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BOUNDARY CROSSING IDENTITIES FOR DIFFUSIONS HAVING THE TIME INVERSION PROPERTY

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ABSTRACT. We review and study a one-parameter family of functional transformations, denoted by $(S^{(\beta)})_{\beta \in \mathbb{R}}$, which, in the case $\beta < 0$, provides a path realization of bridges associated to the family of diffusion processes enjoying the time inversion property. This family includes the Brownian motion, Bessel processes with a positive dimension and their conservative h -transforms. By means of these transformations, we derive an explicit and simple expression which relates the law of the boundary crossing times for these diffusions over a given function f to those over the image of f by the mapping $S^{(\beta)}$, for some fixed $\beta \in \mathbb{R}$. We give some new examples of boundary crossing problems for the Brownian motion and the family of Bessel processes. We also provide, in the Brownian case, an interpretation of the results obtained by the standard method of images and establish connections between the exact asymptotics for large time of the densities corresponding to various curves of each family.

1. INTRODUCTION

Let $X := (X_t, t \geq 0)$ be a E -valued, where $E = \mathbb{R}$ or $\mathbb{R}_+ = [0, \infty)$, conservative, in the sense that it has an infinite life-time, 2-self-similar homogenous diffusion enjoying the time inversion property in the sense of Shiga and Watanabe [39]. We denote by $(\mathbb{P}_x)_{x \in E}$ the family of probability measures of X which act on $\mathcal{C}(\mathbb{R}_+, E)$, the space of continuous functions from \mathbb{R}_+ into E , such that $\mathbb{P}_x(X_0 = x) = 1$. The standard notation \mathcal{F}_t is used for the σ -algebra generated by the process X up to time t and we write simply $\mathcal{F} = \mathcal{F}_\infty$. The above assumptions mean that, for any $x \in E$ and $c > 0$,

the law of $(c^{-1}X_{c^2t}, t \geq 0; \mathbb{P}_{cx})$ is \mathbb{P}_x (2-self-similarity)

and the process

$$(1) \quad i(X) := (tX_{1/t}, t > 0; \mathbb{P}_x) \quad (\text{time inversion})$$

is a homogeneous conservative diffusion. It is well known that the class of processes considered therein consists of Brownian motion and Bessel processes of positive dimensions,

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see Watanabe [41]. Next, let $f \in \mathcal{C}(\mathbb{R}_+, E)$, such that $f(0) \neq X_0$ and set

$$T^f = \inf \{s > 0; X_s = f(s)\}$$

with the usual convention that $\inf\{\emptyset\} = +\infty$. The study of the distribution of the stopping time T^f is known as the boundary crossing or first passage problem. Unfortunately, the explicit determination of these distributions is only attainable for a few specific functions. In this paper, we suggest a new method which allows to relate, in a simple and explicit manner, the law of T^f with the ones of the family of stopping times $(T^{f^{(\beta)}})_{\beta \in \mathbb{R}}$ with $f^{(\beta)} := S^{(\beta)}(f)$ and

$$(2) \quad \begin{aligned} S^{(\beta)} : \mathcal{C}(\mathbb{R}_+, E) &\rightarrow \mathcal{C}([0, \zeta^{(\beta)}), E) \\ f &\rightarrow (1 + \beta.) f \left(\frac{\cdot}{1 + \beta.} \right) \end{aligned}$$

where

$$\zeta^{(\beta)} = \begin{cases} 1/|\beta|, & \text{if } \beta < 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The results extend to h -transforms of the prescribed processes leading to conservative diffusions.

In the case $\beta < 0$, our methodology has a simple description. Indeed, as a consequence of the time inversion property, Pitman and Yor [34] showed that the law of the process $(S^{(\beta)}(X)_t, 0 \leq t < \zeta^{(\beta)})$ corresponds to the law of a bridge associated to X . Thus, one may relate, by means of a deterministic time change, the law of T^f with the first crossing time of the curve $f^{(\beta)}$ by the bridge $(S^{(\beta)}(X)_t, 0 \leq t < \zeta^{(\beta)})$. Finally, our identities are then obtained by using the construction of the law of bridges of Markov processes as a Doob h -transform of their laws, see Fitzsimmons et al. [16]. We also show that similar devices can be implemented in the case $\beta > 0$.

The motivations for investigating the boundary crossing problem are of both practical and theoretical importance. Indeed, on the one hand, such studies were originally motivated by their connections to sequential analysis, see e.g. Robbins and Siegmund [37] and Smirnov-Kolmogorov test, see e.g. Lerche [26]. On the other hand, this problem has found many applications in several fields of sciences, such as mathematical physics [27], neurology [23], epidemiology [28] and mathematical finance [5] and [30].

Amongst general results, in the Brownian setting, Strassen [40] proved that if f is continuously differentiable then the distribution of T^f is absolutely continuous with a continuous probability density function. Moreover, for elementary curves, the density is known explicitly for linear, square root and parabolic functions. For these curves, specific techniques proved to be efficient and we refer to Table 1 in Section 2 for a description of these cases. Besides, the distribution is known for some concave curves solving implicit equations obtained by the celebrated standard method of images, see Lerche [26]. For

some recent investigations, we refer to Pötzelberger and Wang [35] and Borovkov and Novikov [7] for numerical approximations of the density, Peskir [32], [33] for the study of the small time behavior of the density and Kendall et al. [21] for statistical applications.

The remaining part of the paper is organized as follows. In the next Section we recall some recent results regarding diffusions which enjoy the time inversion property. Section 3 is devoted to the statement of our main results and their proofs. Section 4 is concerned with a detailed study of the Brownian motion case. In particular, we present several new explicit examples of the boundary crossing problem. We also show how our results translate and agree with the standard method of images. Finally, in Section 5, we treat the example of Bessel processes and characterize the distribution of hitting times of straight lines. We also mention that, in the Brownian motion case, the boundary crossing identity (7), stated below, has been published without proofs in the note [1].

2. PRELIMINARIES

Let us recall that E is either \mathbb{R}^+ or \mathbb{R} and $X := (X_t, t \geq 0)$ is a 2-self-similar conservative homogenous diffusion enjoying the time inversion property. In recent papers, Gallardo and Yor [18] and Lawi [24] characterized the class of self-similar Markov processes (which might have càdlàg paths) satisfying the time inversion property in terms of their semi-groups when the latter are assumed to be absolutely continuous and twice differentiable. More precisely, they showed that if we write $\mathbb{P}_x(X_t \in dy) = p_t(x, y)dy$ the transitions densities have the following form

$$p_t(x, y) = \frac{c}{\sqrt{t}} \Phi\left(\frac{xy}{t}\right) \left(\frac{y}{\sqrt{t}}\right)^{2\nu+1} e^{-\frac{x^2+y^2}{2t}}, \quad t > 0, x, y \in E,$$

where the reals ν and c are related to the function $\Phi : E \rightarrow \mathbb{R}_+$ as follows.

- (1) If $E = \mathbb{R}$, then X is a Brownian motion, $\Phi(y) = e^y$ and necessarily $\nu = -1/2$ and $c = 1/\sqrt{2\pi}$.
- (2) If $E = \mathbb{R}^+$ then X is Bessel process of dimension δ for some $\delta > 0$, $c = 1$, $\nu = \frac{\delta}{2} - 1$ and $\Phi(y) = y^{-\nu} I_\nu(y)$ where I_ν is the modified Bessel function of the first kind of index ν which admits the following power series representation, see e.g. [25, Section 5.2],

$$I_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(z/2)^{2n}}{n! \Gamma(\nu + n + 1)}, \quad |z| < \infty, \quad |\arg(z)| < \pi.$$

Moreover, it is well known that, in this case, the state 0 is

- an entrance boundary if $\delta \geq 2$,
- a reflecting boundary if $0 < \delta < 2$.

Next, recalling that, for every fixed $y \in E$, the mapping $x \mapsto \Phi(yx)$ is $\frac{y^2}{2}$ -excessive, see e.g. [6], we can define a new family of probability measures as follows

$$(3) \quad d\mathbb{P}_x^{(y)}|_{\mathcal{F}_t} = \frac{\Phi(yX_t)}{\Phi(yx)} e^{-\frac{1}{2}y^2t} d\mathbb{P}_x|_{\mathcal{F}_t}, \quad t > 0.$$

Note, from the definition of Φ , that $\mathbb{P}_x^{(0)} = \mathbb{P}_x$. Moreover, for any $y \in E$, $(\mathbb{P}_x^{(y)})_{x \in E}$ is precisely the family of probability measures of the process $i(X)$ when X starts at y . We also recall that $i(X)$ is conservative and satisfies the time inversion property (1), see [18]. It follows that $i(X)$ has an absolutely continuous semi-group with densities given, for any $t > 0$ and $x, a \in E$, by

$$(4) \quad p_t^{(y)}(x, a) = \frac{1}{\sqrt{t}} \frac{\Phi(ya)}{\Phi(yx)} \Phi\left(\frac{ax}{t}\right) \left(\frac{a}{\sqrt{t}}\right)^{2\nu+1} e^{-\frac{1}{2}(ty^2+(x^2+a^2)/t)}.$$

Note, in particular, the following:

- (1) If $E = \mathbb{R}$ then $i(X)$ is a Brownian motion with drift y .
- (2) If $E = \mathbb{R}^+$ then $i(X)$ is the so-called Bessel process in the wide sense introduced by Watanabe [41].

Now, from the definition of the family of mappings $(S^{(\beta)})_{\beta \geq 0}$, see (2), we observe that for any $f \in \mathcal{C}(\mathbb{R}^+, \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, we have $S^{(0)}(f) = f$ and $S^{(\alpha)} \circ S^{(\beta)} = S^{(\alpha+\beta)}$ and so $(S^{(\beta)})_{\beta \geq 0}$ is a semi-group on $\mathcal{C}(\mathbb{R}^+, \mathbb{R})$. Moreover, the family of mappings $(S^{(\beta)})_{\beta < 0}$ naturally appears in the process of construction of bridges associated to X . To explain the connection, for any fixed $T > 0$ and $x \in E$, let us denote by $(\mathbb{P}_{x,z,T}^{(y)}, z \in E)$ a regular version of the family of conditional probability distributions $(\mathbb{P}_x^{(y)}(\cdot | X_T = z), z \in E)$. That is, for some $z \in E$, $\mathbb{P}_{x,z,T}^{(y)}$ is the law of the bridge of length T associated to X , between x and z . This, according to Fitzsimmons and al. [16], can be obtained from $\mathbb{P}_x^{(y)}$ as the following Doob h -transform

$$(5) \quad \mathbb{P}_{x,z,T}^{(y)} = \frac{p_{T-s}^{(y)}(X_s, z)}{p_T^{(y)}(x, z)} \mathbb{P}_x^{(y)} \quad \text{on } \mathcal{F}_s, \quad s < T.$$

Fitzsimmons [17] observed that these laws remain invariant under changes of probability measures of type (3). Finally, we recall that Pitman and Yor [34, Theorem 5.8], see also [18, Theorem 1], showed, by means of the time inversion property that, for any $x, z \in E$, the processes

$$(6) \quad \{X_u, 0 \leq u < T; \mathbb{P}_{x,T,z,T}^{(y)}\} \text{ and } \{S^{(-1/T)}(X)_u, 0 \leq u < T; \mathbb{P}_x^{(z)}\}$$

have the same law. Note that, in the Brownian case, (6) can also be seen from the unique decomposition, as a semi-martingale, in its own filtration, of the Brownian bridge of length

$T = -\frac{1}{\beta} > 0$, between 0 and 0. Indeed, for any $0 \leq t < T$, we can write

$$S^{(-1/T)}(X)_t = (T-t) \int_0^t \frac{d\tilde{X}_s}{T-s}, \quad X_t = T \int_0^{\frac{Tt}{T-t}} \frac{d\tilde{X}_s}{T-s}, \quad t < T,$$

where \tilde{X} is a Brownian motion on $[0, \zeta^{(-\beta)}]$ with respect to the filtration generated by $(S^{(-\frac{1}{T})}(X)_t, 0 \leq t < T)$. Thus, we have

$$S^{(-1/T)}(X)_t = \tilde{X}_t - \int_0^t \frac{S^{(-1/T)}(X)_s}{T-s} ds, \quad t < T.$$

which coincides with the canonical decomposition of the standard Brownian bridge.

3. MAIN RESULTS AND PROOFS

We keep the notation and setting of the previous section and assume, throughout this section, that β is some fixed real. We proceed by pointing out that the mapping $S^{(\beta)}$ can be defined similarly on the space of probability measures. For instance, in the absolutely continuous case, we associate to $\mu(dt) = h(t)dt$ the image $S^{(\beta)}(\mu)(dt) = h^{(\beta)}(t)dt$ where we recall that $h^{(\beta)}(t) := S^{(\beta)}(h)(t)$. We are now ready to state our main result.

Theorem 3.1. *Let $f \in \mathcal{C}(\mathbb{R}_+, E)$. Then, for any $x, y \in E$ such that $f(0) \neq x$, and $t < \zeta^{(\beta)}$, we have*

$$\begin{aligned} \mathbb{P}_x^{(y)}(T^{f^{(\beta)}} \in dt) &= (1 + \beta t)^{\nu-2} \frac{\Phi(yf^{(\beta)}(t))}{\Phi(yf^{(\beta)}(t)/(1 + \beta t))} e^{-\frac{\beta}{2} \frac{y^2 t^2}{1 + \beta t}} e^{-\frac{\beta}{2} \frac{f^{(\beta)}(t)^2}{1 + \beta t} + \frac{\beta}{2} x^2} \\ &\times S^{(\beta)}\left(\mathbb{P}_x^{(y)}(T^f \in dt)\right). \end{aligned}$$

The particular case $y = 0$ yields

$$(7) \quad \mathbb{P}_x\left(T^{f^{(\beta)}} \in dt\right) = (1 + \beta t)^{\nu-2} e^{-\frac{\beta}{2} \frac{f^{(\beta)}(t)^2}{1 + \beta t} + \frac{\beta}{2} x^2} \times S^{(\beta)}\left(\mathbb{P}_x\left(T^f \in dt\right)\right).$$

Theorem 3.1 simplifies when the focus is on straight lines. Indeed, if we consider a constant function $f \equiv a$ where $a \neq 0$ then, with $\beta = b/a$, for some $b \in \mathbb{R}$, we have $f^{(b/a)}(t) = a + bt$, $t < \zeta^{(b/a)}$. Note that if $b < 0$ then the support of T^{a+b} is $(0, -b/a)$. The previous result reads as follows.

Corollary 3.2. *For any fixed $a, b, x \in E$ such that $a \neq x$ and all $t < \zeta^{(b/a)}$, we have*

$$\begin{aligned} \mathbb{P}_x^{(y)}\left(T^{a+b} \in dt\right) &= (1 + bt/a)^{\nu-2} \frac{\Phi(y(a + bt))}{\Phi(ya)} \exp -\frac{b}{2} \left(a + bt + \frac{t^2 y^2}{a + bt} - \frac{x^2}{a} \right) \\ &\times S^{(b/a)}\left(\mathbb{P}_x^{(y)}(T^a \in dt)\right). \end{aligned}$$

In particular, when $y = 0$, we obtain

$$\begin{aligned} \mathbb{P}_x\left(T^{a+b} \in dt\right) &= (1 + bt/a)^{\nu-2} \exp -\frac{b}{2} (a + bt - x^2/a) \\ &\times S^{(b/a)}\left(\mathbb{P}_x(T^a \in dt)\right). \end{aligned}$$

Before proving Theorem 3.1 we need to prepare two intermediate results. To this end, let us denote by $H^{(\beta,f)}$ the first time when $S^{(\beta)}(X)$ crosses f i.e.

$$H^{(\beta,f)} = \inf \left\{ 0 < s < \zeta^{(\beta)}; S^{(\beta)}(X)_s = f(s) \right\}.$$

The aim of the next result is to relate the stopping times $H^{(\beta,f)}$ and $T^{f^{(\beta)}}$.

Lemma 3.3. *The identities*

$$H^{(-\beta,f)} = \frac{T^{f^{(\beta)}}}{1 + \beta T^{f^{(\beta)}}} \quad \text{and} \quad T^{f^{(\beta)}} = \frac{H^{(-\beta,f)}}{1 - \beta H^{(-\beta,f)}}$$

hold almost surely. In particular, we have $\{H^{(-\beta,f)} < \zeta^{(-\beta)}\} = \{T^{f^{(\beta)}} < \zeta^{(\beta)}\}$.

Proof. From the definition of $S^{(\beta)}(X)$ and by using a deterministic time change, we get

$$\begin{aligned} H^{(-\beta,f)} &= \inf \left\{ 0 < s < \zeta^{(-\beta)}; X_{\frac{s}{1-\beta s}} = \frac{f(s)}{1-\beta s} \right\} \\ &= \inf \left\{ 0 < s < \zeta^{(-\beta)}; X_{\frac{s}{1-\beta s}} = S^{(\beta)}(f) \left(\frac{s}{1-\beta s} \right) \right\} \\ &= \frac{T^{f^{(\beta)}}}{1 + \beta T^{f^{(\beta)}}}. \end{aligned}$$

The second identity is obtained in a similar way. The last statement follows then by observing that, for any $\beta \in \mathbb{R}$, we have $\zeta^{(\beta)} = \frac{\zeta^{(-\beta)}}{1-\beta\zeta^{(-\beta)}}$ in the limiting sense. \square

The image by the mapping $S^{(\beta)}$ of any homogeneous Markov process is clearly a non-homogeneous Markov process. However, the time inversion property allows to connect the law of X and that of $S^{(\beta)}(X)$ via a simple time-space h -transform. To be more precise, we state the following result.

Lemma 3.4. *Let $x, y \in E$ and $\beta \in \mathbb{R}$. Under $\mathbb{P}_x^{(y)}$, the process $X^{(\beta)} = S^{(\beta)}(X)$, defined on $[0, \zeta^{(\beta)})$, is a non-homogeneous strong Markov process. Its law, denoted by $\mathbb{Q}_x^{y,\beta}$, is absolutely continuous with respect to $\mathbb{P}_x^{(y)}$ with Radon-Nikodym derivative $M_x^{y,\beta}(t, X_t)$, $t < \zeta^{(\beta)}$, given by*

$$M_x^{y,\beta}(t, X_t) = (1 + \beta t)^{-\nu-1} \frac{\Phi(yX_t/(1 + \beta t))}{\Phi(yX_t)} e^{\frac{\beta}{2} \frac{y^2 t^2}{1+\beta t} + \frac{\beta}{2} \frac{X_t^2}{1+\beta t} - \frac{\beta}{2} x^2}.$$

In particular, for $y = 0$, we have

$$M_x^{0,\beta}(t, X_t) = (1 + \beta t)^{-\nu-1} e^{\frac{\beta}{2} \frac{X_t^2}{1+\beta t} - \frac{\beta}{2} x^2}.$$

Proof. We start with the case $\beta < 0$. Using successively the identities (6) and (5), we obtain, for any bounded measurable functional F and $t < \zeta^{(\beta)}$, that

$$\begin{aligned} \mathbb{E}_x^{(y)}[F(S^{(\beta)}(X)_s, s \leq t)] &= \mathbb{E}_x^{(y)}[F(X_s, s \leq t) \mid X_{-1/\beta} = -y/\beta] \\ &= \mathbb{E}_x^{(y)} \left[\frac{p_{-1/\beta-t}^{(y)}(X_t, -y/\beta)}{p_{-1/\beta}^{(y)}(x, -y/\beta)} F(X_s, s \leq t) \right] \\ &= \mathbb{E}^{(y)} \left[M_x^{y,\beta}(t, X_t) F(X_s, s \leq t) \right] \end{aligned}$$

where the last line follows from (4). Using the fact that, for any $c, d \in E$, $\frac{\Phi(cy)}{\Phi(dy)} \rightarrow 1$ as $y \rightarrow 0$, we get the stated result for $y = 0$. Next, we treat the case $\beta > 0$ and assume, first, that X is a Bessel process of dimension $\delta > 0$ and $y = 0$. Using the Itô formula, we see that the process $(M_x^{0,\beta}(t, X_t), t \geq 0)$ is a \mathbb{P}_x -local martingale. We now show that it is a true martingale. Recall that the squared Bessel process X^2 is the unique solution to the stochastic differential equation (for short sde)

$$(8) \quad dX_t^2 = 2X_t dB_t + \delta dt, \quad X_0^2 = x^2,$$

where B is a Brownian motion, see [36]. Furthermore, we need to introduce the processes Y and \tilde{Y} which are defined, for a fixed $t \geq 0$, by

$$Y_t = (1 + \beta t)\tilde{Y}_t = S^{(\beta)}(X)_t.$$

It follows from (8) and by performing a deterministic time change, that \tilde{Y} solves the sde

$$d\tilde{Y}_t^2 = \frac{\delta}{(1 + \beta t)^2} dt + \frac{2\tilde{Y}_t}{1 + \beta t} d\gamma_t$$

where $(\gamma_t, t \geq 0)$ is a Brownian motion with respect to the filtration generated by the process \tilde{Y} . Similarly, Y^2 satisfies

$$(9) \quad \begin{aligned} dY_t^2 &= d\left((1 + \beta t)^2 \tilde{Y}_t^2\right) \\ &= 2Y_t \left(d\gamma_t + \frac{\beta}{1 + \beta t} Y_t dt \right) + \delta dt. \end{aligned}$$

Now, from (8) and by Girsanov theorem, the law of the unique solution to (9) is obtained from the law of X^2 by a change of probability measure using the local martingale $(M_x^{0,\beta}(t, X_t), t \geq 0)$. The proof of the claim, in the case $y = 0$, is completed by invoking the conservativeness property of Y^2 , which implies that $(M_x^{0,\beta}(t, X_t), t \geq 0)$ is a martingale.

Finally, to recover the case $y \neq 0$, we use (3) which gives

$$\begin{aligned}
d\mathbb{Q}_x^{y,\beta}|_{\mathcal{F}_t} &= \frac{\Phi(yX_t/(1+\beta t))}{\Phi(yx)} e^{-\frac{1}{2}\frac{ty^2}{1+\beta t}} d\mathbb{Q}_x^{0,\beta}|_{\mathcal{F}_t} \\
&= \frac{\Phi(yX_t/(1+\beta t))}{\Phi(yx)} e^{-\frac{1}{2}\frac{ty^2}{1+\beta t}} M_x^{0,\beta}(t, X_t) d\mathbb{P}_x|_{\mathcal{F}_t} \\
&= \frac{\Phi(yX_t/(1+\beta t))}{\Phi(yX_t)} e^{\frac{\beta}{2}\frac{y^2 t^2}{1+\beta t}} M_x^{0,\beta}(t, X_t) d\mathbb{P}_x^{(y)}|_{\mathcal{F}_t} \\
&= M_x^{y,\beta}(t, X_t) d\mathbb{P}_x^{(y)}|_{\mathcal{F}_t}.
\end{aligned}$$

This completes the proof in the Bessel case. The same arguments work for the Brownian case and this completes the proof. \square

We shall now return back to the proof of Theorem 3.1. Lemmas 3.3 and 3.4, when combined with the optional stopping theorem, allow us to write, for any $\lambda > 0$,

$$\begin{aligned}
\mathbb{E}_x^{(y)} \left[e^{-\lambda T^{f^{(\beta)}}} \mathbb{I}_{\{T^{f^{(\beta)}} < \zeta^{(\beta)}\}} \right] &= \mathbb{E}_x^{(y)} \left[e^{-\lambda \frac{H^{(-\beta, f)}}{1-\beta H^{(-\beta, f)}}} \mathbb{I}_{\{H^{(-\beta, f)} < \zeta^{(-\beta)}\}} \right] \\
&= \mathbb{E}_x^{(y)} \left[e^{-\lambda \frac{T^f}{1-\beta T^f}} M_x^{y,-\beta}(T^f, X_{T^f}) \mathbb{I}_{\{T^f < \zeta^{(-\beta)}\}} \right] \\
&= \int_0^{\zeta^{(-\beta)}} e^{-\lambda \frac{t}{1-\beta t}} M_x^{y,-\beta}(t, f(t)) \mathbb{P}_x^{(y)}(T^f \in dt) \\
&= \int_0^{\zeta^{(\beta)}} \frac{e^{-\lambda u}}{(1+\beta u)^3} M_x^{y,-\beta}(u/(1+\beta u), f(u/(1+\beta u))) \\
&\quad \times S^{(\beta)}(\mathbb{P}_x^{(y)}(T^f \in du))
\end{aligned}$$

where we have performed the change of variables $t/(1-\beta t) = u$. Simplifying the above expression and by the injectivity of the Laplace transform, we get our first assertion. The second statement follows by letting $y \rightarrow 0$.

4. BROWNIAN MOTION

In this paragraph, we take $\mathbb{E} = \mathbb{R}$. Thus, X is a Brownian motion. Thanks to the homogeneity property, in this case, it is clearly enough to consider the case $X_0 = 0$ and we simply write \mathbb{P} for \mathbb{P}_0 . To simplify the discussion, except in Subsection 4.3, we assume that $f \in C^1(\mathbb{R}_+, E)$, with $f(0) \neq 0$, which implies that the studied distributions are absolutely continuous with respect to the Lebesgue measure with continuous densities, see [40]. We write then $\mathbb{P}(T^f \in dt) = p^f(t) dt$ and read from Theorem 3.1 the identity

$$(10) \quad p^{f^{(\beta)}}(t) = \frac{1}{(1+\beta t)^{3/2}} e^{-\frac{1}{2}\frac{\beta}{1+\beta t} f^{(\beta)}(t)^2} p^f\left(\frac{t}{1+\beta t}\right), \quad t < \zeta^{(\beta)}.$$

Observe, when $\beta > 0$, the asymptotic behavior

$$(11) \quad p^{f^{(\beta)}}(t) \sim (\beta t)^{-3/2} e^{-\frac{1}{2} \frac{\beta}{1+\beta t} f^{(\beta)^2}(t)} p^f(1/\beta) \quad \text{as } t \rightarrow \infty,$$

where $h(t) \sim g(t)$ as $t \rightarrow \zeta$ means that $h(t)/g(t) \rightarrow 1$ as $t \rightarrow \zeta$, for some $\zeta \in [0, \infty]$. It seems natural to examine the images of some curves by the mapping $S^{(\beta)}$. We gathered in Table 1 the images of the most studied curves. The fact that $S^{(\beta)}$ preserves straight lines

	f	$f^{(\beta)}$	References
(1)	$a + bt$	$a + (b + a\beta)t$	[4]
(2)	$\sqrt{1 + 2bt}$	$\sqrt{(1 + \beta t)(1 + (\beta + 2b)t)}$	[8], [29]
(3)	$a + bt^2, ab > 0$	$a(1 + \beta t) + bt^2/(1 + \beta t)$	[19], [38]
(4)	$\frac{a}{2} - \frac{t}{a} \ln \left(\frac{b + \sqrt{b^2 + 4b_1 e^{-\frac{a^2}{t}}}}{2} \right)$	$\frac{a(1+\beta t)}{2} - \frac{t}{a} \ln \left(\frac{b + \sqrt{b^2 + 4\hat{b}_1 e^{-\frac{a^2}{t}}}}{2} \right)$	[9]

TABLE 1. Image by $S^{(\beta)}$ of f and the corresponding references where the distribution of T^f is studied with $a, b \in \mathbb{R}$, $b_1 > -b^2/4$ and $\hat{b} = b_1 e^{-a^2\beta}$.

is well known, see for instance [34]. More generally, we observe that if $g_{\beta, \alpha}(\cdot) = (1 + \beta \cdot)^\alpha$, for some reals α and β , then we have $g_{\beta, \alpha}^{(-\beta)} = g_{-\beta, 1-\alpha}$. In particular, if we take $f \equiv a$ and $\beta = b/a$, for some reals a and b , then $f^{(b/a)}(t) = a + bt$, $t < \zeta^{(b/a)}$. An immediate application of Corollary 3.2, combined with

$$\mathbb{P}(T^a \in dt) = \frac{|a|}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}} dt, \quad t > 0,$$

yields then the following well known Bachelier-Lévy formula

$$\mathbb{P}\left(T^{a+b} \in dt\right) = \frac{|a|}{\sqrt{2\pi t^3}} e^{-ba - \frac{b^2}{2}t - \frac{a^2}{2t}} dt.$$

4.1. Some new examples. We now consider the boundary crossing problem associated to two families of curves consisting of the square root of second order polynomials and the reciprocal of affine functions.

4.1.1. First, we consider the distribution of the stopping time

$$T_a^{(\lambda_1, \lambda_2)} = \inf \left\{ s > 0; X_s = a\sqrt{(1 + \lambda_1 s)(1 + \lambda_2 s)} \right\}$$

where a and $\lambda_1 < \lambda_2$ are some fixed reals. We do not consider the case $\lambda_1 = \lambda_2$ which can be studied in a elementary way, with the extra cost of making use of the strong Markov property when $\lambda_1 < 0$. First, consider the case $\lambda_2 = 0$ and, to simplify the notation, set $\lambda_1 = \lambda$ and $T_a^{(\lambda, 0)} = T_a^{(\lambda)}$. This is the setting of the classical example studied by Breiman

in [8]. $T_a^{(\lambda)}$ is related to the hitting time of a constant level by an Ornstein-Uhlenbeck process and we refer to Alili et al. [2] for a recent survey on this topic. That is, with

$$U_t = e^{-\lambda t/2} \int_0^t e^{\lambda s/2} dX_s, \quad t \geq 0,$$

and $H_a = \inf\{s > 0; U_s = a\}$, we have

$$(12) \quad T_a^{(\lambda)} \stackrel{(d)}{=} \lambda^{-1} \left(e^{\lambda H_a} - 1 \right)$$

where $\stackrel{(d)}{=}$ stands for the identity in distribution. By symmetry, it is enough to consider the case where a is positive. We proceed by recalling that the distribution of H_a is given, see for instance [2], by

$$\mathbb{P}(H_a \in dt) = -\frac{1}{2} \lambda e^{-\lambda a^2/4} \sum_{n=1}^{\infty} \frac{D_{\nu_{n,-a\sqrt{\lambda}}}(0)}{D_{\nu_{n,-a\sqrt{\lambda}}}^{(\nu)}(-a\sqrt{\lambda})} e^{-\lambda \nu_{n,-a\sqrt{\lambda}} t/2}, \quad t > 0,$$

where we used the notation $D_{\nu_{n,b}}^{(\nu)}(b) = \frac{\partial D_{\nu}(b)}{\partial \nu} \Big|_{\nu=\nu_{n,b}}$ and $(\nu_{j,b})_{j \geq 0}$ stands for the ordered sequence of the positive zeros of the parabolic function $\nu \rightarrow D_{\nu}(b)$. By means of the identity (12), we get that

$$\mathbb{P}\left(T_a^{(\lambda)} \in dt\right) = \frac{1}{1 + \lambda t} \mathbb{P}(H_a \in d \cdot) \Big|_{\cdot = \frac{1}{\lambda} \log(1 + \lambda t)} dt, \quad t > 0.$$

Next, we assume that $\lambda_1 < \lambda_2$. Thus, the support of $T_a^{(\lambda_1, \lambda_2)}$ is $[0, \zeta^{(\lambda_1)})$ if λ_1 is positive and is the positive real line otherwise. We have

$$S^{(\lambda_1)} \left(\sqrt{1 + (\lambda_2 - \lambda_1) \cdot} \right) = \sqrt{(1 + \lambda_2 \cdot)(1 + \lambda_1 \cdot)}.$$

We are now ready to use Theorem 3.1 and write

$$\mathbb{P}\left(T_a^{(\lambda_1, \lambda_2)} \in dt\right) = \frac{1}{(1 + \lambda_1 t)^{5/2}} e^{-\frac{1}{2} \lambda_1 (1 + \lambda_2 t)} S^{(\lambda_1)} \left(\mathbb{P}\left(T_a^{(\lambda_2 - \lambda_1)} \in dt\right) \right), \quad t < \zeta^{(\lambda_1)}.$$

4.1.2. We are now interested in computing the distribution of the stopping time $T^{h^{(\beta)}}$ defined by

$$T^{h^{(\beta)}} = \inf \left\{ 0 < s < \zeta^{(\beta)}; X_s = \frac{1}{1 + \beta s} \right\}$$

where β is some real. To this end, we recall that Groeneboom [19] has computed the density of $T^{\tilde{h}}$ with $\tilde{h}(t) = 1 + \beta^2 t^2$ as follows

$$\mathbb{P}(T^{\tilde{h}} \in dt) = 2(\beta^2 c)^2 e^{-\frac{2}{3} \beta^4 t^3} \sum_{k=0}^{\infty} \frac{Ai(z_k + 2c\beta^2)}{Ai'(z_k)} e^{-z_k t} dt, \quad t > 0,$$

where $(z_k)_{k \geq 0}$ is the decreasing sequence of negative zeros of the Airy function, see e.g. [25], and we have set $c = (2\beta^2)^{-\frac{1}{3}}$. Next, by means of the Cameron-Martin formula, we obtain,

with $h(t) = (1 - \beta t)^2 = \tilde{h}(t) - 2\beta t$,

$$\mathbb{P}(T^h \in dt) = 2(\beta^2 c)^2 e^{2\beta - 2\beta^2 t(1 + \frac{2}{3}\beta^2 t^2 - \beta t)} \sum_{k=0}^{\infty} \frac{Ai(z_k + 2c\beta^2)}{Ai'(z_k)} e^{-z_k t} dt, \quad t > 0.$$

Finally, since $h^{(\beta)} = S^{(\beta)}(h)$, we obtain, by applying Theorem 3.1, that

$$\mathbb{P}(T^{h^{(\beta)}} \in dt) = \frac{2(\beta^2 c)^2 e^{2\beta}}{(1 + \beta t)^{3/2}} e^{-2\beta^2 t(1 + \frac{1}{4\beta} + \beta t + \frac{2}{3}\beta^2 t^2)} \sum_{k=0}^{\infty} \frac{Ai(z_k + 2c\beta^2)}{Ai'(z_k)} e^{-z_k \frac{t}{1 + \beta t}} dt, \quad t < \zeta^{(\beta)}.$$

We complete the example by stating the following asymptotic result

$$\mathbb{P}(T^{h^{(\beta)}} \in dt) \sim \left(2(\beta^2 c)^2 e^{2\beta} \sum_{k=0}^{\infty} \frac{Ai(z_k + 2c\beta^2)}{Ai'(z_k)} e^{-\frac{z_k}{\beta}} \right) e^{-\frac{4}{3}\beta^4 t^3} dt \quad \text{as } t \rightarrow \infty$$

which holds provided that $\beta > 0$.

4.2. Interpretation of the mapping S^β via the method of images. We aim to describe the impact of our methodology to the so-called standard method of images. To this end, let us assume that $X_0 = 0$ and set

$$h(t, x) dx = \mathbb{P}\left(T^f > t, X_t \in dx\right).$$

We need to impose some conditions on the the curve f in order to be able to apply the standard method of images. That is, we assume that

- f is infinitely often continuously differentiable,
- $f(t)/t$ is monotone decreasing,
- f is concave.

Note that these properties are also satisfied by $f^{(\beta)}$ when $\beta > 0$. The function h is characterized as being the unique solution to the heat equation, see Lerche [26, Chap. I.1],

$$\frac{\partial h}{\partial t} = \frac{1}{2} \frac{\partial^2 h}{\partial x^2} \quad \text{on } \{(t, x) \in \mathbb{R}^+ \times \mathbb{R}; x \leq f(t)\},$$

with the boundary conditions

$$h(t, f(t)) = 0, \quad h(0, \cdot) = \delta_0(\cdot) \quad \text{on }]-\infty, f(0^+)]$$

where δ_0 stands for the Dirac function at 0 and $f(0^+) = \lim_{t \searrow 0} f(t)$. The standard method of images assumes that h is known, whilst f is unknown, and is given by

$$(13) \quad h(t, x) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{x^2}{2t}} - \int_0^\infty e^{-\frac{(x-s)^2}{2t}} F(ds) \right)$$

where $F(ds)$ is some positive σ -finite measure satisfying $\int_0^\infty e^{-\frac{\epsilon s^2}{2}} F(ds) < \infty$, for all $\epsilon > 0$. In [26], it is shown that if f is the unique root of the equation $h(t, x) = 0$ in the unknown x for a fixed $t \geq 0$, then we have

$$(14) \quad \mathbb{P}\left(T^f \in dt\right) = \frac{dt}{\sqrt{2\pi t^3}} \int_0^\infty s e^{-\frac{(s-f(t))^2}{2t}} F(ds), \quad t > 0.$$

Note that the implicit equation $h(t, x) = 0$ may be written as

$$(15) \quad \int_0^\infty e^{-\frac{s^2}{2t} + s\frac{x}{t}} F(ds) = 1.$$

With $F(ds)$ replaced by $F(ds)e^{-\mu s}$, $\mu > 0$, the unique solution to (15) is $f(t) + \mu t$ for a fixed positive real t and this is easily checked by the Cameron-Martin formula. In the same spirit, the identity (10) has the following interpretation.

Proposition 4.1. *If $\beta > 0$ then we have the following representation*

$$\mathbb{P}\left(T^{f^{(\beta)}} \in dt\right) = \frac{dt}{\sqrt{2\pi t^3}} \int_0^\infty s e^{-\frac{(s-f^{(\beta)}(t))^2}{2t}} e^{-\beta \frac{s^2}{2}} F(ds), \quad t > 0.$$

Furthermore, the σ -finite measure corresponding to the curve $f^{(\beta)}$ is $F^{(\beta)}(ds) := e^{-\beta s^2/2} F(ds)$ and

$$\mathbb{P}\left(T^{f^{(\beta)}} > t, X_t \in dx\right) = \frac{dx}{\sqrt{2\pi t}} \left(e^{-\frac{x^2}{2t}} - \int_0^\infty e^{-\frac{(x-s)^2}{2t}} e^{-\beta \frac{s^2}{2}} F(ds) \right).$$

Proof. Noting, from (14), that

$$\mathbb{P}\left(T^{f^{(\beta)}} \in dt\right) = \frac{dt}{\sqrt{2\pi t^3}} \int_0^\infty s e^{-\frac{(s-f^{(\beta)}(t))^2}{2t}} F^{(\beta)}(ds)$$

and by using (10) we obtain that

$$\begin{aligned} \frac{1}{\sqrt{2\pi t^3}} \int_0^\infty s e^{-\frac{(s-f^{(\beta)}(t))^2}{2t}} F^{(\beta)}(ds) &= \frac{1}{(1+\beta t)^{3/2}} e^{-\frac{1}{2} \frac{\beta}{1+\beta t} f^{(\beta)}(t)^2} p^f\left(\frac{t}{1+\beta t}\right) \\ &= \frac{1}{\sqrt{2\pi t^3}} e^{-\frac{1}{2} \frac{\beta}{1+\beta t} f^{(\beta)}(t)^2} \int_0^\infty s e^{-\frac{(s-f(t)/(1+\beta t))^2}{2t/(1+\beta t)}} F(ds) \\ &= \frac{1}{\sqrt{2\pi t^3}} \int_0^\infty s e^{-\frac{(s-f^{(\beta)}(t))^2}{2t}} e^{-\beta \frac{s^2}{2}} F(ds). \end{aligned}$$

This proves the first assertion and suggests, but does not prove, that the σ -finite measure corresponding to the curve $f^{(\beta)}$ is $F^{(\beta)}(ds)$. Next, with

$$h^{(\beta)}(t, x) = \frac{1}{\sqrt{2\pi t}} \left(e^{-\frac{x^2}{2t}} - \int_0^\infty e^{-\frac{(x-s)^2}{2t}} e^{-\beta s^2/2} F(ds) \right),$$

we aim to solve $h^{(\beta)}(t, x) = 0$ for each fixed $t > 0$. Now, setting $x = f(t)$ and replacing x and t , respectively, by $f(t)$ and $t/(1+\beta t)$ in (15), we find that $\int_0^\infty e^{-\frac{s^2}{2t} + s\frac{f^{(\beta)}(t)}{t}} e^{-\beta s^2/2} F(ds) = 1$. It follows that $f^{(\beta)}(t)$ solves the implicit equation $h^{(\beta)}(t, x) = 0$, $t > 0$, which finishes the proof. \square

4.3. Large asymptotics for the density of $T^{f^{(\beta)}}$. Following Anderson and Pitt [3], we consider the asymptotic of the density of the distribution of $T^{f^{(\beta)}}$ as time gets close to $\zeta^{(\beta)}$. To start with, we exceptionally assume in this subsection that $f \in \mathcal{C}^1((0, \infty), \mathbb{R}_+)$. So the distribution of T^f has a continuous density with respect to the Lebesgue measure on $(0, \infty)$. Assume that the distribution of T^f is defective. That is $r = \mathbb{P}(T^f < \infty) < 1$ and, in this case, we say that f is transient. By the classical Kolmogorov-Erdős-Petrovski theorem, see [14], we know that if $t^{-1/2}f(t)$ is increasing for sufficiently large t , then f is transient if and only if the integral test

$$(16) \quad \int_1^\infty t^{-3/2} f(t) e^{-f^2(t)/2t} dt < \infty$$

holds. Clearly, if $\beta > 0$ and $0 < \beta f(1/\beta) < \infty$ then $f^{(\beta)}$ does not satisfy (16) but the general formula (11) holds. Now, we consider the more interesting case $\beta < 0$ and examine the asymptotic of the density of the distribution of $T^{f^{(\beta)}}$ as $t \rightarrow -1/\beta$. We need to work under the following three conditions borrowed from [3]:

- f is increasing, concave, twice differentiable on $(0, \infty)$ and of regular variations at ∞ with index $\alpha \in [1/2, 1)$,
- $f(t)/\sqrt{t}$ is monotonic increasing at ∞ , and $f(t)/t$ is convex and decreases to 0 for sufficiently large t ,
- There exist positive constants $c < 1$ and c' such that we have the inequalities $tf'(t) < cf(t)$ and $|t^2 f''(t)| \leq c' f(t)$ for a sufficiently large enough t .

The behavior at ∞ imposed in the above conditions is granted in the examples where f behaves like

$$f(t) = Ct^a (\log t)^b (\log \log t)^c (\log \log \log t)^d$$

with $1/2 \leq a < 1$, for large t . It is clear that these conditions are not always preserved by the family $S^{(\beta)}$. Equipped with this, we are now ready to state the following result.

Proposition 4.2. *Assume that f is transient and satisfies the above conditions. Then, for any $\beta < 0$, we have*

$$p^{f^{(\beta)}}(t) \sim \frac{1}{\sqrt{2\pi|\beta|^3}} (1-r) \tilde{f}(\beta, t) \quad \text{as } t \rightarrow \zeta^{(\beta)}$$

where

$$\tilde{f}(\beta, t) = f\left(\frac{t}{1+\beta t}\right) - \frac{t}{1+\beta t} f'\left(\frac{t}{1+\beta t}\right).$$

Proof. We read from Theorem 1 that

$$(17) \quad p^f(t) \sim (1-r) \frac{f(t) - tf'(t)}{\sqrt{2\pi t^3/2}} e^{-f^2(t)/2t} \quad \text{as } t \rightarrow \infty.$$

Moreover, as $t \rightarrow \infty$, $f(t)/t \downarrow 0$ and hence

$$e^{-\frac{\beta}{2(1+\beta t)} f^{(\beta)^2}(t) - \frac{1+\beta t}{2t} f^2(t/(1+\beta t))} = e^{-f^{(\beta)^2}(t)/2t} \sim 1.$$

The proof is then completed by combining (17) with (10). \square

Remark 4.3. In the defective case, the distribution of the last crossing time $\tilde{T}^f = \sup\{s > 0; X_s = f(s)\}$ is shown to have an atom at 0 and its asymptotic as $t \rightarrow 0$ is determined in [40]. Similar questions can be treated for $\tilde{T}^{f^{(\beta)}} = \sup\{s > 0; X_s = f^{(\beta)}(s)\}$ using this method.

5. BESSEL PROCESSES AND STRAIGHT LINES

We investigate here the case when X is a Bessel process of dimension $\delta > 0$ and we refer to Revuz and Yor [36, Chap. XI] for a concise treatment of these processes. For $\delta \geq 2$, and $x > 0$, the Bessel process of dimension δ is the unique solution to

$$X_t = B_t + x + \frac{\delta - 1}{2} \int_0^t \frac{ds}{X_s}, \quad t > 0,$$

where B is a standard Brownian motion. For $0 < \delta < 2$, X is defined as the square root of the unique non-negative solution of (8). We recall that 0 is an entrance boundary when $\delta \geq 2$ and a reflecting boundary otherwise. We denote by \mathbb{P}_x^ν (resp. \mathbb{E}_x^ν) the law (resp. the expectation operator) of a Bessel process of index $\nu = \delta/2 - 1$, starting at $x > 0$. The semi-group of X^2 is characterized by

$$(18) \quad \mathbb{E}_x^\nu \left[e^{-\lambda X_t^2} \right] = (1 + 2\lambda t)^{-\delta/2} e^{-\frac{\lambda x^2}{1+2\lambda t}}, \quad \lambda, t \geq 0.$$

The densities of the semi-group of X , with respect to the Lebesgue measure, are

$$p_t^\nu(x, y) = \frac{y}{t} \left(\frac{y}{x} \right)^\nu e^{-\frac{x^2+y^2}{2t}} I_\nu \left(\frac{xy}{t} \right), \quad t, x, y > 0.$$

For a given $f \in \mathcal{C}(\mathbb{R}_+, E)$ let us keep the notation used in the introduction and Section 3. Observe, that if $y = 0$ then Theorem 3.1 reads

$$\mathbb{P}_x^\nu \left(T^{f^{(\beta)}} \in dt \right) = (1 + \beta t)^{\nu-2} e^{-\frac{\beta}{2} \frac{f^{(\beta)}(t)^2}{1+\beta t} + \frac{\beta}{2} x^2} S^{(\beta)} \left(\mathbb{P}_x^\nu(T^f \in dt) \right)$$

for all $t < \zeta^{(\beta)}$. We end up our discussion by computing the distribution of the hitting time by X of a straight line $a + b \cdot$ where $a > 0$ and b is a real. We keep the notation used in Corollary 3.2 and the reader is reminded about observations preceding its statement. To the best of our knowledge, the problem of the determination of the distribution of $T^{a+b \cdot}$, which was raised in [34] for the case $a \neq 0$, remained open. Recall that the law of T^a , which corresponds to $b = 0$, is characterized by

$$(19) \quad \mathbb{E}_x^\nu \left[e^{-\frac{\lambda^2}{2} T^a} \right] = \begin{cases} \frac{x^{-\nu} I_\nu(x\lambda)}{a^{-\nu} I_\nu(a\lambda)}, & x \leq a, \\ \frac{x^{-\nu} K_\nu(x\lambda)}{a^{-\nu} K_\nu(a\lambda)}, & x \geq a, \end{cases}$$

for any $\lambda \geq 0$, where K_ν is the modified Bessel functions of the second kind of index ν , see for instance Borodin and Salminen [6]. Observe that when $x > a$ the distribution of T^a is defective and $\mathbb{P}_x^\nu(T^a < \infty) = (a/x)^{2\nu}$. It is also well known that if $x < a$ then we have

$$(20) \quad \mathbb{P}_x^\nu(T^a \in dt) = \sum_{k=1}^{\infty} \frac{x^{-\nu} j_{\nu,k} J_\nu(j_{\nu,k} \frac{x}{a})}{a^{2-\nu} J_{\nu+1}(j_{\nu,k})} e^{-j_{\nu,k}^2 t/2a^2} dt, \quad t > 0,$$

where $(j_{\nu,k})_{k \geq 1}$ is the ordered sequence of positive zeros of the Bessel function of the first kind $J_\nu(\cdot)$, see [6]. Furthermore, if $a = 0$ and $b > 0$ then T^b and $1/\sup\{s > 0; X_s = b\}$ have the distribution which was determined in [34] by making use of the time inversion property. Now, we are ready to state the following result.

Theorem 5.1. *For $0 \leq x < a$ and $b \in \mathbb{R}$, we have for any $t < \zeta^{(b/a)}$*

$$\mathbb{P}_x^\nu(T^{a+b} \in dt) = \frac{e^{\frac{b}{2a}(a^2-x^2) + \frac{b^2}{2}t}}{(1 + \frac{b}{a}t)^{\nu+2}} \sum_{k=1}^{\infty} \frac{x^{-\nu} j_{\nu,k} J_\nu(j_{\nu,k} \frac{x}{a})}{a^{2-\nu} J_{\nu+1}(j_{\nu,k})} e^{-j_{\nu,k}^2 \frac{t}{2a(a+bt)}} dt.$$

For any $x \geq 0$ and $a, b > 0$, we have

$$\mathbb{E}_x^\nu \left[e^{-\frac{\lambda^2}{2} T^{a-b}} \right] = \begin{cases} e^{\frac{b}{2a}(a^2-x^2)} \frac{x^{-\nu}}{a^{-\nu}} \int_0^\infty \frac{I_\nu(\sqrt{2}xu)}{I_\nu(\sqrt{2}au)} p_{b/2a}^\nu(\lambda_b, u) du, & x \leq a, \\ e^{\frac{b}{2a}(a^2-x^2)} \frac{x^{-\nu}}{a^{-\nu}} \int_0^\infty \frac{K_\nu(\sqrt{2}xu)}{K_\nu(\sqrt{2}au)} p_{b/2a}^\nu(\lambda_b, u) du, & x \geq a, \end{cases}$$

where $\lambda_b = \sqrt{(\lambda^2 + b^2)/2}$, $\lambda \in \mathbb{R}$.

Proof. The first statement results from a combination of Corollary 3.2 and relation (20). Next, using Lemmae 3.3 and 3.4, with $\beta = -b/a$, we can write

$$\begin{aligned} & \mathbb{E}_x^\nu \left[e^{-\frac{\lambda^2}{2} T^{a-b}}; T^{a-b} < \frac{a}{b} \right] \\ &= \mathbb{E}_x^\nu \left[e^{-\frac{\lambda^2}{2} \frac{H(b/a, a)}{1 + \frac{b}{a} H(b/a, a)}}; H(b/a, a) < \infty \right] \\ &= \mathbb{E}_x^\nu \left[e^{-\frac{\lambda^2}{2} \frac{T^a}{1 + \frac{b}{a} T^a}} \left(1 + \frac{b}{a} T^a\right)^{-\delta/2} e^{\frac{b}{2a} \left(\frac{a^2}{1 + \frac{b}{a} T^a} - x^2\right)} \right] \\ &= e^{\frac{b}{2a}(a^2-x^2)} \mathbb{E}_x^\nu \left[e^{-\frac{\lambda_b^2 T^a}{1 + \frac{b}{a} T^a}} \left(1 + \frac{b}{a} T^a\right)^{-\delta/2} \right] \\ &= e^{\frac{b}{2a}(a^2-x^2)} \int_0^\infty p_{b/2a}^\nu(\lambda_b, u) \mathbb{E}_x^\nu \left[e^{-u^2 T^a} \right] du \end{aligned}$$

where the last line follows from (18). It remains to use (19) to conclude. \square

Remark 5.2. The process $(Y_t := X_t + bt, t \geq 0)$ is a non-homogeneous Markov process and solves the sde

$$Y_t = B_t + \frac{\delta - 1}{2} \int_0^t \frac{ds}{Y_s - bs} + bt, \quad t \geq 0.$$

This is to be distinguished from a Bessel process with a "naive" drift b introduced in [42] and defined as a solution to

$$Z_t = B_t + \frac{\delta - 1}{2} \int_0^t \frac{ds}{Z_s} + bt, \quad t \geq 0.$$

Remark 5.3. The Bessel process of dimension $\delta = 1$ is, in fact, the reflected Brownian motion. It follows that the associated first hitting times can be interpreted as double barrier hitting times. That is, with f as above, the time when a Brownian motion B hits one of the curves $x \pm f(\cdot)$, i.e. $\inf\{s > 0; B_s = x \pm f(s)\}$.

6. CONCLUDING REMARKS AND SOME COMMENTS

6.1. It is plain that the results of Theorem 3.1 can be readily extended to any h -transform of the process X , but we need to take care of the life-times of the involved processes in the Bessel case. In the Brownian setting, one gets similar results for the process $X_t^\epsilon = X_t + y\epsilon t$, $t \geq 0$, where ϵ is an independent symmetric Bernoulli random variable taking values in $\{-1, 1\}$. Observe that $i(X^\epsilon)$ is a strong Markov process. However, because the latter starts at the random point $y\epsilon$, $X^{(\epsilon)}$ does not satisfy the time inversion property (1).

6.2. Assuming that X is a 2-self-similar strong Markov process with càdlàg paths (i.e. with possible jumps) enjoying the time inversion property, then Theorem 5.8 in [34] or Theorem 1 in [18] allow to extend Lemma 3.3 and Lemma 3.4 and an analogue of Theorem 3.1 can be stated. However, we did not succeed to construct examples of \mathbb{R}_+ -valued processes enjoying the time inversion property other than Bessel processes in the usual or wide sense. This is the reason why the setting is restricted to the continuous one. We refer to [31] where the second author characterizes, through its Mellin transform, the law of the first passage time above the square root boundary for spectrally negative positive 2-self-similar Markov processes.

6.3. In [12], Durbin considered the studied boundary crossing problem for continuous gaussian processes and showed that, in the absolute-continuous case, the problem reduces to the computation of a conditional expectation. In particular, for the Brownian motion, if f is continuously differentiable and $f(0) \neq 0$ then

$$\mathbb{P}\left(T^f \in dt\right) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{f^2(t)}{2t}} h(t) dt$$

where

$$h(t) = \lim_{s \nearrow t} \frac{1}{t-s} \mathbb{E}\left[B_s - f(s); T^f > s \mid B_t = f(t)\right], \quad t > 0.$$

There seems to be no known way to compute the function h . It is shown in [13] that this method is in agreement with the standard method of images. With the obvious notation, it is clearly possible to relate $h^{(\beta)}$ to h .

6.4. We learned from Kendall [20] an intuitive interpretation of Durbin's expression involving the family of local times of X at f denoted by $(L_t^{X=f}, t \geq 0)$. Indeed, observe that Durbin's formula can be rewritten as $\mathbb{P}(T^f \in dt) = h(t)\mathbb{E}[dL_t^{X=f}]$, $t \geq 0$. The latter when integrated over $[0, a]$, yields $\mathbb{P}(T^f < a) = \mathbb{E}\left[\int_0^a h(s)dL_s^{X=f}\right]$. Such a decomposition is not unique and many can be constructed from the above one. A natural non trivial one to consider is motivated by the following observation found in [20]. If $g: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ solves $\mathbb{E}\left[\int_t^a g(s, a)dL_s^{X=f} | X_t = f(t)\right] = 1$, for $0 < t < a$, then clearly $\mathbb{E}\left[\int_0^a g(s, a)dL_s^{X=f}\right] = \mathbb{P}(T^f < a)$. However, it is not clear how to express g in terms of h . W. Kendall told the first author that the above interpretation could be interesting for further investigations for the studied boundary crossing problem.

6.5. In the Brownian case, it is interesting to analyze the impact of our identities on some integral equations satisfied by the studied densities. However, some of them lead to obvious facts explainable by change of variables. As an example, we observe that if f is positive and does not vanish then X , when started at $x > f(0)$, must hit f before reaching 0. The strong Markov property then gives

$$\frac{x}{\sqrt{2\pi t^3}}e^{-\frac{x^2}{2t}} = \int_0^t \mathbb{P}_x(T^f \in dr) \frac{f(r)}{\sqrt{2\pi(t-r)^3}}e^{-\frac{f(r)^2}{2(t-r)}}, \quad t > 0.$$

This is easily shown to be in accordance with the result stated in Theorem 3.1. For the above and other classical integral equations we refer to [12], [15], [26] and also to [32] for some more recent ones.

6.6. The technics used in the example treated in Subsection 4.1, for Brownian motion, can be applied to Bessel processes and square root curves. The required results, for that end, can be found in Delong [11] or in Yor [42]. This law is connected via a deterministic time change to the one of the first passage time to a fixed level by the radial norm of a δ -dimensional Ornstein-Uhlenbeck process, where δ is some positive integer. Due to the stationarity property, the law of the first passage times can be expressed as infinite convolutions of mixture of exponential distributions, see Kent [22].

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