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Increasing the applicability of Steffensen's method ☆

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ABSTRACT

We present an original alternative to the majorant principle of Kantorovich to study the semilocal convergence of Steffensen's method when it is applied to solve nonlinear systems which are differentiable. This alternative allows choosing starting points from which the convergence of Steffensen's method is guaranteed, but it is not from the majorant principle. Moreover, this study extends the applicability of Steffensen's method to the solution of nonlinear systems which are nondifferentiable and improves a previous result given by the authors.

1. Introduction

The solution of nonlinear systems of the form $F(x) = 0$, where $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ and D is a non-empty open convex domain, is a common problem appearing in science and engineering. It is well-known that finding exact solutions of $F(x) = 0$ is difficult and we then usually use iterative methods to approximate the solutions of $F(x) = 0$. Between them, Newton's method is one of the most used methods, whose local order of convergence is two. The application of Newton's method,

$$\begin{cases} x_0 \in D, \\ x_{n+1} = x_n - \left[F'(x_n)\right]^{-1} F(x_n), \quad n \geq 0, \end{cases}$$

needs to evaluate F' in each step x_n. Then, the iterative methods that do not use derivatives [4,11] are specially interesting if the evaluation of F' is expensive or F' does not exist.

It is common to approximate derivatives by divided differences, so that iterative methods that use divided differences instead of derivatives are obtained. Remember that the operator $[u, v; F] \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, where $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ denotes the space of the bounded linear functions from $\mathbb{R}^m$ into $\mathbb{R}^m$ and $u, v \in \mathbb{R}^m$ $(u \neq v)$, is a divided difference of first order of the function F in the points u and v $(u \neq v)$ if $[u, v; F](u-v) = F(u)-F(v)$.

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Between the iterative methods that use divided differences in their algorithms, we can emphasise Steffensen's method,

$$\begin{cases} x_0 \in D, \\ x_{n+1} = x_n - \left[x_n, x_n + F(x_n); F\right]^{-1} F(x_n), \quad n \geq 0, \end{cases}$$

whose main interest lies in that the approximation of the derivative $F'(x_n)$, that appears in each step of Newton's method, by the divided difference $[x_n, x_n + F(x_n); F]$ is good enough to keep the quadratic convergence [7,10] and, therefore, keeps the computational efficiency [12] of Newton's method.

In general, a problem posed by iterative methods that use divided differences is that their applicability is limited because the domains of starting points, from which the convergence of the methods is guaranteed, are small. So, our first aim in this paper is to improve the domain of starting points for Steffensen's method. Moreover, taking into account that the majority of the semilocal convergence results, that appear in the mathematical literature for Steffensen's method, require that the function F is differentiable [1,5,2–4,6], although the application of this method does not obviously need to evaluate derivatives of F, our second aim in this paper is to obtain a new semilocal convergence result for Steffensen's method in which the function F is not required to be differentiable, so that we can then extend the application of this method to solve nonlinear and nondifferentiable systems. We illustrate all the above with two nonlinear systems that arise from the discretisation of nonlinear integral equations of mixed Hammerstein type. In addition, we compare our result of semilocal convergence with other ones already known for the differentiable and non-differentiable cases and see that ours improves them.

Throughout the paper we denote $\overline{B(x,\varrho)} = \{y \in \mathbb{R}^m; \|y - x\| \leq \varrho\}$ and $B(x,\varrho) = \{y \in \mathbb{R}^m; \|y - x\| < \varrho\}$.

2. Starting condition for the divided difference of the first order

To prove the semilocal convergence of Steffensen's method, it is usually required the following condition for the divided difference of the function F (see [2,4,6]):

$$\big\|[x,y;F] - [u,v;F]\big\| \leq K\big(\|x-u\| + \|y-v\|\big), \quad K \geq 0,\ x,y,u,v \in D,\ x \neq y,\ u \neq v, \tag{1}$$

that can be relaxed by the condition

$$\big\|[x,y;F] - [u,v;F]\big\| \leq \omega\big(\|x-u\|, \|y-v\|\big), \quad x,y,u,v \in D,\ x \neq y,\ u \neq v, \tag{2}$$

where $\omega : \mathbb{R}_+ \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is a non-decreasing function in the two arguments. It is clear that (2) is reduced to (1) if $\omega(s,t) = K(s+t)$. In this case, F is differentiable, since $\omega(0,0) = 0$ (see [11]).

On the other hand, we consider nonlinear integral equations of mixed Hammerstein type of the form

$$\varphi(s) = \phi(s) + \int_a^b G(s,t) H\big(t, \varphi(t)\big)\, dt, \quad s \in [a,b], \tag{3}$$

where $-\infty < a < b < +\infty$, the kernel G is the Green function in $[a,b] \times [a,b]$, ϕ and H are known functions and φ is a solution to be determined, and use a process of discretisation to transform Eq. (3) into a finite dimensional problem by approximating the integral of (3) by a Gauss–Legendre quadrature formula with m nodes:

$$\int_a^b q(t)\, dt \simeq \sum_{i=1}^m w_i q(t_i),$$

where the nodes t_i and the weights w_i are determined.

If we denote the approximations of $\varphi(t_i)$ and $\phi(t_i)$ by φ_i and ϕ_i, respectively, with $i = 1, 2, \dots, m$, then Eq. (3) is transformed into the following nonlinear system:

$$\varphi_i = \phi_i + \sum_{j=1}^{m} a_{ij} H(t_j, \varphi_j), \quad j = 1, 2, \dots, m, \tag{4}$$

where

$$a_{ij} = w_j G(t_i, t_j) = \begin{cases} w_j \frac{(b-t_i)(t_j-a)}{b-a}, & j \le i, \\ w_j \frac{(b-t_j)(t_i-a)}{b-a}, & j > i. \end{cases}$$

After that, system (4) can be written as

$$F(f) \equiv f - g - A\,h = 0, \qquad F : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \tag{5}$$

where

$$f = (\varphi_1, \varphi_2, \dots, \varphi_m)^T, \qquad g = (\phi_1, \phi_2, \dots, \phi_m)^T, \qquad A = (a_{ij})_{i,j=1}^m,$$
$$h = \big(H(t_1, \varphi_1), H(t_2, \varphi_2), \dots, H(t_m, \varphi_m)\big)^T.$$

As in $\mathbb{R}^m$ we can consider divided difference of first order that does not need that the function F is differentiable (see [11]), we then use the divided difference of first order given by $[u, v; F] = ([u, v; F]_{ij})_{i,j=1}^m \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, where

$$[u, v; F]_{ij} = \frac{1}{u_j - v_j}\big(F_i(u_1, \dots, u_j, v_{j+1}, \dots, v_m) - F_i(u_1, \dots, u_{j-1}, v_j, \dots, v_m)\big),$$

$u = (u_1, u_2, \dots, u_m)^T$ and $v = (v_1, v_2, \dots, v_m)^T$.

Next, if we consider, for example, that nonlinear system (5) is of the form

$$F(f) \equiv f - g - A(\delta\nu + \mu\theta) = 0, \qquad F : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \tag{6}$$

where

$$\nu = \big(\varphi_1^2, \varphi_2^2, \dots, \varphi_m^2\big)^T, \qquad \theta = \big(|\varphi_1|, |\varphi_2|, \dots, |\varphi_m|\big)^T,$$

$\delta, \mu \in \mathbb{R}$ and $\mu \neq 0$, it is obvious that the function F defined in (6) is nonlinear and nondifferentiable. Moreover, $[u, v; F] = I - (\delta B + \mu C)$, where $B = (b_{ij})_{i,j=1}^m$ with $b_{ij} = a_{ij}(u_j + v_j)$ and $C = (c_{ij})_{i,j=1}^m$ with $c_{ij} = a_{ij} \frac{|u_j| - |v_j|}{u_j - v_j}$. In addition,

$$\big\|[x, y; F] - [u, v; F]\big\| \le L + K\big(\|x - u\| + \|y - v\|\big) \quad \text{with } L = 2|\mu|\|A\| \text{ and } K = |\delta|\|A\|.$$

Observe then that if the divided difference of first order of the function F satisfies a condition of type

$$\big\|[x, y; F] - [u, v; F]\big\| \le L + K\big(\|x - u\| + \|y - v\|\big); \quad L, K \ge 0;\ x, y, u, v \in D;\ x \neq y;\ u \neq v, \tag{7}$$

in $\mathbb{R}^m$, we can solve equations where the function F is nondifferentiable, as for example Eq. (6).

Notice that condition (7) includes the cases in which the function F is differentiable ($L = 0$) and nondifferentiable ($L \neq 0$).

3. Semilocal convergence of Steffensen's method

Moreover, for the sequence $\{x_n\}$ given by Steffensen's method, it is clear that every divided difference of first order $[x_k, x_k + F(x_k); F]$ exists, except for $x_k = x_k + F(x_k)$, in which case it is evident that x_k is a solution of $F(x) = 0$, so that $x_n = x_k$, for all $n \geq k$, and then the sequence $\{x_n\}$ converges to $x_k \equiv x^*$, which is a solution of $F(x) = 0$.

We start with a technical lemma which is used later. The proof is immediate from the algorithm of Steffensen's method and the definition of the divided difference of first order.

Lemma 1. *Let $\{x_n\}$ be the sequence given by Steffensen's method. If $x_{m-1} \neq x_m$ with $x_{m-1}, x_m \in D$, then*

$$F(x_m) = \big([x_m, x_{m-1}; F] - A_{m-1}\big)(x_m - x_{m-1}), \quad \text{where } A_{m-1} = \big[x_{m-1}, x_{m-1} + F(x_{m-1}); F\big].$$

Next, we present a new semilocal convergence result for Steffensen's method. For this, given $x_0, x_0 + F(x_0) \in D$, we observe that $x_0 \neq x_0 + F(x_0)$, since x_0 is a solution of $F(x) = 0$ in other case, and consequently, $x_n = x_0$ for all $n \in \mathbb{N}$. After that, suppose the following conditions:

(C_1) $\|F(x_0)\| \leq \delta$,
(C_2) $A_0^{-1} = [x_0, x_0 + F(x_0); F]^{-1}$ exists and is such that $\|A_0^{-1}\| \leq \beta$,
(C_3) $\|[x, y; F] - [u, v; F]\| \leq L + K(\|x - u\| + \|y - v\|)$; $L, K \geq 0$; $x, y, u, v \in D$; $x \neq y$; $u \neq v$.

Next, we present the new semilocal convergence result for Steffensen's method under conditions (C_1)–(C_3).

Theorem 2. *Let $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a nonlinear function defined on a non-empty open convex domain D. Suppose that conditions (C_1)–(C_3) are satisfied. If the equation*

$$t = \frac{\beta\delta(1 - \beta(L + K(2t + \delta)))}{1 - \beta(L + K(2t + \delta)) - M} + M\delta, \tag{8}$$

where $M = \beta(L + K\delta(1 + \beta))$, has at least one positive real root and the smallest positive real root, denoted by R, satisfies

$$M + \beta\big(L + K(2R + \delta)\big) < 1 \tag{9}$$

and $B(x_0, R) \subset D$, then Steffensen's method, starting at x_0, is well-defined and converges to a solution x^* of $F(x) = 0$. Moreover, the solution x^* and the iterates x_n belong to $\overline{B(x_0, R)}$ and x^* is unique in $\overline{B(x_0, R)}$.

Proof. We begin by proving that the sequence given by Steffensen's method is well-defined and $x_n \in B(x_0, R) \subset D$ for all $n \in \mathbb{N}$. Note that the smallest positive real root R of Eq. (8) satisfies

$$R = \frac{\beta\delta}{1 - P} + M\delta, \tag{10}$$

where $P = \frac{M}{1-\beta(L+K(2R+\delta))} \in (0, 1)$, as we can see easily from (9).

From (C_1)–(C_2), it follows that x_1 is well-defined and

$$\|x_1 - x_0\| \leq \big\|A_0^{-1}\big\|\big\|F(x_0)\big\| \leq \beta\delta < R,$$

as a consequence of (10). Thus, $x_1 \in B(x_0, R)$. Moreover, by Lemma 1, it follows $F(x_1) = ([x_1, x_0; F] - A_0)(x_1 - x_0)$, so that

$$\|F(x_1)\| \leq \big(L + K\big(\|x_1 - x_0\| + \|F(x_0)\|\big)\big)\|x_1 - x_0\| \leq M\delta.$$

Furthermore, again by (10),

$$\|x_1 + F(x_1) - x_0\| \leq \|x_1 - x_0\| + \|F(x_1)\| \leq \beta\delta + M\delta < R$$

and $x_1 + F(x_1) \in B(x_0, R)$.

Next, by (C_3), we see

$$\|I - A_0^{-1}A_1\| \leq \beta\big(L + K(2R + \delta)\big) < 1,$$

and, by the Banach lemma, the operator A_1^{-1} exists and

$$\|A_1^{-1}\| \leq \frac{\beta}{1 - \beta(L + K(2R + \delta))}.$$

Consequently, since $P < 1$, we have

$$\|x_2 - x_1\| \leq \|A_1^{-1}\|\|F(x_1)\| \leq P\|x_1 - x_0\| < \beta\delta,$$

$$\|x_2 - x_0\| \leq \|x_2 - x_1\| + \|x_1 - x_0\| \leq (1 + P)\|x_1 - x_0\| < \frac{\beta\delta}{1 - P} < R.$$

Thus, $x_2 \in B(x_0, R)$. In addition, as

$$\|F(x_2)\| \leq \big(L + K\big(\|x_2 - x_1\| + \|F(x_1)\|\big)\big)\|x_2 - x_1\| \leq M\delta,$$

it is clear that $x_2 + F(x_2) \in B(x_0, R)$, since

$$\|x_2 + F(x_2) - x_0\| \leq \|x_2 - x_0\| + \|F(x_2)\| < \frac{\beta\delta}{1 - P} + M\delta = R.$$

After that, if we assume that

- the operator A_n^{-1} exists and $\|A_n^{-1}\| \leq \frac{\beta}{1-\beta(L+K(2R+\delta))}$,
- $\|F(x_n)\| \leq (L + K(\delta(1 + \beta)))\|x_n - x_{n-1}\|$,
- $\|x_{n+1} - x_n\| \leq P\|x_n - x_{n-1}\| \leq P^n\|x_1 - x_0\| < \beta\delta$,

provided that $A_i = [x_i, x_i + F(x_i); F]$ is invertible and $x_{i+1}, x_{i+1} + F(x_{i+1}) \in B(x_0, R)$, for all $i = 1, 2, \ldots, n - 1$, it follows in the same way that A_{n+1}^{-1} exists,

$$\|A_{n+1}^{-1}\| \leq \frac{\beta}{1 - \beta(L + K(2R + \delta))},$$

$$\|F(x_{n+1})\| < \big(L + K\delta(1 + \beta)\big)\|x_{n+1} - x_n\|,$$

$$\|x_{n+2} - x_{n+1}\| \leq P^{n+1}\|x_1 - x_0\| < \beta\delta,$$

$$\|x_{n+2} - x_0\| \leq \frac{1 - P^{n+2}}{1 - P}\|x_1 - x_0\| < \frac{\beta\delta}{1 - P} < R,$$

$$\|x_{n+2} + F(x_{n+2}) - x_0\| < \frac{\beta\delta}{1 - P} + M\delta = R,$$

by (10) and $P < 1$, so that $x_{n+1}, x_{n+1} + F(x_{n+1}) \in B(x_0, R)$ for all $n \in \mathbb{N}$.

Once we have seen that the sequence defined by Steffensen's method is well-defined, we see that it is a Cauchy sequence. Indeed, since

$$\|x_{n+j}-x_n\| \le \sum_{i=1}^{j}\|x_{n+i}-x_{n+i-1}\| < \sum_{i=1}^{j}P^{n+i-1}\|x_1-x_0\| < \frac{P^n}{1-P}\|x_1-x_0\|,$$

for $j \ge 1$, and $P < 1$, it is clear that $\{x_n\}$ is a Cauchy sequence. In consequence, $\{x_n\}$ is convergent. Now, if $\lim_{n\to\infty} x_n = x^*$, it follows that $F(x^*) = 0$ from the continuity of the function F, since

$$\big\|F(x_n)\big\| \le \big(L + K\delta(1+\beta)\big)\|x_n - x_{n-1}\|$$

and $\|x_n - x_{n-1}\| \to 0$, by letting $n \to \infty$.

Finally, we prove the uniqueness of the solution x^* in $\overline{B(x_0,R)}$. Suppose that y^* is another solution of $F(x) = 0$ in $\overline{B(x_0,R)}$. If $A = [y^*, x^*; F]$ is invertible, it follows that $x^* = y^*$, since $A(y^* - x^*) = F(y^*) - F(x^*)$. By the Banach lemma, we have that A is invertible, since

$$\big\|I - A_0^{-1}A\big\| \le \big\|A_0^{-1}\big\|\|A_0 - A\| \le \beta\big(L + K(2R+\delta)\big) < 1.$$

The proof is complete. □

4. Domain of parameters

Once we have proved the semilocal convergence of Steffensen's method, our next aim is to see what the domain of parameters associated to Theorem 2 is. We define the domain of parameters associated to Theorem 2 for Steffensen's method as the region of a plane whose points allow us to guarantee the convergence of Steffensen's method from the conditions imposed in Theorem 2.

For the last, we transform Eq. (8) into the following equivalent quadratic equation

$$2K\beta t^2 + \big(M(1-2K\delta\beta) + \beta\big(L + K\delta(1-2\beta)\big) - 1\big)t - \delta\big(M^2 + (L\beta + K\delta\beta - 1)(M+\beta)\big) = 0, \quad (11)$$

and see when (11) has positive real roots in the following lemma.

Lemma 3. *If*

$$\delta\big(M^2 + (L\beta + K\beta\delta - 1)(M+\beta)\big) < 0, \quad (12)$$

$$M(1-2K\beta\delta) + \beta\big(L + K\delta(1-2\beta)\big) + \sqrt{-8K\beta\delta\big(M^2 + (L\beta + K\beta\delta - 1)(M+\beta)\big)} < 1, \quad (13)$$

then Eq. (11) has two positive real roots. Moreover, if we denote the smallest one by

$$R = \frac{1}{4K\beta}\big(1 - M(1-2K\beta\delta) - \beta\big(L + K\delta(1-2\beta)\big) - \sqrt{\Delta}\,\big), \quad (14)$$

where $\Delta = (M(1-2K\beta\delta)+\beta(L+K\delta(1-2\beta))-1)^2 + 8K\beta\delta(M^2+(L\beta+K\beta\delta-1)(M+\beta))$, *then condition (9) of Theorem 2 is satisfied if*

$$1 - L\beta - (1 + 2K\delta\beta)M - K\delta\beta(1+2\beta) + \sqrt{\Delta} > 0 \quad (15)$$

holds.

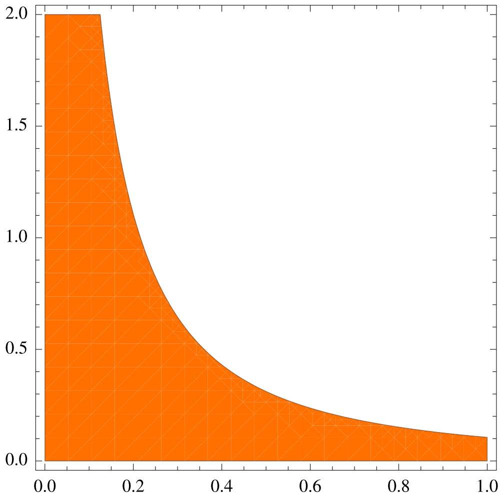

Fig. 1. Domain of parameters of Steffensen's method associated to Corollary 4 when $L = 0$ (differentiable case).

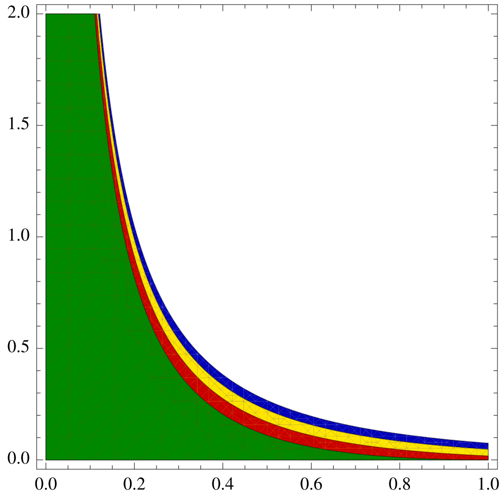

Fig. 2. Domains of parameters of Steffensen's method associated to Corollary 4 when $L = \frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{10}$ (green, red, yellow and blue, respectively; nondifferentiable case). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Note that we can also consider the existence of a double root in the previous lemma. For this, just consider no strict inequalities.

We emphasise that we do not consider that Eq. (11) has one positive real root and one negative real root because, in this case, the positive real root does not satisfy condition (15).

In addition, we formulate the following result, whose proof follows easily from Theorem 2.

Corollary 4. *Let $F : D \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a nonlinear function defined on a non-empty open convex domain D. Suppose that conditions (C_1)–(C_3) are satisfied. Moreover, if (12), (13) and (15) hold and $B(x_0, R) \subset D$, where R is defined in (14), then Steffensen's method, starting at x_0, is well-defined and converges to a solution x^* of $F(x) = 0$. Furthermore, the solution x^* and the iterates x_n belong to $\overline{B(x_0, R)}$ and x^* is unique in $\overline{B(x_0, R)}$.*

After that, taking into account $x = \beta$ and $y = K\delta$, from conditions (12), (13) and (15), we can define the domain of parameters associated to Corollary 4 for Steffensen's method as the region of the xy-plane whose points satisfy conditions (12), (13) and (15), so that the convergence of Steffensen's method is guaranteed from the hypotheses imposed in Corollary 4. For this, we colour the values of the parameters that satisfy conditions (12), (13) and (15) in the xy-plane. Note that initial conditions (C_1)–(C_2), required to the initial approximation x_0, define the parameters δ and β, and condition (C_3), required to the operator F, defines the fixed parameters L and K.

Next, we draw the domain of parameters associated to Corollary 4. We distinguish two cases: the differentiable one, $L = 0$, Fig. 1, and the non-differentiable one, $L \neq 0$, Fig. 2. In view of these two figures, we can say that the domain of parameters is bigger as the value of L decreases.

5. A differentiable case

If we consider the study of the semilocal convergence result under the majorant principle of Kantorovich, we obtain the following semilocal convergence result (see [8]).

Theorem 5. *Let $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a once continuously differentiable function defined on a non-empty open convex domain D. Suppose the conditions*

$$\|F(x_0)\| \le \delta; \qquad \|[F'(x_0)]^{-1}\| \le \gamma; \qquad \|F'(x) - F'(y)\| \le \lambda\|x-y\|, \quad \lambda \ge 0,\ x,y,\in D,$$

are satisfied. Moreover, if

$$\lambda\gamma\delta \le 2, \qquad \delta\ell^2 Q \le \frac{1}{2}, \tag{16}$$

where $\ell = \frac{2\gamma}{2-\lambda\gamma\delta}$ and $Q = \lambda(1+\frac{1}{\ell})$, hold and $B(x_0, t^*+\delta) \subset D$, where $t^* = \frac{1-\sqrt{1-2\delta\ell^2 Q}}{\ell Q}$, then Steffensen's method, starting at x_0, is well-defined and converges to a solution x^* of $F(x) = 0$.

Now, our next aim is to compare the domains of parameters of Steffensen's method associated to Corollary 4 and Theorem 5 and see which one is bigger. For this, we have to represent the same values in the axes of the xy-plane where the domains of parameters are drawn. Then, we have to write γ and λ based on β and K, so that we represent the values of the inverses of the same divided differences in the x-axis and, in the y-axis, the product $K\delta$. So, we proceed as follows.

On the one hand, as F is differentiable, $F'(x) = [x,x;F]$ and then

$$\|I - A_0^{-1}F'(x_0)\| \le \|A_0^{-1}\|\|A_0 - F'(x_0)\| \le K\beta\|F(x_0)\| = K\beta\delta,$$

by the Banach lemma on invertible operators, we have that $[F'(x_0)]^{-1} \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$ exists and is such that

$$\|[F'(x_0)]^{-1}\| \le \frac{\beta}{1-K\beta\delta},$$

provided that $K\beta\delta < 1$. In this case, $\gamma = \frac{\beta}{1-K\beta\delta}$.

On the other hand, as

$$\|F'(x) - F'(y)\| = \|[x,x;F] - [y,y;F]\| \le 2K\|x-y\|,$$

it follows $\lambda = 2K$.

In consequence, the conditions given in (16) are satisfied if

$$2K\beta\delta \le 1 \quad \text{and} \quad 4(K\beta\delta)^2 - 4K\beta\delta\bigl(2 + \beta(1-2K\delta)\bigr) + 1 \ge 0.$$

In addition, we can compare the domains of parameters associated to Corollary 4 and Theorem 5. We can see in Fig. 3 that the domain of parameters associated to Corollary 4 is bigger than that associated to Theorem 5. Therefore, we improve the domain of parameters of Steffensen's method, which is obtained by the classical technique of the majorant principle of Kantorovich, by the novelty technique developed in this work.

We illustrate the above-mentioned with a nonlinear differentiable system. We see that we can guarantee the semilocal convergence of Steffensen's method by Theorem 2, but we cannot do that by Theorem 5.

We consider Eq. (3) with $a = 0$, $b = 1$, $\phi(s) = 2$ and $H(t,\varphi(t)) = \frac{3}{4}\varphi(t)^2$. Next, we use a process of discretisation and transform it into the following nonlinear system (see Section 2):

$$F(f) \equiv f - g - \frac{3}{4}A\nu = 0, \qquad F : \mathbb{R}^8 \longrightarrow \mathbb{R}^8, \tag{17}$$

where

$$f = (\varphi_1, \varphi_2, \dots, \varphi_8)^T, \qquad g = (2,2,\dots,2)^T, \qquad A = (a_{ij})_{i,j=1}^8, \qquad \nu = \bigl(\varphi_1^2, \varphi_2^2, \dots, \varphi_8^2\bigr)^T.$$

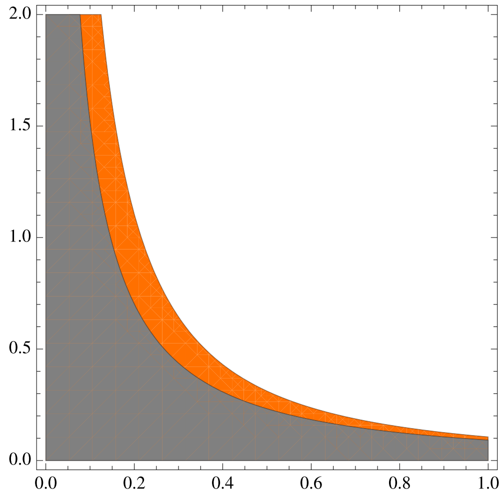

Fig. 3. Domains of parameters of Steffensen's method associated to Corollary 4 (orange) and Theorem 5 (grey). (For interpretation of the references to colour in this figure legend, the reader is referred to the web version of this article.)

Table 1
Numerical solution x^* of system (17).

i	x_i^*	i	x_i^*	i	x_i^*	i	x_i^*
1	2.042845...	3	2.422455...	5	2.576941...	7	2.206427...
2	2.206427...	4	2.576941...	6	2.422455...	8	2.042845...

In addition, see again Section 2, the divided difference of first order of F is

$$[u, v; F] = I - \frac{3}{4} A \operatorname{diag}\{u_1 + v_1, u_2 + v_2, \dots, u_8 + v_8\}.$$

Now, if we choose $x_0 = (\frac{7}{5}, \frac{7}{5}, \dots, \frac{7}{5})^T$ as starting point and the max-norm, we obtain $\delta = 0.7816\dots$, $\gamma = 1.3335\dots$ and $\lambda = 0.1853\dots$. Consequently, the second condition of (16) is not satisfied, since $\delta\ell^2 Q = 0.5295\dots > \frac{1}{2}$, so that we cannot guarantee the convergence of Steffensen's method by Theorem 5. However, we can do it by Theorem 2, since Eq. (8), which is reduced to

$$\frac{(0.2272\dots)(t - 2.5669\dots)(t - 1.6860\dots)}{(0.2272\dots)t - (0.7135\dots)} = 0,$$

has two positive real roots and the smallest one, $R = 1.6860\dots$, satisfies condition (9), since

$$M + \beta\big(L + K(2R + \delta)\big) = 0.6695\dots < 1,$$

where $M = 0.1976\dots$, $L = 0$ and $K = 0.0926\dots$. In this case, we can use Steffensen's method to approximate a solution of (17). In Table 1 we can see the numerical approximation $x^* = (x_1^*, x_2^*, \dots, x_8^*)^T$ of the solution after five iterations of Steffensen's method when the stopping criterion $\|x_n - x_{n-1}\| < 10^{-16}$ is used. In Table 2, we show the errors $\|x_n - x^*\|$ obtained with the same stopping criterion. Notice that the vector shown in Table 1 is a good approximation of a solution of system (17), since $\|F(x^*)\| \leq \text{constant} \times 10^{-16}$. See the sequence $\{\|F(x_n)\|\}$ in Table 2. Moreover, by Theorem 2, the existence of solution is guaranteed in the ball $B(x_0, 1.6860\dots)$.

6. A non-differentiable case

We consider a nondifferentiable system and see that Theorem 2 guarantees the semilocal convergence of Steffensen's method. We also see that the study of the semilocal convergence of Steffensen's method given

Table 2
Absolute errors obtained by Steffensen's method and $\|F(x_n)\|$ for system (17).

n	$\|x^* - x_n\|$	$\|F(x_n)\|$
0	$1.1769\ldots$	$7.8163\ldots \times 10^{-1}$
1	$2.2469\ldots \times 10^{-1}$	$1.3937\ldots \times 10^{-1}$
2	$8.8921\ldots \times 10^{-3}$	$5.4331\ldots \times 10^{-3}$
3	$1.4371\ldots \times 10^{-5}$	$8.7842\ldots \times 10^{-6}$
4	$3.7504\ldots \times 10^{-11}$	$2.29284\ldots \times 10^{-11}$

in [9], for equations $F(x) = 0$ such that F is nondifferentiable, is more restrictive, since it requires that $\|F(x)\|$ is bounded in D, as we see below.

Theorem 6. *(See [9].) Let $F : \Omega \subseteq \mathbb{R}^m \to \mathbb{R}^m$ be a function defined on a non-empty open convex domain D. Suppose that conditions (C_1)–(C_3) are satisfied and $\|F(x)\| \le \sigma$ in D. If the equation*

$$(\sigma - t)\left(1 - \frac{\beta(L + K(\delta\beta + \sigma))}{1 - \beta(L + K(2t + \delta))}\right) + \delta\beta = 0, \tag{18}$$

has at least one positive real root and the smallest positive real root, denoted by ρ, satisfies

$$\beta\bigl(2L + K\bigl(\delta(\beta + 1) + \sigma + 2\rho\bigr)\bigr) < 1 \tag{19}$$

and $B(x_0, \rho) \subset D$, then Steffensen's method, starting at x_0, is well-defined and converges to a solution x^* of $F(x) = 0$. Moreover, the solution x^* and the iterates x_n belong to $\overline{B(x_0, \rho)}$ and x^* is unique in $\overline{B(x_0, \rho)}$.

We consider Eq. (3) with $a = 0$, $b = 1$, $\phi(s) = 2$ and $H(t, \varphi(t)) = \frac{1}{2}(\varphi(t)^2 + |\varphi(t)|)$. Next, we use a process of discretisation and transform it into the following nonlinear and nondifferentiable system (see Section 2):

$$F(f) \equiv f - g - \frac{1}{2}A(\nu + \theta) = 0, \qquad F : \mathbb{R}^8 \longrightarrow \mathbb{R}^8, \tag{20}$$

where

$$f = (\varphi_1, \varphi_2, \ldots, \varphi_8)^T, \qquad g = (2, 2, \ldots, 2)^T, \qquad A = (a_{ij})_{i,j=1}^8,$$
$$\nu = \left(\varphi_1^2, \varphi_2^2, \ldots, \varphi_8^2\right)^T, \qquad \theta = \bigl(|\varphi_1|, |\varphi_2|, \ldots, |\varphi_8|\bigr)^T.$$

We consider again the divided difference given in Section 2, since it does not need that the function F is differentiable in $\mathbb{R}^8$, and obtain

$$[u, v; F] = I - \frac{1}{2}A\,\mathrm{diag}\left\{u_1 + v_1 + \frac{|u_1| - |v_1|}{u_1 - v_1}, u_2 + v_2 + \frac{|u_2| - |v_2|}{u_2 - v_2}, \ldots, u_8 + v_8 + \frac{|u_8| - |v_8|}{u_8 - v_8}\right\}.$$

Moreover,

$$\bigl\|F(x)\bigr\| \le 2 + \|x\| + \frac{1}{2}\|A\|\|x\|\bigl(\|x\| + 1\bigr) = \psi\bigl(\|x\|\bigr),$$

so that $\|F(x)\|$ is not bounded, since ψ is an increasing function. Therefore, Theorem 6 cannot be applied.

To solve the last problem and apply Theorem 6, an alternative is to locate previously a solution of Eq. (20) in some domain and look for a bound for $\|F(x)\|$ in it. For this, we choose the max-norm and taking into account that $\|A^{-1}\| = 0.1235\ldots$ and a solution x^* of (20) satisfies $x^* = g + \frac{1}{2}A(\nu + \theta)$, we have

$$\bigl\|x^*\bigr\| \le 2 + \frac{1}{2}\|A\|\bigl\|x^*\bigr\|\bigl(\bigl\|x^*\bigr\| + 1\bigr),$$

Table 3
Numerical solution x^* of system (20).

i	x_i^*	i	x_i^*	i	x_i^*	i	x_i^*
1	$2.039360\ldots$	3	$2.383586\ldots$	5	$2.521279\ldots$	7	$2.188708\ldots$
2	$2.188708\ldots$	4	$2.521279\ldots$	6	$2.383586\ldots$	8	$2.039360\ldots$

Table 4
Absolute errors obtained by Steffensen's method and $\|F(x_n)\|$ for system (20).

n	$\|x^* - x_n\|$	$\|F(x_n)\|$
0	$3.6063\ldots \times 10^{-1}$	$3.5949\ldots \times 10^{-1}$
1	$2.3907\ldots \times 10^{-3}$	$1.6324\ldots \times 10^{-3}$
2	$6.7249\ldots \times 10^{-7}$	$4.6622\ldots \times 10^{-7}$
3	$5.0981\ldots \times 10^{-14}$	$3.5423\ldots \times 10^{-14}$

since $\|\nu\| \leq \|x^*\|^2$ and $\|\theta\| \leq \|x^*\|$, so that we obtain $\|x^*\| \in [0, \varrho_1] \cup [\varrho_2, +\infty]$, where $\varrho_1 = 2.5648\ldots$ and $\varrho_2 = 12.6217\ldots$ are the two positive real roots of the scalar equation $t - 2 - \frac{\|A\|}{2} t(t+1) = 0$.

From the last, we can consider $F : \Omega \subset \mathbb{R}^8 \to \mathbb{R}^8$ with $D = \{x \in \mathbb{R}^8;\ \|x\| < 4\}$, since $\varrho_1 < 4 < \varrho_2$, so that $\|F(x)\| \leq 7.2355\ldots = \sigma$.

If we choose $x_0 = (\frac{12}{5}, \frac{12}{5}, \ldots, \frac{12}{5})^T$ as starting point, we obtain $\delta = 0.3594\ldots$, $\beta = 1.5115\ldots$ and condition (19) is not satisfied, since

$$\beta\big(2L + K\big(\delta(\beta+1) + \sigma + 2\rho\big)\big) = 2.5255\ldots \geq 1,$$

where $\rho = 7.4537\ldots$ is the smallest positive real root of Eq. (18), which is reduced to

$$\frac{(0.1867\ldots)(t - 7.4537\ldots)(t + 0.3895\ldots)}{(0.7796\ldots) - (0.1867\ldots)t} = 0.$$

Therefore, we cannot guarantee the semilocal convergence of Steffensen's method by Theorem 6, even locating a solution previously. However, we can do it by Theorem 2, since Eq. (8), that is

$$\frac{(0.1867\ldots)(t - 2.2255\ldots)(t - 1.1385\ldots)}{(0.1867\ldots)t - (0.5085\ldots)} = 0,$$

has two positive real roots and the smallest one, $R = 1.1385\ldots$, satisfies condition (9), since

$$M + \beta\big(L + K(2R + \delta)\big) = 0.7040\ldots < 1,$$

where $M = 0.2710\ldots$, $L = 0.1235\ldots$ and $K = 0.0617\ldots$. Now, we can use Steffensen's method to approximate a solution of (20). In Table 3 we can see the numerical approximation $x^* = (x_1^*, x_2^*, \ldots, x_8^*)^T$ of the solution after four iterations of Steffensen's method when the stopping criterion $\|x_n - x_{n-1}\| < 10^{-16}$ is used. In Table 4, we show the errors $\|x^* - x_n\|$ obtained with the same stopping criterion. Notice that the vector shown in Table 3 is a good approximation of a solution of system (20), since $\|F(x^*)\| \leq \text{constant} \times 10^{-16}$. See the sequence $\{\|F(x_n)\|\}$ in Table 4. Moreover, by Theorem 2, the existence and uniqueness of solution is guaranteed in the ball $\overline{B(x_0, 1.1385\ldots)}$.

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