum_{j\neq i}\bar{s}_{j,n-1}\middle|\,F_{i,n}\right)=\frac{\bar{\Gamma}_{n-1}}{\bar{\Lambda}_{n-1}}\bar{s}_{i,n-1}\ ,$$

which is eq. (3.22) with $\phi_n=(M-1)^{-1}\bar{\Lambda}_{n-1}^{-1}\bar{\Gamma}_{n-1}$. ■

3.C.2 Equilibrium

We proceed in three steps. In the first step, we derive a recursive expression for the price deviation induced by traders' suboptimal play. In the second step, we derive the traders' optimality conditions. In the third step, we compute market maker updates.

Step 1: Price deviation

We show that the price deviation induced by suboptimal play of a given trader i has the following recursive structure:

$$p_n-p_n'=\left(p_{n-1}-p_{n-1}'\right)\left[1-\theta^{-1}\hat{G}\left(M-1\right)\beta_n\lambda_n\right]+\hat{G}\lambda_n\beta_n\bar{s}_{i,n-1}-\lambda_n x_{i,n}'\ . \quad (3.37)$$

Indeed, let $y'_n = \sum_{j\neq i} x_{j,n} + x'_{i,n} + u_n$ and t'_n be the aggregate order flow and the market maker's update when trader i deviates to $x'_{i,n}$. By eq. (3.19),

$$t_n = \sum_{k=1}^{n} \zeta_k y_k \ .$$

Similarly $t'_n = \sum_{k=1}^{n} \zeta_k y'_n$. Therefore, by eq. (3.13),

$$\begin{aligned}
\bar{s}_{j,n-1} - \bar{s}'_{j,n-1} &= \left(\bar{s}_{j,0} - t_{n-1}\right) - \left(\bar{s}_{j,0} - t'_{n-1}\right) \\
&= \sum_{k=1}^{n-1} \zeta_k y'_k - \sum_{k=1}^{n-1} \zeta_k y_k \\
&= \frac{1}{\theta}\left(\sum_{k=1}^{n-1} \lambda_k y'_k - \sum_{k=1}^{n-1} \lambda_k y_k\right) \\
&= \frac{1}{\theta}\left(p'_{n-1} - p_{n-1}\right) \qquad (3.38)
\end{aligned}$$

where the third line follows by eq. (3.20), and the fourth line follows because eq. (3.18) implies that $p_n = \sum_{k=1}^{n} \lambda_k y_k$. Thus, by eq. (3.13) and eq. (3.19),

$$\bar{s}_{i,n} = \bar{s}_{i,0} - t_n = \bar{s}_{i,n-1} - (t_n - t_{n-1}) = \bar{s}_{i,n-1} - \zeta_n y_n \ .$$

Substituting for the equilibrium order flow and taking expectations yields:

$$\begin{aligned}
E\left(\bar{s}_{i,n} \middle| F_{i,n}\right) &= \bar{s}_{i,n-1} - \frac{\hat{G}\beta_n \lambda_n}{\theta}\left[\bar{s}_{i,n-1} + E\left(\sum_{j\neq i} \bar{s}_{i,n-1} \middle| F_{i,n}\right)\right] \\
&= \left[1 - \frac{\hat{G}\beta_n \lambda_n}{\theta}\left(1 + (M-1)\phi_n\right)\right] \bar{s}_{i,n-1} \ , \qquad (3.39)
\end{aligned}$$

where the last line follows by eq. (3.29). Using the equilibrium strategy in eq. (3.17) and the price recursion in eq. (3.18), we find that the price deviation has the following expression:

$$\begin{aligned}
p_n - p'_n &= p_{n-1} - p'_{n-1} + \lambda_n \left(y_n - y'_n\right) \\
&= p_{n-1} - p'_{n-1} + \lambda_n \left[\sum_{j\neq i} \hat{G}\beta_n \left(\bar{s}_{j,n-1} - \bar{s}'_{j,n-1}\right) + \hat{G}\beta_n \bar{s}_{i,n-1} - x'_{i,n}\right] \ .
\end{aligned}$$

Substituting for $\left(\bar{s}_{j,n-1} - \bar{s}'_{j,n-1}\right)$ from eq. (3.38) in the previous equation gives eq. (3.37).

Step 2: Traders' behavior

First, we show that the value function in eq. (3.23) and the strategy in eq. (3.24) are mutually consistent. Trader i faces the following recursive problem:

$$\begin{aligned} W_{i,n-1} &= \max_{x'_{i,n}} E\left[\left(f-p_n\right)x'_{i,n}+W_{i,n}\right| F_{i,n}\right] \\ &= \max_{x'_{i,n}} E\left[\left(f-p_{n-1}-\lambda_n x'_{i,n}-\lambda_n \textstyle\sum_{j\neq i} x_{j,n}\right)x'_{i,n}+W_{i,n}\middle| F_{i,n}\right] \end{aligned}$$

Given the trading strategy conjectured in eq. (3.23), the optimality conditions of the previous problem lead to:

$$\begin{aligned} 0 \; = \; & E\left(f-p'_{n-1}\right| F_{i,n}) - \hat{G}\beta_n\lambda_n E\left(\textstyle\sum_{j\neq i}\bar{s}'_{j,n-1}\middle| F_{i,n}\right) \\ & -2\lambda_n x'_{i,n}-\lambda_n\psi_n E\left(\bar{s}_{i,n}\right| F_{i,n}) - 2\lambda_n\mu_n E\left(p_n-p'_n\right| F_{i,n}) \text{ (first order conditions);} \end{aligned}$$

and

$$-\lambda_n+\lambda_n^2\mu_n<0 \text{ (second order conditions).}$$

Because $\left(\bar{s}_{j,n-1},\left(p_{n-1}-p'_{n-1}\right)\right)\in F_{i,n}$, the first order conditions can be reorganized as follows:

$$\begin{aligned} & 0 = E\left(f-p_{n-1}\right| F_{i,n}) + \left(p_{n-1}-p'_{n-1}\right) - \hat{G}\beta_n\lambda_n\textstyle\sum_{j\neq i}\left(\bar{s}'_{j,n-1}-\bar{s}_{j,n-1}\right) \\ & -\hat{G}\beta_n\lambda_n E\left(\textstyle\sum_{j\neq i}\bar{s}_{j,n-1}\middle| F_{i,n}\right) - 2\lambda_n x'_{i,n} - \lambda_n\psi_n E\left(\bar{s}_{i,n}\right| F_{i,n}) - 2\lambda_n\mu_n E\left(p_n-p'_n\right| F_{i,n}) \;. \end{aligned}$$

By replacing eqs. (3.37), (3.38) and (3.39) in the previous equation, and by rearranging terms, we obtain eq. (3.24), where γ_n is as in eq. (3.27) and

$$\beta_n = \frac{\eta_n - \hat{G}^{-1}\lambda_n\psi_n}{\lambda_n\left[1+\left(1-\theta^{-1}\lambda_n\psi_n\right)\left(1+\left(M-1\right)\phi_n\right)\right]}\;. \tag{3.40}$$

Next, we use eq. (3.24), and find that the expected profit of a single auction is:

$$\begin{aligned} E\left[\left(f-p_n\right)x'_{i,n}\right| F_{i,n}\right] \; = \; & \hat{G}^2\beta_n\left[\eta_n-\beta_n\lambda_n\left(1+\left(M-1\right)\phi_n\right)\right]\bar{s}^2_{i,n-1} \\ & +\gamma_n\left[1-\lambda_n\left(\gamma_n+\theta^{-1}\hat{G}\left(M-1\right)\beta_n\right)\right]\left(p_{n-1}-p'_{n-1}\right)^2 \\ & +\left\{\gamma_n\left(\eta_n-2\beta_n\lambda_n\right)+\beta_n\left[1-\left(M-1\right)\lambda_n\left(\gamma_n\phi_n+\theta^{-1}\hat{G}\beta_n\right)\right]\right\} \\ & \qquad \times\hat{G}\bar{s}_{i,n-1}\left(p_{n-1}-p'_{n-1}\right)\;. \end{aligned}$$

By taking the conditional expectation of the value function in eq. (3.23) leaves:

$$E\left(W_{i,n}\middle|\,F_{i,n}\right)=\alpha_n E\left(\bar{s}_{i,n}^2\middle|\,F_{i,n}\right)+\psi_n\left(p_n-p_n'\right)E\left(\bar{s}_{i,n}\middle|\,F_{i,n}\right)+\mu_n\left(p_n-p_n'\right)^2+\delta_n\ .$$

By eqs. (3.37) and (3.39), both $E\left(\bar{s}_{i,n}|F_{i,n}\right)$ and $\left(p_n-p_n'\right)$ are linear in $\left(p_{n-1}-p_{n-1}'\right)$ and $\bar{s}_{i,n-1}$. To identify all coefficients of the value function, we are therefore left with finding the conditional expectation $E(\bar{s}_{i,n}^2|F_{i,n})$. By eqs. (3.13) and (3.19),

$$\begin{aligned}E\left(\bar{s}_{i,n}^2\middle|\,F_{i,n}\right)&=\bar{s}_{i,n-1}^2+\zeta_n^2E\left(y_n^2\middle|\,F_{i,n}\right)-2\zeta_n\bar{s}_{i,n-1}E\left(y_n\middle|\,F_{i,n}\right)\\&=\left[1-\theta^{-1}\hat{G}\beta_n\lambda_n\left(1+\left(M-1\right)\phi_n\right)\right]^2\bar{s}_{i,n-1}^2\\&\quad+\theta^{-1}\lambda_n^2\sigma_u^2+\theta^{-1}\hat{G}^2\beta_n^2\lambda_n^2 var\left(\textstyle\sum_{j\neq i}\bar{s}_{j,n-1}\middle|\,F_{i,n}\right)\ ,\end{aligned}\tag{3.41}$$

where

$$\begin{aligned}var\left(\textstyle\sum_{j\neq i}\bar{s}_{j,n-1}\middle|\,F_{i,n}\right)&=var\left(\textstyle\sum_{j\neq i}\bar{s}_{j,n-1}\middle|\,\bar{s}_{i,n-1}\right)\\&=E\left[\textstyle\sum_{j\neq i}\bar{s}_{j,n-1}-\left(M-1\right)\phi_n\bar{s}_{i,n-1}\right]^2\\&=var\left(\textstyle\sum_{j\neq i}\bar{s}_{j,0}\middle|\,F_{M+1,n-1}\right)-\left(M-1\right)^2\phi_n^2 var\left(\bar{s}_{i,0}\middle|\,F_{M+1,n-1}\right)\\&=M\left[\Lambda_{n-1}+\left(M-1\right)\Omega_{n-1}\right]-\left[1+\left(M-1\right)^2\phi_n^2\right]\bar{\Lambda}_{n-1}-2\bar{\Gamma}_{n-1}\ ,\end{aligned}\tag{3.42}$$

where the first equality follows by the arguments utilized to show lemma 1 in appendix B. Finally, by plugging eq. (3.42) into eq. (3.41), and using the expression for the expected profits and the expectation of the value function, gives the coefficients α_n, μ_n, ψ_n and δ_n in eq. (3.28).

Step 3: Market maker updates

Finally, we turn to consider the market maker's problem. By plugging the equilibrium trades in eq. (3.17) into the order flow in eq. (3.2) gives

$$y_n=\textstyle\sum_{j=1}^M\hat{G}\beta_n\bar{s}_{i,n-1}+u_n=\hat{G}M\beta_n\left(\frac{1}{M}\sum_{j=1}^M\left(\bar{s}_{i,0}-t_{n-1}\right)\right)+u_n\ .$$

where the second line follows from the market maker's update in eq. (3.13). By the definition of $\bar{s}$, and the equality $t_{n-1} = \theta^{-1} p_{n-1}$ in eq. (3.12),

$$y_n = \hat{G}M\beta_n \left(\bar{s} - \frac{p_{n-1}}{\theta}\right) + u_n = \frac{\hat{G}M\beta_n}{\theta}\left(\theta\bar{s} - p_{n-1}\right) + u_n \,. \tag{3.43}$$

By eqs. (3.43) and (3.4), and the Law of Iterated Expectations,

$$cov\left(f, y_n \middle| F_{M+1,n-1}\right) = cov\left(\theta\bar{s}, y_n \middle| F_{M+1,n-1}\right) \,.$$

Therefore,

$$\begin{aligned} cov\left(f, y_n \middle| F_{M+1,n-1}\right) &= cov\left(\theta\bar{s} - p_{n-1}, y_n \middle| F_{M+1,n-1}\right) \\ &= \frac{\hat{G}M\beta_n}{\theta} var\left(\theta\bar{s} - p_{n-1} \middle| F_{M+1,n-1}\right) \\ &= \frac{\hat{G}M\beta_n}{\theta}\sigma^2_{f,n-1} \,, \end{aligned} \tag{3.44}$$

where the first line follows because $p_{n-1} \in F_{M+1,n-1}$, the second line is obtained by using the expression of the aggregate order flow in eq. (3.43), and the third line is due to the expression of the residual variance in eq. (3.14). We now express the recursion of Λ_n in terms of equilibrium parameters. By using the order flow in eq. (3.43),

$$\begin{aligned} c_n &\equiv cov\left(s_{i,n-1}, y_n \middle| F_{M+1,n-1}\right) \\ &= \hat{G}\beta_n cov\left(s_{i,n-1}, \textstyle\sum_{i=1}^{M} s_{i,n-1} \middle| F_{M+1,n-1}\right) \\ &= \hat{G}\beta_n \left(\Lambda_{n-1} + (M-1)\,\Omega_{n-1}\right) \\ &= \frac{M\hat{G}\beta_n}{\theta^2}\sigma^2_{f,n-1} \,, \end{aligned}$$

where the third line follows by the expression of the residual variances in eq. (3.14), and the last line holds by the expression of $\sigma^2_{f,n}$ in eq. (3.15). Therefore, by the above expression for c_n, and eq. (3.20),

$$\Lambda_n = \Lambda_{n-1} - \zeta_n c_n = \Lambda_{n-1} - \frac{M\hat{G}\beta_n\lambda_n}{\theta^3}\sigma^2_{f,n-1} \,. \tag{3.45}$$

Again by the above expression for c_n, and eq. (3.20),

$$\lambda_n = \theta\zeta_n = \frac{\theta c_n}{var\left(y_n \middle| F_{M+1,n-1}\right)} = \frac{M\hat{G}\beta_n\sigma^2_{f,n-1}}{\theta \cdot var\left(y_n \middle| F_{M+1,n-1}\right)} \,.$$

But

$$\begin{aligned} var\left(y_n \middle| F_{M+1,n-1}\right) &= \left(\theta^{-1}\hat{G}M\beta_n\right)^2 var\left(\theta\bar{s} - p_{n-1} \middle| F_{M+1,n-1}\right) + \sigma_u^2 \\ &= \left(\theta^{-1}\hat{G}M\beta_n\right)^2 \sigma_{f,n}^2 + \sigma_u^2 . \end{aligned}$$

Therefore, the price sensitivity in eq. (3.18) can be represented as:

$$\lambda_n = \frac{\theta M \hat{G}\beta_n \sigma_{f,n-1}^2}{\left(\hat{G}M\beta_n\right)^2 \sigma_{f,n-1}^2 + \theta^2\sigma_u^2} . \tag{3.46}$$

After n trading rounds, the full information fundamental value has residual variance given by:

$$\sigma_{f,n}^2 = \sigma_{f,n-1}^2 - \lambda_n cov\left(f, y_n | F_{M+1,n-1}\right) = \left(1 - \theta^{-1}\beta_n\lambda_n\hat{G}M\right)\sigma_{f,n-1}^2 , \tag{3.47}$$

where we have used eq. (3.44). We plug eq. (3.46) into eq. (3.47) and obtain:

$$\sigma_{f,n}^2 = \frac{\theta^2\sigma_u^2\sigma_{f,n-1}^2}{\left(\hat{G}M\beta_n\right)^2 \sigma_{f,n-1}^2 + \theta^2\sigma_u^2} . \tag{3.48}$$

By rearranging the previous expression and using eq. (3.46) gives an alternative expression for λ_n:

$$\lambda_n = \frac{\hat{G}M\beta_n}{\theta}\frac{\sigma_{f,n}^2}{\sigma_u^2} ,$$

or equivalently eq. (3.26). By solving eq. (3.48) for $\sigma_{f,n-1}^2$ gives

$$\sigma_{f,n-1}^2 = -\frac{\theta^2\sigma_u^2\sigma_{f,n}^2}{\left(\hat{G}M\beta_n\right)^2 \sigma_{f,n}^2 - \theta^2\sigma_u^2} ,$$

which together with eq. (3.45) imply

$$\begin{aligned} \Lambda_{n-1} - \Lambda_n &= -\frac{\sigma_u^4}{\left(\hat{G}M\beta_n\right)^2 \sigma_{f,n}^2 - \theta^2\sigma_u^2}\lambda_n^2 \\ &= -\frac{\lambda_n^2\sigma_u^2\sigma_{f,n}^2}{\theta^2\left(\lambda_n\sigma_u^2 - \sigma_{f,n}^2\right)} , \end{aligned} \tag{3.49}$$

where the last equality follows because $\left(\hat{G}M\beta_n\right)^2 \sigma_{f,n}^2 = \theta^2\lambda_n^2\sigma_u^2\sigma_{f,n}^{-2}$ [thanks to eq. (3.26)]. Also, eqs. (3.16c) and (3.16e) imply that

$$\frac{\bar{\Gamma}_{n-1}}{\bar{\Lambda}_{n-1}} = \frac{\bar{\Gamma}_n + (M-1)(\Lambda_{n-1} - \Lambda_n)}{\bar{\Lambda}_n + (\Lambda_{n-1} - \Lambda_n)} . \tag{3.50}$$

Furthermore, by eqs. (3.29)–(3.30):

$$\frac{\bar{\Gamma}_{n-1}}{\bar{\Lambda}_{n-1}} = (M-1)\,\phi_n = \frac{\hat{G}M}{\theta}\eta_n - 1$$

Substituting for β_n as in eq. (3.40) gives:

$$\theta^2\sigma_u^2\lambda_n^2\left[1 + \left(1 - \theta^{-1}\lambda_n\psi_n\right)\left(1 + (M-1)\,\phi_n\right)\right] = \left(\hat{G}\eta_n - \lambda_n\psi_n\right) M\sigma_{f,n}^2$$

Substituting eqs. (3.29)–(3.30) in the previous equation leaves

$$\theta^2\sigma_u^2\lambda_n^2\left[1 + \left(1 - \theta^{-1}\lambda_n\psi_n\right)\left(1 + \frac{\bar{\Gamma}_{n-1}}{\bar{\Lambda}_{n-1}}\right)\right] = \left[\frac{\theta}{M}\left(1 + \frac{\bar{\Gamma}_{n-1}}{\bar{\Lambda}_{n-1}}\right) - \lambda_n\psi_n\right] M\sigma_{f,n}^2 .$$

Substituting eq. (3.49) together with eq. (3.50) in the last equality, and rearranging terms, gives the quartic equation $F(\lambda_n) = 0$ in eq. (3.25). To show that eq. (3.25) admits a unique positive solution, note that the constant and the coefficient of λ_n^4 are both positive, and that the coefficient of λ_n^2 is negative.[10] On the other hand, the sign of ψ_n determines the sign of the terms in λ_n^3 and λ_n. However, regardless of whether ψ_n is positive or negative, there are only two sign changes. By Descartes' rule, eq. (3.25) has at most two real positive roots. Note from eq. (3.47) that $\sigma_{f,n}^2 < \sigma_{f,n-1}^2$ is equivalent to $\theta^{-1}\beta_n\lambda_n\hat{G}M < 1$. Using eq. (3.26) this restriction becomes $\lambda_n^2 < \sigma_{f,n}^2\sigma_u^{-2}$ or equivalently $\lambda_n < \sigma_{f,n}\sigma_u^{-1}$. By eq. (3.25), $F(\lambda = 0) = \frac{\theta}{M}\left[\Lambda_n + (M-1)\,\Omega_n\right] > 0$, $F\left(\lambda_n = \sigma_{f,n}\sigma_u^{-1}\right) = -\frac{\theta}{M^2}\left(\Lambda_n + (M-1)\,\Omega_n\right) < 0$ and $F(\lambda = +\infty) = +\infty$; hence, there is one and only one positive root between 0 and $\sigma_{f,n}\sigma_u^{-1}$. ■

3.C.3 Further results

Derivation of eq. (3.31). Taking the first lag in eq. (3.47) and substituting the result into eq. (3.45) yields:

$$\Lambda_{n-1} = \Lambda_n + \frac{\hat{G}\beta_n\lambda_n}{\theta}\left[\Lambda_{n-1} + (M-1)\,\Omega_{n-1}\right]$$

Since $\Omega_{n-1} - \Omega_n = \Lambda_{n-1} - \Lambda_n$ as in eq. (3.16*a*), and $\Lambda_n - \Omega_n$ is constant,

$$\Omega_{n-1} = \Omega_n + \frac{\hat{G}\beta_n\lambda_n}{\theta}\left(\Lambda_{n-1} - \Omega_{n-1} + M\Omega_{n-1}\right) = \frac{\theta\Omega_n + \hat{G}\beta_n\lambda_n\left(\Lambda_{n-1} - \Omega_{n-1}\right)}{\theta - M\hat{G}\beta_n\lambda_n}$$

and

$$\begin{aligned}
\Lambda_{n-1} &= \Lambda_n + \Omega_{n-1} - \Omega_n \\
&= \Lambda_n + \hat{G}\beta_n\lambda_n\frac{M\Omega_n + \left(\Lambda_{n-1} - \Omega_{n-1}\right)}{\theta - M\hat{G}\beta_n\lambda_n} \\
&= \frac{\theta\Lambda_n - \hat{G}\left(M-1\right)\beta_n\lambda_n\left(\Lambda_{n-1} - \Omega_{n-1}\right)}{\theta - M\hat{G}\beta_n\lambda_n}\,. \blacksquare
\end{aligned}$$

Derivation of eq. (3.32). We have:

$$\begin{aligned}
var\left(t_n|F_{M+1,n-1}\right) &= \zeta_n^2 var\left(y_n|F_{M+1,n-1}\right) \\
&= \left(\frac{\zeta_n}{\theta}\right)^2\left[\left(\hat{G}M\beta_n\right)^2\sigma_{f,n-1}^2 + \theta^2\sigma_u^2\right] \\
&= \frac{\zeta_n^2}{\theta}\frac{\hat{G}M\beta_n}{\lambda_n}\sigma_{f,n-1}^2 \\
&= \frac{\hat{G}M\beta_n\lambda_n}{\theta^3}\sigma_{f,n-1}^2 \\
&= \Lambda_{n-1} - \Lambda_n\,.
\end{aligned}$$

where the second line follows by eq. (3.43) and the equality $\sigma_{f,n-1}^2 = var\left(\theta\bar{s} - p_{n-1}|F_{M+1,n-1}\right)$; the third and fourth lines follow by eq. (3.46) and eq. (3.20); finally, the last line holds by eq. (3.45). Eq. (3.32) then follows by eq. (3.16c). $\blacksquare$

Notes

[1] As an example, Admati and Pfleiderer (1986) introduce a compensation scheme based on signal precision, whereas Brennan and Chordia (1993) consider a scheme based on the (information) buyer's trading activity.

[2] For example, it is well-known that uncertainty about future demand does not lead to profitable information sharing under Cournot competition [see Gal-Or (1985)]. In a related paper [see Colla and Mele (2004)], we have shown that (1) if firms face convex production costs, the previous conclusion does not hold even with positive correlations between private signals, and (2) even in the presence of linear costs, information sharing constitutes an equilibrium when signals are negatively correlated.

[3] We shall make an abuse in notation and write $G \in A$ for $G \in A \bigcap \mathbb{N}$, where A is some set and $\mathbb{N}$ denotes the set of integers. A similar abuse in notation will occur for other objects related to G –such as some traders located on some specific place in the circle.

[4] Naturally, $\bar{\rho}\left(k, \frac{M-1}{2}\right) \equiv \bar{\Omega}_0\left(k, \frac{M-1}{2}\right) / \bar{\Lambda}_0\left(\frac{M-1}{2}\right) = 1$ for all k. This fact does not imply that our information structure is isomorphic to another one in which $\rho \equiv \Omega_0/\Lambda_0 = 1$ because complete information sharing [occurring at $G = (M-1)/2$] also obviously tilts the average signals variance.

[5] We have actually covered many more cases than those we selectively present here. For space reasons, these additional results are only available at the authors' websites.

[6] Information sharing is a less robust phenomenon in the static case. In the appendix, we show that in the static case, information sharing only arises in correspondence of a severely restricted range of negative values of ρ (see proposition B1 in appendix B).

[7] The effects on $\bar{\rho}_n(k)$ depicted in figures 3.10–3.12 do not seem to be widely depending on N.

[8] In fact, Back, Cao and Willard (2000) show theoretically that in its diffusion limit (obtained with $N = \infty$), the basic Foster and Viswanathan (1996) model generates an infinite adverse selection problem at the very end of the trading period. Numerical results (obtained with high values of N) suggest that such a phenomenon disappears in our information-sharing model. It would be interesting to confirm our numerical findings with the help of a fully articulated continuous-time model.

[9] The combination $(N = 10, \rho = 0.1)$ does not correspond to cases in which information sharing is optimal. We report this case because it simplifies our description of the results in this section. Our conclusions in this section are of course independent of the specific pictures we are reporting.

[10] Let $\rho_n = \Omega_n/\Lambda_n$ be the correlation coefficient between individual signals. Since $|\rho_n| \leq 1$, then $\Lambda_n - \Omega_n \geq 0$ and the coefficient for λ_n^4 is non-negative. Moreover $\Lambda_n + (M-1)\,\Omega_n$ is positive due to (3.5), and a fortiori $2\Lambda_n + (M-1)\,\Omega_n > 0$ since $\Lambda_n > 0$.

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4

A Portfolio-Based Evaluation of Affine Term Structure Models

4.1 Introduction

The term structure of interest rates has always been a fashionable topic among financial economists, and the related literature is extremely wide even by historical standards. Duffie and Kan (1996) have generalized previous works and shown that the completely affine class (CA hereafter) nests celebrated models such as Vasicek (1977) and Cox, Ingersoll and Ross (1985) as specific cases. Within the CA class the product between the volatility matrix and the price of risk vector is linear in the state variables. Furthermore the variance of the log state price deflator is linear in the state variables as well. Duffee (2002) has recently proposed an extension to the essentially affine (EA hereafter) class. The key feature of EA models is that the variance of the log state price deflator is not affine in the state variables. This opens the way to model time variation in the price of risk not associated with time variation in the volatility matrix. Comparing the three specifications Duffee (2002) describes as the best ones in his work, it emerges that the EA class performs better than the CA class in forecasting bond yields. One important advantage of the EA structure is that it can produce expected excess returns on 2-year and 10-year bonds which widely fluctuate between positive and negative values. On the other hand, the proposed CA specification produces strictly positive expected excess return to 2-year and 10-year bonds all over the sample period. Therefore EA models are able to reconcile the small unconditional sample mean of bond excess returns with their high standard deviation. For instance, one EA parametrization predicts expected excess returns to the 10-year bond lower than -20% for some months during the 1970s and

1980s. We notice that while predictions arising from EA models seem to be compatible with the mentioned empirical evidence, they might be difficult to justify from a general equilibrium perspective. It is a matter of great concern that at some stage investors might choose to hold long term bonds with such negative expected excess returns.

Dai and Singleton (2003) evaluate various multi-factor models with respect to their ability in fitting the main stylized facts in bond prices' dynamics. Among their main findings are: (1) three-factor models do a better job than two-factor models in matching the persistence in yields' volatility and (2) a three-factor CA Gaussian model can replicate the negative correlation between the yields changes and the slope of the term structure, while multi-factor CIR (1985) models[1] are unable to do so. Taken together, these two findings imply that coming up with a model able to simultaneously fit the dynamics of both the mean and the volatility of yields can be a hard job.

We propose an evaluation of CA and EA term structure models as forecasting tools within a multi-period optimal portfolio problem. Our paper is motivated by three concerns: (1) a statistical evaluation of forecasting models can produce different results from those arising from a financial metric; (2) for a risk averse investor is important the ability to predict not only the mean of excess returns but also their risk, and (3) it is useful to understand what portfolio policy is prescribed by a term structure model in order to evaluate its practical applicability.

The first point is simple. Many estimation techniques pin down parameters minimizing the sum of squared errors, which results in high weights on extreme observations. This is optimal only for specific loss functions used by the researcher, and produces well-known results such that the model's conditional mean coincides with the optimal forecast. However it is easy to think of loss functions which would not select parameters by minimizing the sum of squared errors. Aït-Sahalia and Brandt (2001) suggest to select the model specification based on the first order conditions for utility maximization, thus by-passing the estimation phase of the statistical forecasting model. In such a way variables and parameters are explicitly relevant for portfolio choices. Another example is Christoffersen and Diebold (1997) that solves the conditional forecasting problem for some specific loss functions of the decision-maker. In several interesting cases the authors show how the conditional mean is not the optimal forecast, which should instead stem from (a mixture of) the first two conditional moments. A corollary to this general argument is that a decision-maker may favor a forecasting model which does not provide good performance from a particular statistical standpoint like sum of squared errors minimization. There are several examples –for both the stock and bond market– confirming that even partial ability to explain the total variability of a given variable may not prevent using a model for active asset allocation.[2] Breen, Glosten and Jagannathan (1989) consider a fairly simple forecasting model for stock returns based solely on the interest rate level. The model is characterized by a coefficient of determination between 2% and 8% –depending on the sample period– and may be employed in a stock/bond static active asset allocation problem to produce a 2% return. Barberis (2000) documents that a simple predictive equation for stock returns based on the dividend yield -with a R^2 as low as 1%-, may deeply affect static and dynamic portfolio choices for long term investors. Kandel and Stambaugh (1996) study a similar problem focusing on short term investors within a Bayesian setup. Campbell and Viceira (2002) also concentrate on a dynamic asset allocation problem based on a conditional forecasting model and resort to an approximate solution, while Brennan, Schwartz and Lagnado (1997) finds the optimal dynamic portfolio solving the representative investor's first-order conditions. Handa and Tiwari (2000) use the evidence on US stock return predictability as an input for one-period Bayesian portfolio optimization and find the gains from active asset allocation being unstable over time once transaction costs are taken into account. The predictability problem is studied with Bayesian methodologies by Avramov (2002) as well. A recent example in the bond literature is Frauendorfer and Schürle (2001) which finds conflicting results in an application to the Swiss bond market. The authors show that the one-factor Vasicek (1977) model is preferred to both the one-factor CIR (1985) model and two-factor models from a statistical standpoint (log likelihood maximization) as well as based on its financial performance, even though two-factor models seem to better characterize yields' empirical distribution. Applications of dynamic programming models to bonds have been provided by Brennan, Schwartz and Lagnado (1997), Brennan and Xia (2000), Campbell and Viceira (2001) and Walder (2002). In setting and solving the optimization problem we rely on a scenario generation approach along the path set forth by Zenios and various coauthors, among which Zenios (1993, 1995), Zenios and Kang (1993), Beltratti, Laurent and Zenios (2004), and more recently Jobst and Zenios (2001a). However our model is explicitly forward-looking because the investor solves a two-period problem –as described next in this introduction–, whereas the cited papers consider static portfolio choices. We choose to work with the scenario generation approach for several reasons, among which are: (1) it accounts for the introduction of transaction costs and (2) it is flexible in allowing for a practically intuitive minimization of the expected shortfall, and does not require a tractable utility function of the isoelastic form –which is much harder to use in applications.

As to the second point above –the importance of predicting both mean excess returns and risk–, the conflicting results reported above on the opportunity to simultaneously match means and variances reinforce the idea that it may be interesting to evaluate term structure models from a portfolio choice perspective. The portfolio optimization setting provides a natural way to take into account the ability of a forecasting model to predict both expected returns and a risk indicator. Furthermore it naturally embeds a metric to evaluate the efficiency of the model, which can be tested for its contribution to the improvement in the objective function to be optimized.

Finally, for practical applications it is important to characterize the dynamic portfolio choices suggested by a given model. Investors prefer models which are relatively stable and well balanced in terms of different asset classes' holdings. Models requiring strong fluctuations in asset shares are undesirable for various reasons. From a psychological standpoint, investors may be unwilling to act very aggressively in the short run. Moreover in practice investors incur transactions costs for portfolio turnover, and there is evidence that in most cases high turnover generates costs which severely damage portfolio performance. An important part of our results will therefore be devoted to dynamically characterize portfolio choices arising from the theoretical models in the presence of transaction costs.

Our empirical application hinges on the following steps. First we select the empirical version of dynamic term structure models, comparing three alternative specifications involving both EA and CA multi-factor models. We then use the selected models to generate scenarios for the state vector. The scenarios are selected in such a way as to be compatible with various moments of the distribution function associated with a given model. We determine scenarios such that the means, variances and covariances at each node match the corresponding moments of the joint density function for the state variables. At each node we then use the model to evaluate the prices of discount bonds for any maturity. Given current discount bonds prices, we are therefore able to compute the rate of return on each available bond for any possible future scenario under consideration. After scenarios have been generated, we write and solve a two-period optimization problem. The basic time unit is the quarter, so that a two-period problem corresponds to a six-month horizon. The investor knows that after three months there will be a chance to modify the chosen asset allocation. We compute the wealth implied by repeated application[3] of our dynamic portfolio optimization problem to scenarios generated by the different term structure models, taking into account transaction costs. We finally test the resulting trading strategies to evaluate whether the results are statistically significant.

While we confirm Duffee's (2002) result about the superiority of EA over CA models, we show that the financial metric reverses the ranking of the models within the EA class. Duffee (2002) has shown that a homoskedastic model has a better forecasting performance than a model which allows for heteroskedasticity of bond returns. On the other hand we find that the latter model allows for a better use of a dynamic portfolio strategy. The interpretation is that allowing for time-varying higher moments may more than compensate for lower forecasting power of the distribution's mean. Performance tests do not show that the excess return produced by the model is significantly positive, even though the superiority of the heteroskedastic EA model is confirmed in terms of average performance. Moreover the shortfalls produced by the two EA models are in general statistically significant. As for the motivation to our exercise, on the basis of our results we claim that: (1) the statistical evaluation of CA and EA models produces different results from those arising from our financial metric, i.e. the performance of a dynamic bond allocation model which uses as inputs scenarios generated from the statistical models; (2) the ability to predict not only excess returns' means but also their risk is crucial for a risk averse investor and explains the difference between the statistical and the financial evaluations.

After this Introduction the plan of the chapter is as follows. Section 4.2 provides a general introduction to the affine term structure class of models. Section 4.3 discusses scenario generation via lattice methods. Section 4.4 introduces the portfolio model and the dynamic optimization, while section 4.5 presents the results. Section 4.6 concludes.

4.2 Term structure modelling within the affine class

4.2.1 The affine framework

In what follows we briefly review the building blocks in the dynamic term structure models (DTSMs hereafter) literature. The interested reader may refer to Dai and Singleton (2000, 2003), Duffee (2002), Duffie and Kan (1996), and Fisher and Gilles (1996) for a thorough discussion on both modelling and estimation techniques.

Consider an economy whose state is described by n variables (also called risk factors) $\mathbf{X}_t = (X_{1,t}, ..., X_{n,t})^\top$ following the diffusion:

$$d\mathbf{X}_t = \boldsymbol{\mu}(\mathbf{X}_t)\,dt + \sigma(\mathbf{X}_t)\,d\mathbf{W}_t \,, \tag{4.1}$$

where $\boldsymbol{\mu}(\mathbf{X}_t)$ is $n \times 1$ vector, $\sigma(\mathbf{X}_t)$ is $n \times n$ matrix and the complete probability space $(\Omega, \mathcal{F}, P)$ with the augmented filtration $\{\mathcal{F}_t : t \geq 0\}$ is generated by n standard Brownian motions $\mathbf{W}_t = (W_{1,t}, ..., W_{n,t})^\top$. Using no-arbitrage arguments one can write the risk factor dynamics in (4.1) under the equivalent martingale measure Q as:

$$d\mathbf{X}_t = \boldsymbol{\mu}^Q(\mathbf{X}_t)\,dt + \sigma(\mathbf{X}_t)\,d\mathbf{W}_t^Q \,,$$

where $\mathbf{W}_t^Q = \left(W_{1,t}^Q, ..., W_{n,t}^Q\right)^\top$ is a vector of standard Brownian motions under the risk-neutral measure Q. The $n \times 1$ vector of market prices of risk $\boldsymbol{\Lambda}(\mathbf{X}_t)$ allows to move from the physical probability measure to the risk-neutral one, given that the drift term under Q is $\boldsymbol{\mu}^Q(\mathbf{X}_t) = \boldsymbol{\mu}(\mathbf{X}_t) - \sigma(\mathbf{X}_t)\,\boldsymbol{\Lambda}(\mathbf{X}_t)$. Therefore building a DTSM boils down to specify $(r_t, \boldsymbol{\mu}_t, \boldsymbol{\Lambda}_t, \sigma_t)$ as functions of the state vector $\mathbf{X}_t$, where r_t denotes the riskless rate.

A DTSM is (exponential-) affine[3] if and only if:

$$r(\mathbf{X}_t)\,, \boldsymbol{\mu}(\mathbf{X}_t)\,, \sigma(\mathbf{X}_t)\,\sigma(\mathbf{X}_t)^\top \text{ and } \sigma(\mathbf{X}_t)\,\boldsymbol{\Lambda}(\mathbf{X}_t) \text{ are affine in } \mathbf{X}_t \,. \tag{4.2}$$

Note that affinity in r_t is equivalent to write the riskless rate as

$$r(\mathbf{X}_t) = \delta_0 + \boldsymbol{\delta}^\top \mathbf{X}_t \,, \tag{4.3}$$

where δ_0 is scalar and $\boldsymbol{\delta}$ is $n \times 1$. The key feature of the (exponential-) affine class is that bond prices take the form[5]:

$$P(\mathbf{X}_t, \tau) = \exp\left(A(\tau) - \mathbf{B}(\tau)^\top \mathbf{X}_t\right) \tag{4.4}$$

where $A(\tau)$ is scalar, $\mathbf{B}(\tau) = (B_1(\tau), ..., B_n(\tau))^\top$, and τ is the bond's residual time to maturity. No-arbitrage condition on the price dynamics results in the coefficient $A(\tau)$ and the factor loadings $\mathbf{B}(\tau)$ obeying a PDE with boundary condition $P(\mathbf{X}_t, 0) = 1$:

$$-\dot{A}(\tau) + \dot{\mathbf{B}}(\tau)^\top \mathbf{X}_t - r(\mathbf{X}_t) - \boldsymbol{\mu}^Q(\mathbf{X}_t)^\intercal \mathbf{B}(\tau) + \frac{1}{2}\mathbf{B}(\tau)^\top \sigma(\mathbf{X}_t)\sigma(\mathbf{X}_t)^\top \mathbf{B}(\tau) = 0 \tag{4.5a}$$

$$A(0) = 0 \quad \text{and} \quad \mathbf{B}(0) = \mathbf{0} \tag{4.5b}$$

Duffie and Kan (1996) show that within the affine class the diffusion $\sigma(\mathbf{X}_t)$ has the following structure:

$$\sigma(\mathbf{X}_t) = \Sigma S(\mathbf{X}_t) \ , \tag{4.6}$$

where Σ is $n \times n$ constant matrix and S_t is $n \times n$ diagonal with typical element[5]:

$$[S(\mathbf{X}_t)]_{ii} = \sqrt{\alpha_i + \boldsymbol{\beta}_i^\top \mathbf{X}_t} \ , \tag{4.7}$$

where α_i is scalar and $\boldsymbol{\beta}_i$ is $n \times 1$ vector, i.e. $\boldsymbol{\beta}_i = (\beta_{i1}, ..., \beta_{in})^\top$. For notational purposes, it is convenient to stack the α_i's into the $n \times 1$ vector $\boldsymbol{\alpha}$ and the $\boldsymbol{\beta}_i$'s vectors in the $n \times n$ matrix $\beta = (\boldsymbol{\beta}_1, ..., \boldsymbol{\beta}_N)^\top$ such that

$$S(\mathbf{X}_t) S(\mathbf{X}_t)^\top = \operatorname{diag}(\boldsymbol{\alpha}) + \operatorname{diag}(\beta \mathbf{X}_t) \ ,$$

where diag($\boldsymbol{\alpha}$) (resp. diag($\beta\mathbf{X}_t$)) is a diagonal matrix obtained by inserting the vector $\boldsymbol{\alpha}$ (the vector $\beta\mathbf{X}_t$) into the diagonal.

We now turn to specify the function $\boldsymbol{\Lambda}(\mathbf{X}_t)$, since the representation for the market price of risk turns out to be the key difference between CA and EA models. In the CA class, the market price of risk takes the form:

$$\boldsymbol{\Lambda}(\mathbf{X}_t) = S(\mathbf{X}_t)\boldsymbol{\lambda}_1 \ , \tag{4.8}$$

where λ_1 is $n \times 1$ and $S(\mathbf{X}_t)$ is given in eq. (4.7). Therefore the instantaneous variance of the market price of risk is affine in $\mathbf{X}_t$, since $\mathbf{\Lambda}(\mathbf{X}_t)^\top \mathbf{\Lambda}(\mathbf{X}_t) = \sum_{i=1}^n \lambda_{1i}^2 \alpha_i + \sum_{i=1}^n \lambda_1^\top \text{diag}([\beta]_i)\, \lambda_1 X_{i,t}$.

On the other hand in the EA class the market price of risk is:

$$\mathbf{\Lambda}(\mathbf{X}_t) = S(\mathbf{X}_t)\,\lambda_1 + S^-(\mathbf{X}_t)\,\lambda_2 \mathbf{X}_t \ , \tag{4.9}$$

where λ_2 is $n \times n$ and $S^-(\mathbf{X}_t)$ is $n \times n$ diagonal with typical element[7]:

$$[S^-(\mathbf{X}_t)]_{ii} = \begin{cases} \left(\alpha_i + \beta_i^\top \mathbf{X}_t\right)^{-1/2} & \text{if } \inf_{\mathbf{X}_t}\left(\alpha_i + \beta_i^\top \mathbf{X}_t\right) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Provided that EA models satisfy the affinity condition (4.2) bond prices are affine [see eq. (4.4)] and the PDE (4.5a) still holds together with boundary condition (4.5b). However the instantaneous variance of the market price of risk $\mathbf{\Lambda}(\mathbf{X}_t)^\top \mathbf{\Lambda}(\mathbf{X}_t)$ is *not* affine in $\mathbf{X}_t$ for $\lambda_2 \neq 0$. This feature -arising from the richer specification for the market price of risk in eq. (4.9)- gives EA models higher flexibility in capturing the time variation in the price of risk.

4.2.2 Bond pricing within the CA and EA class

Starting from an affine form for the riskless rate as in eq. (4.3), we specialize the affine condition on the physical drift in (4.1) as

$$\boldsymbol{\mu}(\mathbf{X}_t) = (K\boldsymbol{\theta} - K\mathbf{X}_t) \ , \tag{4.10}$$

where K is $n \times n$ and $\boldsymbol{\theta}$ is $n \times 1$. Without loss of generality we normalize throughout the matrix Σ to be the identity matrix, such that equation (4.6) yields $\sigma(\mathbf{X}_t) = S(\mathbf{X}_t)$. It follows that the instantaneous variance is $\sigma(\mathbf{X}_t)\,\sigma(\mathbf{X}_t)^\top = S(\mathbf{X}_t)\,S(\mathbf{X}_t)^\top = \text{diag}(\boldsymbol{\alpha}) + \sum_{i=1}^n \text{diag}([\beta]_i)\, X_{i,t}$ and the PDE (4.5a) becomes:

$$\begin{aligned} &-\dot{A}(\tau) + \dot{\mathbf{B}}(\tau)^\top \mathbf{X}_t - r(\mathbf{X}_t) - (K\boldsymbol{\theta} - K\mathbf{X}_t - S(\mathbf{X}_t)\,\Lambda(\mathbf{X}_t))^\top \mathbf{B}(\tau) + \\ &\qquad + \frac{1}{2}\sum_{i=1}^n B_i^2(\tau)\,\alpha_i + \frac{1}{2}\left(\sum_{i=1}^n B_i^2(\tau)\,\beta_i^\top\right)\mathbf{X}_t = 0 \end{aligned} \tag{4.11}$$

Recall that eq. (4.11) holds for both CA and EA models. Within the CA class the market price of risk is given in (4.8), such that $S(\mathbf{X}_t)\,\mathbf{\Lambda}(\mathbf{X}_t) = \mathbf{\Psi} + \Phi\mathbf{X}_t$, where $\mathbf{\Psi}$ is

$n \times 1$ with $\Psi_i = \alpha_i \lambda_i$, and Φ is $n \times n$ with typical row $\Phi_i = \lambda_i \boldsymbol{\beta}_i^\top$. The CA specification (4.8) thus results in breaking down the PDE (4.11) into $n+1$ ODEs:

$$\dot{A}(\tau) = -\delta_0 - \mathbf{B}(\tau)^\top (K\boldsymbol{\theta} - \boldsymbol{\Psi}) + \frac{1}{2}\sum_{i=1}^{n} B_i^2(\tau)\,\alpha_i \tag{4.12a}$$

$$\dot{\mathbf{B}}(\tau) = \boldsymbol{\delta} - (K + \Phi)^\top \mathbf{B}(\tau) - \frac{1}{2}\left(\sum_{i=1}^{n} B_i^2(\tau)\,\boldsymbol{\beta}_i\right) \tag{4.12b}$$

On the other hand in the EA class the market price of risk is defined in eq. (4.9). In this case:

$$S(\mathbf{X}_t)\,\boldsymbol{\Lambda}(\mathbf{X}_t) = S^2(\mathbf{X}_t)\,\boldsymbol{\lambda}_1 + I^- \lambda_2 \mathbf{X}_t\ , \tag{4.13}$$

where $I^- \equiv S(\mathbf{X}_t)\,S^-(\mathbf{X}_t)$ is $n \times n$ diagonal with typical element:

$$[I^-]_{ii} = \begin{cases} 1 & \text{if } \inf_{\mathbf{X}_t}\left(\alpha_i + \boldsymbol{\beta}_i^\top \mathbf{X}_t\right) > 0 \\ 0 & \text{otherwise} \end{cases}$$

Plugging eq. (4.13) back into the PDE (4.11) gives the following ODEs:

$$\dot{A}(\tau) = -\delta_0 - B(\tau)^\top (K\boldsymbol{\theta} - \boldsymbol{\Psi}) + \frac{1}{2}\sum_{i=1}^{n} B_i^2(\tau)\,\alpha_i \tag{4.14a}$$

$$\dot{\mathbf{B}}(\tau) = \boldsymbol{\delta} - \left(K + \Phi + I^-\lambda_2\right)^\top \mathbf{B}(\tau) - \frac{1}{2}\left(\sum_{i=1}^{n} B_i^2(\tau)\,\boldsymbol{\beta}_i\right) \tag{4.14b}$$

Note that the EA class nests the CA class since for $\lambda_2 = 0$ the specification for the market price of risk (4.9) reduces to eq. (4.8). It follows that the ODEs (4.14a-4.14b) for EA models coincide with their CA counterpart (4.12a-4.12b).

4.2.3 Forecasting future yields: the first two conditional moments

In the previous section it has been clarified that bond prices at each time t are a function of the current state vector. On the other hand the portfolio model requires forecasts of (scenarios of) future prices. In order to predict future bond prices (or equivalently yields), one needs to forecast the future state vector $\{\mathbf{X}_T\}_{T>t}$ conditional on the information available at time t. Closed form formulas for the first two conditional moments are given in Duffee (2002). The idea is to find the first two moments of a linear transformation of the state vector $\mathbf{X}_t$. More specifically, assume that the matrix K in eq. (4.10) can be diagonalized:

$$K = NDN^{-1}\ ,$$

where D is $n \times n$ diagonal with the $i-$th eigenvalue as the typical element and N is the $n \times n$ matrix with the eigenvector associated with the $i-$th eigenvalue in its $i-$th column. Further consider the transformation:

$$\mathbf{X}_t^* = N^{-1}\mathbf{X}_t \; . \tag{4.15}$$

Then it can be shown [see Duffee (2002), p. 439-442] that for $T > t$:

$$\begin{aligned} E\left(\mathbf{X}_T^*|\mathbf{X}_t^*\right) &= \boldsymbol{\theta}^* + \exp\left(-D\left(T-t\right)\right)\left(\mathbf{X}_t^* - \boldsymbol{\theta}^*\right) && (4.16a) \\ var\left(\mathbf{X}_T^*|\mathbf{X}_t^*\right) &= b_0 + \textstyle\sum_{i=1}^n b_i X_{i,t}^* && (4.16b) \end{aligned}$$

where $\boldsymbol{\theta}^* = N^{-1}\boldsymbol{\theta}$, and the (j,k)-th element in the matrices b_0 and b_i, $i = 1, ..., n$ is defined as follows:

$$\begin{aligned} [b_0]_{jk} &= \frac{1}{[D]_{jj} + [D]_{kk}}\left(1 - e^{-(T-t)([D]_{jj}+[D]_{kk})}\right)\left([G_0]_{jk} + \textstyle\sum_{i=1}^n \theta_i^* [G_i]_{jk}\right) \\ &\quad - \textstyle\sum_{i=1}^n [b_i]_{jk}\, \theta_i^* \\ [b_i]_{jk} &= \frac{[G_i]_{jk}}{[D]_{jj} + [D]_{kk} - [D]_{ii}}\left(e^{-(T-t)[D]_{ii}} - e^{-(T-t)([D]_{jj}+[D]_{kk})}\right) \end{aligned}$$

and $G_0 = \Sigma^*\text{diag}(\boldsymbol{\alpha})\,\Sigma^{*\top}$, $G_i = \Sigma^*\text{diag}([\beta^*]_i)\,\Sigma^{*\top}$, $\Sigma^* \equiv N^{-1}\Sigma$ and $\beta^* \equiv \beta N$. The first two conditional moments for the state vectors are obtained from eqs. (4.16a) and (4.16b) reversing the transformation (4.15):

$$E\left(\mathbf{X}_T|\mathbf{X}_t\right) = NE\left(\mathbf{X}_T^*|\mathbf{X}_t^*\right) \tag{4.17a}$$

$$var\left(\mathbf{X}_T|\mathbf{X}_t\right) = Nvar\left(\mathbf{X}_T^*|\mathbf{X}_t^*\right)N^\top \tag{4.17b}$$

Once equipped with a DTSM parametrization for: (1) the riskfree rate [eq. (4.3)]; (2) the physical drift [eq. (4.10)]; (3) the volatility matrix [eqs. (4.6) and (4.7)] and (4) the market price of risk –either as in (4.8) or (4.9)–, one can use eqs. (4.17a) and (4.17b) together with the appropriate ODEs –either eqs. (4.12a) and (4.12b) or eqs. (4.14a) and (4.14b)– to compute the first two conditional moments of future bond prices.

4.3 Scenario generation via lattice method

For pricing purposes one needs to approximate the diffusion processes for the risk factors. Lattice methods have been extensively used in the financial literature within derivative asset pricing. In the aftermath of the binomial tree in Cox, Ross and Rubinstein (1979), several authors devised lattices as useful tools in pricing claims. The basic strategy consists of building a grid with sizes of the jumps and associated probabilities obtained equating the first two moments of the underlying diffusion to those of the approximating distribution. Nelson and Ramaswamy (1990) use binomial processes in approximating several univariate distributions. In a multivariate setting one has to ensure the convergence of the approximating distribution to the true distribution by matching the variables' covariances as well. Boyle, Evnine and Gibbs (1989) generalize the binomial lattice in Cox, Ross and Rubinstein (1979) allowing each state variable to be proxied by a binomial process. Their method can be easily extended to n variables, yielding processes with 2^n jumps. Kamrad and Ritchken (1991) build on the work in Boyle, Evnine and Gibbs (1989) allowing for a horizontal jump, i.e. the possibility that the state vector does not move from its current value. In a n-dimensional setting their model would result in $2^n + 1$ jumps.

Litterman and Scheinkman (1991) and more recently Chapman and Pearson (2001) have shown that three factors are able to explain the majority of Treasury bond movements. Like Ahn, Dittmar and Gallant (2002), Dai and Singleton (2000) and Duffee (2002) we therefore set $n = 3$ in specifying the DTSMs for our dynamic portfolio problem. Given that the portfolio model we implement (see section 4.4 for further details) takes expected bond prices for two future dates as inputs, we need a lattice for the state vector at time $t + 1$ and $t + 2$. Similarly to Boyle, Evnine and Gibbs (1989) we choose a binomial tree in the three dimensions for the state variables, thus resulting in 8 nodes for the state vector at time $t + 1$ and 64 nodes at time $t + 2$.

Before discussing the three dimensional case, we describe our approximation method with two state variables. The binomial lattice approximating each state variable $\{X_i\}_{i=1,2}$ is represented in figure 4.1. The starting point is the observation at time t of values for $X_{1,t}$ and $X_{2,t}$. At time $t + 1$ each state variable $X_{i,t+1}$ can either move up to $X^u_{i,t+1}$ with probability p_i or down to $X^d_{i,t+1}$ with probability $1 - p_i$ (see panel A in figure 4.1). In general both the values for the vector $\mathbf{X}_{t+1}$ and the transition probabilities depend on the current time t, possibly via the current state $\mathbf{X}_t$. The quantities

FIGURE 4.1. One-dimensional lattice.

$$\begin{array}{ccc} & & X^u_{i,t+1} \\ & p_i \nearrow & \\ X_{i,t} & & \\ & 1-p_i \searrow & \\ & & X^d_{i,t+1} \end{array}$$

Panel A

$$\begin{array}{ccc} & & E\left(X_{i,t+1}|\mathbf{X}_t\right)+\sqrt{var\left(X_{i,t+1}|\mathbf{X}_t\right)} \\ & 1/2 \nearrow & \\ X_{i,t} & & \\ & 1/2 \searrow & \\ & & E\left(X_{i,t+1}|\mathbf{X}_t\right)-\sqrt{var\left(X_{i,t+1}|\mathbf{X}_t\right)} \end{array}$$

Panel B

Each state variable X_i moves along a binomial tree over time. The general form of such a binomial tree is depicted in Panel A. Panel B reports our specification with equal jump probabilities as described in section 4.3.

$\left\{X^u_{i,t+1}, X^d_{i,t+1}, p_i\right\}_{i=1,2}$ are chosen in order to match the first two conditional moments for X_1 and X_2. This amounts to solve the following system:

$$p_1 X^u_{1,t+1} + (1-p_1)\, X^d_{1,t+1} = E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_1 \tag{4.18a}$$

$$p_2 X^u_{2,t+1} + (1-p_2)\, X^d_{2,t+1} = E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_2 \tag{4.18b}$$

$$p_1\left(X^u_{1,t+1} - E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_1\right)^2 + (1-p_1)\left(X^d_{1,t+1} - E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_1\right)^2 = var\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_{11} \tag{4.18c}$$

$$p_2\left(X^u_{2,t+1} - E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_2\right)^2 + (1-p_2)\left(X^d_{2,t+1} - E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_2\right)^2 = var\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_{22} \tag{4.18d}$$

where $E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_i$ and $var\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_{ii}$ are respectively the $i-$th element of the conditional expectation vector and the (i,i) element of the conditional variance-covariance matrix. In order to pin down the values for $\left\{X^u_{i,t+1}, X^d_{i,t+1}, p_i\right\}_{i=1,2}$ in the system (4.18a-4.18d) we impose equal transition probabilities, i.e. $p_1 = p_2 = 1/2$, thus making the number of unknowns equal to the number of equations.[8] Let ΔX^u_i be the deviation of the value $X^u_{i,t+1}$ from the conditional mean, i.e. $\Delta X^u_i \equiv X^u_{i,t+1} - E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_i$ (ΔX^d_i is defined similarly). Plugging $p_i = 1/2$ in the system (4.18a-4.18d) gives, for $i = 1,2$:

$$\Delta X^u_{i,t} = -\Delta X^d_{i,t} \tag{4.19a}$$

$$\left(\Delta X^u_{i,t}\right)^2 + \left(\Delta X^d_{i,t}\right)^2 = 2var\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_{ii} \tag{4.19b}$$

Equation (4.19a) imposes symmetry around the conditional mean. Substituting for $\Delta X_i = \Delta X^u_i$ into eq. (4.19b) gives the value for the deviation at time $t+1$ as $\Delta X_i = \sqrt{var\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_{ii}}$ such that the future values for the state variables become (see panel

TABLE 4.1. State vector values and probabilities (two-dimensional case).

value for $\mathbf{X}_{t+1}$	probability
$\left\{X^u_{1,t+1}, X^u_{2,t+1}\right\}$	π_1
$\left\{X^u_{1,t+1}, X^d_{2,t+1}\right\}$	π_2
$\left\{X^d_{1,t+1}, X^u_{2,t+1}\right\}$	π_3
$\left\{X^d_{1,t+1}, X^d_{2,t+1}\right\}$	π_4

Nodes for the state vector are reported in the first column. Each state variable $\{X_i\}_{i=1,2}$ evolves along a binomial tree. With two state variables the state vector takes four values at time $t+1$. The second column reports the associated probabilities.

B in figure 4.1):

$$X^u_{i,t+1} = E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_i + \sqrt{var\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_{ii}} \tag{4.20a}$$

$$X^d_{i,t+1} = E\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_i - \sqrt{var\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)_{ii}} \tag{4.20b}$$

We now turn to describe the tree for the state vector $\mathbf{X}_t$. When both state variables are approximated through a binomial lattice as in (4.20a) and (4.20b), the state vector (and thus the price of zero-coupon bonds) can be proxied by a 4-jump process. Let the tuple $\{X_{1,t}, X_{2,t}\}$ denote the values for the state variables at time t. Table 4.1 reports the values –and associated probabilities $\pi_1, ..., \pi_4$– that the joint distribution for the state vector can take at time $t+1$, while we give a graphical representation of our procedure in figure 4.2.

Given the state vector values at time $t+1$ as in eqs. (4.20a,4.20b), the probability vector $(\pi_1, ..., \pi_4)$ has to be chosen in order to replicate the marginal distributions for the state variables $\{X_1, X_2\}$ *and* the conditional covariance between the two processes. Matching the marginal distributions and imposing that the probabilities sum up to unity amounts to set the system: $\pi_1 + \pi_2 = 1/2$, $\pi_3 + \pi_4 = 1/2$, $\pi_1 + \pi_3 = 1/2$, $\pi_2 + \pi_4 = 1/2$ and $\sum_{i=1}^4 \pi_i = 1$, which simplifies to

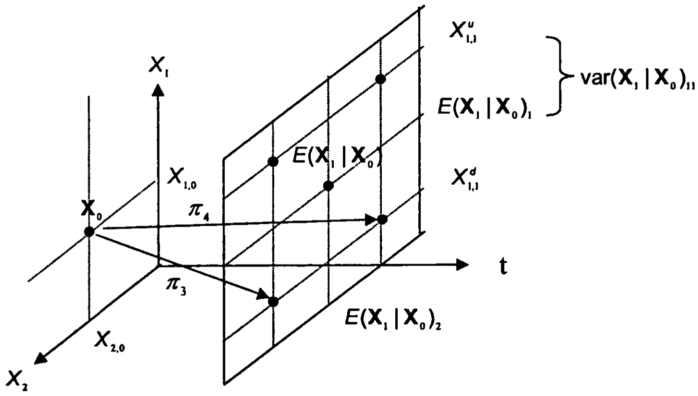
$$\pi_1 = \pi_1 \tag{4.21a}$$

$$\pi_2 = 1/2 - \pi_1 \tag{4.21b}$$

$$\pi_3 = 1/2 - \pi_1 \tag{4.21c}$$

$$\pi_4 = \pi_1 \tag{4.21d}$$

FIGURE 4.2. Two-dimensional lattice.



Each state variable $\{X_i\}_{i=1,2}$ moves along a binomial tree over time like in figure 4.1. The whole state vector's dynamics is given by a two-dimensional lattice (values and probabilities are in table 4.1). Figure 4.2 displays the main features of our lattice generation method in section 4.3.

Matching the conditional covariance requires further that

$$\pi_1 - \pi_2 - \pi_3 + \pi_4 = \rho_{12}\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right) \ , \tag{4.21e}$$

where $\rho_{12}\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)$ denotes the conditional correlation between the two variables, i.e. for $i \neq j$

$$\rho_{ij}\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right) \equiv \left[var\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)\right]_{ij} / \left(\Delta X_{i,t} \Delta X_{j,t}\right) \quad , \quad i \neq j \ . \tag{4.22}$$

Thus the comovement between the two state variables allows to pin down one solution for the probability vector $(\pi_1, ..., \pi_4)$ in the system (4.21a-4.21e):

$$\begin{aligned} \pi_1 &= \pi_4 = \frac{1}{4}\left(1 + \rho_{12}\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)\right) \\ \pi_2 &= \pi_3 = \frac{1}{4}\left(1 - \rho_{12}\left(\mathbf{X}_{t+1}|\mathbf{X}_t\right)\right) \end{aligned}$$

Notice that the probabilities $\{\pi_i\}_{i=1}^4$ are well defined, in that they all lie between zero and one.

With the two-dimensional case in mind, we now discuss our procedure for the three-factor setup (the multi-factor DTSMs are described in subsection 4.5.2). Each diffusion is proxied by a binomial lattice as in figure 4.1-panel B with equal transition probabilities and up/down jumps given by eqs. (4.20a,4.20b). Since we are dealing with three state variables, the whole state vector $\mathbf{X}_{t+1}$ follows an 8-jump process as summarized in table 4.2.

Proceeding as before, the probabilities $\{\pi_i\}_{i=1}^{8}$ are to be chosen in order to match the marginal distributions for the state variables $\{X_{i,t}\}_{i=1}^{3}$ and the conditional covariances between the processes $X_{i,t}$ and $X_{j,t}$, $i \neq j$. Equivalently to the system (4.21a-4.21d) one gets:

$$\sum_{i=1}^{8} \pi_i = 1 \tag{4.24a}$$

$$\pi_1 + \pi_2 + \pi_3 + \pi_4 = 1/2 \tag{4.24b}$$

$$\pi_1 + \pi_2 + \pi_5 + \pi_6 = 1/2 \tag{4.24c}$$

$$\pi_1 + \pi_3 + \pi_5 + \pi_7 = 1/2 \tag{4.24d}$$

Let $\rho_{13}(\cdot)$ and $\rho_{23}(\cdot)$ be defined as in eq. (4.22). Matching the conditional covariances results in the three additional constraints:

$$(\pi_1 + \pi_2 - \pi_3 - \pi_4 - \pi_5 - \pi_6 + \pi_7 + \pi_8) = \rho_{12}(\mathbf{X}_{t+1}|\mathbf{X}_t) \tag{4.24e}$$

$$(\pi_1 - \pi_2 + \pi_3 - \pi_4 - \pi_5 + \pi_6 - \pi_7 + \pi_8) = \rho_{13}(\mathbf{X}_{t+1}|\mathbf{X}_t) \tag{4.24f}$$

$$(\pi_1 - \pi_2 - \pi_3 + \pi_4 + \pi_5 - \pi_6 - \pi_7 + \pi_8) = \rho_{23}(\mathbf{X}_{t+1}|\mathbf{X}_t) \tag{4.24g}$$

The set of equations (4.24a-4.24g) describes a linear system in the probability vector $\boldsymbol{\pi} \equiv (\pi_1, ..., \pi_8)$ that admits infinitely many solutions. This indeterminacy arises in the literature when dealing with approximations for $n > 2$ variables [see Boyle, Evnine and Gibbs (1989) and He (1990)]. Following Boyle, Evnine and Gibbs (1989) we impose one additional constraint –suggested by the analysis for the two dimensional case– in order to get a unique solution. Recall that with two state variables the prob-

TABLE 4.2. State vector values and probabilities (three-dimensional case).

value for $\mathbf{X}_{t+1}$	probability
$\{X^u_{1,t+1}, X^u_{2,t+1}, X^u_{3,t+1}\}$	π_1
$\{X^u_{1,t+1}, X^u_{2,t+1}, X^d_{3,t+1}\}$	π_2
$\{X^u_{1,t+1}, X^d_{2,t+1}, X^u_{3,t+1}\}$	π_3
$\{X^u_{1,t+1}, X^d_{2,t+1}, X^d_{3,t+1}\}$	π_4
$\{X^d_{1,t+1}, X^u_{2,t+1}, X^u_{3,t+1}\}$	π_5
$\{X^d_{1,t+1}, X^u_{2,t+1}, X^d_{3,t+1}\}$	π_6
$\{X^d_{1,t+1}, X^d_{2,t+1}, X^u_{3,t+1}\}$	π_7
$\{X^d_{1,t+1}, X^d_{2,t+1}, X^d_{3,t+1}\}$	π_8

Nodes for the state vector are reported in the first column. Each state variable $\{X_i\}_{i=1,2,3}$ evolves along a binomial tree. With three state variables the state vector takes eight values at time $t + 1$. The second column reports the associated probabilities.

abilities of the extreme values for both the state variables were the same, i.e. $\pi_1 = \pi_4$. Thus we constrain the "all-up-node" $\{X^u_{1,t+1}, X^u_{2,t+1}, X^u_{3,t+1}\}$ and the "all-down-node" $\{X^d_{1,t+1}, X^d_{2,t+1}, X^d_{3,t+1}\}$ to be equally likely:

$$\pi_1 = \pi_8 \ . \tag{4.24h}$$

Then the system (4.24a-4.24h) admits the following solution:

$$\begin{aligned}
\pi_1 &= \pi_8 = \frac{1}{8}(1+\rho_{12}+\rho_{13}+\rho_{23}) \\
\pi_2 &= \pi_7 = \frac{1}{8}(1+\rho_{12}-\rho_{13}-\rho_{23}) \\
\pi_3 &= \pi_6 = \frac{1}{8}(1-\rho_{12}+\rho_{13}-\rho_{23}) \\
\pi_4 &= \pi_5 = \frac{1}{8}(1-\rho_{12}-\rho_{13}+\rho_{23})
\end{aligned}$$

Since in our empirical study we model the state vector's dynamics from time t up to $t+2$, we repeat the procedure outlined above in each of the 8 nodes resulting at time $t+1$. Notice that even though the solution for the probability vector is now unique, nothing guarantees that the probabilities $\{\pi_i\}_{i=1}^{8}$ lie between zero and one. A sufficient –albeit not necessary– condition for this to happen is that the correlations among all state variables are all below 1/3 in absolute value. However the approximating procedure we implement via the system (4.24a)–(4.24h) yields defined probabilities for all the DTSMs we consider in our empirical application.

4.4 Portfolio model

We describe a multi-period portfolio model which takes as inputs the state vector forecasts produced by the DTSMs introduced in section 4.2 (see subsection 4.5.2 for further details) via the lattice procedure outlined in section 4.3. Following the structure proposed by Beltratti, Consiglio and Zenios (1998), investment decisions are measured in terms of dollars of face value –rather than percentages invested in the various assets. The model describes the choices of an investor with a T-period horizon and facing N risky assets and cash at the initial time $t = 0$.

We use the following notation:

S_t set of scenarios (nodes) anticipated at time t; s_t is a generic element of the set S_t

L_t set of paths obtained combining scenarios from the sets $S_0, S_1, ..., S_{t-1}$ ($L_t \subseteq S_0 \times S_1 \times ... \times S_{t-1}$); l_t is the information structure at time t ($l_t \in L_t$) which describes the path of nodes from time 0 until time t ($t \times 1$ vector). At $t = 0$ no path is defined because all the information is available for that period

π_{l_t} probability of path l_t

$\mathbf{P}_t(l_t)$ prices of assets at t, a function of the path between 0 and t ($N \times 1$ vector)

$\rho_t(l_t)$ one-period risk-free rate at t

γ transaction costs on sales

δ transaction costs on purchases

c_0 initial liquidity

$\mathbf{B}_0$ initial holdings ($N \times 1$ vector)

$\vec{1} = (1, ..., 1)$ the unit vector of dimension ($1 \times N$)

At each time period the choice variable is denoted by the $N \times 1$ vector $\mathbf{Z}_t(l_t)$, giving the quantity to be kept in the portfolio for each asset. Knowledge of $\mathbf{Z}_t(l_t)$ together with the initial holdings $\mathbf{B}_0$ gives the quantities that should be bought [$\mathbf{Q}_t(l_t)$, $N \times 1$ vector] or sold [$\mathbf{Y}_t(l_t)$, $N \times 1$ vector], and in turn determines the amount of cash [$v_t(l_t)$]. No short sales are allowed. Notice that at $t = 0$ all the prices are known, and there is only one node so that the control variable does not depend on the path.

At time $t = 0$ the value resulting from the original cash and from the sale of the assets in the portfolio must be equal to the amount invested in liquidity and that invested for increasing other assets (cashflow accounting), so that:

$$c_0 + (\mathbf{P}_0^\top - \gamma\vec{1})\mathbf{Y}_0 = (\mathbf{P}_0^\top + \delta\vec{1})\mathbf{Q}_0 + v_0 \ . \tag{4.25}$$

Moreover each asset must satisfy an inventory balance constraint

$$B_0^i + Q_0^i = Y_0^i + Z_0^i \quad , \quad \forall i \in N \ . \tag{4.26}$$

Decisions made at $t = 1, ..., T$ depend on the information structure l_t and on previous investment decisions. Similarly to what happens at the initial period $t = 0$ [see eq.

(4.25)], the increase in asset holdings must be equal to the income generated by the assets and the value generated by sales. Therefore for each path l_t one has:

$$\rho_{t-1}(l_{t-1})v_{t-1}(l_{t-1})+\left[\mathbf{P}_t^\top(l_t)-\gamma\vec{1}\right]\mathbf{Y}_t(l_t)=\left[\mathbf{P}_t^\top(l_t)+\delta\vec{1}\right]\mathbf{Q}_t(l_t)+v_t(l_t) \quad , \quad \forall l_t \in L_t \ . \tag{4.27}$$

Moreover for each path l_t the amount of each asset sold or kept in the portfolio must be equal to the amount bought or held from the previous period [see eq. (4.26)]:

$$Z_{t-1}^i(l_{t-1})+Q_t^i(l_t)=Y_t^i(l_t)+Z_t^i(l_t) \quad , \quad \forall i \in N \quad \text{and} \quad \forall l_t \in L_t \ . \tag{4.28}$$

We now describe the objective function. Let $V_{P_t}(l_t)$ be the value of the portfolio at time $t = 1, ..., T$ computed for each path l_t. $V_{P_t}(l_t)$ depends on the portfolio composition, the risky assets' value, and the amount of liquidity at time t as follows:

$$V_{P_t}(l_t)=v_t(l_t)+\mathbf{P}_t^\top(l_t)\mathbf{Z}_t(l_t) \quad , \quad \forall l_t \in L_t \ .$$

Denoting with $R_P(l_T)$ the portfolio's rate of return in the final period T computed for each path l_t, one has:

$$R_P(l_T)=\frac{V_{P_T}(l_T)-V_{P_0}}{V_{P_0}} \quad , \quad \forall l_t \in L_t \ ,$$

where $V_{P_0} \equiv c_0 + \mathbf{P}_0^\top \mathbf{B}_0$ denotes the initial portfolio value. Let us define for each $t = 1, ..., T$ and for each path l_t the one period shortfall associated with the portfolio held at time t:

$$SF_t(l_t)=\begin{cases} e^{\tau_t}V_{P_{t-1}}(l_{t-1})-V_{P_t}(l_t) & \text{if} \quad e^{\tau_t}V_{P_{t-1}}(l_{t-1})>V_{P_t}(l_t) \\ 0 & \text{otherwise} \end{cases} \quad , \quad \forall l_t \in L_t \tag{4.29}$$

where τ_t is the minimum required return for the period $(t-1, t]$. Consistently with eq. (4.29), we represent the opportunity cost to an investor as the case in which the value of the portfolio in t is lower than the value of the portfolio in the preceding period multiplied by the factor e^{τ_t}. The required return τ_t could be thought of as the increase in liabilities' value or the competitors' expected rate of return. Alternative formulations can be easily implemented. For example Chevalier and Ellison (1997) have shown that the penalty (in terms of assets under management) fund managers bear when underperforming their competitors is not particularly sensitive to the underperformance's magnitude, while the reward increases strongly with the overperformance. We could easily model this stylized fact by defining a fixed shortfall in case of underperformance and a negative and increasing shortfall in case of overperformance. At time T we define for each path l_T the cumulative shortfall:

$$SF_{cum}(l_T) = \begin{cases} e^{\tau_{cum}} V_{P_0} - V_{P_T}(l_T) & \text{if} \quad e^{\tau_{cum}} V_{P_0} > V_{P_T}(l_T) \\ \\ 0 & \text{otherwise} \end{cases} \quad , \quad \forall l_T \in L_T \tag{4.30}$$

where τ_{cum} is the minimum return over the whole time horizon. Given cost measures for shortfalls each period, $\{k_t\}_{t=1,..,T}$, and for the cumulative shortfall, k_{cum}, the investor wants to minimize a cost function based on the expected shortfall:

$$TSF_T = k_1 \sum_{l_1 \in L_1} \pi_{l_1} SF_1(l_1) + ... + k_T \sum_{l_T \in L_T} \pi_{l_T} SF_T(l_T) + k_{cum} \sum_{l_T \in L_T} \pi_{l_T} SF_{cum}(l_T) \tag{4.31}$$

under the constraints given by obtaining a return equal to the target $\bar{R}$ in each final node

$$R_P(l_T) = \bar{R} \quad , \quad \forall l_T \in L_T \ , \tag{4.32}$$

and by the cashflow accounting and inventory balance constraints (4.25-4.28). From the definition of the cost function it emerges that the higher the parameters τ_t and τ_{cum}, the higher the shortfall associated with a given portfolio return. Minimizing the objective function (4.31) given the constraint (4.32) amounts to risk minimization given a specific target return. This constrained minimization may be interpreted as an alternative to the standard efficient frontier methodology in the presence of asymmetric risk. It has been used in a one-period optimization problem by Zenios and Kang (1993), which shows that this approach is a linear programming problem reformulation of the mean-absolute deviation (MAD) model originally due to Konno and Yamazaki (1991).

This objective function has recently been used within a static optimization setup by Beltratti, Laurent and Zenios (2004) and by Jobst and Zenios (2001b).

There is a growing interest in the use of the shortfall in portfolio optimization, given recent attention to value-at-risk measures as well as portfolio optimization with non-normal assets. Of course non-normal returns do not require the use of the shortfall, and other risk measures like the standard deviation could be considered instead. However there are reasons to prefer the expected shortfall. Indeed, the expected shortfall corresponds to the conditional loss and to the conditional value-at-risk, see Embrechts, Klupperlberg and Mikosch (2003). Jobst and Zenios (2001b) use the expected shortfall framework in an effort to embed credit risk into asset management, and compare mean-absolute deviation portfolio optimization with expected shortfall portfolio optimization. Bogentoft, Romeijn and Uryasev (2001) utilize the expected shortfall for pension funds' asset/liability management, while Topaloglou, Vladimirou and Zenios (2002) employ it for an international portfolio choice problem claiming that "it can be used to exercise some control on the lower tail of the return distribution and thus, it is a suitable risk measure for skewed distributions" [Topaloglou, Vladimirou and Zenios (2002), p. 6]. Moreover, in practical applications, portfolio managers may be more sensitive to risk measures approximating the expected amount of the loss, rather than to a measure like the standard deviation which is independent of the portfolio value and the potential loss.

Our analysis is therefore closer to the computation of (a sequence of) specific points on a shortfall-expected return efficient frontier, rather than to the computation of (a sequence of) optimal portfolios for a specific utility function. Of course we could introduce a parameter representing the relative weights of the target return and the expected shortfall in the investor's utility, and choose one optimal portfolio for each time period. Instead, we compute a portfolio for each of the four target returns we consider in the empirical application (see subsection 4.5.3). Therefore our findings are not directly comparable to recent contributions like Wachter (2003) –showing that increasing risk aversion forces the investor to buy the bond with the same maturity as the consumption horizon–, because we are not solving an optimal consumption-investment problem. Also, our results may be more relevant to short-run investors rather than long-run investors, as the latter are concerned with building portfolios hedging the state variables and may deviate from the simple static efficient portfolios [see Sangvinatsos and Wachter (2003) for a recent application to bond portfolios].

It is also important to compare EA and CA models in terms of their ability to form efficient portfolios because in such a way a weak link with explicit utility maximization is established. It is well known that different affine DTSMs are compatible with different assumptions on the price of risk,[9] and therefore with different representative investor's utility functions supporting the prices in general equilibrium. We are therefore minimizing the risk of preferring one model to the other as the result of an unintended similarity between the implicit utility function embedded in each DTSM and the explicit utility function evaluated in our portfolio optimization model.

4.5 Data, implementation and results

4.5.1 Dataset description

Our analysis is based on data from McCulloch and Kwon (1993) and Bliss (1997) available from Duffee's homepage at http://faculty.haas.berkeley.edu/duffee/affine.htm. The dataset consists of month-end yields on zero-coupon US Treasury bonds with maturities of 3 and 6 months and 1, 2, 5, and 10 years for the sample period January 1952-December 1998. This dataset is the standard for many studies of the bond market, see for example Duffee (2002). We have obtained prices[10] for other maturities by interpolation with the method proposed by Nelson and Siegel (1987). Table 4.3 contains summary statistics about excess return data, i.e. continuosly compounded with respect to the overnight rate, used in this study. Between 1952 and 1998 the average monthly excess return increases with the bond maturity, except for the 10 years bond. A similar result also holds for the subperiod 1952-1992 used in several studies by other researchers, who also report a similar finding [see for example Campbell (2000)]. Returns' standard deviation is also increasing, but much more sharply. As a consequence the Sharpe ratio is decreasing across maturities. Ilmanen (1996) reports similar results over the period 1970-1994. Table 4.3 moreover shows that all excess returns are characterized by non-normal distributions, especially at the short end of the term structure. Non-normality justifies the use of the expected shortfall in our dynamic portfolio optimization as discussed in section 4.4. Finally, table 4.3 reports the average yield curve in the two samples, displaying the standard concavity and upward slope.

TABLE 4.3. Data set summary statistics.

maturity	3M	6M	1Y	2Y	5Y	10Y
mean excess return, 1952-1998	0.56	0.83	0.91	0.86	2.87	-0.57
mean excess return, 1952-1991	0.61	0.88	0.97	0.69	1.66	-1.60
standard deviation, 1952-1998	1.54	2.90	5.62	10.18	21.46	33.44
standard deviation, 1952-1991	1.65	3.11	6.00	10.72	21.86	34.06
Sharpe ratio, 1952-1998	0.36	0.29	0.16	0.08	0.13	-0.02
skewness, 1952-1998	1.68	1.89	0.74	-0.05	-0.49	0.30
kurtosis, 1952-1998	11.49	16.02	12.04	9.15	5.76	2.82
average yield, 1952-1998	5.46	5.69	5.90	6.14	6.46	6.67
average yield, 1952-1991	5.64	5.88	6.08	6.28	6.55	6.72

The data set consists of month-end yields on zero-coupon U.S. Treasury bonds (available from Duffee's homepage at http://faculty.haas.berkeley.edu/duffee/affine.htm) over the sample period January 1952-December 1998 for selected maturities (first row). For each maturity we use continuously compounded excess returns over the overnight rate and report their mean (second and third row), standard deviation (fourth and fifth row), Sharpe ratio (sixth row), skewness (seventh row) and kurtosis (eight row). Average yields are reported in the last two rows. All numbers are in percentage per year. Some of the statistics are computed over the subsample January 1952-December 1991 for comparison with previous studies.

It is of course possible that expected bond returns are larger than historical returns due to unexpected inflation episodes during the 1970s and the 1980s –a point made by Campbell (2000). However measurement error issues about bond returns go beyond the scope of the paper, which is concerned with comparing alternative DTSMs from a dynamic portfolio choice standpoint. It is instead crucial that the data set we use here corresponds to the one employed for econometric estimation of these models.

4.5.2 Three-factor DTSMs specification

We follow Dai and Singleton (2000) and Duffee (2002), and define a $CA_m(n)$ model as a completely affine model with n state variables of which m affect the instantaneous variance of the state vector (similarly, $EA_m(n)$ defines an essentially affine model with n state variables of which m affect the instantaneous variance of the state vector). Setting $n = 3$ the general specification of a three-factor model is given by:

$$r_t = \delta_0 + \delta_1 X_{1t} + \delta_2 X_{2,t} + \delta_3 X_{3,t}$$

$$d\begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} = \left[\begin{pmatrix} (K\theta)_1 \\ (K\theta)_2 \\ (K\theta)_3 \end{pmatrix} - \begin{pmatrix} k_{11} & k_{12} & k_{13} \\ k_{21} & k_{22} & k_{23} \\ k_{31} & k_{32} & k_{33} \end{pmatrix}\begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix}\right] dt + S\left(\mathbf{X}_t\right) d\mathbf{W}_t$$

$$[S(\mathbf{X}_t)]_{ii} = \sqrt{\alpha_i + \begin{pmatrix} \beta_{i1} & \beta_{i2} & \beta_{i3} \end{pmatrix} \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix}}$$

$$\mathbf{\Lambda}(\mathbf{X}_t) = S(\mathbf{X}_t) \begin{pmatrix} \lambda_{1(1)} \\ \lambda_{1(2)} \\ \lambda_{1(3)} \end{pmatrix} + S^{-}(\mathbf{X}_t) \begin{pmatrix} \lambda_{2(11)} & \lambda_{2(12)} & \lambda_{2(13)} \\ \lambda_{2(21)} & \lambda_{2(22)} & \lambda_{2(23)} \\ \lambda_{2(31)} & \lambda_{2(32)} & \lambda_{2(33)} \end{pmatrix} \begin{pmatrix} X_{1,t} \\ X_{2,t} \\ X_{3,t} \end{pmatrix} \tag{4.33}$$

When bringing the DTSM in (4.33) to the data, restrictions on the parameters are needed in order to get an admissible specification, i.e. to yield positive conditional variances $[S(\mathbf{X}_t)]_{ii}$ [see Dai and Singleton (2000)]. Further to these restrictions, Duffee (2002) reports preferred specifications resulting from a new estimation of the model (4.33) after setting to zero all the parameters whose absolute t-statistic do not exceed one. We consider the three preferred models estimated in Duffee (2002) and use his estimated parameters. We refer the reader to Duffee (2002) for a description of the estimation methodology and to his homepage for the parameters' estimates.

The first model considered by Duffee (2002) is $EA_0(3)$, which rules out heteroskedasticity in the state variables.[11] The second model is $EA_1(3)$ and leaves some margin to capture heteroskedasticity by means of one state variable. The third model, $CA_2(3)$, is the richest among those we consider in modelling the dynamics of volatility of interest rates. Table 4.4 reports the canonical representation as well as the restrictions imposed by Duffee (2002) for each DTSM.

4.5.3 Dynamic portfolio model

In our application we consider an investor with a two-period horizon. Technically, our model belongs to the class of two-stage stochastic programming problems, see Golub, Holmer and McKendall (1995). Let $s_t = (t, j)$ denote the scenario at time t and node j. Given that scenarios are generated by binomial trees for the three state variables as described in section 2, we have $S_0 \equiv s_0 = (0,0)$, $S_1 = \{(1,j)\}_{j=1,\dots,8}$, $S_2 = \{(2,j)\}_{j=1,\dots,64}$. We think of each period as one quarter. One quarter and two quarter rates are used to determine the riskless rate respectively for the first and the second period. The risky portfolio is composed of $N = 39$ assets (indexed by subscript i) corresponding to zero-coupon bonds with maturity larger than three months, i.e. $i = 1$

TABLE 4.4. Three-factor DTSMs parameterization.

	$EA_0\,(3)$	$EA_1\,(3)$	$CA_2\,(3)$
K	$\begin{bmatrix} k_{11} & 0 & 0 \\ \mathbf{0} & k_{22} & 0 \\ k_{31} & \mathbf{0} & k_{33} \end{bmatrix}$	$\begin{bmatrix} k_{11} & 0 & 0 \\ k_{21} & k_{22} & k_{23} \\ \mathbf{0} & \mathbf{0} & k_{33} \end{bmatrix}$	$\begin{bmatrix} k_{11} & k_{21} & 0 \\ k_{21} & k_{22} & 0 \\ k_{31} & k_{32} & k_{33} \end{bmatrix}$
$\theta^{\top}$	$\begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \theta_1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} \theta_1 & \theta_2 & 0 \end{bmatrix}$
$\alpha^{\top}$	$\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$
β	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ \beta_{21} & 0 & 0 \\ \beta_{31} & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \end{bmatrix}$
$\lambda_1^{\top}$	$\begin{bmatrix} \lambda_{1(1)} & \lambda_{1(2)} & \lambda_{1(3)} \end{bmatrix}$	$\begin{bmatrix} \lambda_{1(1)} & \lambda_{1(2)} & \mathbf{0} \end{bmatrix}$	$\begin{bmatrix} \lambda_{1(1)} & \mathbf{0} & \lambda_{1(3)} \end{bmatrix}$
λ_2	$\begin{bmatrix} \mathbf{0} & \lambda_{2(12)} & \mathbf{0} \\ \mathbf{0} & \lambda_{2(22)} & \mathbf{0} \\ \lambda_{2(31)} & \lambda_{2(32)} & \lambda_{2(33)} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ \lambda_{2(21)} & \mathbf{0} & \lambda_{2(23)} \\ \mathbf{0} & \mathbf{0} & \lambda_{2(33)} \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$
$[S(\mathbf{X}_t)]_{11}$	1	$\sqrt{X_{1,t}}$	$\sqrt{X_{1,t}}$
$[S(\mathbf{X}_t)]_{22}$	1	$\sqrt{1+\beta_{21}X_{1,t}}$	$\sqrt{X_{2,t}}$
$[S(\mathbf{X}_t)]_{33}$	1	$\sqrt{1+\beta_{31}X_{1,t}}$	$\sqrt{1+X_{1,t}}$

Relevant parameters for admissible DTSMs are reported. The general specification for three-factor DTSMs is given in (4.33). Restrictions on parameters are needed in order to guarantee admissibility as in Dai and Singleton (2000); boldface numbers denote parameters constrained in Duffee's (2002) preferred specifications.

denotes the six-month bond, $i = 2$ denotes the nine-month bond and so on until the ten-year bond ($i = 39$). The asset prices are obtained via the DTSMs in subsection 4.5.2 solving the relevant ODEs [either (4.12a-4.12b) or (4.14a-4.14b)], which are themselves a function of the state variables obtained through the discrete approximation in section 4.3.[12] Therefore we get $P_0^{\top} = \{P(0,0,i)\}_{i=1,\dots,39}$, $P_1^{\top}(s_0,(1,1)) = \{P(1,1,i)\}_{i=1,\dots,39}$, $P_1^{\top}(s_0,(1,2)) = \{P(1,2,i)\}_{i=1,\dots,39}$ and so on.

The other parameters characterizing our asset markets are chosen in the following way: $\gamma = \delta = 5$ basis points[13] for transaction costs and initial cash equal to 100. Within repeated application of the dynamic portfolio problem to each quarter during our sample period, we choose time-dependent parameters for the cost function (4.31) and the expected return constraint (4.32). We set the target return in eq. (4.32) equal to (various multiples of) the average yield on the current term structure (denoted by $\hat{R}_t$). This choice describes an investor with the following characteristics: (1) he expects yields to be constant, so that expected returns corresponds to the yields currently obtained on the bonds and (2) he wants to achieve at least -by using the quantitative model- a return related to the average yield conditions in the bond market for each quarter. Given that the term structure is on average concave and upward sloping [see Ilmanen (1996)] this is equivalent to setting a target return which is in line with the short end of the term structure.

In the empirical analysis we repeat the computations for four different target return levels: $0.5\hat{R}_t$, $0.6\hat{R}_t$, $0.7\hat{R}_t$ and $0.8\hat{R}_t$. This way we are able to compare portfolios for individuals with different risk-aversion parameters. The choice of these parameters is motivated by the fact that values lower than $0.5\hat{R}_t$ resulted in portfolio almost entirely invested in cash with very low asset turnover, thus making the asset allocation exercise uninteresting. When actually solving the portfolio problem, we do not have any guarantee that the optimizer is able to find a solution under all circumstances. In order to clarify this issue, assume that at a given time t the average yield, i.e. across the term structure at the same time, gives $\hat{R}_t = 5\%$ and we choose the multiple 0.5. Thus the target return is equal to 2.5%. If the optimizer is not able to find any portfolio achieving the 2.5% target, we decrease the target step by step until a solution is achieved. The occurrence of such target correction is quite rare[14] for multiples up to $\bar{R}_t/\hat{R}_t = 0.8$, which motivates our target returns' choice.

Finally, for each model we set $\tau_1 = \tau_2 = \ln\left(\hat{R}_t\right)$, $\tau_{cum} = 2\ln\left(\hat{R}_t\right)$ and $k_1 = k_2 = k_{cum} = 1$. The first choice reflects the hypothesis that the minimum required return corresponds to the average yield of the term structure, while the second is a normalization aimed at describing a case where all the shortfalls are equally important for the decision-maker.

4.5.4 Results

Duffee (2002) contains figures for the instantaneous expected excess returns on the 10-year bond predicted by the three models described in subsection 4.5.2. These pictures are useful to motivate our study from an empirical point of view [see Duffee (2002), figures 1-3]. They show very clearly how large is the difference in the expected returns on holding long-term bonds between the two DTSMs classes. The $CA_2\,(3)$ model generates positive excess returns over the full sample, while the models belonging to the EA class –particularly $EA_0\,(3)$– produce returns which fluctuate between positive and negative. Positive risk premia seem to be more compatible with potential general equilibrium explanations of asset returns –arising in equilibrium assuming risk aversion on the part of investor. However a model predicting both positive and negative expected excess returns could be very useful as an input to active asset management if the signals about future returns are the right ones.[15]

TABLE 4.5. State vector summary statistics

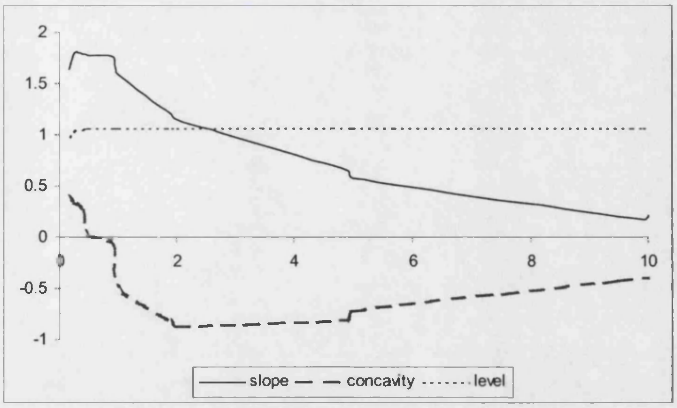
Panel A: $EA_0\,(3)$				
state variable	mean	std.dev	min	max
X_1	0.180	0.755	-1.945	3.106
X_2	0.004	0.351	-1.388	1.402
X_3	0.486	2.579	-3.592	7.687
Panel B: $EA_1\,(3)$				
state variable	mean	std.dev	min	max
X_1	22.694	12.550	2.881	56.364
X_2	16.609	17.093	-14.296	88.144
X_3	-0.663	1.074	-5.025	3.168
Panel C: $CA_2\,(3)$				
state variable	mean	std.dev	min	max
X_1	19.308	9.757	3.768	43.380
X_2	9.616	5.202	0.358	26.911
X_3	1.318	2.987	-4.056	14.825

The table reports mean (first column), standard deviation (second column), minimum and maximum (third and fourth columns) for several three-factor DTSMs. The models are described in section 4.5.2.

The three models also differ with respect to their implications for volatility: simply recall from subsection 4.5.2 that $EA_0\,(3)$ is homoskedastic and predicts constant conditional volatility, while both $EA_1\,(3)$ and $CA_2\,(3)$ allow for time varying conditional volatility. This evidence is therefore supportive towards our effort in this study: comparing models as inputs for asset allocation is particularly interesting when the candidates are characterized by similar statistical performance, but produce very different outputs in terms of financial choices. In this case the relevant outputs are the conditional predictive density functions, characterized for example in terms of the conditional mean and variance –which we have shown differ across models. Under such conditions a financial decision-maker would be unsure about which of the three DTSMs is best from a statistical point of view, and at the same time realize that following one or another would produce different portfolio structures over time.

Table 4.5 reports descriptive statistics for the state variables[16] for the three DTSMs in subsection 4.5.2. Note that $EA_0\,(3)$ does not impose any restriction on the state vector's sign since the β's are all set equal to zero. Thus it is not surprising that the state variables take negative values (see panel A). For the matrix $S\left(\mathbf{X}_t\right)$ of conditional second moments to be well defined in model $EA_1\,(3)$ [respectively $EA_2\,(3)$] one needs X_1 (respectively X_1 and X_2) to be positive everywhere: table 4.5 shows that this is actually the case (panels B and C respectively). The interpretation of the three factors follows standard analyses, see for example Bühler and Zimmermann (1996). For each DTSM we have run a linear regression of the yields (on relevant maturities) on the state vector; the resulting coefficients are displayed in figure 4.3 for $CA_2\,(3)$. The same

FIGURE 4.3. Interpretation of the state variables.



For the model CA_2 (3) the yields of the bonds with the 12 maturities included in the dataset have been regressed on the estimated state variables and the resulting coefficients have been plotted. Each line in the figure reports the 12 estimated coefficients of the yields of the 12 maturities (from one month to ten year) on each of the three state variables. The lines have been defined as "slope", "concavity" and "level" to interpret the results. "Slope" is a negatively inclined function interpreted as a shock affecting yields of the bonds with different maturities in a way negatively related to maturity. "Concavity" reports a function with a weak effect on both short and long yields and a strong effects on yields of intermediate maturities. "Level" represents a shock that affects in a similar way all yields.

qualitative behaviour arises from the other two EA models and we do not include the relevant figures for reasons of space. Figure 4.3 confirms the typical result that the three state variables may be interpreted as a level, slope and concavity shock to the term structure of interest rates.

Table 4.6 reports summary statistics for the total portfolio shortfall for the three models[17]. The numbers in the table refer to the minimized value of the ex-ante total shortfall. This means that the shortfall is computed based on the asset prices considered in the generated scenarios, rather than on the realized prices. Suppose for example that at time t the optimizer selects –based on scenarios stemming from a given DTSM– portfolio $\mathbf{Z}_t(l_t)$ for the path l_t and portfolios $\mathbf{Z}_t(l_{t+1})$ for each l_{t+1}. We can compute at each time $t+1$ and $t+2$ the portfolio's expected shortfall considering whether the expected portfolio value –given respectively by $V_{P_{t+1}}(l_{t+1})$ and $V_{P_{t+2}}(l_{t+2})$– lies below the minimum required return [see eqs. (4.29-4.30)]. The total shortfall, i.e. the minimized value for TSF_{t+2} in eq. (4.31), may differ from the one we would get for the actual ex-post shortfall because: (1) actual bond prices at time t, $t+1$ and $t+2$ may be different from the ones implied by the DTSM, and (2) one can only compute one quarterly ex-post shortfall since the asset allocation at time $t+1$ is not implemented –more precisely the ex-post shortfall associated with the asset allocation $\mathbf{Z}_t(l_t)$, but not the one arising from the portfolio choice $\mathbf{Z}_{t+1}(l_{t+1})$. Differences across DTSMs are

TABLE 4.6. Ex-ante shortfall analysis (total shortfall).

Panel A: EA_0 (3)				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	4.131	4.116	4.337	4.075
std dev	2.085	2.298	2.862	3.054
min	0.861	0.750	0.624	0.415
max	9.477	9.341	9.658	8.982
Panel B: EA_1 (3)				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	**3.107**	**3.363**	**3.665**	**3.762**
std dev	**1.257**	**1.660**	2.213	**3.021**
min	0.859	0.818	0.621	0.413
max	6.462	7.167	7.750	15.672
Panel C: CA_2 (3)				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	3.308	3.526	3.721	4.285
std dev	1.403	1.743	**2.172**	3.818
min	0.860	0.830	0.621	0.413
max	7.206	7.597	7.816	20.329

For each DTSM and risk-aversion parameter (given by the ratio $\bar{R}_t/\hat{R}_t$ in the first row) the mean (second row), standard deviation (third row), minimum (fourth row) and maximum (fifth row) of the total ex-ante shortfall is reported. The shortfall is computed on the basis of the theoretical bond prices produced by the various models.

entirely due to differences in predictive density functions because all the three models are used by the same decision maker characterized by model-invariant parameters k_1, k_2, k_{cum}, τ_1, τ_2 and τ_{cum}. Of course this comparison is preliminary to an analysis of the actual (ex-post) shortfall, which we report in table 4.8. The model with the most cautious approach would otherwise be selected by the ex-ante shortfall approach, regardless of its true relevance for dynamic portfolio choices. However we emphasize that the ex-ante analysis is useful to characterize the DTSMs in terms of their relative attitude towards risk. Moreover, a comparison of the relative merits of the three DTSMs in both the ex-ante and ex-post shortfalls may be informative to the extent that such rankings do not change when moving from ex-ante to ex-post considerations. Analysing how the results change with modifications in the inputs, i.e. the prices, represents an important form of robustness check.

Table 4.6 shows some interesting results. First, models $EA_1(3)$ and CA_2 (3) present a positive relation between the average shortfall and the target return. This is consistent with the efficient markets hypothesis view that greater expected return can be obtained only by bearing more risk. The same does not strictly happen to $EA_0(3)$. The most relevant result in table 4.6 from the point of view of our study is that –contrary to the forecasting results in Duffee (2002)– $EA_0(3)$ is not the best model in terms of shortfall minimization for each target return. The best model is $EA_1(3)$, since it always produces the lowest average total shortfall. The same ranking among the three models

TABLE 4.7. Ex-ante shortfall analysis (first quarter shortfall).

	Panel A: EA_0 (3)			
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	1.929	1.885	1.919	1.710
std dev	1.016	1.092	1.306	1.294
min	0.299	0.269	0.251	0.168
max	4.733	4.687	4.879	4.414
	Panel B: EA_1 (3)			
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	**1.453**	**1.540**	**1.622**	**1.530**
std dev	**0.636**	**0.807**	1.016	**1.116**
min	0.273	0.248	0.222	0.158
max	3.240	3.611	3.937	3.757
	Panel C: CA_2 (3)			
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	1.539	1.602	1.627	1.747
std dev	0.707	0.848	**0.980**	1.419
min	0.258	0.300	0.261	0.156
max	3.516	3.819	3.901	7.015

For each DTSM and risk-aversion parameter (given by the ratio $\bar{R}_t/\hat{R}_t$ in the first row) the mean (second row), standard deviation (third row), minimum (fourth row) and maximum (fifth row) of the first quarter ex-ante shortfall is reported. The shortfall is computed on the basis of the theoretical bond prices produced by the various models.

obtains if one looks at the shortfall for the first period only (see table 4.7). Results for the second period shortfall and the cumulative shortfall –not shown for reasons of space– are similar. This is important to evaluate the robustness of our results, since the ordering does not depend only on one of the three shortfall measures included in our objective function. Tables 4.6 and 4.7 show the relevance in predicting accurately both the first and the higher order moments of the state vector distribution for portfolio allocation purposes. Not only $EA_1(3)$ produces the lowest average shortfall, but it is also characterized by the least volatile shortfall in three out of four cases. $EA_0(3)$ is generally the worse model.[18] The objective function (4.31) aims at minimizing the level of the total shortfall at time t. However when considering a sequence of asset allocations, investors may regard a DTSM as valuable if it produces shortfalls that are relatively stable through time. $EA_0(3)$ may be the best in terms of point forecast of returns, but a dynamic asset allocation policy requires forecasting time-varying higher moments and this is impossible for a homoskedastic model. This illustrates in very simple terms the main point of our paper.

Descriptive statistics for the ex-post, i.e. computed on the basis of actual market prices, quarterly shortfall are reported in table 4.8. These results are more relevant for practical applications given that the investor relying to a DTSM has to evaluate its performance on the basis of actual measures, i.e. on market prices and not on

TABLE 4.8. Ex-post shortfall analysis.

	Panel A: $EA_0\,(3)$			
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	1.715	1.739	1.727	**1.554**
std dev	1.265	1.349	1.511	**1.589**
min	0.0	0.0	0.0	0.0
max	9.838	9.518	9.536	9.207
	Panel B: $EA_1\,(3)$			
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	**1.529**	**1.627**	1.653	**1.554**
std dev	**1.046**	**1.204**	**1.412**	2.328
min	0.0	0.0	0.0	0.0
max	7.60	8.342	8.952	8.363
	Panel C: $CA_2\,(3)$			
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
mean	1.576	1.649	**1.647**	1.704
std dev	1.164	1.298	1.428	2.478
min	0.0	0.0	0.0	0.0
max	9.235	9.681	9.355	28.260

For each DTSM and risk-aversion parameter (given by the ratio $\bar{R}_t/\hat{R}_t$ in the first row) the mean (second row), standard deviation (third row), minimum (fourth row) and maximum (fifth row) of the quarterly ex-post shortfall is reported. The shortfall is computed on the basis of the actual bond prices.

theoretical prices. Moreover for dynamic optimization models this exercise is even more important than in other cases, since the actual implementation of the portfolio strategy only regards the first of the two periods included in the planning horizon. As usual, when coming to the second period, the model is re-set and the new policy for the first of the two planning periods is implemented in place of the second period policy. In other words the optimal policy is actually implemented only for one quarter. Table 4.8 confirms the ex-ante analysis. $EA_0(3)$ –the best model in terms of point forecast of returns– is not the best one in terms of driving a dynamic asset allocation policy which requires forecasting time-varying higher moments. In fact, this latter task cannot be achieved by a homoskedastic model like $EA_0(3)$. Another relevant result here is that the ex-post shortfall is very close to its ex-ante counterpart for all the models and this confirms the statistical validity of the three DTSMs at the quarterly horizon. This is not surprising as the three specifications have been selected as the best ones by Duffee (2002) from a wide set of potential candidates and are therefore guaranteed to track the data well. An investor using the DTSMs we consider here can be confident that ex-ante and ex-post measures generally coincide.[19] Recall that the higher the target return, the more unlikely is the optimizer to achieve the threshold $\bar{R}_t$ (see footnote 14). As such we regard cases with a relatively low target return as the most interesting ones, in that they guarantee a fair comparison across different models. In general $EA_1(3)$

is still the best model given that it produces the lowest and least volatile shortfall in three cases, thus confirming our ex-ante analysis.[20]

Table 4.9 reports summary statistics for average percentage portfolio weights, splitting the risky assets in short-term maturities (between 6 months and 2 years), medium-term maturities (between 27 months and 7 years) and long-term maturities (between 87 months and 10 years). The goal here is to see whether it is possible to understand the dynamic portfolio policy in a simple way. The results show that model $CA_2(3)$ has a tendency to decrease the share of cash for higher target returns in order to increase investment in short-term bonds and long-term bonds, with a relative underweight of the medium term part of the term structure. $EA_0(3)$ and $EA_1(3)$ decrease the share invested in cash as well as the target return increases, even though in a less marked fashion. Moreover, they have a tendency to substitute cash with short-term bonds but do not show any clear policy of increasing the share invested in medium and long-term bonds. This policy is coherent with the ratio between expected return and risk historically produced by bonds belonging to the various maturities, as described in table 4.3.

Finally, table 4.10 reports values for the performance test. The trading strategies associated with the DTSMs can be evaluated relying to the literature on performance measurement based on information about portfolio holdings. This literature tries to devise powerful performance measures by exploiting the information contained in mutual funds' portfolios. These measures are usually hard to compute in practice due to scarcity of information about funds' actual holdings. However we are comparing trading strategies which are known to us, and we are able to describe the portfolio chosen by each strategy every period. We refer to two of the simplest (and earliest) measures proposed in the literature by Grinblatt and Titman (1993):

$$ESM_t = \sum_{n=1}^{N} w_{n,t} \left(r_{n,t} - \frac{\sum_{t=1}^{T} r_{n,t}}{T} \right) \tag{4.34}$$

$$PCM_t = \sum_{n=1}^{N} r_{n,t}(w_{n,t} - w_{n,t-1}) \tag{4.35}$$

The former, the event study measure, computes the product between the portfolio weights at the beginning of time t and the difference between time t returns and the average return –which is taken as an estimate of the unconditional mean return. The

TABLE 4.9. Portfolio allocation.

Panel A: $EA_0\,(3)$				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
cash	87.02	87.35	88.51	86.99
	29.41	28.81	25.36	23.98
short	6.89	8.17	8.64	10.42
	20.70	23.23	22.95	23.69
med	3.22	2.57	1.31	1.14
	13.59	11.11	6.41	5.31
long	2.87	1.91	1.54	1.45
	9.13	6.78	4.96	3.80
Panel B: $EA_1\,(3)$				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
cash	82.27	82.41	81.99	77.73
	34.53	33.90	33.33	34.15
short	8.19	11.16	13.60	17.48
	23.28	27.26	30.49	33.04
med	7.06	4.48	2.61	3.28
	19.94	12.92	8.08	9.86
long	2.47	1.94	1.79	1.51
	7.16	5.69	4.55	3.52
Panel C: $CA_2\,(3)$				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
cash	81.13	80.58	76.78	69.27
	34.46	33.57	36.17	38.04
short	4.44	7.88	13.18	20.10
	13.58	20.21	28.45	35.13
med	12.10	8.78	4.90	2.06
	27.57	20.95	13.03	6.83
long	2.34	2.77	5.13	8.56
	6.24	8.19	16.39	21.35

Descriptive statistics for the asset allocation implemented by the DTSMs. For each DTSM and risk aversion parameter (given by the ratio $\bar{R}_t/\hat{R}_t$ in the first row) portfolio weights are reported: average weight (first row), and standard deviation (second row).The asset classes other than cash are defined as short-term maturities (between 6 months and 2 years), medium-term maturities (between 27 months and 7 years) and long-term maturities (between 87 months and 10 years). All numbers are in percentage.

second measure, the portfolio change measure, is the product between the change in portfolio holdings and the returns on the various assets. The idea is that a good strategy on average increases weights on those assets producing the highest future return. Notice that PCM_t can also be interpreted as the return on a zero-cost portfolio. The two measures can be computed each period over a given sample, and inference can be performed via a standard t-statistic. [21] From table 4.10 it emerges that no model produces a performance test that is significantly different from zero. An indication about the superiority of model $EA_1\,(3)$ can however be retrieved from table 4.10, since $EA_1\,(3)$ produces a positive performance value in seven out of eight cases, while the other two DTSMs either alternate between positive and negative signs or generally produce negative performance measures.

TABLE 4.10. Performance measurement test.

Panel A: $EA_0\,(3)$				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
ESM	-1.213	-0.905	-0.603	-0.062
	(0.415)	(0.433)	(0.393)	(0.882)
PCM	-0.629	-0.537	-0.434	0.020
	(0.305)	(0.175)	(0.150)	(0.875)
Panel B: $EA_1\,(3)$				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
ESM	0.446	0.209	0.281	0.210
	(0.736)	(0.838)	(0.684)	(0.583)
PCM	0.156	-0.003	0.021	0.116
	(0.541)	(0.988)	(0.889)	(0.273)
Panel C: $CA_2\,(3)$				
$\bar{R}_t/\hat{R}_t$	0.5	0.6	0.7	0.8
ESM	0.047	-0.008	0.075	0.252
	(0.966)	(0.992)	(0.894)	(0.624)
PCM	-0.028	-0.107	-0.096	0.282
	(0.911)	(0.629)	(0.625)	(0.116)

Descriptive statistics for measuring the performance of the asset allocation strategies implemented by the DTSMs. For each DTSM and risk aversion parameter (given by the ratio $\bar{R}_t/\hat{R}_t$ in the first row) two tests are reported: ESM is event study measure, looking at the ability of the model to produce time period returns larger than the unconditional returns, PCM is portfolio change measure looking at the ability of the model to increase portfolio weights in those assets with the highest return. All numbers are in percentage. The numbers in parentheses are the p-values of each statistic.

4.6 Conclusion

We have used theoretical affine DTSMs in the context of active bond portfolio management. We have selected three interesting DTSMs and employed them to produce forecasts for the future values of the relevant state variables. Starting from the theoretical moments of the state variables of the models, we have introduced binomial approximations to come up with discrete scenarios for the future state variables. From the theoretical asset pricing relations we have computed the bond prices for various maturities at the relevant future dates and the consequent returns. We have used these returns as inputs in a portfolio optimization problem where an investor with a six month horizon takes into account the possibility to rebalance after one quarter. The optimizer selects the optimal portfolio each quarter of our sample period. As usual in the context of these problems, only the first stage of the optimal solution is actually implemented. The sequence of optimal portfolios is then evaluated in terms of financial properties.

The goals of the exercise were the following: (1) provide new evidence on the usefulness of DTSMs for active asset allocation, and (2) accumulate evidence on the relative merits of various affine models from a financial metric standpoint, rather than from that of the standard statistical metric. Both goals are in our opinion important. There is a growing debate on the validity of several DTSMs, especially in light of their low performance at replicating basic stylized facts in the bond market, like the low average return and the high unconditional volatility. Moreover the debate has so far been concentrated on a statistical evaluation of the models, while we push towards a financial evaluation. Also, there is an ongoing discussion on the possibility of implementing active asset allocation in efficient markets, and most analyses have focused on the stock market. We try to shift the attention towards the bond market. While still preliminary due to the specific sample we have used, we feel that our results are useful to the overall debates about the validity of theoretical DTSMs and the possibility of using quantitative methods for active asset allocation.

We believe our results are interesting. In particular, the superiority of EA over CA models claimed by Duffee (2002) also holds from the point of view of dynamic portfolio optimization, even though our application does not simply concentrate on expected returns but also take risk into consideration. However evaluating the models by means of a financial metric rather a purely statistical metric reverses the order of the models within the EA class. The interpretation is that allowing for time-varying higher moments may more than compensate for lower forecasting power of the distribution's mean. Performance tests do not allow to show that the excess return produced by the model is significantly positive, even though the superiority of the heteroskedastic EA model is confirmed in terms of average performance. Moreover, the shortfalls produced by the two EA models are in general statistically significant.

At least three directions of future research seem promising to us. The first lies in a comparison of the two-period dynamic optimization model with the standard one-period optimization model. The former is clearly superior to the latter in theory, even though in practice errors in the production of scenarios may negatively affect the model's operational performance. This issue is interesting to analyze in the context of our current data set and theoretical models for the term structure of interest rates. The second direction of future research hinges on testing the relevance of affine term structure models for balanced portfolios involving both stocks and bonds. Recent theoretical advances in theoretical pricing of bonds and stocks are very promising steps in this direction. Finally, our methodology can be used in order to evaluate the quadratic DTSMs in Ahn, Dittmar and Gallant (2002), thus establishing a comparison between different DTSMs classes and an alternative perspective –based on financial performance– to Brandt and Chapman's (2002) recent work.

Notes

[1] Duffee (2002) provides similar evidence within the EA class: the three-factor EA version of the Gaussian model outperforms the multi-factor CIR.

[2] Moreover, the forecasting specifications in the models mentioned in the text are not the best ones in their respective categories. For example, an extended model which includes the term spread and the default spread as supplement to the dividend yield achieves higher values of coefficient of determination.

[3] The second point of the planning horizon serves as a theoretical reference only, since the investor will never actually implement the second stage decision.

[4] Duffie and Kan (1996) do not carry out their analysis neither in terms of $\boldsymbol{\mu}(\mathbf{X}_t)$ nor $\boldsymbol{\Lambda}(\mathbf{X}_t)$; on the other hand they postulate an affine form for the short rate $r(\mathbf{X}_t)$, the risk adjusted drift $\boldsymbol{\mu}^Q(\mathbf{X}_t)$ and the diffusion $\sigma(\mathbf{X}_t)\,\sigma(\mathbf{X}_t)^\top$. Thus with respect to Duffie and Kan's (1996) requirement, condition (4.2) is more restrictive in that imposes affinity in $\sigma(\mathbf{X}_t)\,\boldsymbol{\Lambda}(\mathbf{X}_t)$ as well. It follows however that under (4.2) the risk-adjusted drift $\boldsymbol{\mu}^Q(\mathbf{X}_t)$ is affine as well.

[5] Equivalently, bond yields $y_{t,\tau} \equiv Y(\mathbf{X}_t, \tau)$ are affine in the state vector:

$$Y(\mathbf{X}_t, \tau) \equiv -\frac{\ln(P(\mathbf{X}_t, \tau))}{\tau} = \frac{1}{\tau}\left(-A(\tau) + \mathbf{B}(\tau)^\top \mathbf{X}_t\right)$$

[6] The requirement of a diagonal diffusion matrix is needed for identification of parameters in estimation. Some affine models cannot be represented via a diagonal diffusion matrix, implying that the canonical representation in Dai and Singleton (2000) cannot be imposed. In particular, only affine models with more than three factors might not be characterized by a diagonal diffusion matrix [see Cheridito, Filipović and Kimmel (2003) and the references therein]. In such cases, one should consider the diagonalization requirement (4.7) as a primitive condition on the DTSM, rather than one of its properties.

[7] The specification (4.9) is provided by Duffee (2002), and the expression for the S^- matrix formalizes the idea that the market price of risk goes to zero together with the volatility of the corresponding state variable. However this restriction can be relaxed without impairing no-arbitrage conditions as in Cheridito, Filipović and Kimmel (2003).

[8] Our procedure differs from Boyle, Evnine and Gibbs (1989), since they assume values for $\mathbf{X}_{t+1}$ that are symmetric around the current value $\mathbf{X}_t$, i.e. $X^u_{i,t+1} = X_{i,t} + \Delta X_{i,t}$ and $X^d_{i,t+1} = X_{i,t} - \Delta X_{i,t}$, and then solve (4.18a-4.18d) for $\{p_i, \Delta X_{i,t}\}_{i=1,2}$. On the other hand our method is closer to He (1990), where a trinomial tree with equal probabilities for every state is employed in approximating each diffusion for the two-dimensional case.

[9] See Duffee (2002) for a comparison of the hypotheses maintained in CA and EA models.

[10] More specifically, we use the original data from McCulloch and Kwon (1993) over the sample period January 1952-February 1992, which includes 29 maturities ranging from 1 month

to 10 years. We interpolate yields from Duffee's homepage for the remaining period (March 1992-December 1998).

[11] For $EA_0(3)$ the model restrictions (see table 4.4) allow to write the first two conditional moments as $E(\mathbf{X}_T^*|\mathbf{X}_t^*) = \exp(-D(T-t))\mathbf{X}_t^*$ and $var(\mathbf{X}_T^*|\mathbf{X}_t^*) = b_0$ respectively, where $[b_0]_{jk} = \frac{1}{[D]_{jj}+[D]_{kk}}\left(1-\exp\left(-(T-t)\left([D]_{jj}+[D]_{kk}\right)\right)\right)[G_0]_{jk}$ and $G_0 = \Sigma^*\Sigma^{*\top}$. Therefore the conditional variance-covariance matrix does not depend on the current state vector $\mathbf{X}_t$ under this specification.

[12] It may be useful to notice that the probabilities determined in section 4.3 are to be interpreted as true probabilities, not as risk neutral probabilities. We do not need risk neutral probabilities because pricing is carried out by the theoretical DTSM. We can therefore use the model to properly study a portfolio problem that has to be applied to the real world.

[13] Over the period 1972-1997, Driessen, Melenberg and Nijman (1999) report a 1.6 basis point average transaction cost (measured by the bid-ask spread midpoint) for Treasury bills with maturities from 1 month to 9 months. Transaction costs increase with the maturity from 0.6 b.p. for the 1 month T-bill up to approximately 3 b.p. for the 9-month maturity. Given that: (1) bonds are more volatile than short-maturity T-bills and (2) the T-bill bid-ask spread increases with the time to maturity, it is reasonable to assume that transactions costs on long-maturity bonds are higher than the average 1.6 b.p. T-bill bid-ask spread. Further, since our sample dates back to 1952 and comprises U.S. Treasury bonds with maturity up to 10 years, we assume 5 b.p transaction costs.

[14] For example $CA_2(3)$ does not reach the target return in six cases with the 0.8 multiple. We therefore decrease the target to $0.75\hat{R}_t$ (3 asset allocations), $0.7\hat{R}_t$ (2 asset allocations) and $0.6\hat{R}_t$ (1 asset allocation). The other DTSMs are characterized by similar corrections.

[15] It is useful to point out for completeness that similar figures for other maturities do not show such a clear cut outcome, in that also $CA_2(3)$ generates negative expected returns from time to time as well. This implies that $CA_2(3)$ may as well suggest time-varying dynamic asset allocation decisions depending on the state variables' values.

[16] For a given DTSM parametrization, we invert the relationship between yields and the state vector [derived from eq. (4.4)]. As in Duffee (2002) we consider bonds with maturities 6M, 2Y and 10Y as measured without error.

[17] In order to evaluate the robustness of our results to the structure of the lattice for the state variables we have compared the one-period shortfall obtained from our 8 scenario lattice to the shortfall obtained from a finer grid involving 512 scenarios, i.e. 8 values for the state vector every month, over 30 randomly chosen dates. The results show that the mean shortfall obtained by the fine lattice is not statistically different from the mean shortfall obtained from the coarse lattice.

[18] We have performed a two-sided test on the difference in mean shortfalls (both total and first quarter) across DTSMs. From the results –not reported for reasons of space– it emerges

that the difference in the ex-ante shortfall across models is significant in most cases, both for the total and the first quarter shortfall. In particular the shortfall for $EA_0\,(3)$ is always statistically different at usual significance levels from the other two DTSMs in all cases but for the multiple 0.8. On the other hand, when comparing $EA_1\,(3)$ with $CA_2\,(3)$, the test for the equality of average ex-ante shortfalls does not reject the null hypothesis. Taken together with tables 4.7 and 4.8 this means that the merits of $EA_1\,(3)$ relative to $EA_0\,(3)$ are significant for values of $\bar{R}_t/\hat{R}_t$ up to (and including) 0.7, and the performance of $CA_2\,(3)$ is very close to $EA_1\,(3)$ over the same range. On the other hand for $\bar{R}_t/\hat{R}_t = 0.8$ the ex-ante shortfall of the three DTSMs is almost indistinguishable.

[19] The test statistic for the equality of means between the ex-ante (first quarter) and ex-post shortfall does not reject the null hypothesis at usual significance levels in all but one case (the average ex-post shortfall is statistically different from the ex-ante shortfall at 10% statistical level for $EA_0\,(3)$ and $\bar{R}_t/\hat{R}_t = 0.5$).

[20] It is worth stressing that we are simulating the choice of an investor who is comparing the three DTSMs from the point of view of their in-sample performance. In our view the investor would be happy to choose the model producing the most efficient portfolio performance. We are not making the alternative exercise [see Pastor (2000)] of assuming that one model is true and compute the utility cost for an investor using another model. On the other hand we are simply claiming that an investor would choose the model, among the three considered here, that achieves the target return with the lowest average shortfall.

[21] Solnik (1993) generalizes the event study measure and allows for time-varying portfolio volatility (associated with changing portfolio composition) by normalizing each measure with its standard devation, i.e. $w_t' V w_t$ where V is the unconditional variance covariance matrix. We have also computed the performance measure with this heteroskedasticity correction, as well as a third version with an historical estimate of the conditional variance-covariance matrix, $w_t' V_t w_t$. However the results are not qualitatively different from the simple version that we described in the text, so we have chosen not to report all the results for reasons of space.

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