ELSEVIER Journal of Computational and Applied Mathematics 83 (1997) 131–135

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

Letter

Relaxing convergence conditions for an inverse-free Jarratt-type approximation

J.A. Ezquerro*, M.A. Hernández

Department of Mathematics and Computation, University of La Rioja, C/Luis de Ulloa s/n, 26004, Logroño, Spain

Received 15 November 1996; received in revised form 20 March 1997

Abstract

We consider an inverse-free Jarratt-type approximation of order four in a Banach space (Argyros et al., 1996). We establish a convergence theorem by using recurrence relations. The purpose of this note is to relax convergence conditions and give an example where our convergence theorem can be applied but not the other ones.

Keywords: Nonlinear equations in Banach spaces; Fourth-order method; Recurrence relations

AMS classification: 65J15

1. Introduction

Although multipoint iteration functions are not used much in practice, one interesting family of this type of functions, studied by Traub [8], is very efficient when the equation to be solved is such that the evaluation of the first derivative is rapid compared to the function. An example of this occurs when the function is defined by an integral.

A modification of Newton's method for solving nonlinear equations of the type $F(x)=0$ was recently introduced by Argyros et al. [3]. They studied a new inverse-free approximation scheme defined, for all $n \geqslant 0$, by

$$y_n = x_n - F'(x_n)^{-1}F(x_n),$$

$$H(x_n, y_n) = F'(x_n)^{-1}[F'(x_n + \tfrac{2}{3}(y_n - x_n)) - F'(x_n)],$$

$$x_{n+1} = y_n - \tfrac{3}{4}H(x_n, y_n)[I - \tfrac{3}{2}H(x_n, y_n)](y_n - x_n), \tag{1}$$

* Corresponding author. E-mail: jezquer@siur.unirioja.es.

PII S0377-0427(97)00102-7

where F is a nonlinear operator defined on a convex domain Ω of a Banach space X with values in a Banach space Y. It has been assumed that F has a first-order continuous Fréchet derivative on Ω, $F'(x_0)^{-1}$ exists for $x_0 \in \Omega$.

A Kantorovich-like convergence theorem is given in [3], where it was shown that the previous method converges locally with order four, under the regularity assumptions

$$\|F''(x)\| \leqslant L_1, \quad \|F'''(x)\| \leqslant L_2, \quad \|F'''(x) - F'''(y)\| \leqslant L_3\|x - y\|, \quad x, y \in \Omega. \tag{2}$$

On the other hand, convergence studies, where the majorant principle [5] is applied for one-point iterations of order three, are based on conditions of the form (see [1, 2, 4, 10])

$$\|F'''(x)\| \leqslant K_1 \quad \text{or} \quad \|F''(x) - F''(y)\| \leqslant K_2\|x - y\|, \quad x, y \in \Omega. \tag{3}$$

In [7], Smale presented a new concept of point estimation. Instead of the region conditions in the Newton–Kantorovich theorem, he got the convergence of the Newton method for analytic maps from the data at one point. In Smale-like theorems for iterative methods, it is supposed that the conditions

$$\frac{1}{j!}\|\Gamma_0 F^{(j)}(x_0)\|\,\|\Gamma_0 F(x_0)\|^{j-1} \leqslant h^{j-1}, \quad j \geqslant 2, \tag{4}$$

are satisfied at x_0. The h constant is different for each method (see [7, 11]).

In this note, we give another convergence theorem for operator equations. In order to provide the convergence of (1), it is only assumed

$$\|F'(x) - F'(y)\| \leqslant K\|x - y\|, \quad x, y \in \Omega, \tag{5}$$

instead of (2), (3) or (4). Observe that we can apply a method of order four (1) under the same condition (5) as for Newton's method (see [9]). Finally, an example where the conditions (2)–(4) but not (5) fail is presented.

We denote $\overline{B(x,r)} = \{y \in X;\ \|y - x\| \leqslant r\}$ and $B(x,r) = \{y \in X;\ \|y - x\| < r\}$.

2. A convergence theorem

Theorem 2.1. *Let F be a nonlinear once Fréchet-differentiable operator in an open convex domain Ω of a Banach space X with values in a Banach space Y. Let $x_0 \in \Omega$ and suppose that $\Gamma_0 = F'(x_0)^{-1}$ exists. Let us assume*

$$\|\Gamma_0\| \leqslant a, \quad \|y_0 - x_0\| \leqslant b, \quad \|F'(x) - F'(y)\| \leqslant K\|x - y\|, \quad x, y \in \Omega.$$

If $a_0 = abK < s = 0.300637\ldots$ (s is the smallest positive root of polynomial $q(x) = 2x^6 + 3x^5 + 8x^4 - 5x^3 - 8x^2 - 24x + 8$) and $\overline{B(x_0,r)} \subset \Omega$, where

$$r = \frac{1 + (a_0/2)(1 + a_0)}{1 - M_0}\|y_0 - x_0\|,$$

$M_0 = (a_0(8 + 8a_0 + 5a_0^2 + 2a_0^3 + a_0^4))/(8(1 - b_0))$ and $b_0 = a_0(1 + (a_0/2)(1 + a_0))$, then sequence $\{x_n\}$ given by (1) is well-defined, $x_n, y_n \in B(x_0, r)$, for all $n \geqslant 0$, and converges R-quadratically at

least to a solution $x^* \in \overline{B(x_0,r)}$ *of the equation* $F(x)=0$. *Moreover, the solution* x^* *is unique in* $B(x_0,(2/aK)-r)\cap\Omega$.

Proof. Let us define the following two real sequences for all $n\geqslant 1$:

$$a_0=abK,\quad b_n=a_n(1+(a_n/2)(1+a_n)),\quad a_n=\frac{a_{n-1}^2(8+8a_{n-1}+5a_{n-1}^2+2a_{n-1}^3+a_{n-1}^4)}{8(1-b_{n-1})^2}.$$

Under the hypotheses mentioned above, the existence of x_1 is guaranteed and $\|x_1-x_0\|\leqslant(1+(a_0/2)(1+a_0))\|y_0-x_0\|$. Besides Γ_1 exists and $\|\Gamma_1\|\leqslant a/(1-b_0)$ by the Banach lemma. So x_2 is defined, and taking into account the Taylor's formula

$$F(x_1)=-\tfrac{3}{4}[F'(x_0+\tfrac{2}{3}(y_0-x_0))-F'(x_0)][I-\tfrac{3}{2}H(x_0,y_0)](y_0-x_0)$$
$$+\int_{x_0}^{x_1}[F'(x)-F'(x_0)]\,dx,$$

it follows from (1) that

$$\|y_1-x_1\|\leqslant M_0\|y_0-x_0\|,\quad K\|\Gamma_1\|\|y_1-x_1\|\leqslant a_1,$$
$$\|x_2-x_1\|\leqslant\left(1+\frac{a_1}{2}(1+a_1)\right)\|y_1-x_1\|.$$

Furthermore, as $a_0<s$, we have $a_1<a_0$. Therefore, by applying mathematical induction on n, we can replace x_1 by x_2, x_2 by x_3 and, in general, x_{n-1} by x_n to obtain that there exists $\Gamma_n=F'(x_n)^{-1}$, $\|\Gamma_n\|\leqslant(\|\Gamma_{n-1}\|)/(1-b_{n-1})$,

$$\|y_n-x_n\|\leqslant M_{n-1}\|y_{n-1}-x_{n-1}\|=\frac{a_{n-1}(8+8a_{n-1}+5a_{n-1}^2+2a_{n-1}^3+a_{n-1}^4)}{8(1-b_{n-1})}\|y_{n-1}-x_{n-1}\|,$$
$$K\|\Gamma_n\|\|y_n-x_n\|\leqslant a_n,$$
$$\|x_{n+1}-x_n\|\leqslant\left(1+\frac{a_n}{2}(1+a_n)\right)\|y_n-x_n\|,$$

$a_n<\gamma^{2n-1}a_{n-1}$, where $\gamma=a_1/a_0$, and $b_n<b_{n-1}$. In addition, $M_{n-1}\leqslant\gamma^{2^{n-2}}M_{n-2}$.

Consequently,

$$\|x_{n+1}-x_n\|\leqslant\left(1+\frac{a_0}{2}(1+a_0)\right)M_{n-1}\|y_{n-1}-x_{n-1}\|$$
$$\leqslant\cdots\leqslant\left(1+\frac{a_0}{2}(1+a_0)\right)\prod_{j=0}^{n-1}M_j\|y_0-x_0\|,$$

where $\prod_{j=0}^{n-1}M_j<(1-b_0)^n\gamma^{2^n-1}$. Therefore, sequence (1) is a Cauchy sequence, since $b_0<1$ and $\gamma<1$ as a consequence of the fact that $a_0<s$, and then sequence (1) converges to $x^*\in\overline{B(x_0,r)}$. We now deduce that $F(x^*)=0$ from the continuity of F and taking into account that $\|y_n-x_n\|\leqslant\prod_{j=0}^{n-1}M_j\|y_0-x_0\|\to 0$ when $n\to\infty$.

It now follows that sequence (1) converges at least with R-order two [6] to x^* from the estimation

$$\|x^* - x_n\| \leqslant \left(1 + \frac{a_0}{2}(1 + a_0)\right) \frac{b}{\gamma b_0} \gamma^{2^n}.$$

Finally, to show uniqueness, let us assume that $y^* \in B(x_0, (2/ak) - r) \cap \Omega$ is another solution of $F(x) = 0$. Following the technique given by Argyros and Chen in [2], we observe

$$0 = F(y^*) - F(x^*) = \int_0^1 F'(x^* + t(y^* - x^*))\,\mathrm{d}t(y^* - x^*).$$

Then we have to prove that the operator $\int_0^1 F'(x^* + t(y^* - x^*))\,\mathrm{d}t$ is invertible and consequently $y^* = x^*$. Indeed, from

$$\begin{aligned} \|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|\,\mathrm{d}t &\leqslant aK \int_0^1 \|x^* + t(y^* - x^*) - x_0\|\,\mathrm{d}t \\ &\leqslant aK \int_0^1 ((1-t)\|x^* - x_0\| + t\|y^* - x_0\|)\,\mathrm{d}t \\ &< \frac{aK}{2}\left(r + \frac{2}{aK} - r\right) = 1, \end{aligned}$$

it follows that $[\int_0^1 F'(x^* + t(y^* - x^*))\,\mathrm{d}t]^{-1}$ exists. □

3. Example

We provide an example where assumptions (2)–(4) fail but the conditions of Theorem 2.1 are fulfilled.

Let us consider the system of equations $G(x, y) = 0$ where $G : (-1, 1) \times (-1, 1) \to \mathbb{R}^2$ such that

$$G(x, y) = (x^3 \ln x^2 + 2y - 1/16,\ x(y - 2)).$$

If we choose $x_0 = (0, 0)$, we observe that $G^{(j)}(x_0)$ is not defined for $j \geqslant 3$ and then (4) is not satisfied. Moreover, G does not satisfy conditions (2) and (3).

On the other hand, we can apply Theorem 2.1, since

$$a = \|\Gamma_0\|_\infty = 1/2, \quad b = \|y_0 - x_0\|_\infty = 1/32, \quad k = 10$$

and, consequently, $a_0 = abK = 0.15625 < s = 0.300637\ldots$. As a result, we only study the convergence for this system of equations under the hypotheses of Theorem 2.1.

References

[1] M. Altman, Concerning the method of tangent hyperbolas for operator equations, Bull. Acad. Pol. Sci., Ser. Sci. Math. Ast. Phys. 9 (1961) 633–637.

[2] I.K. Argyros, D. Chen, Results on the Chebyshev method in Banach spaces, Proyecciones 12 (2) (1993) 119–128.

[3] I.K. Argyros, D. Chen, Q. Qian, An inverse-free Jarratt type approximation in a Banach space, J. Approx. Theory Appl. 12 (1) (1996) 19–30.

[4] J.M. Gutiérrez, M.A. Hernández, A family of Chebyshev–Halley type methods in Banach spaces, Bull. Aust. Math. Soc. 55 (1) (1997) 113–130.
[5] L.V. Kantorovich, G.P. Akilov, Functional Analysis, Pergamon Press, Oxford, 1982.
[6] F.A. Potra, V. Pták, Nondiscrete Induction and Iterative Processes, Pitman, New York, 1984.
[7] S. Smale, Newton's method estimates from data at one point, Proc. Conf. in Honor of Gail Young, vol. 2, Laramie, New York, 1986, pp. 1–16.
[8] J.F. Traub, Iterative Methods for the Solution of Equations, Prentice-Hall, Englewood Cliffs, NJ, 1964.
[9] T. Yamamoto, A method for finding sharp error bounds for Newton's method under the Kantorovich assumption, Numer. Math. 49 (1986) 203–220.
[10] T. Yamamoto, On the method of tangent hyperbolas in Banach spaces, J. Comput. Appl. Math. 21 (1988) 75–86.
[11] S. Zheng, Point estimates for Halley's iteration, Acta. Math. Appl. Sinica 14 (1991) 376–383.