

# Modeling, simulation, and control of the spacecraft attitude dynamics

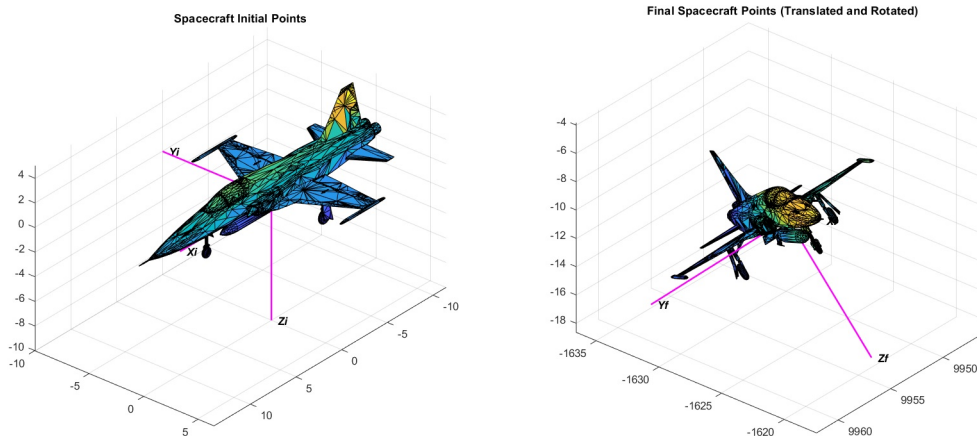
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## Abstract

*Based on the three-dimensional dynamics of a rigid body and Newton's laws, the simplified dynamics of a spacecraft is studied and described through the systematical representation, mathematical modeling and also by a block diagram representation, to finally simulate the spacecraft dynamics in the Matlab programming environment called Simulink. It is paramount to be able to identify and recognize the attitude (often represented with the Euler angles) and position variables like the degrees of freedom (DOF) of the system and also the linear behavior. All this to conclude up about the non-linear behavior presented by the accelerations, velocities, positions and Euler angles (attitude) when those mentioned are plotted against time. In addition to this, the linearized system is found in order to facilitate the control analysis and stability analysis, at using linear analysis tools of Simulink and concepts like controllability and observability, reaching the point of determining under the previous concepts to proceed with the control design phase. Lastly, an uncertainty and sensitivity analysis is realized, by means the Monte-Carlo and the Linear regression method (in Simulink too), to find the torque like critical model input, since it has the greatest effect on the response variables in the system; and thus finally, to implement the Linear Quadratic Regulator (LQR) controller, at using the `lqr` Matlab function.*



**Keywords:** Attitude; mathematical model; block diagram; simulation; linearization, stability; controllability and observability; uncertainty and sensitivity; linear and angular momentum conservation; Euler angles; Monte-Carlo Method; Linear Quadratic Regulator; Simulink.

## I. INTRODUCTION

**I**N this research, the dynamics of a spacecraft is studied, it is based on the dynamics of a rigid body in three dimensions, this through the laws of conservation of angular and linear momentum,

to then acquired with these, information about the attitude (Euler angles) and about the position to have finally information about the 6DOF (Degrees Of Freedom), that is, about the three rotations ( $\phi$ ,  $\theta$ ,  $\psi$ ) and the three translations ( $x$ ,  $y$ ,  $z$ ).

Therefore, motivated to apply and articulate what

has been learned about modeling and mathematical simulation, the mathematical and physical knowledge acquired in mechanical engineering and learn about the aerospace branch; making a description of both the system and the mathematical model is wanted; furthermore is required a block diagram representation using the Matlab programming environment called Simulink, to finally perform mathematical simulations in it and graphically show the dynamics of the system in question.

Lastly, the linearized system is represented mathematically, in order to ease its future optimal control (specifically implementing the Linear-Quadratic Regulator (LQR) controller with the *lqr* Matlab function) through a Simulink's tool called Linear analysis tool, likewise in order to analyze its stability and, of course at the end, the uncertainty and sensitivity of the system, using Simulink tools.

## II. METHODS

This research, starts with a search of information about the rigid bodies dynamics in three dimensions, going through the systems in which the dynamics of spacecraft it is often analyzed until arriving at such mathematical models of those systems.

### i. System description

For this research, a dynamic system of 6DOF is taken in a three-dimensional space, in the Earth's orbit ignoring the earth's oblateness, this simplifying the mathematical model as much as possible. In addition, external disturbing effects are disregarded, such as the gravitational gradient (given because the center of mass of the spacecraft is not aligned with its center of gravity), atmospheric drag (given in low orbit), electromagnetic effects of third bodies (at least the earth's one) and solar pressure (caused by the sun's electromagnetic radiation in low orbit).

Likewise, internal disturbing effects such as those produced by fluid movements and crew movements (given the case) are neglected. [3].

Then, knowing the above, both the system inputs and outputs are shown in tables 1 and 2.

### ii. Mathematical model description

To begin to describe the system mathematically, it is necessary to define the frame of reference, in this case, the fixed body, then it is here where the

**Table 1:** *System inputs*

Inputs	Variable
Inertia tensor	$\vec{I}$
Mass	$M$
Torque	$\vec{T}_{(x,y,z)}$
Force	$\vec{F}_{(x,y,z)}$
Initial angular velocity	$\vec{\omega}_{0(x,y,z)}$
Initial linear velocity	$\vec{v}_{0(x,y,z)}$
Initial Euler angles	$\phi_0, \theta_0, \psi_0$
Initial position	$x_0, y_0, z_0$

**Table 2:** *System outputs*

Outputs	Variable
Angular acceleration	$\vec{\ddot{\omega}}$
Linear acceleration	$\vec{\ddot{v}}$
Angular velocity	$\vec{\omega}$
Linear velocity	$\vec{v}$
Euler angles change rate	$\dot{\phi}, \dot{\theta}, \dot{\psi}$
Euler angles	$\phi, \theta, \psi$
Position	$x, y, z$

mathematical description begins.

Being brief and concise, starting with the Newton's second Law, it is known that for a particle  $F=ma$ :

$$\vec{F} = \sum_i \vec{F}_i = m \frac{d\vec{v}}{dt} \quad (1)$$

Based on the above, the concept of Linear momentum is introduced, who is represented by the mathematical equation 2.

This Linear momentum (L), depends on the frame of reference, but when is used in any inertial frame, it is conserved what means that the total linear momentum does not change. Newton's second law is valid only if the force and velocity are defined in an inertial coordinate system, remembering that if a coordinate system is not accelerating or rotating, this one is Inertial.

$$\vec{L} = m\vec{v} \quad (2)$$

Using Calculus, the previous concept can be extended to rigid bodies by integration over all particles, when the concept is translated to the angular ambit and so is obtained:

$$\vec{M} = \sum_i \vec{M}_i = \frac{d\vec{H}}{dt} \quad (3)$$

Equivalent to the linear momentum, here is introduced the angular momentum concept, whose behaves as its homologous, when only the type of movement is changed to the angular motion, and it is represented by this equation:

$$\vec{H} = \vec{I}\vec{\omega} \quad (4)$$

After that, it is important to define a new essential concept to understand the angular momentum, called Inertia. The inertia is the resistance of any object to any change in its motion, including its speed and direction.

Also is important to understand the physical concept known as Moment of inertia or Rotational inertia (of a rigid body in this study case), which is a tensor that determines the torque needed for a desired angular acceleration about a rotational axis (eq. 5). It depends on the body's mass distribution and the axis in question.

$$\vec{I} = \begin{bmatrix} I_{xx} & -I_{xy} & -I_{xz} \\ -I_{yx} & I_{yy} & -I_{yz} \\ -I_{zx} & -I_{zy} & I_{zz} \end{bmatrix} \quad (5)$$

The moment of inertia matrix (I) is fixed in the body frame. However, it is necessary to remember that Newton's second law only applies for inertial frames:

$$F = m \frac{d\vec{v}}{dt} + m\vec{\omega} \times \vec{v} \quad (6)$$

And again, equivalent to the linear momentum, the angular momentum and the inertia moment matrix (I) are fixed in the body frame like its homologous (linear momentum):

$$M = \frac{d\vec{H}}{dt} + \vec{\omega} \times \vec{H} \quad (7)$$

Thus, simplifying it determines that the forces moment regarding the gravity center is represented by this following equation:

$$\vec{T} = \vec{I}\vec{\omega} + \vec{\omega} \times \vec{I}\vec{\omega} \quad (8)$$

Having reached this point, the aim probably will be to find some way to get the Euler angles by known physical information like angular velocity; and due to that, the linear transformation represented by the equation 9 acquire a sense.

$$\vec{\omega} = \begin{bmatrix} 1 & 0 & -\sin \theta \\ 0 & \cos \phi & \cos \theta \sin \phi \\ 0 & -\sin \phi & \cos \phi \cos \theta \end{bmatrix} \times \begin{bmatrix} \dot{\phi} \\ \dot{\theta} \\ \dot{\psi} \end{bmatrix} \quad (9)$$

Euler angles are particularly useful to describe the motion of a body that rotates about a fixed point, such as a gyroscope or, a body that rotates about its center of mass, such as an aircraft or spacecraft, like it was found in the bibliography. Unfortunately, there is no standard formulation nor standard notation for Euler angles; thus, it is chosen to follow, one used in the bibliography (see the following figure). [1], [2] and [5] <sup>1</sup>

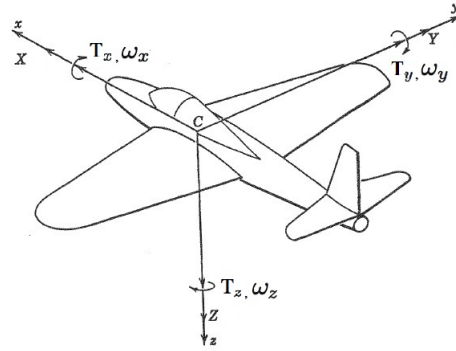


Figure 1: Euler angles (figure from [1])

To identify the new positions of the principal axes as a result of angular displacement through the three Euler angles ( $\phi$ ,  $\theta$  and  $\psi$ ), going through the following series of coordinate rotations Z-Y-X (or the so-called 3-2-1) rotation sequence or ( $\gamma$ ,  $\beta$  and  $\alpha$ ) like is proposed in the bibliography:

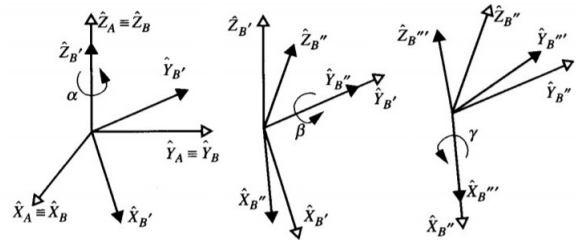


Figure 2: Series of coordinate rotations (figure from [7])

First, the object's (spacecraft) points are rotated since an initial system ( $x, y, z$ ) into a  $x', y', z'$  system through a rotation (right-handed rotation) angle  $\psi$

<sup>1</sup>For aircraft and spacecraft motion a slightly different one is used.

about the,  $z'$  axis on the x-y plane. This angle denoted by  $\psi$  is called Yaw. The aforementioned is shown in equation 10.

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \psi & \sin \psi & 0 \\ -\sin \psi & \cos \psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [T1] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (10)$$

The resulting  $x'$  and  $y'$  coordinates remain in the x, y plane. Now the angle from the  $x''$  axis to the body frame  $x'$  axis is the angle in question. The right-handed rotation is about the  $y'$  axis and after this rotation, the  $y''$  axis remains coincident with the  $y'$  axis. This angle denoted by  $\theta$  is called pitch. The aforementioned is shown in equation 11.

$$\begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = \begin{bmatrix} \cos \theta & 0 & -\sin \theta \\ 0 & 1 & 0 \\ \sin \theta & 0 & \cos \theta \end{bmatrix} \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = [T2] \begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} \quad (11)$$

Finally, the previous coordinates is rotated (right-handed rotation) about  $x''$  axis through an angle who is the angle from the  $y''$  axis (or  $z''$  axis). The  $x'''$  axis is coincident with the  $x''$  axis. This angle denoted by  $\phi$  is called Roll. The aforementioned is shown in equation 12.

$$\begin{bmatrix} x''' \\ y''' \\ z''' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos \phi & \sin \phi \\ 0 & -\sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} = [T3] \begin{bmatrix} x'' \\ y'' \\ z'' \end{bmatrix} \quad (12)$$

At the end, the final "Euler" Transformation is:

$$\begin{bmatrix} x'''' \\ y'''' \\ z'''' \end{bmatrix} = [T3][T2][T1] \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad (13)$$

For more understanding about the mathematics applied here, look at references <sup>2</sup>.

### iii. Block diagrams

This system is composed by 4 big subsystems (like is possible to see in the following figures), which are going to be described here:

In the first subsystem (see fig. 3) are calculated the angular velocity and the angular acceleration. The angular acceleration is obtained using the Inertia tensor and the torque (see eq. 8). After that, by integration and with the initial condition  $\omega_0$ , is obtained the angular velocity.

<sup>2</sup>All formulas are obtained from [1], [2] and [4]

Figure 3: Subsystem 1

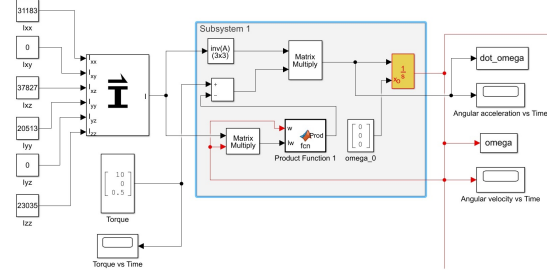
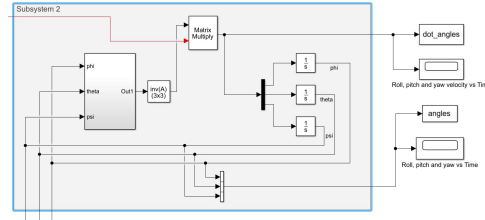
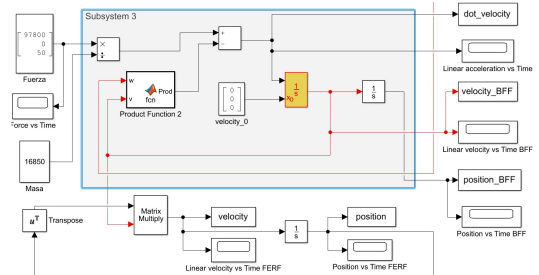


Figure 4: Subsystem 2



In the second subsystem (see fig. 4) are calculated the attitude or the Euler angles ( $\phi$ ,  $\theta$ ,  $\psi$ ) and its rate of change. First, the change of rate was found through the equation 9 (multiplying the angular velocity by the inverse of the A matrix). And finally, the attitude was calculated integrating its rate of change with an initial attitude ( $\phi_0$ ,  $\theta_0$ ,  $\psi_0$ ).

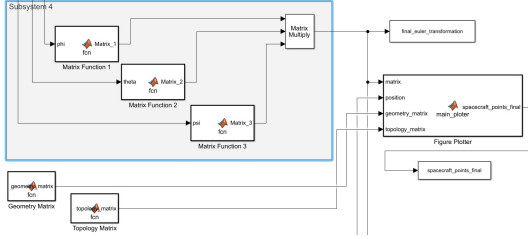
Figure 5: Subsystem 3



In the third subsystem (see fig. 5) are calculated the linear acceleration, the linear velocity, and position. Starting from the force, mass, and the cross product between the angular velocity and the linear velocity, with the aim of adding it arithmetically, the linear acceleration is found. After that, by integration is calculated the linear velocity using the initial linear velocity  $v_0$  both in the body fixed frame. So later, by means of the final "Euler" transformation matrix or well know as Direction Cosine Matrix (DCM) is calculated the linear acceleration and lin-

ear velocity in the flat Earth reference frame, and so, finally by integration again, find the spacecraft position (see eq. 6) later plotted.

**Figure 6:** Subsystem 4



In the fourth subsystem (see fig. 6) is calculated the Final Euler transformation matrix (see eq. 13), this one calculated multiplying the series of matrices regarding with the coordinate rotation systems proposed previously (see eqs. 10, 11 and 12). After that, in the Matlab function is plotted the 3D model geometry of the spacecraft (which is taken from [11]) using geometric functions of computer-aided geometric design, such as translational and rotational functions and other rigid functions to built an object (in this case a spacecraft), using geometrical and topological matrices.<sup>3</sup>

#### iv. Simulation methods

In this context, it should be noted that the following units correspond to the international system (SI). The initial conditions (tab. 3) and simulation parameters (tab. 4) that were used in Simulink, were the following:

**Table 3:** Initial Conditions

Initial condition	Variable	Value
Initial angular velocity	$\vec{\omega}_0(x,y,z)$	[0; 0; 0]
Initial linear velocity	$\vec{v}_0(x,y,z)$	[0; 0; 0]
Initial Euler angles	$\phi_0, \theta_0, \psi_0$	[0; 0; 0]
Initial position	$x_0, y_0, z_0$	[0; 0; 0]

These are the simulation parameters used to get in Simulink a dynamics simulation of a spacecraft with the target of acquiring information about its 6DOF (Euler angles and position) during 60 seconds; using the physical parameters (that are shown in table 5) for a McDonnell Douglas / Boeing F / A - 18 (aircraft).

<sup>3</sup>For more information, look at the bibliography [4] and [6].

**Table 4:** Simulation parameters

Simulation parameters	Value
Initial simulation time	0
Final simulation time	60
Type of solver	Fixed-step
Solver	Ode1 (Euler)
Fixed-step size (sample time)	auto

**Table 5:** Physical properties

Physical property	Value
Inertia Tensor	[Look at the footnote]
Mass	16850
Torque	[10; 0; 0.5]
Force	[97800; 0; 50]

4

#### v. Mathematical model linearization

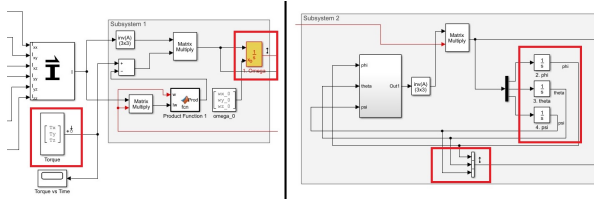
The aforementioned dynamic system was linearized about a stationary point of linearization (which corresponds to the initial conditions of the model expressed in the simulation methods) to get a system described in the form of the equation 14; all this to simplify the future design phase of its controller (if it is possible) and also with the aim of analyzing the system stability. This point of linearization was chosen due to the numerical and operational simplification that this one offers.

$$\begin{cases} \dot{x} = Ax + Bu \\ y = Cx + Du \end{cases} \quad (14)$$

The input variables of the model were each torque's component ( $T_x, T_y, T_z$ ) and the output variables were each attitude angle ( $\phi, \theta, \psi$ ) and each angular velocity component ( $\omega_x, \omega_y, \omega_z$ ) (see next figure). It should be noted that for this linearization only the part of the system that modeled the dynamics of the spacecraft attitude was taken.

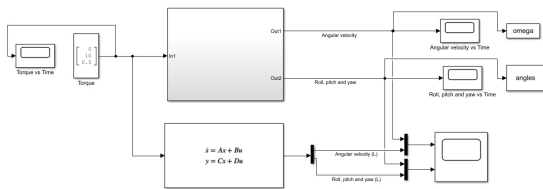
In this case, the linearization was made through the Simulink's tool called Linear Analysis Tool after choosing the corresponded inputs and outputs for this system, expressing the initial conditions as the

<sup>4</sup>Inertia values are given by  $I_{xx} = 31183, I_{xy} = 0, I_{xz} = 37827, I_{yy} = 20513, I_{yz} = 0, I_{zz} = 23035$ .

**Figure 7: Inputs and outputs selection for the linear model**


point of linearization and without variation of parameters, all that using a step plot; which saves the manual work of expanding in Taylor series, around a stationary point, to later built the Jacobian matrix.

After that, the original system (non-linear) and the linear one are compared, all this to know the approach level of the linear system (regarding the original one) and then to know how good was the linearization, this by means of the graphics comparison of angular velocity and attitude (states).

**Figure 8: Comparison between linear and non-linear systems**


## vi. Stability analysis

The stability criterion for linear systems says that being  $\lambda$  the vector of the eigenvalues of the matrix  $A$ , then the system is stable if for all the eigenvalues  $\lambda$  are satisfied the equation 15:

$$\text{Re}(\lambda) < 0 \quad (15)$$

If at least one of the eigenvalues is positive, the system is unstable ( $\text{Re}(\lambda) > 0$ ) and finally, if for all the eigenvalues, it is true that its real part is equal to zero, the system is critically stable ( $\text{Re}(\lambda) = 0$ ).

## vii. Controllability and observability analysis

As above was mentioned, it is desired to implement a system control (one of the main reasons for having linearized the system) and before going to design the controller, it will be introduced two important criteria that it is recommended to analyze before this, to reach a solution of optimal control.

Such criteria are controllability and observability, they are dual of the same problem, and its analysis determines early the existence of a design solution according to design parameters and objectives.

- Controllability means that it is possible by admissible entries, to change the states of any initial value to any other final value in a time interval, the controllability does not mean to stay there, but only to reach that state. The controllability matrix is given as:

$$R = [B \quad AB \quad A^2B \quad \dots \quad A^{n-1}B] \quad (16)$$

Each linear system is controllable if the control matrix has a maximum range.

- Observability means that from the outputs of a system is possible to know the behavior of the entire system. If a system is non-observable, it means that some of its states cannot be determined by output sensors.

The observability matrix is given as:

$$O = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} \quad (17)$$

Each linear system is observable if the observability matrix has a range = n or if it is a full rank matrix.

## viii. Uncertainty and sensitivity analysis

Now, the next step is to proceed with the uncertainty and sensitivity analysis given that:

- The uncertainty analysis determines qualitatively how an uncertainty in the model inputs is reflected in an uncertainty in the outputs. For this analysis, the Monte-Carlo Method will be used, which consists of simulating the dynamic behavior of the system for randomly chosen values or uncertain entries using 100 samples of parameters or inputs generated assuming a uniform distribution, and then performing a statistical analysis of the results obtained, often a frequency histogram. All this using the Simulink software tool called Sensitivity analysis, with  $\vec{T}_x$ ,  $\vec{T}_y$ ,  $\vec{T}_z$ ,  $I_{xx}$ ,  $I_{xy}$ ,  $I_{xz}$ ,

$I_{yy}$ ,  $I_{yz}$ ,  $I_{zz}$ ,  $\vec{\omega}_{0(x)}$ ,  $\vec{\omega}_{0(y)}$  and  $\vec{\omega}_{0(z)}$  as parameters and with Step response envelope as the requirement.

The Ranges of Monte-Carlo simulation parameters were made up according to clues given by literature about the current spacecraft analyzed and some of them (initial angular velocity) selected due to the model sensitivity with the aim of avoiding singularities in the sensitivity analysis in Simulink (units correspond to the SI):

**Table 6:** Range of Monte-Carlo simulation parameters

Model input	Min. value	Max. value
$I_{xx}$	-50000	50000
$I_{xy}$	-50000	50000
$I_{xz}$	-50000	50000
$I_{yy}$	-50000	50000
$I_{yz}$	-50000	50000
$I_{zz}$	-50000	50000
$\vec{T}_x$	-100	100
$\vec{T}_y$	-100	100
$\vec{T}_z$	-100	100
$\vec{\omega}_{0(x)}$	0	eps
$\vec{\omega}_{0(y)}$	0	eps
$\vec{\omega}_{0(z)}$	0	eps

- The sensitivity analysis quantifies the effect of the inputs or parameters on the uncertainty in the outputs. This analysis aims to identify the input variables which have the greatest effect over the output variables (critical parameters). For this analysis, the chosen method is the Global sensitivity analysis based on Linear regression. In this method, a linear regression is done between the response variable and the input variables, in the form of equation 19 for linear systems, that comes from 18:

$$y = \beta_0 + \sum_{i=1}^n \beta_i u_i \quad (18)$$

So:

$$S_i = \beta_i \quad (19)$$

This Standardized regression coefficient will be obtained by computing the statics of the sensitivity analysis, the same as the Linear correlation coefficient (another statistical index too)

help to understand better the sensitivity analysis.

## ix. Control design

Finally, after going through a successful analysis of controllability and observability, and of course having confirmed that the system is controllable and observable, it proceeds to design the controller. In this case, optimal control has been chosen, exactly the Linear Quadratic Regulator (LQR) controller.

The LQR controller has the aim of reducing the energy present in the system and minimizing this criterion, obtaining so a linear control; thus an optimal control like that offers the path of the control variables that minimize a cost function that tells about the aforementioned system energy [8].

Firstly, it is necessary to find a gain  $K$  for feedback in the system which minimizes the following cost function:

$$J = \int_0^{\infty} (x^T Q x + u^T R u) dt \quad (20)$$

To solve the previous equation, two matrices ( $Q$  and  $R$ ) are necessary; those matrices must be positive semi-definite and positive definite, respectively. Those parameters are related with the relative importance of the control effort ( $u$ ) and error (deviation from 0), respectively, in the cost function (see eq. 20) which is seeking to optimize.

$$Q = C^T C \quad (21)$$

$$R = \rho I \quad (22)$$

In this case, to assume the previous equations (eqs. 21 and 22) is a good choice [9], adding that in the simplest case  $\rho$  is equal to 1, nevertheless in this case, this value is finding out by trial and error until reaching the desired behavior [10], which is a setting time of approximately 7 seconds regardless of the overshoot in this case to simplify the control.

After that, to find the  $K$  optimal (gain), it is necessary to solve the Algebraic Riccati Equation (ARE) (eq. 23) or use the lqr Matlab function just like this: lqr(A,B,Q,R).

$$A^T P + P A - P B R^{-1} B^T P + Q = 0 \quad (23)$$

Thus, the optimal  $K$  is equal to:

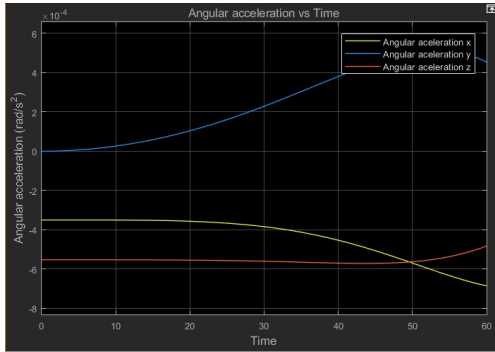
$$K_{opt} = R^{-1} B^T P \quad (24)$$

### III. RESULTS AND DISCUSSION

#### i. Simulation of the model

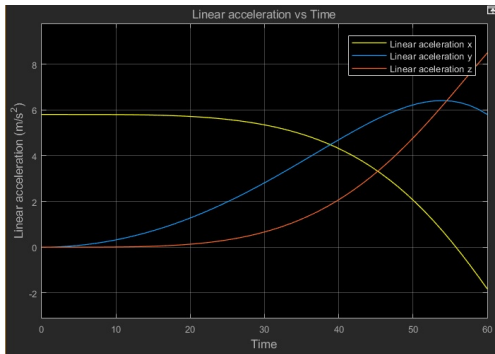
The results obtained were the following, clarifying beforehand that due to the model simplifying and the nature of the engineering branch who analyze this problem (high accuracy), there is no verification with the data taken from reality:

**Figure 9:** Angular acceleration vs time



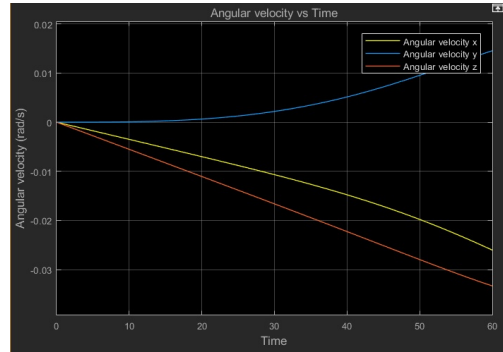
As shown in figure 9 is possible to observe the non-constant behavior of the angular acceleration, it due to the changes in the angular velocity who influence in the angular acceleration equations (feeding back) which cancel the constant behavior related to the constant torque and also with the constant value of the Inertia matrix.

**Figure 10:** Linear acceleration vs time



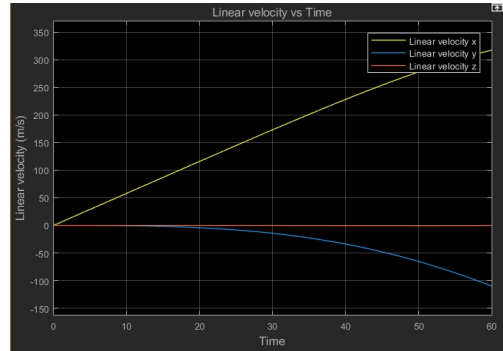
As shown in figure 10 is possible to appreciate the non-constant behavior of the linear acceleration, it given by the non-constant angular velocity who affects in an important way it variable despite of the constant force present in the system and also it due to the assumption of the spacecraft constant mass; which finally can be understood in a non-constant linear acceleration of the system.

**Figure 11:** Angular velocity vs time



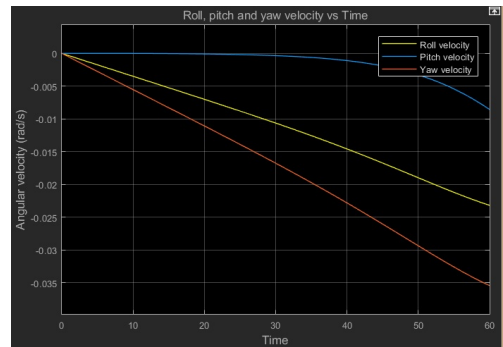
As can be seen in figure 11, the angular velocity behavior changes at a non-constant rate (but it is quite similar) because it is obtained by integration from the angular acceleration which changes with small oscillations.

**Figure 12:** Linear velocity vs time



As can be seen in figure 12, the linear velocity (in flat earth reference frame) behavior changes at a non-constant rate (but it is quite similar) because it is obtained by integration from the linear acceleration whose changes with small oscillations.

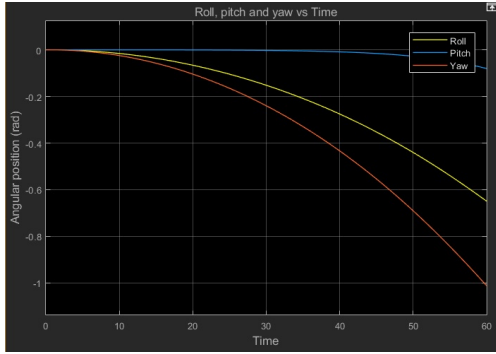
**Figure 13:** Attitude velocity vs time





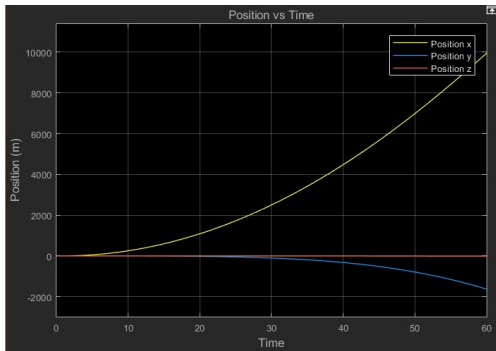
In the case of the attitude velocity (see fig. 13) the rate of change of roll and yaw angles are almost linear (similar) while the rate of change of pitch angle has a parabolic behavior and all of those changes of rate are decreasing.

Figure 14: Attitude vs time



As shown in figure 14 the Euler angles has a parabolic behavior and also are decreasing continuously as indicated by them changes of rate.

Figure 15: Position vs time



Finally, as can be seen in figure 15, the spacecraft position (in flat earth reference frame) present parabolic or quadratic behavior too, given by the integration realized to the linear velocity to obtain this one. This occurs in its components even though, in the z component it is not easily observable.

## ii. Mathematical model linearization

On the other hand, the linearization done in Simulink gave the followings parameters (figs. 25, 26, 27 and 28) and behavior plot of the linear model (see fig. 16), it should be noted that the following units correspond to the international system (SI):

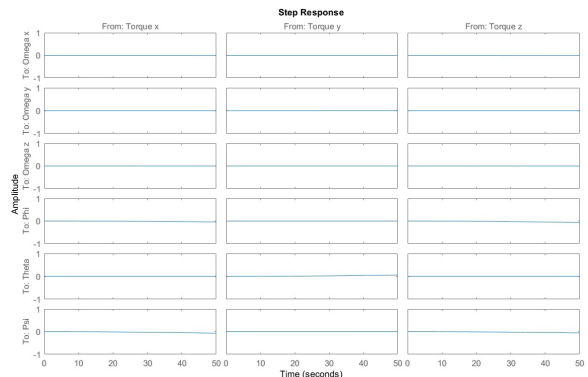
$$A = \begin{bmatrix} \omega_x & \omega_y & \omega_z & \phi & \theta & \psi \\ \omega_x & 0 & 0 & 0 & 0 & 0 \\ \omega_y & 0 & 0 & 0 & 0 & 0 \\ \omega_z & 0 & 0 & 0 & 0 & 0 \\ \phi & 1 & 0 & 0 & 0 & 0 \\ \theta & 0 & 1 & 0 & 0 & 0 \\ \psi & 0 & 0 & 1 & 0 & 0 \end{bmatrix} \quad (25)$$

$$B = \begin{bmatrix} T_x & T_y & T_z \\ \omega_x & -3.233e-05 & 0 & -5.308e-05 \\ \omega_y & 0 & 4.875e-05 & 0 \\ \omega_z & -5.308e-05 & 0 & -4.376e-05 \\ \phi & 0 & 0 & 0 \\ \theta & 0 & 0 & 0 \\ \psi & 0 & 0 & 0 \end{bmatrix} \quad (26)$$

$$C = \begin{bmatrix} \omega_x & \omega_y & \omega_z & \phi & \theta & \psi \\ \omega_x & 1 & 0 & 0 & 0 & 0 \\ \omega_y & 0 & 1 & 0 & 0 & 0 \\ \omega_z & 0 & 0 & 1 & 0 & 0 \\ \phi & 0 & 0 & 0 & 1 & 0 \\ \theta & 0 & 0 & 0 & 0 & 1 \\ \psi & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (27)$$

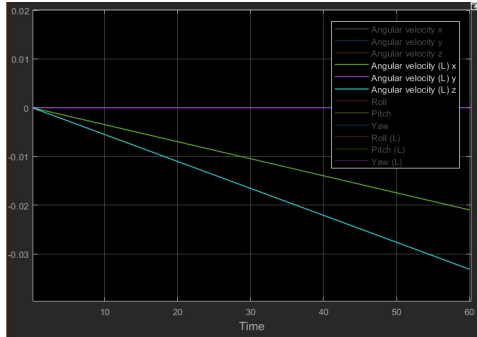
$$D = \begin{bmatrix} T_x & T_y & T_z \\ \omega_x & 0 & 0 & 0 \\ \omega_y & 0 & 0 & 0 \\ \omega_z & 0 & 0 & 0 \\ \phi & 0 & 0 & 0 \\ \theta & 0 & 0 & 0 \\ \psi & 0 & 0 & 0 \end{bmatrix} \quad (28)$$

Figure 16: Step plot - Torque vs attitude and angular velocity in the linear model



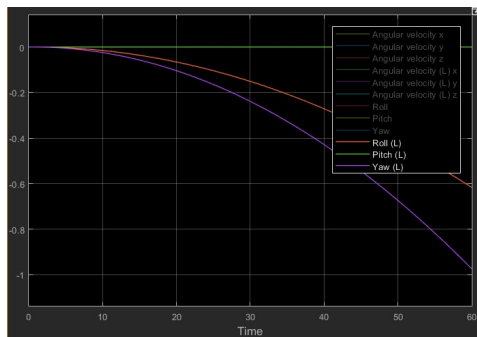
The comparison between the non-linear (original) and the linear system, can be done using figures 17 and 18, shown below.

**Figure 17:** Angular velocity of the linear system



In figure 17 is possible to notice that the angular velocity behavior in both cases are pretty similar (see figure 11) and at first sight, the numerical values are very similar despite the constant value at "zero" in the linearized y component (see figure 11 again).

**Figure 18:** Attitude of the linear system



In figure 18 is easy to notice that the attitude of the spacecraft (roll, pitch, and yaw) have similar behavior (linear and non-linear) (see figure 14), furthermore, at first sight the numerical values are pretty similar despite of the constant value at "zero" in the pitch angle (see figure 18).

Consequently, from the above and knowing that the critical variables to control are the attitude, it is noticed that a good approximation was obtained.

### iii. Stability analysis

On the other hand, after having compared the behavior of the non-linear and linear systems, the analysis of stability of the linear system is done, this under the aforementioned eigenvalue criteria.

$$eig(A) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (29)$$

As a result of the stability analysis for the current system, the system is critically stable as can be seen in equation 29, according to the mentioned stability criterion.

### iv. Controllability and observability analysis

Additionally, the ranks of the controllability (R) and observability (O) matrices, required for the analysis with their same name, are shown below in equations 30 and 31 respectively:

$$rank(R) = 1 \quad (30)$$

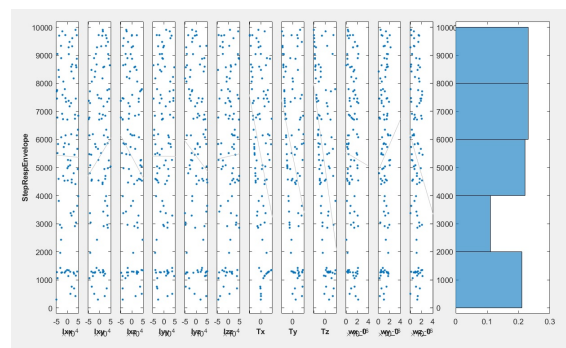
$$rank(O) = 1 \quad (31)$$

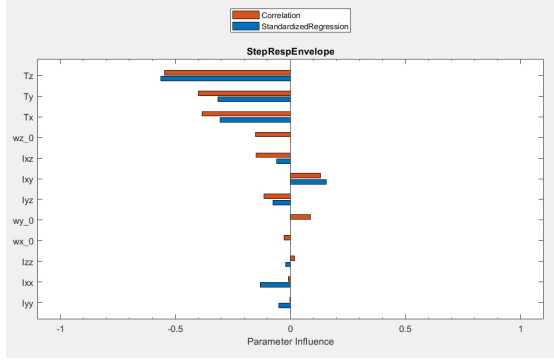
What the previously result mean is that this linear system is controllable and observable, what means or determine the existence of a design solution according to design parameters and objectives.

### v. Uncertainty and sensitivity analysis

Subsequently, the uncertainty and sensitivity of the original Simulink system are analyzed, using the Monte-Carlo Method and by the Linear Regression Method, respectively, figures 19 and 20:

**Figure 19:** Step response envelope and histogram (Monte-Carlo method)



**Figure 20:** Standardized regression coefficient and Linear correlation coefficient


From the results of the analysis of uncertainty and sensitivity of the system, it can be clearly perceived that the input variables or parameters that represent the greatest uncertainty for the system are the torques ( $T_x$ ,  $T_y$  and  $T_z$ ), due to its high standardized regression coefficient ( $\beta_i$ ) and linear correlation coefficient ( $r$ ), as can be seen in figure 20 (For more detailed about these, look at the table 7).

**Table 7:** Standardized regression coefficient and Linear correlation coefficient (table)

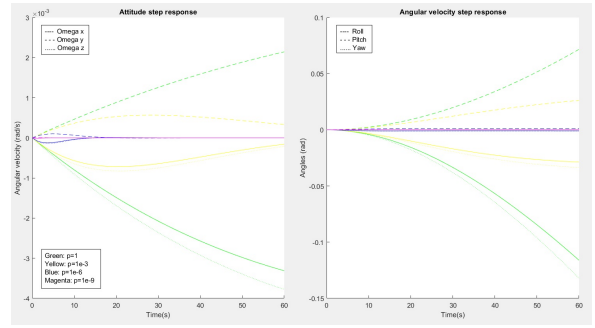
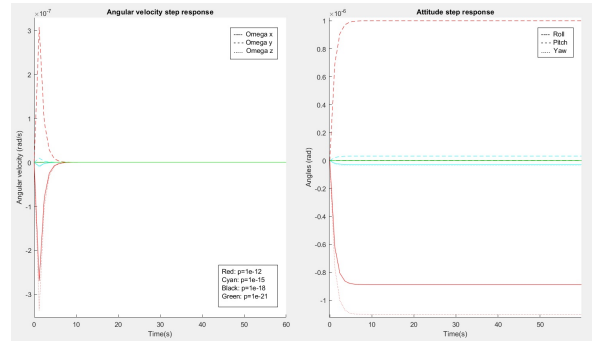
Model input	$\beta_i$	$r$
$T_x$	-0.3067	-0.3847
$T_y$	-0.3139	-0.3989
$T_z$	-0.5633	-0.5479
$I_{xx}$	-0.1309	-0.0087
$I_{xy}$	0.1554	0.1315
$I_{xz}$	-0.0596	-0.1487
$I_{yy}$	-0.0502	-0.0014
$I_{yz}$	-0.0765	-0.1160
$I_{zz}$	-0.0209	0.0182
$\vec{\omega}_0(x)$	0	-0.0281
$\vec{\omega}_0(y)$	0	0.0873
$\vec{\omega}_0(z)$	0	-0.1529

## vi. Control design

Finally, subsequent of have confirmed that the system was controllable and observable, the results of the control design phase using the LQR controller, gave the following Q matrix:

$$Q = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (32)$$

The trial and error process to find out the  $\rho$  value necessary to build the R matrix is shown in figures 21 and 22.

**Figure 21:** Step response simulation - Torque vs attitude and angular velocity in the linear model with LQR controller using different  $\rho$  values ( $1 - 10^{-9}$ )

**Figure 22:** Step response simulation - Torque vs attitude and angular velocity in the linear model with LQR controller using different  $\rho$  values ( $10^{-12} - 10^{-21}$ )


There (see figs. 21 and 22), can be observed simulations of the attitude and angular velocity in the linear model with LQR controller, responding to torque at using a step signal; all of that done for different  $\rho$  values in the range of 1 to  $10^{-21}$  with a step of  $10^{-3}$ .

Thanks to the step response simulation graphics, was possible find an approximate  $\rho$  value of  $10^{-11}$  based on the setting time of approximately 7 seconds and so finally, build the R matrix (see eq. 33).

$$R = \begin{bmatrix} 1e-11 & 0 & 0 \\ 0 & 1e-11 & 0 \\ 0 & 0 & 1e-11 \end{bmatrix} \quad (33)$$

The result of using the `lqr` Matlab function to solve the ARE with its respective arguments (A,B,Q,R) was the following K matrix or gain (see eq. 34) and the new eigenvalues or poles of the system in closed loop at using the LQR controller (see eq. 35):

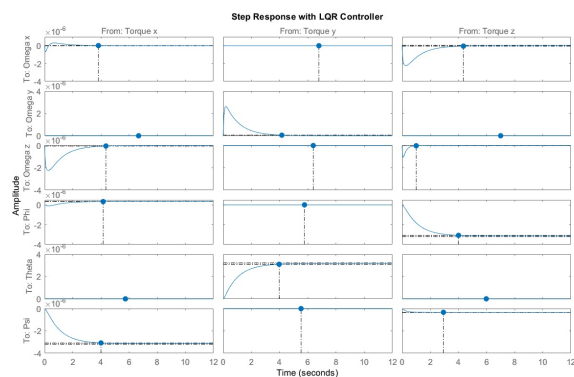
$$K' = \begin{bmatrix} 6.202e4 & -8.262e-12 & -3.494e5 \\ -1.413e-11 & 3.361e5 & -7.171e-12 \\ -3.494e5 & 1.801e-11 & -1.323e4 \\ 3.387e4 & 1.65e-10 & -3.144e5 \\ -3.965e-11 & 3.162e5 & -4.017e-11 \\ -3.144e5 & 1.52e-10 & -3.387e4 \end{bmatrix} \quad (34)$$

5

$$\text{eig}(A - B * K) = \begin{bmatrix} -28.969 \\ -4.744 \\ -1.023 \\ -1.001 \\ -15.383 \\ -1.002 \end{bmatrix} \quad (35)$$

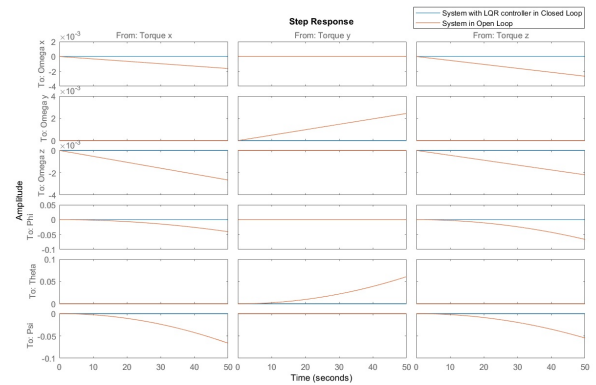
Consequently, with the previously defined parameters, are obtained plots such as the system response with the LQR controller in closed loop to the torque, using a step signal (see fig. 23); in addition to the comparison between the system in open loop and closed loop (see fig. 24).

**Figure 23:** Step plot - Torque vs attitude and angular velocity in the linear model with LQR controller



<sup>5</sup>The controller gain or K matrix is presented like K transposed (K') due to K does not fit in the available width to show it.

**Figure 24:** Step plot - Torque vs attitude and angular velocity in the linear model in closed loop vs in open loop



## IV. CONCLUSIONS

- In conclusion, in this case of study has been learned, applied and articulated some knowledge about mathematical modeling and simulation, mathematics and physics as other concepts and theories used in mechanical engineering, at realize the systematic, mathematical and in block diagram representation, all of this using the programming environment called Simulink, these with the target of represent a simplified spacecraft attitude dynamics.
- Furthermore, the simulation executed in Simulink presented an expected behavior (non-linear), since the mathematical model is highly non-linear. None of the accelerations, velocities, positions nor Euler angles (attitude) presented a linear behavior when those were plotted against time like it was supposed since the beginning.
- Additionally, the linearization that was made to the system in Simulink was good, since a good approximation to the non-linear model was achieved in the attitude of the linear model, so adding that the system is controllable and observable, the design of the control could be implemented.
- As a result of the uncertainty and sensitivity analysis carried out in Simulink, using the Monte-Carlo method and the linear regression method, the torques were identified as the input variables that have the greatest effect

on the response variables for this system, these being so the critical model inputs if it is desired to reduce the variability of the system results.

- Finally, as a result of the control design, relatively good control was obtained because of the set point offset (approximately  $1 * 10^{-6}$  for the angular velocity and  $3 * 10^{-6}$  for the attitude, both in the step response simulation) and also due to the overshoot presented in the three components of the angular velocity in the aforementioned simulation.

#### i. Recommendations for a future work

- With the aim of improve this or futures papers about this topic, like recommendation, it will be fine to have in mind external disturbing effects who were disregarded (like the aforementioned in the system description), to add complexity and accuracy like is demanded by the current engineering and science and so to be allowed to use experimental data to validate the simulation.
- Furthermore, is valid recommend to use a type of control with integral action to achieve eliminate the set-point offset like the Linear-Quadratic-Integral (LQI) control.

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