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A construction procedure of iterative methods with cubical convergence II: Another convergence approach

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Abstract

We extend the analysis of convergence of the iterations considered in Ezquerro et al. [Appl. Math. Comput. 85 (1997) 181] for solving nonlinear operator equations in Banach spaces. We establish a different Kantorovich-type convergence theorem for this family and give some error estimates in terms of a real parameter $\alpha \in [-5, 1)$. © 1998 Elsevier Science Inc. All rights reserved.

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1. Introduction

Here we are concerned with the problem of approximating a locally unique solution x^* of the equation

$$F(x) = 0 \tag{1}$$

in a Banach space X, where F is a nonlinear operator defined on some convex subset Ω of X with values in a Banach space Y.

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We continue the analysis initiated in Ref. [1] of a new parameter-based iteration following the basic idea of continuation methods [2,3]. We defined a homotopy $\alpha J_1(x) + (1-\alpha)J_0(x)$, where $\alpha \in [0,1]$, between two operators J_0 and J_1 and designed the iterative process

$$x_{\alpha,n+1} = \alpha J_1(x_{\alpha,n}) + (1-\alpha)J_0(x_{\alpha,n}), \quad n \geqslant 0, \tag{2}$$

where $x_{\alpha,n+1} = J_0(x_{\alpha,n})$ is the Chebyshev method [4,5] and $x_{\alpha,n+1} = J_1(x_{\alpha,n})$ is the convex acceleration of Newton's method [6,7].

In Ref. [1], we studied the convergence of Eq. (2) for $\alpha \in [0, 1]$. The aim of this paper is to get new iterative processes from the previous homotopy. For that, we extend the values of the parameter α by using a different technique that consists in decomposing iteration (2) (see Refs. [4,6,8]), and so we analyse the convergence of the family (2) of iterates for $\alpha \in [-5, 1)$. Under Newton–Kantorovich assumptions, we give an existence-uniqueness theorem and provide error bound expressions depending on α.

If $x_0 = x_{\alpha,0} \in \Omega$, we can define Eq. (2) as

$$\begin{aligned} y_{\alpha,n} &= x_{\alpha,n} - F'(x_{\alpha,n})^{-1}F(x_{\alpha,n}), \\ x_{\alpha,n+1} &= y_{\alpha,n} + \tfrac{1}{2}L_F(x_{\alpha,n})G_\alpha(x_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n}), \end{aligned} \tag{3}$$

where $L_F(x)$ is the linear operator defined by

$$L_F(x) = F'(x)^{-1}F''(x)F'(x)^{-1}F(x), \quad x \in X$$

(if $F'(x)^{-1}$ exists), $G_\alpha(x) = I + \alpha L_F(x)H(x), H(x) = (I - L_F(x))^{-1}, I$ is the identity operator on X and $\alpha \in [-5, 1)$. The first and second Fréchet derivatives of F evaluated at $x = x_{\alpha,n}$ are denoted by $F'(x_{\alpha,n})$ and $F''(x_{\alpha,n})$. Note that $F'(x_{\alpha,n})$ is a linear operator whereas $F''(x_{\alpha,n})$ is a bilinear operator for all $n \geqslant 0$. We prove that if the sequence $\{x_{\alpha,n}\}$ defined by Eq. (3) converges to a limit $x^* \in \Omega$, then x^* is a zero of Eq. (1).

Let us denote

$$\overline{B(x,r)} = \{x' \in X; \|x' - x\| \leqslant r\} \quad \text{and} \quad B(x,r) = \{x' \in X; \|x' - x\| < r\}.$$

2. A Kantorovich-type convergence in Banach spaces

Our convergence analysis will show that under standard Newton–Kantorovich assumptions, we have convergence to a zero x^* of Eq. (1).

Following Argyros [8], it is assumed that

(i) There exists a continuous linear operator $\Gamma_0 = F'(x_0)^{-1}, x_0 \in \Omega$. Moreover $\|\Gamma_0\| \leqslant \beta$.

(ii) $\|F''(x)\| \leqslant M$ for $x \in \Omega$.

(iii) $\|F''(x) - F''(y)\| \leqslant N\|x - y\|$ for $x, y \in \Omega$.

(iv) $\|\Gamma_0 F(x_0)\| \leqslant \eta$.

(v) Let the equation

$$g(t) \equiv \frac{k}{2}t^2 - \frac{t}{\beta} + \frac{\eta}{\beta} = 0, \tag{4}$$

where

$$(\mathrm{v}_1)\quad \frac{1-3\alpha}{1-\alpha}M^2 + \frac{N}{3\beta} \leqslant k^2 \quad \text{if } \alpha \in [-5,-1],$$

$$(\mathrm{v}_2)\quad 2\left(M^2 + \frac{N}{3(1-\alpha)\beta}\right) \leqslant k^2 \quad \text{if } \alpha \in (-1,0),$$

$$(\mathrm{v}_3)\quad M^2 + \frac{N}{3\beta} \leqslant k^2 \quad \text{if } \alpha = 0,$$

$$(\mathrm{v}_4)\quad \frac{2}{1-\alpha}\left((1+\alpha)M^2 + \frac{N}{3\beta}\right) \leqslant k^2 \quad \text{if } \alpha \in (0,1).$$

Assume that this equation has two positive roots t^* and t^{**} ($t^* \leqslant t^{**}$) or equivalently $2k\beta\eta \leqslant 1$.

Let us define the scalar sequence $\{t_{\alpha,n}\}$ for all $\alpha \in [-5, 1)$ by

$$t_0 = t_{\alpha,0}, \quad s_{\alpha,n} = t_{\alpha,n} - \frac{g(t_{\alpha,n})}{g'(t_{\alpha,n})}, \quad n \geqslant 0,$$

$$t_{\alpha,n+1} = P_\alpha(t_{\alpha,n}) = s_{\alpha,n} + \frac{1}{2}L_g(t_{\alpha,n})\left(1 + \frac{\alpha L_g(t_{\alpha,n})}{1 - L_g(t_{\alpha,n})}\right)(s_{\alpha,n} - t_{\alpha,n}), \quad n \geqslant 0, \tag{5}$$

where g is the polynomial defined in Eq. (4) and $L_g(t) = g(t)g''(t)/g'(t)^2$.

In Lemma 2.1, we prove that the sequence $\{t_{\alpha,n}\}$ defined by Eq. (5) is increasing and converges cubically to t^* for all $\alpha \in [-5, 1)$.

Lemma 2.1. *Let g be the polynomial defined in Eq. (4) Then sequence (5) is increasing and converges cubically to t^* for all $\alpha \in [-5, 1)$.*

Proof. Note that

$$P'_\alpha(t) = \frac{L_{g'}(t)^2}{2(1 - L_{g'}(t))^2}\left[(1-\alpha)(1 - L_g(t))^2(3 - L_{g'}(t)) + \alpha(L_g(t) - L_{g'}(t))\right]$$
$$\geqslant 0$$

in $[0, t^*]$. Then by mathematical induction on n, it follows that $t_n \leqslant t^*, n \geqslant 0$.

On the other hand, it is easy to show that $t_n \leqslant t_{n+1}$ for all $n \in \mathbb{N}$ and consequently the proof is completed. □

We now show that $\{t_{\alpha,n}\}$ is a majorizing sequence of $\{x_{\alpha,n}\}$ (see Ref. [9]).

Theorem 2.2. *Let $F:\Omega\subseteq X\to Y$. Let us assume that the nonlinear operator F is twice Fréchet differentiable on Ω. Assume that conditions (i)–(v) are satisfied and $\overline{B(y_{\alpha,0},t^*-\eta)}\subset\Omega$. Then the iterations generated by Eq. (3) are well defined for all $n\geqslant 0$ and converge to a zero $x^*\in B(x_0,t^*)$ of Eq. (1) for all $\alpha\in[-5,1)$. Moreover $x_{\alpha,n},y_{\alpha,n}\in\overline{B(x_0,t^*)}$, for all $n\geqslant 0$. The limit x^* is the unique solution of Eq. (1) in $B(x_0,t^{**})\cap\Omega$. Furthermore the following error estimates are true for all $n\geqslant 0$:*

$$\|x^*-x_{\alpha,n}\|\leqslant t^*-t_{\alpha,n}\quad\text{and}\quad\|x^*-y_{\alpha,n}\|\leqslant t^*-s_{\alpha,n}.$$

Besides we have:

(a) When $t^*<t^{**}$, let $\lambda_\alpha=2(1-\alpha)$ and $\theta_\alpha=(t^*/t^{**})\sqrt{\lambda_\alpha}$. Hence

(a$_1$) *If $\alpha\in[-5,0)$ and* $k\beta\eta<\dfrac{2\sqrt{\lambda_\alpha}}{(1+\sqrt{\lambda_\alpha})^2}\leqslant 0.485$,

$$t^*-t_{\alpha,n}\sim\frac{(t^{**}-t^*)\theta_\alpha^{3^n}}{\sqrt{\lambda_\alpha}-\theta_\alpha^{3^n}},\quad n\geqslant 0,$$

where $\theta_\alpha<1$.

(a$_2$) *If $\alpha\in\left[0,\frac{1}{2}\right)$ and* $k\beta\eta<\dfrac{2\sqrt{\lambda_\alpha}}{(1+\sqrt{\lambda_\alpha})^2}<0.5$,

$$t^*-t_{\alpha,n}\sim\frac{(t^{**}-t^*)\theta_\alpha^{3^n}}{\sqrt{\lambda_\alpha}-\theta_\alpha^{3^n}},\quad n\geqslant 0,$$

where $\theta_\alpha<1$.

(a$_3$) *If $\alpha\in[\frac{1}{2},1)$,*

$$t^*-t_{\alpha,n}\sim\frac{(t^{**}-t^*)\theta_\alpha^{3^n}}{\sqrt{\lambda_\alpha}-\theta_\alpha^{3^n}},\quad n\geqslant 0,$$

where $\lambda_\alpha\leqslant 1$ and $\theta_\alpha<1$.

(b) When $t^*=t^{**}$,

$$t^*-t_{\alpha,n}=t^*\left(\frac{3-\alpha}{8}\right)^n,\quad n\geqslant 0.$$

We first need the following results.

Lemma 2.3. *Let $F:\Omega\subseteq X\to Y$. Let us assume that the nonlinear operator F is twice Fréchet differentiable on Ω, the iterations $\{x_{\alpha,n}\}$ generated by Eq. (3) belong to Ω and $F'(x_{\alpha,n})^{-1}$ exists for all $n\geqslant 0$. Then we have for $n\geqslant 0$:*

$$
\begin{aligned}
F(x_{\alpha,n+1}) = & \int_0^1 F''(y_{\alpha,n} + t(x_{\alpha,n+1} - y_{\alpha,n}))(x_{\alpha,n+1} - y_{\alpha,n})^2 (1-t)\,\mathrm{d}t \\
& + \int_0^1 F''(x_{\alpha,n} + t(y_{\alpha,n} - x_{\alpha,n}))(x_{\alpha,n+1} - y_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n})\,\mathrm{d}t \\
& + \int_0^1 F''(x_{\alpha,n} + t(y_{\alpha,n} - x_{\alpha,n}))(I - G_\alpha(x_{\alpha,n}))(y_{\alpha,n} - x_{\alpha,n})^2 (1-t)\,\mathrm{d}t \\
& + \int_0^1 [F''(x_{\alpha,n} + t(y_{\alpha,n} - x_{\alpha,n})) \\
& - F''(x_{\alpha,n})] G_\alpha(x_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n})^2 (1-t)\,\mathrm{d}t.
\end{aligned}
$$

Proof. To prove this equality, we note that

$$
\begin{aligned}
F(x_{\alpha,n+1}) = & F(x_{\alpha,n+1}) - F(y_{\alpha,n}) - F'(y_{\alpha,n})(x_{\alpha,n+1} - y_{\alpha,n}) + F(y_{\alpha,n}) \\
& + F'(y_{\alpha,n})(x_{\alpha,n+1} - y_{\alpha,n}) \\
= & \int_{y_{\alpha,n}}^{x_{\alpha,n+1}} F''(x)(x_{\alpha,n+1} - x)\,\mathrm{d}x + F(y_{\alpha,n}) + F'(y_{\alpha,n})(x_{\alpha,n+1} - y_{\alpha,n}).
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
F(y_{\alpha,n}) & = \int_{x_{\alpha,n}}^{y_{\alpha,n}} F''(x)(y_{\alpha,n} - x)\,\mathrm{d}x + F(x_{\alpha,n}) + F'(x_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n}) \\
& = \int_0^1 F''(x_{\alpha,n} + t(y_{\alpha,n} - x_{\alpha,n}))(y_{\alpha,n} - x_{\alpha,n})^2 (1-t)\,\mathrm{d}t
\end{aligned}
$$

and

$$
F'(y_{\alpha,n})(x_{\alpha,n+1} - y_{\alpha,n}) = \int_{x_{\alpha,n}}^{y_{\alpha,n}} F''(x)(x_{\alpha,n+1} - y_{\alpha,n})\,\mathrm{d}x + F'(x_{\alpha,n})(x_{\alpha,n+1} - y_{\alpha,n}).
$$

As $F''(x_{\alpha,n})$ is a symmetric bilinear operator, we see that

$$
F'(y_{\alpha,n})(x_{\alpha,n+1} - y_{\alpha,n}) = -\tfrac{1}{2} F''(x_{\alpha,n}) G_\alpha(x_{\alpha,n})(y_{\alpha,n} - x_{\alpha,n})^2
$$

and the proof is complete. □

Lemma 2.4. *The sequence $\{t_{\alpha,n}\}$ defined by Eq. (5) is a majorizing sequence of $\{x_{\alpha,n}\}$ defined by Eq. (3), i.e.*

$$\|x_{\alpha,n+1} - x_{\alpha,n}\| \leqslant t_{\alpha,n+1} - t_{\alpha,n}, \quad n \geqslant 0.$$

Proof. It suffices to show by mathematical induction that the following items are true for all $n \geqslant 0$.

$[\mathrm{I}_n]$ $\quad x_{\alpha,n} \in \overline{B(x_0, t_{\alpha,n})}$.

$[\mathrm{II}_n]$ $\quad \|\Gamma_{\alpha,n}\| \leqslant -\dfrac{1}{g'(t_{\alpha,n})}$,

$[\mathrm{III}_n]$ $\quad \|y_{\alpha,n} - x_{\alpha,n}\| \leqslant s_{\alpha,n} - t_{\alpha,n}$,

$[\mathrm{IV}_n]$ $\quad y_{\alpha,n} \in \overline{B(x_0, s_{\alpha,n})}$,

$[\mathrm{V}_n]$ $\quad \|x_{\alpha,n+1} - y_{\alpha,n}\| \leqslant t_{\alpha,n+1} - s_{\alpha,n}$.

For $\alpha \in [-5, -1]$, it is easy to check $[\mathrm{I}_0] - [\mathrm{V}_0]$ from the initial conditions (i)–(v). Now assume that the above statements are true for a fixed $n \geqslant 1$. Then $[\mathrm{I}_{n+1}]$ follows immediately.

Notice that

$$I - \Gamma_0 F'(x_{\alpha,n+1}) = \int_0^1 \Gamma_0 F''(x_0 + t(x_{\alpha,n+1} - x_0))(x_{\alpha,n+1} - x_0)\, \mathrm{d}t,$$

so

$$\|I - \Gamma_0 F'(x_{\alpha,n+1})\| \leqslant \beta k \|x_{\alpha,n+1} - x_0\| \leqslant \beta k t^* < 1,$$

and, by the Banach lemma [9] on inversion of operators, $\Gamma_{\alpha,n+1}$ exists and

$$\|\Gamma_{\alpha,n+1}\| \leqslant \frac{\|\Gamma_0\|}{1 - \|I - \Gamma_0 F'(x_{\alpha n+1})\|} \leqslant \frac{\beta}{1 - \beta k \|x_{\alpha,n+1} - x_0\|} \leqslant \frac{-1}{g'(t_{\alpha,n+1})}.$$

So $[\mathrm{II}_{n+1}]$ is also true.

By using Lemma 2.3, the Altman lemma [10] and taking into account that $\|L_F(x_{\alpha,n+1})\| \leqslant L_g(t_{\alpha,n+1})$, we can estimate $F(x_{\alpha,n+1})$ to obtain

$$\begin{aligned}
\|F(x_{\alpha,n+1})\| &\leqslant \frac{M}{2}\|x_{\alpha,n+1} - y_{\alpha,n}\|^2 + M\|x_{\alpha,n+1} - y_{\alpha,n}\|\|y_{\alpha,n} - x_{\alpha,n}\| \\
&\quad + \frac{M}{2}\|I - G_\alpha(x_{\alpha,n})\|\|y_{\alpha,n} - x_{\alpha,n}\|^2 + \frac{N}{6}\|G_\alpha(x_{\alpha,n})\|\|y_{\alpha,n} - x_{\alpha,n}\|^3 \\
&\leqslant \frac{M}{2}(t_{\alpha,n+1} - s_{\alpha,n})^2 + M\|L_F(x_{\alpha,n})\|[1 - (1+\alpha)\|L_F(x_{\alpha,n})\|] \\
&\quad \times \frac{\|y_{\alpha,n} - x_{\alpha,n}\|^2}{2(1 - L_g(t_{\alpha,n}))} + M\alpha\|L_F(x_{\alpha,n})\| \frac{\|y_{\alpha,n} - x_{\alpha,n}\|^2}{2(1 - L_g(t_{\alpha,n}))}
\end{aligned}$$

$$+\frac{N}{6}[1-(1+\alpha)\|L_F(x_{\alpha,n})\|]\frac{\|y_{\alpha,n}-x_{\alpha,n}\|^3}{1-L_g(t_{\alpha,n})}$$
$$\leqslant \frac{M}{2}(t_{\alpha,n+1}-s_{\alpha,n})^2+\left[\left(\frac{N}{3}-\frac{M^2}{g'(t_{\alpha,n})}\right)\right.$$
$$\left.\times(1-(1+\alpha)L_g(t_{\alpha,n}))+\frac{\alpha M^2}{g'(t_{\alpha,n})}\right]\frac{(s_{\alpha,n}-t_{\alpha,n})^3}{2(1-L_g(t_{\alpha,n}))}$$
$$\leqslant \frac{k}{2}(t_{\alpha,n+1}-s_{\alpha,n})^2-\left(\frac{1-3\alpha}{2}M^2+\frac{1-\alpha}{6\beta}N\right)$$
$$\times\frac{(s_{\alpha,n}-t_{\alpha,n})^3}{2g'(t_{\alpha,n})(1-L_g(t_{\alpha,n}))}\leqslant\frac{k}{2}(t_{\alpha,n+1}-s_{\alpha,n})^2-\frac{k^2(1-\alpha)}{2g'(t_{\alpha,n})}(s_{\alpha,n}-t_{\alpha,n})^3.$$

Consequently, it is satisfied

$$\|F(x_{\alpha,n+1})\|\leqslant g(t_{\alpha,n+1}) \tag{6}$$

and

$$\|y_{\alpha,n+1}-x_{\alpha,n+1}\|\leqslant\|\Gamma_{\alpha,n+1}\|\|F(x_{\alpha,n+1})\|\leqslant\frac{g(t_{\alpha,n+1})}{g'(t_{\alpha,n+1})}=s_{\alpha,n+1}-t_{\alpha,n+1}.$$

Then [III_{n+1}] holds. Now, [IV_{n+1}] and [V_{n+1}] follow easily. The cases $\alpha\in(-1,0)$ and $\alpha\in(0,1)$ are similar to the case mentioned above.

Finally, for $\alpha=0$, we deduce that

$$\|F(x_{\alpha,n+1})\|\leqslant\frac{M}{2}(t_{\alpha,n+1}-s_{\alpha,n})^2-\left(M^2+\frac{N}{3\beta}\right)\frac{(s_{\alpha,n}-t_{\alpha,n})^3}{2g'(t_{\alpha,n})}$$
$$\leqslant\frac{k}{2}(t_{\alpha,n+1}-s_{\alpha,n})^2-\frac{k^2}{2g'(t_{\alpha,n})}(s_{\alpha,n}-t_{\alpha,n})^3=g(t_{\alpha,n+1})$$

and [I_{n+1}] – [V_{n+1}] following analogously to the case $\alpha\in[-5,-1]$. □

Proof of Theorem 2.2. It follows from Lemma 2.4 that the sequence $\{t_{\alpha,n}\}$ defined by Eq. (5) majorizes the sequence $\{x_{\alpha,n}\}$ given by Eq. (3). Therefore the convergence of $\{t_{\alpha,n}\}$ implies the convergence of $\{x_{\alpha,n}\}$ to a limit x^*. Letting $n\to\infty$ in Eq. (6), we infer that $F(x^*)=0$. Moreover

$$\|x_{\alpha,n}-y_{\alpha,0}\|\leqslant\|x_{\alpha,n}-y_{\alpha,n-1}\|+\|y_{\alpha,n-1}-x_{\alpha,n-1}\|+\cdots+\|x_{\alpha,1}-y_{\alpha,0}\|$$
$$\leqslant(t_{\alpha,n}-s_{\alpha,n-1})+(s_{\alpha,n-1}-t_{\alpha,n-1})+\cdots+(t_{\alpha,1}-s_{\alpha,0})$$
$$=t_{\alpha,n}-\eta\leqslant t^*-\eta,$$

and similarly

$$\|y_{\alpha,n}-y_{\alpha,0}\|\leqslant s_{\alpha,n}-\eta\leqslant t^*-\eta.$$

Now it follows for $p\geqslant 0$,

$$\|x_{\alpha,n+p}-x_{\alpha,n}\|\leqslant t_{\alpha,n+p}-t_{\alpha,n}\quad\text{and}\quad\|x_{\alpha,n+p}-y_{\alpha,n}\|\leqslant t_{\alpha,n+p}-s_{\alpha,n}$$

and, by letteing $p \to \infty$ we get

$$\|x^* - x_{\alpha,n}\| \leqslant t^* - t_{\alpha,n} \quad \text{and} \quad \|x^* - y_{\alpha,n}\| \leqslant t^* - s_{\alpha,n}, \qquad n \geqslant 0.$$

To prove the uniqueness, we assume that there exists another solution z^* of Eq. (1) in $B(x_0, t^{**}) \cap \Omega$. Following Argyros and Chen [4] and using the estimate

$$\begin{aligned}
\|\Gamma_0\| \int_0^1 \|F'(x^* + t(z^* - x^*)) - F'(x_0)\| \, dt
&\leqslant \beta M \int_0^1 \|x^* + t(z^* - x^*) - x_0\| \, dt \\
&\leqslant \beta M \int_0^1 ((1-t)\|x^* - x_0\| + t\|z^* - x_0\|) \, dt \\
&< \frac{\beta M}{2}(t^* + t^{**}) \leqslant 1.
\end{aligned}$$

we deduce that the linear operator $\int_0^1 F'(x^* + t(z^* - x^*)) \, dt$ is invertible. It follows from the approximation

$$\int_0^1 F'(x^* + t(z^* - x^*))(z^* - x^*) \, dt = F(z^*) - F(x^*) = 0,$$

that $x^* = z^*$. Finally, see Ref. [1] to get the error bounds. □

Remark. Note that the value of $\alpha = 1$ has been omitted. For $\alpha = 1$ the iteration defined by Eq. (3) is reduced to the convex acceleration of Newton's method. In [11] it has been shown that it is a third-order method in general. However if this method is applied to polynomials of degree two, the order of the method is four. Consequently, it is not possible to find a second degree polynomial which majorizes an operator satisfying (i)–(iv) with $N \neq 0$. An analysis of the convergence of the convex acceleration of Newton's method is also made in Ref. [11] by taking polynomials of degree three.

3. Application

Let us consider the Chandrasekhar integral equation cited in Ref. [4]

$$F(x)(s) = 1 - x(s) + \frac{1}{4} x(s) \int_0^1 \frac{s}{s+t} x(t) \, dt$$

in the space $X = C[0,1]$ of all continuous functions on the interval [0, 1] with the norm

$$\|x\| = \max_{s\in[0,1]} |x(s)|.$$

Let $x_0 = x_0(s) = 1$ for Theorem 2.2. Use the definition of the first and second Fréchet derivatives of the operator F to obtain (see Ref. [4])

$$\beta = \|F'(x_0)^{-1}\| = 1.53039421,$$

$$M = \frac{1}{2}\max_{s\in[0,1]}\left|\int_0^1 \frac{s}{s+t}\,\mathrm{d}t\right| = \ln\sqrt{2} = 0.34657359,$$

$$N = 0, \quad \eta = \|F'(x_0)^{-1}F(x_0)\| = 0.2651971.$$

If we choose $\alpha = -\frac{1}{3}$ then $k = M$. As $k\beta\eta = 0.1406590 < \frac{1}{2}$, Eq. (9) becomes

$$0.173287t^2 - 0.653426t + 0.173287 = 0.$$

This equation has two positive roots: $t^* = 0.287049$ and $t^{**} = 3.48372$. Then there is a solution $x^* \in \{u \in C[0,1]; \|u-1\| \leqslant 0.287049\}$ of the equation $F(x) = 0$. Besides the solution is unique in $\{u \in C[0,1]; \|u-1\| < 3.48372\}$ and the error bound is

$$\|x^* - x_{-\frac{1}{3},n}\| \sim \frac{3.19668(0.1345538)^{3^n}}{1.6329932 - (0.1345538)^{3^n}}.$$

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