Doubly Stochastic Point Processes in Reinsurance and the Pricing of Catastrophe Insurance Derivatives

by

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Contents

Contents (continued)

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Abstract

This dissertation presents pricing models for stop-loss reinsurance contracts for catastrophic events and for catastrophe insurance derivatives.

We use *doubly stochastic Poisson process* or the *Cox process* for the claim arrival process for catastrophic events. The *shot noise process* is able to measure the frequency, magnitude and time period needed to determine the effect of the catastrophe. This process is used for the claim intensity function within the Cox process. The Cox process with shot noise intensity is examined by piecewise deterministic Markov process theory.

We apply the Cox process incorporating the shot noise process as its intensity to price stop-loss catastrophe reinsurance contracts and catastrophe insurance derivatives. In order to calculate *fair prices* for reinsurance contracts and catastrophe insurance derivatives we need to assume that there is an absence of arbitrage opportunities in the market. This can be achieved by using an *equivalent martingale probability measure* in our pricing models. The Esscher transform is used to change probability measure.

The dissertation also shows how to estimate the parameters of claim intensity using the likelihood function. In order to estimate the distribution of claim intensity, state estimation is employed as well. Since the claim intensity is not observable we filter it out on the basis of the number of claims, i.e. we employ the *Kalman-Bucy filter*. We also derive pricing formulae for stop-loss reinsurance contracts for catastrophic events using the distribution of claim intensity that is obtained by the Kalman-Bucy filter. Both estimations are essential in pricing stop-loss reinsurance contracts and catastrophe insurance derivatives.

Introduction

1. General

The principal aim of this dissertation is practical reinsurance problem solving. Namely, to create models for the pricing of catastrophe reinsurance contracts and catastrophe insurance derivatives. As most of our references ignore the effect of interest rates we will do so as well.

Insurance companies have traditionally used reinsurance contracts to hedge themselves against losses from catastrophic events. During the last decade, the world has experienced a higher level of catastrophic events both in terms of frequency and severity. Some of the recent catastrophes are Hurricane Andrew (USA 1992) and the Kobe earthquake (Japan 1995) (see Booth (1997)). This has had a marked effect on the reinsurance market. Such events have impacted the profitability and capital bases of reinsurance companies some of which have withdrawn from the market and others have reduced the level of catastrophe cover they are willing to provide.

In the early 1990s, some believed that there was undercapacity provided by the reinsurance market. Some investment banks, particularly US banks, recognised the opportunities that existed in the reinsurance market. Through their large capital bases the investment banks were able to offer alternative reinsurance products. This caused reinsurance companies to reassess their strategies for the type of products offered to the market.

The Chicago Board of Trade introduced catastrophe insurance futures and catastrophe insurance options on futures traded on a quarterly basis (Jan-Mar, Apr-June, July-Sep and Oct-Dec) in December 1992. The CBOT devised a loss ratio index as the underlying instrument for catastrophe insurance futures and options contracts. The Insurance Service Office calculates the index from loss data reported by at least 25 selected companies (see CBOT (1994, 1995a, 1995b)). The loss ratio index is the reported losses incurred in a given quarter and reported by the end of the following quarter, L_t, divided by one fourth of the premiums received in the previous year, Π, i.e. $\frac{L_t}{\Pi}$.

The value of the insurance futures, F_t, at maturity t, is the nominal contract value, US$25,000, times the loss ratio index capped at 2, i.e.

$$F_t = 25{,}000 \;\times\; Min\left(\frac{L_t}{\Pi},\; 2\right).$$

The CBOT capped the maximum loss ratio at 200% in order to limit the credit risk from unexpected huge losses and to make the contract look like a non-proportional reinsurance policy. However, to date there has not been an incident where the maximum loss ratio has been reached; the highest estimated loss ratio being 179% for Hurricane Andrew. Therefore ignoring the maximum loss ratio, the value of the catastrophe insurance call options on futures, P_t, at maturity t is given by

$$\mathrm{P}_t = Max(F_t - E,\; 0) = (F_t - E)^+ = \left(25{,}000 \times \frac{L_t}{\Pi} - E\right)^+ = \frac{25{,}000}{\Pi}(L_t - \mathrm{B})^+$$

where E is the exercise price and $\mathrm{B} = \dfrac{\Pi E}{25{,}000}$.

There has been discussion and research into the possibility of using catastrophe insurance futures and options contracts rather than conventional reinsurance contracts (see Lomax & Lowe (1994), Smith (1994), Ryan (1994), Sutherland (1995), Kielholz & Durrer (1997) and Smith, Canelo & Di Dio (1997)). The competitiveness in the reinsurance market emphasises the need for an appropriate pricing model for reinsurance contracts and catastrophe insurance derivatives. It is common practice for most references to ignore the effect of interest rates. We will also adopt this approach.

In insurance modelling, the Poisson process has been used as a claim arrival process. Extensive discussion of the Poisson process, from both applied and theoretical viewpoints, can be found in Cramér (1930), Cox & Lewis (1966), Bühlmann (1970), Cinlar (1975), and Medhi (1982). However, there has been a significant volume of literature that questions the appropriateness of the Poisson process in insurance modelling (see Seal (1983) and Beard *et al.*(1984)) and more specifically for rainfall modelling (see Smith (1980) and Cox & Isham (1986)).

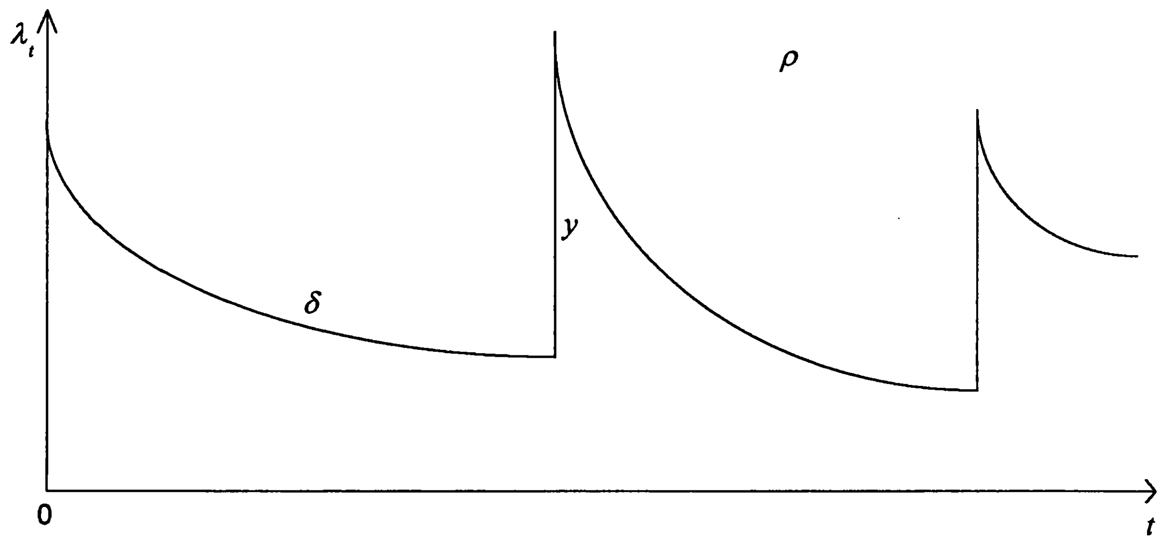
As catastrophic events occur periodically, the assumption that resulting claims occur in terms of the Poisson process is inadequate. Therefore an alternative point process needs to be used to generate the claim arrival process. We will employ a **doubly stochastic Poisson process,** or the **Cox process,** (see Cox (1955), Bartlett (1963), Serfozo (1972), Grandell (1976, 1991), Bremaud (1981) and Lando (1994)).

Claims arising from catastrophic events depend on the intensity of natural disasters (e.g. flood, windstorm, hail, earthquake). By intensity we mean the frequency of claims resulting from the natural disaster.

The **shot noise process** measures the impact of catastrophic events (see Cox & Isham (1980,1986) and Klüppelberg & Mikosch (1995)). As time passes, the shot noise process decreases as more and more claims are settled. This decrease continues until another catastrophe occurs which will result in a positive jump in the shot noise process. The shot noise process is particularly useful in the claim arrival process for catastrophic events as it mirrors the nature of such events. Therefore we will use it as a claim intensity function to generate the Cox process.

We will adopt the shot noise process used by Cox & Isham (1980):

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i < t}} y_i e^{-\delta(t-s_i)}$$



where:

i catastrophe

λ_0 initial value of λ

y_i jump size of catastrophe i (i.e. magnitude of contribution of catastrophe i to intensity) where $E(y_i) < \infty$

s_i time at which catastrophe i occurs where $s_i < t < \infty$

δ exponential decay which never reaches zero

ρ the number of catastrophes in time period t.

The **piecewise deterministic Markov process theory** developed by Davis (1984) is a powerful mathematical tool for examining non-diffusion models. Using this theory, Dassios (1987) examined the Cox process incorporating the shot noise process as its intensity. In Chapter 1 we present definitions and important properties of the Cox and shot noise processes with the aid of piecewise deterministic process theory. In Chapter 2 this theory is used to calculate the distribution of the number of claims and the mean of the claims. These are important factors in the pricing of any reinsurance product.

Harrison & Kreps (1979) and Harrison & Pliska (1981) launched the approach for the pricing and analysis of movements of the financial derivatives whose prices are determined by the price of the underlying assets. Their mathematical framework originates from the idea of risk-neutral, or non-arbitrage, valuation of Cox & Ross (1976).

A reinsurance contract is similar to a financial derivative in that its value is determined by the underlying claim arrival process. Sondermann (1991) introduced the non-arbitrage approach for the pricing of reinsurance contracts. He proved that if there is no arbitrage opportunities in the market, reinsurance premiums are calculated by the expectation of their value at maturity with respect to a new probability measure and not with respect to the original probability measure. This new probability measure is called the **equivalent martingale probability measure**. The equivalent martingale probability measure is not unique in general. It is not the purpose of this thesis to decide which one to use.

One of the methods to change the probability measure is the **Esscher transform**. Gerber & Shiu (1996) priced derivatives using the Esscher transform to go from the original probability measure to the equivalent martingale probability measure. In Chapter 3 we use this approach for the pricing catastrophe reinsurance contract and catastrophe insurance derivatives (i.e. catastrophe insurance futures and catastrophe insurance options on futures). For the reason already explained, the Cox process with shot noise intensity is used as the claim arrival process.

We need to solve two estimation problems for the pricing of a reinsurance contract and catastrophe insurance derivatives. Firstly, the parameters of the shot noise process i.e. the rate of occurrence of catastrophe, the size of catastrophe and the rate of decay (settlement) should be estimated. In chapter 4 we present parametric estimation techniques for the estimation of the three parameters in the shot noise process. Secondly, the distribution of the claim intensity should be obtained. Since the claim intensity is unobservable it has to be "filtered out" from the observed number of claims at various times. So chapter 5 deals with state estimation to derive the distribution of the claim intensity. One of the methods used is the Kalman-Bucy filter. We derive pricing models for catastrophe reinsurance contracts using the distribution of the claim intensity that is obtained by the Kalman-Bucy filter. The Laplace transform of distribution of λ_t is obtained assuming that we know the times of catastrophe jumps and claim points. We also examine the Laplace transform of distribution of λ_t assuming that the number of claims in a fixed time interval is known. The idea and discussion of state estimation can be found in Karr (1991).

2. Definitions and conditions

Let us look at a brief outline of the concepts of **generator** (see Davis (1984)), **martingale** (see Gerber (1979) and Øksendal (1992)) and **equivalent martingale probability measure** (see Sondermann (1991)) that we will use in this dissertation.

The process $X_t = (n_t, \zeta_t)$ takes values from the set $\mathrm{L} \times \mathrm{E}$ where L is a discrete set and $\mathrm{E} \subset \Re^n$. In any interval $[0, t]$, ζ_t can be discontinuous at only a finite number of points $t_1, t_2, \cdots, t_n$ where $0 \le t_i \le t$. In theorem 5.5 of Davis (1984), the discontinuity can also occur when a boundary is reached. However this is not applicable here. The **generator** A of the process is an operator on functions $f : \mathrm{L} \times \mathrm{E} \to \Re$ and its domain contains functions f satisfying the following conditions:

(i) For all $n \in \mathrm{L}$, the function f is absolutely continuous on ζ;

(ii) $E\left\{ \left| f(X_{t_i}) - f(X_{t_i^-}) \right| \right\} < \infty$.

We define the generator A acting on any function f belonging to its domain. Then $\mathrm{A}f$ is the unique function that makes $f(X_t)-\int_0^t \mathrm{A}f(X_s)ds$ is an $\Im_t$-martingale. We will use the term '**martingale**' to refer to martingale with respect to the natural filtration i.e. $\Im_t$-martingale.

The generator of the process $\{X_t\}$ acting on a function $f(X)$ belonging to its domain as described above is also given by:

$$\mathrm{A}f(X_t)=\lim_{h\downarrow 0}\frac{E[f(X_{t+h})|X_t=x]-f(X_t)}{h}.$$

$\mathrm{A}f(X_t)$ is the expected increment of the process $\{X_t\}$ between t and $t+h$, given the history X_t at time t. From this interpretation the following inversion formula is plausible:

$$E[f(X_{t+h})|X_t=x]-f(X_t)=\int_0^h E[\mathrm{A}f(X_s)]ds.$$

Therefore if we are able to find a function f such that $\mathrm{A}f=0$ then $f(X_t)$ is an $\Im_t$-martingale. In this thesis, it will be very important to solve the equation $\mathrm{A}f=0$ for various circumstances.

Let (Ω, F, P) be a probability space with information structure F. The information structure F is the filtration, i.e. $F=\{\Im_t,\ t\in[0,\ T]\}$. F consists of σ-algebra's $\Im_t$ on Ω, for any point t in the time interval [0, T], representing the information available at time t. Then a probability measure P^* is called an **equivalent martingale probability measure** if:

- $P^*[A]=0$ iff $P[A]=0$, for any $A\in\Im_t$;
- the Radon-Nikodym derivative $\frac{dP^*}{dP}$ belongs to $L_2(\Omega,\Im_t,P)$;
- a specified process+ R_t is a martingale under P^*, i.e.

$$E^*[R_t|\Im_s]=R_s,\quad P^*-\text{a.s.}$$

for any $0\le s\le t\le T$, where E^* denotes the expectation with respect to P^*.

+ Note: In our case, a specified process R_t is premiums minus claims, i.e. $R_t=PR_t-C_t$.

3. Overview

We now conclude our introduction by giving an overview of this dissertation. Let $\aleph_i$ be the claim amount then the total loss excess over b, which is a retention limit, up to time t is

$$\left(\sum_{i=1}^{N_t}\aleph_i - b\right)^+$$

where N_t is the number of claims up to time t and $\left(\sum_{i=1}^{N_t}\aleph_i - b\right)^+ = Max\left(\sum_{i=1}^{N_t}\aleph_i - b,\ 0\right)$.

Let $C_t = \sum_{i=1}^{N_t}\aleph_i$ be the total amount of claims up to time t. Then

$$\left(\sum_{i=1}^{N_t}\aleph_i - b\right)^+ = (C_t - b)^+.$$

When $\aleph_i = 1$, C_t becomes N_t. Therefore the stop-loss reinsurance premium at time 0 is

$$E\left\{(N_t - b)^+\right\} \text{ or } E\left\{(C_t - b)^+\right\}.$$

Catastrophe insurance futures and options introduced by CBOT are standardised reinsurance instruments. If we assume that $L_t = C_t$, ignoring the effect of interest rates, the price of the insurance futures at time 0 is

$$E\left[25{,}000 \times Min\left(\frac{C_t}{\Pi}, 2\right)\right]$$

and ignoring the maximum loss ratio, the price of the insurance call option on futures at time 0 is

$$\frac{25{,}000}{\Pi}E\left[(C_t - \mathrm{B})^+\right].$$

If we substitute 'b' with 'B' in the formula of the stop-loss reinsurance premium at time 0 the two formula are equivalent.

The probability distribution for N_t or C_t used in this dissertation is based on the following hierarchical specification:

- a Poisson process with constant intensity ρ is used to generate catastrophe times s_i. The random variable M_t is the total number of catastrophes up to time t.

- the positive random variable y_i specifies the size of the catastrophe at time s_i. The distribution of y_i is denoted by $G(y_i)$.
- the intensity decreases with the decay rate δ, which is a constant, between s_{i-1} and s_i until another catastrophe jump y_i occurs at s_i.
- given (s_i, y_i, δ), we define the positive function

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i < t}} y_i e^{-\delta(t-s_i)}$$

where λ_0 is the initial value of λ. This is the shot noise process and also piecewise deterministic Markov process. We also define the aggregated process $X_t = \int_0^t \lambda_s ds$ which is the accumulated intensity.
- the function λ_t is used as the claim intensity of a Poisson process to generate the number of claims N_t.
- given a claim, the claim size distribution is $H(u)$. The total amount of claims up to time t is denoted by C_t.

The unknown parameters are ρ, y and δ which are the rate of occurrence of catastrophe jump, the size of catastrophe and the rate of decay respectively. In the time dependent case δ, ρ and $G(y)$ vary with time t. When λ_t is time homogeneous these do not vary with time t.

In chapter 1, we examine the shot noise process λ_t as it mirrors the nature of catastrophic events. Other process may be applied to measure the impact of catastrophic events. Also the shot noise process itself can be applied in financial field such as interest rate modelling, bond pricing and bond option pricing.

In chapter 2, doubly stochastic Poisson process is discussed as a claim arrival process N_t where the claim intensity function is the shot noise process λ_t. By changing the claim intensity function in Poisson process, we can derive a variety of doubly stochastic Poisson processes. In insurance modelling, if a homogeneous or non-homogeneous Poisson process are not appropriate as a claim arrival process, doubly stochastic Poisson process can be considered as an alternative by selecting proper claim intensity function.

The results in chapters 1 and 2 are very important in the pricing of any reinsurance products. Even if we change the claim intensity function in Poisson process, we can easily obtain important results using the approach used in these chapters.

In chapter 3, we assume that there are no arbitrage opportunities in the market to obtain fair premiums for stop-loss reinsurance contracts and fair prices for insurance derivatives. This can be achieved by using an equivalent martingale probability measure in our pricing models. The Esscher transform is used to change probability measure. If the market is complete, there is only one equivalent martingale probability measure and only one fair premium or price can be obtained for each contracts. However we have more than one equivalent martingale probability measures as the market is not complete. We present quite a flexible family of martingales that can be used to change measure; however it is not the purpose of this dissertation to decide which one to use.

In chapters 4 and 5, we solve two estimation problems for the pricing of reinsurance contracts and insurance derivatives. Firstly, we suggest estimating parameters of λ_t using the likelihood function and the distribution of λ_t via the Kalman-Bucy filter. The λ_t is standardised by subtracting its mean and dividing by the variance leading to Z_t. The claim arrival process N_t is similarly approximated as well as C_t, which is the total amount of claims up to time t, leading to W_t and U_t. For large ρ, the standardised process Z_t becomes an Ornstein-Uhlenbeck process and the standardised processes W_t and U_t become Brownian motions. The assumption of $\rho \to \infty$ implies that catastrophes are a common events. Therefore this approach can be used for the pricing of reinsurance contracts when we expect quite a few claims in the near future. Secondly, in order to estimate the parameters and distribution of λ_t, direct and realistic approaches are presented. For estimation of parameters and the Laplace transform of distribution of λ_t we assume that we know the catastrophe time s_i as well as the claim point t_j. We examine another method of obtaining the Laplace transform of the distribution of λ_t assuming that the number of claims in a fixed time interval is known.

The direct approach provides us with accurate pricing as it is realistic. However it is not always practical to use given the complexity of the equations involved. A more practical approach is to use the transformed and approximated approach, where we assume $\rho \to \infty$. However we should examine whether the third moments of the catastrophe size y and the claim size u exist; if it does not exist the central limit theorem does not provide an accurate result.

It would be interesting to empirically test our models and estimation methods derived. The relevant data would need to be obtained from reinsurance companies or CBOT. In that case, the numerical examples illustrated in chapters 3, 4 and 5 may be useful.

1. Doubly Stochastic Poisson Process, Shot Noise Process and Aggregated Process

1.1 Doubly stochastic Poisson process

Claims arising from catastrophic events depend on the intensity of such natural disasters. Therefore the intensity means the frequency of claims resulting from the natural disaster.

In order to calculate the price for catastrophe reinsurance contracts and insurance derivatives, the *claim arrival process* needs to be determined. A homogeneous Poisson process can be used as a claim arrival process. Under this approach the *claim intensity function* is assumed to be constant. Another approach is to use a non-homogeneous Poisson process where the claim intensity function is assumed to be a non-random function of time. However, both these process do not adequately explain the phenomena of catastrophes as mentioned in the introduction.

Under *doubly stochastic Poisson process*, or *the Cox process*, the claim intensity function is assumed to be stochastic. The Cox process is more appropriately used as a claim arrival process as it allows for the assumption that catastrophic events occur periodically. However, little work has been done to further develop this assumption in an insurance context. We will now proceed to examine doubly stochastic Poisson process as the claim arrival process.

Doubly stochastic Poisson process provides flexibility by letting the intensity not only depend on time but also allowing it to be a stochastic process. Therefore doubly stochastic Poisson process can be viewed as a two step randomisation procedure. A process λ_t is used to generate another process N_t by acting as its intensity. That is, N_t is a point process conditional on λ_t which itself is a stochastic process (if λ_t is deterministic then N_t is a Poisson process).

Many alternative definitions of a doubly stochastic Poisson process can be given. We will offer the one adopted by Bremaud (1981).

Definition 1.1.1. Let N_t be a point process adopted to a history $\Im_t$ and let λ_t be a non-negative process. Suppose that λ_t is $\Im_t$-measurable, $t \geq 0$ and that

$$\int_0^t \lambda_s ds < \infty \quad \text{almost surely} \quad \text{(no explosions)}.$$

If for all $0 \leq t_1 \leq t_2$ and $u \in \Re$

$$E\left\{e^{iu(N_{t_2}-N_{t_1})} \middle| \Im_{t_1}\right\} = e^{(e^{iu}-1)\int_{t_1}^{t_2} \lambda_s ds} \tag{1.1.1}$$

then N_t is called a $\Im_t$-doubly stochastic Poisson process with intensity λ_t.

□

In this dissertation we will take $\Im_t$ to be the natural filtration of the probability space.

(1.1.1) gives us

$$\Pr\left\{N_{t_2} - N_{t_1} = k \middle| \lambda_s; t_1 \leq s \leq t_2\right\} = \frac{e^{-\int_{t_1}^{t_2} \lambda_s ds}\left(\int_{t_1}^{t_2} \lambda_s ds\right)^k}{k!} \tag{1.1.2}$$

and

$$E\left\{\theta^{N_{t_2}-N_{t_1}} \middle| \lambda_s; t_1 \leq s \leq t_2\right\} = e^{-(1-\theta)\int_{t_1}^{t_2} \lambda_s ds} \tag{1.1.3}$$

so

$$E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\left\{E\left(\theta^{N_{t_2}-N_{t_1}} \middle| \lambda_s; t_1 \leq s \leq t_2\right)\right\} = E\{e^{-(1-\theta)\int_{t_1}^{t_2} \lambda_s ds}\}. \tag{1.1.4}$$

If λ_t is a Markov process and A_λ is the generator of the process (λ_t, t) then the generator of the process (N_t, λ_t, t) acting on a function $f(n,\lambda,t)$ belonging to its domain is given by

$$\mathrm{A} f(n,\lambda,t) = \lambda\{f(n+1,\lambda,t) - f(n,\lambda,t)\} + \mathrm{A}_\lambda f(n,\lambda,t). \tag{1.1.5}$$

Clearly, for $f(n,\lambda,t)$ to belong to the domain of the generator A, it is essential that $f(n,\lambda,t)$ belongs to the domain of A_λ for all n.

Notice that the generator A acting on a function $\theta^n f(\lambda,t)$ gives

$$\mathrm{A}\, \theta^n f(\lambda,t) = -\lambda(1-\theta)\theta^n f(\lambda,t) + \theta^n \mathrm{A}_\lambda f(\lambda,t). \tag{1.1.6}$$

Now consider the process

$$X_t = \int_0^t \lambda_s ds \quad \textit{(the aggregated process)};$$

the generator A of the process (X_t, λ_t, t) acting on a function $f(x,\lambda,t)$ belonging to its domain is given by

$$\mathrm{A} f(x,\lambda,t) = \lambda \frac{\partial f}{\partial x} + \mathrm{A}_\lambda f(x,\lambda,t). \tag{1.1.7}$$

Clearly, for $f(x,\lambda,t)$ to belong to the domain of the generator A, it is essential that $f(x,\lambda,t)$ belongs to the domain of A_λ for all x and that it is differentiable w.r.t. x. Trying a function of the form $e^{-(1-\theta)x} f(\lambda,t)$ then

$$\mathrm{A}\,[e^{-(1-\theta)x} f(\lambda,t)] = -\lambda(1-\theta)e^{-(1-\theta)x} f(\lambda,t) + e^{-(1-\theta)x}\mathrm{A}_\lambda f(\lambda,t). \tag{1.1.8}$$

By the aggregated process we can easily find that (1.1.4) is

$$E\left(\theta^{N_{t_2}-N_{t_1}}\right) = E\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}\}. \tag{1.1.9}$$

(1.1.6) and (1.1.8) as well as the relationships (1.1.2), (1.1.3) and (1.1.4) suggest that the problem of finding the distribution of N_t, the point process, is equivalent to the problem of finding the distribution of X_t, the aggregated process. For example (1.1.9) means that we just have to find the p.g.f. (probability generating function) of N_t to retrieve the m.g.f. (moment generating function) of X_t and vice versa.

1.2 Shot noise process with time dependent parameters

The shot noise process measures the impact of catastrophic events. As it mirrors their nature, the shot noise process can be used as a claim intensity function in the claim arrival process. In this section we examine a shot noise process λ_t.

We are now going to generalise the shot noise process by allowing three parameters to depend on time. Therefore we assume that $\delta(t)$, $\rho(t)$ and $G(y;t)$ are all Riemann integrable functions of t and are all positive. Furthermore the rate of jump arrivals, $\rho(t)$, is bounded on all intervals [0, t) (no explosions). $\delta(t)$ is the rate of decay but we assume $\delta(t)=\delta$ throughout the rest of this thesis. The distribution function of jump sizes for all t is $G(y;t)$ $(y>0)$. If the jump size distribution is exponential, its density is $g(y;t)=(\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\alpha+\gamma e^{\delta t}>0$ (i.e. $\gamma>-\alpha e^{-\delta t}$), a special case that will be quite useful later.

The generator of the process (λ_t, t) acting on a function $f(\lambda,t)$ belonging to its domain is given by

$$\mathrm{A}f(\lambda,t)=\frac{\partial f}{\partial t}-\delta(t)\lambda\frac{\partial f}{\partial \lambda}+\rho(t)\{\int_0^\infty f(\lambda+y,t)dG(y;t)-f(\lambda,t)\}. \qquad (1.2.1)$$

For this process we can derive the Laplace transform of the distribution of λ_t at any given time t given λ_0.

Theorem 1.2.1 Let λ_t as defined. Then

$$e^{-ve^{\Delta(t)}\lambda_t}e^{\int_0^t \rho(s)[1-\hat{g}\{ve^{\Delta(s)};s\}]ds} \qquad \left(\hat{g}(k;s)=\int_0^\infty e^{-ky}dG(y;s)\right)$$

is a martingale for all $v\geq 0$, where $\Delta(t)=\int_0^t \delta(s)ds$.

Proof

From (1.2.1), $f(\lambda,t)$ has to satisfy $\mathrm{A}f=0$ for it to be a martingale. Setting $e^{-A(t)\lambda}e^{R(t)}$ we get the equation

$$-\lambda A'(t)+R'(t)+\delta(t)\lambda A(t)+\rho(t)[\hat{g}\{A(t);t\}-1]=0 \qquad (1.2.2)$$

and solving (1.2.2) we obtain

$$A(t) = ve^{\int_0^t \delta(s)ds} \quad \text{and} \quad R(t) = \int_0^t \rho(s)[1-\hat{g}\{ve^{\int_0^s \delta(u)du};s\}]ds$$

and the result follows.

□

We now will use the martingale found in theorem 1.2.1.

Theorem 1.2.2 Let λ_t as defined. Then

$$E\{e^{-v\lambda_t}|\lambda_0\} = e^{-v\lambda_0 e^{-\Delta(t)}} e^{-\int_0^t \rho(s)[1-\hat{g}\{ve^{-\Delta(t)}e^{\Delta(s)};s\}]ds} \quad . \tag{1.2.3}$$

<u>**Proof**</u>

From theorem 1.2.1, for a fixed time t^* and a fixed constant $v^* \geq 0$, we have

$$E\left\{e^{-v^* e^{\Delta(t^*)}\lambda_{t^*}}|\lambda_0\right\} = e^{-v^*\lambda_0} e^{-\int_0^{t^*} \rho(s)[1-\hat{g}\{v^* e^{\Delta(s)};s\}]ds}$$

and setting $v^* = ve^{-\Delta(t^*)}$ we have

$$E\{e^{-v\lambda_{t^*}}|\lambda_0\} = e^{-v\lambda_0 e^{-\Delta(t^*)}} e^{-\int_0^{t^*} \rho(s)[1-\hat{g}\{ve^{-\Delta(t^*)}e^{\Delta(s)};s\}]ds} \quad . \tag{1.2.4}$$

Since (1.2.4) holds for an arbitrary fixed t^*, it holds for all $t \geq 0$ and the theorem is proved.

□

We can also obtain the asymptotic (stationary) distribution from theorem 1.2.2. In this context we interpret it as the limit when $t \to -\infty$. In other words, if we know λ at '$-\infty$' and no information between '$-\infty$' to present time t, '$-\infty$' asymptotic distribution of λ_t can be used as the distribution of λ_t.

In order to find the '$-\infty$' asymptotic distribution let us start with lemma.

Lemma 1.2.3 Let's assume that $\delta(t) = \delta$, $\lim_{t\to-\infty}\rho(t) = \rho$ and $\lim_{t\to-\infty}\mu_1(t) = \mu_1$. Then

$$\int_{-\infty}^{t} \rho(s)[1-\hat{g}\{ve^{-\delta(t-s)};s\}]ds < \infty$$

where $\mu_1(t) = \int_0^\infty ydG(y;t) = E(y;t)$ and $\hat{G}(u;t) = \dfrac{1-\hat{g}(u;t)}{u}$.

Proof

When $\delta(t) = \delta$ the exponential part of the second term in the right hand side in (1.2.3) is

$$\int_{-\infty}^{t} \rho(s)[1-\hat{g}\{ve^{-\delta(t-s)};s\}]ds$$

$$= \int_0^\infty ve^{-\delta s}\rho(t-s)\frac{1-\hat{g}\{ve^{-\delta s};t-s\}}{ve^{-\delta s}}ds = \int_0^\infty ve^{-\delta s}\rho(t-s)\hat{G}(ve^{-\delta s};t-s)ds$$

but

$$\hat{G}(ve^{-\delta s};t-s) = \int_0^\infty e^{-ve^{\delta s}y}\{1-G(y;t-s)\}dy < \int_0^\infty \{1-G(y;t-s)\}dy = \mu_1(t-s). \quad (1.2.5)$$

From (1.2.5)

$$\int_{-\infty}^{t} \rho(s)[1-\hat{g}\{ve^{-\delta(t-s)};s\}]ds = \int_0^\infty ve^{-\delta s}\rho(t-s)\hat{G}(ve^{-\delta s};t-s)ds < \int_0^\infty ve^{-\delta s}\rho(t-s)\mu_1(t-s)ds < \infty.$$

□

Corollary 1.2.4 Let $\delta(t) = \delta$ and assume that $\lim_{t\to-\infty}\rho(t) = \rho$ and $\lim_{t\to-\infty}\mu_1(t) = \mu_1$ where $\mu_1(t) = \int_0^\infty ydG(y;t)$. Then the '$-\infty$' asymptotic distribution of λ_t has Laplace transform

$$e^{-\int_{-\infty}^{t_1}\rho(s)[1-\hat{g}\{ve^{-\delta(t_1-s)};s\}]ds}. \quad (1.2.6)$$

Proof

From theorem 1.2.2 and let $\delta(t) = \delta$, then

$$E\left\{e^{-v\lambda_{t_1}}\middle|\lambda_{t_0}\right\} = e^{-v\lambda_{t_0}e^{-\delta(t_1-t_0)}}e^{-\int_{t_0}^{t_1}\rho(s)[1-\hat{g}\{ve^{-\delta(t_1-s)};s\}]ds} \quad (1.2.7)$$

letting $t_0 \to -\infty$ in (1.2.7) then

$$E(e^{-v\lambda_{t_1}}) = e^{-\int_{-\infty}^{t_1}\rho(s)[1-\hat{g}\{ve^{-\delta(t_1-s)};s\}]ds}.$$

In order to use $E(e^{-\nu\lambda_{t_1}})$ as the '$-\infty$' asymptotic distribution, it should be shown that the integrated value, $\int_{-\infty}^{t_1} \rho(s)[1-\hat{g}\{\nu e^{-\delta(t_1-s)};s\}]ds$, is smaller than ∞. Otherwise the '$-\infty$' asymptotic distribution approaches zero. In lemma 1.2.3 we proved that the integrated value is smaller than ∞ and so the result follows.

□

Theorem 1.2.5 Let $\delta(t)=\delta$ and the jump size distribution be exponential i.e. $g(y;t)=(\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$. Assuming that $\rho(t)=\rho\dfrac{\alpha}{\alpha+\gamma e^{\delta t}}$ + then

$$E\left\{e^{-\nu\lambda_{t_1}}\middle|\lambda_{t_0}\right\}=e^{-\nu\lambda_0 e^{-\delta(t_1-t_0)}}\left(\frac{\gamma e^{\delta t_0}+\alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0}+\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_0}+\nu e^{-\delta(t_1-t_0)}+\alpha}{\gamma e^{\delta t_0}+(\nu+\alpha)e^{-\delta(t_1-t_0)}}\right)^{\frac{\rho}{\delta}}. \qquad (1.2.8)$$

__Proof__

Use theorem 1.2.2 and

$$\hat{g}\{\nu e^{-\delta(t_1-s)};s\}=\int_0^\infty e^{-\nu e^{-\delta(t_1-s)}y}dG(y;s)=\int_0^\infty e^{-\nu e^{-\delta(t_1-s)}y}(\alpha+\gamma e^{\delta s})e^{-(\alpha+\gamma e^{\delta s})y}dy=\alpha+\gamma e^{\delta s}\int_0^\infty e^{-\{\alpha+\gamma e^{-\delta s}+\nu e^{-\delta(t_1-s)}\}y}dy$$

$$=\alpha+\gamma e^{\delta s}\left[-\frac{e^{-\{\alpha+\gamma e^{-\delta s}+\nu e^{-\delta(t_1-s)}\}y}}{\alpha+\gamma e^{-\delta s}+\nu e^{-\delta(t_1-s)}}\right]_0^\infty=\frac{\alpha+\gamma e^{-\delta s}}{\alpha+\gamma e^{-\delta s}+\nu e^{-\delta(t_1-s)}}.$$

Therefore

$$\int_{t_0}^{t_1}\rho(s)[1-\hat{g}\{\nu e^{-\delta(t_1-s)};s\}]ds=\alpha\rho\int_{t_0}^{t_1}\frac{1}{\alpha+\gamma e^{\delta s}}ds-\alpha\rho\int_{t_0}^{t_1}\frac{1}{\alpha+(\gamma+\nu e^{-\delta t_1})e^{\delta s}}ds$$

$$=\frac{\rho}{\delta}\ln\left(\frac{\gamma e^{\delta t_0}+\alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0}+\alpha}\right)-\frac{\rho}{\delta}\ln\left(\frac{\gamma e^{\delta t_0}+\nu e^{-\delta(t_1-t_0)}+\alpha}{\gamma e^{\delta t_0}+(\nu+\alpha)e^{-\delta(t_1-t_0)}}\right).$$

Hence

$$e^{-\nu\lambda_0 e^{-\delta(t_1-t_0)}}e^{-\int_{t_0}^{t_1}\rho(s)[1-\hat{g}\{\nu e^{-\delta(t_1-s)};s\}]ds}=e^{-\nu\lambda_0 e^{-\delta(t_1-t_0)}}\left(\frac{\gamma e^{\delta t_0}+\alpha e^{-\delta(t_1-t_0)}}{\gamma e^{\delta t_0}+\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_0}+\nu e^{-\delta(t_1-t_0)}+\alpha}{\gamma e^{\delta t_0}+(\nu+\alpha)e^{-\delta(t_1-t_0)}}\right)^{\frac{\rho}{\delta}}.$$

□

Similarly, the '$-\infty$' asymptotic distribution can be obtained from theorem 1.2.5.

+ Note: The reason for this particular assumption will become apparent later when we change the probability measure.

Corollary 1.2.6 Let $\delta(t)=\delta$ and the jump size distribution be exponential i.e. $g(y;t)=(\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$. Assuming that $\rho(t)=\rho\dfrac{\alpha}{\alpha+\gamma e^{\delta t}}$ then the '$-\infty$' asymptotic distribution of λ_t has Laplace transform

$$\left(\frac{\gamma+\alpha e^{-\delta t_1}}{\gamma+(v+\alpha)e^{-\delta t_1}}\right)^{\frac{\rho}{\delta}}. \tag{1.2.9}$$

Proof

We let $t_0\to-\infty$ in (1.2.8) and the corollary follows immediately.

□

Now let us evaluate the mean and variance of λ_t assuming that λ_0 is given.

Theorem 1.2.7 Let λ_t be a generalised shot noise process. Assuming that we know λ_0,

$$E(\lambda_t|\lambda_0)=\lambda_0 e^{-\delta t}+e^{-\delta t}\int_0^t e^{\delta s}\rho(s)\mu_1(s)ds \tag{1.2.10}$$

where $\delta(t)=\delta$.

Proof

The generator of the process λ_t acting on a function $f(\lambda)$ is given by

$$\mathrm{A}f(\lambda)=-\delta(t)\lambda\frac{\partial f}{\partial\lambda}+\rho(t)[\int_0^\infty f(\lambda+y)dG(y;t)-f(\lambda)]. \tag{1.2.11}$$

If we set $f(\lambda)=\lambda$, then

$$\mathrm{A}\,\lambda=-\delta\lambda+\mu_1(t)\rho(t) \tag{1.2.12}$$

where $\delta(t)=\delta$ and $\mu_1(t)=\int_0^\infty y dG(y;t)$.

(If the jump size distribution is exponential i.e. $g(y;t)=(\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$, $\mu_1(t)=\dfrac{1}{\alpha+\gamma e^{\delta t}}$).

From $E(\lambda_t|\lambda_0)-\lambda_0=E[\int_0^t\{\mathrm{A}f(\lambda_s)|\lambda_0\}ds]$

$$E(\lambda_t|\lambda_0)=\lambda_0-\delta\int_0^t E(\lambda_s|\lambda_0)+\int_0^t\rho(s)\mu_1(s)ds.$$

Differentiate w.r.t t

$$\frac{dE(\lambda_t|\lambda_0)}{dt}=-\delta E(\lambda_t|\lambda_0)+\mu_1(t)\rho(t).$$

Solving the differential equation

$$E(\lambda_t|\lambda_0)=\lambda_0 e^{-\delta t}+e^{-\delta t}\int_0^t e^{\delta s}\mu_1(s)\rho(s).$$

□

Lemma 1.2.8 Let λ_t as defined. Assuming that we know λ_0,

$$E\left(\lambda_{t_1}^2|\lambda_{t_0}\right)=\lambda_{t_0}^2 e^{-2\delta(t_1-t_0)}+2e^{-2\delta t_1}\int_{t_0}^{t_1} e^{2\delta s}\mu_1(s)\rho(s)E(\lambda_s|\lambda_{t_0})ds+e^{-2\delta t_1}\int_{t_0}^{t_1} e^{2\delta s}\mu_2(s)\rho(s)ds. \tag{1.2.13}$$

__Proof__

From (1.2.11) and set $f(\lambda)=\lambda^2$ then

$$\mathrm{A}\,\lambda^2=-2\delta(t)\lambda^2+2\mu_1(t)\rho(t)\lambda+\mu_2(t)\rho(t)$$

where $\mu_2(t)=\int_0^\infty y^2 dG(y;t)$.

(If the jump size distribution is exponential i.e. $g(y;t)=(\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$, $\mu_1(t)=\dfrac{1}{\alpha+\gamma e^{\delta t}}$).

From $E(\lambda_{t_1}^2|\lambda_{t_0})-\lambda_{t_0}^2=E[\int_{t_0}^{t_1}(\mathrm{A}\,\lambda_s^2|\lambda_{t_0})ds]$

$$E(\lambda_{t_1}^2|\lambda_{t_0})=\lambda_{t_0}^2-2\delta\int_{t_0}^{t_1}E(\lambda_s^2|\lambda_{t_0})ds+2\int_{t_0}^{t_1}\mu_1(s)\rho(s)E(\lambda_s|\lambda_{t_0})ds+\int_{t_0}^{t_1}\mu_2(s)\rho(s)ds.$$

Differentiate w.r.t t_1

$$\frac{dE\left(\lambda_{t_1}^2|\lambda_{t_0}\right)}{dt_1}=-2\delta E\left(\lambda_{t_1}^2|\lambda_{t_0}\right)+2\mu_1(t_1)\rho(t_1)E\left(\lambda_{t_1}|\lambda_{t_0}\right)+\mu_2(t_1)\rho(t_1).$$

Multiply by $e^{2\delta t_1}$, then

$$\frac{d}{dt_1}\left[e^{2\delta t_1}E\left(\lambda_{t_1}^2|\lambda_{t_0}\right)\right]=e^{2\delta t_1}\left[2\mu_1(t_1)\rho(t_1)E\left(\lambda_{t_1}|\lambda_{t_0}\right)+\mu_2(t_1)\rho(t_1)\right].$$

Solving the differential equation

$$E\left(\lambda_{t_1}^2|\lambda_{t_0}\right)=\lambda_{t_0}^2 e^{-2\delta(t_1-t_0)}+2e^{-2\delta t_1}\int_{t_0}^{t_1} e^{2\delta s}\mu_1(s)\rho(s)E(\lambda_s|\lambda_{t_0})ds+e^{-2\delta t_1}\int_{t_0}^{t_1} e^{2\delta s}\mu_2(s)\rho(s)ds.$$

□

Corollary 1.2.9 Let λ_t as defined. Assuming that we know λ_0,

$$Var\left(\lambda_{t_1} \middle| \lambda_{t_0}\right) = e^{-2\delta t_1} \int_{t_0}^{t_1} e^{2\delta s} \mu_2(s) \rho(s) ds. \tag{1.2.14}$$

__Proof__

$$Var\left(\lambda_{t_1} \middle| \lambda_{t_0}\right) = E\left(\lambda_{t_1}^2 \middle| \lambda_{t_0}\right) - E\left\{\left(\lambda_{t_1} \middle| \lambda_{t_0}\right)\right\}^2.$$

Therefore (1.2.14) follows immediately from (1.2.10) and (1.2.13).

□

Similarly, the '$-\infty$' asymptotic mean and variance can be obtained from theorem 1.2.7 and corollary 1.2.9.

Corollary 1.2.10 Let λ_t as defined. Furthermore if λ_t is '$-\infty$' asymptotic then

$$E(\lambda_{t_1}) = e^{-\delta t_1} \int_{-\infty}^{t_1} e^{\delta s} \rho(s) \mu_1(s) ds. \tag{1.2.15}$$

__Proof__

From (1.2.10)

$$E\left(\lambda_{t_1} \middle| \lambda_{t_0}\right) = \lambda_{t_0} e^{-\delta(t_1 - t_0)} + e^{-\delta t_1} \int_{t_0}^{t_1} e^{\delta s} \rho(s) \mu_1(s) ds. \tag{1.2.16}$$

Letting $t_0 \to -\infty$ in (1.2.16) and the result follows immediately.

□

Corollary 1.2.11 Let λ_t as defined. Assume that we know λ_{t_0} and $\delta(t) = \delta$. Furthermore if λ_t is '$-\infty$' asymptotic then

$$Var\left(\lambda_{t_1}\right) = e^{-2\delta t_1} \int_{-\infty}^{t_1} e^{2\delta s} \mu_2(s) \rho(s) ds \tag{1.2.17}$$

__Proof__

Letting $t_0 \to -\infty$ in (1.2.14) and the result follows immediately.

□

1.3 Time homogeneous shot noise process

We are now going to simplify the shot noise process described in the previous section by allowing the parameters to be homogeneous in time. Therefore the shot noise process, where the decay is exponential δ, which is a constant, can never reach 0. The frequency of jump arrivals follows a Poisson distribution with ρ. We will have generally distributed jump sizes with distribution function $G(y)$ $(y>0)$. If the jump size distribution is exponential its density is $g(y)=\alpha e^{-\alpha y}$, $y>0$, $\alpha>0$.

The results in this section can also be found in Dassios (1987).

The generator of the process (λ_t, t) acting on a function $f(\lambda,t)$ belonging to its domain is given by

$$\mathrm{A}f(\lambda,t)=\frac{\partial f}{\partial t}-\delta\lambda\frac{\partial f}{\partial\lambda}+\rho\{\int_0^\infty f(\lambda+y,t)dG(y)-f(\lambda,t)\}. \tag{1.3.1}$$

For this process we can find the Laplace transform of the distribution of λ_t at any given time t given the distribution of λ_0. We begin with a related theorem that is also used by Dassios (1987).

Theorem 1.3.1 Let $Z_t=\lambda_t e^{\delta t}$. Then

$$e^{-vZ_t}e^{\rho\int_0^t\{1-\hat{g}(ve^{\delta s})\}ds} \qquad \left(\hat{g}(v)=\int_0^\infty e^{-vy}dG(y)\right)$$

is a martingale for all $v\geq 0$.

<u>**Proof**</u>

The result follows immediately from theorem 1.2.1.

□

We will now use the martingale found in theorem 1.3.1.

Theorem 1.3.2 Let λ_t as defined. Then

$$E\{e^{-v\lambda_t}|\lambda_0\}=e^{-v\lambda_0e^{-\delta t}}e^{-\rho\int_0^t\{1-\hat{g}(ve^{-\delta s})\}ds}. \tag{1.3.2}$$

Proof

The result follows immediately from theorem 1.2.2.

□

Also the asymptotic (stationary) distribution can be obtained from theorem 1.3.2.

Corollary 1.3.3 The asymptotic distribution of λ_t has Laplace transform

$$e^{-\rho\int_0^\infty \{1-\hat{g}(ve^{-\delta s})\}ds}$$

which can also be written as

$$e^{-\frac{\rho}{\delta}\int_0^v \hat{G}(u)du}$$

where $\hat{G}(u)=\dfrac{1-\hat{g}(u)}{u}$.

Proof

Let $t\to\infty$ in (1.3.2) and the corollary follows immediately.

□

Theorem 1.3.4 Let the jump size distribution be exponential i.e. $g(y)=\alpha e^{-\alpha y}$, $y>0$, $\alpha>0$. Then

$$E\{e^{-v\lambda_t}|\lambda_0\}=e^{-v\lambda_0 e^{-\delta t}}\left(\frac{\alpha+ve^{-\delta t}}{\alpha+v}\right)^{\frac{\rho}{\delta}}. \tag{1.3.3}$$

Proof

Use theorem 1.3.2 and

$$\hat{g}(ve^{-\delta s})=\int_0^\infty e^{-ve^{-\delta s}y}dG(y)=\int_0^\infty e^{-ve^{-\delta s}y}\alpha e^{-\alpha y}dy=\alpha\int_0^\infty e^{-(\alpha+ve^{-\delta s})y}dy=\alpha\left[-\frac{e^{-(\alpha+ve^{-\delta s})y}}{\alpha+ve^{-\delta s}}\right]_0^\infty$$

$$=\frac{\alpha}{\alpha+ve^{-\delta s}}.$$

Therefore

$$\int_0^t\{1-\hat{g}(ve^{-\delta s})\}ds=\int_0^t(1-\frac{\alpha}{\alpha+ve^{-\delta s}})ds=\int_0^t(\frac{ve^{-\delta s}}{\alpha+ve^{-\delta s}})ds=\frac{1}{\delta}\int_0^t(\frac{\delta e^{-\delta s}}{\frac{\alpha}{v}+e^{-\delta s}})ds$$

$$=\frac{1}{\delta}\ln\left[\frac{\frac{\alpha}{v}+1}{\frac{\alpha}{v}+e^{-\delta t}}\right].$$

Hence

$$e^{-\rho\int_0^t \{1-\hat{g}(ve^{-\delta s})\}ds} = e^{-\frac{\rho}{\delta}\ln\left(\frac{\alpha+v}{\alpha+ve^{-\delta t}}\right)} = \left(\frac{\alpha+ve^{-\delta t}}{\alpha+v}\right)^{\frac{\rho}{\delta}}.$$

□

Similarly, the asymptotic (stationary) distribution can be obtained from theorem 1.3.4

Corollary 1.3.5 Let the jump size distribution be exponential i.e. $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$. The asymptotic distribution of λ_t has Laplace transform

$$\left(\frac{\alpha}{\alpha+v}\right)^{\frac{\rho}{\delta}}.$$

Proof

Let $t \to \infty$ in (1.3.3) and the corollary follows immediately.

□

Now let us evaluate the mean and variance of λ_t assuming that λ_0 is given.

Theorem 1.3.6 Let λ_t as defined. Assuming that we know λ_0 then

$$E(\lambda_t|\lambda_0) = \frac{\mu_1\rho}{\delta} + (\lambda_0 - \frac{\mu_1\rho}{\delta})e^{-\delta t}. \qquad (1.3.4)$$

Proof

The result follows immediately from theorem 1.2.7.

□

Lemma 1.3.7 Let λ_t as defined. Assuming that we know λ_0 then

$$E(\lambda_t^2|\lambda_0) = \lambda_0^2 e^{-2\delta t} + \frac{2\mu_1\rho}{\delta}(\lambda_0 - \frac{\mu_1\rho}{\delta})(e^{-\delta t} - e^{-2\delta t}) + (\frac{\mu_1^2\rho^2}{\delta^2} + \frac{\mu_2\rho}{2\delta})(1 - e^{-2\delta t}). \qquad (1.3.5)$$

Proof

The result follows immediately from lemma 1.2.8.

□

Corollary 1.3.8 Let λ_t as defined. Assuming that we know λ_0 then

$$Var(\lambda_t|\lambda_0) = (1 - e^{-2\delta t})\frac{\mu_2 \rho}{2\delta}. \tag{1.3.6}$$

Proof

$$Var(\lambda_t|\lambda_0) = E(\lambda_t^2|\lambda_0) - \{E(\lambda_t|\lambda_0)\}^2.$$

Therefore (1.3.6) follows immediately from (1.3.5) and (1.3.4).

□

Similarly, the asymptotic (stationary) mean and variance can be obtained from theorem 1.3.6 and corollary 1.3.8.

Corollary 1.3.9 Let N_t, λ_t as defined. Furthermore if λ_t is stationary, that is λ_0 has the stationary distribution then

$$E(\lambda_t) = \frac{\mu_1 \rho}{\delta}. \tag{1.3.7}$$

Proof

Let $t \to \infty$ in (1.3.4) and the corollary follows immediately.

□

Corollary 1.3.10 Let λ_t as defined. If λ_t is stationary then

$$Var(\lambda_t) = \frac{\mu_2 \rho}{2\delta}. \tag{1.3.8}$$

Proof

Let $t \to \infty$ in (1.3.6) and the corollary follows immediately.

□

1.4 Aggregated process

From (1.1.4), we see that $E(N_{t_2} - N_{t_1}) = E(\int_{t_1}^{t_2} \lambda_s ds)$. In other words, the mean number of points in a fixed time interval of length $t_2 - t_1$ is determined by the expected value of $\int_{t_1}^{t_2} \lambda_s ds$. It will be of interest to find the mean and variance of the aggregated process X_t.

As we have seen, the integrated value of the shot noise process λ_t is the aggregated process X_t, i.e. $X_t = \int_0^t \lambda_s ds$. Throughout this section, we assume that the shot noise process is time homogeneous.

Now let us evaluate the mean and variance of $\int_{t_1}^{t_2} \lambda_s ds$ assuming that λ_0 is given.

Theorem 1.4.1 Let λ_t, X_t and N_t as defined. Assuming that we know λ_0, then

$$E(\int_0^t \lambda_s ds | \lambda_0) = E(X_t|\lambda_0) = E(N_t|\lambda_0) = \lambda_0 \frac{1-e^{-\delta t}}{\delta} + \frac{\mu_1 \rho}{\delta}(t - \frac{1-e^{-\delta t}}{\delta}). \tag{1.4.1}$$

<u>**Proof**</u>

From (1.1.3)

$$E(N_t|\lambda_s,\ 0 \le s \le t) = \int_0^t \lambda_s ds. \tag{1.4.2}$$

$$E(N_t) = E\{E(N_t|\lambda_s)\} = E(\int_0^t \lambda_s ds) = \int_0^t E(\lambda_s) ds. \tag{1.4.3}$$

Conditioning on λ_0 in (1.4.3), then

$$E(N_t|\lambda_0) = E(\int_0^t \lambda_s ds | \lambda_0) = \int_0^t E(\lambda_s|\lambda_0) ds. \tag{1.4.4}$$

Therefore (1.4.1) follows immediately from (1.3.4).

□

Lemma 1.4.2 Let N_t, λ_t as defined. Assuming that we know λ_0 then

$$E(N_t\lambda_t|\lambda_0)=\frac{\mu_1^2\rho^2}{\delta^2}t+\frac{\mu_1\rho}{\delta}(\lambda_0-\frac{\mu_1\rho}{\delta})te^{-\delta t}+\frac{\mu_2\rho}{2\delta^2}+\frac{\mu_1\rho}{\delta^2}(\lambda_0-\frac{\mu_1\rho}{\delta})$$
$$+(\frac{\lambda_0^2}{\delta}+\frac{2\mu_1^2\rho^2}{\delta^3}-\frac{3\mu_1\rho}{\delta^2}\lambda_0-\frac{\mu_2\rho}{\delta^2})e^{-\delta t}-(\frac{\lambda_0^2}{\delta}+\frac{\mu_1^2\rho^2}{\delta^3}-\frac{2\mu_1\rho}{\delta^2}\lambda_0-\frac{\mu_2\rho}{2\delta^2})e^{-2\delta t}. \quad (1.4.5)$$

Proof

The generator of (N_t,λ_t) acting on $f(n,\lambda)$ is given by

$$\mathrm{A}f(n,\lambda)=\lambda[f(n+1,\lambda)-f(n,\lambda)]-\delta\lambda\frac{\partial f}{\partial\lambda}+\rho[\int_0^\infty f(n,\lambda+y)dG(y)-f(n,\lambda)]. \quad (1.4.6)$$

If we set $f(n,\lambda)=n\lambda$ in (1.4.6) then

$$\mathrm{A}\,n\lambda=\lambda^2-\delta n\lambda+\rho\mu_1 n.$$

Therefore

$$E(N_t\lambda_t|\lambda_0)=-\delta\int_0^t E(N_s\lambda_s|\lambda_0)ds+\int_0^t E(\lambda_s^2|\lambda_0)ds+\mu_1\rho\int_0^t E(N_s|\lambda_0)ds.$$

Differentiate w.r.t t

$$\frac{dE(N_t\lambda_t|\lambda_0)}{dt}=-\delta E(N_t\lambda_t|\lambda_0)+E(\lambda_t^2|\lambda_0)+\mu_1\rho E(N_t|\lambda_0).$$

Solving the differential equation

$$E(N_t\lambda_t|\lambda_0)=e^{-\delta t}\{\int_0^t e^{\delta s}E(\lambda_s^2|\lambda_0)ds+\mu_1\rho\int_0^t e^{\delta s}E(N_s|\lambda_0)ds\}. \quad (1.4.7)$$

The result follows when we substitute (1.3.5) and (1.4.1) in (1.4.7).

□

Lemma 1.4.3 Let λ_t and X_t as defined. Assuming that we know λ_0, then

$$E\{(\int_0^t \lambda_s ds)^2|\lambda_0\}=E(X_t^2|\lambda_0)$$
$$=\frac{\mu_1^2\rho^2}{\delta^2}t^2+(\frac{2\mu_1\rho}{\delta^2}\lambda_0+\frac{\mu_2\rho}{\delta^2}-\frac{2\mu_1^2\rho^2}{\delta^3})t-(\frac{2\mu_1\rho}{\delta^2}\lambda_0-\frac{2\mu_1^2\rho^2}{\delta^3})te^{-\delta t}$$
$$+(\frac{2\lambda_0^2}{\delta^2}+\frac{2\mu_1^2\rho^2}{\delta^4}-\frac{4\mu_1\rho}{\delta^3}\lambda_0-\frac{2\mu_2\rho}{\delta^3})(1-e^{-\delta t})-(\frac{\lambda_0^2}{\delta^2}+\frac{\mu_1^2\rho^2}{\delta^4}-\frac{2\mu_1\rho}{\delta^3}\lambda_0-\frac{\mu_2\rho}{2\delta^3})(1-e^{-2\delta t}). \quad (1.4.8)$$

Proof

The second moment of $\int_0^t \lambda_s ds$, given λ_0, becomes

$$E\{(\int_0^t \lambda_s ds)^2 | \lambda_0\} = E\{\int_0^t \lambda_s ds \cdot \int_0^t \lambda_r dr | \lambda_0\} = 2\int_0^t \int_0^s E(\lambda_r \lambda_s | \lambda_0) dr ds \tag{1.4.9}$$

where $r \le s$.

Also

$$E(N_s \lambda_s | \lambda_r; 0 \le r \le s) = \lambda_s E(N_s | \lambda_r; 0 \le r \le s). \tag{1.4.10}$$

From (1.4.2)

$$E(N_s \lambda_s | \lambda_r; 0 \le r \le s) = \lambda_s \int_0^s \lambda_r dr$$

$$E(N_s \lambda_s) = \int_0^s E(\lambda_r \lambda_s) dr. \tag{1.4.11}$$

Conditioning on λ_0 in (1.4.11), then

$$E(N_s \lambda_s | \lambda_0) = \int_0^s E(\lambda_r \lambda_s | \lambda_0) dr.$$

Hence (1.4.9) becomes

$$E\{(\int_0^t \lambda_s ds)^2 | \lambda_0\} = 2\int_0^t E(N_s \lambda_s | \lambda_0) ds. \tag{1.4.12}$$

The result follows when we set (1.4.5) in (1.4.12).

□

Corollary 1.4.4 Let λ_t and X_t as defined. Assuming that we know λ_0, then

$$Var(\int_0^t \lambda_s ds | \lambda_0) = Var(X_t | \lambda_0) = \frac{\mu_2 \rho}{\delta^2} t - \frac{2\mu_2 \rho}{\delta^3}(1 - e^{-\delta t}) + \frac{\mu_2 \rho}{2\delta^3}(1 - e^{-2\delta t}). \tag{1.4.13}$$

Proof

$$Var(\int_0^t \lambda_s ds | \lambda_0) = E\{(\int_0^t \lambda_s ds)^2 | \lambda_0\} - \{E(\int_0^t \lambda_s ds | \lambda_0)\}^2.$$

Therefore (1.4.13) follows immediately from (1.4.8) and (1.4.1).

□

Let us also try to obtain the asymptotic (stationary) mean and variance of $\int_{t_1}^{t_2} \lambda_s ds$.

Corollary 1.4.5 Let λ_t and X_t as defined. Furthermore if λ_t is stationary then

$$E(\int_0^t \lambda_s ds) = E(X_t) = E(N_t) = \frac{\mu_1 \rho}{\delta} t. \tag{1.4.14}$$

Proof

Since λ_t is stationary, if we set (1.3.7) in (1.4.3) then the result follows immediately.

□

Lemma 1.4.6 Let λ_t, X_t as defined and λ_t be stationary. Then

$$E(\lambda_t X_t) = \frac{\mu_2 \rho}{2\delta} + \frac{\mu_1^2 \rho^2}{\delta^2} t - \frac{\mu_2 \rho}{2\delta} e^{-\delta t}. \tag{1.4.15}$$

Proof

The aggregated process $X_t = \int_0^t \lambda_s ds$. Therefore

$$E(\lambda_t X_t) = E(\lambda_t \int_0^t \lambda_s ds) = E(\int_0^t \lambda_t \lambda_s ds) = \int_0^t E(\lambda_t \lambda_s) ds, \quad s < t \tag{1.4.16}$$

where

$$\begin{aligned} E(\lambda_t \lambda_s | \lambda_u,\ s \le u) &= \lambda_s E(\lambda_t | \lambda_u,\ s \le u) = \lambda_s E(\lambda_t | \lambda_s) \\ E(\lambda_t \lambda_s) &= E\{E(\lambda_t \lambda_s | \lambda_u,\ s \le u)\} = E\{\lambda_s E(\lambda_t | \lambda_s)\}. \end{aligned} \tag{1.4.17}$$

From (1.3.4)

$$E(\lambda_t \lambda_s) = E\{\lambda_s E(\lambda_t | \lambda_s)\} = E[\lambda_s \{\frac{\mu_1 \rho}{\delta} + (\lambda_s - \frac{\mu_1 \rho}{\delta}) e^{-\delta(t-s)}\}] = E[\lambda_s \frac{\mu_1 \rho}{\delta} + (\lambda_s^2 - \frac{\mu_1 \rho}{\delta} \lambda_s) e^{-\delta(t-s)}]$$

$$= \frac{\mu_1 \rho}{\delta} E(\lambda_s) + e^{-\delta(t-s)} E(\lambda_s^2 - \frac{\mu_1 \rho}{\delta} \lambda_s) = \frac{\mu_1 \rho}{\delta} E(\lambda_s) + e^{-\delta(t-s)} \{E(\lambda_s^2) - \frac{\mu_1 \rho}{\delta} E(\lambda_s)\}.$$

Since λ_t is stationary, from (1.3.7) and (1.3.8)

$$\begin{aligned} &= \frac{\mu_1 \rho}{\delta} \frac{\mu_1 \rho}{\delta} + e^{-\delta(t-s)} \{(\frac{\mu_1 \rho}{\delta})^2 + \frac{\mu_2 \rho}{2\delta} - \frac{\mu_1 \rho}{\delta} \frac{\mu_1 \rho}{\delta}\} \\ &= (\frac{\mu_1 \rho}{\delta})^2 + (\frac{\mu_2 \rho}{2\delta}) e^{-\delta(t-s)}. \end{aligned} \tag{1.4.18}$$

Hence

$$E(\lambda_t X_t) = \int_0^t E(\lambda_t \lambda_s)ds = \int_0^t \{(\frac{\mu_1\rho}{\delta})^2 + (\frac{\mu_2\rho}{2\delta})e^{-\delta(t-s)}\}ds = (\frac{\mu_1\rho}{\delta})^2 t + (\frac{\mu_2\rho}{2\delta})e^{-\delta t}\int_0^t e^{\delta s}ds$$

$$= (\frac{\mu_1\rho}{\delta})^2 t + (\frac{\mu_2\rho}{2\delta})e^{-\delta t}\frac{1}{\delta}\int_0^t \delta e^{\delta s}ds = (\frac{\mu_1\rho}{\delta})^2 t + (\frac{\mu_2\rho}{2\delta^2})e^{-\delta t}(e^{\delta t} - 1)$$

$$= \frac{\mu_2\rho}{2\delta^2} + (\frac{\mu_1\rho}{\delta})^2 t - \frac{\mu_2\rho}{2\delta^2}e^{-\delta t}.$$

□

Lemma 1.4.7 Let λ_t, X_t as defined and λ_t be stationary. Then

$$E\{(\int_0^t \lambda_s ds)^2\} = E(X_t^2) = 2\int_0^t E(\lambda_s X_s)ds = (\frac{\mu_1\rho}{\delta})^2 t^2 + (\frac{\mu_2\rho}{\delta^2})t + \frac{\mu_2\rho}{\delta^3}e^{-\delta t} - \frac{\mu_2\rho}{\delta^3}. \quad (1.4.19)$$

<u>**Proof**</u>

The aggregated process $X_t = \int_0^t \lambda_s ds$. Therefore

$$E\{(\int_0^t \lambda_s ds)^2\} = E(X_t^2) = E\{\int_0^t \lambda_s ds \int_0^t \lambda_r dr\} = E\{2\int_0^t\int_0^s \lambda_r \lambda_s drds\} = 2\int_0^t\int_0^s E(\lambda_r \lambda_s)drds.$$

From (1.4.16)

$$E\{(\int_0^t \lambda_s ds)^2\} = 2\int_0^t E(\lambda_s X_s)ds. \quad (1.4.20)$$

Hence from lemma 1.4.6

$$E\{(\int_0^t \lambda_s ds)^2\} = E(X_t^2) = 2\int_0^t E(\lambda_s X_s)ds = 2\int_0^t (\frac{\mu_2\rho}{2\delta^2} + \frac{\mu_1^2\rho^2}{\delta^2}s - \frac{\mu_2\rho}{2\delta^2}e^{-\delta s})ds$$

$$= 2\{\frac{\mu_2\rho}{2\delta^2}t + (\frac{\mu_1\rho}{\delta})^2\frac{1}{2}t^2 - \frac{\mu_2\rho}{2\delta^2}(-\frac{1}{\delta})(e^{-\delta t} - 1)\}$$

$$= (\frac{\mu_1\rho}{\delta})^2 t^2 + (\frac{\mu_2\rho}{\delta^2})t + \frac{\mu_2\rho}{\delta^3}e^{-\delta t} - \frac{\mu_2\rho}{\delta^3}.$$

□

Corollary 1.4.8 Let λ_t and X_t as defined. If λ_t is stationary then

$$Var(\int_0^t \lambda_s ds) = Var(X_t) = \frac{\mu_2\rho}{\delta^2}t + \frac{\mu_2\rho}{\delta^3}e^{-\delta t} - \frac{\mu_2\rho}{\delta^3}. \quad (1.4.21)$$

Proof

$$Var(\int_0^t \lambda_s ds) = E\{(\int_0^t \lambda_s ds)^2\} - \{E(\int_0^t \lambda_s ds)\}^2. \tag{1.4.22}$$

Since λ_t is stationary, if we set (1.4.19) and (1.4.14) in (1.4.22) then the result follows immediately.

□

2. The Cox Process with Shot Noise Intensity

This chapter deals with the Cox process incorporating the shot noise process as its intensity function.

In section 1, Laplace transforms of the distributions of $N_{t_2} - N_{t_1}$ and $X_{t_2} - X_{t_1} = \int_{t_1}^{t_2} \lambda_s ds$ will be obtained. We will then specialise our analysis of this Laplace transform where the jump size distribution is exponential. We will also examine the asymptotic behaviour of X_t and N_t where λ_t is assumed to be stationary in both these cases. As a matter of interest we discuss the higher order properties of X_t such as the joint distribution of $X_{t_2} - X_{t_1}, X_{t_3} - X_{t_2}, \cdots, X_{t_n} - X_{t_{n-1}}$ at specified times $t_1, t_2, \cdots, t_n$.

The mean, variance and covariance of $N_{t_2} - N_{t_1}$, assuming that λ_t is stationary, will be evaluated in section 2. As a matter of interest we also discuss the higher order properties of N_t such as the joint distribution of $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \cdots, N_{t_n} - N_{t_{n-1}}$ at specified times $t_1, t_2, \cdots, t_n$.

In section 3, the distribution of the intensity at point times, λ_T, where T is a time at which a point of N_t has occurred, will be derived assuming that λ_t is stationary. It will be of interest to compare the distribution of the intensity of point times, λ_T with the distribution of λ_t.

We look at the interarrival time between points and its distribution in section 4, in particular we will examine $\tau_1 \equiv \inf\{t: N_t = 1 | N_0 = 0\}$ assuming that λ_t is time homogeneous. The mean and variance of interarrival time between points will be shown when stationary has been achieved.

2.1 The distribution of number of points in a fixed time interval

2.1.1 Time homogeneous case

In this section we assume a time homogeneous shot noise process; this is discussed in detail in section 3 of Chapter 1. Therefore λ_t is a shot noise process with rate of decay δ, rate of jump arrivals ρ and jump size distribution function $G(y)$ $(y > 0)$. If the jump size distribution is exponential its density is given by $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$.

Now we will obtain the distribution of N_t which is a doubly stochastic Poisson process with intensity function λ_t. To do so we will first examine the aggregated process X_t and then use the relationship (1.1.9) to deduce results about N_t. We will therefore evaluate the joint Laplace transform of the distribution of X_t and N_t at time t.

The generator of the process (X_t, λ_t, t) acting on a function $f(x,\lambda,t)$ belonging to its domain is given by

$$\mathrm{A}f(x,\lambda,t) = \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho[\int_0^\infty f(x,\lambda+y,t)dG(y) - f(x,\lambda,t)]. \qquad (2.1.1)$$

Also the generator of (N_t, λ_t, t) acting on $f(n,\lambda,t)$ is given by

$$\mathrm{A}f(n,\lambda,t) = \frac{\partial f}{\partial t} + \lambda[f(n+1,\lambda,t) - f(n,\lambda,t)] - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho[\int_0^\infty f(n,\lambda+y,t)dG(y) - f(n,\lambda,t)].$$

(2.1.2)

Let us begin with a theorem also used by Dassios (1987). Our proof is somewhat shorter.

Theorem 2.1.1 Let X_t and λ_t as defined above and evolving up to a fixed time t^*. We will assume that the constants k_1 and k_2 are such that $k_1 \geq 0$ and $k_2 \geq -k_1 e^{-\delta t^*}$; then

$$e^{-k_1\delta X_t} e^{-(k_1+k_2e^{\delta t})\lambda_t} e^{\rho\int_0^t \{1-\hat{g}(k_1+k_2e^{\delta s})\}ds} \qquad \left(\hat{g}(v) = \int_0^\infty e^{-vy}dG(y)\right) \qquad (2.1.3)$$

is a martingale.

<u>**Proof**</u>

Define $W_t = \delta X_t + \lambda_t$ and $Z_t = \lambda_t e^{\delta t}$, then the generator of the process (W_t, Z_t, t) acting on a function $f(w,z,t)$ is given by

$$\mathrm{A}f(w,z,t) = \frac{\partial f}{\partial t} + \rho[\int_0^\infty f(w+y, z+ye^{\delta t}, t)dG(y) - f(w,z,t)]. \qquad (2.1.4)$$

and $f(w,z,t)$ has to satisfy $\mathrm{A}f = 0$ for $f(W_t, Z_t, t)$ to be a martingale. Setting $e^{-k_1 w}e^{-k_2 z}h(t)$ we get the equation

$$h'(t) - \rho[1 - \hat{g}(k_1 + k_2e^{\delta t})]h(t) = 0. \qquad (2.1.5)$$

$e^{-k_1 w}e^{-k_2 z}h(t)$ belongs to the domain of the generator because of our choice of k_1, k_2; the function is bounded for all $t \leq t^*$ and our process evolves up to time t^* only.

Solving (2.1.5)

$$h(t) = \mathrm{K}e^{\rho\int_0^t \{1-\hat{g}(k_1+k_2e^{\delta s})\}ds} \tag{2.1.6}$$

where K is an arbitrary constant.

Therefore

$$e^{-k_1W_t}e^{-k_2Z_t}e^{\rho\int_0^t \{1-\hat{g}(k_1+k_2e^{\delta s})\}ds}$$

is a martingale and hence (2.1.3) is a martingale.

□

Theorem 2.1.2 Let the jump size distribution be exponential i.e. $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$. Then

$$e^{-k_1\delta X_t}e^{-(k_1+k_2e^{\delta t})\lambda_t}e^{\rho t}\{\frac{k_2+(\alpha+k_1)e^{-\delta t}}{k_2+\alpha+k_1}\}^{\frac{\alpha\rho}{\delta(\alpha+k_1)}}. \tag{2.1.7}$$

is a martingale.

Proof

Since the jump size distribution is exponential i.e. $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$, from theorem 2.1.1

$$\hat{g}(k_1+k_2e^{\delta s}) = \int_0^\infty e^{-(k_1+k_2e^{\delta s})y}dG(y) = \int_0^\infty e^{-(k_1+k_2e^{\delta s})y}\alpha e^{-\alpha y}dy = \alpha\int_0^\infty e^{-\{\alpha+(k_1+k_2e^{\delta s})\}y}dy = \alpha\left[-\frac{e^{-\{\alpha+(k_1+k_2e^{\delta s})\}y}}{\alpha+(k_1+k_2e^{\delta s})}\right]_0^\infty$$

$$= \frac{\alpha}{\alpha+k_1+k_2e^{\delta s}}.$$

Therefore

$$\int_0^t \{1-\hat{g}(k_1+k_2e^{\delta s})\}ds = \int_0^t (1-\frac{\alpha}{\alpha+k_1+k_2e^{\delta s}})ds = t - \frac{\alpha}{\delta(\alpha+k_1)}\int_0^t \{\frac{\delta e^{-\delta s}}{e^{-\delta s}+\frac{k_2}{\alpha+k_1}}\}ds$$

$$= t - \frac{\alpha}{\delta(\alpha+k_1)}\ln\left[\frac{k_2+\alpha+k_1}{k_2+(\alpha+k_1)e^{-\delta t}}\right].$$

Hence

$$e^{\rho\int_0^t \{1-\hat{g}(k_1+k_2e^{\delta s})\}ds} = e^{\rho\left\{t-\frac{\alpha}{\delta(\alpha+k_1)}\ln(\frac{k_2+\alpha+k_1}{k_2+(\alpha+k_1)e^{-\delta t}})\right\}} = e^{\rho t}e^{-\frac{\alpha\rho}{\delta(\alpha+k_1)}\ln(\frac{k_2+\alpha+k_1}{k_2+(\alpha+k_1)e^{-\delta t}})}$$

$$= e^{\rho t}\{\frac{k_2+(\alpha+k_1)e^{-\delta t}}{k_2+\alpha+k_1}\}^{\frac{\alpha\rho}{\delta(\alpha+k_1)}}. \tag{2.1.8}$$

Set (2.1.8) in (2.1.3) and the result follows immediately.

□

Corollary 2.1.3 Let $\nu_1 \geq 0,\ \nu_2 \geq 0,\ \nu \geq 0,\ 0 \leq \theta \leq 1$ and $t_1,\ t_2$ be fixed times. Then

$$E\{e^{-\nu_1(X_{t_2}-X_{t_1})}e^{-\nu_2\lambda_{t_2}} \big| X_{t_1},\lambda_{t_1}\} = E\{e^{-\nu_1(X_{t_2}-X_{t_1})}e^{-\nu_2\lambda_{t_2}} \big| \lambda_{t_1}\} = E\{e^{-\nu_1\int_{t_1}^{t_2}\lambda_s ds} e^{-\nu_2\lambda_{t_2}} \big| \lambda_{t_1}\}$$
$$= e^{-\{\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta s}\}]ds} \tag{2.1.9}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}} \big| N_{t_1},\lambda_{t_1}\} = E\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}e^{-\nu\lambda_{t_2}} \big| X_{t_1},\lambda_{t_1}\} = E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}} \big| \lambda_{t_1}\}$$
$$= e^{-\{\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})e^{-\delta s}\}]ds}. \tag{2.1.10}$$

__**Proof**__

We set $k_1 = \frac{\nu_1}{\delta},\ k_2 = (\nu_2 - \frac{\nu_1}{\delta})e^{-\delta t_2},\ t^* \geq t_2$ in theorem 2.1.1 and (2.1.9) follows immediately.

(2.1.10) follows from (2.1.9) and (1.1.9).

□

Corollary 2.1.4 Let $\nu_1 \geq 0,\ \nu_2 \geq 0,\ \nu \geq 0,\ 0 \leq \theta \leq 1$ and $t_1,\ t_2$ be fixed times. Let the jump size distribution be exponential, i.e. $g(y) = \alpha e^{-\alpha y},\ y > 0,\ \alpha > 0$. Then

$$E\{e^{-\nu_1(X_{t_2}-X_{t_1})}e^{-\nu_2\lambda_{t_2}} \big| X_{t_1},\lambda_{t_1}\} = E\{e^{-\nu_1(X_{t_2}-X_{t_1})}e^{-\nu_2\lambda_{t_2}} \big| \lambda_{t_1}\} = E\{e^{-\nu_1\int_{t_1}^{t_2}\lambda_s ds} e^{-\nu_2\lambda_{t_2}} \big| \lambda_{t_1}\}$$
$$= e^{-\{\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho(t_2-t_1)} \{\frac{(\nu_2-\frac{\nu_1}{\delta})e^{-\delta(t_2-t_1)}+\alpha+\frac{\nu_1}{\delta}}{(\nu_2+\alpha)e^{-\delta(t_2-t_1)}}\}^{\frac{\alpha\rho}{\delta\alpha+\nu_1}} \tag{2.1.11}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}} \big| N_{t_1},\lambda_{t_1}\} = E\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}e^{-\nu\lambda_{t_2}} \big| X_{t_1},\lambda_{t_1}\} = E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}} \big| \lambda_{t_1}\}$$
$$= e^{-\{\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho(t_2-t_1)} \{\frac{(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}+\alpha+\frac{1-\theta}{\delta}}{(\nu+\alpha)e^{-\delta(t_2-t_1)}}\}^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \tag{2.1.12}$$

Proof

We set $k_1 = \frac{v_1}{\delta}$, $k_2 = (v_2 - \frac{v_1}{\delta})e^{-\delta t_2}$, $t^* \geq t_2$ in theorem 2.1.2 and (2.1.11) follows immediately.

(2.1.12) follows from (2.1.11) and (1.1.9).

□

Corollary 2.1.5 Let X_t, N_t, λ_t as defined; then

$$E\{e^{-v(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\} = E\{e^{-v\int_{t_1}^{t_2}\lambda_s ds}\big|\lambda_{t_1}\} = e^{-\frac{v}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\frac{v}{\delta}(1-e^{-\delta s})\}]ds} \tag{2.1.13}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\big|\lambda_{t_1}\} = E\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\} = e^{-\frac{1-\theta}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\frac{1-\theta}{\delta}(1-e^{-\delta s})\}]ds}. \tag{2.1.14}$$

Proof

If we set $v_2 = 0$, $v = 0$ in (2.1.9) and (2.1.10) then (2.1.13) and (2.1.14) follow.

□

Corollary 2.1.6 Let X_t, N_t, λ_t as defined. Let the jump size distribution be exponential i.e. $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$. Then

$$E\{e^{-v(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\} = E\{e^{-v\int_{t_1}^{t_2}\lambda_s ds}\big|\lambda_{t_1}\} = E\{e^{-v\int_0^{t_2-t_1}\lambda_s ds}\big|\lambda_{t_1}\} = e^{-\frac{v}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho(t_2-t_1)} \left\{\frac{\alpha + \frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})}{\alpha e^{-\delta(t_2-t_1)}}\right\}^{\frac{\alpha\rho}{\delta\alpha+v}} \tag{2.1.15}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\big|\lambda_{t_1}\} = E\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\} = e^{-\frac{1-\theta}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho(t_2-t_1)} \left\{\frac{\alpha + \frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}{\alpha e^{-\delta(t_2-t_1)}}\right\}^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \tag{2.1.16}$$

Proof

If we set $\nu_2 = 0$, $\nu = 0$ in (2.1.11) and (2.1.12) then (2.1.15) and (2.1.16) follow.

□

Corollary 2.1.7 Let X_t, N_t, λ_t as defined. Furthermore if λ_t is stationary then

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\} = E\{e^{-\nu\int_{t_1}^{t_2}\lambda_s ds}\} = e^{-\frac{\nu\rho}{\delta}\int_0^{t_2-t_1}\hat{G}\{\frac{\nu}{\delta}(1-e^{-\delta s})\}ds} \tag{2.1.17}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\} = E\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}\} = e^{-\frac{(1-\theta)\rho}{\delta}\int_0^{t_2-t_1}\hat{G}\{\frac{1-\theta}{\delta}(1-e^{-\delta s})\}ds} \tag{2.1.18}$$

where $\hat{G}(u) = \dfrac{1-\hat{g}(u)}{u}$.

Proof

From corollary 2.1.5

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\} = E\{E\{e^{-\nu(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\}\} = E[e^{-\frac{\nu}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}e^{-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\frac{\nu}{\delta}(1-e^{-\delta s})\}]ds}]$$

$$= E[e^{-\frac{\nu}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}]e^{-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\frac{\nu}{\delta}(1-e^{-\delta s})\}]ds}.$$

From corollary 1.3.3, when λ_t is stationary it has a distribution with Laplace transform

$$e^{-\frac{\rho}{\delta}\int_0^{\nu}\hat{G}(u)du}.$$

Therefore

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\} = e^{-\frac{\rho}{\delta}\int_0^{\frac{\nu}{\delta}\{1-e^{-\delta(t_2-t_1)}\}}\hat{G}(u)du}\cdot e^{-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\frac{\nu}{\delta}(1-e^{-\delta s})\}]ds} = e^{-\rho\int_0^{t_2-t_1}[\{\frac{1-\hat{g}\{\frac{\nu}{\delta}(1-e^{-\delta s})\}}{1-e^{-\delta s}}\}e^{-\delta s}]ds}\cdot e^{-\rho\int_0^{t_2-t_1}[1-\hat{g}\{\frac{\nu}{\delta}(1-e^{-\delta s})\}]ds}$$

$$= e^{-\rho\int_0^{t_2-t_1}[\frac{1-\hat{g}\{\frac{\nu}{\delta}(1-e^{-\delta s})\}}{1-e^{-\delta s}}]ds} = e^{-\frac{\nu\rho}{\delta}\int_0^{t_2-t_1}\hat{G}\{\frac{\nu}{\delta}(1-e^{-\delta s})\}ds}.$$

Similarly (2.1.18) follows.

□

Corollary 2.1.8 Let X_t, N_t, λ_t as defined. Let the jump size distribution be exponential i.e. $g(y)=\alpha e^{-\alpha y}$, $y>0$, $\alpha>0$. Furthermore if λ_t is stationary then

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\}=E\{e^{-\nu\int_{t_1}^{t_2}\lambda_s ds}\}=E\{e^{-\nu\int_{0}^{t_2-t_1}\lambda_s ds}\}$$

$$=e^{-\rho(t_2-t_1)}\{\frac{\alpha}{\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}\}^{\frac{\rho}{\delta}}\{\frac{\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}{\alpha e^{-\delta(t_2-t_1)}}\}^{\frac{\alpha\rho}{\delta\alpha+\nu}}$$

$$=\{\frac{\alpha e^{-\delta(t_2-t_1)}}{\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}\}^{\frac{\rho}{\delta}}\{\frac{\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}{\alpha e^{-\delta(t_2-t_1)}}\}^{\frac{\alpha\rho}{\delta\alpha+\nu}} \qquad (2.1.19)$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\}=E\{e^{-(1-\theta)(X_{t_2}-X_{t_1})}\}$$

$$=e^{-\rho(t_2-t_1)}\{\frac{\alpha}{\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}\}^{\frac{\rho}{\delta}}\{\frac{\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}{\alpha e^{-\delta(t_2-t_1)}}\}^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}$$

$$=\{\frac{\alpha e^{-\delta(t_2-t_1)}}{\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}\}^{\frac{\rho}{\delta}}\{\frac{\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}{\alpha e^{-\delta(t_2-t_1)}}\}^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \qquad (2.1.20)$$

__Proof__

From corollary 2.1.6

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\}=E\{E\{e^{-\nu(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\}\}=E[e^{-\frac{\nu}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}e^{-\rho(t_2-t_1)}\{\frac{\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}{\alpha e^{-\delta(t_2-t_1)}}\}^{\frac{\alpha\rho}{\delta\alpha+\nu}}]$$

$$=E[e^{-\frac{\nu}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}]e^{-\rho(t_2-t_1)}\{\frac{\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}{\alpha e^{-\delta(t_2-t_1)}}\}^{\frac{\alpha\rho}{\delta\alpha+\nu}}.$$

Therefore, since λ_t is stationary and from corollary 1.3.5, (2.1.19) follows immediately. Similarly (2.1.20) follows.

□

We will now close this section by evaluating the joint Laplace transform of the distribution of $X_{t_2}-X_{t_1}, X_{t_3}-X_{t_2},\cdots,X_{t_n}-X_{t_{n-1}}$ ($n-1$ successive increments). Using the important corollary 2.1.3 and 2.1.5 we can derive higher order properties of X_t.

Lemma 2.1.9 Let X_t, λ_t as defined. $\nu_{n-1},\nu_{n-2},\cdots\cdots,\nu_1$ and γ_0 are non-negative constants. Then

$$E\{e^{-\nu_{n-1}(X_{t_2}-X_{t_1})}e^{-\nu_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-\nu_1(X_{t_n}-X_{t_{n-1}})}e^{-\gamma_0\lambda_{t_n}}\big|\lambda_{t_1}\}=\phi_1\phi_2\cdots\cdots\phi_{n-1}e^{-\{\frac{\nu_{n-1}}{\delta}+(\gamma_{n-2}-\frac{\nu_{n-1}}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}$$

(2.1.21)

where $\phi_i = e^{-\rho \int_0^{t_{n-i+1}-t_{n-i}} [1-\hat{g}\{\frac{v_i}{\delta}+(\gamma_{i-1}-\frac{v_i}{\delta})e^{-\delta s}\}]ds}$, $\gamma_i = \frac{v_i}{\delta} + K_i$ and $K_i = (\gamma_{i-1} - \frac{v_i}{\delta})e^{-\delta(t_{n-i+1}-t_{n-i})}$ for $i = 1,2,\cdots,n-1$.

Proof

$$E\{e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_1(X_{t_n}-X_{t_{n-1}})}e^{-\gamma_0\lambda_{t_n}} \mid \lambda_s; 0 \le s \le t_{n-1}\}$$

$$= e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_2(X_{t_{n-1}}-X_{t_{n-2}})}E\{e^{-v_1(X_{t_n}-X_{t_{n-1}})}e^{-\gamma_0\lambda_{t_n}} \mid \lambda_{t_{n-1}}\}. \tag{2.1.22}$$

Therefore from corollary 2.1.3

$$= e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_2(X_{t_{n-1}}-X_{t_{n-2}})}e^{-\{\frac{v_1}{\delta}+(\gamma_0-\frac{v_1}{\delta})e^{-\delta(t_n-t_{n-1})}\}\lambda_{t_{n-1}}} e^{-\rho \int_0^{t_n-t_{n-1}} [1-\hat{g}\{\frac{v_1}{\delta}+(\gamma_0-\frac{v_1}{\delta})e^{-\delta s}\}]ds}. \tag{2.1.23}$$

Put $\phi_1 = e^{-\rho \int_0^{t_n-t_{n-1}} [1-\hat{g}\{\frac{v_1}{\delta}+(\gamma_0-\frac{v_1}{\delta})e^{-\delta s}\}]ds}$, $\gamma_1 = \frac{v_1}{\delta} + K_1$ and $K_1 = (\gamma_0 - \frac{v_1}{\delta})e^{-\delta(t_n-t_{n-1})}$ in (2.1.23), then

$$= \phi_1 e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_2(X_{t_{n-1}}-X_{t_{n-2}})}e^{-\gamma_1\lambda_{t_{n-1}}}. \tag{2.1.24}$$

Hence

$$E\{e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_1(X_{t_n}-X_{t_{n-1}})}e^{-\gamma_0\lambda_{t_n}}\}$$

$$= \phi_1 E\{e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_2(X_{t_{n-1}}-X_{t_{n-2}})}e^{-\gamma_1\lambda_{t_{n-1}}}\}. \tag{2.1.25}$$

Taking the same procedure from (2.1.22) to (2.1.25) recursively, then

$$E\{e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_1(X_{t_n}-X_{t_{n-1}})}e^{-\gamma_0\lambda_{t_n}}\}$$

$$= \phi_1\phi_2\cdots\cdots\phi_{n-2}E[e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-\{\frac{v_{n-2}}{\delta}+(\gamma_{n-3}-\frac{v_{n-2}}{\delta})e^{-\delta(t_3-t_2)}\}\lambda_{t_2}}]. \tag{2.1.26}$$

(2.1.21) follows where we condition λ_{t_1} in (2.1.26).

□

Corollary 2.1.10 Let X_t, λ_t as defined. $v_{n-1}, v_{n-2}, \cdots\cdots, v_1$ and γ_0 are non-negative constants. Then

$$E\{e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_1(X_{t_n}-X_{t_{n-1}})}\big|\lambda_{t_1}\} = \phi_1'\phi_2'\cdots\cdots\phi_{n-1}'e^{-\{\frac{v_{n-1}}{\delta}+(\gamma_{n-2}'-\frac{v_{n-1}}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} \tag{2.1.27}$$

where $\phi_1' = e^{-\rho\int_0^{t_n-t_{n-1}}[1-\hat{g}\{\frac{v_1}{\delta}(1-e^{-\delta s})\}]ds}$, $\gamma_1' = \frac{V_1}{\delta}+K_1'$, $K_1' = -\frac{V_1}{\delta}e^{-\delta(t_n-t_{n-1})}$ and

$$\phi_i' = e^{-\rho\int_0^{t_{n-i+1}-t_{n-i}}[1-\hat{g}\{\frac{v_i}{\delta}+\{\frac{v_{i-1}}{\delta}(1-e^{-\delta(t_{n-i+2}-t_{n-i+1})})-\frac{v_i}{\delta}\}e^{-\delta s}\}]ds},$$

$$\gamma_i' = \frac{V_i}{\delta}+K_i' = \frac{V_i}{\delta}+[\frac{V_{i-1}}{\delta}\{1-e^{-\delta(t_{n-i+2}-t_{n-i+1})}\}-\frac{V_i}{\delta}]e^{-\delta(t_{n-i+1}-t_{n-i})} \;,\quad K_i' = (\gamma_{i-1}'-\frac{V_i}{\delta})e^{-\delta(t_{n-i+1}-t_{n-i})}$$

for $i = 2,3,\cdots,n-1$.

Proof

Set $\gamma_0 = 0$ in (2.1.22) and take the same procedures as lemma 2.1.9, using corollary 2.1.5 then the result follows.

□

2.1.2 Time dependent parameters

In this section we assume a shot noise process that is dependent on time; this is discussed in detail in section 2 of Chapter 1. Therefore λ_t is a generalised shot noise process with rate of decay $\delta(t)$, rate of jump arrivals $\rho(t)$ and jump size distribution function $G(y;t)$ $(y>0)$. If the jump size distribution is exponential its density is $g(y;t) = (\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\gamma > -\alpha e^{-\delta t}$.

The generator of the process (X_t, λ_t, t) acting on a function $f(x,\lambda,t)$ belonging to its domain is given by

$$\mathrm{A}f(x,\lambda,t) = \frac{\partial f}{\partial t}+\lambda\frac{\partial f}{\partial x}-\delta(t)\lambda\frac{\partial f}{\partial \lambda}+\rho(t)[\int_0^\infty f(x,\lambda+y,t)dG(y;t)-f(x,\lambda,t)]. \tag{2.1.28}$$

Also the generator of (N_t, λ_t, t) acting on $f(n,\lambda,t)$ is given by

$$\mathrm{A} f(n,\lambda,t) = \frac{\partial f}{\partial t} + \lambda[f(n+1,\lambda,t) - f(n,\lambda,t)] - \delta(t)\lambda\frac{\partial f}{\partial \lambda} + \rho(t)[\int_0^\infty f(n,\lambda+y,t)dG(y;t) - f(n,\lambda,t)].$$

(2.1.29)

Similar to the previous section, let us evaluate the Laplace transform of the distribution of X_t and N_t at time t.

Theorem 2.1.11 Let X_t and λ_t as defined. Also consider constants k and v such that $k \geq 0$ and $v \geq 0$; then

$$e^{-vX_t} \cdot e^{-\{ke^{\Delta(t)} - ve^{\Delta(t)}\int_0^t e^{-\Delta(r)}dr\}\lambda_t} \cdot e^{\int_0^t \rho(s)[1-\hat{g}\{ke^{\Delta(s)} - ve^{\Delta(s)}\int_0^s e^{-\Delta(r)}dr;s\}]ds} \qquad \left(\hat{g}(u;s) = \int_0^\infty e^{-uy}dG(y;s)\right)$$

(2.1.30)

is a martingale where $\Delta(t) = \int_0^t \delta(s)ds$.

<u>Proof</u>

From (2.1.28) $f(x,\lambda,t)$ has to satisfy $\mathrm{A}f = 0$ for it to be a martingale. Setting $f = e^{-vx}e^{-A(t)\lambda}e^{R(t)}$ we get the equation

$$-\lambda A'(t) + R'(t) - \lambda v + \delta(t)\lambda A(t) + \rho(t)[\hat{g}\{A(t);t\} - 1] = 0 \qquad (2.1.31)$$

and solving (2.1.31) we get

$$A(t) = ke^{\Delta(t)} - ve^{\Delta(t)}\int_0^t e^{-\Delta(r)}dr \text{ and } R(t) = \int_0^t \rho(s)[1 - \hat{g}\{ke^{\Delta(s)} - ve^{\Delta(s)}\int_0^s e^{-\Delta(r)}dr;s\}]ds.$$

Put $\Delta(t) = \int_0^t \delta(s)ds$ and the result follows.

□

Theorem 2.1.12 Let the jump size distribution be exponential i.e. $g(y;t) = (\alpha + \gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y > 0$, $\gamma > -\alpha e^{-\delta t}$ and $\rho(t) = \rho\frac{\alpha}{\alpha + \gamma e^{\delta t}}$. Assuming that $\delta(t) = \delta$, then

$$
e^{-\nu X_t} e^{-\{\frac{\nu}{\delta}+(k-\frac{\nu}{\delta})e^{\delta t}\}\lambda_t} \left(\frac{\gamma+\alpha}{\gamma+\alpha e^{-\delta t}} \right)^{\frac{\rho}{\delta}} \left(\frac{(\gamma+k-\frac{\nu}{\delta})+(\alpha+\frac{\nu}{\delta})e^{-\delta t}}{\gamma+k+\alpha} \right)^{\frac{\alpha\rho}{\delta\alpha+\nu}} \tag{2.1.32}
$$

is a martingale.

Proof

This theorem can be proved in a similar method to theorem 2.1.2. In this case, if we set $\rho(t)=\rho\dfrac{\alpha}{\alpha+\gamma e^{\delta t}}$ and $g(y;t)=(\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$ the result follows.

□

Corollary 2.1.13 Assume that $\delta(t)=\delta$. Let $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\nu \geq 0$, $0 \leq \theta \leq 1$. Then

$$
E\{e^{-\nu_1(X_{t_2}-X_{t_1})}e^{-\nu_2\lambda_{t_2}} \,|\, X_{t_1},\lambda_{t_1}\} = e^{-\{\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta(t_2-s)};s\}]ds} \tag{2.1.33}
$$

and

$$
E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}} \,|\, N_{t_1},\lambda_{t_1}\} = e^{-\{\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-s)};s\}]ds} . \tag{2.1.34}
$$

Proof

(2.1.33) follows immediately where we set $\nu=\nu_1$, $k=\dfrac{\nu_1}{\delta}+(\nu_2-\dfrac{\nu_1}{\delta})e^{-\delta t_2}$ in theorem 2.1.11. (2.1.34) follows from (2.1.33) and (1.1.9).

□

Corollary 2.1.14 Let $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\nu \geq 0$, $0 \leq \theta \leq 1$ and the jump size distribution be exponential i.e. $g(y;t)=(\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$. Assuming that $\delta(t)=\delta$ then

$$
E\{e^{-\nu_1(X_{t_2}-X_{t_1})}e^{-\nu_2\lambda_{t_2}} \,|\, X_{t_1},\lambda_{t_1}\} = e^{-\{\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha} \right)^{\frac{\rho}{\delta}} \left(\frac{\gamma e^{\delta t_1}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta(t_2-t_1)}+\alpha+\frac{\nu_1}{\delta}}{\gamma e^{\delta t_1}+(\nu_2+\alpha)e^{-\delta(t_2-t_1)}} \right)^{\frac{\alpha\rho}{\delta\alpha+\nu_1}} \tag{2.1.35}
$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}}\big|N_{t_1},\lambda_{t_1}\} = e^{-\{\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_1}+(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}+\alpha+\frac{1-\theta}{\delta}}{\gamma e^{\delta t_1}+(\nu+\alpha)e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}.$$

(2.1.36)

Proof

(2.1.35) follows immediately if we set $\nu=\nu_1$, $k=\frac{\nu_1}{\delta}+(\nu_2-\frac{\nu_1}{\delta})e^{-\delta t_2}$ in theorem 2.1.12. (2.1.36) follows from (2.1.35) and (1.1.9).

□

Corollary 2.1.15 Let X_t, N_t as before. Assume that $\delta(t)=\delta$; then

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\} = e^{-\frac{\nu}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{\nu}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds} \tag{2.1.37}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\big|\lambda_{t_1}\} = e^{-\frac{1-\theta}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds}. \tag{2.1.38}$$

Proof

If we set $\nu_2=0$, $\nu=0$ in (2.1.33) and (2.1.34) then (2.1.37) and (2.1.38) follow.

□

Corollary 2.1.16 Let X_t, N_t as defined and the jump size distribution be exponential i.e. $g(y;t)=(\alpha+\gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y>0$, $\gamma>-\alpha e^{-\delta t}$. Assuming that $\delta(t)=\delta$ then

$$E\{e^{-\nu(X_{t_2}-X_{t_1})}\big|\lambda_{t_1}\} = e^{-\frac{\nu}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{\nu}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+\nu}}$$

(2.1.39)

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\big|\lambda_{t_1}\} = e^{-\frac{1-\theta}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}.$$

(2.1.40)

Proof

If we set $v_2 = 0,\ \ v = 0$ in (2.1.35) and (2.1.36) then (2.1.39) and (2.1.40) follow.

□

Corollary 2.1.17 Let X_t, N_t as defined and λ_t is '$-\infty$' asymptotic. Assuming that $\delta(t) = \delta$ then

$$E\{e^{-v(X_{t_2}-X_{t_1})}\} = e^{-\int_{-\infty}^{t_1}\rho(s)[1-\hat{g}\{\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})e^{-\delta(t_1-s)};s\}]ds}\ e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{v}{\delta}e^{-\delta(t_2-s)};s\}]ds} \tag{2.1.41}$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\} = e^{-\int_{-\infty}^{t_1}\rho(s)[1-\hat{g}\{\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})e^{-\delta(t_1-s)};s\}]ds}\ e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{1-\theta}{\delta}e^{-\delta(t_2-s)};s\}]ds}\ . \tag{2.1.42}$$

Proof

From corollary 2.1.15

$$E\{e^{-v(X_{t_2}-X_{t_1})}\} = E[e^{-\frac{v}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}]e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{v}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds}\ .$$

From corollary 1.2.4, when λ_t is '$-\infty$' asymptotic it has a distribution with Laplace transform

$$e^{-\int_{\infty}^{t_1}\rho(s)[1-\hat{g}\{ve^{-\delta(t_1-s)};s\}]ds}\ .$$

Therefore

$$E\{e^{-v(X_{t_2}-X_{t_1})}\} = E[e^{-\frac{v}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}]e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{v}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds}$$

$$= e^{-\int_{-\infty}^{t_1}\rho(s)[1-\hat{g}\{\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})e^{-\delta(t_1-s)};s\}]ds}\ e^{-\int_{t_1}^{t_2}\rho(s)[1-\hat{g}\{\frac{v}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds}\ .$$

Similarly (2.1.42) follows.

□

Corollary 2.1.18 Let X_t, N_t as defined, λ_t is '$-\infty$' asymptotic and the jump size distribution be exponential i.e. $g(y;t) = (\alpha + \gamma e^{\delta t})e^{-(\alpha+\gamma e^{\delta t})y}$, $y > 0$, $\gamma > -\alpha e^{-\delta t}$. Assuming that $\delta(t) = \delta$ then

$$E\{e^{-v(X_{t_2}-X_{t_1})}\} = \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})}\right)^{\frac{\rho}{\delta}} \left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}} \quad (2.1.43)$$

and

$$E\{\theta^{(N_{t_2}-N_{t_1})}\} = \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}\right)^{\frac{\rho}{\delta}} \left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{1-\theta}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}. \quad (2.1.44)$$

Proof

From corollary 2.1.16 and corollary 1.2.6

$$E\{e^{-v(X_{t_2}-X_{t_1})}\} = E[e^{-\frac{v}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}]\left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}} \left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}}$$

$$= \left(\frac{\gamma+\alpha e^{-\delta t_1}}{\gamma+\{\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})+\alpha\}e^{\delta t_1}}\right)^{\frac{\rho}{\delta}} \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha}\right)^{\frac{\rho}{\delta}} \left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}}$$

$$= \left(\frac{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})}\right)^{\frac{\rho}{\delta}} \left(\frac{\gamma e^{\delta t_1}+\alpha+\frac{v}{\delta}(1-e^{-\delta(t_2-t_1)})}{\gamma e^{\delta t_1}+\alpha e^{-\delta(t_2-t_1)}}\right)^{\frac{\alpha\rho}{\delta\alpha+v}}.$$

Similarly (2.1.44) follows.

□

We will now close this section by evaluating the joint Laplace transform of the distribution of $X_{t_2}-X_{t_1}, X_{t_3}-X_{t_2}, \cdots, X_{t_n}-X_{t_{n-1}}$ ($n-1$ successive increments). Using the important corollary 2.1.13 and 2.1.15 we can derive higher order properties of X_t.

Lemma 2.1.19 Let X_t, λ_t as defined. $v_{n-1}, v_{n-2}, \cdots\cdots, v_1$ and γ_0 are non-negative constants. Then

$$E\{e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_1(X_{t_n}-X_{t_{n-1}})}e^{-\gamma_0\lambda_{t_n}}\Big|\lambda_{t_1}\} = \varphi_1\varphi_2\cdots\cdots\varphi_{n-1}e^{-\{\frac{v_{n-1}}{\delta}+(\gamma_{n-2}-\frac{v_{n-1}}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} \tag{2.1.45}$$

where $\varphi_i = e^{-\int_{t_{n-i}}^{t_{n-i+1}}\rho(s)[1-\hat{g}\{\frac{v_i}{\delta}+(\gamma_{i-1}-\frac{v_i}{\delta})e^{-\delta(t_{n-i+1}-s)};s\}]ds}$, $\gamma_i = \dfrac{v_i}{\delta}+K_i$ and $K_i = (\gamma_{i-1}-\dfrac{v_i}{\delta})e^{-\delta(t_{n-i+1}-t_{n-i})}$

for $i = 1,2,\cdots,n-1$.

Proof

The result follows if we take the same procedures as lemma 2.1.9, using corollary 2.1.13.

□

Corollary 2.1.20 Let X_t, λ_t as defined. $v_{n-1}, v_{n-2},\cdots\cdots, v_1$ and γ_0 are non-negative constants. Then

$$E\{e^{-v_{n-1}(X_{t_2}-X_{t_1})}e^{-v_{n-2}(X_{t_3}-X_{t_2})}\cdots\cdots e^{-v_1(X_{t_n}-X_{t_{n-1}})}\Big|\lambda_{t_1}\} = \varphi_1'\varphi_2'\cdots\cdots\varphi_{n-1}'e^{-\{\frac{v_{n-1}}{\delta}+(\gamma_{n-2}'-\frac{v_{n-1}}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} \tag{2.1.46}$$

where $\varphi_1' = e^{-\int_{t_{n-1}}^{t_n}\rho(s)[1-\hat{g}\{\frac{v_1}{\delta}+(1-e^{-\delta(t_n-s)};s)\}]ds}$, $\gamma_1' = \dfrac{v_1}{\delta}+K_1'$, $K_1' = -\dfrac{v_1}{\delta}e^{-\delta(t_n-t_{n-1})}$ and

$$\varphi_i' = e^{-\int_{t_{n-i}}^{t_{n-i+1}}\rho(s)[1-\hat{g}\{\frac{v_i}{\delta}+\{\frac{v_{i-1}}{\delta}(1-e^{-\delta(t_{n-i+2}-t_{n-i+1})})-\frac{v_i}{\delta}\}e^{-\delta(t_{n-i+1}-s)};s\}]ds},$$

$$\gamma_i' = \frac{v_i}{\delta}+K_i' = \frac{v_i}{\delta}+[\frac{v_{i-1}}{\delta}\{1-e^{-\delta(t_{n-i+2}-t_{n-i+1})}\}-\frac{v_i}{\delta}]e^{-\delta(t_{n-i+1}-t_{n-i})} \;,\quad K_i' = (\gamma_{i-1}'-\frac{v_i}{\delta})e^{-\delta(t_{n-i+1}-t_{n-i})}$$

for $i = 2,3,\cdots,n-1$.

Proof

Set $\gamma_0 = 0$ in (2.1.22) and take the same procedures as lemma 2.1.19, using corollary 2.1.15 then the result follows.

□

2.2 The distribution of the increments of N_t

We have found the p.g.f. of $N_{t_2} - N_{t_1}$ (refer to corollary 2.1.7) assuming that λ_t is stationary. Therefore moments, cumulants etc. can be expressed in terms of p.g.f. of $N_{t_2} - N_{t_1}$. However, it is not easy to find the mean and variance of $N_{t_2} - N_{t_1}$ in terms of the p.g.f. of $N_{t_2} - N_{t_1}$ as the first derivative of $E\{\theta^{(N_{t_2} - N_{t_1})}\}$ w.r.t θ is very complicated. So we will evaluate the mean, variance and covariance of $N_{t_2} - N_{t_1}$ from the Poisson properties assuming that λ_t is stationary. We will also evaluate the higher order properties of N_t such as the joint Laplace transform of the distribution of $N_{t_2} - N_{t_1}, N_{t_3} - N_{t_2}, \cdots, N_{t_n} - N_{t_{n-1}}$ at specified times $t_1, t_2, \cdots, t_n$.

2.2.1 Time homogeneous case

2.2.1.1 Mean number of points in a fixed time interval

From (1.4.3)

$$E(N_{t_1+h} - N_{t_1}) = E\{E(N_{t_1+h} - N_{t_1}|\lambda_s)\} = E(\int_{t_1}^{t_1+h} \lambda_s ds) = \int_{t_1}^{t_1+h} E(\lambda_s) ds. \tag{2.2.1}$$

Since we have obtained $E(\lambda_t)$ when λ_t is stationary, the mean of the number of points in a fixed time interval, $E(N_{t_1+h} - N_{t_1})$ can be easily found.

Theorem 2.2.1 Let N_t as defined. Also let λ_t be time homogeneous and stationary then

$$E(N_{t_1+h} - N_{t_1}) = \frac{\mu_1 \rho}{\delta} h. \tag{2.2.2}$$

<u>**Proof**</u>

The result follows immediately if we set (1.3.7) in (2.2.1).

□

From (2.2.2) we can see that $E(N_{t_1+h} - N_{t_1})$ is a function of time h i.e. $E(N_{t_1+h} - N_{t_1})$ is linear in h when λ_t is stationary.

2.2.1.2 The variance of number of points in a fixed time interval

From (1.1.3)

$$Var(N_{t_1+h}-N_{t_1}|\lambda_s,\ t_1\le s\le t_1+h)=\int_{t_1}^{t_1+h}\lambda_s ds. \tag{2.2.3}$$

and

$$E\{(N_{t_1+h}-N_{t_1})^2|\lambda_s,\ t_1\le s\le t_1+h\}=\{E(N_{t_1+h}-N_{t_1}|\lambda_s,\ t_1\le s\le t_1+h)\}^2+Var(N_{t_1+h}-N_{t_1}|\lambda_s,\ t_1\le s\le t_1+h)$$
$$=(\int_{t_1}^{t_1+h}\lambda_s ds)^2+\int_{t_1}^{t_1+h}\lambda_s ds. \tag{2.2.4}$$

From (2.2.4)

$$E\{(N_{t_1+h}-N_{t_1})^2\}=E[E\{(N_{t_1+h}-N_{t_1})^2|\lambda_s,\ t_1\le s\le t_1+h\}]=E\{(\int_{t_1}^{t_1+h}\lambda_s ds)^2\}+E(\int_{t_1}^{t_1+h}\lambda_s ds)$$
$$=E\{(\int_{t_1}^{t_1+h}\lambda_s ds)^2\}+\int_{t_1}^{t_1+h}E(\lambda_s)ds. \tag{2.2.5}$$

Therefore

$$Var(N_{t_1+h}-N_{t_1})=E\{(N_{t_1+h}-N_{t_1})^2\}-\{E(N_{t_1+h}-N_{t_1})\}^2$$
$$=E\{(\int_{t_1}^{t_1+h}\lambda_s ds)^2\}+\int_{t_1}^{t_1+h}E(\lambda_s)ds-\{\int_{t_1}^{t_1+h}E(\lambda_s)ds\}^2. \tag{2.2.6}$$

Theorem 2.2.2 Let N_t, λ_t as defined and λ_t be stationary then

$$Var(N_{t_1+h}-N_{t_1})=\frac{\mu_1\rho}{\delta}h+\frac{\mu_2\rho}{\delta^2}(h+\frac{e^{-\delta h}-1}{\delta}). \tag{2.2.7}$$

<u>**Proof**</u>

The result follows immediately if we use (1.4.19) and (1.3.7) in (2.2.6).

□

We can find a interesting result from theorem 2.2.1 and theorem 2.2.2. In the case of homogeneous or non-homogeneous Poisson process, the mean and variance of number points in a fixed time interval are the same (see (1.1.3)). However in the case of doubly stochastic Poisson process, they are different i.e. $E(N_{t_1+h}-N_{t_1})=\frac{\mu_1\rho}{\delta}h$ and $Var(N_{t_1+h}-N_{t_1})=\frac{\mu_1\rho}{\delta}h+\frac{\mu_2\rho}{\delta^2}(h+\frac{e^{-\delta h}-1}{\delta})$.

2.2.1.3 The covariance of the number of points in two fixed time intervals

To obtain the covariance of the number of points in two fixed time intervals, $Cov(N_{t_1+h} - N_{t_1}, N_{t_2+h} - N_{t_2})$, we will need to examine a lemma.

Lemma 2.2.3 Let λ_t as defined and be stationary. Then

$$E(\int_0^h \int_{mh}^{(m+1)h} \lambda_u \lambda_s du ds) = (\frac{\mu_1 \rho}{\delta})^2 h^2 + \frac{\mu_2 \rho}{2\delta} e^{-\delta(m-1)h} (\frac{1-e^{-\delta h}}{\delta})^2 \tag{2.2.8}$$

where $m = t_2 - t_1$ and $t_1 \le t_1 + h \le t_2 \le t_2 + h$.

Proof

$$E(\int_{t_1}^{t_1+h} \lambda_s ds \cdot \int_{t_2}^{t_2+h} \lambda_s ds) = E(\int_{t_1}^{t_1+h} \int_{t_2}^{t_2+h} \lambda_u \lambda_s du ds) = \int_{t_1}^{t_1+h} \int_{t_2}^{t_2+h} E(\lambda_u \lambda_s) du ds, \quad s < u. \tag{2.2.9}$$

Since λ_t is stationary

$$\int_{t_1}^{t_1+h} \int_{t_2}^{t_2+h} E(\lambda_u \lambda_s) du ds = \int_0^h \int_{mh}^{(m+1)h} E(\lambda_u \lambda_s) du ds \tag{2.2.10}$$

where $m = t_2 - t_1$ and $t_1 \le t_1 + h \le t_2 \le t_2 + h$.

From (1.4.17) and (1.4.18)

$$E(\lambda_u \lambda_s) = E\{E(\lambda_u \lambda_s | \lambda_w, s \le w)\} = E\{\lambda_s E(\lambda_u | \lambda_s)\} = (\frac{\mu_1 \rho}{\delta})^2 + (\frac{\mu_2 \rho}{2\delta}) e^{-\delta(u-s)}. \tag{2.2.11}$$

The result follows if we set (2.2.11) in (2.2.10).

□

Theorem 2.2.4 Let N_t, λ_t as defined and λ_t be stationary then

$$Cov(N_{t_1+h} - N_{t_1}, N_{t_2+h} - N_{t_2}) = (\frac{\mu_2 \rho}{2\delta}) e^{-\delta(m-1)h} (\frac{1-e^{-\delta h}}{\delta})^2 \tag{2.2.12}$$

where $m = t_2 - t_1$ and $t_1 \le t_1 + h \le t_2 \le t_2 + h$.

Proof

$$Cov\{(N_{t_1+h} - N_{t_1}), (N_{t_2+h} - N_{t_2}) | \lambda_s, t_1 \le s \le t_2 + h\} = 0.$$

Therefore

$$E\{(N_{t_1+h}-N_{t_1})(N_{t_2+h}-N_{t_2})|\lambda_s,\ t_1\le s\le t_2+h\}=E(N_{t_1+h}-N_{t_1}|\lambda_s,\ t_1\le s\le t_2+h)E(N_{t_2+h}-N_{t_2}|\lambda_s,\ t_1\le s\le t_2+h)$$
$$=\int_{t_1}^{t_1+h}\lambda_s ds\cdot\int_{t_2}^{t_2+h}\lambda_s ds.$$

and

$$E\{(N_{t_1+h}-N_{t_1})(N_{t_2+h}-N_{t_2})\}=E[E\{(N_{t_1+h}-N_{t_1})(N_{t_2+h}-N_{t_2})|\lambda_s,\ t_1\le s\le t_2+h\}]$$
$$=E(\int_{t_1}^{t_1+h}\lambda_s ds\cdot\int_{t_2}^{t_2+h}\lambda_s ds)=E(\int_{t_1}^{t_1+h}\int_{t_2}^{t_2+h}\lambda_u\lambda_s duds).$$

Hence

$$Cov\{(N_{t_1+h}-N_{t_1}),(N_{t_2+h}-N_{t_2})\}=E\{(N_{t_1+h}-N_{t_1})(N_{t_2+h}-N_{t_2})\}-E(N_{t_1+h}-N_{t_1})E(N_{t_2+h}-N_{t_2})$$
$$=E(\int_{t_1}^{t_1+h}\int_{t_2}^{t_2+h}\lambda_u\lambda_s duds)-(\frac{\mu_1\rho}{\delta})^2h^2.$$

(2.2.13)

Since λ_t is stationary

$$Cov\{(N_{t_1+h}-N_{t_1}),(N_{t_2+h}-N_{t_2})\}=Cov\{(N_h-N_0),(N_{t_2-t_1+h}-N_{t_2-t_1})\}=Cov\{(N_h-N_0),(N_{mh+h}-N_{mh})\}$$
$$=Cov\{(N_h-N_0),(N_{(m-1)h}-N_{mh})\}=E(\int_0^h\int_{mh}^{(m+1)h}\lambda_u\lambda_s duds)-(\frac{\mu_1\rho}{\delta})^2h^2$$

(2.2.14)

where $m=t_2-t_1$ and $t_1\le t_1+h\le t_2\le t_2+h$.

If we set (2.2.8) in (2.2.14) then (2.2.12) follows immediately.

□

2.2.1.4 The joint distribution of $N_{t_2}-N_{t_1},N_{t_3}-N_{t_2},\cdots,N_{t_n}-N_{t_{n-1}}$

We will now evaluate the joint Laplace transform of the distribution of $N_{t_2}-N_{t_1},N_{t_3}-N_{t_2},\cdots,N_{t_n}-N_{t_{n-1}}$ ($n-1$ successive increments). Using the important corollary 2.1.3 and 2.1.5 we can derive higher order properties of N_t

Lemma 2.2.5 Let N_t, λ_t as defined. $\theta_{n-1},\theta_{n-2},\cdots\cdots,\theta_1$ and γ_0 are non-negative constants. Then

$$E\{\theta_{n-1}^{N_{t_2}-N_{t_1}}\theta_{n-2}^{N_{t_3}-N_{t_2}}\cdots\cdots\theta_1^{N_{t_n}-N_{t_{n-1}}}e^{-\gamma_0\lambda_{t_n}}\Big|\lambda_{t_1}\}=\phi_1\phi_2\cdots\cdots\phi_{n-1}e^{-\{\frac{1-\theta_{n-1}}{\delta}+(\gamma_{n-2}-\frac{1-\theta_{n-1}}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}$$

(2.2.15)

where $\phi_i = e^{-\rho \int_0^{t_{n-i+1}-t_{n-i}} [1-\hat{g}\{\frac{1-\theta_i}{\delta}+(\gamma_{i-1}-\frac{1-\theta_i}{\delta})e^{-\delta s}\}]ds}$, $\gamma_i = \frac{1-\theta_i}{\delta} + K_i$ and $K_i = (\gamma_{i-1} - \frac{1-\theta_i}{\delta})e^{-\delta(t_{n-i+1}-t_{n-i})}$

for $i = 1,2,\cdots,n-1$.

Proof

(1.1.9) implies

$$E\{\theta_{n-1}^{N_{t_2}-N_{t_1}}\theta_{n-2}^{N_{t_3}-N_{t_2}}\cdots\cdots\theta_1^{N_{t_n}-N_{t_{n-1}}}\} = E\{e^{-(1-\theta_{n-1})(X_{t_2}-X_{t_1})}e^{-(1-\theta_{n-2})(X_{t_3}-X_{t_2})}\cdots\cdots e^{-(1-\theta_1)(X_{t_n}-X_{t_{n-1}})}\}. \tag{2.2.16}$$

Therefore the result follows from (2.2.16) and (2.1.21).

□

Corollary 2.2.6 Let N_t, λ_t as defined. $\theta_{n-1},\theta_{n-2},\cdots\cdots,\theta_1$ and γ_0 are non-negative constants. Then

$$E\{\theta_{n-1}^{N_{t_2}-N_{t_1}}\theta_{n-2}^{N_{t_3}-N_{t_2}}\cdots\cdots\theta_1^{N_{t_n}-N_{t_{n-1}}}\,\big|\,\lambda_{t_1}\} = \phi_1'\phi_2'\cdots\cdots\phi_{n-1}'e^{-\{\frac{1-\theta_{n-1}}{\delta}+(\gamma_{n-2}'-\frac{1-\theta_{n-1}}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} \tag{2.2.17}$$

where $\phi_1' = e^{-\rho\int_0^{t_n-t_{n-1}}[1-\hat{g}\{\frac{1-\theta_1}{\delta}(1-e^{-\delta s})\}]ds}$, $\gamma_1' = \frac{1-\theta_1}{\delta} + K_1'$, $K_1' = -\frac{1-\theta_1}{\delta}e^{-\delta(t_n-t_{n-1})}$ and

$$\phi_i' = e^{-\rho\int_0^{t_{n-i+1}-t_{n-i}}[1-\hat{g}\{\frac{1-\theta_i}{\delta}+\{\frac{1-\theta_{i-1}}{\delta}(1-e^{-\delta(t_{n-i+2}-t_{n-i+1})})-\frac{1-\theta_i}{\delta}\}e^{-\delta s}\}]ds},$$

$$\gamma_i' = \frac{1-\theta_i}{\delta} + K_i' = \frac{1-\theta_i}{\delta} + [\frac{1-\theta_{i-1}}{\delta}\{1-e^{-\delta(t_{n-i+2}-t_{n-i+1})}\} - \frac{1-\theta_i}{\delta}]e^{-\delta(t_{n-i+1}-t_{n-i})},$$
$$K_i' = (\gamma_{i-1}' - \frac{1-\theta_i}{\delta})e^{-\delta(t_{n-i+1}-t_{n-i})}$$

for $i = 2,3,\cdots,n-1$.

Proof

If we set $\gamma_0 = 0$ in (2.2.15) then the result follows.

□

2.2.2 Time dependent parameters

2.2.2.1 Mean number of points in a fixed time interval

Since we have obtained $E(\lambda_t)$ when λ_t is '$-\infty$' asymptotic, the mean of the number of points in a fixed time interval, $E(N_{t_1+h} - N_{t_1})$, can be easily found.

Theorem 2.2.7 Let N_t, λ_t as defined and λ_t be '$-\infty$' asymptotic then

$$E(N_{t_1+h} - N_{t_1}) = \int_{t_1}^{t_1+h} e^{-\delta s} \int_{-\infty}^{s} e^{\delta u} \rho(u) \mu_1(u) du ds \tag{2.2.18}$$

where $-\infty < u < t_1 < s < t_1 + h$.

Proof

From (2.2.1)

$$E(N_{t_1+h} - N_{t_1}) = \int_{t_1}^{t_1+h} E(\lambda_s) ds. \tag{2.2.19}$$

If we set (1.2.15) in (2.2.19) then (2.2.18) follows immediately.

□

2.2.2.2 The variance of number of points in a fixed time interval

We start with lemmas that are very useful when trying to find variance of number of points in a fixed time interval, $Var(N_{t_1+h} - N_{t_1})$.

Lemma 2.2.8 Let λ_t, X_t as defined and λ_t be '$-\infty$' asymptotic. Then

$$E(\lambda_{t_1}^2) = e^{-2\delta t_1} \left[\int_{-\infty}^{t_1} e^{\delta u} \mu_1(u) \rho(u) du \right]^2 + e^{-2\delta t_1} \int_{-\infty}^{t_1} e^{2\delta s} \mu_2(s) \rho(s) ds. \tag{2.2.20}$$

Proof

Letting $t_0 \to -\infty$ in (1.2.13) and from (1.2.15),

$$E(\lambda_{t_1}^2) = 2e^{-2\delta t_1} \int_{-\infty}^{t_1} e^{2\delta s} \mu_1(s) \rho(s) \cdot e^{-\delta s} \int_{-\infty}^{s} e^{\delta u} \mu_1(u) \rho(u) du ds + e^{-2\delta t_1} \int_{-\infty}^{t_1} e^{2\delta s} \mu_2(s) \rho(s) ds \tag{2.2.21}$$

where $-\infty < u < s < t_1$.

Letting $c(s)=\int_{-\infty}^{s} e^{\delta u}\mu_1(u)\rho(u)du$, then

$$\int_{-\infty}^{t_1} e^{2\delta s}\mu_1(s)\rho(s)\cdot e^{-\delta s}\int_{-\infty}^{s} e^{\delta u}\mu_1(u)\rho(u)duds=\int_{-\infty}^{t_1}\int_{-\infty}^{s} e^{\delta u}\mu_1(u)\rho(u)du\, e^{\delta s}\mu_1(s)\rho(s)ds$$

$$=\int_{-\infty}^{t_1}\int_{-\infty}^{s} e^{\delta u}\mu_1(u)\rho(u)du\, e^{\delta s}\mu_1(s)\rho(s)ds=\int_{-\infty}^{t_1} c(s)c'(s)ds=\frac{1}{2}c^2(t_1)=\frac{1}{2}\left[\int_{-\infty}^{t_1} e^{\delta u}\mu_1(u)\rho(u)du\right]^2. \quad (2.2.22)$$

Therefore from (2.2.22) we have

$$E(\lambda_{t_1}^2)=e^{-2\delta t_1}\left[\int_{-\infty}^{t_1} e^{\delta u}\mu_1(u)\rho(u)du\right]^2+e^{-2\delta t_1}\int_{-\infty}^{t_1} e^{2\delta s}\mu_2(s)\rho(s)ds. \quad (2.2.23)$$

□

Lemma 2.2.9 Let λ_t, X_t as defined and λ_t be '$-\infty$' asymptotic. Then

$$E\left(\lambda_{t_1}X_{t_1}\right)=e^{-\delta t_1}\int_0^{t_1} e^{-\delta s}\int_{-\infty}^{s} e^{2\delta u}\rho(u)\mu_2(u)duds+e^{-\delta t_1}\int_{-\infty}^{t_1} e^{\delta u}\rho(u)\mu_1(u)du\int_0^{t_1} e^{-\delta s}\int_{-\infty}^{s} e^{\delta u}\rho(u)\mu_1(u)duds. \quad (2.2.24)$$

Proof

The aggregated process $X_t=\int_0^t \lambda_s ds$. Therefore from (1.4.16)

$$E\left(\lambda_{t_1}X_{t_1}\right)=\int_0^{t_1} E(\lambda_{t_1}\lambda_s)ds,\quad s<t_1. \quad (2.2.25)$$

Conditioning on λ_{t_0} in (1.4.17) and from theorem 1.2.7

$$E\left(\lambda_{t_1}\lambda_s\middle|\lambda_{t_0}\right)=e^{-\delta(t_1-s)}E(\lambda_s^2|\lambda_{t_0})+e^{-\delta t_1}\int_s^{t_1} e^{\delta u}\rho(u)\mu_1(u)du\cdot E(\lambda_s|\lambda_{t_0}). \quad (2.2.26)$$

Letting $t_0\to-\infty$ in (2.2.26) and from (2.2.21) and (1.2.15)

$$E(\lambda_{t_1}\lambda_s)=e^{-\delta t_1}e^{-\delta s}\int_{-\infty}^{s} e^{2\delta u}\rho(u)\mu_2(u)du+e^{-\delta t_1}e^{-\delta s}\int_{-\infty}^{t_1} e^{\delta u}\rho(u)\mu_1(u)du\int_{-\infty}^{s} e^{\delta u}\rho(u)\mu_1(u)du. \quad (2.2.27)$$

If we set (2.2.27) in (2.2.25) then (2.2.24) follows immediately.

□

Lemma 2.2.10 Let λ_t, X_t as defined and λ_t be '$-\infty$' asymptotic. Then

$$E\{(\int_0^{t_1}\lambda_s ds)^2\}=2\left[\int_0^{t_1}e^{-\delta s}\int_0^{s}e^{-\delta u}\int_{-\infty}^{u}e^{2\delta v}\rho(v)\mu_2(v)dvduds+\int_0^{t_1}e^{-\delta s}\int_{-\infty}^{s}e^{\delta v}\rho(v)\mu_1(v)dv\int_0^{s}e^{-\delta u}\int_{-\infty}^{u}e^{\delta v}\rho(v)\mu_1(v)dvduds\right]. \tag{2.2.28}$$

__Proof__

The aggregated process $X_t=\int_0^t\lambda_s ds$. Therefore from (1.4.20)

$$E\{(\int_0^{t_1}\lambda_s ds)^2\}=2\int_0^{t_1}E(\lambda_s X_s)ds. \tag{2.2.29}$$

The result follows if we set (2.2.24) in (2.2.29).

□

Theorem 2.2.11 Let N_t, λ_t as defined and λ_t be '$-\infty$' asymptotic then

$$\begin{aligned}Var(N_{t_1+h}-N_{t_1})&=2\left[\int_0^{h}e^{-\delta s}\int_0^{s}e^{-\delta u}\int_{-\infty}^{u}e^{2\delta v}\rho(v)\mu_2(v)dvduds+\int_0^{h}e^{-\delta s}\int_{-\infty}^{s}e^{\delta v}\rho(v)\mu_1(v)dv\int_0^{s}e^{-\delta u}\int_{-\infty}^{u}e^{\delta v}\rho(v)\mu_1(v)dvduds\right]\\&+\int_0^{h}e^{-\delta s}\int_{-\infty}^{s}e^{\delta u}\rho(u)\mu_2(u)duds-\left\{\int_0^{h}e^{-\delta s}\int_{-\infty}^{s}e^{\delta u}\rho(u)\mu_2(u)duds\right\}^2.\end{aligned} \tag{2.2.30}$$

__Proof__

From (2.2.6) and since λ_t is '$-\infty$' asymptotic

$$\begin{aligned}Var(N_{t_1+h}-N_{t_1})&=E\{(N_{t_1+h}-N_{t_1})^2\}-\{E(N_{t_1+h}-N_{t_1})\}^2=E\{(N_h-N_0)^2\}-\{E(N_h-N_0)\}^2\\&=E\{(\int_0^h\lambda_s ds)^2\}+\int_0^h E(\lambda_s)ds-\{\int_0^h E(\lambda_s)ds\}^2.\end{aligned}$$

The result follows immediately from (2.2.28) and (1.2.15).

□

2.2.2.3 The covariance of the number of points in two fixed time intervals

To obtain the covariance of the number of points in two fixed time intervals, $Cov(N_{t_1+h}-N_{t_1},N_{t_2+h}-N_{t_2})$, we will need to prove the following lemma.

Lemma 2.2.12 Let λ_t as defined and be '$-\infty$' asymptotic. Then

$$E(\int_{t_1}^{t_1+h}\int_{t_2}^{t_2+h}\lambda_u\lambda_s duds)=\int_0^h e^{-\delta s}\int_{-\infty}^{s}e^{2\delta v}\rho(v)\mu_2(v)dv\int_{mh}^{(m+1)h}e^{-\delta u}duds+\int_0^h e^{-\delta s}\int_{-\infty}^{s}e^{\delta v}\rho(v)\mu_1(v)dv\int_{mh}^{(m+1)h}e^{-\delta u}\int_{-\infty}^{u}e^{\delta v}\rho(v)\mu_1(v)dvduds \tag{2.2.31}$$

where $m=t_2-t_1$ and $s<u$.

Proof

From (2.2.9)

$$E(\int_{t_1}^{t_1+h}\lambda_s ds\cdot\int_{t_2}^{t_2+h}\lambda_s ds)=\int_{t_1}^{t_1+h}\int_{t_2}^{t_2+h}E(\lambda_u\lambda_s)duds \quad , \quad s<u. \tag{2.2.32}$$

Since λ_t is '$-\infty$' asymptotic,

$$E(\int_{t_1}^{t_1+h}\lambda_s ds\cdot\int_{t_2}^{t_2+h}\lambda_s ds)=\int_{t_1}^{t_1+h}\int_{t_2}^{t_2+h}E(\lambda_u\lambda_s)duds=\int_{0}^{h}\int_{mh}^{(m+1)h}E(\lambda_u\lambda_s)duds \tag{2.2.33}$$

where $m=t_2-t_1$ and $t_1\le t_1+h\le t_2\le t_2+h$.

The result follows if we set (2.2.27) in (2.2.33).

□

Theorem 2.2.13 Let N_t, λ_t as defined and λ_t be '$-\infty$' asymptotic then

$$Cov(N_{t_1+h}-N_{t_1},N_{t_2+h}-N_{t_2})$$
$$=\int_0^h e^{-\delta s}\int_{-\infty}^{s}e^{2\delta v}\rho(v)\mu_2(v)dv\int_{mh}^{(m+1)h}e^{-\delta u}duds+\int_0^h e^{-\delta s}\int_{-\infty}^{s}e^{\delta v}\rho(v)\mu_1(v)dv\int_{mh}^{(m+1)h}e^{-\delta u}\int_{-\infty}^{u}e^{\delta v}\rho(v)\mu_1(v)dvduds-\left\{\int_0^h e^{-\delta s}\int_{-\infty}^{s}e^{\delta v}\rho(v)\mu_1(v)dvds\right\}^2 \tag{2.2.34}$$

where $m=t_2-t_1$ and $t_1\le t_1+h\le t_2\le t_2+h$.

Proof

Since λ_t is '$-\infty$' asymptotic

$$Cov\{(N_{t_1+h}-N_{t_1}),(N_{t_2+h}-N_{t_2})\}=Cov\{(N_h-N_0),(N_{(m-1)h}-N_{mh})\}$$

where $m=t_2-t_1$ and $t_1\le t_1+h\le t_2\le t_2+h$.

From (2.2.13) and (2.2.19)

$$Cov\{(N_{t_1+h}-N_{t_1}),(N_{t_2+h}-N_{t_2})\} = E\{(N_h-N_0)(N_{(m+1)h}-N_{mh})\} - E(N_h-N_0)E(N_{(m+1)h}-N_{mh})$$

$$= \int_0^h \int_{mh}^{(m+1)h} E(\lambda_u\lambda_s)duds - \left\{\int_0^h E(\lambda_s)ds\right\}^2. \tag{2.2.35}$$

The result follows immediately if we set (2.2.31) and (2.2.18) in (2.2.35).

□

2.2.2.4 The joint distribution of $N_{t_2}-N_{t_1}, N_{t_3}-N_{t_2}, \cdots, N_{t_n}-N_{t_{n-1}}$

We will now evaluate the joint Laplace transform of the distribution of $N_{t_2}-N_{t_1}, N_{t_3}-N_{t_2}, \cdots, N_{t_n}-N_{t_{n-1}}$ ($n-1$ successive increments). Using the important corollary 2.1.13 and 2.1.15 we can derive higher order properties of N_t

Lemma 2.2.14 Let N_t, λ_t as defined. $\theta_{n-1}, \theta_{n-2}, \cdots\cdots, \theta_1$ and γ_0 are non-negative constants. Then

$$E\{\theta_{n-1}^{N_{t_2}-N_{t_1}}\theta_{n-2}^{N_{t_3}-N_{t_2}}\cdots\cdots\theta_1^{N_{t_n}-N_{t_{n-1}}}e^{-\gamma_0\lambda_{t_n}}\big|\lambda_{t_1}\} = \varphi_1\varphi_2\cdots\cdots\varphi_{n-1}e^{-\{\frac{1-\theta_{n-1}}{\delta}+(\gamma_{n-2}-\frac{1-\theta_{n-1}}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} \tag{2.2.36}$$

where $\varphi_i = e^{-\int_{t_{n-i}}^{t_{n-i+1}}\rho(s)[1-\hat{g}\{\frac{1-\theta_i}{\delta}+(\gamma_{i-1}-\frac{1-\theta_i}{\delta})e^{-\delta(t_{n-i+1}-s)};s\}]ds}$, $\gamma_i = \dfrac{1-\theta_i}{\delta}+K_i$

and $K_i = (\gamma_{i-1}-\dfrac{1-\theta_i}{\delta})e^{-\delta(t_{n-i+1}-t_{n-i})}$

for $i=1,2,\cdots,n-1$.

<u>**Proof**</u>

(2.2.36) follows from (2.1.45) and (2.2.16).

□

Corollary 2.2.15 Let N_t, λ_t as defined. $\theta_{n-1}, \theta_{n-2}, \cdots\cdots, \theta_1$ and γ_0 are non-negative constants. Then

$$E\{\theta_{n-1}^{N_{t_2}-N_{t_1}}\theta_{n-2}^{N_{t_3}-N_{t_2}}\cdots\cdots\theta_1^{N_{t_n}-N_{t_{n-1}}}\big|\lambda_{t_1}\} = \varphi_1'\varphi_2'\cdots\cdots\varphi_{n-1}'e^{-\{\frac{1-\theta_{n-1}}{\delta}+(\gamma_{n-2}'-\frac{1-\theta_{n-1}}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} \tag{2.2.37}$$

where $\varphi_1' = e^{-\int_{t_{n-1}}^{t_n} \rho(s)[1-\hat{g}\{\frac{1-\theta_1}{\delta}(1-e^{-\delta(t_n-s)};s)\}]ds}$, $\gamma_1' = \frac{1-\theta_1}{\delta} + K_1'$, $K_1' = -\frac{1-\theta_1}{\delta} e^{-\delta(t_n - t_{n-1})}$ and

$$\varphi_i' = e^{-\int_{t_{n-i}}^{t_{n-i+1}} \rho(s)[1-\hat{g}\{\frac{1-\theta_i}{\delta}+\{\frac{1-\theta_{i-1}}{\delta}(1-e^{-\delta(t_{n-i+2}-t_{n-i+1})})-\frac{1-\theta_i}{\delta}\}e^{-\delta(t_{n-i+1}-s)};s\}]ds},$$

$$\gamma_i' = \frac{1-\theta_i}{\delta} + K_i' = \frac{1-\theta_i}{\delta} + [\frac{1-\theta_{i-1}}{\delta}\{1-e^{-\delta(t_{n-i+2}-t_{n-i+1})}\} - \frac{1-\theta_i}{\delta}]e^{-\delta(t_{n-i+1}-t_{n-i})},$$

$$K_i' = (\gamma_{i-1}' - \frac{1-\theta_i}{\delta})e^{-\delta(t_{n-i+1}-t_{n-i})}$$

for $i = 2,3,\cdots,n-1$.

Proof

If we set $\gamma_0 = 0$ in (2.2.36) then the result follows.

□

2.3 The distribution of the intensity at point times

In this section we will examine the Laplace transform of the distribution of the intensity of point times λ_{T_n}. We will assume that the process λ_t is stationary where T_n is the time of the n^{th} point of N_t and λ_{T_n} is the value of λ_t when N_t takes the value n for the first time. The distribution of λ_{T_n} should not be the same as the distribution of λ_t. Since a point occurs at that time this implies that the intensity should be higher than "expected" at other times (see comment at the end of this section).

Let us assume that shot noise process is time homogeneous and start with a lemma also used by Dassios (1987). We will provide our own proof.

Lemma 2.3.1 Let N_t, λ_t as defined. Let A be the generator for the process λ_t and suppose that $f(\lambda)$ is a function belonging to its domain and furthermore that it satisfies

$$\lim_{t\to\infty} E\{f(\lambda_t)\cdot e^{-\int_0^t \lambda_s ds} \,|\lambda_0\} = 0. \tag{2.3.1}$$

If $h(\lambda)$ is such that

$$\lambda\{h(\lambda)-f(\lambda)\}+\mathrm{A}f(\lambda)=0$$
$$\therefore \quad \mathrm{A}f(\lambda)=-\lambda\{h(\lambda)-f(\lambda)\} \tag{2.3.2}$$

then

$$E\{h(\lambda_{\tau_1})|\lambda_0\}=f(\lambda_0). \tag{2.3.3}$$

<u>**Proof**</u>

From (2.3.2)

$$f(\lambda_t)+\int_0^t [\lambda_s\{h(\lambda_s)-f(\lambda_s)\}]ds$$

is a martingale and since $\tau_1 \wedge t$ is a stopping time ($\Pr(\tau_1 \le s)=\Pr(N_s>0)$ and N_s is λ_s-measurable)

$$E\{f(\lambda_{\tau_1\wedge t}|\lambda_0)+E[\int_0^{\tau_1\wedge t}[\lambda_s\{h(\lambda_s)-f(\lambda_s)\}]ds|\lambda_0]=f(\lambda_0). \tag{2.3.4}$$

If we now place a condition on the realisation λ_v, $0\le v\le t$ then the first term of the left-hand side in (2.3.4) is

$$E\{f(\lambda_{\tau_1\wedge t}|\lambda_0)\}=\int_\Omega E\{f(\lambda_{\tau_1\wedge t})|\lambda_v, 0\le v\le t\}\cdot dP(\lambda_v, 0\le v\le t) \tag{2.3.5}$$

and the second term of the left-hand side in (2.3.4) is

$$E[\int_0^{\tau_1\wedge t}[\lambda_s\{h(\lambda_s)-f(\lambda_s)\}]ds|\lambda_0]=\int_\Omega E[\int_0^{\tau_1\wedge t}\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds|\lambda_v,0\le v\le t]\cdot dP(\lambda_v,0\le v\le t) \tag{2.3.6}$$

where $dP(\lambda_v,0\le v\le t)$ is the probability differential of a particular realisation in Ω, the set of all realisations.

Since $\tau_1\wedge t$ is distributed with density $\lambda_r e^{-\int_0^r\lambda_s ds}$ on $(0,t)$ and a mass $e^{-\int_0^t\lambda_s ds}$ at t conditionally on $\lambda_r, 0\le r\le t$ (N_t is doubly stochastic Poisson), we have

$$E\{f(\lambda_{\tau_1\wedge t})|\lambda_v\}=\int_0^t\{f(\lambda_r)\lambda_r e^{-\int_0^r\lambda_s ds}\}dr+f(\lambda_t)e^{-\int_0^t\lambda_s ds} \tag{2.3.7}$$

and

$$E[\int_0^{\tau_1\wedge t}\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds|\lambda_v]=\int_0^t[\int_0^r\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds\Pr(\tau_1=r)]dr+\int_0^t\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds\Pr(\tau_1>t)$$

$$=\int_0^t[\int_0^r\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds\cdot\lambda_r e^{-\int_0^r\lambda_s ds}]dr+\int_0^t\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds e^{-\int_0^t\lambda_s ds}.$$

Integrating by parts

$$E[\int_0^{\tau_1\wedge t}\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds|\lambda_v]=\int_0^t[-\int_s^t-\lambda_r e^{-\int_0^r\lambda_s ds}dr]\cdot\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds+\int_0^t\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds e^{-\int_0^t\lambda_s ds}$$

$$=\int_0^t(e^{-\int_0^s\lambda_s ds}-e^{-\int_0^t\lambda_s ds})\cdot\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds+\int_0^t\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds e^{-\int_0^t\lambda_s ds}=\int_0^t\lambda_s\{h(\lambda_s)-f(\lambda_s)\}e^{-\int_0^s\lambda_s ds}ds.$$

Put $s=r$

$$E[\int_0^{\tau_1\wedge t}\lambda_s\{h(\lambda_s)-f(\lambda_s)\}ds|\lambda_v]=\int_0^t\lambda_r\{h(\lambda_r)-f(\lambda_r)\}e^{-\int_0^r\lambda_s ds}dr. \tag{2.3.8}$$

Therefore (2.3.4) becomes

$$E\{f(\lambda_{\tau_1\wedge t}|\lambda_0)+E[\int_0^{\tau_1\wedge t}[\lambda_s\{h(\lambda_s)-f(\lambda_s)\}]ds|\lambda_0]$$

$$= \int_{\Omega} [\int_0^t f(\lambda_r)\lambda_r e^{-\int_0^r \lambda_s ds} dr + f(\lambda_t) e^{-\int_0^t \lambda_s ds}] dP(\lambda_v, 0 \le v \le t)$$

$$+ \int_{\Omega} [\int_0^t \lambda_r \{h(\lambda_r) - f(\lambda_r)\} e^{-\int_0^r \lambda_s ds} dr] dP(\lambda_v, 0 \le v \le t)$$

$$= \int_{\Omega} f(\lambda_t) e^{-\int_0^t \lambda_s ds} dP(\lambda_v, 0 \le v \le t) + \int_{\Omega} [\int_0^t \lambda_r h(\lambda_r) e^{-\int_0^r \lambda_s ds} dr] dP(\lambda_v, 0 \le v \le t)$$

$$= E\{f(\lambda_t) e^{-\int_0^t \lambda_s ds} | \lambda_0\} + \int_{\Omega} [\int_0^t h(\lambda_r)\lambda_r e^{-\int_0^r \lambda_s ds} dr] dP(\lambda_v, 0 \le v \le t) = f(\lambda_0).$$

Letting $t \to \infty$ the first term in the left-hand side tends to 0 from (2.3.1) and the second to $E\{h(\lambda_{\tau_1}) | \lambda_0\}$ ($\because$ $\lambda_r e^{-\int_0^r \lambda_s ds}$ is a density) we therefore have $E\{h(\lambda_{\tau_1}) | \lambda_0\} = f(\lambda_0)$.

□

Now let us derive the Laplace transform of the distribution of the intensity of point times λ_{T_n} assuming that the process λ_t is stationary.

Theorem 2.3.2 Let T_n be the time of the n^{th} point of N_t. Assume μ_1 is the first moment of G and that it exists. When the process λ_t is stationary

$$H(v) = E(e^{-v\lambda_{T_1}}) = \frac{\hat{G}(v)}{\mu_1} \cdot e^{-\frac{\rho}{\delta}\int_0^v \hat{G}(u)du} . \tag{2.3.9}$$

Proof

We will use lemma 2.3.1 which implies that if $f(\lambda)$ and $h(\lambda)$ are such that

$$\lambda\{h(\lambda) - f(\lambda)\} + \mathrm{A}f(\lambda) = 0$$

$$\text{i.e. } \lambda\{h(\lambda) - f(\lambda)\} - \delta\lambda f'(\lambda) + \rho\{\int_0^\infty f(\lambda + y) dG(y) - f(\lambda)\} = 0 \tag{2.3.10}$$

and (2.3.1) is satisfied then by starting the process from T_i

$$E\{h(\lambda_{T_{i+1}}) | \lambda_{T_i}\} = f(\lambda_{T_i}). \tag{2.3.11}$$

We will use

$$f(\lambda)=\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\}\cdot e^{-\nu\lambda}=\lambda e^{-\nu\lambda}-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda}.$$

$f(\lambda)$ clearly satisfies (2.3.1) and substituting into (2.3.10)

$$\lambda\{h(\lambda)-\lambda e^{-\nu\lambda}+\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda}\}-\delta\lambda[1\cdot e^{-\nu\lambda}+\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\}(-\nu)e^{-\nu\lambda}]$$
$$=-\rho\lambda[\int_0^\infty[\{(\lambda+y)-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\}e^{-\nu(\lambda+y)}]dG(y)-\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\}e^{-\nu\lambda}].$$

Therefore

$$\lambda\{h(\lambda)-\lambda e^{-\nu\lambda}+\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda}\}+\delta\nu\lambda\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\}e^{-\nu\lambda}-\delta\lambda e^{-\nu\lambda}$$
$$=-\rho\lambda[e^{-\nu\lambda}\{\lambda\hat{g}'(\nu)+\hat{g}'(\nu)-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\hat{g}(\nu)\}-\{\lambda-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}\}e^{-\nu\lambda}]$$
$$=-\rho\lambda e^{-\nu\lambda}\{\hat{g}(\nu)-1\}.$$

Dividing by λ and simplifying

$$h(\lambda)=\lambda e^{-\nu\lambda}(1-\delta\nu)+\delta e^{-\nu\lambda}-(1-\delta\nu)\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda}+\rho e^{-\nu\lambda}\{1-\hat{g}(\nu)\}$$

and by (2.3.11)

$$E\{h(\lambda_{T_{l+1}})\}=E[E\{h(\lambda_{T_{l+1}})|\lambda_{T_l}\}]=E\{f(\lambda_{T_l})\}$$

then

$$E[\lambda_{T_{i+1}}e^{-\nu\lambda_{T_{i+1}}}(1-\delta\nu)+\delta e^{-\nu\lambda_{T_{i+1}}}-(1-\delta\nu)\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda_{T_{i+1}}}+\rho e^{-\nu\lambda_{T_{i+1}}}\{1-\hat{g}(\nu)\}]=E\{\lambda_{T_i}e^{-\nu\lambda_{T_i}}-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}e^{-\nu\lambda_{T_i}}\}$$
$$(1-\delta\nu)E(\lambda_{T_{i+1}}e^{-\nu\lambda_{T_{i+1}}})+\delta E(e^{-\nu\lambda_{T_{i+1}}})-(1-\delta\nu)\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}E(e^{-\nu\lambda_{T_{i+1}}})+\rho\{1-\hat{g}(\nu)\}E(e^{-\nu\lambda_{T_{i+1}}})=E(\lambda_{T_i}e^{-\nu\lambda_{T_i}})-\frac{\hat{g}'(\nu)}{1-\hat{g}(\nu)}E(e^{-\nu\lambda_{T_i}}).$$
(2.3.12)

When the process is stationary $\lambda_{T_{l+1}}$ and λ_{T_l} have the same distribution whose Laplace transform is denoted by $H(\nu)$; by (2.3.12) we have

$$-(1-\delta v)H'(v)-(1-\delta v)\frac{\hat{g}'(v)}{1-\hat{g}(v)}H(v)+[\delta+\rho\{1-\hat{g}(v)\}]H(v)=-H'(v)-\frac{\hat{g}'(v)}{1-\hat{g}(v)}H(v).$$

$$\delta vH'(v)+\delta v\frac{\hat{g}'(v)}{1-\hat{g}(v)}H(v)+[\delta+\rho\{1-\hat{g}(v)\}]H(v)=0.$$

Dividing by δv

$$H'(v)+\frac{\hat{g}'(v)}{1-\hat{g}(v)}H(v)+\{\frac{1}{v}+\frac{\rho}{\delta}\frac{1-\hat{g}(v)}{v}\}H(v)=0. \qquad (2.3.13)$$

Therefore $H(v)$ is given by solving (2.3.13) subject to

$$H(0)=1 \qquad (2.3.14)$$

and we get

$$H(v)=A_1\frac{1-\hat{g}(v)}{v}\cdot e^{-\frac{\rho}{\delta}\int_0^v \hat{G}(u)du}$$

where A_1 is a constant.

From (2.3.14) $A_1=\frac{1}{\mu_1}$ then

$$H(v)=\frac{1}{\mu_1}\frac{1-\hat{g}(v)}{v}\cdot e^{-\frac{\rho}{\delta}\int_0^v \hat{G}(u)du}=\frac{\hat{G}(v)}{\mu_1}\cdot e^{-\frac{\rho}{\delta}\int_0^v \hat{G}(u)du}. \qquad (2.3.15)$$

□

(2.3.15) provides us with the interesting fact that this is the distribution of the sum of two random variables; one having the stationary (asymptotic) distribution of λ_t as its distribution (see corollary 1.3.3) and the other having density $\frac{G(y)}{\mu_1}$ $(G(y)=1-G(y))$. In other words, the intensity of point times are higher than the intensity at other times.

2.4 The distribution of interarrival times

The distribution of the time between two successive points is also of interest. By τ_k we denote the length of the interval between the $(k-1)^{\text{th}}$ and the k^{th} point. Since the distribution of τ_1 is the same as the distribution of τ_k when stationary has been achieved, we will try to find the distribution of $\tau_1 \equiv \inf\{t: N_t = 1 | N_0 = 0\}$. This section also deals with the derivation of the mean and variance of interarrival time between points assuming that λ_t is stationary.

Let us assume that the shot noise process is time homogeneous.

2.4.1 The distribution of interarrival time between points

From (2.1.14) in corollary 2.1.5

$$E\left(\theta^{N_t} | \lambda_0\right) = e^{-\frac{1-\theta}{\delta}(1-e^{-\delta t})\lambda_0} e^{-\rho \int_0^t [1-\hat{g}\{\frac{1-\theta}{\delta}(1-e^{-\delta s})\}] ds} . \tag{2.4.1}$$

By setting $\theta = 0$ we get the tail of the distribution of τ_1,

$$\Pr(\tau_1 > t | \lambda_0) = \Pr(N_t = 0 | \lambda_0) = e^{-\frac{\lambda_0}{\delta}(1-e^{-\delta t})} e^{-\rho \int_0^t \{1-\hat{g}(\frac{1-e^{-\delta s}}{\delta})\} ds} . \tag{2.4.2}$$

Theorem 2.4.1 Let τ a interarrival time of points of N_t. Assume that 0 is a time at which a point of N_t has occurred. When the process λ_t is stationary,

$$\Pr(\tau > t) = \Pr(\tau_1 > t) = \frac{\hat{G}(\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta t})}{\mu_1} \cdot e^{-\frac{\rho}{\delta}\int_0^t \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}) ds} . \tag{2.4.3}$$

<u>**Proof**</u>

From (2.4.2)

$$\Pr(\tau_1 > t) = E\{\Pr(\tau_1 > t | \lambda_0)\} = E[e^{-\frac{\lambda_0}{\delta}(1-e^{-\delta t})} e^{-\rho \int_0^t \{1-\hat{g}(\frac{1-e^{\delta s}}{\delta})\} ds}] = e^{-\rho \int_0^t \{1-\hat{g}(\frac{1-e^{\delta s}}{\delta})\} ds} E\{e^{-\frac{(1-e^{-\delta t})}{\delta}\lambda_0}\}. \tag{2.4.4}$$

Since 0 is a time at which a point of N_t has occurred and λ_t is stationary, substitute (2.3.15) into (2.4.4) then

$$\Pr(\tau > t) = \Pr(\tau_1 > t) = e^{-\rho\int_0^t \{1-\hat{g}(\frac{1-e^{-\delta s}}{\delta})\}ds} E\{e^{-\frac{(1-e^{-\delta t})}{\delta}\lambda_0}\} = e^{-\rho\int_0^t \{1-\hat{g}(\frac{1-e^{-\delta s}}{\delta})\}ds} \frac{\hat{G}(\frac{1-e^{-\delta t}}{\delta})}{\mu_1} e^{-\frac{\rho}{\delta}\int_0^{\frac{1-e^{-\delta t}}{\delta}} \hat{G}(u)du}$$

$$= \frac{\hat{G}(\frac{1-e^{-\delta t}}{\delta})}{\mu_1} \cdot e^{-\rho\int_0^t [\{\frac{1-\hat{g}(\frac{1-e^{-\delta s}}{\delta})}{1-e^{-\delta s}}\}(1-e^{-\delta s})]ds} \cdot e^{-\rho\int_0^t [\{\frac{1-\hat{g}(\frac{1-e^{-\delta s}}{\delta})}{1-e^{-\delta s}}\}e^{-\delta s}]ds} = \frac{\hat{G}(\frac{1-e^{-\delta t}}{\delta})}{\mu_1} \cdot e^{-\rho\int_0^t \{\frac{1-\hat{g}(\frac{1-e^{-\delta s}}{\delta})}{1-e^{-\delta s}}\}ds}$$

$$= \frac{\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta t})}{\mu_1} \cdot e^{-\rho\int_0^t \{\frac{1-\hat{g}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})}{1-e^{-\delta s}}\}ds} = \frac{\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta t})}{\mu_1} \cdot e^{-\frac{\rho}{\delta}\int_0^t \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}.$$

□

2.4.2 Mean of interarrival time between points

We have found the distribution of interarrival time between points assuming that λ_t is stationary. So by integrating it we can evaluate the mean of the interarrival time between points when λ_t is stationary, i.e.

$$E(\tau) = \int_0^\infty \Pr(\tau > t)dt.$$

Theorem 2.4.2 Let N_t as defined and also τ be a interarrival time of points of N_t. If λ_t is stationary then

$$E(\tau) = \int_0^\infty \Pr(\tau > t)dt = \frac{\delta}{\mu_1 \rho}. \tag{2.4.5}$$

__Proof__

From theorem 2.4.1

$$\int_u^\infty \Pr(\tau > t)dt = \int_u^\infty [\frac{\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta t})}{\mu_1} \cdot e^{-\rho\int_0^t \{\frac{1-\hat{g}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})}{1-e^{-\delta s}}\}ds}]dt = \int_u^\infty [\frac{\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta t})}{\mu_1} \cdot e^{-\frac{\rho}{\delta}\int_0^t \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}]dt$$

$$= \frac{\delta}{\mu_1 \rho}\int_u^\infty [\frac{\rho}{\delta} \cdot \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta t}) \cdot e^{-\frac{\rho}{\delta}\int_0^t \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}]dt.$$

If we differentiate $-e^{-\frac{\rho}{\delta}\int_0^t \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}$ w.r.t t we can obtain $\frac{\rho}{\delta} \cdot \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta t})e^{-\frac{\rho}{\delta}\int_0^t \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}$.

Therefore

$$\int_u^{\infty}\Pr(\tau>t)dt=\frac{\delta}{\mu_1\rho}\int_u^{\infty}[\frac{\rho}{\delta}\cdot\hat{G}(\tfrac{1}{\delta}-\tfrac{1}{\delta}e^{-\delta t})\cdot e^{-\frac{\rho}{\delta}\int_0^t\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}]dt=\frac{\delta}{\mu_1\rho}[e^{-\frac{\rho}{\delta}\int_0^t\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}]_u^{\infty}$$

$$=\frac{\delta}{\mu_1\rho}e^{-\frac{\rho}{\delta}\int_0^u\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}. \tag{2.4.6}$$

Set $u=0$ in (2.4.6) then (2.4.5) follows immediately.

□

Theorem 2.4.2 shows the interesting fact that the expected value of the interarrival time between points is the inverse of the expected value of the intensity of the point process i.e. $E(\tau)=\frac{1}{E(\lambda_t)}=\frac{\delta}{\mu_1\rho}$ (see corollary 1.3.9).

2.4.3 Variance of interarrival time between points

Let us derive the second moment of τ, i.e. $E(\tau^2)$ to obtain the variance of the interarrival time between points when λ_t is stationary, $Var(\tau)$.

Lemma 2.4.3 Let N_t and τ as defined. If λ_t is stationary then

$$E(\tau^2)=2\int_0^{\infty}[u\cdot\frac{\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta u})}{\mu_1}\cdot e^{-\frac{\rho}{\delta}\int_0^u\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}]du. \tag{2.4.7}$$

<u>**Proof**</u>

$$E(\tau^2)=\int_0^{\infty}t^2f(t)dt=\int_0^{\infty}[\{\int_0^t 2udu\}f(t)]dt=\int_0^{\infty}2u\{\int_u^{\infty}f(t)dt\}du=2\int_0^{\infty}u\cdot\Pr(\tau>u)du$$

$$=2\int_0^{\infty}[u\cdot\frac{\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta u})}{\mu_1}\cdot e^{-\frac{\rho}{\delta}\int_0^u\hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}]du.$$

□

Theorem 2.4.4 Let N_t and τ as defined. If λ_t is stationary then

$$Var(\tau) = 2\int_0^\infty [u \cdot \frac{\hat{G}(\frac{1}{\delta} - \frac{1}{\delta}e^{-\delta u})}{\mu_1} \cdot e^{-\frac{\rho}{\delta}\int_0^u \hat{G}(\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s})ds}]du - (\frac{\delta}{\mu_1 \rho})^2. \qquad (2.4.8)$$

__Proof__

$$Var(\tau) = E(\tau^2) - \{E(\tau)\}^2.$$

The result follows from (2.4.5) and (2.4.7)

□

3. Insurance Applications

In this chapter we will apply the Cox process incorporating the shot noise process as its intensity to price stop-loss reinsurance contracts and catastrophe insurance derivatives. In other words, we will use this point process as the claim arrival process. From hereon we will refer to the rate of jump arrivals, the rate of decay and the jump size in the shot noise process as the rate of occurrence of catastrophe, the settlement of claim and the size of catastrophe respectively.

Aase (1994) and Meister (1995) discuss pricing techniques such as the general equilibrium approach and the utility maximisation pricing. The non-arbitrage pricing technique will be employed (see Sondermann (1991) and Cummins & Geman (1995)) in our pricing model.

The assumption of no arbitrage opportunities in the market is equivalent to the existence of an equivalent martingale probability measure. We will examine an equivalent martingale probability measure obtained via the *Esscher* transform (see Gerber & Shiu, 1996). Furthermore, using this equivalent martingale probability measure, the pricing models for two contracts will be established and illustrated through numerical examples. In general, more than one equivalent martingale probability measure exists so we will also show more equivalent martingale probability measures. However it will not the purpose of this thesis to decide which is the appropriate one to use.

3.1 The Esscher transform and change of probability measure

In general, the Esscher transform is defined as a change of probability measure for certain stochastic processes. An Esscher transform of such a process induces an equivalent probability measure on the process. The parameters involved for an Esscher transform are determined so that the price of a random payment in the future is a martingale under the new probability measure. A random payment therefore is calculated as the expectation of that at maturity with respect to the equivalent martingale probability measure (also known as the risk-neutral Esscher measure).

We here offer the definition of the Esscher transform that is adopted from Gerber & Shiu (1996).

Definition 3.1.1 Let X_t be a stochastic process and h^* a real number. For a measurable function *f*, the expectation of the random variable $f(X_t)$ with respect to the equivalent martingale probability measure is

$$E^*[f(X_t)] = E\left[f(X_t)\frac{e^{h^*X_t}}{E\left(e^{h^*X_t}\right)}\right] = \frac{E\left[f(X_t)e^{h^*X_t}\right]}{E\left[e^{h^*X_t}\right]} \tag{3.1.1}$$

where the process $\frac{e^{h^*X_t}}{E\left(e^{h^*X_t}\right)}$ is a martingale.

From definition 3.1.1, we need to obtain a martingale that can be used to define a change of probability measure, i.e. it can be used to define the Radon-Nikodym derivative $\frac{dP^*}{dP}$ where P is the original probability measure and P^* is the equivalent martingale probability measure with parameters involved. This martingale will be used to calculate the fair prices for stop-loss reinsurance contract and catastrophe insurance derivatives.

Theorem 3.1.2 Let N_t, λ_t as defined and $\theta^* \geq 1$. Then

$$\theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds} \tag{3.1.2}$$

is a martingale.

__Proof__

The generator of (N_t, X_t) acting on a function $f(n,x)$ is given by

$$\mathrm{A}f(n,x) = \lambda\frac{\partial f}{\partial x} + \lambda[f(n+1,x) - f(n,x)]$$

and $f(n,x)$ has to satisfy $\mathrm{A}f = 0$ for $f(N_t, X_t)$ to be a martingale. Setting $\theta^{*n} e^{\phi^* \int_0^t \lambda_s ds}$ we get the equation

$$\lambda\phi^* + \lambda(\theta^* - 1) = 0$$

yielding

$$\phi^* = -(\theta^* - 1)$$

Therefore

$$\theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds}$$

is a martingale.

□

Theorem 3.1.3 If $f(\lambda_t, t)$ is a martingale with respect to the equivalent martingale probability measure when it is a martingale with respect to the original probability measure

$$f(\lambda_t, t) \cdot \theta^{*N_t} e^{-(\theta^* - 1)\int_0^t \lambda_s ds} \qquad (3.1.3)$$

is also a martingale with respect to the original probability measure.

Proof

We will use

$$\frac{\theta^{*N_t} e^{-(\theta^* - 1)\int_0^t \lambda_s ds}}{E(\theta^{*N_t} e^{-(\theta^* - 1)\int_0^t \lambda_s ds})}$$

as the Radon-Nikodym derivative to define equivalent martingale probability measure. Hence the expected value of $f(\lambda_t, t)$ with respect to the equivalent martingale probability measure is

$$E^*\{f(\lambda_t, t)\} = \frac{E[f(\lambda_t, t) \cdot \theta^{*N_t} e^{-(\theta^* - 1)\int_0^t \lambda_s ds}]}{E(\theta^{*N_t} e^{-(\theta^* - 1)\int_0^t \lambda_s ds})}. \qquad (3.1.4)$$

In theorem 3.1.2 we found a martingale that is the denominator used in (3.1.4). If we condition on λ_0 and N_0, (3.1.4) becomes

$$E^*\{f(\lambda_t, t) | \lambda_0, N_0\} = E[f(\lambda_t, t) \cdot \theta^{*N_t} e^{-(\theta^* - 1)\int_0^t \lambda_s ds} | \lambda_0, N_0]. \qquad (3.1.5)$$

Conditioning on λ_s such that $0 \le s \le t$ in the right-hand side of (3.1.5)

$$E[f(\lambda_t, t) \cdot \theta^{*N_t} e^{-(\theta^* - 1)\int_0^t \lambda_s ds} | \lambda_0, N_0; \lambda_s, 0 \le s \le t] = f(\lambda_t, t) e^{-(\theta^* - 1)\int_0^t \lambda_s ds} E[\theta^{*N_t} | \lambda_0, N_0; \lambda_s, 0 \le s \le t]. \qquad (3.1.6)$$

Therefore from (1.1.3), (3.1.6) becomes

$$E[f(\lambda_t, t) \cdot \theta^{*N_t} e^{-(\theta^* - 1)\int_0^t \lambda_s ds} | \lambda_0, N_0; \lambda_s, 0 \le s \le t] = f(\lambda_t, t). \qquad (3.1.7)$$

Hence, since $f(\lambda_t, t)$ is a martingale with respect to the original probability measure

$$E[f(\lambda_t,t)\cdot \theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds} |\lambda_0, N_0] = E[E\{f(\lambda_t,t)\cdot \theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds} |\lambda_0, N_0; \lambda_s, 0 \le s \le t\}]$$
$$= E\{f(\lambda_t,t)\} = f(\lambda_0,0). \qquad (3.1.8)$$

Furthermore, from (3.1.5) and (3.1.8)

$$E^*\{f(\lambda_t,t)|\lambda_0, N_0\} = f(\lambda_0,0). \qquad (3.1.9)$$

□

Corollary 3.1.4 Let N_t, λ_t as defined. Consider constants γ^*, θ^* such that $\gamma^* \le 0$ and $\theta^* \ge 1$. Then

$$\theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho \int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\} ds} \qquad (3.1.10)$$

is a martingale.

<u>**Proof**</u>

From theorem 3.1.2 it has already been found that $\theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds}$ is a martingale. Put $\nu = \gamma^*$ in theorem 1.3.1 then $e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho \int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\} ds}$ is also a martingale. Therefore if we set $f(\lambda_t,t) = e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho \int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\} ds}$ in (3.1.3) the corollary has been proved.

□

Now we have quite a flexible family of martingales to use as the Radon-Nikodym derivative.

3.2 Pricing of a stop-loss reinsurance contract for catastrophic events

This section deals with the derivation of the pricing model for stop-loss reinsurance contracts for catastrophic events. Its application in computing the premium is illustrated in section 4 of this chapter.

3.2.1 Constant claim sizes

Ignoring the effect of interest rates, the stop-loss reinsurance premium at time 0 is

$$E[(\sum_{i=1}^{N_t} \aleph_i - b)^+] \tag{3.2.1}$$

where:

$\aleph_i$ claim amount

N_t number of claims up to time t

b retention limit

$(\sum_{i=1}^{N_t} \aleph_i - b)^+ = Max(\sum_{i=1}^{N_t} \aleph_i - b,\ 0)$.

If we assume that $\aleph_i = 1$, then

$$E[(\sum_{i=1}^{N_t} \aleph_i - b)^+] = E\left[(N_t - b)^+\right]. \tag{3.2.2}$$

However, to calculate a premium for a reinsurance contract we need to assume that there is an absence of arbitrage opportunities in the market. This can be achieved by using an equivalent martingale probability measure, P^*, within the pricing model used for calculating premiums for reinsurance contracts. That is

$$RP_b = E^*\left[(N_t - b)^+\right] \tag{3.2.3}$$

where RP_b is the fair reinsurance premium at retention level b and E^* denotes the expectation with respect to P^*.

Therefore

$$RP_b = E^*\left[(N_t - b)^+\right] = \sum_{n=b}^{\infty}(n-b)P^*(n) = \sum_{n=b+1}^{\infty}\sum_{j=b+1}^{n} P^*(n) = \sum_{j=b}^{\infty} P^*(N > j) \tag{3.2.4}$$

where P^* is an equivalent martingale probability measure.

We aim to obtain the stop-loss reinsurance premium at retention level b with respect to the equivalent martingale measure. Let us start with a few technical lemmas.

Lemma 3.2.1 Let N_t as defined. Consider a constant $0 \le \theta \le 1$. Then

$$\sum_{j=0}^{\infty} \theta^j \cdot P(N_t \le j) = \frac{E(\theta^{N_t})}{1-\theta}. \tag{3.2.5}$$

Proof

The p.g.f. of N_t is

$$E(\theta^{N_t}) = \sum_{j=0}^{\infty} \theta^j \cdot P(N_t = j).$$

Similarly

$$\begin{aligned}\sum_{j=0}^{\infty} \theta^j \cdot P(N_t \le j) &= \sum_{j=0}^{\infty} \theta^j \cdot \sum_{i=0}^{j} P(N_t = i) = \sum_{i=0}^{\infty}\sum_{j=i}^{\infty} \theta^j \cdot P(N_t = i) \\ &= \sum_{i=0}^{\infty} \frac{\theta^i}{1-\theta} \cdot P(N_t = i) = \frac{\sum_{i=0}^{\infty} \theta^i \cdot P(N_t = i)}{1-\theta} \\ &= \frac{E(\theta^{N_t})}{1-\theta}.\end{aligned}$$

□

Lemma 3.2.2 Let N_t and θ as defined. Then

$$\sum_{j=0}^{\infty} \theta^j \cdot P(N_t > j) = \frac{1}{1-\theta} - \frac{E(\theta^{N_t})}{1-\theta}. \tag{3.2.6}$$

Proof

$$\sum_{j=0}^{\infty} \theta^j \cdot P(N_t > j) = \sum_{j=0}^{\infty} \theta^j \cdot \{1 - P(N_t \le j)\} = \sum_{j=0}^{\infty} \theta^j - \sum_{j=0}^{\infty} \theta^j \cdot P(N_t \le j).$$

From (3.2.5)

$$= \frac{1}{1-\theta} - \frac{E(\theta^{N_t})}{1-\theta}.$$

□

We will now derive the stop-loss reinsurance premium at retention level b with respect to the equivalent martingale measure.

Theorem 3.2.3 Let N_t as defined. Consider constants θ and b such that $0 \le \theta \le 1$ and $b \ge 0$. Then

$$\sum_{b=0}^{\infty} \theta^b RP_b = \frac{1}{1-\theta} E^*(N_t) - \frac{\theta}{(1-\theta)^2} + \frac{\theta}{(1-\theta)^2} E^*(\theta^{N_t}). \tag{3.2.7}$$

Proof

From (3.2.4)

$$\sum_{b=0}^{\infty} \theta^b RP_b = \sum_{b=0}^{\infty} \theta^b E^*\left[(N_t - b)^+\right] = \sum_{b=0}^{\infty} \theta^b \sum_{j=b}^{\infty} \Pr{}^*(N > j)$$

$$= \sum_{j=0}^{\infty} \frac{1-\theta^{j+1}}{1-\theta} \Pr{}^*(N > j)$$

$$= \frac{1}{1-\theta} \{\sum_{j=0}^{\infty} \Pr{}^*(N > j) - \theta \sum_{j=0}^{\infty} \theta^j \Pr{}^*(N > j)\}.$$

From lemma 3.2.2

$$\sum_{b=0}^{\infty} \theta^b RP_b = \frac{1}{1-\theta} E^*(N_t) - \frac{\theta}{(1-\theta)^2} + \frac{\theta}{(1-\theta)^2} E^*(\theta^{N_t}).$$

□

We can see that $E^*(N_t)$ and $E^*(\theta^{N_t})$ need to be determined in order to obtain stop-loss reinsurance premium with respect to the equivalent martingale measure at retention level b. Firstly we will examine the generator A^* of the process (N_t, λ_t, t) acting on a function $f(n, \lambda, t)$ with respect to the equivalent martingale probability measure.

Lemma 3.2.4 Let λ_t as defined. Assume that $f(n, \lambda, t) = f(\lambda, t)$ for all n and that $e^{v^* \lambda_t}$ is a martingale. Consider a constant v^* such that $v^* \ge 0$. Then

$$A^* f(\lambda, 0) = \frac{A\,\{f(\lambda, 0) e^{-v^* \lambda}\}}{e^{-v^* \lambda}}. \tag{3.2.8}$$

Proof

The generator of the process (λ_t, t) acting on a function $f(\lambda, t)$ with respect to the equivalent martingale probability measure is

$$A^* f(\lambda, 0) = \lim_{t \downarrow 0} \frac{E^*[f(\lambda_t, t) | \lambda] - f(\lambda, 0)}{t}. \tag{3.2.9}$$

We will use $\dfrac{e^{v^*\lambda_t}}{E(e^{v^*\lambda_t})}$ as the Radon-Nikodym derivative to define equivalent martingale probability measure. Hence, the expected value of $f(\lambda_t,t)$ given λ with respect to the equivalent martingale probability measure is

$$E^*\{f(\lambda_t,t)|\lambda\} = \frac{E[f(\lambda_t,t)\cdot e^{-v^*\lambda_t}|\lambda]}{E(e^{-v^*\lambda_t}|\lambda)}. \tag{3.2.10}$$

Since the denominator in (3.2.10) is a martingale, it becomes

$$E^*\{f(\lambda_t,t)|\lambda\} = \frac{f(\lambda,0)\cdot e^{-v^*\lambda} + \int_0^t E[\mathrm{A} f(\lambda_s,s)\cdot e^{-v^*\lambda_s}|\lambda]ds}{e^{-v^*\lambda}}. \tag{3.2.11}$$

Set (3.2.11) in (3.2.9) then

$$\mathrm{A}^* f(\lambda,0) = \frac{1}{e^{-v^*\lambda}} \lim_{t\downarrow 0} \int_0^t \frac{E[\mathrm{A} f(\lambda_s,s)\cdot e^{-v^*\lambda_s}|\lambda]ds}{t}. \tag{3.2.12}$$

Therefore, from Dynkin's formula (see Øksendal (1992)) (3.2.8) follows immediately.

□

Theorem 3.2.5 Let N_t, λ_t as defined. Consider constants γ^*, θ^* such that $\gamma^* \le 0$ and $\theta^* \ge 1$. Then

$$\mathrm{A}^* f(n,\lambda,t) = \frac{\partial f}{\partial t} + \theta^*\lambda\{f(n+1,\lambda,t) - f(n,\lambda,t)\} - \delta\lambda\frac{\partial f}{\partial \lambda} + \rho^*(t)\{\int_0^\infty f(n,\lambda+y,t)dG^*(y;t) - f(n,\lambda,t)\} \tag{3.2.13}$$

where $\rho^*(t) = \rho\hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y}dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

<u>Proof</u>

From corollary 3.1.4 we can use

$$\frac{\theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds} e^{-\gamma^*\lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\}ds}}{E[\theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds} e^{-\gamma^*\lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\}ds}]} \tag{3.2.14}$$

as the Radon-Nikodym derivative to define an equivalent martingale probability measure.

Therefore from lemma 3.2.4

$$A^* f(N_t,\lambda_t,t) = \frac{A\, f(N_t,\lambda_t,t)\cdot \theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\}ds}}{\theta^{*N_t} e^{-(\theta^*-1)\int_0^t \lambda_s ds} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\}ds}}.$$

From (2.1.2), using the generator with respect to the original probability measure,

$$A\, f(n,\lambda,t)\cdot \theta^{*n} e^{-(\theta^*-1)\int_0^t \lambda_s ds} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\}ds}$$

$$= [\frac{\partial f}{\partial t} + \theta^* \lambda \{f(n+1,\lambda,t) - f(n+1,\lambda,t)\}$$

$$- \delta\lambda \frac{\partial f}{\partial \lambda} + \rho\{\int_0^\infty f(n,\lambda+y,t) e^{-\gamma^* e^{\delta t} y} dG(y) - \hat{g}(\gamma^* e^{\delta t}) f(n,\lambda,t)\}]\cdot \theta^{*n} e^{-(\theta^*-1)\int_0^t \lambda_s ds} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\hat{g}(\gamma^* e^{\delta s})\}ds}.$$

(3.2.15)

Therefore

$$A^* f(n,\lambda,t) = \frac{\partial f}{\partial t} + \theta^* \lambda \{f(n+1,\lambda,t) - f(n,\lambda,t)\}$$
$$- \delta\lambda \frac{\partial f}{\partial \lambda} + \rho^*(t)\{\int_0^\infty f(n,\lambda+y,t) dG^*(y;t) - f(n,\lambda,t)\} \quad (3.2.16)$$

where $\rho^*(t) = \rho \hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

□

Theorem 3.2.5 yields the following:

(i) The claim intensity function λ_t has changed to $\theta^* \lambda_t$;

(ii) The rate of jump arrival ρ has changed to $\rho^*(t) = \rho \hat{g}(\gamma^* e^{\delta t})$ (it now depends on time);

(iii) The jump size measure $dG(y)$ has changed to $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ (it now depends on time).

In practice, the reinsurer will calculate the value of a stop-loss contract using $\theta^* > 1$ and $\gamma^* < 0$. This results in the reinsurer assuming that there will be a higher value of claims, a higher value of the damage caused by the catastrophe and more catastrophes occurring in a given period of time. These assumptions are necessary as the reinsurer wants compensation for the risks involved in operating in an imperfect market. The reinsurer would also be aiming to maximise their shareholders' wealth by earning profits rather than operating at breakeven point where premiums are equal to claims.

Now, let us derive the expected value of N_t and the Laplace transform of the distribution of N_t with respect to the equivalent martingale probability measure, i.e. $E^*(N_t)$ and $E^*(\theta^{N_t})$.

Theorem 3.2.6 Let N_t as defined and λ_t be a generalised shot noise process with $\rho^*(t) = \rho\hat{g}(\gamma^* e^{\delta t})$, $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ and $\delta(t) = \delta$. Consider constants γ^*, θ^* such that $\gamma^* \le 0$ and $\theta^* \ge 1$. Then

$$E(N_{t_2} - N_{t_1}) = \int_{t_1}^{t_2} E(\lambda_s) ds = \left(\frac{1 - e^{-\delta(t_2 - t_1)}}{\delta}\right) E(\lambda_{t_1}) + \frac{1}{\delta}\int_{t_1}^{t_2} (1 - e^{-\delta(t_2 - s)}) \rho^*(s) \mu_1^*(s) ds. \tag{3.2.17}$$

and

$$E^*(N_{t_2} - N_{t_1}) = \theta^* \int_{t_1}^{t_2} E(\lambda_s) ds = \theta^* \left[\left(\frac{1 - e^{-\delta(t_2 - t_1)}}{\delta}\right) E(\lambda_{t_1}) + \frac{1}{\delta}\int_{t_1}^{t_2} (1 - e^{-\delta(t_2 - s)}) \rho^*(s) \mu_1^*(s) ds\right]. \tag{3.2.18}$$

__Proof__

From (2.2.1)

$$E[N_{t_2} - N_{t_1}] = \int_{t_1}^{t_2} E(\lambda_s) ds. \tag{3.2.19}$$

Conditioning on λ_{t_1} in (3.2.19) and theorem 1.2.7 gives us

$$\begin{aligned} E(N_{t_2} - N_{t_1} | \lambda_{t_1}) &= \int_{t_1}^{t_2} E(\lambda_s | \lambda_{t_1}) ds = \int_{t_1}^{t_2} [\lambda_{t_1} e^{-\delta(s - t_1)} + e^{-\delta s} \int_{t_1}^{s} e^{\delta u} \rho(u) \mu_1(u) du] ds \\ &= \frac{1 - e^{-\delta(t_2 - t_1)}}{\delta} \lambda_{t_1} + \frac{1}{\delta} \int_{t_1}^{t_2} (1 - e^{-\delta(t_2 - s)}) \rho(s) \mu_1(s) ds. \end{aligned} \tag{3.2.20}$$

Hence

$$E(N_{t_2}-N_{t_1})=\int_{t_1}^{t_2}E(\lambda_s)ds=\left(\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right)E(\lambda_{t_1})+\frac{1}{\delta}\int_{t_1}^{t_2}(1-e^{-\delta(t_2-s)})\rho(s)\mu_1(s)ds \quad (3.2.21)$$

If we set $\rho(s)=\rho^*(s)$ and $\mu_1(s)=\mu_1^*(s)$ in (3.2.21) (3.2.17) follows immediately.

From theorem 3.2.5

$$E^*[N_{t_2}-N_{t_1}]=\theta^*\int_{t_1}^{t_2}E(\lambda_s)ds. \quad (3.2.22)$$

Therefore (3.2.18) follows from (3.2.17) and (3.2.22).

□

Theorem 3.2.7 Let N_t as defined and $\Im_t^N$ be the filtration generated by $\{N_s;\ 0\le s\le t\}$. Also let λ_t be the generalised shot noise process and $\Im_t^\lambda$ be the filtration generated by $\{\lambda_s;\ 0\le s\le t\}$. Consider constants γ^*, θ^* and θ such that $\gamma^*\le 0$, $\theta^*\ge 1$ and $0\le\theta\le 1$. Then

$$E^*(\theta^{N_{t_2}-N_{t_1}}|\Im_{t_1}^{N,\lambda})=E(e^{-\theta^*(1-\theta)\int_{t_1}^{t_2}\lambda_s ds}|\Im_{t_1}^{\lambda})=e^{-\frac{\theta^*(1-\theta)}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}}e^{-\int_{t_1}^{t_2}\rho^*(s)[1-\hat{g}^*\{\frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds}$$

(3.2.23)

where $\rho^*(t)=\rho\hat{g}(\gamma^*e^{\delta t})$, $dG^*(y;t)=\dfrac{e^{-\gamma^*e^{\delta t}y}dG(y)}{\hat{g}(\gamma^*e^{\delta t})}$ and $\hat{g}^*(u;t)=\int_0^\infty e^{-uy}dG^*(y;t)$.

<u>**Proof**</u>

From (3.2.22) and (1.1.4) $E^*(\theta^{N_{t_2}-N_{t_1}})=E(e^{-\theta^*(1-\theta)\int_{t_1}^{t_2}\lambda_s ds})$. Therefore (3.2.23) follows immediately if we set $\nu=\theta^*(1-\theta)$, $\rho(s)=\rho^*(s)$ and $\hat{g}(u;s)=\hat{g}^*(u;s)$ in (2.1.37).

□

We can now substitute $E^*(N_t)$ and $E^*(\theta^{N_t})$ in (3.2.7) and find stop-loss reinsurance premiums at retention level b with respect to the equivalent martingale measure.

Theorem 3.2.8 Let N_t as defined and λ_t be the generalised shot noise process. Consider constants γ^*, θ^* and θ such that $\gamma^*\le 0$, $\theta^*\ge 1$ and $0\le\theta\le 1$. Then

$$\sum_{b=0}^{\infty} \theta^b RP_b = \frac{\theta^*}{1-\theta}\left[\left(\frac{1-e^{-\delta t}}{\delta}\right)E(\lambda_0) + \frac{1}{\delta}\int_0^t \{1-e^{-\delta(t-s)}\}\rho^*(s)\mu_1^*(s)ds\right] - \frac{\theta}{(1-\theta)^2}$$

$$+\frac{\theta}{(1-\theta)^2} e^{-\int_0^t \rho^*(s)[1-\hat{g}^*\{\frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta(t-s)});s\}]ds} E[e^{-\frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta t})\lambda_0}]$$

(3.2.24)

where $\rho^*(t) = \rho\hat{g}(\gamma^* e^{\delta t})$, $\mu_1^*(t) = \int_0^\infty y dG^*(y;t)$ and $\hat{g}^*(u;t) = \int_0^\infty e^{-uy} dG^*(y;t)$.

Proof

From (3.2.7)

$$\sum_{b=0}^{\infty} \theta^b RP_b = \frac{1}{1-\theta}E^*(N_t) - \frac{\theta}{(1-\theta)^2} + \frac{\theta}{(1-\theta)^2}E^*(\theta^{N_t}).$$

Therefore use (3.2.18) and (3.2.23) then the result follows.

□

We will now analyse the above results assuming that the jump size distribution is exponential ($g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$) and that λ_t is '$-\infty$' asymptotic. Since $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$, we can obtain that $g^*(y;t) = (\alpha + \gamma^* e^{\delta t})e^{-(\alpha+\gamma^* e^{\delta t})y}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma^* \le 0$ and $t < \frac{1}{\delta}\ln(-\frac{\alpha}{\gamma^*})$. It is clear that such a model is appropriate in the short term only as it break down for $t \ge \frac{1}{\delta}\ln(-\frac{\alpha}{\gamma^*})$.

Corollary 3.2.9 Let N_t and λ_t as defined. Also let the jump size distribution be exponential, i.e. $g^*(y;t) = (\alpha + \gamma^* e^{\delta t})e^{-(\alpha+\gamma^* e^{\delta t})y}$, $y > 0$, $-\alpha e^{-\delta t} < \gamma^* \le 0$ and $t < \frac{1}{\delta}\ln(-\frac{\alpha}{\gamma^*})$. Consider a constant $\theta^* \ge 1$. Then

$$E^*(N_{t_2} - N_{t_1}) = \theta^*\left[\left(\frac{1-e^{-\delta(t_2-t_1)}}{\delta}\right)E(\lambda_{t_1}) + \frac{1}{\delta}\left\{\frac{\rho}{\alpha}(t_2-t_1) - \frac{\rho}{\delta\alpha}\ln\left(\frac{\gamma^* e^{\delta t_2}+\alpha}{\gamma^* e^{\delta t_1}+\alpha}\right) - \frac{\rho(1-e^{-\delta(t_2-t_1)})}{\delta(\alpha+\gamma^* e^{\delta t_1})}\right\}\right]$$

(3.2.25)

where $0 < t_1 < t_2 < t$.

Proof

The result follows if we set $\rho^*(s) = \rho\frac{\alpha}{\alpha+\gamma^* e^{\delta s}}$ and $\mu_1^*(s) = \frac{1}{\alpha+\gamma^* e^{\delta s}}$ in (3.2.18).

□

Corollary 3.2.10 Let the jump size distribution is exponential. If λ_t is '$-\infty$' asymptotic, the '$-\infty$' asymptotic expected value of λ_{t_1} is

$$\frac{\rho}{\delta(\alpha+\gamma^* e^{\delta t_1})} \tag{3.2.26}$$

where $0 < t_1 < t$.

Proof

The result follows if we set $\rho^*(s) = \rho\frac{\alpha}{\alpha+\gamma^* e^{\delta s}}$ and $\mu_1^*(s) = \frac{1}{\alpha+\gamma^* e^{\delta s}}$ in (1.2.15).

□

Corollary 3.2.11 Let N_t as defined and the jump size distribution be exponential. Consider a constant $\theta^* \geq 1$. If λ_t is '$-\infty$' asymptotic

$$E^*(N_{t_2} - N_{t_1}) = \frac{\theta^*\rho}{\delta\alpha}(t_2 - t_1) - \frac{\theta^*\rho}{\delta^2\alpha}\ln\left(\frac{\gamma^* e^{\delta t_2} + \alpha}{\gamma^* e^{\delta t_1} + \alpha}\right) \tag{3.2.27}$$

where $0 < t_1 < t_2 < t$.

Proof

Since λ_t is '$-\infty$' asymptotic, set (3.2.26) in (3.2.25) and the result follows.

□

Corollary 3.2.12 Let N_t as defined and $\Im_t^N$ be the filtration generated by $\{N_s;\ 0 \leq s \leq t\}$. Also let λ_t be the generalised shot noise process and $\Im_t^\lambda$ be the filtration generated by $\{\lambda_s;\ 0 \leq s \leq t\}$. Consider constants θ^* and θ such that $\theta^* \geq 1$ and $0 \leq \theta \leq 1$. Assuming that the jump size distribution is exponential then

$$E^*(\theta^{N_{t_2}-N_{t_1}} \mid \Im_{t_1}^{N,\lambda}) = E(e^{-\theta^*(1-\theta)\int_{t_1}^{t_2}\lambda_s ds} \mid \Im_{t_1}^\lambda)$$

$$
= e^{-\frac{\theta^*(1-\theta)}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} \left(\frac{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}{\gamma^* e^{\delta t_1} + \alpha} \right)^{\frac{\rho}{\delta}} \left(\frac{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}(1 - e^{-\delta(t_2-t_1)})}{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}} \right)^{\frac{\alpha\rho}{\delta\alpha + \theta^*(1-\theta)}}
$$

(3.2.28)

where $0 < t_1 < t_2 < t$.

Proof

The result follows if we set $\nu = \theta^*(1-\theta)$ in (2.1.39) in corollary 2.1.16.

□

Corollary 3.2.13 Let N_t as defined and the jump size distribution be exponential. Consider constants θ^* and θ such that $\theta^* \geq 1$ and $0 \leq \theta \leq 1$. Furthermore if λ_t is '$-\infty$' asymptotic then

$$
E^*(\theta^{N_{t_2}-N_{t_1}}) = \left(\frac{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}}{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}(1 - e^{-\delta(t_2-t_1)})} \right)^{\frac{\rho}{\delta}} \left(\frac{\gamma^* e^{\delta t_1} + \alpha + \frac{\theta^*(1-\theta)}{\delta}(1 - e^{-\delta(t_2-t_1)})}{\gamma^* e^{\delta t_1} + \alpha e^{-\delta(t_2-t_1)}} \right)^{\frac{\alpha\rho}{\delta\alpha + \theta^*(1-\theta)}}
$$

(3.2.29)

where $0 < t_1 < t_2 < t$.

Proof

Set $\nu = \frac{\theta^*(1-\theta)}{\delta}\{1 - e^{-\delta(t_2-t_1)}\}$ in (1.2.9) and from corollary 3.2.12 the result follows.

□

Corollary 3.2.14 Let N_t, λ_t as defined and the jump size distribution be exponential. Consider constants θ^* and θ such that $\theta^* \geq 1$ and $0 \leq \theta \leq 1$. Then

$$
\sum_{b=0}^{\infty} \theta^b RP_b = \frac{\theta^*}{1-\theta}\left[\left(\frac{1-e^{-\delta t}}{\delta} \right) E(\lambda_0) + \frac{1}{\delta}\left\{ \frac{\rho}{\alpha} t - \frac{\rho}{\delta\alpha} \ln\left(\frac{\gamma^* e^{\delta t} + \alpha}{\gamma^* + \alpha} \right) - \frac{\rho(1-e^{-\delta t})}{\delta(\gamma^* + \alpha)} \right\} \right] - \frac{\theta}{(1-\theta)^2}
$$

$$
+ \frac{\theta}{(1-\theta)^2} \left(\frac{\gamma^* + \alpha e^{-\delta t}}{\gamma^* + \alpha} \right)^{\frac{\rho}{\delta}} \left(\frac{\gamma^* + \alpha + \frac{\theta^*(1-\theta)}{\delta}(1 - e^{-\delta t})}{\gamma^* + \alpha e^{-\delta t}} \right)^{\frac{\alpha\rho}{\delta\alpha + \theta^*(1-\theta)}} \cdot E[e^{-\frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta t})\lambda_0}].
$$

(3.2.30)

Proof

The result follows immediately from theorem 3.2.8, corollary 3.2.9 and corollary 3.2.12.

□

Corollary 3.2.15 Let N_t as defined and the jump size distribution be exponential. Consider constants θ^* and θ such that $\theta^* \geq 1$ and $0 \leq \theta \leq 1$. Furthermore if λ_t is '$-\infty$' asymptotic then

$$\sum_{b=0}^{\infty} \theta^b RP_b = \frac{\theta^*}{1-\theta}\left[\frac{\rho}{\delta\alpha}t - \frac{\rho}{\delta^2\alpha}\ln\left(\frac{\gamma^* e^{\delta t}+\alpha}{\gamma^*+\alpha}\right)\right] - \frac{\theta}{(1-\theta)^2}$$
$$+\frac{\theta}{(1-\theta)^2}\left(\frac{\gamma^*+\alpha e^{-\delta t}}{\gamma^*+\alpha+\frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta t})}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma^*+\alpha+\frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta t})}{\gamma^*+\alpha e^{-\delta t}}\right)^{\frac{\alpha\rho}{\delta\alpha+\theta^*(1-\theta)}}.$$

(3.2.31)

Proof

The result follows immediately from theorem 3.2.8, corollary 3.2.11 and corollary 3.2.13.

□

We will close this section with a lemma that proves that the joint Laplace transform of N_t and λ_t with respect to the equivalent martingale probability measure is equal to an Esscher transform of such a process.

Lemma 3.2.16 Let N_t and λ_t as defined. Consider constants γ^*, θ^* and θ such that $\gamma^* \leq 0$, $\theta^* \geq 1$ and $0 \leq \theta \leq 1$. Then

$$E^*\left(\theta^{N_{t_2}-N_{t_1}} e^{-v\lambda_{t_2}} \middle| \Im_{t_1}^{N,\lambda}\right) = \frac{E\left[\theta^{N_{t_2}-N_{t_1}} e^{-v\lambda_{t_2}} \theta^{*\,N_{t_2}-N_{t_1}} e^{-(\theta^*-1)\int_{t_1}^{t_2}\lambda_s ds} e^{-\gamma^*\lambda_{t_2} e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2}\{1-\hat{g}(\gamma^* e^{\delta s})\}ds} \middle| \Im_{t_1}^{N,\lambda}\right]}{E\left[\theta^{*\,N_{t_2}-N_{t_1}} e^{-(\theta^*-1)\int_{t_1}^{t_2}\lambda_s ds} e^{-\gamma^*\lambda_{t_2} e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2}\{1-\hat{g}(\gamma^* e^{\delta s})\}ds} \middle| \Im_{t_1}^{N,\lambda}\right]}.$$

(3.2.32)

Proof

From theorem 3.2.7, the left-hand side of (3.2.32) is

$$E^*(\theta^{N_{t_2}-N_{t_1}} e^{-\nu\lambda_{t_2}} | \Im_{t_1}^{N,\lambda}) = E(e^{-\theta^*(1-\theta)\int_{t_1}^{t_2}\lambda_s ds} e^{-\nu\lambda_{t_2}} | \Im_{t_1}^{\lambda}).$$

Setting $\nu_1 = \theta^*(1-\theta)$, $\nu_2 = \nu$, $\rho(s) = \rho^*(s)$ and $\hat{g}(u;s) = \hat{g}^*(u;s)$ in (2.1.33), then

$$E^*(\theta^{N_{t_2}-N_{t_1}} e^{-\nu\lambda_{t_2}} | \Im_{t_1}^{N,\lambda}) = e^{-[\frac{\theta^*(1-\theta)}{\delta}+\{\nu-\frac{\theta^*(1-\theta)}{\delta}\}e^{-\delta(t_2-t_1)}]\lambda_{t_1}} e^{-\rho\int_{t_1}^{t_2}\hat{g}(\gamma^* e^{\delta s})ds} e^{\rho\int_{t_1}^{t_2}\hat{g}[\frac{\theta^*(1-\theta)}{\delta}+\{\nu+\gamma^* e^{\delta t_2}-\frac{\theta^*(1-\theta)}{\delta}\}e^{-\delta(t_2-s)}]ds}.$$

(3.2.33)

Conditioning on λ_s such that $t_1 \le s \le t_2$, then the denominator of the right-hand side of (3.2.32) is

$$E[\theta^{*N_{t_2}-N_{t_1}} e^{-(\theta^*-1)\int_{t_1}^{t_2}\lambda_s ds} e^{-\gamma^*\lambda_{t_2}e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2}\{1-\hat{g}(\gamma^* e^{\delta s})\}ds} | \Im_{t_1}^{N,\lambda}; \lambda_s\ ,\ t_1 \le s \le t_2]$$

$$= e^{-(\theta^*-1)\int_{t_1}^{t_2}\lambda_s ds} e^{-\gamma^*\lambda_{t_2}e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2}\{1-\hat{g}(\gamma^* e^{\delta s})\}ds} E[\theta^{*N_{t_2}-N_{t_1}} | \Im_{t_1}^{N,\lambda}; \lambda_s\ ,\ t_1 \le s \le t_2].$$

(3.2.34)

Therefore from (1.1.3)

$$E[\theta^{*N_{t_2}-N_{t_1}} e^{-(\theta^*-1)\int_{t_1}^{t_2}\lambda_s ds} e^{-\gamma^*\lambda_{t_2}e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2}\{1-\hat{g}(\gamma^* e^{\delta s})\}ds} | \Im_{t_1}^{N,\lambda}; \lambda_s\ ,\ t_1 \le s \le t_2] = e^{-\gamma^*\lambda_{t_2}e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2}\{1-\hat{g}(\gamma^* e^{\delta s})\}ds}.$$

Hence

$$E[\theta^{*N_{t_2}-N_{t_1}} e^{-(\theta^*-1)\int_{t_1}^{t_2}\lambda_s ds} e^{-\gamma^*\lambda_{t_2}e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2}\{1-\hat{g}(\gamma^* e^{\delta s})\}ds} | \Im_{t_1}^{N,\lambda}] = e^{\rho\int_{t_1}^{t_2}\{1-\hat{g}(\gamma^* e^{\delta s})\}ds} E(e^{-\gamma^*\lambda_{t_2}e^{\delta t_2}} | \Im_{t_1}^{N,\lambda}).$$

(3.2.35)

Similarly, conditioning on λ_s such that $t_1 \le s \le t_2$, then the numerator of the right-hand side of (3.2.32) is

$$E[(\theta^* \theta)^{N_{t_2}-N_{t_1}} e^{-(\theta^*-1)\int_{t_1}^{t_2} \lambda_s ds} e^{-(\nu+\gamma^* e^{\delta t_2})\lambda_{t_2}} e^{\rho\int_{t_1}^{t_2} \{1-\hat{g}(\gamma^* e^{\delta s})\} ds} \big| \Im_{t_1}^{N,\lambda}]$$

$$= e^{\rho\int_{t_1}^{t_2} \{1-\hat{g}(\gamma^* e^{\delta s})\} ds} E\{e^{-\theta^*(1-\theta)\int_{t_1}^{t_2} \lambda_s ds} e^{-(\nu+\gamma^* e^{\delta t_2})\lambda_{t_2}} \big| \Im_{t_1}^{N,\lambda}\}. \tag{3.2.36}$$

Therefore, from (3.2.35) and (3.2.36), the right-hand side of (3.2.32) is

$$\frac{E[\theta^{N_{t_2}-N_{t_1}} e^{-\nu\lambda_{t_2}} \theta^{*N_{t_2}-N_{t_1}} e^{-(\theta^*-1)\int_{t_1}^{t_2} \lambda_s ds} e^{-\gamma^* \lambda_{t_2} e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2} \{1-\hat{g}(\gamma^* e^{\delta s})\} ds} \big| \Im_{t_1}^{N,\lambda}]}{E[\theta^{*N_{t_2}-N_{t_1}} e^{-(\theta^*-1)\int_{t_1}^{t_2} \lambda_s ds} e^{-\gamma^* \lambda_{t_2} e^{\delta t_2}} e^{\rho\int_{t_1}^{t_2} \{1-\hat{g}(\gamma^* e^{\delta s})\} ds} \big| \Im_{t_1}^{N,\lambda}]} = \frac{E\{e^{-\theta^*(1-\theta)\int_{t_1}^{t_2} \lambda_s ds} e^{-(\nu+\gamma^* e^{\delta t_2})\lambda_{t_2}} \big| \Im_{t_1}^{N,\lambda}\}}{E(e^{-\gamma^* \lambda_{t_2} e^{\delta t_2}} \big| \Im_{t_1}^{N,\lambda})}. \tag{3.2.37}$$

Set $\nu_1 = \theta^*(1-\theta)$, $\nu_2 = \nu+\gamma^* e^{\delta t_2}$ in (2.1.9) for the numerator of (3.2.37) and set $\nu = \gamma^* e^{\delta t_2}$ in (1.3.2) for the denominator of (3.2.37). Then

$$= \frac{e^{-[\frac{\theta^*(1-\theta)}{\delta}+\{\nu+\gamma^* e^{\delta t_2}-\frac{\theta^*(1-\theta)}{\delta}\}e^{-\delta(t_2-t_1)}]\lambda_{t_1}} e^{-\rho\int_{t_1}^{t_2} [1-\hat{g}\{\frac{\theta^*(1-\theta)}{\delta}+(\nu+\gamma^* e^{\delta t_2}-\frac{\theta^*(1-\theta)}{\delta})e^{-\delta(t_2-s)}\}] ds}}{e^{-\gamma^* e^{\delta t_2} e^{-\delta(t_2-t_1)}\lambda_{t_1}} e^{-\rho\int_{t_1}^{t_2} [1-\hat{g}\{\gamma^* e^{\delta t_2} e^{-\delta(t_2-s)}\}] ds}}$$

$$= e^{-[\frac{\theta^*(1-\theta)}{\delta}+\{\nu-\frac{\theta^*(1-\theta)}{\delta}\}e^{-\delta(t_2-t_1)}]\lambda_{t_1}} e^{-\rho\int_{t_1}^{t_2} \hat{g}(\gamma^* e^{\delta s}) ds} e^{\rho\int_{t_1}^{t_2} \hat{g}[\frac{\theta^*(1-\theta)}{\delta}+\{\nu+\gamma^* e^{\delta t_2}-\frac{\theta^*(1-\theta)}{\delta}\}e^{-\delta(t_2-s)}] ds}. \tag{3.2.38}$$

Hence from (3.2.33) and (3.2.38) the lemma has been proved.

□

3.2.2 Random claim sizes

Let us look at the stop-loss reinsurance premium for catastrophic events when the claim size is random. Again, in order to calculate a fair price for a reinsurance contract we need to assume that there is an absence of arbitrage opportunities in the market. This can be achieved by using an equivalent martingale measure P^*.

Let C_t be the total amount of claims up to time t then the fair stop-loss reinsurance premium at time 0 is

$$E^*[(\sum_{i=1}^{N_t}\aleph_i - b)^+] = E^*[(C_t - b)^+] \qquad (3.2.39)$$

where $C_t = \sum_{i=1}^{N_t}\aleph_i$ and all other symbols have previously been defined.

Theorem 3.2.17 Let N_t and C_t as defined. Also let the claim size distribution be gamma, i.e. $h(u) = \dfrac{\beta^\varphi u^{\varphi-1}e^{-\beta u}}{(\varphi-1)!}$, $u > 0,\ \beta > 0,\ \varphi \ge 1$. Then

$$E^*[(C_t - b)^+] = \sum_{n=1}^{\infty} a_n^* \{\frac{n\varphi}{\beta}\int_b^\infty \frac{\beta^{n\varphi+1}c^{n\varphi}e^{-\beta c}}{(n\varphi)!}dc - b\int_b^\infty \frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}dc\} \qquad (3.2.40)$$

where $a_n^* = P^*(N_t = n)$.

__Proof__

Since the claim size distribution is gamma, C_t is also gamma with parameters $n\varphi$ and β given $N_t = n$. Therefore

$$\begin{aligned} E^*[(C_t - b)^+] &= \sum_{n=1}^{\infty} P^*(N_t = n)\int_b^\infty (c-b)\frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}dc \\ &= \sum_{n=1}^{\infty} P^*(N_t = n)\{\frac{n\varphi}{\beta}\int_b^\infty \frac{\beta^{n\varphi+1}c^{n\varphi}e^{-\beta c}}{(n\varphi)!}dc - b\int_b^\infty \frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}dc\}. \end{aligned} \qquad (3.2.41)$$

Set $a_n^* = P^*(N_t = n)$ in (3.2.41), then the result follows immediately.

□

Corollary 3.2.18 Let N_t and C_t as defined and the claim size distribution be exponential, i.e. $h(u)=\beta e^{-\beta u}$, $u>0$, $\beta>0$. Then

$$E^{*}[(C_t-b)^{+}]=\sum_{n=1}^{\infty}a_n^{*}\{\frac{n}{\beta}\int_b^{\infty}\frac{\beta^{n+1}c^{n}e^{-\beta c}}{n!}dc-b\int_b^{\infty}\frac{\beta^{n}c^{n-1}e^{-\beta c}}{(n-1)!}dc\}. \qquad (3.2.42)$$

Proof

The result follows immediately if we set $\varphi=1$ in (3.2.41).

□

3.3 Pricing of catastrophe insurance derivatives

3.3.1 Catastrophe insurance futures

In this section we will derive the model for the pricing of catastrophe insurance futures. Its application in computing the price is illustrated in section 4 of this chapter.

The value of the insurance futures, F_t, at maturity t is given by

$$F_t = 25,000 \times Min\left(\frac{L_t}{\Pi},\, 2\right). \tag{3.3.1}$$

Assuming that $L_t = \sum_{i=1}^{N_t} \aleph_i = C_t$, ignoring the effect of interest rates, the price of the insurance futures at time 0 is

$$E\left[25,000 \times Min\left(\frac{C_t}{\Pi},\, 2\right)\right]. \tag{3.3.2}$$

In order to calculate a fair price for the insurance futures contract, we assume that there is an absence of arbitrage opportunities in the market which can be achieved by using an equivalent martingale probability measure P^*. Therefore

$$F_0 = E^*\left[25,000 \times Min\left(\frac{C_t}{\Pi},\, 2\right)\right] \tag{3.3.3}$$

where F_0 is a fair price for the insurance futures contract.

Now let us derive the price of the insurance futures with respect to the equivalent martingale probability measure.

Theorem 3.3.1 Let F_t , C_t and Π as defined. Then

$$F_0 = 25,000 \times \frac{1}{\Pi}\left[E^*(C_t) - E^*\{(C_t - 2\Pi)^+\}\right] \tag{3.3.4}$$

where $(C_t - 2\Pi)^+ = Max(C_t - 2\Pi,\ 0)$.

Proof

(3.3.3) can be written as

$$F_0 = E^*\left[25{,}000 \times Min\left(\frac{C_t}{\Pi},\ 2\right)\right] = E^*\left[25{,}000 \times \left\{\frac{C_t}{\Pi} - Max\left(\frac{C_t}{\Pi} - 2,\ 0\right)\right\}\right]$$

$$= 25{,}000 \times E^*\left[\left\{\frac{C_t}{\Pi} - Max\left(\frac{C_t}{\Pi} - 2,\ 0\right)\right\}\right] = 25{,}000 \times \frac{1}{\Pi}\left[E^*(C_t) - E^*\{Max(C_t - 2\Pi\,,\ 0)\}\right].$$

(3.3.5)

If we set $(C_t - 2\Pi)^+ = Max(C_t - 2\Pi,\ 0)$ in (3.3.5) then (3.3.4) follows immediately.

□

We will show how to apply theorem 3.3.1 if the claim size distributions are gamma (i.e. $h(u) = \dfrac{\beta^{\varphi} u^{\varphi-1} e^{-\beta u}}{(\varphi-1)!}$, $u > 0,\ \beta > 0,\ \varphi \geq 1$) and exponential (i.e. $h(u) = \beta e^{-\beta u}$, $u > 0$, $\beta > 0$).

Corollary 3.3.2 Let F_t , C_t and Π as defined. Also let the claim size distribution be gamma, i.e. $h(u) = \dfrac{\beta^{\varphi} u^{\varphi-1} e^{-\beta u}}{(\varphi-1)!}$, $u > 0,\ \beta > 0,\ \varphi \geq 1$. Then

$$F_0 = 25{,}000\frac{1}{\Pi}\left[\left\{\sum_{n=1}^{\infty} a_n^* \frac{n\varphi}{\beta}\right\} - \left\{\sum_{n=1}^{\infty} a_n^* \left(\frac{n\varphi}{\beta}\int_{2\Pi}^{\infty} \frac{\beta^{n\varphi+1} c^{n\varphi} e^{-\beta c}}{(n\varphi)!} dc - 2\Pi \int_{2\Pi}^{\infty} \frac{\beta^{n\varphi} c^{n\varphi-1} e^{-\beta c}}{(n\varphi-1)!} dc\right)\right\}\right].$$

(3.3.6)

Proof

From theorem 3.2.17

$$E^*(C_t) = \sum_{n=1}^{\infty} a_n^* \int_0^{\infty} c \frac{\beta^{n\varphi} c^{n\varphi-1} e^{-\beta c}}{(n\varphi-1)!} dc = \sum_{n=1}^{\infty} a_n^* \frac{n\varphi}{\beta} \qquad (3.3.7)$$

and

$$E^*[(C_t - 2\Pi)^+] = \sum_{n=1}^{\infty} a_n^* \left\{\frac{n\varphi}{\beta} \int_{2\Pi}^{\infty} \frac{\beta^{n\varphi+1} c^{n\varphi} e^{-\beta c}}{(n\varphi)!} dc - 2\Pi \int_{2\Pi}^{\infty} \frac{\beta^{n\varphi} c^{n\varphi-1} e^{-\beta c}}{(n\varphi-1)!} dc\right\}. \qquad (3.3.8)$$

The result follows if we set (3.3.7) and (3.3.8) in (3.3.4).

□

Corollary 3.3.3 Let F_t, C_t and Π as defined. Also let the claim size distribution be exponential, i.e. $h(u) = \beta e^{-\beta u}$, $u > 0$, $\beta > 0$. Then

$$F_0 = 25,000 \frac{1}{\Pi}\left[\left\{\sum_{n=1}^{\infty} a_n^* \frac{n}{\beta}\right\} - \left\{\sum_{n=1}^{\infty} a_n^* \left(\frac{n}{\beta}\int_{2\Pi}^{\infty} \frac{\beta^{n+1} c^n e^{-\beta c}}{n!} dc - 2\Pi \int_{2\Pi}^{\infty} \frac{\beta^n c^{n-1} e^{-\beta c}}{(n-1)!} dc\right)\right\}\right]. \qquad (3.3.9)$$

Proof

The result follows if we set $\varphi = 1$ in (3.3.6).

□

3.3.2 Catastrophe insurance options on futures

We will now derive the model for the pricing of catastrophe insurance call options on futures. Ignoring the maximum loss ratio, the value of the catastrophe insurance call options on futures, P_t, at maturity t is given by

$$\mathrm{P}_t = Max(F_t - E,\ 0) = (F_t - E)^+ = \left(25,000 \times \frac{L_t}{\Pi} - E\right)^+ = \frac{25,000}{\Pi}(L_t - \mathrm{B})^+ \qquad (3.3.10)$$

where E is the exercise price and $\mathrm{B} = \dfrac{\Pi E}{25,000}$.

Assuming that $L_t = \sum_{i=1}^{N_t} \aleph_i = C_t$, ignoring the effect of interest rates, the price of the insurance call options on futures at time 0 is

$$\frac{25,000}{\Pi} E\left[(C_t - \mathrm{B})^+\right]. \qquad (3.3.11)$$

If we set $\mathrm{B} = b$ in (3.3.11) and assume that there is an absence of arbitrage opportunities in the market, it can be found that (3.3.11), excluding $\dfrac{25,000}{\Pi}$, is equivalent to (3.2.39). As the formulae are easily can be obtained by substituting b with B in (3.2.40) and (3.2.42) we will omit the details.

3.4 Numerical examples

This section illustrates the calculation of stop-loss reinsurance premiums for catastrophic events and catastrophe insurance futures contracts using the pricing models derived previously. The change of stop-loss reinsurance premiums associated with changes in value of θ^* and γ^* is also examined. The appendix contains the *S-Plus* routines needed for these calculations.

As discussed in section 3.3.2 the pricing model for catastrophe insurance call options on futures is equivalent to that of stop-loss reinsurance premium for catastrophic events. We shall therefore use one example to illustrate the pricing of both these products.

Let us assume that the claim size distribution is gamma, i.e. $h(u)=\dfrac{\beta^{\varphi}u^{\varphi-1}e^{-\beta u}}{(\varphi-1)!}$, $u>0$, $\beta>0$, $\varphi\geq 1$. From (3.2.40) and (3.3.6)

The stop-loss reinsurance premium for catastrophic events

$$E^*[(C_t-b)^+]=\sum_{n=1}^{\infty}a_n^*\{\frac{n\varphi}{\beta}\int_b^{\infty}\frac{\beta^{n\varphi+1}c^{n\varphi}e^{-\beta c}}{(n\varphi)!}dc-b\int_b^{\infty}\frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}dc\}. \tag{3.4.1}$$

The price of catastrophe insurance futures

$$F_0=25{,}000\frac{1}{\Pi}\left[\left\{\sum_{n=1}^{\infty}a_n^*\frac{n\varphi}{\beta}\right\}-\left\{\sum_{n=1}^{\infty}a_n^*\left(\frac{n\varphi}{\beta}\int_{2\Pi}^{\infty}\frac{\beta^{n\varphi+1}c^{n\varphi}e^{-\beta c}}{(n\varphi)!}dc-2\Pi\int_{2\Pi}^{\infty}\frac{\beta^{n\varphi}c^{n\varphi-1}e^{-\beta c}}{(n\varphi-1)!}dc\right)\right\}\right]. \tag{3.4.2}$$

Let us also assume that the jump size distribution is exponential i.e. $g^*(y;t)=(\alpha+\gamma^*e^{\delta t})e^{-(\alpha+\gamma^*e^{\delta t})y}$, $y>0$, $-\alpha e^{-\delta t}<\gamma^*\leq 0$ and $t<\frac{1}{\delta}\ln(-\frac{\alpha}{\gamma^*})$ and that λ_t is '$-\infty$' asymptotic. Consider constants θ^* and θ such that $\theta^*\geq 1$ and $0\leq\theta\leq 1$. From (3.2.29), the p.g.f. of N_t is

$$E^*(\theta^{N_t})=\sum_{n=0}^{\infty}\theta\cdot P^*(N_t=n)=\sum_{n=0}^{\infty}\theta^n a_n^*$$

$$=\left(\frac{\gamma^*+\alpha e^{-\delta t}}{\gamma^*+\alpha+\frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta t})}\right)^{\frac{\rho}{\delta}}\left(\frac{\gamma^*+\alpha+\frac{\theta^*(1-\theta)}{\delta}(1-e^{-\delta t})}{\gamma^*+\alpha e^{-\delta t}}\right)^{\frac{\alpha\rho}{\delta\alpha+\theta^*(1-\theta)}} \tag{3.4.3}$$

The parameter values used to expand (3.4.3) with respect to θ are

$$\theta^* = 1.1\ ,\ \gamma^* = -0.1\ ,\ \alpha = 1\ ,\ \delta = 0.3\ ,\ \rho = 4\ ,\ t = 1.$$

Using these parameter values we can calculate the mean of the claim number in a unit period of time. From (3.2.27)

$$E^*(N_t) = \frac{\theta^*\rho}{\delta\alpha}t - \frac{\theta^*\rho}{\delta^2\alpha}\ln\left(\frac{\gamma^* e^{\delta t} + \alpha}{\gamma^* + \alpha}\right) \approx 16.61.$$

By expanding (3.4.3) using the *MAPLE* algebraic manipulations package we can obtain $a_n^* = P^*(N_t = n)$ which is as follows:

$$E^*(\theta^{N_t}) = \sum_{n=0}^{\infty} \theta \cdot P^*(N_t = n) = \sum_{n=0}^{\infty} \theta^n a_n^* = \left\{\frac{0.64082}{0.9 + 0.95033(1-\theta)}\right\}^{\frac{4.4(1-\theta)}{0.09+0.33(1-\theta)}}$$

$= 0.000014982 + 0.00011628\theta + 0.00048266\theta^2 + 0.0014225\theta^3 + 0.0033355\theta^4 + 0.006615\theta^5 + 0.011523\theta^6 + 0.018086\theta^7 + 0.026045\theta^8 + 0.034881\theta^9 + 0.0439\theta^{11} + 0.052349\theta^{11} + 0.059537\theta^{12} + 0.064932\theta^{13} + 0.068214\theta^{14} + 0.06929\theta^{15} + 0.068273\theta^{16} + 0.065434\theta^{17} + 0.061148\theta^{18} + 0.055831\theta^{19} + 0.049898\theta^{20} + 0.043723\theta^{21} + 0.037616\theta^{22} + 0.031815\theta^{23} + 0.026484\theta^{24} + 0.02172\theta^{25} + 0.017567\theta^{26} + 0.014023\theta^{27} + 0.011056\theta^{28} + 0.0086166\theta^{29} + 0.0066419\theta^{30} + 0.0050667\theta^{31} + 0.0038272\theta^{32} + 0.0028639\theta^{33} + 0.0021241\theta^{34} + 0.0015621\theta^{35} + 0.0011396\theta^{36} + 0.00082497\theta^{37} + 0.00059282\theta^{38} + 0.00042301\theta^{39} + 0.00029981\theta^{40} + 0.00021112\theta^{41} + 0.00014775\theta^{42} + 0.00010279\theta^{43} + 0.000071101\theta^{44} + 0.000048911\theta^{45} + 0.000033469\theta^{46} + 0.000022785\theta^{47} + 0.000015436\theta^{48} + 0.000010407\theta^{49} + 0.000006985\theta^{50} + 0.0000046672\theta^{51} + 0.0000031051\theta^{52} + 0.0000020573\theta^{53} + 0.0000013575\theta^{54} + O(\theta^{55})$.

(3.4.4)

Example 3.4.1

The parameter values used to calculate (3.4.1) are

$$n: 1 \sim 41,\ \varphi = 1,\ \beta = 1,\ b = 0,\ 5,\ 10,\ 16.61,\ 20,\ 25,\ 30,\ 33.22$$

$$E^*(C_t) = E^*(N_t)E(\aleph) = 16.61.$$

By computing (3.4.1) using *S-Plus* the calculation of the stop-loss reinsurance premiums for catastrophic events at each retention level b are shown in Table 3.4.1 (see appendix for *S-Plus* routine).

Table 3.4.1

Retention level b	Reinsurance premiums
0	16.58403
5	11.61916
10	7.06779
16.61	2.833487
20	1.587005
25	0.595824
30	0.1951147
33.22	0.0886971

Example 3.4.2

The parameter values used to calculate (3.4.2) are

$$n: 1 \sim 41, \ \varphi = 1, \ \beta = 1, \ \Pi = 16.61.$$

By computing (3.4.2) using *S-Plus* the calculation of the price of catastrophe insurance futures is as follows (see appendix for *S-Plus* routine):

$$F_0 = \$25{,}000 \times (0.9984363 - 0.005339982) = \$24{,}827.41.$$

Example 3.4.3

We will now examine the effect on stop-loss reinsurance premiums caused by changes in the value of θ^* and γ^*. By expanding (3.4.3) using *MAPLE* at each value of θ^* and γ^* respectively and computing (3.4.1) by *S-Plus*, the calculation of the stop-loss reinsurance premiums for catastrophic events at the retention limit $b = 25$ are shown in Table 3.4.2 and Table 3.4.3 (see appendix for *S-Plus* routine).

Table 3.4.2

θ^*	$\gamma^* = -0.1$
1.0	0.3544252
1.1	0.595824
1.2	0.9299355
1.3	1.366049
1.4	1.90885
1.5	2.558786

Table 3.4.3

γ^*	$\theta^* = 1.1$
0.0	0.3029752
-0.1	0.595824
-0.2	1.207256
-0.3	2.512553
-0.4	5.364622
-0.5	11.65184

3.5 More equivalent martingale probability measures and the distribution of the total amount of claims

So far we have used (3.2.14) as the Radon-Nikodym derivative to define the equivalent martingale probability measure in our pricing model for stop-loss reinsurance contract for catastrophic events and catastrophe insurance derivatives. We will now use an alternative martingale taking into account the claim sizes. After the equivalent martingale probability measure is obtained, the expected value of N_t and the Laplace transform of the distribution of N_t with respect to the equivalent martingale probability measure, i.e. $E^*(N_t)$ and $E^*(\theta^{N_t})$ will be derived. The Laplace transform of the distribution of C_t at time t will also be derived. These can be used to derive the pricing models through the same methods presented in sections 2 and 3 of this chapter.

Let $H(u)$ $(u>0)$ be the claim size distribution function and M_t be the total number of catastrophe jumps up to time t. We will assume that claim points and catastrophe jumps do not occur at the same time.

The generator of the process $(X_t, N_t, C_t, \lambda_t, M_t, t)$ acting on a function $f(x,n,c,\lambda,m,t)$ belonging to its domain is given by

$$\mathrm{A}f(x,n,c,\lambda,m,t)=\frac{\partial f}{\partial t}+\lambda\frac{\partial f}{\partial x}+\lambda[\int_0^\infty f(x,n+1,c+u,\lambda,m,t)dH(u)-f(x,n,c,\lambda,m,t)]$$
$$-\delta\lambda\frac{\partial f}{\partial \lambda}+\rho[\int_0^\infty f(x,n,c,\lambda+y,m+1,t)dG(y)-f(x,n,c,\lambda,m,t)]. \tag{3.5.1}$$

Clearly, for $f(x,n,c,\lambda,m,t)$ to belong to the domain of the generator A, it is essential that $f(x,n,c,\lambda,m,t)$ is differentiable w.r.t. x, c, λ, t for all x, n, c, λ, m, t and that $\left|\int_0^\infty f(\cdot,\lambda+y,\cdot)dG(y)-f(\cdot,\lambda,\cdot)\right|<\infty$ and $\left|\int_0^\infty f(\cdot,c+u,\cdot)dH(u)-f(\cdot,c,\cdot)\right|<\infty$.

The generator of the process (N_t, C_t, t) acting on $f(n,c,t)$ is given by

$$\mathrm{A}f(n,c,t)=\frac{\partial f}{\partial t}+\lambda[\int_0^\infty f(n+1,c+u,t)dH(u)-f(n,c,t)] \tag{3.5.2}$$

and the generator of the process (λ_t, M_t, t) acting on $f(\lambda,m,t)$ is given by

$$\mathrm{A}f(\lambda,m,t)=\frac{\partial f}{\partial t}-\delta\lambda\frac{\partial f}{\partial \lambda}+\rho[\int_0^\infty f(\lambda+y,m+1,t)dG(y)-f(\lambda,m,t)]. \tag{3.5.3}$$

Theorem 3.5.1 Let N_t, C_t and λ_t as defined. Consider constants θ and v such that $0 \le \theta \le 1$ and $v \ge 0$. Then

$$\theta^{N_t} e^{-vC_t} e^{-\{\theta \hat{h}(v)-1\}\int_0^t \lambda_s ds} \qquad \left(\hat{h}(v) = \int_0^\infty e^{-vu} dH(u)\right) \tag{3.5.4}$$

is a martingale.

Proof

From (3.5.2), $f(n,c,t)$ has to satisfy $\mathrm{A}f = 0$ for $f(N_t, C_t, t)$ to be a martingale. Trying $\theta^n e^{-vc} e^{A(t)}$ we get the equation

$$A'(t) + \lambda\{\theta \int_0^\infty e^{-vu} dH(u) - 1\} = 0 \tag{3.5.5}$$

and solving (3.5.5) we get $A(t) = -\{\theta \hat{h}(v) - 1\}\int_0^t \lambda_s ds$ and the result follows.

□

Theorem 3.5.2 Let λ_t and M_t as defined. Consider constants ψ and k such that $0 \le \psi \le 1$ and $k \ge 0$. Then

$$\psi^{M_t} e^{-k\lambda_t e^{\delta t}} e^{\rho \int_0^t \{1-\psi \hat{g}(ke^{\delta s})\} ds} \qquad \left(\hat{g}(v) = \int_0^\infty e^{-vy} dG(y)\right) \tag{3.5.6}$$

is a martingale.

Proof

From (3.5.3), $f(\lambda, m, t)$ has to satisfy $\mathrm{A}f = 0$ for $f(\lambda_t, M_t, t)$ to be a martingale. Trying $\psi^m e^{-\lambda A(t)} e^{R(t)}$ we get the equation

$$-\lambda A'(t) + R'(t) + \delta\lambda A(t) + \rho[\psi \hat{g}\{A(t)\} - 1] = 0 \tag{3.5.7}$$

and solving (3.5.7) we get

$$A(t) = ke^{\delta t} \quad \text{and} \quad R(t) = \rho \int_0^t \{1 - \psi \hat{g}(ke^{\delta s})\} ds$$

and the result follows.

□

Theorem 3.5.3 Let N_t, C_t, λ_t and M_t as defined. Consider constants θ^*, v^*, ψ^* and γ^*, such that $\theta^* \geq 1$, $v^* \leq 0$, $\psi^* \geq 1$ and $\gamma^* \leq 0$. Then

$$\theta^{*N_t} e^{-v^* C_t} e^{-\{\theta^* \hat{h}(v^*)-1\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho \int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds} \tag{3.5.8}$$

is a martingale.

__Proof__

From (3.5.1), $f(x,n,c,\lambda,m,t)$ has to satisfy $\mathrm{A}f = 0$ for $f(X_t, N_t, C_t, \lambda_t, M_t, t)$ to be a martingale. Trying $\theta^{*n} e^{-v^* c} e^{\phi^* x} \psi^{*m} e^{-\gamma^* \lambda e^{\delta t}} e^{A(t)}$ we get the equation

$$A'(t) + \lambda\phi^* + \lambda\{\theta^* \hat{h}(v^*) - 1\} + \rho\{\psi^* \hat{g}(\gamma^* e^{\delta t}) - 1\} = 0 \tag{3.5.9}$$

and solving (3.5.9) we get

$$\phi^* = -\{\theta^* \hat{h}(v^*) - 1\} \quad \text{and} \quad A(t) = \rho\int_0^t \{1 - \psi^* \hat{g}(\gamma^* e^{\delta s})\}ds$$

and the result follows.

□

Now, let us examine the generator A^* of the process $(X_t, N_t, C_t, \lambda_t, M_t, t)$ acting on a function $f(x,n,c,\lambda,m,t)$ with respect to the equivalent martingale probability measure.

Theorem 3.5.4 Let N_t, C_t, λ_t and M_t as defined. Consider constants θ^*, v^*, ψ^* and γ^*, such that $\theta^* \geq 1$, $v^* \leq 0$, $\psi^* \geq 1$ and $\gamma^* \leq 0$. Then

$$\begin{aligned}\mathrm{A}^* f(x,n,c,\lambda,m,t) = {} & \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \theta^* \hat{h}(v^*)\lambda\{\int_0^\infty f(x,n+1,c+u,\lambda,m,t)dH^*(u) - f(x,n,c,\lambda,m,t)\} \\ & -\delta\lambda\frac{\partial f}{\partial \lambda} + \rho^*(t)\{\int_0^\infty f(x,n,c,\lambda+y,m+1,t)dG^*(y;t) - f(x,n,c,\lambda,m,t)\}\end{aligned} \tag{3.5.10}$$

where $dH^*(u) = \dfrac{e^{-v^* u}dH(u)}{\hat{h}(v^*)}$, $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y}dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

Proof

From theorem 3.5.3, we can use

$$\frac{\theta^{*N_t} e^{-v^* C_t} e^{-\{\theta^* \hat{h}(v^*)-1\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds}}{E[\theta^{*N_t} e^{-v^* C_t} e^{-\{\theta^* \hat{h}(v^*)-1\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds}]} \tag{3.5.11}$$

as the Radon-Nikodym derivative to define an equivalent martingale probability measure.

Therefore from lemma 3.2.4

$$A^* f(X_t, N_t, C_t, \lambda_t, M_t, t) = \frac{A\, f(X_t, N_t, C_t, \lambda_t, M_t, t)\cdot \theta^{*N_t} e^{-v^* C_t} e^{-\{\theta^* \hat{h}(v^*)-1\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds}}{\theta^{*N_t} e^{-v^* C_t} e^{-\{\theta^* \hat{h}(v^*)-1\}\int_0^t \lambda_s ds} \psi^{*M_t} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds}}.$$

From (3.5.1), using the generator with respect to the original probability measure,

$$A f(x,n,c,\lambda,m,t)\theta^{*n} e^{-v^* c} e^{-\{\theta^* \hat{h}(v^*)-1\}\int_0^t \lambda_s ds} \psi^{*m} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds}$$

$$= [\frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \lambda\{\theta^* \int_0^\infty f(x,n+1,c+u,\lambda,m,t)e^{-v^* u} dH(u) - \theta^* \hat{h}(v^*) f(x,n,c,\lambda,m,t)\}$$

$$-\delta\lambda\frac{\partial f}{\partial \lambda} + \rho\{\psi^* \int_0^\infty f(x,n,c,\lambda+y,m+1,t)dG(y) - \psi^* \hat{g}(\gamma^* e^{\delta t}) f(x,n,c,\lambda,m,t)\}]$$

$$\cdot \theta^{*n} e^{-v^* c} e^{-\{\theta^* \hat{h}(v^*)-1\}\int_0^t \lambda_s ds} \psi^{*m} e^{-\gamma^* \lambda_t e^{\delta t}} e^{\rho\int_0^t \{1-\psi^* \hat{g}(\gamma^* e^{\delta s})\}ds}.$$

Therefore

$$A^* f(x,n,c,\lambda,m,t) = \frac{\partial f}{\partial t} + \lambda\frac{\partial f}{\partial x} + \theta^* \hat{h}(v^*)\lambda\{\int_0^\infty f(x,n+1,c+u,\lambda,m,t)dH^*(u) - f(x,n,c,\lambda,m,t)\}$$

$$-\delta\lambda\frac{\partial f}{\partial \lambda} + \rho^*(t)\{\int_0^\infty f(x,n,c,\lambda+y,m+1,t)dG^*(y;t) - f(x,n,c,\lambda,m,t)\} \tag{3.5.12}$$

where $dH^*(u) = \frac{e^{-v^* u} dH(u)}{\hat{h}(v^*)}$, $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$ and $dG^*(y;t) = \frac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$.

□

Theorem 3.5.4 yields the following:

(i) The claim intensity function λ_t has changed to $\lambda_t \theta^* \hat{h}(v^*)$;

(ii) The rate of jump arrival ρ has changed to $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$ (it now depends on time);

(iii) The jump size measure $dG(y)$ has changed to $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ (it now depends on time);

(iv) The claim size measure $dH(u)$ has changed to $dH^*(u) = \dfrac{e^{-v^* x} dH(u)}{\hat{h}(v^*)}$.

Let us evaluate the expected value of N_t and the Laplace transform of the distribution of N_t with respect to the equivalent martingale probability measure, i.e. $E^*(N_t)$ and $E^*(\theta^{N_t})$.

Theorem 3.5.5 Let N_t as defined and λ_t be a generalised shot noise process with $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$, $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$ and $\delta(t) = \delta$. Consider constants γ^*, θ^* and ψ^* such that $\gamma^* \le 0$, $\theta^* \ge 1$ and $\psi^* \ge 1$. Then

$$E(N_{t_2} - N_{t_1}) = \int_{t_1}^{t_2} E(\lambda_s) ds = \left(\frac{1 - e^{-\delta(t_2 - t_1)}}{\delta} \right) E(\lambda_{t_1}) + \frac{1}{\delta} \int_{t_1}^{t_2} (1 - e^{-\delta(t_2 - s)}) \rho^*(s) \mu_1^*(s) ds.$$

(3.5.13)

and

$$E^*(N_{t_2} - N_{t_1}) = \theta^* \hat{h}(v^*) \int_{t_1}^{t_2} E(\lambda_s) ds = \theta^* \hat{h}(v^*) \left[\left(\frac{1 - e^{-\delta(t_2 - t_1)}}{\delta} \right) E(\lambda_{t_1}) + \frac{1}{\delta} \int_{t_1}^{t_2} (1 - e^{-\delta(t_2 - s)}) \rho^*(s) \mu_1^*(s) ds \right]$$

(3.5.14)

where $\hat{h}(v) = \int_0^\infty e^{-vu} dH(u)$.

<u>**Proof**</u>

This theorem can be proved in a similar method to theorem 3.2.6. In this case $\rho^*(s) = \rho\psi^* \hat{g}(\gamma^* e^{\delta s})$ and use $E^*[N_{t_2} - N_{t_1}] = \theta^* \hat{h}(v^*) \int_{t_1}^{t_2} E(\lambda_s) ds$ then the results follow.

□

Theorem 3.5.6 Let N_t as defined and $\Im_t^N$ be the filtration generated by $\{N_s;\ 0 \le s \le t\}$. Also let λ_t be the generalised shot noise process and $\Im_t^\lambda$ be the filtration generated by $\{\lambda_s;\ 0 \le s \le t\}$. Consider constants θ, θ^*, v^*, ψ^* and γ^* such that $0 \le \theta \le 1$, $\theta^* \ge 1$, $v^* \le 0$, $\psi^* \ge 1$ and $\gamma^* \le 0$. Then

$$E^*(\theta^{N_{t_2}-N_{t_1}} \big| \Im_{t_1}^{N,\lambda}) = E(e^{-\theta^* \hat{h}(v^*)(1-\theta)\int_{t_1}^{t_2} \lambda_s ds} \big| \Im_{t_1}^{\lambda}) = e^{-\frac{\theta^* \hat{h}(v^*)(1-\theta)}{\delta}\{1-e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\int_{t_1}^{t_2} \rho^*(s)[1-\hat{g}^*\{\frac{\theta^* \hat{h}(v^*)(1-\theta)}{\delta}(1-e^{-\delta(t_2-s)});s\}]ds} \tag{3.5.15}$$

where $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$, $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$, $\hat{g}^*(u;t) = \int_0^\infty e^{-uy} dG^*(y;t)$ and $\hat{h}(v) = \int_0^\infty e^{-vu} dH(u)$.

Proof

From $E^*[N_{t_2} - N_{t_1}] = \theta^* \hat{h}(v^*) \int_{t_1}^{t_2} E(\lambda_s) ds$ and (1.1.4),

$$E^*(\theta^{N_{t_2}-N_{t_1}}) = E(e^{-\theta^* \hat{h}(v^*)(1-\theta)\int_{t_1}^{t_2} \lambda_s ds}).$$

By setting $v = \theta^* \hat{h}(v^*)(1-\theta)$, $\rho(s) = \rho^*(s)$ and $\hat{g}(u;s) = \hat{g}^*(u;s)$ in (2.1.37), (3.5.15) follows immediately.

□

We are now going to close this section with an evaluation of the Laplace transform of the distribution of C_t at time *t*.

3.5.1 Where λ_t is time homogeneous

Corollary 3.5.7 Let N_t, C_t and λ_t as defined. Let $v \ge 0$, $0 \le \theta \le 1$ and t_1, t_2 be fixed times. Then

$$E\{\theta^{N_{t_2}-N_{t_1}} e^{-v(C_{t_2}-C_{t_1})} \big| N_{t_1}, \lambda_{t_1}, C_{t_1}\} = \theta^{N_{t_1}} e^{-vC_{t_1}} e^{\{\theta\hat{h}(v)-1\}\int_{t_1}^{t_2} \lambda_s ds}. \tag{3.5.16}$$

Proof

Since (3.5.4) is a martingale (3.5.16) follows immediately.

□

Corollary 3.5.8 Let N_t and λ_t as defined. Consider constant $\nu \geq 0$. Then

$$E\{e^{-\nu(C_{t_2}-C_{t_1})} \big| C_{t_1}\} = e^{-\nu C_{t_1}} e^{\{\hat{h}(\nu)-1\}\int_{t_1}^{t_2} \lambda_s ds}. \tag{3.5.17}$$

Proof

The result follows if we set $\theta = 1$ in (3.5.16).

□

Corollary 3.5.9 Let N_t and λ_t as defined. Consider constant $\nu \geq 0$. Then

$$E(e^{-\nu C_t} \big| \lambda_0) = e^{-\frac{\{1-\hat{h}(\nu)\}}{\delta}(1-e^{-\delta t})\lambda_0} e^{-\rho\int_0^t [1-\hat{g}\{\frac{1-\hat{h}(\nu)}{\delta}(1-e^{-\delta s})\}]ds}. \tag{3.5.18}$$

Proof

From (3.5.17)

$$E\{e^{-\nu(C_{t_2}-C_{t_1})}\} = E[e^{-\nu C_{t_1}} e^{\{\hat{h}(\nu)-1\}\int_{t_1}^{t_2} \lambda_s ds}]. \tag{3.5.19}$$

Without loss of generality, change the time scale and condition on λ_0, then (3.5.19) can be written as

$$E(e^{-\nu C_t} \big| \lambda_0) = E[e^{-\{1-\hat{h}(\nu)\}\int_0^t \lambda_s ds} \big| \lambda_0]. \tag{3.5.20}$$

Therefore (3.5.18) follows immediately from (2.1.13).

□

Corollary 3.5.10 Let N_t and λ_t as defined. Consider constants $\nu_1 \geq 0$, $\nu_2 \geq 0$. Then

$$E(e^{-\nu_1 C_t} e^{-\nu_2 \lambda_t} \big| \lambda_0) = e^{-[\frac{\{1-\hat{h}(\nu_1)\}}{\delta}+(\nu_2-\frac{\{1-\hat{h}(\nu_1)\}}{\delta})e^{-\delta t}]\lambda_0} e^{-\rho\int_0^t [1-\hat{g}\{\frac{1-\hat{h}(\nu_1)}{\delta}+(\nu_2-\frac{\{1-\hat{h}(\nu_1)\}}{\delta})e^{-\delta s}\}]ds}. \tag{3.5.21}$$

Proof

Multiply $e^{-\nu_2 \lambda_t}$ and set $\nu = \nu_1$ in (3.5.20) then

$$E(e^{-\nu_1 C_t} e^{-\nu_2 \lambda_t} | \lambda_0) = E[e^{-\{1-\hat{h}(\nu_1)\}\int_0^t \lambda_s ds} e^{-\nu_2 \lambda_t} | \lambda_0] \qquad (3.5.22)$$

The corollary follows immediately from (2.1.9).

□

3.5.2 Where the parameters of λ_t are time dependent

Corollary 3.5.11 Let N_t and λ_t as defined. Consider a constant $\nu \geq 0$. Assume that $\delta(t) = \delta$. Then

$$E(e^{-\nu C_t} | \lambda_0) = E[e^{-\{1-\hat{h}(\nu)\}\int_0^t \lambda_s ds} | \lambda_0] = e^{-\frac{\{1-\hat{h}(\nu)\}}{\delta}(1-e^{-\delta t})\lambda_0} e^{-\int_0^t \rho(s)[1-\hat{g}\{\frac{1-\hat{h}(\nu)}{\delta}(1-e^{-\delta(t-s)});s\}]ds} . \qquad (3.5.23)$$

<u>**Proof**</u>
The corollary follows immediately from (3.5.20) and (2.1.37).

□

Corollary 3.5.12 Let N_t and λ_t as defined. Consider constants $\nu_1 \geq 0$, $\nu_2 \geq 0$. Assume that $\delta(t) = \delta$. Then

$$E(e^{-\nu_1 C_t} e^{-\nu_2 \lambda_t} | \lambda_0) = E[e^{-\{1-\hat{h}(\nu_1)\}\int_0^t \lambda_s ds} e^{-\nu_2 \lambda_t} | \lambda_0]$$

$$= e^{-[\frac{\{1-\hat{h}(\nu_1)\}}{\delta}+(\nu_2-\frac{\{1-\hat{h}(\nu_1)\}}{\delta})e^{-\delta t}]\lambda_0} e^{-\int_0^t \rho(s)[1-\hat{g}\{\frac{1-\hat{h}(\nu_1)}{\delta}+(\nu_2-\frac{\{1-\hat{h}(\nu_1)\}}{\delta})e^{-\delta(t-s)};s\}]ds} .$$

(3.5.24)

<u>**Proof**</u>
The corollary follows immediately from (3.5.22) and (2.1.33).

□

Now we can easily evaluate the Laplace transform of the distribution of C_t at time t with respect to equivalent martingale probability measure since λ_t is a time dependent shot noise process with rate of decay δ, rate of jump arrivals $\rho^*(t)$ and jump size distribution function $G^*(y;t)$ $(y > 0)$.

Corollary 3.5.13 Let N_t and λ_t as defined. Consider constants ν, θ^*, ν^*, ψ^* and γ^* such that $\nu \geq 0$, $\theta^* \geq 1$, $\nu^* \leq 0$, $\psi^* \geq 1$ and $\gamma^* \leq 0$. Then

$$E^*(e^{-\nu C_t} | \lambda_0) = E[e^{-\theta^* \hat{h}(\nu^*)\{1-\hat{h}(\nu)\} \int_0^t \lambda_s ds} | \lambda_0] = e^{-\frac{\theta^* \hat{h}(\nu^*)\{1-\hat{h}(\nu)\}}{\delta}(1-e^{-\delta t})\lambda_0} e^{-\int_0^t \rho^*(s)[1-\hat{g}\{\frac{\theta^* \hat{h}(\nu^*)(1-\hat{h}(\nu))}{\delta}(1-e^{-\delta(t-s)});s\}]ds} \tag{3.5.25}$$

where $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$, $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$, $\hat{g}^*(u;t) = \int_0^\infty e^{-uy} dG^*(y;t)$ and $\hat{h}(\nu) = \int_0^\infty e^{-\nu u} dH(u)$.

__Proof__

From $E^*[N_{t_2} - N_{t_1}] = \theta^* \hat{h}(\nu^*) \int_{t_1}^{t_2} E(\lambda_s) ds$ and (2.1.37), the corollary follows immediately.

□

Corollary 3.5.14 Let N_t and λ_t as defined. Consider the constants ν_1, ν_2, θ^*, ν^*, ψ^* and γ^* such that $\nu_1 \geq 0$, $\nu_2 \geq 0$, $\theta^* \geq 1$, $\nu^* \leq 0$, $\psi^* \geq 1$ and $\gamma^* \leq 0$. Then

$$E^*(e^{-\nu_1 C_t} e^{-\nu_2 \lambda_t} | \lambda_0) = E[e^{-\theta^* \hat{h}(\nu^*)\{1-\hat{h}(\nu_1)\} \int_0^t \lambda_s ds} e^{-\nu_2 \lambda_t} | \lambda_0]$$
$$= e^{-[\frac{\theta^* \hat{h}(\nu^*)\{1-\hat{h}(\nu_1)\}}{\delta} + (\nu_2 - \frac{\theta^* \hat{h}(\nu^*)\{1-\hat{h}(\nu_1)\}}{\delta})e^{-\delta t}]\lambda_0} e^{-\int_0^t \rho^*(s)[1-\hat{g}\{\frac{\theta^* \hat{h}(\nu^*)(1-\hat{h}(\nu_1))}{\delta} + (\nu_2 - \frac{\theta^* \hat{h}(\nu^*)(1-\hat{h}(\nu_1))}{\delta})e^{-\delta(t-s)};s\}]ds} \tag{3.5.26}$$

where $\rho^*(t) = \rho\psi^* \hat{g}(\gamma^* e^{\delta t})$, $dG^*(y;t) = \dfrac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}$, $\hat{g}^*(u;t) = \int_0^\infty e^{-uy} dG^*(y;t)$ and $\hat{h}(\nu) = \int_0^\infty e^{-\nu u} dH(u)$.

__Proof__

From $E^*[N_{t_2} - N_{t_1}] = \theta^* \hat{h}(\nu^*) \int_{t_1}^{t_2} E(\lambda_s) ds$ and (2.1.33), the corollary follows immediately.

□

4. Parametric Estimation

In the previous chapter, arbitrary values for the parameters of the shot noise process were used to calculate the stop-loss reinsurance premium and the price of catastrophe insurance futures. The next stage in the pricing process is to estimate these parameters of the time homogeneous shot noise process, i.e. the rate of occurrence of jump ρ, the size of jump y or its distribution and the rate of decay δ should be estimated. In this chapter, using the likelihood function, the parametric estimation for the shot noise process will be presented.

In the first section, we will derive the likelihood function by transforming and approximating λ_t as a normal variable Z_t. In the second section, the maximum likelihood estimators of the three parameters will be obtained by assuming that the times of catastrophes and claims are known.

4.1 Approximation of λ_t as an Ornstein-Uhlenbeck Process

Let $t_1, t_2, \cdots, t_n$ be the epochs at which the claim points occur. The likelihood function conditional on $\{\lambda_s,\ 0 \le s \le t\}$, evolving up to a fixed time t, is given by

$$\lambda_{t_1}\lambda_{t_2}\cdots\cdots\lambda_{t_n}e^{-\int_0^t \lambda_s ds} \tag{4.1.1}$$

where $0 < t_1 < \cdots < t_n < t$.

From (4.1.1) we have the likelihood function that is the expectation with respect to the intensity process $\{\lambda_s,\ 0 \le s \le t\}$, i.e.

$$E(\lambda_{t_1}\lambda_{t_2}\cdots\cdots\lambda_{t_n}e^{-\int_0^t \lambda_s ds}). \tag{4.1.2}$$

We will now assume ρ is large and obtain this expectation by transforming the shot noise process λ_t using

$$Z_t^{(\rho)} = \frac{\lambda_t - E(\lambda_t)}{Var(\lambda_t)}. \tag{4.1.3}$$

In corollary 1.3.9 and 1.3.10, assuming that λ_t is stationary, we have found that $E(\lambda_t) = \frac{\mu_1\rho}{\delta}$ and $Var(\lambda_t) = \frac{\mu_2\rho}{2\delta}$. Therefore (4.1.3) becomes

$$Z_t^{(\rho)} = \frac{\lambda_t - \frac{\mu_1\rho}{\delta}}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \quad \text{i.e.} \quad \lambda_t = \frac{\mu_1\rho}{\delta} + Z_t^{(\rho)}\sqrt{\frac{\mu_2\rho}{2\delta}}. \tag{4.1.4}$$

Now let us start with a proposition used by Ethier & Kurtz (1985).

Proposition 4.1.1 For $n = 1,2,\cdots$, let $\{\Im_t^n\}$ be a filtration and let M_n be an $\{\Im_t^n\}$-local martingale with sample paths in $D_{\Re^d}[0,\infty)$ and $M_n(0) = 0$. Let $A_n = ((A_n^{ij}))$ be symmetric $d \times d$ matrix-valued processes such that A_n^{ij} has sample paths in $D_{\Re}[0,\infty)$ and $A_n(t) - A_n(s)$ is nonnegative definite for $0 \le s < t$. Assume

$$\lim_{n\to\infty} E\left[\sup_{t\le T} \left|A_n^{ij}(t) - A_n^{ij}(t-)\right|\right] = 0,$$

$$\lim_{n\to\infty} E\left[\sup_{t\le T} \left|M_n(t) - M_n(t-)\right|^2\right] = 0,$$

and for $i, j = 1,2,\cdots,d$,

$$M_n^i(t)M_n^j(t) - A_n^{ij}(t)$$

is an $\{\Im_t^n\}$-local martingale.

For each $t \ge 0$ and $i, j = 1,2,\cdots,d$,

$$A_n^{ij}(t) \to c_{ij}(t)$$

in probability where $C = ((c_{ij}))$ is a continuous, symmetric, $d \times d$ matrix-valued function, defined on $[0,\infty)$, satisfying $C(0) = 0$ and $\sum (c_{ij}(t) - c_{ij}(s))\xi_i\xi_j \ge 0$, $\xi \in \Re^d$. Then

$$M_n \Rightarrow X$$

in law where X is a process with independent Gaussian increments such that $X_iX_j - c_{ij}$ are (local) martingales with respect to $\{\Im_t^X\}$.

□

Let us define $J_t = \sum_{i=1}^{M_t} y_i$, $V_t^{(\rho)} = \frac{J_t - \mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}$ and look at a technical lemma.

Lemma 4.1.2 Let J_t and $V_t^{(\rho)}$ as defined and $\rho \to \infty$. Then

$$V_t^{(\rho)} = \frac{J_t - \mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \quad \Rightarrow \quad \sqrt{2\delta}B_t \tag{4.1.5}$$

in law where B_t is a standard Brownian motion.

Proof

The generator of the process $V_t^{(\rho)}$ acting on a function $f(v)$ is given by

$$\mathrm{A}f(v) = -\frac{\mu_1\rho}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\frac{\partial f}{\partial v} + \rho\{\int_0^\infty f(v+\frac{y}{\sqrt{\frac{\mu_2\rho}{2\delta}}})dG(y) - f(v)\}. \tag{4.1.6}$$

Set $f(v) = v^2$. Then

$$\mathrm{A}\,v^2 = -2\frac{\mu_1\rho}{\sqrt{\frac{\mu_2\rho}{2\delta}}}v + \rho\{\int_0^\infty (v+\frac{y}{\sqrt{\frac{\mu_2\rho}{2\delta}}})^2 dG(y) - v^2\} = 2\delta.$$

As mentioned in the introduction, $f(X_t) - \int_0^t \mathrm{A}f(X_s)ds$ is a martingale therefore $\mathrm{A}f$ is the solution to the 'martingale problem'. Hence $\left(V_t^{(\rho)^2}\right) - 2\delta t$ is a martingale.

Therefore from proposition 4.1.1, (4.1.5) follows immediately.

□

Let us look at a theorem that proves that Z_t is a normal variable.

Theorem 4.1.3 Let $Z_t^{(\rho)}$ as defined and $\rho \to \infty$. Then $Z_t^{(\rho)}$ converges in law to Z_t where

$$dZ_t = -\delta Z_t dt + \sqrt{2\delta}dB_t \tag{4.1.7}$$

and B_t is a standard Brownian motion.

Proof

From $\lambda_t = \lambda_0 e^{-\delta t} + J_t - \int_0^t \delta e^{-\delta(t-u)} J_u du$,

$$Z_t^{(\rho)} = \frac{\lambda_t - \frac{\mu_1\rho}{\delta}}{\sqrt{\frac{\mu_2\rho}{2\delta}}}$$

$$= \frac{(\lambda_0 - \frac{\mu_1\rho}{\delta})e^{-\delta t} - \frac{\mu_1\rho}{\delta}(1-e^{-\delta t}) + J_t - \mu_1\rho t + \mu_1\rho t - \int_0^t \delta e^{-\delta(t-u)}\{J_u - \mu_1\rho u\}du - \int_0^t \delta e^{-\delta(t-u)}\mu_1\rho u du}{\sqrt{\frac{\mu_2\rho}{2\delta}}}$$

$$= \frac{(\lambda_0 - \frac{\mu_1\rho}{\delta})e^{-\delta t}}{\sqrt{\frac{\mu_2\rho}{2\delta}}} + \frac{J_t - \mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} - \delta e^{-\delta(t-u)}\int_0^t \frac{J_u - \mu_1\rho u}{\sqrt{\frac{\mu_2\rho}{2\delta}}}du. \tag{4.1.8}$$

Therefore by continuous mapping theorem (see Billingsley (1968)) and lemma 4.1.2, (4.1.8) converges to

$$Z_t = Z_0 e^{-\delta t} + \sqrt{2\delta}\int_0^t e^{-\delta(t-s)}dB_s . \qquad (4.1.9)$$

Hence (4.1.7) follows immediately from (4.1.9).

□

Theorem 4.1.3 has proved that Z_t is a normal variable, which is an Ornstein-Uhlenbeck process. As a result of this, we have obtained $\tilde{\lambda}_t$ which is a Gaussian approximation of λ_t;

$$\tilde{\lambda}_t = \frac{\mu_1\rho}{\delta} + Z_t\sqrt{\frac{\mu_2\rho}{2\delta}} \quad \text{i.e.} \quad Z_t = \frac{\tilde{\lambda}_t - \frac{\mu_1\rho}{\delta}}{\sqrt{\frac{\mu_2\rho}{2\delta}}}. \qquad (4.1.10)$$

Lemma 4.1.4 Let Z_t as defined. Then

$$Z_{t_{k+1}} = Z_{t_k} e^{-\delta(t_{k+1}-t_k)} + \sqrt{2\delta}\int_{t_k}^{t_{k+1}} e^{-\delta(t_{k+1}-u)}dB_u \qquad (4.1.11)$$

and

$$\int_{t_k}^{t_{k+1}} Z_s ds = Z_{t_k}\frac{1-e^{-\delta(t_{k+1}-t_k)}}{\delta} + \sqrt{2\delta}\int_{t_k}^{t_{k+1}}\frac{1-e^{-\delta(t_{k+1}-u)}}{\delta}dB_u \qquad (4.1.12)$$

where $t_k \le t_{k+1}$.

__Proof__

From Ito's lemma and (4.1.7)

$$d(Z_t e^{\delta t}) = \delta Z_t e^{\delta t}dt + e^{\delta t}dZ_t = \delta Z_t e^{\delta t}dt + e^{\delta t}(-\delta Z_t dt + \sqrt{2\delta}dB_t) = \sqrt{2\delta}e^{\delta t}dB_t .$$

Therefore evolving up to a fixed time s

$$Z_s e^{\delta s} - Z_{t_k} e^{\delta t_k} = \int_{t_k}^{s}\sqrt{2\delta}e^{\delta u}dB_u$$

$$Z_s = Z_{t_k} e^{-\delta(s-t_k)} + \sqrt{2\delta}\int_{t_k}^{s} e^{-\delta(s-u)}dB_u . \qquad (4.1.13)$$

If we set $s = t_{k+1}$ in (4.1.13) and integrate it from t_k to t_{k+1} then the result follows immediately.

□

Corollary 4.1.5 Let Z_t as defined. Assuming that we know Z_{t_k}

$$Var\left(Z_{t_{k+1}}\middle|Z_{t_k}\right) = 2\delta\int_{t_k}^{t_{k+1}} e^{-2\delta(t_{k+1}-u)}du \tag{4.1.14}$$

and

$$Cov\left(\int_{t_k}^{t_{k+1}} Z_s ds, Z_{t_{k+1}}\middle|Z_{t_k}\right) = 2\delta\int_{t_k}^{t_{k+1}} \frac{e^{-\delta(t_{k+1}-u)} - e^{-2\delta(t_{k+1}-u)}}{\delta}\,du. \tag{4.1.15}$$

<u>Proof</u>

From (4.1.11)

$$Var\left(Z_{t_{k+1}}\middle|Z_{t_k}\right) = Var\left(Z_{t_k}e^{-\delta(t_{k+1}-t_k)} + \int_{t_k}^{t_{k+1}}\sqrt{2\delta}e^{-\delta(t_{k+1}-u)}dB_u\middle|Z_{t_k}\right) = Var\left(\int_{t_k}^{t_{k+1}}\sqrt{2\delta}e^{-\delta(t_{k+1}-u)}dB_u\middle|Z_{t_k}\right)$$

$$= Var\left(\int_{t_k}^{t_{k+1}}\sqrt{2\delta}e^{-\delta(t_{k+1}-u)}dB_u\right) = E\{(\int_{t_k}^{t_{k+1}}\sqrt{2\delta}e^{-\delta(t_{k+1}-u)}dB_u)^2\} - \{E(\int_{t_k}^{t_{k+1}}\sqrt{2\delta}e^{-\delta(t_{k+1}-u)}dB_u)\}^2$$

$$= E\{(\int_{t_k}^{t_{k+1}}\sqrt{2\delta}e^{-\delta(t_{k+1}-u)}dB_u)^2\} - 0 = 2\delta\int_{t_k}^{t_{k+1}} e^{-2\delta(t_{k+1}-u)}du.$$

From (4.1.11) and (4.1.12)

$$Cov\left(\int_{t_k}^{t_{k+1}} Z_s ds, Z_{t_{k+1}}\middle|Z_{t_k}\right)$$

$$= Cov\left\{Z_{t_k}\frac{1-e^{-\delta(t_{k+1}-t_k)}}{\delta} + \sqrt{2\delta}\int_{t_k}^{t_{k+1}}\frac{1-e^{-\delta(t_{k+1}-u)}}{\delta}\,dB_u,\ Z_{t_k}e^{-\delta(t_{k+1}-t_k)} + \int_{t_k}^{t_{k+1}}\sqrt{2\delta}\,e^{-\delta(t_{k+1}-u)}dB_u\middle|Z_{t_k}\right\}$$

$$= Cov\left\{\sqrt{2\delta}\int_{t_k}^{t_{k+1}}\frac{1-e^{-\delta(t_{k+1}-u)}}{\delta}\,dB_u, \sqrt{2\delta}\int_{t_k}^{t_{k+1}} e^{-\delta(t_{k+1}-u)}dB_u\right\} = 2\delta\int_{t_k}^{t_{k+1}}\frac{e^{-\delta(t_{k+1}-u)} - e^{-2\delta(t_{k+1}-u)}}{\delta}\,du.$$

□

Lemma 4.1.6 Let Z_t as defined and K_k be a constant. Then $\int_{t_k}^{t_{k+1}} Z_s ds - K_k Z_{t_{k+1}}$ and $Z_{t_{k+1}}$ are independent given Z_{t_k} if and only if

$$K_k = \frac{2}{\delta\{1+e^{-\delta(t_{k+1}-t_k)}\}} - \frac{1}{\delta}. \tag{4.1.16}$$

<u>Proof</u>

$$Cov\left(\int_{t_k}^{t_{k+1}} Z_s ds - K_k Z_{t_{k+1}}, Z_{t_{k+1}}\middle|Z_{t_k}\right) = Cov\left(\int_{t_k}^{t_{k+1}} Z_s ds, Z_{t_{k+1}}\middle|Z_{t_k}\right) - K_k Var\left(Z_{t_{k+1}}\middle|Z_{t_k}\right). \tag{4.1.17}$$

We would like to obtain K_k that makes $\int_{t_k}^{t_{k+1}} Z_s ds - \mathrm{K}_k Z_{t_{k+1}}$ and $Z_{t_{k+1}}$ independent given Z_{t_k}. Therefore $Cov(\int_{t_k}^{t_{k+1}} Z_s ds - \mathrm{K}_k Z_{t_{k+1}}, Z_{t_{k+1}} \big| Z_{t_k})$ should be 0. If we set (4.1.17) equal 0 and from (4.1.14) and (4.1.15)

$$Cov(\int_{t_k}^{t_{k+1}} Z_s ds - \mathrm{K}_k Z_{t_{k+1}}, Z_{t_{k+1}} \big| Z_{t_k}) = 2\delta \int_{t_k}^{t_{k+1}} \frac{e^{-\delta(t_{k+1}-u)} - e^{-2\delta(t_{k+1}-u)}}{\delta} du - \mathrm{K}_k 2\delta \int_{t_k}^{t_{k+1}} e^{-2\delta(t_{k+1}-u)} du = 0. \tag{4.1.18}$$

Solving (4.1.18) then

$$\mathrm{K}_k = \frac{2}{\delta\{1+e^{-\delta(t_{k+1}-t_k)}\}} - \frac{1}{\delta}.$$

□

Theorem 4.1.7 Let Z_t as defined. Then

$$E(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}} \int_{t_k}^{t_{k+1}} Z_s ds} \big| Z_{t_k}, Z_{t_{k+1}})$$

$$= e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}[\frac{1}{\delta} - \frac{2}{\delta\{1+e^{\delta(t_{k+1}-t_k)}\}}] Z_{t_k}} \cdot e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}[\frac{2}{\delta\{1+e^{-\delta(t_{k+1}-t_k)}\}} - \frac{1}{\delta}] Z_{t_{k+1}}} \cdot e^{\frac{1}{2}\mu_2\rho \int_{t_k}^{t_{k+1}} \{\frac{1-e^{-\delta(t_{k+1}-u)}}{\delta} - K_k e^{-\delta(t_{k+1}-u)}\}^2 du} . \tag{4.1.19}$$

__**Proof**__

$$E(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}} \int_{t_k}^{t_{k+1}} Z_s ds} \big| Z_{t_k}, Z_{t_{k+1}}) = E(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}} (\int_{t_k}^{t_{k+1}} Z_s ds - K_k Z_{t_{k+1}} + K_k Z_{t_{k+1}})} \big| Z_{t_k}, Z_{t_{k+1}}).$$

From lemma 4.1.6 we have found that $\int_{t_k}^{t_{k+1}} Z_s ds - \mathrm{K}_k Z_{t_{k+1}}$ and $Z_{t_{k+1}}$ are independent given Z_{t_k}. Therefore

$$E(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}} \int_{t_k}^{t_{k+1}} Z_s ds} \big| Z_{t_k}, Z_{t_{k+1}}) = E(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}} (\int_{t_k}^{t_{k+1}} Z_s ds - K_k Z_{t_{k+1}} + K_k Z_{t_{k+1}})} \big| Z_{t_k}, Z_{t_{k+1}}) = E(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}} (\int_{t_k}^{t_{k+1}} Z_s ds - K_k Z_{t_{k+1}})} \big| Z_{t_k}) e^{-\sqrt{\frac{\mu_2\rho}{2\delta}} K_k Z_{t_{k+1}}}. \tag{4.1.20}$$

Hence (4.1.19) follows immediately from (4.1.11), (4.1.12) and (4.1.16).

□

Theorem 4.1.8 Let Z_t as defined.

$$E\left(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_0^{t_1} Z_s ds} e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_1}^{t_2} Z_s ds} \cdots\cdots e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_k}^{t_{k+1}} Z_s ds} \cdots\cdots e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_{n-1}}^{t_n} Z_s ds} e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_n}^{t} Z_s ds} \middle| Z_0, Z_{t_1}, \cdots\cdots Z_{t_n}, Z_t\right)$$

$$= e^{H_0} e^{H_1} \cdots\cdots e^{H_{n-1}} e^{H_n} e^{\frac{\mu_2\rho}{\delta}C_0} e^{\frac{\mu_2\rho}{\delta}C_1} e^{\frac{\mu_2\rho}{\delta}C_2} \cdots\cdots e^{\frac{\mu_2\rho}{\delta}C_n} e^{\frac{\mu_2\rho}{\delta}C_{n+1}} e^{-C_0\tilde{\lambda}_0} e^{-C_1\tilde{\lambda}_{t_1}} e^{-C_2\tilde{\lambda}_{t_2}} \cdots\cdots e^{-C_n\tilde{\lambda}_{t_n}} e^{-C_{n+1}\tilde{\lambda}_t} \tag{4.1.21}$$

where $t_0 = 0 < t_1 < \cdots < t_n < t = t_{n+1}$,

$$H_k = \frac{1}{2}\mu_2\rho\int_{t_k}^{t_{k+1}} \left\{\frac{1-e^{-\delta(t_{k+1}-u)}}{\delta} - K_k e^{-\delta(t_{k+1}-u)}\right\}^2 du \text{ for } k = 0,1,\cdots,n+1,$$

$C_0 = \left[\dfrac{1}{\delta} - \dfrac{2}{\delta\{1+e^{\delta(t_1-0)}\}}\right]$, $C_k = \left[\dfrac{2}{\delta\{1+e^{-\delta(t_k-t_{k-1})}\}} - \dfrac{2}{\delta\{1+e^{\delta(t_{k+1}-t_k)}\}}\right]$ for $k = 1,2,\cdots,n$ and

$C_{n+1} = \left[\dfrac{2}{\delta\{1+e^{-\delta(t-t_n)}\}} - \dfrac{1}{\delta}\right]$.

Proof

The intervals between points are independent and Z_t has the Markov property. Therefore

$$E\left(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_0^{t_1} Z_s ds} e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_1}^{t_2} Z_s ds} \cdots\cdots e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_k}^{t_{k+1}} Z_s ds} \cdots\cdots e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_{n-1}}^{t_n} Z_s ds} e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_n}^{t} Z_s ds} \middle| Z_0, Z_{t_1}, \cdots\cdots Z_{t_n}, Z_t\right)$$

$$= E\left(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_0^{t_1} Z_s ds} \middle| Z_0, Z_{t_1}\right) E\left(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_1}^{t_2} Z_s ds} \middle| Z_{t_1}, Z_{t_2}\right) \cdots E\left(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_k}^{t_{k+1}} Z_s ds} \middle| Z_{t_k}, Z_{t_{k+1}}\right) \cdots E\left(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_{n-1}}^{t_n} Z_s ds} \middle| Z_{t_{n-1}}, Z_{t_n}\right) E\left(e^{-\sqrt{\frac{\mu_2\rho}{2\delta}}\int_{t_n}^{t} Z_s ds} \middle| Z_{t_n}, Z_t\right).$$

From (4.1.19)

$$= e^{-\sqrt{\frac{\mu_2\rho}{\delta}}\left[\frac{1}{\delta} - \frac{2}{\delta\{1+e^{\delta(t_1-0)}\}}\right]Z_0} \cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}\left[\frac{2}{\delta\{1+e^{-\delta(t_1-0)}\}} - \frac{1}{\delta}\right]Z_{t_1}} \cdot e^{\frac{1}{2}\mu_2\rho\int_0^{t_1}\left\{\frac{1-e^{-\delta(t_1-u)}}{\delta} - K_0 e^{-\delta(t_1-u)}\right\}^2 du}$$

$$\cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}\left[\frac{1}{\delta} - \frac{2}{\delta\{1+e^{\delta(t_2-t_1)}\}}\right]Z_{t_1}} \cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}\left[\frac{2}{\delta\{1+e^{-\delta(t_2-t_1)}\}} - \frac{1}{\delta}\right]Z_{t_2}} \cdot e^{\frac{1}{2}\mu_2\rho\int_{t_1}^{t_2}\left\{\frac{1-e^{-\delta(t_2-u)}}{\delta} - K_1 e^{-\delta(t_2-u)}\right\}^2 du}$$

$$\cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}\left[\frac{1}{\delta} - \frac{2}{\delta\{1+e^{\delta(t_3-t_2)}\}}\right]Z_{t_2}} \cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}\left[\frac{2}{\delta\{1+e^{-\delta(t_3-t_2)}\}} - \frac{1}{\delta}\right]Z_{t_3}} \cdot e^{\frac{1}{2}\mu_2\rho\int_{t_2}^{t_3}\left\{\frac{1-e^{-\delta(t_3-u)}}{\delta} - K_2 e^{-\delta(t_3-u)}\right\}^2 du}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}\left[\frac{1}{\delta} - \frac{2}{\delta\{1+e^{\delta(t_{k+1}-t_k)}\}}\right]Z_{t_k}} \cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}\left[\frac{2}{\delta\{1+e^{-\delta(t_{k+1}-t_k)}\}} - \frac{1}{\delta}\right]Z_{t_{k+1}}} \cdot e^{\frac{1}{2}\mu_2\rho\int_{t_k}^{t_{k+1}}\left\{\frac{1-e^{-\delta(t_{k+1}-u)}}{\delta} - K_k e^{-\delta(t_{k+1}-u)}\right\}^2 du}$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$\vdots \qquad\qquad \vdots \qquad\qquad \vdots$$

$$
\cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}[\frac{1}{\delta}-\frac{2}{\delta\{1+e^{\delta(t_n-t_{n-1})}\}}]Z_{t_{n-1}}} \cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}[\frac{2}{\delta\{1+e^{-\delta(t_n-t_{n-1})}\}}-\frac{1}{\delta}]Z_{t_n}} \cdot e^{\frac{1}{2}\mu_2\rho\int_{t_{n-1}}^{t_n}\{\frac{1-e^{-\delta(t_n-u)}}{\delta}-K_{n-1}e^{-\delta(t_n-u)}\}^2du}
$$

$$
\cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}[\frac{1}{\delta}-\frac{2}{\delta\{1+e^{\delta(t-t_n)}\}}]Z_{t_n}} \cdot e^{-\sqrt{\frac{\mu_2\rho}{\delta}}[\frac{2}{\delta\{1+e^{-\delta(t-t_n)}\}}-\frac{1}{\delta}]Z_t} \cdot e^{\frac{1}{2}\mu_2\rho\int_{t_n}^{t}\{\frac{1-e^{-\delta(t-u)}}{\delta}-K_n e^{-\delta(t-u)}\}^2du} .
$$

(4.1.22)

If we set $H_k=\dfrac{1}{2}\mu_2\rho\int_{t_k}^{t_{k+1}}\{\frac{1-e^{-\delta(t_{k+1}-u)}}{\delta}-K_k e^{-\delta(t_{k+1}-u)}\}^2du$ and $Z_{t_k}=(\tilde{\lambda}_{t_k}-\frac{\mu_1\rho}{\delta})\sqrt{\frac{2\delta}{\mu_2\rho}}$ for $k=0,1,\cdots,n+1$ in (4.1.22) then

$$
= e^{H_0}e^{H_1}\cdots\cdots e^{H_{n-1}}e^{H_n}
$$

$$
\cdot e^{-(\tilde{\lambda}_0-\frac{\mu_1\rho}{\delta})[\frac{1}{\delta}-\frac{2}{\delta\{1+e^{\delta(t_1-0)}\}}]}
$$

$$
\cdot e^{-(\tilde{\lambda}_{t_1}-\frac{\mu_1\rho}{\delta})[\frac{2}{\delta\{1+e^{-\delta(t_1-0)}\}}-\frac{2}{\delta\{1+e^{\delta(t_2-t_1)}\}}]} e^{-(\tilde{\lambda}_{t_2}-\frac{\mu_1\rho}{\delta})[\frac{2}{\delta\{1+e^{-\delta(t_2-t_1)}\}}-\frac{2}{\delta\{1+e^{\delta(t_3-t_2)}\}}]}
$$

$$
\vdots \qquad\qquad\qquad \vdots
$$

$$
\vdots \qquad\qquad\qquad \vdots
$$

$$
\cdot e^{-(\tilde{\lambda}_{t_{n-3}}-\frac{\mu_1\rho}{\delta})[\frac{2}{\delta\{1+e^{-\delta(t_{n-3}-t_{n-2})}\}}-\frac{2}{\delta\{1+e^{\delta(t_{n-2}-t_{n-3})}\}}]} e^{-(\tilde{\lambda}_{t_{n-2}}-\frac{\mu_1\rho}{\delta})[\frac{2}{\delta\{1+e^{-\delta(t_{n-2}-t_{n-3})}\}}-\frac{2}{\delta\{1+e^{\delta(t_{n-1}-t_{n-2})}\}}]}
$$

$$
\cdot e^{-(\tilde{\lambda}_{t_{n-1}}-\frac{\mu_1\rho}{\delta})[\frac{2}{\delta\{1+e^{-\delta(t_{n-1}-t_{n-2})}\}}-\frac{2}{\delta\{1+e^{\delta(t_n-t_{n-1})}\}}]} e^{-(\tilde{\lambda}_{t_n}-\frac{\mu_1\rho}{\delta})[\frac{2}{\delta\{1+e^{-\delta(t_n-t_{n-1})}\}}-\frac{2}{\delta\{1+e^{\delta(t-t_n)}\}}]}
$$

$$
\cdot e^{-(\tilde{\lambda}_t-\frac{\mu_1\rho}{\delta})[\frac{2}{\delta\{1+e^{-\delta(t-t_n)}\}}-\frac{1}{\delta}]} .
$$

(4.1.23)

If we set $C_0=\left[\dfrac{1}{\delta}-\dfrac{2}{\delta\{1+e^{\delta(t_1-0)}\}}\right]$, $C_k=\left[\dfrac{2}{\delta\{1+e^{-\delta(t_k-t_{k-1})}\}}-\dfrac{2}{\delta\{1+e^{\delta(t_{k+1}-t_k)}\}}\right]$ for $k=1,2,\cdots,n$ and $C_{n+1}=\left[\dfrac{2}{\delta\{1+e^{-\delta(t-t_n)}\}}-\dfrac{1}{\delta}\right]$ in (4.1.23) the result follows.

□

Now let us examine the likelihood function (4.1.2). We can write (4.1.2) as

$$
E(\lambda_{t_1}\lambda_{t_2}\cdots\cdots\lambda_{t_n}e^{-\int_0^t\lambda_s ds})=E(\lambda_{t_1}\lambda_{t_2}\cdots\cdots\lambda_{t_n}e^{-\int_0^{t_1}\lambda_s ds}\cdot e^{-\int_{t_1}^{t_2}\lambda_s ds}\cdots\cdots e^{-\int_{t_{n-1}}^{t_n}\lambda_s ds}\; e^{-\int_{t_n}^{t}\lambda_s ds}). \quad (4.1.24)
$$

Since we have obtained $\tilde{\lambda}_t$ which is a Gaussian approximation of λ_t, we will use this approximation (see (4.1.10)). We will also assume that the process has reached its stationary state. Then (4.1.24) can be written as

$$E(\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-\int_0^t\tilde{\lambda}_s ds}) = E(\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-\int_0^{t_1}\tilde{\lambda}_s ds}\cdot e^{-\int_{t_1}^{t_2}\tilde{\lambda}_s ds}\cdots\cdots e^{-\int_{t_{n-1}}^{t_n}\tilde{\lambda}_s ds}\,e^{-\int_{t_n}^{t}\tilde{\lambda}_s ds})$$

$$= e^{-\int_0^{t_1}\frac{\mu\varphi}{\delta}ds}\cdot e^{-\int_{t_1}^{t_2}\frac{\mu\varphi}{\delta}ds}\cdots\cdots e^{-\int_{t_{n-1}}^{t_n}\frac{\mu\varphi}{\delta}ds}\,e^{-\int_{t_n}^{t}\frac{\mu\varphi}{\delta}ds}\,E\{\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}E(e^{-\sqrt{\frac{\mu^2\varphi}{2\delta}}\int_0^{t_1}Z_s ds}\cdot e^{-\sqrt{\frac{\mu^2\varphi}{2\delta}}\int_{t_1}^{t_2}Z_s ds}\cdots\cdots e^{-\sqrt{\frac{\mu^2\varphi}{2\delta}}\int_{t_{n-1}}^{t_n}Z_s ds}\,e^{-\sqrt{\frac{\mu^2\varphi}{2\delta}}\int_{t_n}^{t}Z_s ds}\,\big|Z_0, Z_{t_1},\cdots\cdots Z_{t_n}, Z_t)\}.$$

From (4.1.21)

$$= e^{-\int_{t_n}^{t}\frac{\mu\varphi}{\delta}ds}\,e^{H_0}e^{H_1}\cdots e^{H_{n-1}}e^{H_n}e^{\frac{\mu\varphi}{\delta}C_0}e^{\frac{\mu\varphi}{\delta}C_1}e^{\frac{\mu\varphi}{\delta}C_2}\cdots e^{\frac{\mu\varphi}{\delta}C_n}e^{\frac{\mu\varphi}{\delta}C_{n+1}}\cdot E\{\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-C_0\tilde{\lambda}_0}e^{-C_1\tilde{\lambda}_{t_1}}e^{-C_2\tilde{\lambda}_{t_2}}\cdots\cdots e^{-C_n\tilde{\lambda}_{t_n}}e^{-C_{n+1}\tilde{\lambda}_t}\}.$$

Assuming that we know $\tilde{\lambda}_0$ then

$$E(\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-\int_0^t\tilde{\lambda}_s ds}\,\big|\,\tilde{\lambda}_0)$$

$$= e^{H_0+H_1\cdots+H_n}e^{\frac{\mu\varphi}{\delta}(C_0+C_1+\cdots+C_{n+1}-t)}e^{-C_0\tilde{\lambda}_0}\cdot E(\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-C_1\tilde{\lambda}_{t_1}}e^{-C_2\tilde{\lambda}_{t_2}}\cdots\cdots e^{-C_n\tilde{\lambda}_{t_n}}e^{-C_{n+1}\tilde{\lambda}_t}). \tag{4.1.25}$$

From (4.1.25) we can see that $E(\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-C_1\tilde{\lambda}_{t_1}}e^{-C_2\tilde{\lambda}_{t_2}}\cdots\cdots e^{-C_n\tilde{\lambda}_{t_n}}e^{-C_{n+1}\tilde{\lambda}_t})$ needs to be obtained. We will use the m.g.f. to derive it, i.e.

$$E(e^{V'\Lambda}) = E(e^{v_1\tilde{\lambda}_{t_1}}e^{v_2\tilde{\lambda}_{t_2}}\cdots e^{v_n\tilde{\lambda}_{t_n}}e^{v_{n+1}\tilde{\lambda}_t}) = e^{V'\Psi+\frac{1}{2}V'\Sigma V} = e^{\sum_{i=1}^{n+1}v_i\phi_i+\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}v_i v_j\sigma_{ij}} \tag{4.1.26}$$

where

$$V' = [v_1, v_2, \cdots\cdots, v_{n+1}],\quad \Lambda = \begin{bmatrix}\tilde{\lambda}_{t_1}\\ \tilde{\lambda}_{t_2}\\ \vdots\\ \vdots\\ \tilde{\lambda}_t\end{bmatrix},\quad \Psi = \begin{bmatrix}\phi_1\\ \phi_2\\ \vdots\\ \vdots\\ \phi_{n+1}\end{bmatrix} = \begin{bmatrix}E(\tilde{\lambda}_{t_1})\\ E(\tilde{\lambda}_{t_2})\\ \vdots\\ \vdots\\ E(\tilde{\lambda}_t)\end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix}\sigma_{11} & \sigma_{12} & \sigma_{13} & \cdots & \cdots & \sigma_{1,n+1}\\ \sigma_{21} & \sigma_{22} & \sigma_{23} & \cdots & \cdots & \sigma_{2,n+1}\\ \sigma_{31} & \sigma_{32} & \sigma_{33} & \cdots & \cdots & \sigma_{3,n+1}\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ \sigma_{n+1,1} & \sigma_{n+1,2} & \sigma_{n+1,3} & \cdots & \cdots & \sigma_{n+1,n+1}\end{bmatrix}.$$

From (4.1.10)

$$E(\tilde{\lambda}_{t_i}) = E(\frac{\mu_1\rho}{\delta} + Z_{t_i}\sqrt{\frac{\mu_2\rho}{2\delta}}) = \frac{\mu_1\rho}{\delta}.$$

If $i = j$,

$$\sigma_{ii} = \sigma_i^2 = Cov(\tilde{\lambda}_{t_i}, \tilde{\lambda}_{t_i}) = Var(\tilde{\lambda}_{t_i}) = Var(\frac{\mu_1\rho}{\delta} + Z_{t_i}\sqrt{\frac{\mu_2\rho}{2\delta}}) = \frac{\mu_2\rho}{2\delta}.$$

If $i \neq j$,

$$\sigma_{ij} = Cov(\tilde{\lambda}_{t_i}, \tilde{\lambda}_{t_j}) = Cov(\frac{\mu_1\rho}{\delta} + Z_{t_i}\sqrt{\frac{\mu_2\rho}{2\delta}}, \frac{\mu_1\rho}{\delta} + Z_{t_j}\sqrt{\frac{\mu_2\rho}{2\delta}}) = (\frac{\mu_2\rho}{2\delta})e^{-\delta|t_j - t_i|}.$$

Hence

$$\Psi = \begin{bmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \vdots \\ \phi_{n+1} \end{bmatrix} = \begin{bmatrix} E(\tilde{\lambda}_{t_1}) \\ E(\tilde{\lambda}_{t_2}) \\ \vdots \\ \vdots \\ E(\tilde{\lambda}_t) \end{bmatrix} = \begin{bmatrix} \frac{\mu_1\rho}{\delta} \\ \frac{\mu_1\rho}{\delta} \\ \vdots \\ \vdots \\ \frac{\mu_1\rho}{\delta} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} \frac{\mu_2\rho}{2\delta} & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t_2-t_1)} & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t_3-t_1)} & \cdots & \cdots & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t-t_1)} \\ (\frac{\mu_2\rho}{2\delta})e^{-\delta(t_2-t_1)} & \frac{\mu_2\rho}{2\delta} & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t_3-t_2)} & \cdots & \cdots & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t-t_2)} \\ (\frac{\mu_2\rho}{2\delta})e^{-\delta(t_3-t_1)} & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t_3-t_2)} & \frac{\mu_2\rho}{2\delta} & \cdots & \cdots & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t-t_3)} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (\frac{\mu_2\rho}{2\delta})e^{-\delta(t-t_1)} & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t-t_2)} & (\frac{\mu_2\rho}{2\delta})e^{-\delta(t-t_3)} & \cdots & \cdots & \frac{\mu_2\rho}{2\delta} \end{bmatrix}.$$

Lemma 4.1.9 Let $\tilde{\lambda}_t$ as defined. Then

$$E(\tilde{\lambda}_{t_1}\,\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}\,e^{-C_1\tilde{\lambda}_{t_1}}e^{-C_2\tilde{\lambda}_{t_2}}\cdots\cdots e^{-C_n\tilde{\lambda}_{t_n}}e^{-C_{n+1}\tilde{\lambda}_t})$$

$$= A\cdot\{\frac{\sigma_{1,n}}{\phi_n + \sum_{j=1}^{n+1}\sigma_{nj}\nu_j} + \frac{\sigma_{1,n-1}}{\phi_{n-1} + \sum_{j=1}^{n+1}\sigma_{n-1j}\nu_j} + \cdots + \frac{\sigma_{12}}{\phi_2 + \sum_{j=1}^{n+1}\sigma_{2j}\nu_j} + (\phi_1 + \sum_{j=1}^{n+1}\sigma_{1j}\nu_j)\}\cdot e^{\sum_{i=1}^{n+1}\nu_i\phi_i + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}\nu_i\nu_j\sigma_{ij}}\Big|_{\nu_{n+1}=-C_{n+1},\nu_n=-C_n,\cdots,\nu_2=-C_2,\nu_1=-C_1} \tag{4.1.27}$$

where

$$A = \{(\phi_n + \sum_{j=1}^{n+1}\sigma_{nj}\nu_j)(\phi_{n-1} + \sum_{j=1}^{n+1}\sigma_{n-1j}\nu_j)(\phi_{n-2} + \sum_{j=1}^{n+1}\sigma_{n-2j}\nu_j)\cdots\cdots(\phi_3 + \sum_{j=1}^{n+1}\sigma_{3j}\nu_j)(\phi_2 + \sum_{j=1}^{n+1}\sigma_{2j}\nu_j)\}.$$

Proof

By differentiating (4.1.26) w.r.t. $v_1, v_2, \cdots\cdots, v_{n-1}, v_n$

$$\frac{\partial^n}{\partial v_n \partial v_{n-1} \cdots\cdots \partial v_2 \partial v_1} E(e^{V'\Lambda}) = \frac{\partial^n}{\partial v_n \partial v_{n-1} \cdots\cdots \partial v_2 \partial v_1} e^{\sum_{i=1}^{n+1} v_i \phi_i + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} v_i v_j \sigma_{ij}} = E(\tilde{\lambda}_{t_1} \tilde{\lambda}_{t_2} \cdots \tilde{\lambda}_{t_n} e^{-v_1 \tilde{\lambda}_{t_1}} e^{-v_2 \tilde{\lambda}_{t_2}} \cdots e^{-v_n \tilde{\lambda}_{t_n}} e^{-v_{n+1} \tilde{\lambda}_t})$$

$$\begin{aligned}
&= \{(\phi_{n-1} + \sum_{j=1}^{n+1} \sigma_{n-1j} v_j)(\phi_{n-2} + \sum_{j=1}^{n+1} \sigma_{n-2j} v_j) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (\phi_2 + \sum_{j=1}^{n+1} \sigma_{2j} v_j)\sigma_{1n} \\
&\quad + (\phi_n + \sum_{j=1}^{n+1} \sigma_{nj} v_j)(\phi_{n-2} + \sum_{j=1}^{n+1} \sigma_{n-2j} v_j) \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots (\phi_2 + \sum_{j=1}^{n+1} \sigma_{2j} v_j)\sigma_{1n-1} \\
&\quad + (\phi_n + \sum_{j=1}^{n+1} \sigma_{n-1j} v_j)(\phi_{n-1} + \sum_{j=1}^{n=1} \sigma_{n-1j} v_j)(\phi_{n-3} + \sum_{j=1}^{n+1} \sigma_{n-3j} v_j) \cdots\cdots (\phi_2 + \sum_{j=1}^{n+1} \sigma_{2j} v_j)\sigma_{1n-2} \\
&\quad + \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad + \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad + \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad + \quad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \\
&\quad + \{(\phi_n + \sum_{j=1}^{n+1} \sigma_{nj} v_j)(\phi_{n-1} + \sum_{j=1}^{n+1} \sigma_{n-1j} v_j) \cdots\cdots\cdots\cdots (\phi_4 + \sum_{j=1}^{n+1} \sigma_{4j} v_j)(\phi_2 + \sum_{j=1}^{n+1} \sigma_{2j} v_j)\sigma_{13} \\
&\quad + (\phi_n + \sum_{j=1}^{n+1} \sigma_{nj} v_j)(\phi_{n-2} + \sum_{j=1}^{n+1} \sigma_{n-2j} v_j) \cdots\cdots\cdots\cdots\cdots (\phi_4 + \sum_{j=1}^{n+1} \sigma_{4j} v_j)(\phi_3 + \sum_{j=1}^{n+1} \sigma_{3j} v_j)\sigma_{12} \\
&\quad + (\phi_n + \sum_{j=1}^{n+1} \sigma_{n-1j} v_j)(\phi_{n-1} + \sum_{j=1}^{n+1} \sigma_{n-1j} v_j) \cdots\cdots\cdots\cdots (\phi_2 + \sum_{j=1}^{n+1} \sigma_{2j} v_j)(\phi_1 + \sum_{j=1}^{n+1} \sigma_{1j} v_j)\} \cdot e^{\sum_{i=1}^{n+1} v_i \phi_i + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} v_i v_j \sigma_{ij}}.
\end{aligned}$$

(4.1.28)

If we set

$$A = \{(\phi_n + \sum_{j=1}^{n+1} \sigma_{nj} v_j)(\phi_{n-1} + \sum_{j=1}^{n+1} \sigma_{n-1j} v_j)(\phi_{n-2} + \sum_{j=1}^{n+1} \sigma_{n-2j} v_j) \cdots\cdots (\phi_3 + \sum_{j=1}^{n+1} \sigma_{3j} v_j)(\phi_2 + \sum_{j=1}^{n+1} \sigma_{2j} v_j)\}.$$

in (4.1.28) then

$$= A \cdot \{\frac{\sigma_{1,n}}{\phi_n + \sum_{j=1}^{n+1} \sigma_{nj} v_j} + \frac{\sigma_{1,n-1}}{\phi_{n-1} + \sum_{j=1}^{n+1} \sigma_{n-1j} v_j} + \frac{\sigma_{1,n-2}}{\phi_{n-2} + \sum_{j=1}^{n+1} \sigma_{n-2j} v_j} + \cdots\cdots + \frac{\sigma_{12}}{\phi_2 + \sum_{j=1}^{n+1} \sigma_{2j} v_j} + (\phi_1 + \sum_{j=1}^{n+1} \sigma_{1j} v_j)\} \cdot e^{\sum_{i=1}^{n+1} v_i \phi_i + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} v_i v_j \sigma_{ij}}.$$

(4.1.29)

The result follows if we set $v_{n+1} = -C_{n+1}, v_n = -C_n, \cdots\cdots, v_2 = -C_2, v_1 = -C_1$ in (4.1.29).

□

Theorem 4.1.10 Let $\tilde{\lambda}_t$ as defined. Then

$$E(\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-\int_0^t \tilde{\lambda}_s ds}\Big|\tilde{\lambda}_0) = B\cdot e^{H_0+H_1+\cdots H_{n-1}+H_n}\cdot e^{\frac{\mu_1\rho}{\delta}(C_0+C_1+\cdots+C_{n+1}-t)}\cdot e^{-C_0\tilde{\lambda}_0} \tag{4.1.30}$$

where

$$H_k = \frac{1}{2}\mu_2\rho\int_{t_k}^{t_{k+1}}\{\tfrac{1-e^{-\delta(t_{k+1}-u)}}{\delta} - K_k e^{-\delta(t_{k+1}-u)}\}^2 du \text{ for } k = 0,1,\cdots,n+1,$$

$$C_0 = \left[\frac{1}{\delta} - \frac{2}{\delta\{1+e^{\delta(t_1-0)}\}}\right],\ C_k = \left[\frac{2}{\delta\{1+e^{-\delta(t_k-t_{k-1})}\}} - \frac{2}{\delta\{1+e^{\delta(t_{k+1}-t_k)}\}}\right] \text{ for } k = 1,2,\cdots,n,$$

$$C_{n+1} = \left[\frac{2}{\delta\{1+e^{-\delta(t-t_n)}\}} - \frac{1}{\delta}\right],$$

$$A = \{(\phi_n + \sum_{j=1}^{n+1}\sigma_{nj}v_j)(\phi_{n-1} + \sum_{j=1}^{n+1}\sigma_{n-1j}v_j)(\phi_{n-2} + \sum_{j=1}^{n+1}\sigma_{n-2j}v_j)\cdots\cdots(\phi_3 + \sum_{j=1}^{n+1}\sigma_{3j}v_j)(\phi_2 + \sum_{j=1}^{n+1}\sigma_{2j}v_j)\},$$

and

$$B = A\cdot\{\frac{\sigma_{1,n}}{\phi_n + \sum_{j=1}^{n+1}\sigma_{nj}v_j} + \frac{\sigma_{1,n-1}}{\phi_{n-1} + \sum_{j=1}^{n+1}\sigma_{n-1j}v_j} + \frac{\sigma_{1,n-2}}{\phi_{n-2} + \sum_{j=1}^{n+1}\sigma_{n-2j}v_j} + \cdots\cdots + \frac{\sigma_{12}}{\phi_2 + \sum_{j=1}^{n+1}\sigma_{2j}v_j}$$

$$+(\phi_1 + \sum_{j=1}^{n+1}\sigma_{1j}v_j)\}\cdot e^{\sum_{i=1}^{n+1}v_i\phi_i + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}v_iv_j\sigma_{ij}}\Big| v_{n+1} = -C_{n+1}, v_n = -C_n, \cdots, v_2 = -C_2, v_1 = -C_1.$$

Proof

From (4.1.25)

$$E(\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-\int_0^t \tilde{\lambda}_s ds}\Big|\tilde{\lambda}_0)$$

$$= e^{H_0+H_1\cdots+H_n}e^{\frac{\mu_1\rho}{\delta}(C_0+C_1+\cdots+C_{n+1}-t)}e^{-C_0\tilde{\lambda}_0}\cdot E(\tilde{\lambda}_{t_1}\tilde{\lambda}_{t_2}\cdots\cdots\tilde{\lambda}_{t_n}e^{-C_1\tilde{\lambda}_{t_1}}e^{-C_2\tilde{\lambda}_{t_2}}\cdots\cdots e^{-C_n\tilde{\lambda}_{t_n}}e^{-C_{n+1}\tilde{\lambda}_t}).$$

From (4.1.27)

$$= e^{H_0+H_1\cdots+H_n}e^{\frac{\mu_1\rho}{\delta}(C_0+C_1+\cdots+C_{n+1}-t)}e^{-C_0\tilde{\lambda}_0}$$

$$\cdot A\{\frac{\sigma_{1,n}}{\phi_n + \sum_{j=1}^{n+1}\sigma_{nj}v_j} + \frac{\sigma_{1,n-1}}{\phi_{n-1} + \sum_{j=1}^{n+1}\sigma_{n-1j}v_j} + \cdots + \frac{\sigma_{12}}{\phi_2 + \sum_{j=1}^{n+1}\sigma_{2j}v_j} + (\phi_1 + \sum_{j=1}^{n+1}\sigma_{1j}v_j)\}\cdot e^{\sum_{i=1}^{n+1}v_i\phi_i + \frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}v_iv_j\sigma_{ij}}\Big|_{v_{n+1}=-C_{n+1}, v_n=-C_n, \cdots, v_2=-C_2, v_1=-C_1}.$$

Set

$$B = A\cdot\{\frac{\sigma_{1,n}}{\phi_n+\sum_{j=1}^{n+1}\sigma_{nj}v_j}+\frac{\sigma_{1,n-1}}{\phi_{n-1}+\sum_{j=1}^{n+1}\sigma_{n-1j}v_j}+\frac{\sigma_{1,n-2}}{\phi_{n-2}+\sum_{j=1}^{n+1}\sigma_{n-2j}v_j}+\cdots\cdots+\frac{\sigma_{12}}{\phi_2+\sum_{j=1}^{n+1}\sigma_{2j}v_j}$$

$$+(\phi_1+\sum_{j=1}^{n+1}\sigma_{1j}v_j)\}\cdot e^{\sum_{i=1}^{n+1}v_i\phi_i+\frac{1}{2}\sum_{i=1}^{n+1}\sum_{j=1}^{n+1}v_iv_j\sigma_{ij}}\Big|v_{n+1}=-C_{n+1},v_n=-C_n,\cdots,v_2=-C_2,v_1=-C_1$$

then the result follows immediately.

□

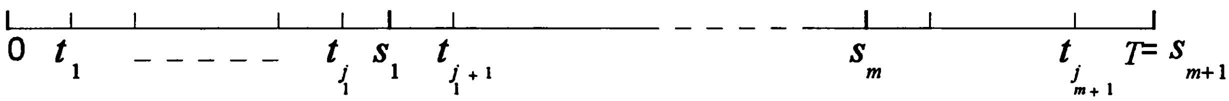
As the likelihood function has been obtained the value of the parameters, i.e. δ, $\frac{\mu_1\rho}{\delta}$, $\frac{\mu_2\rho}{\delta}$ that maximise the likelihood function can be evaluated from (4.1.30) as an estimation procedure.

4.2 Where the times of catastrophes and claims are known: A direct approach

This section deals with obtaining the maximum likelihood estimators of the parameters in the shot noise process λ_t assuming that the times of catastrophe jumps and claim points are known. In other words, we will obtain the maximum likelihood estimators of ρ , δ and y_i on the basis of observed catastrophes and claims. Its application in calculating the estimators of the parameters is illustrated in example 4.2.1 at the end of this section. The appendix contains the *S-Plus* routine needed for this calculation. Note that ρ is not large and therefore the Ornstein-Uhlenbeck process approximation is not used.

The times of the jump occurrence are denoted by $s_1, s_2, \cdots, s_m$ (i.e. m catastrophe jumps) and the times of point occurrence in each jump interarrival time as $t_1, t_2, \cdots, t_{j_1}, t_{j_1+1}, \cdots, t_{j_2}, \cdots\cdots\cdots,$ $t_{j_m+1}, \cdots, t_{j_{m+1}}$ (i.e. j_{m+1} claim points). The intensity function λ decreases with the decay rate δ between the time s_{i-1} and s_i where $i = 2, 3, \cdots, m$ until another catastrophe jump y_i occurs at s_i. Therefore $\lambda_{s_1}, \lambda_{s_2}, \cdots, \lambda_{s_m}$ are regarded as parameters of the problems from which we can estimate the jump sizes as there are not many catastrophes. We observe the claim points between catastrophe jumps while λ decreases with the decay rate δ.

Let's assume that the claim points and catastrophe jumps do not occur at the same time. A time interval $(0, T)$ can be divided by the epochs at which the jumps and points occur. The figure below illustrates the epochs at which the jumps and points occur.



where $T = s_{m+1}$ is a fixed time not the epoch at which the $m+1^{th}$ jump occurs.

We can estimate the rate of jump arrivals by

$$\hat{\rho} = \frac{m}{T}. \tag{4.2.1}$$

Let us begin by deriving the maximum likelihood estimator of λ_{s_i}, $\hat{\lambda}_{s_i}$.

Theorem 4.2.1 Let N_t as defined with intensity λ_t that is time homogeneous. Then

$$\hat{\lambda}_{s_i} = \frac{\hat{\delta}(j_{i+1} - j_i)}{1 - e^{-\hat{\delta}(s_{i+1} - s_i)}} \tag{4.2.2}$$

where $\hat{\delta}$ is the maximum likelihood estimator of δ and $i = 0, 1, \cdots, m$.

Proof

If we know the times of catastrophe jumps and claim points, from (4.1.1), the likelihood function L is given by

$$L = \lambda_0^{j_1} e^{-\delta \sum_{l=1}^{j_1} t_l} e^{-\lambda_0 \frac{1-e^{-\delta s_1}}{\delta}} \cdot \lambda_{s_1}^{j_2 - j_1} e^{-\delta \sum_{l=j_1+1}^{j_2} (t_l - s_1)} e^{-\lambda_{s_1} \frac{1-e^{-\delta(s_2 - s_1)}}{\delta}} \cdots\cdots$$
$$\lambda_{s_{m-1}}^{j_m - j_{m-1}} e^{-\delta \sum_{l=j_{m-1}+1}^{j_m} (t_l - s_{m-1})} e^{-\lambda_{s_{m-1}} \frac{1-e^{-\delta(s_m - s_{m-1})}}{\delta}} \cdot \lambda_{s_m}^{j_{m+1} - j_m} e^{-\delta \sum_{l=j_m+1}^{j_{m+1}} (t_l - s_m)} e^{-\lambda_{s_m} \frac{1-e^{-\delta(s_{m+1} - s_m)}}{\delta}}. \tag{4.2.3}$$

Take the logarithm of (4.2.3) and differentiate with respect to λ_{s_i} $(i = 0, 1, \cdots, m)$; if we equate this to 0 then the maximum likelihood estimator of λ_{s_i} is

$$\hat{\lambda}_{s_i} = \frac{\hat{\delta}(j_{i+1} - j_i)}{1 - e^{-\hat{\delta}(s_{i+1} - s_i)}}.$$

□

Corollary 4.2.2 Let N_t and λ_t as defined in theorem 4.2.1. Then

$$\left.\frac{\partial^2 \ln L}{\partial \lambda_{s_i}^2}\right|_{\substack{\lambda_{s_i} = \hat{\lambda}_{s_i} \\ \delta = \hat{\delta}}} = -\frac{\left\{1 - e^{-\hat{\delta}(s_{i+1} - s_i)}\right\}^2}{\hat{\delta}^2 (j_{i+1} - j_i)} \tag{4.2.4}$$

where $i = 0, 1, \cdots, m$.

Proof

The result follows if we take the logarithm of (4.2.3) and differentiate twice with respect to λ_{s_i} $(i = 0, 1, \cdots, m)$ and set $\hat{\lambda}_{s_i} = \dfrac{\hat{\delta}(j_{i+1} - j_i)}{1 - e^{-\hat{\delta}(s_{i+1} - s_i)}}$.

□

Corollary 4.2.3 Let N_t and λ_t as defined. Then

$$\left.\frac{\partial^2 \ln L}{\partial\lambda_{s_i}\partial\lambda_{s_k}}\right|_{\substack{\lambda_{s_i}=\hat{\lambda}_{s_i}\\ \lambda_{s_k}=\hat{\lambda}_{s_k}}} = 0 \tag{4.2.5}$$

where $i \neq k$.

<u>**Proof**</u>

The result follows if we take the logarithm of (4.2.3) and differentiate with respect to λ_{s_i} and λ_{s_k}.

□

Theorem 4.2.4 Let N_t and λ_t as defined above. Then

$$\left.\frac{\partial \ln L}{\partial\delta}\right|_{\substack{\delta=\hat{\delta}\\ \lambda_{s_i}=\hat{\lambda}_{s_i}\,\text{all}\, i}} = \frac{j_{m+1}}{\hat{\delta}} - j_1 s_1\frac{e^{-\hat{\delta}s_1}}{1-e^{-\hat{\delta}s_1}} - (j_2-j_1)(s_2-s_1)\frac{e^{-\hat{\delta}(s_2-s_1)}}{1-e^{-\hat{\delta}(s_2-s_1)}} - \cdots\cdots - (j_m-j_{m-1})(s_m-s_{m-1})\frac{e^{-\hat{\delta}(s_m-s_{m-1})}}{1-e^{-\hat{\delta}(s_m-s_{m-1})}}$$

$$-(j_{m+1}-j_m)(T-s_m)\frac{e^{-\hat{\delta}(T-s_m)}}{1-e^{-\hat{\delta}(T-s_m)}} - \sum_{i=1}^{j_1} t_i - \sum_{i=j_1+1}^{j_2}(t_i-s_1) - \cdots\cdots - \sum_{i=j_{m-1}+1}^{j_m}(t_i-s_{m-1}) - \sum_{i=j_m+1}^{j_{m+1}}(t_i-s_m). \tag{4.2.6}$$

<u>**Proof**</u>

The result follows if we take the logarithm of (4.2.3) and differentiate with respect to δ and set $\hat{\lambda}_{s_i} = \dfrac{\hat{\delta}(j_{i+1}-j_i)}{1-e^{-\hat{\delta}(s_{i+1}-s_i)}}$ $(i = 0,1,\cdots,m)$.

□

The maximum likelihood estimator of δ, $\hat{\delta}$ can be obtained by equating (4.2.6) to 0.

Corollary 4.2.5 Let N_t and λ_t as defined. Then

$$\left.\frac{\partial^2 \ln L}{\partial\hat{\delta}^2}\right|_{\substack{\delta=\hat{\delta}\\ \lambda_{s_i}=\hat{\lambda}_{s_i}\,\text{all}\, i}} = j_1 s_1\left(\frac{2}{\hat{\delta}}+s_1\right)\frac{e^{-\hat{\delta}s_1}}{1-e^{-\hat{\delta}s_1}} + (j_2-j_1)(s_2-s_1)\left\{\frac{2}{\hat{\delta}}+(s_2-s_1)\right\}\frac{e^{-\hat{\delta}(s_2-s_1)}}{1-e^{-\hat{\delta}(s_2-s_1)}} + \cdots\cdots$$

$$+(j_m-j_{m-1})(s_m-s_{m-1})\left\{\frac{2}{\hat{\delta}}+(s_m-s_{m-1})\right\}\frac{e^{-\hat{\delta}(s_m-s_{m-1})}}{1-e^{-\hat{\delta}(s_m-s_{m-1})}} + (j_{m+1}-j_m)(T-s_m)\left\{\frac{2}{\hat{\delta}}+(T-s_m)\right\}\frac{e^{-\hat{\delta}(T-s_m)}}{1-e^{-\hat{\delta}(T-s_m)}} - \frac{2j_{m+1}}{\hat{\delta}^2}. \tag{4.2.7}$$

Proof

The result follows if we take the logarithm of (4.2.3) and differentiate twice with respect to δ and set $\hat{\lambda}_{s_i} = \dfrac{\hat{\delta}(j_{i+1}-j_i)}{1-e^{-\hat{\delta}(s_{i+1}-s_i)}}$ ($i = 0,1,\cdots,m$).

□

Corollary 4.2.6 Let N_t and λ_t as defined. Then

$$\left.\frac{\partial^2 \ln L}{\partial \hat{\lambda}_{s_i}\, \partial \hat{\delta}}\right|_{\substack{\delta=\hat{\delta} \\ \lambda_{s_i}=\hat{\lambda}_{s_i} \text{all } i}} = \frac{1-e^{-\hat{\delta}(s_{i+1}-s_i)}}{\hat{\delta}^2} - \frac{(s_{i+1}-s_i)e^{-\hat{\delta}(s_{i+1}-s_i)}}{\hat{\delta}} \tag{4.2.8}$$

where $i = 0,1,\cdots,m$.

Proof

The result follows if we take the logarithm in (4.2.3) and differentiate with respect to λ_{s_i} ($i = 0,1,\cdots,m$) and δ.

□

As the maximum likelihood estimation is used, the variance-covariance matrix of the maximum likelihood estimators of $\lambda_0, \lambda_{s_1}, \cdots, \lambda_{s_m}$ and δ (i.e. $\hat{\lambda}_0, \hat{\lambda}_{s_1}, \cdots, \hat{\lambda}_{s_m}$ and $\hat{\delta}$) is given by

$$-\begin{bmatrix} \frac{\partial^2 \ln L}{\partial \lambda_0^2} & 0 & \cdots & \cdots & 0 & \frac{\partial^2 \ln L}{\partial \lambda_0 \partial \delta} \\ 0 & \frac{\partial^2 \ln L}{\partial \lambda_{s_1}^2} & 0 & \cdots & 0 & \frac{\partial^2 \ln L}{\partial \lambda_{s_1} \partial \delta} \\ \vdots & 0 & \ddots & 0 & 0 & \vdots \\ \vdots & \vdots & 0 & \ddots & 0 & \vdots \\ 0 & 0 & 0 & 0 & \frac{\partial^2 \ln L}{\partial \lambda_{s_m}^2} & \frac{\partial^2 \ln L}{\partial \lambda_{s_m} \partial \delta} \\ \frac{\partial^2 \ln L}{\partial \lambda_{s_0} \partial \delta} & \frac{\partial^2 \ln L}{\partial \lambda_{s_1} \partial \delta} & \cdots & \cdots & \frac{\partial^2 \ln L}{\partial \lambda_{s_m} \partial \delta} & \frac{\partial^2 \ln L}{\partial \delta^2} \end{bmatrix}^{-1} \tag{4.2.9}$$

Now let us find the maximum likelihood estimator of jump size y_i, $\hat{y}_i$.

Corollary 4.2.7 Let N_t and λ_t as defined and y_i be the jump size. Then

$$\hat{y}_i = \hat{\lambda}_{s_i} - \hat{\lambda}_{s_{i-1}} e^{-\hat{\delta} s_i} = \frac{\hat{\delta}(j_{i+1} - j_i)}{1 - e^{-\hat{\delta}(s_{i+1} - s_i)}} - \frac{\hat{\delta}(j_i - j_{i-1})}{1 - e^{-\hat{\delta}(s_i - s_{i-1})}} e^{-\hat{\delta} s_i} \tag{4.2.10}$$

where $i = 1, 2, \cdots, m$.

__Proof__

The intensity at s_1 at which the first jump occurred is $\lambda_{s_1} = \lambda_0 e^{-\delta s_1} + y_1$. Therefore

$$\hat{y}_1 = \hat{\lambda}_{s_1} - \hat{\lambda}_0 e^{-\hat{\delta} s_1}.$$

From theorem 4.2.1

$$\hat{y}_1 = \hat{\lambda}_{s_1} - \hat{\lambda}_0 e^{-\hat{\delta} s_1} = \frac{\hat{\delta}(j_2 - j_1)}{1 - e^{-\hat{\delta}(s_2 - s_1)}} - \frac{\hat{\delta} j_1}{1 - e^{-\hat{\delta} s_1}} e^{-\hat{\delta} s_1}.$$

Similarly,

$$\hat{y}_i = \hat{\lambda}_{s_i} - \hat{\lambda}_{s_{i-1}} e^{-\hat{\delta} s_i} = \frac{\hat{\delta}(j_{i+1} - j_i)}{1 - e^{-\hat{\delta}(s_{i+1} - s_i)}} - \frac{\hat{\delta}(j_i - j_{i-1})}{1 - e^{-\hat{\delta}(s_i - s_{i-1})}} e^{-\hat{\delta} s_i}.$$

□

The following example illustrates the calculation of the estimators of parameters by maximum likelihood estimation.

__Example 4.2.1__

The numerical values used to simulate the claim arrival process are $m = 3$, $\delta = 0.1$, $\lambda_0 = 1{,}000$. We will assume that the interarrival time between jumps is exponential with mean 1, i.e. $\rho = 1$ and that the jump size follows exponential with mean 100, i.e. $y \sim Exponential(0.01)$.

S-Plus was used to generate random values and to simulate the claim arrival process. The calculation of the estimators of parameters by maximum likelihood estimation are shown in Table 4.2.1 (see appendix for simulation and estimation).

The distribution of the jump size has not been estimated directly. As we have seen in the previous chapter, to calculate the stop-loss reinsurance premium and the price of catastrophe insurance derivatives, we should use a theoretical distribution for jump size y and what we only need to calculate the prices is the mean of jump sizes. Therefore if the jump size distribution is exponential (i.e. $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$) $\frac{\hat{y}_1 + \hat{y}_2 + \hat{y}_3}{3}$ can be used as $\frac{1}{\hat{\alpha}}$. The reader should bear in mind that the general distribution of jump size y cannot be deduced from 3 catastrophes.

Table 4.2.1 The estimates of parameters calculated by M.L.E.

	Numerical values used for simulation		Estimate of parameters	Standard error
δ	0.1	$\hat{\delta}$	0.070768653	0.04572912
λ_0	1,000	$\hat{\lambda}_0$	976.7606	39.7807772
λ_{s_1}	1,024.994	$\hat{\lambda}_{s_1}$	1,068.861	49.5543917
λ_{s_2}	1,302.009	$\hat{\lambda}_{s_2}$	1,216.297	51.076307
λ_{s_3}	1,230.569	$\hat{\lambda}_{s_3}$	1,173.496	43.732490
$\frac{y_1 + y_2 + y_3}{3}$ Average jump size	179.6427 where $y_1 = 100.3073$ $y_2 = 323.7813$ $y_3 = 114.8395$	$\frac{\hat{y}_1 + \hat{y}_2 + \hat{y}_3}{3}$	186.0299	80.89193
ρ	1	$\hat{\rho}$	0.80386	0.46411

From (4.2.4), (4.2.5), (4.2.7), (4.2.8) and (4.2.9) *the variance-covariance matrix* of the maximum likelihood estimators of $\lambda_0, \lambda_{s_1}, \lambda_{s_2}, \lambda_{s_3}$ and δ (i.e. $\hat{\lambda}_0, \hat{\lambda}_{s_1}, \hat{\lambda}_{s_2}, \hat{\lambda}_{s_3}$ and $\hat{\delta}$) is as follows:

$$\begin{bmatrix} 1582.5102371 & 196.6463007 & 730.386784 & 431.220483 & 0.792275617 \\ 196.6463007 & 2455.6377323 & 478.488780 & 282.499858 & 0.519033205 \\ 730.3867842 & 478.4887797 & 2608.789093 & 1049.265418 & 1.927801296 \\ 431.2204832 & 282.4998580 & 1049.265418 & 1912.530661 & 1.138174217 \\ 0.792275617 & 0.519033205 & 1.927801296 & 1.138174217 & 0.002091152 \end{bmatrix}.$$

5. State Estimation

In practical situations, we observe claims (and perhaps catastrophes) and we want to filter the "noise" out and "estimate" the value of λ_t at any time. This is useful for the pricing of catastrophe reinsurance contracts and catastrophe insurance derivatives in chapter 3 as it helps us estimate the distribution of λ_0 from the past data. Therefore in this chapter we will examine the conditional distribution of λ_t given "observed information".

Firstly, we will try to obtain the Kalman-Bucy filter by transforming and approximating λ_t and N_t as normal variables Z_t and W_t from which we will derive the distribution of Z_t. As a matter of interest we will also examine the pricing of stop-loss reinsurance contracts using the Kalman-Bucy filter. Its application in computing the premium will be illustrated. Secondly, the Laplace transform of distribution of λ_t will be obtained assuming that we know the times of catastrophe jumps and claim points. Finally, we will look at the Laplace transform of distribution of λ_t assuming that the number of claims in a fixed time interval is known.

Let us assume that the shot noise process λ_t is time homogeneous.

5.1 Transformations, approximations and pricing: The Kalman-Bucy filter

5.1.1 Transformations and approximations

Given the observations $\{N_s; 0 \le s \le t\}$, it is required to estimate the value of λ_t at any time t. Therefore "the filtering problem" is to obtain the best estimate λ_t on the basis of the observed process $\{N_s; 0 \le s \le t\}$. We will also assume ρ is large.

We will now find the best estimate of λ_t based on the observations $\{N_s; 0 \le s \le t\}$. We start by transforming the processes λ_t and N_t using

$$Z_t^{(\rho)} = \frac{\lambda_t - \frac{\mu_1 \rho}{\delta}}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \quad \text{i.e.} \quad \lambda_t = \frac{\mu_1 \rho}{\delta} + Z_t^{(\rho)} \sqrt{\frac{\mu_2 \rho}{2\delta}} \tag{5.1.1}$$

and

$$W_t^{(\rho)} = \frac{N_t - \frac{\mu_1 \rho}{\delta} t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \quad \text{i.e.} \quad N_t = \frac{\mu_1 \rho}{\delta} t + W_t^{(\rho)} \sqrt{\frac{\mu_2 \rho}{2\delta}}. \tag{5.1.2}$$

Now let us define $L_t^{(\rho)} = \dfrac{N_t - \int_0^t \lambda_s ds}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} = \dfrac{N_t - X_t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}$ and $Q_t^{(\rho)} = \dfrac{C_t - m_1 N_t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}$ where C_t is the total amount of claims up to time t and $m_1 = \int_0^\infty u dH(u)$. $H(u)$ $(u > 0)$ is the claim size distribution function. Also recall $V_t^{(\rho)} = \dfrac{J_t - \mu_1 \rho t}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}$ and $J_t = \sum_{i=1}^{M_t} y_i$.

Lemma 5.1.1 Let $V_t^{(\rho)}$, $L_t^{(\rho)}$ and $Q_t^{(\rho)}$ as defined and $\rho \to \infty$. Then

$$\begin{bmatrix} V_t^{(\rho)} \\ L_t^{(\rho)} \\ Q_t^{(\rho)} \end{bmatrix} \Rightarrow \begin{bmatrix} \sqrt{2\delta} B_t^{(1)} \\ \sqrt{\frac{2\mu_1}{\mu_2}} B_t^{(2)} \\ \sqrt{k_2 \frac{2\mu_1}{\mu_2}} B_t^{(3)} \end{bmatrix} \tag{5.1.3}$$

in law where $B_t^{(1)}$, $B_t^{(2)}$ and $B_t^{(3)}$ are three independent standard Brownian motions and $k_2 = \int_0^\infty u^2 dH(u) - \left(\int_0^\infty u dH(u) \right)^2$ (the variance of claim size).

__Proof__

The generator of the process $(X_t, N_t, C_t, \lambda_t, J_t, t)$ acting on a function $f(x,n,c,\lambda,j,t)$ is given by

$$\begin{aligned} \mathrm{A} f(x,n,c,\lambda,j,t) &= \frac{\partial f}{\partial t} + \lambda \frac{\partial f}{\partial x} - \delta\lambda \frac{\partial f}{\partial \lambda} + \rho\{\int_0^\infty f(x,n,c,\lambda+y,j+y,t)dG(y) - f(x,n,c,\lambda,j,t)\} \\ &\quad + \lambda\{f(x,n+1,c+u,\lambda,j,t)dH(u) - f(x,n,c,\lambda,j,t)\}. \end{aligned} \tag{5.1.4}$$

Set $f(x,n,c,\lambda,j,t) = \left(\dfrac{n-x}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right)^2$ and $f(x,n,c,\lambda,j,t) = \left(\dfrac{c - m_1 n}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right)^2$. Then

$$\begin{aligned} \mathrm{A}\left(\frac{n-x}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right)^2 &= \lambda 2 \left(\frac{n-x}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right)\left(-\frac{1}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right) + \rho\left\{ \int_0^\infty \left(\frac{n-x}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right)^2 dG(y) - \left(\frac{n-x}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right)^2 \right\} + \lambda\left\{ \left(\frac{n+1-x}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right)^2 - \left(\frac{n-x}{\sqrt{\frac{\mu_2 \rho}{2\delta}}} \right)^2 \right\} \\ &= \frac{2\delta\lambda}{\mu_2 \rho} \end{aligned}$$

and

$$\mathrm{A}\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)^2 = \lambda\left\{\int_0^\infty\left\{\frac{c+u-m_1(n+1)}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right\}^2 dH(u) - \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)^2\right\} = (m_2-m_1^2)\frac{2\delta\lambda}{\mu_2\,\rho}$$

$$= k_2\frac{2\delta\lambda}{\mu_2\,\rho}$$

where $m_1 = \int_0^\infty u\,dH(u)$, $m_2 = \int_0^\infty u^2 dH(u)$ and $k_2 = m_2 - m_1^2$.

As mentioned in the introduction, $\mathrm{A}f$ is the solution to the 'martingale problem'. Hence $\left(L_t^{(\omega)^2}\right) - \int_0^t \frac{2\delta\lambda_s}{\mu_2\,\rho}ds$ and $\left(Q_t^{(\omega)^2}\right) - \int_0^t k_2\frac{2\delta\lambda_s}{\mu_2\,\rho}ds$ are martingales.

By Chebyschev's inequality,

$$\Pr\left\{\left|\int_0^t \frac{2\delta\lambda_s}{\mu_2\,\rho}ds - \frac{2\mu_1}{\mu_2}t\right| > \varepsilon\right\} \le \frac{(\frac{2\delta}{\mu_2})^2 Var(\int_0^t \lambda_s ds)}{\rho^2\varepsilon^2}$$

and

$$\Pr\left\{\left|\int_0^t k_2\frac{2\delta\lambda_s}{\mu_2\,\rho}ds - k_2\frac{2\mu_1}{\mu_2}t\right| > \varepsilon\right\} \le \frac{k_2^2(\frac{2\delta}{\mu_2})^2 Var(\int_0^t \lambda_s ds)}{\rho^2\varepsilon^2}.$$

As can be seen from (1.4.13) (see also (1.4.21)) $Var(\int_0^t \lambda_s ds) = K(t)\rho$. Therefore as $\rho \to \infty$

$$\Pr\left\{\left|\int_0^t \frac{2\delta\lambda_s}{\mu_2\,\rho}ds - \frac{2\mu_1}{\mu_2}t\right| > \varepsilon\right\} \le \frac{(\frac{2\delta}{\mu_2})^2 Var(\int_0^t \lambda_s ds)}{\rho^2\varepsilon^2} = \frac{(\frac{2\delta}{\mu_2})^2 K(t)\rho}{\rho^2\varepsilon^2} \to 0 \qquad (5.1.5)$$

and

$$\Pr\left\{\left|k_2\int_0^t \frac{2\delta\lambda_s}{\mu_2\,\rho}ds - k_2\frac{2\mu_1}{\mu_2}t\right| > \varepsilon\right\} \le \frac{k_2^2(\frac{2\delta}{\mu_2})^2 Var(\int_0^t \lambda_s ds)}{\rho^2\varepsilon^2} = \frac{k_2^2(\frac{2\delta}{\mu_2})^2 K(t)\rho}{\rho^2\varepsilon^2} \to 0. \qquad (5.1.6)$$

Therefore from (5.1.5) and (5.1.6)

$$\int_0^t \frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds \quad \to \quad \frac{2\mu_1}{\mu_2}t$$

and

$$\int_0^t k_2\frac{2\delta}{\mu_2}\frac{\lambda_s}{\rho}ds \quad \to \quad k_2\frac{2\mu_1}{\mu_2}t.$$

in probability. Hence from proposition 4.1.1,

$$L_t^{(\rho)} = \frac{N_t - \int_0^t \lambda_s ds}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \quad \Rightarrow \quad \sqrt{\frac{2\mu_1}{\mu_2}}B_t^{(2)} \tag{5.1.7}$$

and

$$Q_t^{(\rho)} = \frac{C_t - m_1 N_t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \quad \Rightarrow \quad \sqrt{k_2\frac{2\mu_1}{\mu_2}}B_t^{(3)} \tag{5.1.8}$$

in law where $B_t^{(2)}$ and $B_t^{(3)}$ are two independent standard Brownian motions.

Set $f(x,n,c,\lambda,j,t) = \left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)$, $f(x,n,c,\lambda,j,t) = \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)$ and $f(x,n,c,\lambda,j,t) = \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)$. Then

$$\begin{aligned}
&\mathcal{A}\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right) = \left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(-\frac{\mu_1\rho}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right) + \lambda\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(-\frac{1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\\
&+\rho\left\{\int_0^\infty \left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j+y-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)dG(y) - \left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\right\} + \lambda\left\{\left(\frac{n+1-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right) - \left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\right\}\\
&= -\frac{\mu_1\rho}{\frac{\mu_2\rho}{2\delta}}(n-x) - \frac{\lambda}{\frac{\mu_2\rho}{2\delta}}(j-\mu_1\rho t) + \rho\frac{\mu_1}{\frac{\mu_2\rho}{2\delta}}(n-x) + \frac{\lambda}{\frac{\mu_2\rho}{2\delta}}(j-\mu_1\rho t)\\
&= 0,
\end{aligned} \tag{5.1.9}$$

$$\begin{aligned}
&\mathcal{A}\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right) = \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(-\frac{\mu_1\rho}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right) + \rho\left\{\int_0^\infty \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j+y-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)dG(y) - \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\right\}\\
&+\lambda\left\{\int_0^\infty \left\{\frac{c+u-m_1(n+1)}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right\}\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)dH(u) - \left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\right\}
\end{aligned}$$

$$=\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(-\frac{\mu_1\rho}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)+\rho\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{\mu_1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)+\lambda\left\{\left(\frac{m_1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)-\left(\frac{m_1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{j-\mu_1\rho t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\right\}$$

$$=0, \tag{5.1.10}$$

and

$$\mathrm{A}\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)=\lambda\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(-\frac{1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)+\lambda\left\{\int_0^\infty\left\{\frac{c+u-m_1(n+1)}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right\}\left(\frac{n+1-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)dH(u)-\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\right\}$$

$$=\lambda\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(-\frac{1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)+\lambda\left\{\left(\frac{c-m_1 n}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)+\left(\frac{m_1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)+\left(\frac{m_1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)-\left(\frac{m_1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{n-x}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)-\left(\frac{m_1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\left(\frac{1}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right)\right\}$$

$$=0. \tag{5.1.11}$$

Therefore (5.1.3) follows immediately from lemma 4.1.2, (5.1.7), (5.1.8), (5.1.9), (5.1.10) and (5.1.11).

□

Let us now define $U_t^{(\rho)}=\dfrac{C_t-m_1\frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}$ and prove the main result of this section.

Theorem 5.1.2 Let $Z_t^{(\rho)}$, $W_t^{(\rho)}$ and $U_t^{(\rho)}$ as defined and $\rho\to\infty$. Then $Z_t^{(\rho)}$, $W_t^{(\rho)}$ and $U_t^{(\rho)}$ converge in law to Z_t, W_t and U_t where

$$dZ_t=-\delta Z_t dt+\sqrt{2\delta}dB_t^{(1)} \tag{5.1.12}$$

$$dW_t=Z_t dt+\sqrt{\frac{2\mu_1}{\mu_2}}dB_t^{(2)} \tag{5.1.13}$$

$$dU_t=m_1 dW_t+\sqrt{k_2\frac{2\mu_1}{\mu_2}}dB_t^{(3)}=m_1 Z_t dt+\sqrt{m_2\frac{2\mu_1}{\mu_2}}dB_t^{(4)} \tag{5.1.14}$$

where $B_t^{(1)}$, $B_t^{(2)}$, $B_t^{(3)}$ are three independent standard Brownian motions and $B_t^{(4)}=\dfrac{m_1\sqrt{\frac{2\mu_1}{\mu_2}}B_t^{(2)}+\sqrt{k_2\frac{2\mu_1}{\mu_2}}B_t^{(3)}}{\sqrt{(m_1^2+k_2)\frac{2\mu_1}{\mu_2}}}$ (also a standard Brownian motion).

Proof

First, note that (5.1.12) has already been proved from theorem 4.1.3.

$W_t^{(\rho)}$ and $U_t^{(\rho)}$ can be written as

$$W_t^{(\rho)} = \frac{N_t - \frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} = \frac{N_t - \int_0^t \lambda_s ds + \int_0^t \lambda_s ds - \int_0^t \frac{\mu_1\rho}{\delta} ds}{\sqrt{\frac{\mu_2\rho}{2\delta}}} = \frac{N_t - \int_0^t \lambda_s ds}{\sqrt{\frac{\mu_2\rho}{2\delta}}} + \int_0^t \frac{\lambda_s - \frac{\mu_1\rho}{\delta}}{\sqrt{\frac{\mu_2\rho}{2\delta}}} ds \qquad (5.1.15)$$

and

$$U_t^{(\rho)} = \frac{C_t - m_1\frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} = \frac{C_t - m_1 N_t + m_1 N_t - m_1\frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} = \frac{C_t - m_1 N_t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} + m_1\left(\frac{N_t - \frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}\right). \qquad (5.1.16)$$

Therefore by continuous mapping theorem (see Billingsley (1968)), lemma 5.1.1 and theorem 4.1.3, (5.1.15) and (5.1.16) converge to

$$W_t = \int_0^t Z_s ds + \sqrt{\frac{2\mu_1}{\mu_2}} B_t^{(2)} \qquad (5.1.17)$$

and

$$U_t = m_1 W_t + \sqrt{k_2\frac{2\mu_1}{\mu_2}} B_t^{(3)}. \qquad (5.1.18)$$

Set $dW_t = Z_t dt + \sqrt{\frac{2\mu_1}{\mu_2}} dB_t^{(2)}$ in differential form of (5.1.18) then

$$dU_t = m_1 dW_t + \sqrt{k_2\frac{2\mu_1}{\mu_2}} dB_t^{(3)} = m_1 Z_t dt + m_1\sqrt{\frac{2\mu_1}{\mu_2}} dB_t^{(2)} + \sqrt{k_2\frac{2\mu_1}{\mu_2}} dB_t^{(3)}. \qquad (5.1.19)$$

Since the sum of two independent standard Brownian motions is also a standard Brownian motion (5.1.19) becomes

$$dU_t = m_1 Z_t dt + \sqrt{(m_1^2 + k_2)\frac{2\mu_1}{\mu_2}} dB_t^{(4)} = m_1 Z_t dt + \sqrt{m_2\frac{2\mu_1}{\mu_2}} dB_t^{(4)}$$

where $B_t^{(4)} = \dfrac{m_1\sqrt{\frac{2\mu_1}{\mu_2}} B_t^{(2)} + \sqrt{k_2\frac{2\mu_1}{\mu_2}} B_t^{(3)}}{\sqrt{(m_1^2 + k_2)\frac{2\mu_1}{\mu_2}}}$.

□

The theorem 5.1.2 has proved that Z_t, W_t and U_t are normal variables. As a result of this, we obtained $\tilde{\lambda}_t$, $\tilde{N}_t$ and $\tilde{C}_t$ which are Gaussian approximations of λ_t, N_t and C_t;

$$\tilde{\lambda}_t = \frac{\mu_1\rho}{\delta} + Z_t\sqrt{\frac{\mu_2\rho}{2\delta}} \quad \text{i.e.} \quad Z_t = \frac{\tilde{\lambda}_t - \frac{\mu_1\rho}{\delta}}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \tag{5.1.20}$$

$$\tilde{N}_t = \frac{\mu_1\rho}{\delta}t + W_t\sqrt{\frac{\mu_2\rho}{2\delta}} \quad \text{i.e.} \quad W_t = \frac{\tilde{N}_t - \frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}} \tag{5.1.21}$$

$$\tilde{C}_t = m_1\frac{\mu_1\rho}{\delta}t + U_t\sqrt{\frac{\mu_2\rho}{2\delta}} \quad \text{i.e.} \quad U_t = \frac{\tilde{C}_t - m_1\frac{\mu_1\rho}{\delta}t}{\sqrt{\frac{\mu_2\rho}{2\delta}}}. \tag{5.1.22}$$

5.1.2 The Kalman-Bucy filter and the distribution of Z_t

We will derive the conditional distribution of Z_t, given $\{W_s; 0 \le s \le t\}$, by the Kalman-Bucy filter where

$$dZ_t = -\delta Z_t dt + \sqrt{2\delta}dB_t^{(1)} \tag{5.1.23}$$

and

$$dW_t = Z_t dt + \sqrt{\frac{2\mu_1}{\mu_2}}dB_t^{(2)}. \tag{5.1.24}$$

Let us begin with a proposition used by Øksendal (1992).

Proposition 5.1.3 The solution $\hat{Z}_t = E(Z_t | W_s; 0 \le s \le t)$ of the 1-dimensional linear filtering problem

$$dZ_t = F(t)Z_t dt + C(t)dB_t^{(1)}; \; F(t), C(t) \in \Re \tag{5.1.25}$$

$$dW_t = G(t)Z_t dt + D(t)dB_t^{(2)}; \; G(t), D(t) \in \Re \tag{5.1.26}$$

satisfies the stochastic differential equation

$$d\hat{Z}_t = \{F(t) - \frac{G^2(t)S(t)}{D^2(t)}\}\hat{Z}_t dt + \frac{G(t)S(t)}{D^2(t)}dW_t; \; \hat{Z}_0 = E(Z_0) \tag{5.1.27}$$

where

$S(t) = E\left\{\left(Z_t - \hat{Z}_t\right)^2\right\}$ satisfies the Riccati equation

$$\frac{dS}{dt} = 2F(t)S(t) - \frac{G^2(t)}{D^2(t)}S^2(t) + C^2(t), \; S(0) = E\left[\{Z_0 - E(Z_0)\}^2\right] = Var(Z_0). \tag{5.1.28}$$

Proof

From theorem 6.10 in chapter IV of Øksendal (1992), the result follows immediately.

□

Theorem 5.1.4 Let (Z_t, W_t) be a two-dimensional normal process satisfying the system of equations of (5.1.23) and (5.1.24). Then the estimate of Z_t based on the observed $\{W_s; 0 \le s \le t\}$ is

$$\hat{Z}_t = E\left(Z_t \middle| W_s; 0 \le s \le t\right) = e^{\int_0^t H(s)ds} \hat{Z}_0 + \frac{\mu_2}{2\mu_1} \int_0^t e^{\int_s^t H(u)du} S(s) dW_s \qquad (5.1.29)$$

where

$$S(0) = a^2,$$

$$S(s) = \frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}\left\{1 + \frac{a^2 + \frac{2\delta\mu_1}{\mu_2} + \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2 + \frac{2\delta\mu_1}{\mu_2} - \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}} e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s}\right\}}{\frac{a^2 + \frac{2\delta\mu_1}{\mu_2} + \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2 + \frac{2\delta\mu_1}{\mu_2} - \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}} e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s} - 1} - 2\delta\frac{\mu_1}{\mu_2}$$

(5.1.30)

and

$$H(s) = -\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}\left\{1 + \frac{a^2 + \frac{2\delta\mu_1}{\mu_2} + \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2 + \frac{2\delta\mu_1}{\mu_2} - \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}} e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s}\right\}}{\frac{2\mu_1}{\mu_2}\left\{\frac{a^2 + \frac{2\delta\mu_1}{\mu_2} + \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2 + \frac{2\delta\mu_1}{\mu_2} - \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}} e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s} - 1\right\}}. \qquad (5.1.31)$$

Proof

Let $S(0) = a^2$. Then from (5.1.28) the Riccati equation has the solution

$$S(t)=\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}\left\{1+\frac{a^2+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}t}\right\}}{\frac{a^2+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}t}-1}-2\delta\frac{\mu_1}{\mu_2}.$$

(5.1.32)

Therefore from (5.1.27) (5.1.32) offers the solution for $\hat{Z}_t$ of the form

$$\hat{Z}_t=E(Z_t|W_s;0\le s\le t)=e^{\int_0^t H(s)ds}\hat{Z}_0+\frac{\mu_2}{2\mu_1}\int_0^t e^{\int_s^t H(u)du}S(s)dW_s$$

where

$$H(s)=-\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}\left\{1+\frac{a^2+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s}\right\}}{\frac{2\mu_1}{\mu_2}\left\{\frac{a^2+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{a^2+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s}-1\right\}}.$$

□

Corollary 5.1.5 Let W_t as defined. Then if Z_t is stationary the estimate of Z_t based on the observation $\{W_s; 0\le s\le t\}$ is

$$\hat{Z}_t=E(Z_t|W_s;0\le s\le t)=\frac{\mu_2}{2\mu_1}\int_0^t e^{\int_s^t H(u)du}S(s)dW_s \tag{5.1.33}$$

where

$$S(s)=\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}\left\{1+\frac{1+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{1+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s}\right\}}{\frac{1+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{1+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}s}-1}-2\delta\frac{\mu_1}{\mu_2}$$

(5.1.34)

and $$H(u) = -\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}\left\{1+\frac{1+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{1+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}u}\right\}}{\frac{2\mu_1}{\mu_2}\left\{\frac{1+\frac{2\delta\mu_1}{\mu_2}+\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{1+\frac{2\delta\mu_1}{\mu_2}-\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}e^{\frac{\sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}}{\frac{\mu_1}{\mu_2}}u}-1\right\}} \quad (5.1.35)$$

Proof

Since Z_t is stationary, $E(Z_0)=0$. Therefore from (5.1.27) and (5.1.28), $\hat{Z}_0 = 0$ and $S(0) = a^2 = Var(Z_0) = \frac{(\sqrt{2\delta})^2}{2\delta} = 1$.

Hence put $\hat{Z}_0 = 0$ in (5.1.29) and $a^2 = 1$ in (5.1.30) and (5.1.31) then the result follows immediately.

□

Corollary 5.1.6 Let Z_t and W_t as defined. Then the estimate of Z_t conditionally on $\{W_s; -\infty \le s \le t\}$ is

$$\hat{Z}_t = E\left(Z_t \middle| W_s; -\infty \le s \le t\right) = e^{-\beta t}\hat{Z}_0 - \left\{\delta - \frac{\mu_2}{2\mu_1}\sqrt{\frac{2\mu_1}{2\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}\right\}e^{-\beta t}\int_0^t e^{\beta s}dW_s \quad (5.1.36)$$

where $\beta = \sqrt{\delta\left(\delta + \frac{\mu_2}{\mu_1}\right)}$.

Proof

For large value of s, we have obtained $S(s) = -\frac{2\delta\mu_1}{\mu_2} + \sqrt{\frac{2\mu_1}{\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+1\right)}$. Therefore from (5.1.27) this offers the solution for $\hat{Z}_t$ of the form

$$\hat{Z}_t = E\left(Z_t \middle| W_s; -\infty \le s \le t\right) = e^{-\beta t}\hat{Z}_0 - \left\{\delta - \frac{\mu_2}{2\mu_1}\sqrt{\frac{2\mu_1}{2\mu_2}}\sqrt{\delta\left(\frac{2\delta\mu_1}{\mu_2}+2\right)}\right\}e^{-\beta t}\int_0^t e^{\beta s}dW_s$$

where $\beta = \sqrt{\delta\left(\delta + \frac{\mu_2}{\mu_1}\right)}$.

□

We have obtained $E(Z_t | W_s; 0 \le s \le t) = \hat{Z}_t$. Now let us try to find the distribution of Z_t.

Corollary 5.1.7 Let Z_t ,W_t, $\hat{Z}_t$ and $S(t)$ as defined. Then

$$E(e^{-\gamma Z_t} | W_s; 0 \le s \le t) = e^{-\gamma \hat{Z}_t + \frac{1}{2}\gamma^2 S(t)}. \tag{5.1.37}$$

Proof

From theorem 5.1.4 and the fact that Z_t is normally distributed, given $W_s; 0 \le s \le t$, with $Var(Z_t | W_s; 0 \le s \le t) = S(t)$ the result follows immediately.

□

5.1.3 Pricing of a stop-loss reinsurance contract using the Kalman-Bucy filter

We have transformed and approximated λ_t and N_t as normal variables Z_t and W_t from which we have obtained the distribution of Z_t. As a matter of interest let us derive the pricing model for stop-loss reinsurance contracts for catastrophic events using normal variables Z_t and W_t.

5.1.3.1 Constant claim sizes

Ignoring the effect of interest rates, the stop-loss reinsurance premium at time t is

$$E[(\sum_{i=1}^{N_T - N_t} \aleph_i - b)^+] \tag{5.1.38}$$

where:

$\aleph_i$ claim amount,

$N_T - N_t$ number of claims between time T and t,

b retention limit.

$(\sum_{i=1}^{N_T - N_t} \aleph_i - b)^+ = Max(\sum_{i=1}^{N_T - N_t} \aleph_i - b,\ 0)$,

If we assume that $\aleph_i = 1$, then

$$E[(\sum_{i=1}^{N_T - N_t} \aleph_i - b)^+] = E\left[\{(N_T - N_t) - b\}^+\right]. \tag{5.1.39}$$

Since we have obtained $\tilde{N}_t$ which is a Gaussian approximation of N_t, we will use this approximation (see (5.1.21)). Therefore set $\tilde{N}_t = \frac{\mu_1\rho}{\delta}t + W_t\sqrt{\frac{\mu_2\rho}{2\delta}}$ in (5.1.39) then

$$E\left[\{(\tilde{N}_T - \tilde{N}_t) - b\}^+\right] = E\left[\left\{\sqrt{\frac{\mu_2\rho}{2\delta}}(W_T - W_t) + \frac{\mu_1\rho}{\delta}(T-t) - b\right\}^+\right]. \tag{5.1.40}$$

From (5.1.40), we can see that mean and variance of $W_T - W_t$ need to be determined to obtain stop-loss reinsurance premium. Therefore let us derive the expected value and variance of $W_T - W_t$.

Lemma 5.1.8 Let Z_t ,W_t, $\hat{Z}_t$ and $S(t)$ as defined. Then

$$E(W_T - W_t | W_s; 0 \le s \le t) = \frac{1-e^{-\delta(T-t)}}{\delta}\hat{Z}_t \tag{5.1.41}$$

and

$$Var(W_T - W_t | W_s; 0 \le s \le t)$$
$$= \frac{1}{\delta^2}\left\{\left(1-e^{-\delta(T-t)}\right)^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right\} + 2\left(\frac{1}{\delta} + \frac{\mu_1}{\mu_2}\right)(T-t). \tag{5.1.42}$$

<u>Proof</u>

From (5.1.23) and (5.1.24)

$$Z_s = Z_t e^{-\delta(s-t)} + \sqrt{2\delta}\int_t^s e^{-\delta(s-u)} dB_u^{(1)};\ t \le s \tag{5.1.43}$$

and

$$W_T - W_t = \int_t^T Z_s ds + \sqrt{\frac{2\mu_1}{\mu_2}}\int_t^T dB_s^{(2)}. \tag{5.1.44}$$

Set (5.1.43) in (5.1.44) then

$$W_T - W_t = \frac{1-e^{-\delta(T-t)}}{\delta} Z_t + \sqrt{2\delta}\int_t^T \frac{1-e^{-\delta(T-u)}}{\delta} dB_u^{(1)} + \sqrt{\frac{2\mu_1}{\mu_2}}\int_t^T dB_s^{(2)}. \tag{5.1.45}$$

Take expectation in (5.1.45) then (5.1.41) follows immediately.

$$Var(W_T - W_t | W_s; 0 \le s \le t) = E\left\{(W_T - W_t)^2 | W_s; 0 \le s \le t\right\} - E\left\{(W_T - W_t | W_s; 0 \le s \le t)\right\}^2. \tag{5.1.46}$$

Therefore (5.1.42) follows immediately from (5.1.45) and (5.1.41).

□

We can now find stop-loss reinsurance premium at time t based on the observations $\{W_s; 0 \le s \le t\}$.

Theorem 5.1.9 Let Z_t, W_t, $\hat{Z}_t$ and $S(t)$ as defined. Then

$$E\left[\{(\tilde{N}_T - \tilde{N}_t) - b\}^+ | W_s; 0 \le s \le t\right] = \left[\sqrt{\frac{\mu_2 \rho \Sigma}{4\delta\pi}} e^{-\frac{1}{2}B^2} + \left\{\sqrt{\frac{\mu_2 \rho}{2\delta}} \mathrm{M} + \frac{\mu_1 \rho}{\delta}(T-t) - b\right\} \Phi(-B)\right] \quad (5.1.47)$$

where

$$\mathrm{M} = E(W_T - W_t | W_s; 0 \le s \le t) = \frac{1 - e^{-\delta(T-t)}}{\delta} \hat{Z}_t,$$

$$\Sigma = Var(W_T - W_t | W_s; 0 \le s \le t)$$

$$= \frac{1}{\delta^2}\left\{\left(1 - e^{-\delta(T-t)}\right)^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right\} + 2\left(\frac{1}{\delta} + \frac{\mu_1}{\mu_2}\right)(T-t),$$

$B = \dfrac{A - \mathrm{M}}{\sqrt{\Sigma}}$, $A = \dfrac{b - \frac{\mu_1 \rho}{\delta}(T-t)}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}$ and $\Phi(\cdot)$ is the cumulative normal distribution function.

__Proof__

Conditioning on $W_s; 0 \le s \le t$ and set $W = W_T - W_t$ in (5.1.40), then

$$E\left[\{(\tilde{N}_T - \tilde{N}_t) - b\}^+ | W_s; 0 \le s \le t\right] = E\left[\left\{\sqrt{\frac{\mu_2 \rho}{2\delta}}(W_T - W_t) + \frac{\mu_1 \rho}{\delta}(T-t) - b\right\}^+ \middle| W_s; 0 \le s \le t\right]$$

$$= \int_A^\infty \left\{\sqrt{\frac{\mu_2 \rho}{2\delta}}\omega + \frac{\mu_1 \rho}{\delta}(T-t) - b\right\} \frac{1}{\sqrt{2\pi\Sigma}} e^{-\frac{1}{2}\frac{(\omega - \mathrm{M})^2}{\Sigma}} d\omega \quad (5.1.48)$$

where

$$\mathrm{M} = E(W_T - W_t | W_s; 0 \le s \le t) = \frac{1 - e^{-\delta(T-t)}}{\delta} \hat{Z}_t,$$

$$\Sigma = Var(W_T - W_t | W_s; 0 \le s \le t)$$

$$= \frac{1}{\delta^2}\left\{\left(1 - e^{-\delta(T-t)}\right)^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right\} + 2\left(\frac{1}{\delta} + \frac{\mu_1}{\mu_2}\right)(T-t),$$

and $A = \dfrac{b - \frac{\mu_1 \rho}{\delta}(T-t)}{\sqrt{\frac{\mu_2 \rho}{2\delta}}}$.

Set $x = \frac{\omega - \mathrm{M}}{\sqrt{\Sigma}}$ in (5.1.48) and put $B = \frac{A - \mathrm{M}}{\sqrt{\Sigma}}$ then (5.1.47) follows immediately.

□

Let us evaluate the fair stop-loss reinsurance premium at time t assuming that there is an absence of arbitrage opportunities in the market. This can be achieved by using an equivalent martingale probability measure P^*. By theorem 3.2.5 we have found that the claim intensity function λ_t has changed to $\theta^* \lambda_t$ and the rate of jump arrival ρ and the jump size y in the shot noise process depend on time with respect to the equivalent martingale probability measure, i.e.

$$\lambda_t \to \theta^* \lambda_t;$$
$$\rho \to \rho^*(t) = \rho \hat{g}(\gamma^* e^{\delta t});$$
$$dG(y) \to dG^*(y;t) = \frac{e^{-\gamma^* e^{\delta t} y} dG(y)}{\hat{g}(\gamma^* e^{\delta t})}.$$

For simplicity, let us assume that $\theta^* = (1+\theta)$ and $\gamma^* = 0$ where $0 \le \theta \le 1$. Then

$$\lambda_t \to (1+\theta)\lambda_t. \qquad (5.1.49)$$

Corollary 5.1.10 Let Z_t, W_t, $\hat{Z}_t$ and $S(t)$ as defined. Then

$$E^*\left[\{(\tilde{N}_T - \tilde{N}_t) - b\}^+ \middle| W_s; 0 \le s \le t\right]$$
$$= \left[\sqrt{\frac{(1+\theta)^2 \mu_2 \rho \Sigma^*}{4\delta\pi}} e^{-\frac{1}{2}B^{*2}} + \left\{\sqrt{\frac{(1+\theta)^2 \mu_2 \rho}{2\delta}} \mathrm{M}^* + \frac{(1+\theta)\mu_1 \rho}{\delta}(T-t) - b\right\} \Phi(-B^*)\right] \qquad (5.1.50)$$

where

E^* is the expectation with respect to equivalent martingale probability measure P^*,

$\mathrm{M}^* = \frac{1 - e^{-\delta(T-t)}}{\delta}(1+\theta)\hat{Z}_t$,

$\Sigma^* = \frac{1}{\delta^2}\left\{\left(1 - e^{-\delta(T-t)}\right)^2 (1+\theta)^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right\} + 2\left\{\frac{1}{\delta} + \frac{\mu_1}{(1+\theta)\mu_2}\right\}(T-t)$,

$B^* = \frac{A^* - \mathrm{M}^*}{\sqrt{\Sigma^*}}$, $A^* = \frac{b - \frac{(1+\theta)\mu_1\rho}{\delta}(T-t)}{\sqrt{\frac{(1+\theta)^2 \mu_2 \rho}{2\delta}}}$ and $\Phi(\cdot)$ is the cumulative normal distribution function.

Proof

If we replace μ_1, μ_2, $\hat{Z}_t$ and $S(t)$ with $(1+\theta)\mu_1$, $(1+\theta)^2\mu_2$, $(1+\theta)\hat{Z}_t$, and $(1+\theta)^2 S(t)$ respectively in (5.1.47) then (5.1.50) follows.

□

5.1.3.2 Random claim sizes

Let $C_T - C_t$ be the total amount of claims between time T and t. Then from (5.1.39), the stop-loss reinsurance premium at time t is

$$E[(\sum_{i=1}^{N_T-N_t}\aleph_i - b)^+] = E\left[\{(C_T - C_t) - b\}^+\right] \tag{5.1.51}$$

where all symbols have previously been defined. Since we have obtained $\tilde{C}_t$ which is a Gaussian approximation of C_t, we will use this approximation (see (5.1.22)). Therefore set $\tilde{C}_t = m_1 \frac{\mu_1\rho}{\delta} t + U_t\sqrt{\frac{\mu_2\rho}{2\delta}}$ in (5.1.51) then

$$E\left[\{(\tilde{C}_T - \tilde{C}_t) - b\}^+\right] = E\left[\left\{\sqrt{\frac{\mu_2\rho}{2\delta}}(U_T - U_t) + m_1\frac{\mu_1\rho}{\delta}(T-t) - b\right\}^+\right]. \tag{5.1.52}$$

From (5.1.52), we can see that mean and variance of $U_T - U_t$ need to be determined to obtain stop-loss reinsurance premium. Therefore let us derive the expected value and variance of $U_T - U_t$.

Lemma 5.1.11 Let Z_t, W_t, U_t and $\hat{Z}_t$ as defined. Then

$$E\left(U_T - U_t \middle| W_s; 0 \le s \le t\right) = m_1 \frac{1-e^{-\delta(T-t)}}{\delta}\hat{Z}_t. \tag{5.1.53}$$

and

$$Var\left(U_T - U_t \middle| W_s; 0 \le s \le t\right)$$
$$= \left(\frac{m_1}{\delta}\right)^2 \left\{\left(1-e^{-\delta(T-t)}\right)^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right\} + 2\left(\frac{m_1^2}{\delta} + \frac{m_2\mu_1}{\mu_2}\right)(T-t). \tag{5.1.54}$$

Proof

From (5.1.14)

$$U_T - U_t = m_1 \int_t^T Z_s ds + \sqrt{m_2 \frac{2\mu_1}{\mu_2}} \int_t^T dB_s^{(4)}. \tag{5.1.55}$$

Set (5.1.43) in (5.1.55) then

$$U_T - U_t = m_1 \frac{1-e^{-\delta(T-t)}}{\delta} Z_t + m_1\sqrt{2\delta}\int_t^T \frac{1-e^{-\delta(T-u)}}{\delta} dB_u^{(1)} + \sqrt{m_2 \frac{2\mu_1}{\mu_2}}\int_t^T dB_s^{(4)}. \quad (5.1.56)$$

Take expectation in (5.1.56) then (5.1.53) follows immediately.

$$Var\left(U_T - U_t \middle| W_s; 0 \le s \le t\right) = E\left\{(U_T - U_t)^2 \middle| W_s; 0 \le s \le t\right\} - E\left\{\left(U_T - U_t \middle| W_s; 0 \le s \le t\right)\right\}^2. \quad (5.1.57)$$

Therefore (5.1.54) follows immediately from (5.1.56) and (5.1.53).

□

We can now find stop-loss reinsurance premium at time t based on the observations $\{W_s; 0 \le s \le t\}$.

Theorem 5.1.12 Let Z_t, W_t, $\hat{Z}_t$ and $S(t)$ as defined. Then

$$E\left[\{(\tilde{C}_T - \tilde{C}_t) - b\}^+ \middle| W_s; 0 \le s \le t\right] = \left[\sqrt{\frac{\mu_2\rho\Psi}{4\delta\pi}} e^{-\frac{1}{2}L^2} + \left\{\sqrt{\frac{\mu_2\rho}{2\delta}}\Gamma + \frac{m_1\mu_1\rho}{\delta}(T-t) - b\right\}\Phi(-L)\right] \quad (5.1.58)$$

where

$$\Gamma = E\left(U_T - U_t \middle| W_s; 0 \le s \le t\right) = m_1 \frac{1-e^{-\delta(T-t)}}{\delta}\hat{Z}_t,$$

$$\Psi = Var\left(U_T - U_t \middle| W_s; 0 \le s \le t\right)$$

$$= \left(\frac{m_1}{\delta}\right)^2 \left\{\left(1-e^{-\delta(T-t)}\right)^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right\} + 2\left(\frac{m_1^2}{\delta} + \frac{m_2\mu_1}{\mu_2}\right)(T-t),$$

$L = \dfrac{K-\Gamma}{\sqrt{\Psi}}$, $K = \dfrac{b - m_1\frac{\mu_1\rho}{\delta}(T-t)}{\sqrt{\frac{\mu_2\rho}{2\delta}}}$ and $\Phi(\cdot)$ is the cumulative normal distribution function.

<u>Proof</u>

Conditioning on $W_s; 0 \le s \le t$ and set $U = U_T - U_t$ in (5.1.52), then

$$E\left[\{(\tilde{C}_T - \tilde{C}_t) - b\}^+ \middle| W_s; 0 \le s \le t\right] = E\left[\left\{\sqrt{\frac{\mu_2\rho}{2\delta}}(U_T - U_t) + m_1\frac{\mu_1\rho}{\delta}(T-t) - b\right\}^+ \middle| W_s; 0 \le s \le t\right]$$

$$= \int_K^\infty \left\{\sqrt{\frac{\mu_2\rho}{2\delta}}\upsilon + m_1\frac{\mu_1\rho}{\delta}(T-t) - b\right\}\frac{1}{\sqrt{2\pi\Psi}} e^{-\frac{1}{2}\frac{(\upsilon-\Gamma)^2}{\Psi}} d\upsilon \quad (5.1.59)$$

where

$\Gamma = E\left(U_T - U_t \middle| W_s; 0 \le s \le t\right) = m_1 \frac{1-e^{-\delta(T-t)}}{\delta} \hat{Z}_t,$

$\Psi = Var\left(U_T - U_t \middle| W_s; 0 \le s \le t\right)$

$= \left(\frac{m_1}{\delta}\right)^2 \left\{\left(1-e^{-\delta(T-t)}\right)^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right\} + 2\left(\frac{m_1^2}{\delta} + \frac{m_2\mu_1}{\mu_2}\right)(T-t),$

and $K = \dfrac{b - m_1 \frac{\mu_1\rho}{\delta}(T-t)}{\sqrt{\frac{\mu_2\rho}{2\delta}}}.$

Set $y = \dfrac{\upsilon - \Gamma}{\sqrt{\Psi}}$ in (5.1.59) and put $L = \dfrac{K-\Gamma}{\sqrt{\Psi}}$ then (5.1.58) follows immediately.

□

Let us evaluate the fair stop-loss reinsurance premium at time t assuming that there is an absence of arbitrage opportunities in the market. This can be achieved by using an equivalent martingale probability measure P^*.

Corollary 5.1.13 Let Z_t ,W_t, $\hat{Z}_t$ and $S(t)$ as defined. Then

$$E^*\left[\{(\tilde{C}_T - \tilde{C}_t) - b\}^+ \middle| W_s; 0 \le s \le t\right]$$
$$= \left[\sqrt{\frac{(1+\theta)^2 \mu_2 \rho \Psi^*}{4\delta\pi}} e^{-\frac{1}{2}L^{*2}} + \left\{\sqrt{\frac{(1+\theta)^2 \mu_2 \rho}{2\delta}} \Gamma^* + \frac{m_1(1+\theta)\mu_1\rho}{\delta}(T-t) - b\right\} \Phi\left(-L^*\right)\right]$$

(5.1.60)

where

E^* is the expectation with respect to equivalent martingale probability measure P^*,

$\Gamma^* = m_1 \frac{1-e^{-\delta(T-t)}}{\delta}(1+\theta)\hat{Z}_t,$

$\Psi^* = \left(\frac{m_1}{\delta}\right)^2 \left\{\left(1-e^{-\delta(T-t)}\right)^2 (1+\theta)^2 S(t) - e^{-2\delta(T-t)} + 4e^{-\delta(T-t)} - 3\right\} + 2\left\{\frac{m_1^2}{\delta} + \frac{m_2\mu_1}{(1+\theta)\mu_2}\right\}(T-t),$

$L^* = \dfrac{K^* - \Gamma^*}{\sqrt{\Psi^*}}$, $K^* = \dfrac{b - m_1 \frac{(1+\theta)\mu_1\rho}{\delta}(T-t)}{\sqrt{\frac{(1+\theta)^2 \mu_2 \rho}{2\delta}}}$ and $\Phi(\cdot)$ is the cumulative normal distribution function.

Proof

If we replace μ_1, μ_2, $\hat{Z}_t$ and $S(t)$ with $(1+\theta)\mu_1$, $(1+\theta)^2\mu_2$, $(1+\theta)\hat{Z}_t$, and $(1+\theta)^2 S(t)$ respectively in (5.1.58) then (5.1.60) follows.

□

The following example illustrates the calculation of stop-loss reinsurance premiums using the pricing model derived previously.

Example 5.1.1

The numerical values used to simulate the claim arrival process are $\delta = 0.5$, $\lambda_0 = 200$. We will assume that $\rho = 100$ i.e. the interarrival time between jumps is exponential with mean 0.01 and that the jump size follows exponential with mean 1, i.e. $y \sim$ *Exponential*(1). *S-Plus* was used to generate random values and to simulate the claim arrival process. The numerical values used to calculate (5.1.29) and (5.1.60) are

$$\hat{Z}_0 = 0,\ S(0) = 0,\ \theta = 0.1,\ \mu_1 = 1,\ \mu_2 = 2,\ m_1 = 1,\ m_2 = 3,\ t = 1,\ T = 2,$$

$$b = 0,\ 190,\ 200,\ 210,\ 220,\ 230,\ 240,\ 250,$$

$$E^*(C_T - C_t) = E^*(N_T - N_t)E(\aleph) = \frac{(1+\theta)\mu_1\rho}{\delta}m_1 = 220.$$

By computing (5.1.29) and (5.1.60) using *MAPLE* and *S-Plus*, where $\hat{Z}_1 = 0.5579152$, the calculation of the stop-loss reinsurance premiums at each retention level b are shown in Table 5.1.1 (see appendix for simulation and pricing).

Table 5.1.1

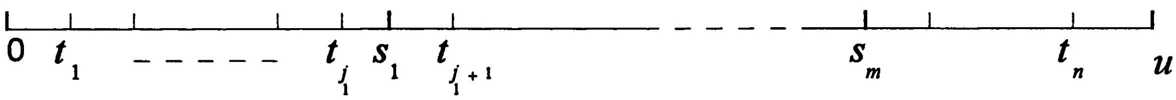
Retention level b	Reinsurance premiums
0	227.512939
190	38.767164
200	30.049486
210	22.209532
220	15.521060
230	10.171605
240	6.202363
250	3.494186

5.2 Where the times of catastrophes and claims are known

In this section we will derive the Laplace transform of distribution of λ_t assuming that the times of catastrophe jumps and claim points are known. In other words, we will obtain the Laplace transform of distribution of λ_t by conditioning on observed catastrophes and claims .

The times of the jump occurrence are denoted by $s_1, s_2, \cdots, s_m$ (i.e. m catastrophe jumps) and the times of point occurrence by $t_1, t_2, \cdots, t_{j_1}, t_{j_1+1}, \cdots, t_{j_2}, \cdots\cdots\cdots, t_{j_m+1}, \cdots, t_n$ (i.e. n claim points). The intensity function λ decreases with the decay rate δ between the time s_{i-1} and s_i where $i = 2, 3, \cdots, m$ until another catastrophe jump y_i occurs at s_i. Note that in the most general case in this section jump sizes $y_1, y_2, \cdots, y_m$ do not have to be independent. We observe the claim points between catastrophe jumps while λ decreases with the decay rate δ.

Let's assume that claim points and catastrophe jumps do not occur at the same time. A time interval $(0, u)$ can be divided by the epochs at which the jumps and points occur. The figure below illustrates the epochs at which the jumps and points occurred.



Let us start with a lemma.

Lemma 5.2.1 Let λ_t evolving up to a fixed time u $(t \le u)$, $y_1, y_2, \cdots, y_{m-1}$, $s_1, s_2, \cdots, s_m$ and $t_1, t_2, \cdots, t_n$ as defined. Assuming that we know λ_0 then

$$E(e^{-v\lambda_u}\lambda_{t_1}\lambda_{t_2}\cdots\lambda_{t_n}e^{-\int_0^u \lambda_s ds} \mid \lambda_0) = \mathrm{K}_1 \int_0^\infty\int_0^\infty\cdots\int_0^\infty h_1(y_1, y_2, \cdots, y_m) g_{1_{Y_1,Y_2,\cdots,Y_m}}(y_1, y_2, \cdots, y_m) dy_1 dy_2 \cdots dy_m. \tag{5.2.1}$$

where

$$\mathrm{K}_1 = e^{-v\lambda_0 e^{-\delta u}} \lambda_0^{j_1} e^{-\delta\{\sum_{i=1}^{j_1} t_i + \sum_{i=j_1+1}^{j_2}(t_i - s_1) + \cdots\cdots + \sum_{i=j_{m-1}+1}^{j_m}(t_i - s_{m-1}) + \sum_{i=j_m+1}^{n}(t_i - s_m)\}} e^{-\lambda_0 \frac{1-e^{-\delta u}}{\delta}}, \tag{5.2.2}$$

$$
\begin{aligned}
&h_1(y_1, y_2, \cdots, y_m) \\
&= (\lambda_0 e^{-\delta s_1} + y_1)^{j_2 - j_1} e^{-\{ v e^{-\delta(u-s_1)} + \frac{1-e^{-\delta(u-s_1)}}{\delta} \} y_1} \\
&\quad \cdot (\lambda_0 e^{-\delta s_2} + y_1 e^{-\delta(s_2 - s_1)} + y_2)^{j_3 - j_2} e^{-\{ v e^{-\delta(u-s_2)} + \frac{1-e^{-\delta(u-s_2)}}{\delta} \} y_2} \\
&\qquad \vdots \\
&\qquad \vdots \\
&\cdot (\lambda_0 e^{-\delta s_{m-1}} + y_1 e^{-\delta(s_{m-1}-s_1)} + y_2 e^{-\delta(s_{m-1}-s_2)} + \cdots + y_{m-2} e^{-\delta(s_{m-1}-s_{m-2})} + y_{m-1})^{j_m - j_{m-1}} e^{-\{ v e^{-\delta(u-s_{m-1})} + \frac{1-e^{-\delta(u-s_{m-1})}}{\delta} \} y_{m-1}} \\
&\cdot (\lambda_0 e^{-\delta s_m} + y_1 e^{-\delta(s_m-s_1)} + y_2 e^{-\delta(s_m-s_2)} + \cdots + y_{m-2} e^{-\delta(s_m-s_{m-2})} + y_{m-1} e^{-\delta(s_m-s_{m-1})} + y_m)^{n - j_m} e^{-\{ v e^{-\delta(u-s_m)} + \frac{1-e^{-\delta(u-s_m)}}{\delta} \} y_m},
\end{aligned}
\tag{5.2.3}
$$

and

$g_{1_{Y_1,Y_2,\cdots,Y_m}}(y_1, y_2, \cdots, y_m)$ is the m-dimensional joint probability density function of jump size $Y_1, Y_2, \cdots, Y_{m-1}, Y_m$.

<u>Proof</u>

The shot noise process λ_t evolving up to a fixed time t is

$$\lambda_t = \lambda_0 e^{-\delta t} + \sum_{\substack{\text{all } i \\ s_i < t}} y_i e^{-\delta(t-s_i)}. \tag{5.2.4}$$

Therefore

$$
\begin{aligned}
&\lambda_{t_{j_{m-1}+1}} \lambda_{t_{j_{m-1}+2}} \cdots \lambda_{t_{j_m}} \\
&\qquad = (\lambda_0 e^{-\delta s_{m-1}} + y_1 e^{-\delta(s_{m-1}-s_1)} + y_2 e^{-\delta(s_{m-1}-s_2)} + \cdots + y_{m-2} e^{-\delta(s_{m-1}-s_{m-2})} + y_{m-1})^{j_m - j_{m-1}} e^{-\delta \sum_{i=j_{m-1}+1}^{j_m} (t_i - s_{m-1})}.
\end{aligned}
\tag{5.2.5}
$$

and

$$e^{-\int_{s_{m-1}}^{s_m} \lambda_s ds} = e^{-(\lambda_0 e^{-\delta s_{m-1}} + y_1 e^{-\delta(s_{m-1}-s_1)} + y_2 e^{-\delta(s_{m-1}-s_2)} + \cdots + y_{m-2} e^{-\delta(s_{m-1}-s_{m-2})} + y_{m-1}) \frac{1-e^{-\delta(s_m - s_{m-1})}}{\delta}}. \tag{5.2.6}$$

The result follows if we use (5.2.5) and (5.2.6) for the left-hand side of (5.2.1).

□

Corollary 5.2.2 Let λ_t evolving up to a fixed time u $(t \le u)$, $y_1, y_2, \cdots, y_{m-1}$, $s_1, s_2, \cdots, s_m$ and $t_1, t_2, \cdots, t_n$ as defined. Assuming that we know λ_0 then

$$E(\lambda_{t_1} \lambda_{t_2} \cdots \lambda_{t_n} e^{-\int_0^u \lambda_s ds} | \lambda_0) = \mathrm{K}_2 \int_0^\infty \int_0^\infty \cdots \int_0^\infty h_2(y_1, y_2, \cdots, y_m) g_{2_{Y_1,Y_2,\cdots,Y_m}}(y_1, y_2, \cdots, y_m) dy_1 dy_2 \cdots dy_m \tag{5.2.7}$$

where

$$K_2 = \lambda_0^{j_1} e^{-\delta\{\sum_{i=1}^{j_1} t_i + \sum_{i=j_1+1}^{j_2}(t_i - s_1) + \cdots\cdots + \sum_{i=j_{m-1}+1}^{j_m}(t_i - s_{m-1}) + \sum_{i=j_m+1}^{n}(t_i - s_m)\}} e^{-\lambda_0 \frac{1-e^{-\delta u}}{\delta}}, \tag{5.2.8}$$

$$\begin{aligned}
& h_2(y_1, y_2, \cdots, y_m) \\
& = (\lambda_0 e^{-\delta s_1} + y_1)^{j_2 - j_1} e^{-\frac{1-e^{-\delta(u-s_1)}}{\delta} y_1} \\
& \cdot (\lambda_0 e^{-\delta s_2} + y_1 e^{-\delta(s_2 - s_1)} + y_2)^{j_3 - j_2} e^{-\frac{1-e^{-\delta(u-s_2)}}{\delta} y_2} \\
& \qquad \vdots \\
& \qquad \vdots \\
& \cdot (\lambda_0 e^{-\delta s_{m-1}} + y_1 e^{-\delta(s_{m-1}-s_1)} + y_2 e^{-\delta(s_{m-1}-s_2)} + \cdots + y_{m-2} e^{-\delta(s_{m-1}-s_{m-2})} + y_{m-1})^{j_m - j_{m-1}} e^{-\frac{1-e^{-\delta(u-s_{m-1})}}{\delta} y_{m-1}} \\
& \cdot (\lambda_0 e^{-\delta s_m} + y_1 e^{-\delta(s_m - s_1)} + y_2 e^{-\delta(s_m - s_2)} + \cdots + y_{m-2} e^{-\delta(s_m - s_{m-2})} + y_{m-1} e^{-\delta(s_m - s_{m-1})} + y_m)^{n - j_m} e^{-\frac{1-e^{-\delta(u-s_m)}}{\delta} y_m},
\end{aligned} \tag{5.2.9}$$

and

$g_{2_{Y_1,Y_2,\cdots,Y_m}}(y_1, y_2, \cdots, y_m)$ is the m-dimensional joint probability density function of jump size $Y_1, Y_2, \cdots, Y_{m-1}, Y_m$.

Proof

The result follows immediately if we set $\nu = 0$ in (5.2.1).

□

Now let us obtain the Laplace transform of the distribution of intensity λ_t.

Theorem 5.2.3 Let λ_t evolving up to a fixed time u as defined. Also N_t as defined and $\Im_t^N$ be the filtration generated by $\{N_s;\ 0 \le s \le t\}$. Let M_t be the total number of jumps up to time t and $\Im_t^M$ be the filtration generated by $\{M_s;\ 0 \le s \le t\}$. Assuming that we know λ_0 then

$$E(e^{-\nu\lambda_u} | \Im_u^{N,M}, \lambda_0) = \frac{e^{-\nu\lambda_0 e^{-\delta u}} \int_0^\infty \int_0^\infty \cdots \int_0^\infty h_1(y_1, y_2, \cdots, y_m) g_{1_{Y_1,Y_2,\cdots,Y_m}}(y_1, y_2, \cdots, y_m) dy_1 dy_2 \cdots dy_m}{\int_0^\infty \int_0^\infty \cdots \int_0^\infty h_2(y_1, y_2, \cdots, y_m) g_{2_{Y_1,Y_2,\cdots,Y_m}}(y_1, y_2, \cdots, y_m) dy_1 dy_2 \cdots dy_m}. \tag{5.2.10}$$

Proof

If we assume that we know intensity λ from 0 to u i.e. λ_ω where $0 \le \omega \le u$. Assuming that we know when the jumps and points occur then

$$E(e^{-v\lambda_u} | \Im_u^{N,M}; \lambda_\omega, 0 \le \omega \le u)$$
$$= \lim_{\substack{\varepsilon_1 \downarrow 0 \\ \varepsilon_2 \downarrow 0 \\ \vdots \\ \varepsilon_n \downarrow 0 \\ \varsigma_1 \downarrow 0 \\ \varsigma_2 \downarrow 0 \\ \vdots \\ \varsigma_m \downarrow 0}} E(e^{-v\lambda_u} | T_1 \in [t_1, t_1 + \varepsilon_1), \cdots, T_n \in [t_n, t_n + \varepsilon_n), u < T_{n+1}, S_1 \in [s_1, s_1 + \varsigma_1), \cdots, S_m \in [s_m, s_m + \varsigma_m), u < S_{m+1}; \lambda_\omega, 0 \le \omega \le u)$$

where T_i is interarrival time between claims.

$$= \frac{E\{e^{-v\lambda_u} I(T_1 \in dt_1, T_2 \in dt_2, \cdots, T_n \in dt_n, u < T_{n+1}, S_1 \in ds_1, S_2 \in ds_2, \cdots, S_m \in ds_m, u < S_{m+1}) | \lambda_\omega, 0 \le \omega \le u\}}{\Pr(T_1 \in dt_1, T_2 \in dt_2, \cdots, T_n \in dt_n, u < T_{n+1}, S_1 \in ds_1, S_2 \in ds_2, \cdots, S_m \in ds_m, u < S_{m+1} | \lambda_\omega, 0 \le \omega \le u)}. \tag{5.2.11}$$

We can find that the denominator of (5.2.11) is a joint probability density function of $T_1, T_2, \cdots, T_n$. Therefore

$$E(e^{-v\lambda_u} | \Im_u^{N,M})$$
$$= \frac{E[E\{e^{-v\lambda_u} I(T_1 \in dt_1, T_2 \in dt_2, \cdots, T_n \in dt_n, u < T_{n+1}, S_1 \in ds_1, S_2 \in ds_2, \cdots, S_m \in ds_m, u < S_{m+1}) | \lambda_\omega, 0 \le \omega \le u\}]}{E[\lambda_{t_1} \lambda_{t_2} \cdots\cdots \lambda_{t_n} e^{-\int_0^u \lambda_s ds}]}. \tag{5.2.12}$$

Conditioning on $Y_1, Y_2, \cdots, Y_{m-1}, Y_m$ in the numerator of (5.2.12) then

$$E[E_{Y_1,\cdots,Y_m}[E\{e^{-v\lambda_u} I(T_1 \in dt_1, \cdots, T_n \in dt_n, u < T_{n+1}, S_1 \in ds_1, \cdots, S_m \in ds_m, u < S_{m+1}) | Y_1, \cdots, Y_m\}] | \lambda_\omega, 0 \le \omega \le u]$$
$$= E[E_{Y_1,\cdots,Y_m}\{e^{-v\lambda_u} \Pr(T_1 \in dt_1, \cdots, T_n \in dt_n, u < T_{n+1}, S_1 \in ds_1, \cdots, S_m \in ds_m, u < S_{m+1} | Y_1, \cdots, Y_m)\} | \lambda_\omega, 0 \le \omega \le u]$$
$$= E[e^{-v\lambda_u} E_{Y_1,\cdots,Y_m}\{\Pr(T_1 \in dt_1, \cdots, T_n \in dt_n, u < T_{n+1}, S_1 \in ds_1, \cdots, S_m \in ds_m, u < S_{m+1} | Y_1, \cdots, Y_m)\} | \lambda_\omega, 0 \le \omega \le u]$$
$$= E[e^{-v\lambda_u} \Pr(T_1 \in dt_1, \cdots, T_n \in dt_n, u < T_{n+1}, S_1 \in ds_1, \cdots, S_m \in ds_m, u < S_{m+1} | \lambda_\omega, 0 \le \omega \le u)]. \tag{5.2.13}$$

We can also find that the second part of (5.2.13) is the joint probability density function of $T_1, T_2, \cdots, T_n$. Therefore

$$= E[e^{-v\lambda_u} \lambda_{t_1} \lambda_{t_2} \cdots\cdots \lambda_{t_n} e^{-\int_0^u \lambda_s ds}]. \tag{5.2.14}$$

Hence

$$E(e^{-\nu\lambda_u}\big|\Im_u^{N,M}) = \frac{E[e^{-\nu\lambda_u}\lambda_{t_1}\lambda_{t_2}\cdots\cdots\lambda_{t_n}e^{-\int_0^u \lambda_s ds}]}{E[\lambda_{t_1}\lambda_{t_2}\cdots\cdots\lambda_{t_n}e^{-\int_0^u \lambda_s ds}]}. \tag{5.2.15}$$

The result follows immediately from (5.2.1) and (5.2.7) if we place a condition on λ_0 in (5.2.15).

□

We will close this section with a corollary that illustrates the use of theorem 5.2.3 when the jump size distribution is exponential i.e. $g(y_1) = \alpha e^{-\alpha y_1}$, $y_1 > 0$, $\alpha > 0$ and there is only one jump.

Corollary 5.2.4 Let λ_t as defined and evolving up to a fixed time u. Assume that there is only one jump, it occurs at s_1 and that there are n points at $t_1, t_2, \cdots, t_n$ where $0 < t_1 < s_1 < t_n < u$. Let the jump size distribution be exponential i.e. $g(y_1) = \alpha e^{-\alpha y_1}$, $y_1 > 0$, $\alpha > 0$. Then

$$E(e^{-\nu\lambda_u}\big|\Im_u^{N,M},\lambda_0) = \frac{\{\alpha+\frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1}\int_{\lambda_0 e^{-\delta s_1}}^{\infty}\frac{\{\alpha+\nu e^{-\delta(u-s_1)}+\frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1}p^{n-j}e^{-\{\alpha+\nu e^{-\delta(u-s_1)}+\frac{1-e^{-\delta(u-s_1)}}{\delta}\}p}}{\Gamma(n-j+1)}dp}{\{\alpha+\nu e^{-\delta(u-s_1)}+\frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1}\int_{\lambda_0 e^{-\delta s_1}}^{\infty}\frac{\{\alpha+\frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1}p^{n-j}e^{-\{\alpha+\frac{1-e^{-\delta(u-s_1)}}{\delta}\}p}}{\Gamma(n-j+1)}dp}. \tag{5.2.16}$$

<u>Proof</u>

As there is only one jump, the numerator of (5.2.10) becomes

$$e^{-\nu\lambda_0 e^{-\delta u}}\int_0^\infty\int_0^\infty\cdots\int_0^\infty h_1(y_1,y_2,\cdots,y_m)g_{1_{Y_1,Y_2,\cdots,Y_m}}(y_1,y_2,\cdots,y_m)dy_1dy_2\cdots dy_m$$

$$= e^{-\nu\lambda_0 e^{-\delta u}}\int_0^\infty(\lambda_0 e^{-\delta s_1}+y_1)^{n-j_1}e^{-\{\nu e^{-\delta(u-s_1)}+\frac{1-e^{-\delta(u-s_1)}}{\delta}\}y_1}dG(y_1).$$

Set $g(y_1) = \alpha e^{-\alpha y_1}$, then

$$= \alpha e^{\lambda_0 e^{-\delta s_1}\{\alpha + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}} \frac{\Gamma(n-j+1)}{\{\alpha + ve^{-\delta(u-s_1)} + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1}}$$

$$\cdot \int_{\lambda_0 e^{-\delta s_1}}^{\infty} \frac{\{\alpha + ve^{-\delta(u-s_1)} + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1} p^{n-j} e^{-\{\alpha + ve^{-\delta(u-s_1)} + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}P}}{\Gamma(n-j+1)} dp \qquad (5.2.17)$$

where

$$\frac{\{\alpha + ve^{-\delta(u-s_1)} + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1} p^{n-j} e^{-\{\alpha + ve^{-\delta(u-s_1)} + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}P}}{\Gamma(n-j+1)} \sim Gamma\left(n-j+1\,,\ \{\alpha + ve^{-\delta(u-s_1)} + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}\right).$$

Setting $v = 0$ in (5.2.17), then the denominator of (5.2.10) becomes

$$\int_0^{\infty}\int_0^{\infty}\cdots\int_0^{\infty} h_2(y_1, y_2, \cdots, y_m) g_{2_{Y_1,Y_2,\cdots,Y_m}}(y_1, y_2, \cdots, y_m) dy_1 dy_2 \cdots dy_m$$

$$= \alpha e^{\lambda_0 e^{-\delta s_1}\{\alpha + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}} \frac{\Gamma(n-j+1)}{\{\alpha + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1}} \int_{\lambda_0 e^{-\delta s_1}}^{\infty} \frac{\{\alpha + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1} p^{n-j} e^{-\{\alpha + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}P}}{\Gamma(n-j+1)} dp. \qquad (5.2.18)$$

where

$$\frac{\{\alpha + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}^{n-j+1} p^{n-j} e^{-\{\alpha + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}P}}{\Gamma(n-j+1)} \sim Gamma\left(n-j+1\,,\ \{\alpha + \frac{1-e^{-\delta(u-s_1)}}{\delta}\}\right).$$

The result follows immediately from (5.2.17) and (5.2.18).

□

5.3 Where the number of claims in a fixed time interval is known

This section deals with the derivation of the Laplace transform of the distribution of intensity λ_t assuming that the number of claims in a fixed time interval is known. We do not have any information about catastrophes and also the times of claim point occurrence. Therefore we will obtain the Laplace transform of the distribution of intensity λ_t by conditioning the number of claims between a time interval for which information is available.

Let us assume that the shot noise process λ_t is time homogeneous.

Lemma 5.3.1 Let N_t and λ_t as defined. Let $\nu \geq 0$, $0 \leq \theta \leq 1$, $0 \leq n \leq r < \infty$ and t_1, t_2 be fixed times. Then

$$E\{e^{-\nu\lambda_{t_2}} I(N_{t_2} - N_{t_1} = n)\} = \frac{1}{n!} E[\{\frac{d^n}{d\theta^n} E(\theta^{N_{t_2}-N_{t_1}} e^{-\nu\lambda_{t_2}} \big| N_{t_1}, \lambda_{t_1})\}\big|_{\theta=0}]. \tag{5.3.1}$$

<u>**Proof**</u>

$$E(\theta^{N_{t_2}-N_{t_1}} e^{-\nu\lambda_{t_2}}) = \sum_{i=0}^{r} E\{\theta^i I(N_{t_2} - N_{t_1} = i) e^{-\nu\lambda_{t_2}}\} = E\{E(\theta^{N_{t_2}-N_{t_1}} e^{-\nu\lambda_{t_2}} \big| N_{t_1}, \lambda_{t_1})\}. \tag{5.3.2}$$

The result follows immediately if we differentiate (5.3.2) n times w.r.t. θ and set $\theta = 0$.

□

Corollary 5.3.2 Let N_t and λ_t as defined and $0 \leq \theta \leq 1$. Then

$$\Pr(N_{t_2} - N_{t_1} = n) = \frac{1}{n!} E[\{\frac{d^n}{d\theta^n} E(\theta^{N_{t_2}-N_{t_1}} \big| N_{t_1}, \lambda_{t_1})\}\big|_{\theta=0}]. \tag{5.3.3}$$

<u>**Proof**</u>

The result follows immediately if we set $\nu = 0$ in (5.3.1).

□

Now let us derive the Laplace transform of the distribution of intensity λ_t.

Theorem 5.3.3 Let N_t and λ_t as defined. Let $\nu \geq 0$, $0 \leq \theta \leq 1$ and t_1, t_2 be fixed times. Then

$$E(e^{-\nu\lambda_{t_2}} \mid N_{t_2} - N_{t_1} = n) = \frac{E[\{\frac{d^n}{d\theta^n} E(\theta^{N_{t_2}-N_{t_1}} e^{-\nu\lambda_{t_2}} \mid N_{t_1}, \lambda_{t_1})\}|_{\theta=0}]}{E[\{\frac{d^n}{d\theta^n} E(\theta^{N_{t_2}-N_{t_1}} \mid N_{t_1}, \lambda_{t_1})\}|_{\theta=0}]}. \tag{5.3.4}$$

Proof

$E(e^{-\nu\lambda_{t_2}} \mid N_{t_2} - N_{t_1} = n)$ can be written as

$$E(e^{-\nu\lambda_{t_2}} \mid N_{t_2} - N_{t_1} = n) = \frac{E\{e^{-\nu\lambda_{t_2}} I(N_{t_2} - N_{t_1} = n)\}}{\Pr(N_{t_2} - N_{t_1} = n)}. \tag{5.3.5}$$

The result follows from (5.3.1) and (5.3.3).

□

We will close this section with a corollary that illustrates the use of theorem 5.3.3 when the jump size distribution is exponential i.e. $g(y) = \alpha e^{-\alpha y}$, $y > 0$, $\alpha > 0$ and there is only one claim in a fixed time interval.

Corollary 5.3.4 Let N_t and λ_t as defined. Let $\nu \geq 0$, $0 \leq \theta \leq 1$ and t_1, t_2 be fixed times. Assume that there is only one claim between the interval (t_1, t_2), i.e. $N_{t_2} - N_{t_1} = 1$. Let the jump size distribution be exponential i.e. $g(y) = \alpha e^{-\alpha y_1}$, $y > 0$, $\alpha > 0$. Assuming that we know the Laplace transform of the distribution of λ_{t_1} then

$$E(e^{-\nu\lambda_{t_2}} \mid N_{t_2} - N_{t_1} = 1) =$$

$$\frac{\left[H_1\left(\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta s}\right)\left\{\frac{\alpha\rho}{(\delta\alpha+1)^2}\ln\left(\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right)-\frac{\alpha\rho(1-e^{-\delta s})}{\delta(\delta\alpha+1)(\nu+\alpha)e^{-\delta s}}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{-1}\right\}+H_1^{(1)}\frac{1-e^{-\delta s}}{\delta}\right]\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}}}{\left[H_1\left(\frac{1}{\delta}(1-e^{-\delta s})\right)\left\{\frac{\alpha\rho}{(\delta\alpha+1)^2}\ln\left(\frac{\alpha+\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}}{\alpha e^{-\delta s}}\right)-\frac{\alpha\rho(1-e^{-\delta s})}{\delta(\delta\alpha+1)\alpha e^{-\delta s}}\left\{\frac{\alpha+\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}}{\alpha e^{-\delta s}}\right\}^{-1}\right\}+H_1^{(1)}\frac{1-e^{-\delta s}}{\delta}\right]\left\{\frac{\alpha+\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}}{\alpha e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}}} \tag{5.3.6}$$

where $H_1(\nu)$ denotes the Laplace transform of the distribution of λ_{t_1}, $H_1^{(1)} = -H_1'\left(\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta s}\right)$ and $s = t_2 - t_1$.

Proof

From (2.1.12),

$$
\begin{aligned}
&E\{\theta^{(N_{t_2}-N_{t_1})}e^{-\nu\lambda_{t_2}} \mid N_{t_1},\lambda_{t_1}\}\\
&= e^{-\{\frac{1-\theta}{\delta}+(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho(t_2-t_1)} \left\{\frac{(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}+\alpha+\frac{1-\theta}{\delta}}{(\nu+\alpha)e^{-\delta(t_2-t_1)}}\right\}^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}\\
&= e^{-\{\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho(t_2-t_1)} e^{-\lambda_{t_1}\{\frac{1}{\delta}(1-e^{-\delta(t_2-t_1)})\theta\}} \left\{\frac{(\nu-\frac{1-\theta}{\delta})e^{-\delta(t_2-t_1)}+\alpha+\frac{1-\theta}{\delta}}{(\nu+\alpha)e^{-\delta(t_2-t_1)}}\right\}^{\frac{\alpha\rho}{\delta\alpha+(1-\theta)}}\\
&= e^{-\{\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho(t_2-t_1)} e^{-\lambda_{t_1}(\nu+\alpha)e^{-\delta(t_2-t_1)}\theta'} \left\{\frac{\nu e^{-\delta(t_2-t_1)}-\frac{1}{\delta}e^{-\delta(t_2-t_1)}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta(t_2-t_1)}}-\theta'\right\}^{L}\\
&= \mathrm{K}e^{k\theta'}(a-\theta')^{\frac{b}{c-\theta'}}
\end{aligned}
$$

(5.3.7)

where

$$
\theta' = \frac{1-e^{-\delta(t_2-t_1)}}{\delta(\nu+\alpha)e^{-\delta(t_2-t_1)}}\theta, \quad L = \frac{\dfrac{\alpha\rho\{1-e^{-\delta(t_2-t_1)}\}}{\delta(\nu+\alpha)e^{-\delta(t_2-t_1)}}}{\dfrac{(\delta\alpha+1)\{1-e^{-\delta(t_2-t_1)}\}}{\delta(\nu+\alpha)e^{-\delta(t_2-t_1)}}-\theta'},
$$

$$
\mathrm{K} = e^{-\{\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta(t_2-t_1)}\}\lambda_{t_1}} e^{-\rho(t_2-t_1)},\ k=\lambda_{t_1}(\nu+\alpha)e^{-\delta(t_2-t_1)},\ a=\frac{\nu e^{-\delta(t_2-t_1)}-\frac{1}{\delta}e^{-\delta(t_2-t_1)}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta(t_2-t_1)}},
$$

$$
b=\frac{\alpha\rho\{1-e^{-\delta(t_2-t_1)}\}}{\delta(\nu+\alpha)e^{-\delta(t_2-t_1)}} \text{ and } c=\frac{(\delta\alpha+1)\{1-e^{-\delta(t_2-t_1)}\}}{\delta(\nu+\alpha)e^{-\delta(t_2-t_1)}}.
$$

(5.3.8)

Using *MAPLE*, expanding (5.3.7) with respect to θ without a constant K then

$$
\begin{aligned}
&e^{k\theta'}(a-\theta')^{\frac{b}{c-\theta'}} = e^{\frac{b}{c}\ln a}\\
&+\left\{e^{\frac{b}{c}\ln a}b\frac{-c+(\ln a)a}{c^2a}+ke^{\frac{b}{c}\ln a}\right\}\theta'\\
&+\left\{ke^{\frac{b}{c}\ln a}b\frac{-c+(\ln a)a}{c^2a}+\frac{1}{2}e^{\frac{b}{c}\ln a}b\frac{-2c^2a-c^3+2(\ln a)ca^2+bc^2-2bc(\ln a)a+b(\ln^2 a)a^2}{c^4a^2}+\frac{1}{2}k^2e^{\frac{b}{c}\ln a}\right\}\theta'^2\\
&+\left\{\begin{aligned}&\frac{1}{2}ke^{\frac{b}{c}\ln a}b\frac{-2c^2a-c^3+2(\ln a)ca^2+bc^2-2bc(\ln a)a+b(\ln^2 a)a^2}{c^4a^2}+\frac{1}{2}k^2e^{\frac{b}{c}\ln a}b\frac{-c+(\ln a)a}{c^2a}+\frac{1}{6}k^3e^{\frac{b}{c}\ln a}\\&+\frac{1}{6}e^{\frac{b}{c}\ln a}b\frac{-3c^4a-6c^3a^2-2c^5+6(\ln a)c^2a^3+6bc^3a+3bc^4-12bc^2(\ln a)a^2-3bc^3(\ln a)a+6bc(\ln^2 a)a^3-b^2c^3+3b^2c^2(\ln a)a-3b^2c(\ln^2 a)a^2+b^2(\ln^3 a)a^3}{c^6a^3}\end{aligned}\right\}\theta'^3\\
&+\left\{\begin{aligned}&\frac{1}{6}ke^{\frac{b}{c}\ln a}b\frac{-3c^4a-6c^3a^2-2c^5+6(\ln a)c^2a^3+6bc^3a+3bc^4-12bc^2(\ln a)a^2-3bc^3(\ln a)a+6bc(\ln^2 a)a^3-b^2c^3+3b^2c^2(\ln a)a-3b^2c(\ln^2 a)a^2+b^2(\ln^3 a)a^3}{c^6a^3}\\&+\frac{1}{4}k^2e^{\frac{b}{c}\ln a}b\frac{-2c^2a-c^3+2(\ln a)ca^2+bc^2+3bc^4-2bc(\ln a)+b(\ln^2 a)a^2}{c^4a^2}+\frac{1}{6}k^3e^{\frac{b}{c}\ln a}b\frac{-c+(\ln a)a}{c^2a}+\frac{1}{24}e^{\frac{b}{c}\ln a}\frac{b\mathrm{A}}{c^8a^4}+\frac{1}{24}k^4e^{\frac{b}{c}\ln a}\end{aligned}\right\}\theta'^4\\
&+O(\theta'^5)
\end{aligned}
$$

(5.3.9)

where

$$A=-24c^4a^3-6c^7+11bc^6-6b^2c^5+b^3c^4+24(\ln a)c^3a^4-8c^6a-12c^5a^2+24bc^5a+36bc^4a^2-12b^2c^4a$$
$$+b^3(\ln^4 a)a^4-72bc^3(\ln a)a^3-24bc^4a^2(\ln a)-8bc^5(\ln a)a+36bc^2(\ln^2 a)a^4+36b^2c^3(\ln a)a^2+12b^2c^4(\ln a)a$$
$$-36b^2c^2(\ln^2 a)a^3-6b^2c^3(\ln^2 a)a^2+12b^2c(\ln^3 a)a^4-4b^3c^3(\ln a)a+6b^3c^2(\ln^2 a)a^2-4b^3c(\ln^3 a)a^3.$$

Therefore, from (5.3.9) and $N_{t_2}-N_{t_1}=1$, the numerator of (5.3.4) becomes

$$E[\{\frac{d}{d\theta}E(\theta^{N_{t_2}-N_{t_1}}e^{-\nu\lambda_{t_2}}\big|N_{t_1},\lambda_{t_1})\}\big|_{\theta=0}]$$
$$=E[\{\frac{d}{d\theta}\mathrm{K}e^{k\theta}(a-\theta)^{\frac{b}{c-\theta}}\}\big|_{\theta=0}]=E\left[\mathrm{K}\left\{e^{\frac{b}{c}\ln a}b\frac{-c+(\ln a)a}{c^2a}+ke^{\frac{b}{c}\ln a}\right\}\frac{1-e^{-\delta s}}{\delta(\nu+\alpha)e^{-\delta s}}\right].\tag{5.3.10}$$

Substitute (5.3.8) into (5.3.10) and put $s=t_2-t_1$ then

$$=e^{-\rho s}$$
$$\left[E\left[e^{-(\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta s})\lambda_{t_1}}\left\{-\frac{\alpha\rho}{\delta\alpha+1}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}-1}\frac{1-e^{-\delta s}}{\delta(\nu+\alpha)e^{-\delta s}}+\frac{\alpha\rho}{(\delta\alpha+1)^2}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}}\ln\left(\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right)\right\}+\frac{1-e^{-\delta s}}{\delta}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}}\lambda_{t_1}\right]\right]$$

$$=e^{-\rho s}$$
$$\left[E\left\{e^{-(\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta s})\lambda_{t_1}}\right\}\left\{-\frac{\alpha\rho}{\delta\alpha+1}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}-1}\frac{1-e^{-\delta s}}{\delta(\nu+\alpha)e^{-\delta s}}+\frac{\alpha\rho}{(\delta\alpha+1)^2}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}}\ln\left(\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right)\right\}+\frac{1-e^{-\delta s}}{\delta}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}}\cdot E\left\{\lambda_{t_1}e^{-(\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta s})\lambda_{t_1}}\right\}\right].\tag{5.3.11}$$

Since we know the Laplace transform of the distribution of λ_{t_1}, we have

$$=e^{-\rho s}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}}$$
$$\cdot\left[H_1\left(\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta s}\right)\left\{\frac{\alpha\rho}{(\delta\alpha+1)^2}\ln\left(\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right)-\frac{\alpha\rho(1-e^{-\delta s})}{\delta(\delta\alpha+1)(\nu+\alpha)e^{-\delta s}}\left\{\frac{(\nu-\frac{1}{\delta})e^{-\delta s}+\alpha+\frac{1}{\delta}}{(\nu+\alpha)e^{-\delta s}}\right\}^{-1}\right\}+H_1^{(1)}\frac{1-e^{-\delta s}}{\delta}\right].\tag{5.3.12}$$

where $H_1(\nu)$ denotes the Laplace transform of the distribution of λ_{t_1}, $H_1^{(1)}=-H_1'\left(\frac{1}{\delta}+(\nu-\frac{1}{\delta})e^{-\delta s}\right)$.

Setting $v=0$ in (5.3.12), then the denominator of (5.3.4) becomes

$$E[\{\frac{d}{d\theta}E(\theta^{N_{t_2}-N_{t_1}}|N_{t_1},\lambda_{t_1})\}|_{\theta=0}]$$

$$=e^{-\rho s}\left\{\frac{\alpha+\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}}{\alpha e^{-\delta s}}\right\}^{\frac{\alpha\rho}{\delta\alpha+1}}$$

$$\cdot\left[H_1\left(\frac{1}{\delta}(1-e^{-\delta s})\right)\left\{\frac{\alpha\rho}{(\delta\alpha+1)^2}\ln\left(\frac{\alpha+\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}}{\alpha e^{-\delta s}}\right)-\frac{\alpha\rho(1-e^{-\delta s})}{\delta(\delta\alpha+1)\alpha e^{-\delta s}}\left\{\frac{\alpha+\frac{1}{\delta}-\frac{1}{\delta}e^{-\delta s}}{\alpha e^{-\delta s}}\right\}^{-1}\right\}+H_1^{(1)}\frac{1-e^{-\delta s}}{\delta}\right].$$

(5.3.13)

The result follows immediately from (5.3.12) and (5.3.13).

□

6. Conclusions

As a claim intensity function for doubly stochastic Poisson process, we have examined the shot noise process. We have looked at doubly stochastic Poisson process incorporating the shot noise process as its intensity for the claim arrival process for catastrophic events.

We have derived pricing formulae for stop-loss reinsurance contracts for catastrophic events and for catastrophe insurance derivatives applying doubly stochastic Poisson process incorporating the shot noise process as its intensity. We have also presented pricing formulae for stop-loss reinsurance contracts for catastrophic events using the Kalman-Bucy filter. For these pricing models, it has been assumed that there are no arbitrage opportunities in the market. This has been achieved by using an equivalent martingale probability measure via the Esscher transform.

Having established the pricing models, it has turned out two estimations, i.e. estimation of the parameters of the claim intensity and estimation of the distribution of claim intensity were essential. We have shown how to estimate the parameters of the claim intensity using likelihood function. For estimation of the distribution of claim intensity, we have employed state estimation, one of which was by the Kalman-Bucy filter.

Appendix

1. The calculations and *S-Plus* routine for example 3.4.1

```
> an
 [1] 0.00011628 0.00048266 0.00142250 0.00333550 0.00661500 0.01152300
 [7] 0.01808600 0.02604500 0.03488100 0.04390000 0.05234900 0.05953700
[13] 0.06493200 0.06821400 0.06929000 0.06827300 0.06543400 0.06114800
[19] 0.05583100 0.04989800 0.04372300 0.03761600 0.03181500 0.02648400
[25] 0.02172000 0.01756700 0.01402300 0.01105600 0.00861660 0.00664190
[31] 0.00506670 0.00382720 0.00286390 0.00212410 0.00156210 0.00113960
[37] 0.00082497 0.00059282 0.00042301 0.00029981 0.00021112

> sum(an)
[1] 0.9995107

> 0.9995107+0.000014982
[1] 0.9995257

> n
[1] 41

> g1
[1] 1

> beta1
[1] 1

> b1  (Retention limit)
[1] 0.00 5.00 10.00 16.61 20.00 25.00 30.00 33.22

> b2
function(b1)
{
        z1 <- 1:n
```

```
        for(i in 1:n) {
                a1 <- (g1 * i) + 1
                aa1 <- 1 - pgamma(b1, a1)
                aa2 <- 1 - pgamma(b1, (a1 - 1))
                z1[i] <- an[i] * (((aa1 * (a1 - 1))/beta1) - (b1 * aa2))
        }
        return(sum(z1))
}

> b2(b1[i])  (Reinsurance premium)
[1] 16.58403 11.61916 7.06779 2.833487 1.587005 0.595824 0.1951147 0.0886971
```

2. The calculations and *S-Plus* routine for example 3.4.2

```
> an
 [1] 0.00011628 0.00048266 0.00142250 0.00333550 0.00661500 0.01152300
 [7] 0.01808600 0.02604500 0.03488100 0.04390000 0.05234900 0.05953700
[13] 0.06493200 0.06821400 0.06929000 0.06827300 0.06543400 0.06114800
[19] 0.05583100 0.04989800 0.04372300 0.03761600 0.03181500 0.02648400
[25] 0.02172000 0.01756700 0.01402300 0.01105600 0.00861660 0.00664190
[31] 0.00506670 0.00382720 0.00286390 0.00212410 0.00156210 0.00113960
[37] 0.00082497 0.00059282 0.00042301 0.00029981 0.00021112

> sum(an)
[1] 0.9995107

> 0.9995107+0.000014982
[1] 0.9995257

> n
[1] 41

> g1
[1] 1

> beta1
[1] 1

> b1
[1] 0.00 5.00 10.00 16.61 20.00 25.00 30.00 33.22

> b2
function(b1)
{
        z1 <- 1:n
        for(i in 1:n) {
                a1 <- (g1 * i) + 1
```

```
		aa1 <- 1 - pgamma(b1, a1)
		aa2 <- 1 - pgamma(b1, (a1 - 1))
		z1[i] <- an[i] * (((aa1 * (a1 - 1))/beta1) - (b1 * aa2))
	}
	return(sum(z1))
}

> b3
function(b1)
{
	z1 <- 1:n
	for(i in 1:n) {
		a1 <- (g1 * i)
		z1[i] <- (1/b1) * an[i] * (a1/beta1)
	}
	return(sum(z1))
}

> b3(b1[4])
[1] 0.9984363

> bb2[8]<-b2(b1[8])/b1[4]
> bb2[8]
[1] 0.005339982

> futu<-25000*(b3(b1[4])-bb2[8])  (Catastrophe futures price)
> futu
[1] 24827.41
```

3. The calculations and *S-Plus* routine for example 3.4.3

```
> g1
[1] 1

> beta1
[1] 1

> b1 (Retention limit)
[1] 25.00

> b2
function(b1)
{
        z1 <- 1:n
        for(i in 1:n) {
                a1 <- (g1 * i) + 1
                aa1 <- 1 - pgamma(b1, a1)
                aa2 <- 1 - pgamma(b1, (a1 - 1))
                z1[i] <- an[i] * (((aa1 * (a1 - 1))/beta1) - (b1 * aa2))
        }
        return(sum(z1))
}
```

3.1 $\theta^* = 1.0$ and $\gamma^* = -0.1$

```
> an
 [1] 0.00022918 0.00090937 0.00256150 0.00574030 0.01087900 0.01810700
 [7] 0.02715300 0.03735600 0.04779100 0.05745400 0.06543800 0.07108200
[13] 0.07403700 0.07427900 0.07205100 0.06779100 0.06204000 0.05535700
[19] 0.04825800 0.04117900 0.03444900 0.02829500 0.02284600 0.01815500
[25] 0.01421400 0.01097400 0.00836150 0.00629300 0.00468130 0.00344420
[31] 0.00250780 0.00180800 0.00129130 0.00091405 0.00064155 0.00044666
[37] 0.00030858 0.00021162
```

> sum(an)
[1] 0.9995349

> 0.9995349+0.000030886
[1] 0.9995658

> n
[1] 38

> b2(b1[1]) (Reinsurance premium)
[1] 0.3544252

3.2 $\theta^* = 1.1$ and $\gamma^* = -0.1$

> an
 [1] 0.00011628 0.00048266 0.00142250 0.00333550 0.00661500 0.01152300
 [7] 0.01808600 0.02604500 0.03488100 0.04390000 0.05234900 0.05953700
[13] 0.06493200 0.06821400 0.06929000 0.06827300 0.06543400 0.06114800
[19] 0.05583100 0.04989800 0.04372300 0.03761600 0.03181500 0.02648400
[25] 0.02172000 0.01756700 0.01402300 0.01105600 0.00861660 0.00664190
[31] 0.00506670 0.00382720 0.00286390 0.00212410 0.00156210 0.00113960
[37] 0.00082497 0.00059282 0.00042301 0.00029981 0.00021112

> sum(an)
[1] 0.9995107

> 0.9995107+0.000014982
[1] 0.9995257

> n
[1] 41

> b2(b1[1]) (Reinsurance premium)
[1] 0.595824

3.3 $\theta^* = 1.2$ and $\gamma^* = -0.1$

```
> an
 [1] 0.000060714 0.00026207 0.00080323 0.001959   0.0040414  0.0073238
 [7] 0.011959    0.017919   0.02497    0.032703   0.040582   0.048034
[13] 0.054522    0.059616   0.063031   0.064647   0.064496   0.062741
[19] 0.059636    0.055488   0.050619   0.04534    0.039926   0.034604
[25] 0.02955     0.024885   0.020684   0.016981   0.013781   0.011062
[31] 0.0087872   0.006912   0.0053864  0.0041604  0.0031864  0.0024208
[37] 0.0018251   0.0013659  0.0010151  0.00074928 0.00054953 0.00040055
[43] 0.00029023  0.00020909

> sum(an)
[1] 0.9994842

> 0.9994842+0.000007524
[1] 0.9994917

> n
[1] 44

> b2(b1[1])   (Reinsurance premium)
[1] 0.9299355
```

3.4 $\theta^* = 1.3$ and $\gamma^* = -0.1$

```
> an
 [1] 3.2563e-05 1.4545e-04 4.6138e-04 1.1647e-03 2.4871e-03 4.6660e-03
 [7] 7.8881e-03 1.2237e-02 1.7657e-02 2.3946e-02 3.0773e-02 3.7721e-02
[13] 4.4343e-02 5.0218e-02 5.4993e-02 5.8422e-02 6.0375e-02 6.0839e-02
[19] 5.9905e-02 5.7742e-02 5.4570e-02 5.0639e-02 4.6199e-02 4.1485e-02
[25] 3.6704e-02 3.2025e-02 2.7581e-02 2.3463e-02 1.9730e-02 1.6410e-02
[31] 1.3508e-02 1.1010e-02 8.8914e-03 7.1168e-03 5.6485e-03 4.4472e-03
[37] 3.4746e-03 2.6949e-03 2.0755e-03 1.5878e-03 1.2069e-03 9.1174e-04
[43] 6.8468e-04 5.1125e-04 3.7967e-04 2.8047e-04 2.0613e-04
```

```
> sum(an)
[1] 0.9994578

> 0.9994578+0.0000039001
[1] 0.9994617

> n
[1] 47

> b2(b1[1])   (Reinsurance premium)
[1] 1.366049
```

3.5 $\theta^{*} = 1.4$ and $\gamma^{*} = -0.1$

```
> an
 [1] 1.7904e-05 8.2427e-05 2.6951e-04 7.0137e-04 1.5442e-03 2.9869e-03
 [7] 5.2068e-03 8.3297e-03 1.2395e-02 1.7337e-02 2.2978e-02 2.9052e-02
[13] 3.5228e-02 4.1152e-02 4.6489e-02 5.0948e-02 5.4318e-02 5.6470e-02
[19] 5.7367e-02 5.7050e-02 5.5630e-02 5.3264e-02 5.0141e-02 4.6459e-02
[25] 4.2415e-02 3.8189e-02 3.3939e-02 2.9794e-02 2.5855e-02 2.2192e-02
[31] 1.8852e-02 1.5859e-02 1.3217e-02 1.0918e-02 8.9433e-03 7.2672e-03
[37] 5.8601e-03 4.6910e-03 3.7289e-03 2.9444e-03 2.3100e-03 1.8012e-03
[43] 1.3961e-03 1.0761e-03 8.2483e-04 6.2893e-04 4.7714e-04 3.6022e-04
[49] 2.7067e-04 2.0245e-04

> sum(an)
[1] 0.9994294

> 0.9994294+0.0000020808
[1] 0.9994315

> n
[1] 50
```

```
> b2(b1[1])   (Reinsurance premium)
[1] 1.90885
```

3.6 $\theta^{*} = 1.5$ **and** $\gamma^{*} = -0.1$

```
> an
 [1] 1.0075e-05 4.7649e-05 1.6006e-04 4.2797e-04 9.6818e-04 1.9245e-03
 [7] 3.4476e-03 5.6683e-03 8.6694e-03 1.2463e-02 1.6980e-02 2.2069e-02
[13] 2.7509e-02 3.3037e-02 3.8370e-02 4.3233e-02 4.7390e-02 5.0657e-02
[19] 5.2913e-02 5.4108e-02 5.4253e-02 5.3416e-02 5.1708e-02 4.9270e-02
[25] 4.6258e-02 4.2832e-02 3.9147e-02 3.5344e-02 3.1543e-02 2.7846e-02
[31] 2.4329e-02 2.1049e-02 1.8043e-02 1.5330e-02 1.2916e-02 1.0796e-02
[37] 8.9543e-03 7.3729e-03 6.0286e-03 4.8965e-03 3.9516e-03 3.1694e-03
[43] 2.5272e-03 2.0036e-03 1.5799e-03 1.2393e-03 9.6719e-04 7.5116e-04
[49] 5.8065e-04 4.4681e-04 3.4231e-04 2.6115e-04 1.9841e-04

> sum(an)
[1] 0.9994037
> 0.9994037+0.00000114
[1] 0.9994048

> n
[1] 53

> b2(b1[1])   (Reinsurance premium)
[1] 2.558786
```

3.7 $\gamma^{*} = -0.0$ **and** $\theta^{*} = 1.1$

```
> an
 [1] 0.00028698 0.00111750 0.00309080 0.00680370 0.01267100 0.02073400
 [7] 0.03057600 0.04138100 0.05209800 0.06165000 0.06913600 0.07396100
[13] 0.07588700 0.07501500 0.07171100 0.06650800 0.06000800 0.05279800
[19] 0.04539500 0.03821000 0.03153600 0.02555800 0.02036600 0.01597400
[25] 0.01234500 0.00940970 0.00707950 0.00526160 0.00386560 0.00280920
```

```
[31] 0.00202050 0.00143910 0.00101560 0.00071035 0.00049272 0.00033903
[37] 0.00023151

> sum(an)
[1] 0.9994914

> 0.9994914+0.000039944
[1] 0.9995313

> n
[1] 37

> b2(b1[1])    (Reinsurance premium)
[1] 0.3029752
```

3.8 $\gamma^* = -0.2$ **and** $\theta^* = 1.1$

```
> an
 [1] 0.00003774 0.00016795 0.00053041 0.00133230 0.00282920 0.00527530
 [7] 0.00885960 0.01364800 0.01954700 0.02630100 0.03352200 0.04074100
[13] 0.04746900 0.05326600 0.05778100 0.06078700 0.06219300 0.06203100
[19] 0.06044100 0.05763600 0.05387700 0.04944100 0.04459700 0.03958700
[25] 0.03461700 0.02984700 0.02539700 0.02134300 0.01772600 0.01456000
[31] 0.01183400 0.00952320 0.00759140 0.00599730 0.00469750 0.00364950
[37] 0.00281330 0.00215260 0.00163540 0.00123400 0.00092503 0.00068909
[43] 0.00051025 0.00037564 0.00027501 0.00020026

> sum(an)
[1] 0.999491

> 0.999491+0.0000045336
[1] 0.9994955

> n
[1] 46
```

```
> b2(b1[1])  (Reinsurance premium)
[1] 1.207256
```

3.9 $\gamma^* = -0.3$ and $\theta^* = 1.1$

```
> an
 [1] 8.8706e-06 4.2635e-05 1.4534e-04 3.9384e-04 9.0186e-04 1.8125e-03
 [7] 3.2798e-03 5.4414e-03 8.3904e-03 1.2151e-02 1.6661e-02 2.1781e-02
[13] 2.7289e-02 3.2918e-02 3.8376e-02 4.3379e-02 4.7676e-02 5.1069e-02
[19] 5.3429e-02 5.4696e-02 5.4878e-02 5.4042e-02 5.2302e-02 4.9803e-02
[25] 4.6710e-02 4.3190e-02 3.9404e-02 3.5500e-02 3.1604e-02 2.7822e-02
[31] 2.4233e-02 2.0895e-02 1.7845e-02 1.5101e-02 1.2669e-02 1.0541e-02
[37] 8.7015e-03 7.1288e-03 5.7982e-03 4.6835e-03 3.7580e-03 2.9963e-03
[43] 2.3744e-03 1.8705e-03 1.4652e-03 1.1415e-03 8.8471e-04 6.8219e-04
[49] 5.2346e-04 3.9978e-04 3.0393e-04 2.3005e-04

> sum(an)
[1] 0.9993227

> 0.9993227+0.00000098605
[1] 0.9993237

> n
[1] 52

> b2(b1[1])  (Reinsurance premium)
[1] 2.512553
```

3.10 $\gamma^* = -0.4$ and $\theta^* = 1.1$

```
> an
 [1] 1.2588e-06 6.6004e-06 2.4529e-05 7.2415e-05 1.8055e-04 3.9485e-04
 [7] 7.7702e-04 1.4013e-03 2.3477e-03 3.6922e-03 5.4965e-03 7.7967e-03
[13] 1.0596e-02 1.3860e-02 1.7515e-02 2.1454e-02 2.5543e-02 2.9632e-02
[19] 3.3564e-02 3.7191e-02 4.0378e-02 4.3018e-02 4.5029e-02 4.6366e-02
```

```
[25] 4.7013e-02 4.6986e-02 4.6325e-02 4.5092e-02 4.3365e-02 4.1230e-02
[31] 3.8779e-02 3.6100e-02 3.3280e-02 3.0397e-02 2.7520e-02 2.4705e-02
[37] 2.2000e-02 1.9442e-02 1.7054e-02 1.4854e-02 1.2851e-02 1.1046e-02
[43] 9.4354e-03 8.0113e-03 6.7629e-03 5.6774e-03 4.7406e-03 3.9379e-03
[49] 3.2549e-03 2.6773e-03 2.1920e-03 1.7866e-03 1.4499e-03 1.1716e-03
[55] 9.4289e-04 7.5582e-04 6.0354e-04 4.8014e-04 3.8059e-04 3.0063e-04
[61] 2.3665e-04 1.8567e-04

> sum(an)
[1] 0.9993604

> 0.9993604+0.00000012815
[1] 0.9993605

> n
[1] 62

> b2(b1[1])   (Reinsurance premium)
[1] 5.364622
```

3.11 $\gamma^* = -0.5$ and $\theta^* = 1.1$

```
> an
 [1] 7.3820e-08 4.2887e-07 1.7641e-06 5.7589e-06 1.5863e-05 3.8298e-05
 [7] 8.3138e-05 1.6528e-04 3.0502e-04 5.2813e-04 8.6503e-04 1.3493e-03
[13] 2.0155e-03 2.8960e-03 4.0184e-03 5.4021e-03 7.0559e-03 8.9760e-03
[19] 1.1145e-02 1.3531e-02 1.6092e-02 1.8771e-02 2.1508e-02 2.4233e-02
[25] 2.6879e-02 2.9377e-02 3.1664e-02 3.3686e-02 3.5398e-02 3.6764e-02
[31] 3.7762e-02 3.8381e-02 3.8623e-02 3.8498e-02 3.8027e-02 3.7238e-02
[37] 3.6166e-02 3.4847e-02 3.3323e-02 3.1636e-02 2.9825e-02 2.7931e-02
[43] 2.5989e-02 2.4033e-02 2.2093e-02 2.0192e-02 1.8354e-02 1.6594e-02
[49] 1.4926e-02 1.3359e-02 1.1899e-02 1.0549e-02 9.3105e-03 8.1815e-03
[55] 7.1590e-03 6.2387e-03 5.4151e-03 4.6821e-03 4.0332e-03 3.4616e-03
[61] 2.0605e-03 2.5232e-03 2.1434e-03 1.8148e-03 1.5317e-03 1.2888e-03
[67] 1.0811e-03 9.0428e-04 7.5420e-04 6.2728e-04 5.2029e-04 4.3041e-04
```

```
[73] 3.5514e-04 2.9229e-04 2.3997e-04 1.9653e-04 1.6058e-04 1.3091e-04
[79] 1.0647e-04 8.6410e-05 6.9975e-05 5.6547e-05 4.5601e-05 3.6699e-05
[85] 2.9477e-05 2.3629e-05 1.8906e-05 1.5099e-05 1.2036e-05 9.5773e-06
[91] 7.6075e-06 6.0323e-06 4.7753e-06 3.7738e-06 2.9776e-06 2.3456e-06
[97] 1.8448e-06 1.4487e-06 1.1359e-06

> sum(an)
[1] 0.9990944

> 0.9990944+0.0000000067754
[1] 0.9990944

> n
[1] 99

> b2(b1[1])      (Reinsurance premium)
[1] 11.65184
```

4. The calculations and *S-Plus* routine for example 4.2.1

4.1 Simulation

```
> ttt<-rexp(4,1)
> ttt
[1] 0.7834462 0.4673357 1.5443291 0.9398078

> ttt<-round(ttt,3)
> ttt
[1] 0.783 0.467 1.544 0.940

> ttt<-array(ttt)
> ttt
  [1]   [2]   [3]  [4]
 0.783 0.467 1.544 0.94

> tttt<-cumsum(ttt)
  [1]  [2]   [3]   [4]
 0.783 1.25 2.794 3.734

> tttm<-sum(ttt)
> tttm
[1] 3.734

> ti<-ttt*5000
> ti
 [1]  [2]  [3]  [4]
 3915 2335 7720 4700

> tim<-cumsum(ti)
> tim
 [1]  [2]   [3]   [4]
 3915 6250 13970 18670
```

```
> yy<-rexp(4,0.01)
> yy
[1] 100.3073 323.7813 114.8395 119.3889

> yy<-array(yy)
> yy
   [1]      [2]      [3]      [4]
 100.3073 323.7813 114.8395 119.3889

> (yy[1]+yy[2]+yy[3])/3
[1] 179.6427

> yy0<-1000
> dd<-0.1

> jj1<-c(0:ti[1])
> BB<-((1-exp(-dd/5000))/dd)
> AA0<-exp(-dd*jj1/5000)
> intens0<-yy0*AA0*BB
> NN0<-rpois(ti[1],intens0)

> jj2<-c(0:ti[2])
> yy1<-yy0*exp(-dd*ttt[1])+yy[1]
> AA1<-exp(-dd*jj2/5000)
> intens1<-yy1*AA1*BB
> NN1<-rpois(ti[2],intens1)

> jj3<-c(0:ti[3])
> yy2<-yy1*exp(-dd*ttt[2])+yy[2]
> AA2<-exp(-dd*jj3/5000)
> intens2<-yy2*AA2*BB
> NN2<-rpois(ti[3],intens2)

> jj4<-c(0:ti[4])
> yy3<-yy2*exp(-dd*ttt[3])+yy[3]
```

```
> AA3<-exp(-dd*jj4/5000)
> intens3<-yy3*AA3*BB
> NN3<-rpois(ti[4],intens3)

> intensi<-c(intens0,intens1,intens2,intens3)

> NNn<-c(NN0,NN1,NN2,NN3)

> Aa<-sum(NNn)
> Aa
[1] 845

> Bb<-sum(NNn^2)
> Bb
[1] 1001

> XX<-(Bb-Aa)/2
> XX
[1] 78

> YY<-(2*Aa-Bb)
> YY
[1] 689

> XX*2+YY
[1] 845

> Aaa<-order(NNn)

> Hh<-1:(Aa-XX)
> for (i in 1:(Aa-XX)) {Hh[i]<-Aaa[tim[4]+1-i]}

> gg<-1:XX
> for (i in 1:XX) {gg[i]<-Hh[i]-1}
```

```
> ff<-(XX+1):(XX+YY)
> for (i in (XX+1):(XX+YY)) {ff[i-6]<-Hh[i]-0.5}

> Hhh<-1:6
> for (i in 1:6) {Hhh[i]<-Hh[i]}
> kk<-c(ff,gg,Hhh)

> AAa2<-1:3324
> for (i in 1:3324) {AAa2[i]<-Aaa[i+14974]}

> AAa3<-1:361
> for (i in 1:361) {AAa3[i]<-Aaa[i+18298]}

> AAa4<-1:11
> for (i in 1:11) {AAa4[i]<-Aaa[i+18659]}

> u2<-1:3324
> for (i in 1:3324) {u2[i]<-AAa2[i]-0.5}

> u3<-1:361
> for (i in 1:361) {u3[i]<-AAa3[i]-1}

> u4<-1:11
> for (i in 1:11) {u4[i]<-AAa4[i]-1}

> uu4<-1:11
> for (i in 1:11) {uu4[i]<-AAa4[i]-0.5}

> kk<-c(u2,u3,AAa3,u4,uu4,AAa4)

> tpo<-sort(kk)
> tpoi<-tpo/5000
> tpoi
 [1] 0.0001 0.0007 0.0009 0.0047 0.0055 0.0061 0.0101 0.0105 0.0106 0.0108
[11] 0.0121 0.0135 0.0143 0.0153 0.0159 0.0165 0.0179 0.0193 0.0201 0.0205
```

[21] 0.0223 0.0239 0.0255 0.0257 0.0265 0.0277 0.0283 0.0287 0.0291 0.0301
[31] 0.0309 0.0316 0.0318 0.0319 0.0322 0.0324 0.0333 0.0343 0.0347 0.0349
[41] 0.0351 0.0353 0.0367 0.0369 0.0383 0.0387 0.0412 0.0414 0.0421 0.0423
[51] 0.0432 0.0434 0.0447 0.0465 0.0493 0.0509 0.0510 0.0512 0.0513 0.0527
[61] 0.0543 0.0551 0.0575 0.0593 0.0601 0.0627 0.0645 0.0647 0.0671 0.0675
[71] 0.0677 0.0709 0.0711 0.0723 0.0751 0.0752 0.0754 0.0756 0.0758 0.0767
[81] 0.0771 0.0812 0.0814 0.0851 0.0857 0.0891 0.0895 0.0907 0.0909 0.0927
[91] 0.0929 0.0941 0.0959 0.0965 0.0967 0.0969 0.0973 0.0975 0.0979 0.0985
[101] 0.0987 0.1013 0.1021 0.1025 0.1033 0.1041 0.1051 0.1053 0.1071 0.1072
[111] 0.1074 0.1075 0.1085 0.1089 0.1091 0.1093 0.1097 0.1101 0.1106 0.1108
[121] 0.1110 0.1112 0.1173 0.1190 0.1192 0.1197 0.1201 0.1205 0.1207 0.1229
[131] 0.1247 0.1249 0.1251 0.1253 0.1255 0.1261 0.1267 0.1287 0.1295 0.1297
[141] 0.1305 0.1309 0.1321 0.1323 0.1329 0.1330 0.1332 0.1344 0.1346 0.1347
[151] 0.1377 0.1409 0.1427 0.1433 0.1442 0.1444 0.1445 0.1457 0.1485 0.1493
[161] 0.1505 0.1557 0.1567 0.1585 0.1623 0.1635 0.1645 0.1647 0.1649 0.1671
[171] 0.1677 0.1699 0.1729 0.1733 0.1746 0.1748 0.1751 0.1760 0.1762 0.1765
[181] 0.1767 0.1781 0.1795 0.1797 0.1807 0.1835 0.1853 0.1857 0.1861 0.1867
[191] 0.1875 0.1883 0.1888 0.1890 0.1899 0.1925 0.1957 0.1965 0.1971 0.1974
[201] 0.1976 0.1981 0.1989 0.1997 0.2001 0.2029 0.2031 0.2067 0.2082 0.2084
[211] 0.2084 0.2086 0.2101 0.2121 0.2129 0.2143 0.2157 0.2165 0.2167 0.2185
[221] 0.2201 0.2210 0.2212 0.2219 0.2221 0.2235 0.2241 0.2243 0.2265 0.2317
[231] 0.2327 0.2331 0.2341 0.2373 0.2383 0.2387 0.2393 0.2407 0.2409 0.2411
[241] 0.2435 0.2445 0.2451 0.2461 0.2465 0.2489 0.2549 0.2578 0.2580 0.2593
[251] 0.2605 0.2621 0.2623 0.2639 0.2655 0.2683 0.2685 0.2699 0.2703 0.2707
[261] 0.2717 0.2728 0.2730 0.2737 0.2743 0.2747 0.2761 0.2763 0.2765 0.2783
[271] 0.2785 0.2811 0.2821 0.2831 0.2843 0.2849 0.2853 0.2859 0.2863 0.2869
[281] 0.2893 0.2919 0.2933 0.2941 0.2947 0.2963 0.2969 0.2980 0.2982 0.2983
[291] 0.2989 0.3007 0.3013 0.3033 0.3039 0.3041 0.3053 0.3057 0.3064 0.3066
[301] 0.3073 0.3095 0.3105 0.3157 0.3172 0.3174 0.3191 0.3197 0.3243 0.3245
[311] 0.3255 0.3261 0.3271 0.3281 0.3287 0.3297 0.3319 0.3325 0.3331 0.3333
[321] 0.3359 0.3361 0.3385 0.3386 0.3388 0.3391 0.3403 0.3407 0.3409 0.3413
[331] 0.3433 0.3443 0.3444 0.3446 0.3451 0.3459 0.3462 0.3464 0.3483 0.3489
[341] 0.3513 0.3516 0.3518 0.3527 0.3529 0.3531 0.3559 0.3567 0.3579 0.3583
[351] 0.3596 0.3598 0.3603 0.3604 0.3606 0.3649 0.3681 0.3705 0.3713 0.3724
[361] 0.3726 0.3729 0.3764 0.3766 0.3775 0.3787 0.3793 0.3795 0.3807 0.3818

[371] 0.3820 0.3847 0.3863 0.3865 0.3905 0.3907 0.3919 0.3927 0.3931 0.3944
[381] 0.3946 0.3965 0.3967 0.3969 0.3979 0.3989 0.4013 0.4023 0.4041 0.4053
[391] 0.4063 0.4069 0.4075 0.4077 0.4079 0.4081 0.4101 0.4103 0.4109 0.4123
[401] 0.4127 0.4145 0.4181 0.4189 0.4193 0.4197 0.4203 0.4205 0.4213 0.4243
[411] 0.4250 0.4252 0.4263 0.4285 0.4287 0.4311 0.4315 0.4329 0.4365 0.4367
[421] 0.4377 0.4381 0.4385 0.4411 0.4413 0.4417 0.4419 0.4421 0.4433 0.4453
[431] 0.4471 0.4473 0.4479 0.4501 0.4525 0.4531 0.4535 0.4555 0.4583 0.4585
[441] 0.4593 0.4603 0.4647 0.4679 0.4681 0.4713 0.4723 0.4735 0.4737 0.4753
[451] 0.4755 0.4763 0.4768 0.4770 0.4783 0.4797 0.4803 0.4811 0.4817 0.4829
[461] 0.4835 0.4849 0.4887 0.4889 0.4891 0.4919 0.4921 0.4935 0.4957 0.4969
[471] 0.4981 0.4983 0.5001 0.5015 0.5021 0.5029 0.5033 0.5041 0.5045 0.5055
[481] 0.5061 0.5089 0.5117 0.5151 0.5171 0.5175 0.5181 0.5183 0.5209 0.5223
[491] 0.5249 0.5253 0.5273 0.5281 0.5297 0.5303 0.5314 0.5316 0.5321 0.5325
[501] 0.5341 0.5359 0.5361 0.5365 0.5387 0.5389 0.5397 0.5414 0.5416 0.5423
[511] 0.5477 0.5480 0.5482 0.5485 0.5492 0.5494 0.5495 0.5519 0.5531 0.5537
[521] 0.5539 0.5541 0.5548 0.5550 0.5561 0.5575 0.5585 0.5601 0.5604 0.5606
[531] 0.5625 0.5627 0.5633 0.5641 0.5645 0.5653 0.5659 0.5673 0.5679 0.5683
[541] 0.5701 0.5709 0.5713 0.5755 0.5759 0.5769 0.5771 0.5777 0.5781 0.5787
[551] 0.5789 0.5815 0.5819 0.5829 0.5843 0.5849 0.5852 0.5854 0.5855 0.5887
[561] 0.5891 0.5897 0.5911 0.5925 0.5931 0.5935 0.5943 0.5949 0.5971 0.5993
[571] 0.5995 0.5999 0.6011 0.6015 0.6017 0.6019 0.6035 0.6037 0.6069 0.6077
[581] 0.6089 0.6126 0.6128 0.6135 0.6139 0.6163 0.6165 0.6175 0.6197 0.6200
[591] 0.6202 0.6217 0.6225 0.6227 0.6251 0.6257 0.6313 0.6321 0.6343 0.6359
[601] 0.6361 0.6367 0.6375 0.6393 0.6397 0.6403 0.6415 0.6489 0.6491 0.6493
[611] 0.6495 0.6505 0.6513 0.6515 0.6523 0.6525 0.6537 0.6541 0.6545 0.6549
[621] 0.6565 0.6573 0.6579 0.6589 0.6601 0.6603 0.6621 0.6645 0.6661 0.6668
[631] 0.6670 0.6671 0.6699 0.6719 0.6721 0.6732 0.6734 0.6735 0.6749 0.6841
[641] 0.6855 0.6859 0.6873 0.6879 0.6881 0.6889 0.6893 0.6913 0.6929 0.6937
[651] 0.6945 0.6947 0.6949 0.6951 0.6963 0.6973 0.6979 0.6983 0.6997 0.6999
[661] 0.7007 0.7009 0.7015 0.7039 0.7067 0.7071 0.7077 0.7079 0.7081 0.7083
[671] 0.7099 0.7105 0.7135 0.7143 0.7151 0.7171 0.7173 0.7175 0.7177 0.7179
[681] 0.7225 0.7227 0.7257 0.7265 0.7273 0.7283 0.7289 0.7311 0.7327 0.7347
[691] 0.7349 0.7367 0.7377 0.7384 0.7386 0.7390 0.7392 0.7433 0.7437 0.7439
[701] 0.7461 0.7475 0.7480 0.7482 0.7485 0.7496 0.7498 0.7502 0.7504 0.7508
[711] 0.7510 0.7511 0.7541 0.7553 0.7555 0.7573 0.7581 0.7605 0.7606 0.7608

[721] 0.7609 0.7627 0.7628 0.7630 0.7641 0.7680 0.7682 0.7700 0.7702 0.7705
[731] 0.7706 0.7708 0.7713 0.7715 0.7725 0.7731 0.7737 0.7739 0.7773 0.7787
[741] 0.7797 0.7811 0.7823 0.7825[+] 0.7843 0.7849 0.7851 0.7855 0.7857 0.7859
[751] 0.7867 0.7871 0.7887 0.7895 0.7903 0.7905 0.7921 0.7933 0.7937 0.7949
[761] 0.7957 0.7967 0.7973 0.7993 0.7999 0.8009 0.8011 0.8013 0.8021 0.8039
[771] 0.8045 0.8047 0.8069 0.8081 0.8085 0.8119 0.8151 0.8161 0.8164 0.8166
[781] 0.8166 0.8168 0.8205 0.8215 0.8225 0.8241 0.8245 0.8267 0.8279 0.8288
[791] 0.8290 0.8303 0.8305 0.8307 0.8310 0.8312 0.8317 0.8325 0.8359 0.8379
[801] 0.8383 0.8391 0.8393 0.8397 0.8403 0.8409 0.8417 0.8431 0.8433 0.8439
[811] 0.8441 0.8449 0.8453 0.8477 0.8483 0.8485 0.8489 0.8493 0.8503 0.8505
[821] 0.8513 0.8517 0.8527 0.8535 0.8536 0.8538 0.8565 0.8569 0.8573 0.8594
[831] 0.8596 0.8597 0.8659 0.8665 0.8667 0.8695 0.8737 0.8739 0.8745 0.8751
[841] 0.8753 0.8761 0.8767 0.8773 0.8791 0.8793 0.8816 0.8818 0.8821 0.8826
[851] 0.8828 0.8829 0.8835 0.8911 0.8923 0.8959 0.8965 0.8967 0.8971 0.8977
[861] 0.8979 0.8983 0.8989 0.8997 0.9007 0.9017 0.9020 0.9022 0.9033 0.9041
[871] 0.9073 0.9085 0.9093 0.9103 0.9115 0.9117 0.9119 0.9125 0.9135 0.9147
[881] 0.9161 0.9163 0.9191 0.9207 0.9211 0.9221 0.9231 0.9235 0.9241 0.9247
[891] 0.9255 0.9257 0.9259 0.9273 0.9289 0.9290 0.9292 0.9297 0.9306 0.9308
[901] 0.9319 0.9327 0.9333 0.9335 0.9359 0.9363 0.9367 0.9368 0.9370 0.9373
[911] 0.9380 0.9382 0.9405 0.9417 0.9421 0.9439 0.9451 0.9453 0.9459 0.9460
[921] 0.9462 0.9471 0.9495 0.9505 0.9507 0.9515 0.9537 0.9553 0.9558 0.9560
[931] 0.9569 0.9571 0.9585 0.9587 0.9609 0.9619 0.9622 0.9624 0.9629 0.9647
[941] 0.9665 0.9667 0.9669 0.9675 0.9703 0.9717 0.9721 0.9731 0.9739 0.9751
[951] 0.9767 0.9771 0.9795 0.9851 0.9873 0.9875 0.9885 0.9891 0.9895 0.9897
[961] 0.9909 0.9917 0.9929 0.9945 0.9953 0.9959 0.9987 1.0015 1.0019 1.0037
[971] 1.0062 1.0064 1.0065 1.0073 1.0101 1.0111 1.0127 1.0155 1.0185 1.0187
[981] 1.0211 1.0214 1.0216 1.0235 1.0246 1.0248 1.0251 1.0255 1.0263 1.0271
[991] 1.0275 1.0281 1.0315 1.0317 1.0325 1.0331 1.0341 1.0345 1.0347 1.0349
[1001] 1.0369 1.0371 1.0379 1.0401 1.0403 1.0409 1.0415 1.0425 1.0427 1.0429
[1011] 1.0437 1.0443 1.0451 1.0467 1.0497 1.0519 1.0521 1.0524 1.0526 1.0529
[1021] 1.0539 1.0553 1.0555 1.0560 1.0562 1.0563 1.0567 1.0581 1.0591 1.0601
[1031] 1.0607 1.0609 1.0611 1.0613 1.0623 1.0627 1.0637 1.0643 1.0663 1.0665
[1041] 1.0669 1.0675 1.0681 1.0685 1.0693 1.0699 1.0733 1.0777 1.0783 1.0785

[+] Note: $j_1 = 744$ and $s_1 = 0.783$.

[1051] 1.0795 1.0797 1.0805 1.0825 1.0843 1.0845 1.0849 1.0865 1.0874 1.0876
[1061] 1.0877 1.0879 1.0891 1.0917 1.0931 1.0959 1.0968 1.0970 1.0981 1.0997
[1071] 1.1007 1.1009 1.1023 1.1033 1.1043 1.1049 1.1051 1.1067 1.1069 1.1071
[1081] 1.1091 1.1095 1.1101 1.1105 1.1107 1.1149 1.1151 1.1159 1.1165 1.1177
[1091] 1.1179 1.1191 1.1197 1.1199 1.1205 1.1207 1.1211 1.1213 1.1224 1.1226
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[2861] 2.6315 2.6319 2.6331 2.6361 2.6369 2.6379 2.6387 2.6388 2.6390 2.6397

[2871] 2.6407 2.6415 2.6431 2.6441 2.6451 2.6453 2.6463 2.6465 2.6469 2.6477

[2881] 2.6481 2.6501 2.6515 2.6519 2.6521 2.6523 2.6541 2.6545 2.6551 2.6555

[2891] 2.6581 2.6599 2.6637 2.6675 2.6683 2.6689 2.6695 2.6699 2.6713 2.6721

[2901] 2.6727 2.6733 2.6743 2.6753 2.6773 2.6777 2.6781 2.6797 2.6811 2.6829

[2911] 2.6841 2.6843 2.6847 2.6855 2.6875 2.6901 2.6915 2.6931 2.6933 2.6935

[2921] 2.6937 2.6943 2.6965 2.6979 2.6985 2.7029 2.7031 2.7053 2.7059 2.7061

[2931] 2.7067 2.7068 2.7070 2.7071 2.7086 2.7088 2.7091 2.7109 2.7116 2.7118

[2941] 2.7137 2.7147 2.7155 2.7159 2.7161 2.7175 2.7190 2.7192 2.7193 2.7203

[2951] 2.7205 2.7207 2.7209 2.7213 2.7231 2.7236 2.7238 2.7261 2.7265 2.7283

[2961] 2.7293 2.7295 2.7303 2.7311 2.7329 2.7348 2.7350 2.7357 2.7369 2.7379

[2971] 2.7381 2.7397 2.7413 2.7433 2.7443 2.7465 2.7469 2.7481 2.7495 2.7497

[2981] 2.7505 2.7525 2.7537 2.7545 2.7561 2.7581 2.7629 2.7639 2.7655 2.7677

[2991] 2.7685 2.7703 2.7711 2.7733 2.7741 2.7743 2.7745 2.7763 2.7779 2.7788

[3001] 2.7790 2.7795 2.7807 2.7809 2.7811 2.7817 2.7827 2.7831 2.7843 2.7871

[3011] 2.7887 2.7893 2.7933 2.7935+ 2.7943 2.7945 2.7947 2.7960 2.7962 2.7971

[3021] 2.7973 2.7991 2.7995 2.8017 2.8025 2.8029 2.8033 2.8035 2.8058 2.8060

[3031] 2.8073 2.8083 2.8091 2.8095 2.8103 2.8109 2.8119 2.8125 2.8131 2.8135

[3041] 2.8139 2.8153 2.8165 2.8169 2.8195 2.8201 2.8204 2.8206 2.8207 2.8211

[3051] 2.8219 2.8233 2.8237 2.8241 2.8245 2.8249 2.8293 2.8299 2.8309 2.8325

[3061] 2.8335 2.8337 2.8351 2.8387 2.8391 2.8397 2.8399 2.8405 2.8410 2.8412

[3071] 2.8435 2.8437 2.8439 2.8449 2.8451 2.8456 2.8458 2.8467 2.8477 2.8535

[3081] 2.8545 2.8575 2.8577 2.8579 2.8597 2.8623 2.8625 2.8629 2.8635 2.8649

[3091] 2.8655 2.8657 2.8661 2.8670 2.8671 2.8672 2.8677 2.8683 2.8685 2.8687

[3101] 2.8701 2.8731 2.8737 2.8746 2.8748 2.8767 2.8775 2.8791 2.8803 2.8827

+ Note: $j_3 = 3014$ and $s_3 = 2.794$.

[3111] 2.8845 2.8851 2.8853 2.8857 2.8865 2.8891 2.8907 2.8917 2.8927 2.8929
[3121] 2.8930 2.8932 2.8935 2.8951 2.8959 2.8961 2.8967 2.8977 2.8978 2.8980
[3131] 2.8981 2.8983 2.8992 2.8994 2.9021 2.9025 2.9029 2.9033 2.9061 2.9065
[3141] 2.9069 2.9079 2.9089 2.9095 2.9099 2.9107 2.9109 2.9115 2.9123 2.9135
[3151] 2.9153 2.9157 2.9163 2.9179 2.9187 2.9190 2.9192 2.9203 2.9213 2.9217
[3161] 2.9221 2.9237 2.9241 2.9249 2.9265 2.9271 2.9283 2.9287 2.9293 2.9309
[3171] 2.9311 2.9313 2.9318 2.9320 2.9321 2.9322 2.9324 2.9341 2.9351 2.9353
[3181] 2.9355 2.9375 2.9379 2.9387 2.9389 2.9395 2.9415 2.9421 2.9433 2.9435
[3191] 2.9451 2.9453 2.9455 2.9463 2.9465 2.9467 2.9485 2.9503 2.9505 2.9507
[3201] 2.9509 2.9515 2.9521 2.9541 2.9545 2.9547 2.9559 2.9569 2.9581 2.9589
[3211] 2.9604 2.9606 2.9622 2.9624 2.9639 2.9653 2.9655 2.9659 2.9661 2.9679
[3221] 2.9687 2.9711 2.9713 2.9715 2.9735 2.9737 2.9741 2.9743 2.9748 2.9750
[3231] 2.9759 2.9781 2.9795 2.9797 2.9807 2.9817 2.9823 2.9829 2.9833 2.9839
[3241] 2.9849 2.9855 2.9859 2.9867 2.9875 2.9881 2.9906 2.9908 2.9919 2.9923
[3251] 2.9949 2.9954 2.9956 2.9957 2.9963 2.9975 2.9991 3.0017 3.0019 3.0023
[3261] 3.0029 3.0041 3.0050 3.0052 3.0073 3.0081 3.0093 3.0097 3.0111 3.0131
[3271] 3.0167 3.0173 3.0177 3.0191 3.0209 3.0233 3.0241 3.0269 3.0285 3.0287
[3281] 3.0295 3.0297 3.0298 3.0300 3.0311 3.0317 3.0321 3.0325 3.0327 3.0331
[3291] 3.0341 3.0343 3.0347 3.0357 3.0365 3.0381 3.0383 3.0397 3.0399 3.0429
[3301] 3.0439 3.0449 3.0451 3.0473 3.0477 3.0483 3.0491 3.0492 3.0494 3.0497
[3311] 3.0503 3.0505 3.0513 3.0523 3.0527 3.0539 3.0549 3.0553 3.0559 3.0565
[3321] 3.0575 3.0579 3.0585 3.0594 3.0595 3.0596 3.0603 3.0613 3.0617 3.0633
[3331] 3.0638 3.0640 3.0644 3.0646 3.0660 3.0662 3.0665 3.0681 3.0687 3.0691
[3341] 3.0694 3.0696 3.0707 3.0719 3.0720 3.0722 3.0723 3.0727 3.0735 3.0763
[3351] 3.0773 3.0775 3.0783 3.0791 3.0803 3.0809 3.0811 3.0813 3.0837 3.0849
[3361] 3.0857 3.0859 3.0871 3.0893 3.0897 3.0905 3.0915 3.0916 3.0918 3.0922
[3371] 3.0924 3.0929 3.0943 3.0959 3.0963 3.0971 3.0975 3.0991 3.0999 3.1001
[3381] 3.1002 3.1004 3.1008 3.1010 3.1027 3.1069 3.1075 3.1099 3.1105 3.1139
[3391] 3.1143 3.1153 3.1165 3.1173 3.1181 3.1182 3.1184 3.1199 3.1214 3.1216
[3401] 3.1223 3.1250 3.1252 3.1253 3.1273 3.1281 3.1283 3.1287 3.1289 3.1295
[3411] 3.1296 3.1298 3.1300 3.1302 3.1332 3.1334 3.1334 3.1336 3.1345 3.1391
[3421] 3.1397 3.1419 3.1425 3.1433 3.1520 3.1522 3.1533 3.1535 3.1543 3.1549
[3431] 3.1581 3.1586 3.1588 3.1592 3.1594 3.1595 3.1612 3.1614 3.1631 3.1634
[3441] 3.1636 3.1640 3.1642 3.1642 3.1644 3.1653 3.1663 3.1679 3.1701 3.1705
[3451] 3.1721 3.1723 3.1731 3.1747 3.1759 3.1763 3.1765 3.1793 3.1795 3.1797

[3461] 3.1799 3.1807 3.1817 3.1821 3.1829 3.1841 3.1845 3.1847 3.1849 3.1865
[3471] 3.1866 3.1868 3.1885 3.1892 3.1894 3.1909 3.1933 3.1939 3.1953 3.1957
[3481] 3.1963 3.1979 3.1985 3.1987 3.2007 3.2019 3.2029 3.2031 3.2054 3.2056
[3491] 3.2065 3.2089 3.2093 3.2099 3.2115 3.2117 3.2118 3.2120 3.2125 3.2141
[3501] 3.2155 3.2157 3.2167 3.2181 3.2189 3.2195 3.2197 3.2203 3.2214 3.2216
[3511] 3.2219 3.2222 3.2224 3.2229 3.2231 3.2235 3.2254 3.2256 3.2273 3.2275
[3521] 3.2297 3.2305 3.2314 3.2316 3.2325 3.2329 3.2333 3.2367 3.2375 3.2377
[3531] 3.2383 3.2385 3.2403 3.2409 3.2413 3.2417 3.2423 3.2426 3.2428 3.2449
[3541] 3.2457 3.2459 3.2473 3.2489 3.2495 3.2503 3.2504 3.2506 3.2519 3.2525
[3551] 3.2534 3.2536 3.2537 3.2543 3.2550 3.2552 3.2559 3.2582 3.2584 3.2587
[3561] 3.2593 3.2597 3.2619 3.2631 3.2633 3.2634 3.2636 3.2653 3.2668 3.2670
[3571] 3.2677 3.2685 3.2693 3.2697 3.2709 3.2711 3.2715 3.2717 3.2721 3.2773
[3581] 3.2775 3.2783 3.2804 3.2806 3.2817 3.2837 3.2866 3.2868 3.2871 3.2877
[3591] 3.2893 3.2901 3.2903 3.2915 3.2925 3.2928 3.2930 3.2951 3.2955 3.2973
[3601] 3.2981 3.2989 3.2997 3.3023 3.3039 3.3047 3.3057 3.3059 3.3061 3.3063
[3611] 3.3077 3.3079 3.3107 3.3108 3.3110 3.3125 3.3141 3.3143 3.3161 3.3197
[3621] 3.3201 3.3203 3.3215 3.3219 3.3239 3.3244 3.3246 3.3258 3.3260 3.3269
[3631] 3.3281 3.3287 3.3301 3.3308 3.3310 3.3319 3.3321 3.3323 3.3337 3.3350
[3641] 3.3352 3.3365 3.3375 3.3379 3.3383 3.3385 3.3417 3.3423 3.3431 3.3435
[3651] 3.3438 3.3440 3.3441 3.3444 3.3446 3.3450 3.3452 3.3468 3.3470 3.3487
[3661] 3.3505 3.3507 3.3511 3.3531 3.3547 3.3553 3.3557 3.3561 3.3575 3.3577
[3671] 3.3581 3.3583 3.3597 3.3599 3.3600 3.3602 3.3604 3.3606 3.3625 3.3643
[3681] 3.3655 3.3675 3.3677 3.3689 3.3693 3.3703 3.3705 3.3711 3.3725 3.3744
[3691] 3.3746 3.3789 3.3793 3.3809 3.3821 3.3833 3.3839 3.3845 3.3851 3.3855
[3701] 3.3861 3.3869 3.3871 3.3875 3.3899 3.3939 3.3947 3.3953 3.3959 3.3964
[3711] 3.3966 3.3967 3.3975 3.3979 3.3980 3.3982 3.3985 3.4027 3.4033 3.4041
[3721] 3.4047 3.4051 3.4053 3.4075 3.4077 3.4079 3.4080 3.4082 3.4091 3.4105
[3731] 3.4107 3.4114 3.4116 3.4125 3.4141 3.4151 3.4157 3.4161 3.4166 3.4168
[3741] 3.4171 3.4179 3.4191 3.4199 3.4203 3.4205 3.4219 3.4238 3.4240 3.4245
[3751] 3.4253 3.4259 3.4261 3.4265 3.4273 3.4291 3.4297 3.4301 3.4303 3.4319
[3761] 3.4325 3.4327 3.4337 3.4351 3.4353 3.4365 3.4386 3.4388 3.4395 3.4403
[3771] 3.4411 3.4419 3.4433 3.4451 3.4463 3.4473 3.4479 3.4480 3.4482 3.4487
[3781] 3.4497 3.4503 3.4517 3.4519 3.4521 3.4530 3.4532 3.4553 3.4555 3.4561
[3791] 3.4617 3.4647 3.4651 3.4677 3.4679 3.4689 3.4693 3.4707 3.4713 3.4715
[3801] 3.4717 3.4727 3.4733 3.4753 3.4803 3.4809 3.4814 3.4816 3.4840 3.4842

```
[3811] 3.4847 3.4849 3.4853 3.4867 3.4869 3.4871 3.4873 3.4884 3.4886 3.4901
[3821] 3.4907 3.4917 3.4929 3.4942 3.4944 3.4947 3.4965 3.4989 3.4993 3.4994
[3831] 3.4996 3.4996 3.4998 3.5015 3.5047 3.5055 3.5071 3.5077 3.5089 3.5091
[3841] 3.5105 3.5121 3.5151 3.5169 3.5183 3.5195 3.5198 3.5200 3.5203 3.5205
[3851] 3.5207 3.5211 3.5213 3.5225 3.5261 3.5269 3.5273 3.5289 3.5313 3.5327
[3861] 3.5329 3.5343 3.5357 3.5371 3.5375 3.5389 3.5417 3.5427 3.5433 3.5435
[3871] 3.5445 3.5451 3.5467 3.5483 3.5487 3.5501 3.5506 3.5508 3.5523 3.5535
[3881] 3.5539 3.5550 3.5552 3.5565 3.5566 3.5568 3.5615 3.5617 3.5627 3.5643
[3891] 3.5645 3.5653 3.5657 3.5661 3.5663 3.5669 3.5675 3.5683 3.5691 3.5695
[3901] 3.5703 3.5707 3.5723 3.5743 3.5751 3.5757 3.5767 3.5775 3.5787 3.5805
[3911] 3.5807 3.5809 3.5831 3.5845 3.5847 3.5848 3.5850 3.5863 3.5883 3.5887
[3921] 3.5915 3.5921 3.5925 3.5927 3.5939 3.5949 3.5953 3.5963 3.5965 3.5975
[3931] 3.5977 3.5983 3.5989 3.6002 3.6004 3.6009 3.6023 3.6031 3.6055 3.6070
[3941] 3.6072 3.6075 3.6081 3.6082 3.6084 3.6099 3.6119 3.6127 3.6147 3.6161
[3951] 3.6163 3.6167 3.6171 3.6175 3.6225 3.6227 3.6229 3.6233 3.6243 3.6249
[3961] 3.6251 3.6255 3.6257 3.6269 3.6287 3.6289 3.6301 3.6321 3.6335 3.6343
[3971] 3.6356 3.6358 3.6359 3.6361 3.6368 3.6370 3.6371 3.6377 3.6383 3.6388
[3981] 3.6390 3.6399 3.6405 3.6417 3.6429 3.6439 3.6451 3.6487 3.6497 3.6509
[3991] 3.6513 3.6524 3.6526 3.6537 3.6559 3.6565 3.6567 3.6579 3.6587 3.6599
[4001] 3.6603 3.6605 3.6607 3.6631 3.6637 3.6643 3.6650 3.6652 3.6659 3.6665
[4011] 3.6667 3.6669 3.6679 3.6685 3.6689 3.6699 3.6701 3.6705 3.6732 3.6734
[4021] 3.6737 3.6743 3.6750 3.6752 3.6755 3.6759 3.6761 3.6771 3.6791 3.6811
[4031] 3.6847 3.6861 3.6879 3.6885 3.6899 3.6907 3.6935 3.6945 3.6953 3.6969
[4041] 3.6973 3.6985 3.6993 3.6995 3.7001 3.7009 3.7011 3.7029 3.7043 3.7056
[4051] 3.7058 3.7073 3.7085 3.7089 3.7095 3.7103 3.7113 3.7115 3.7135 3.7155
[4061] 3.7157 3.7159 3.7167 3.7181 3.7185 3.7213 3.7215 3.7223 3.7233 3.7234
[4071] 3.7236 3.7253 3.7263 3.7269 3.7271 3.7283 3.7287 3.7297 3.7307+
```

4.2 Estimation

```
> tpoi1<-1:744
> for (i in 1:744) {tpoi1[i]<-tpoi[i]}
```

+ Note: $j_4 = 4079$ and $T = 3.732$.

```
> tpoi2<-1:491
> for(i in 1:491) {tpoi2[i]<-tpoi[i+744]}

> tpoi3<-1:1779
> for(i in 1:1779) {tpoi3[i]<-tpoi[i+1235]}

> tpoi4<-1:1065
> for(i in 1:1065) {tpoi4[i]<-tpoi[i+3014]}

> stp1<-sum(tpoi1)
> stp1
[1] 289.0196

> stp2<-sum(tpoi2-tttt[1])
> stp2
[1] 115.8695

> stp3<-sum(tpoi3-tttt[2])
> stp3
[1] 1348.608

> stp4<-sum(tpoi4-tttt[3])
> stp4
[1] 491.4479

> f1
function(d1)
{
	{a1 <- 4079/d1
	aa1 <- (744 * 0.783 * exp(-0.783 * d1))/(1 - exp(-0.783 * d1))
	aa2 <- (491 * 0.467 * exp(-0.467 * d1))/(1 - exp(-0.467 * d1))
	aa3 <- (1779 * 1.544 * exp(-1.544 * d1))/(1 - exp(-1.544 * d1))
	aa4 <- (1065 * 0.938 * exp(-0.938 * d1))/(1 - exp(-0.938 * d1))
	a2 <- stp1 + stp2 + stp3 + stp4
	z1 <- a1 - aa1 - aa2 - aa3 - aa4 - a2
```

```
}
        return(z1)
}

> d1
[1] 0.08 0.09 0.10 0.11 0.12

> f1(0.070768653)
[1] 1.118724e-07

> yy0
[1] 1000

> yy1
[1] 1024.994

> yy2
[1] 1302.009

> yy3
[1] 1230.569

> l0<-(dd1*(jj1-jj0))/(1-exp(-dd1*(s1-s0)))
> l0
[1] 976.7606

> l1<-(dd1*(jj2-jj1))/(1-exp(-dd1*(s2-s1)))
> l1
[1] 1068.861

> l2<-(dd1*(jj3-jj2))/(1-exp(-dd1*(s3-s2)))
> l2
[1] 1216.297
```

```
> l3<-(dd1*(jj4-jj3))/(1-exp(-dd1*(s4-s3)))
> l3
[1] 1173.496

> v0<-(1-exp(-dd1*(s1-s0)))^2/(dd1^2*(jj1-jj0))
> v0
[1] 0.0007798241

> v1<-(1-exp(-dd1*(s2-s1)))^2/(dd1^2*(jj2-jj1))
> v1
[1] 0.0004297727

> v2<-(1-exp(-dd1*(s3-s2)))^2/(dd1^2*(jj3-jj2))
> v2
[1] 0.001202532

> v3<-(1-exp(-dd1*(s4-s3)))^2/(dd1^2*(jj4-jj3))
> v3
[1] 0.0007733688

> co0<--((1-exp(-dd1*(s1-s0)))/dd1^2-((s1-s0)*exp(-dd1*(s1-s0))/dd1))
> co0
[1] -0.2954522

> co1<--((1-exp(-dd1*(s2-s1)))/dd1^2-((s2-s1)*exp(-dd1*(s2-s1))/dd1))
> co1
[1] -0.1066715

> co2<--((1-exp(-dd1*(s3-s2)))/dd1^2-((s3-s2)*exp(-dd1*(s3-s2))/dd1))
> co2
[1] -1.108596

> co3<--((1-exp(-dd1*(s4-s3)))/dd1^2-((s4-s3)*exp(-dd1*(s4-s3))/dd1))
> co3
[1] -0.4209299
```

```
> v4<--(((jj1-jj0)*(s1-s0)*(2/dd1+s1-s0)*exp(-dd1*(s1-s0))/(1-exp(-dd1*(s1-s0))))
 ++((jj2-jj1)*(s2-s1)*(2/dd1+s2-s1)*exp(-dd1*(s2-s1))/(1-exp(-dd1*(s2-s1))))
 ++((jj3-jj2)*(s3-s2)*(2/dd1+s3-s2)*exp(-dd1*(s3-s2))/(1-exp(-dd1*(s3-s2))))
 ++((jj4-jj3)*(s4-s3)*(2/dd1+s4-s3)*exp(-dd1*(s4-s3))/(1-exp(-dd1*(s4-s3))))
 +-(2*jj4/dd1^2))
> v4
[1] 1867.722

> matr<-matrix(matr,5,5)
> matr
              [,1]          [,2]        [,3]          [,4]         [,5]
[1,]  0.0007798241  0.0000000000  0.000000000  0.0000000000   -0.2954522
[2,]  0.0000000000  0.0004297727  0.000000000  0.0000000000   -0.1066715
[3,]  0.0000000000  0.0000000000  0.001202532  0.0000000000   -1.1085961
[4,]  0.0000000000  0.0000000000  0.000000000  0.0007733688   -0.4209299
[5,] -0.2954522389 -0.1066714766 -1.108596093 -0.4209298537 1867.7215633

> cova<-solve(matr)
> cova
             [,1]         [,2]        [,3]        [,4]        [,5]
[1,] 1582.5102371  196.6463007  730.386784  431.220483 0.792275617
[2,]  196.6463007 2455.6377323  478.488780  282.499858 0.519033205
[3,]  730.386784   478.488780  2608.789093 1049.265418 1.927801296
[4,]  431.220483   282.499858  1049.265418 1912.530661 1.138174217
[5,]    0.792275617  0.519033205  1.927801296  1.138174217 0.002091152

> yye1<-l1-l0*exp(-dd1*s1)
> yye1
[1] 144.7525

> yye2<-l2-l1*exp(-dd1*s2)
> yye2
[1] 237.9266
```

```
> yye3<-l3-l2*exp(-dd1*s3)
> yye3
[1] 175.4105

> (yye1+yye2+yye3)/3
[1] 186.0299
```

5. The calculations and *S-Plus* routine for example 5.1.1

5.1 Simulation

```
> te<-rexp(150,100)
> tte<-1:100
> for (i in 1:100) {tte[i]<-te[i]}

> tte<-round(tte,4)
> tte
  [1] 0.0122 0.0040 0.0099 0.0035 0.0031 0.0091 0.0033 0.0710 0.0033 0.0125
 [11] 0.0024 0.0222 0.0134 0.0057 0.0237 0.0063 0.0212 0.0026 0.0142 0.0195
 [21] 0.0001 0.0066 0.0037 0.0089 0.0010 0.0058 0.0045 0.0059 0.0188 0.0158
 [31] 0.0101 0.0012 0.0387 0.0031 0.0026 0.0069 0.0072 0.0051 0.0032 0.0144
 [41] 0.0004 0.0128 0.0102 0.0001 0.0058 0.0238 0.0037 0.0032 0.0181 0.0159
 [51] 0.0027 0.0409 0.0045 0.0162 0.0228 0.0120 0.0009 0.0059 0.0150 0.0042
 [61] 0.0019 0.0126 0.0208 0.0023 0.0080 0.0051 0.0067 0.0073 0.0018 0.0118
 [71] 0.0060 0.0079 0.0167 0.0079 0.0013 0.0060 0.0263 0.0121 0.0055 0.0051
 [81] 0.0113 0.0037 0.0119 0.0064 0.0367 0.0033 0.0011 0.0014 0.0042 0.0347
 [91] 0.0019 0.0120 0.0029 0.0005 0.0043 0.0079 0.0033 0.0086 0.0022 0.0244

> cumsum(tte)
  [1] 0.0122 0.0162 0.0261 0.0296 0.0327 0.0418 0.0451 0.1161 0.1194 0.1319
 [11] 0.1343 0.1565 0.1699 0.1756 0.1993 0.2056 0.2268 0.2294 0.2436 0.2631
 [21] 0.2632 0.2698 0.2735 0.2824 0.2834 0.2892 0.2937 0.2996 0.3184 0.3342
 [31] 0.3443 0.3455 0.3842 0.3873 0.3899 0.3968 0.4040 0.4091 0.4123 0.4267
 [41] 0.4271 0.4399 0.4501 0.4502 0.4560 0.4798 0.4835 0.4867 0.5048 0.5207
 [51] 0.5234 0.5643 0.5688 0.5850 0.6078 0.6198 0.6207 0.6266 0.6416 0.6458
 [61] 0.6477 0.6603 0.6811 0.6834 0.6914 0.6965 0.7032 0.7105 0.7123 0.7241
 [71] 0.7301 0.7380 0.7547 0.7626 0.7639 0.7699 0.7962 0.8083 0.8138 0.8189
 [81] 0.8302 0.8339 0.8458 0.8522 0.8889 0.8922 0.8933 0.8947 0.8989 0.9336
 [91] 0.9355 0.9475 0.9504 0.9509 0.9552 0.9631 0.9664 0.9750 0.9772 1.0016

> y<-rexp(100,1)
> y<-array(y)
```

```
> y
  [1] 2.844531 0.6437058 0.1532211 0.1419299 2.136895  0.9931174 0.6334041
  [8] 3.089103 0.846644 0.1783297 0.0999909 0.3013416 0.04383538 0.2155333
 [15] 2.575489 0.562017 1.639105 1.942076 0.5436354 0.004495321 0.006376755
 [22] 0.7757282 1.028568 0.6781959 2.755949 0.3756035 1.400532 0.3567848
 [29] 0.479679 0.5749752 3.214524 1.117339 0.008017103 0.4736012 0.2124399
 [36] 3.445639 1.455314 0.3378651 0.2721739 0.000110719 1.136713 0.02692588
 [43] 1.316751 2.508186 0.1748482 0.4527609 1.515758 0.6491918 1.393431
 [50] 1.750964 0.3921691 1.440703 3.064295 1.212635 0.1391625 3.968223
 [57] 0.1834681 0.3241229 0.06615166 3.33812 0.1966045 0.857705 0.7827508
 [64] 1.444248 1.69832 0.538809 2.462212 3.799613 0.1554118 1.029232
 [71] 0.5837805 0.6370158 2.938962 0.630229 0.198576 1.973318 0.1792413
 [78] 1.084332 0.6133536 0.4536421 1.173732 7.341652 2.585211 0.1193903
 [85] 0.02439253 1.11829 1.92124 1.712752 0.6441057 2.026121 0.9926892
 [92] 0.2372974 0.06095418 0.1912166 1.051397 0.5888466 3.651955 1.472982
 [99] 1.738937 1.762146

> y0
[1] 200
> d
[1] 0.5
> B0<-((1-exp(- d * tte[1]))/d)
> B0
[1] 0.01216287
> inten0<-y0*B0
> inten0
[1] 2.432573
> N0<-rpois(1,inten0)
> N0
[1] 3

> y1<-y0*exp(-d*tte[1])+y[1]
> y1
[1] 201.6282
> A1<-exp(-d*tte[1])
```

```
> B1<-((1-exp(-d*tte[2]))/d)
> inten1<-y1*A1*B1
> inten1
[1] 0.8008072
> N1<-rpois(1,inten1)
> N1
[1] 2

> gene1<-function(i){
+ y1[i]<-y1[i-1]* exp(-d*tte[i])+y[i]
+ return(y1[i])
+ }

> y1[1]<-201.6282
> for(i in 2:99){
+ y1[i]<-gene1(i)
+ }

> y1
 [1] 201.6282 201.8691 201.0255 200.8159 202.6418 202.7150 203.0142 199.0227
 [9] 199.5412 198.4763 198.3383 196.4503 195.1823 194.8423 195.1226 195.0709
[17] 194.6532 196.3424 195.4969 193.6046 193.6013 193.7392 194.4097 194.2247
[25] 196.8835 196.6890 197.6475 197.4220 196.0547 195.0869 197.3187 198.3177
[33] 194.5252 194.6975 194.6570 197.4322 198.1780 198.0112 197.9668 196.5467
[41] 197.6441 196.4101 196.7277 199.2261 198.8240 196.9248 198.0766 198.4091
[49] 198.0150 198.1980 198.3228 195.7490 198.3733 197.9856 195.8806 198.6770
[57] 198.7711 198.5097 197.0926 200.0173 200.0240 199.6255 198.3429 199.5592
[65] 200.4609 200.4891 202.2808 205.3435 205.3142 205.1356 205.1049 204.9333
[73] 206.1682 205.9857 206.0504 207.4065 204.8762 204.7248 204.7759 204.7081
[81] 204.7285 211.6917 213.0211 212.4599 208.6212 209.3956 211.2017 212.7666
[89] 212.9644 211.3275 212.1195 211.0879 210.8430 210.9815 211.5798 211.3345
[97] 214.6380 215.1901 216.6924

> gene<-function(i){
+ A <- exp(- d * tte[i])
```

```
+ B<-((1-exp( - d * tte[i+1]))/d)
+ inten<-gene1(i)*A*B
+ return(inten)
+ }

> gene2<-function(i){
+ Cl[i]<-rpois(1,gene(i))
+ return(Cl[i])
+ }

> Cl<-1:99

> Cl[1]<-2
> for(i in 2:99){
+ Cl[i]<-gene2(i)
+ }
 > Cl
 [1] 2 1 3 1 1 2 17 0 0 0 3 4 0 3 1 3 0 3 3 0 4 0 3 0 1
[26] 2 1 2 3 0 1 9 0 0 2 1 2 0 3 0 3 2 0 3 5 0 0 6 5 0
[51] 5 0 7 7 4 1 2 3 1 0 4 7 0 3 1 0 0 0 2 3 1 3 3 0 0
[76] 5 1 0 0 4 3 1 0 9 0 0 0 2 9 1 5 0 1 0 1 1 2 1 4

> cla<-1:100
> cla[1]<-3
> for (i in 2:99){
+ }
> for (i in 2:100){
+ cla[i]<-Cl[i-1]}

> cla
  [1] 3 2 1 3 1 1 2 17 0 0 0 3 4 0 3 1 3 0 3 3 0 4 0 3 0
[26] 1 2 1 2 3 0 1 9 0 0 2 1 2 0 3 0 3 2 0 3 5 0 0 6 5
[51] 0 5 0 7 7 4 1 2 3 1 0 4 7 0 3 1 0 0 0 2 3 1 3 3 0
[76] 0 5 1 0 0 4 3 1 0 9 0 0 0 2 9 1 5 0 1 0 1 1 2 1 4
```

```
> cl
 [1] 3 2 4 2 3 2 3 2 3 2 3 1 0 1 1 2 3 0 2 1 1 2 1 2 2 2 4 2 2 3 1 1 2 1 2 2 3
[38] 2 1 2 3 2 1 3 2 4 2 2 1 4 2 4 0 2 1 2 0 5 3 4 2 4 3 3 1 4 3 4 3 1 0 1 4 1
[75] 2 3 1 2 2 1 1 0 4 4 0 2 2 3 2 2 3 3 2 4 3 2 2 2 2 2

> sum(cl)
[1] 215
```

5.2 Pricing

```
> ge1<-function(i,v){
+ h[i]<-exp(21.457-1.0*log(6.5451*10^8*exp(2.2361*v)+2.5*10^8)+1.118*v)
+ return(h[i])
+ }

> ge2<-function(i,v){
+ s[i]<-((1.118*(1.0-2.618*exp(2.2361*v))/(-2.618*exp(2.2361*v)-1.0))-0.5)
+ return(s[i])
+ }

> hhh
 [1] 0.01 0.02 0.03 0.04 0.05 0.06 0.07 0.08 0.09 0.10 0.11 0.12 0.13 0.14 0.15
[16] 0.16 0.17 0.18 0.19 0.20 0.21 0.22 0.23 0.24 0.25 0.26 0.27 0.28 0.29 0.30
[31] 0.31 0.32 0.33 0.34 0.35 0.36 0.37 0.38 0.39 0.40 0.41 0.42 0.43 0.44 0.45
[46] 0.46 0.47 0.48 0.49 0.50 0.51 0.52 0.53 0.54 0.55 0.56 0.57 0.58 0.59 0.60
[61] 0.61 0.62 0.63 0.64 0.65 0.66 0.67 0.68 0.69 0.70 0.71 0.72 0.73 0.74 0.75
[76] 0.76 0.77 0.78 0.79 0.80 0.81 0.82 0.83 0.84 0.85 0.86 0.87 0.88 0.89 0.90
[91] 0.91 0.92 0.93 0.94 0.95 0.96 0.97 0.98 0.99 1.00

> ge3<-function(i,v){
+ a[i]<-((1/14.14214)*(v-200*0.01))
+ return(a[i])
+ }
```

```
> zz1<-1:100
> for (i in 1:100){
+ zz1[i]<-ge1(1,hhh[i])*ge2(1,hhh[i])*ge3(1,cl[i])
+ }
> zz1
  [1] 0.001608527 0.000000000 0.009467405 0.000000000 0.007725902
  [6] 0.000000000 0.010583998 0.000000000 0.013307409 0.000000000
 [11] 0.015896063 -0.017139969 -0.036700755 -0.019527412 -0.020671221
 [16] 0.000000000 0.022859886 -0.047810325 0.000000000 -0.025898823
 [21] -0.026847758 0.000000000 -0.028651360 0.000000000 0.000000000
 [26] 0.000000000 0.063782202 0.000000000 0.000000000 0.034011355
 [31] -0.034661342 -0.035283673 0.000000000 -0.036447172 0.000000000
 [36] 0.000000000 0.037996460 0.000000000 -0.038904213 0.000000000
 [41] 0.039716393 0.000000000 -0.040436912 0.040764026 0.000000000
 [46] 0.082708843 0.000000000 0.000000000 -0.042087773 0.084587143
 [51] 0.000000000 0.085300151 -0.085603463 0.000000000 -0.043054198
 [56] 0.000000000 -0.086483887 0.129938041 0.043368587 0.086820206
 [61] 0.000000000 0.086903369 0.043452597 0.043440811 -0.043416734
 [66] 0.086761533 0.043333301 0.086549451 0.043205419 -0.043125755
 [71] -0.086072197 -0.042936808 0.085656470 -0.042710724 0.000000000
 [76] 0.042450227 -0.042307894 0.000000000 0.000000000 -0.041836322
 [81] -0.041665278 -0.082975584 0.082608288 0.082229215 -0.081838898
 [86] 0.000000000 0.000000000 0.040302809 0.000000000 0.000000000
 [91] 0.039644530 0.039416927 0.000000000 0.077901509 0.038712596
 [96] 0.000000000 0.000000000 0.000000000 0.000000000 0.000000000

> szz1<-sum(zz1)
> szz1
[1] 0.5579152

> ge8<-function(i){
pm <- (((121/pi)^(1/2)) * (3.3576^(1/2)) * exp(-0.5 * ((3.5081 * 0.01 * i - 7.9815)^2))) +
(((242^(1/2)) * 0.48295 + 220 - i) * pnorm( - (3.5081 * 0.01 * i - 7.9815)))
return(pm)
}
```

```
> pm<-1:10
> for (i in 1:10){
+ pm[i]<-ge8(bb1[i])}

> bb1 (Retention limit)
 [1]   0 190 200 210 220 230 240 250

> pm (Reinsurance premium)
 [1] 227.512939  38.767164  30.049486  22.209532  15.521060
 [6] 10.171605   6.202363   3.494186
```

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