A Study of Quartic K3 Surfaces with a $(\mathbb{Z}/2\mathbb{Z})^4$ Action

By

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Contents

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Declaration

Chapter 5 is a joint work with Marco Streng (Leiden) and an extended version of it has appeared as a paper [BS15]. Except for that chapter, I declare that, to the best of my knowledge, the material contained in this thesis is my own original work except where otherwise indicated, stated or where it is common knowledge.

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. It has been composed by myself and has not been submitted in any previous application for any degree.

Abstract

The main focus of this thesis is to study the equation

$$A(x^4+y^4+z^4+w^4)+Bxyzw+C(x^2y^2+z^2w^2)+D(x^2z^2+y^2w^2)+E(x^2w^2+y^2z^2)=0.$$

To do so, we view this equation as a family of quartic K3 surfaces in $\mathbb{P}^3_{[x,y,z,w]}$, parametrised by points $[A,B,C,D,E]\in\mathbb{P}^4$. We pursue two directions. First we look at 320 conics that such a K3 surface contains. In particular, we explore the field of definition of these 320 conics and the Monodromy group linked to these conics. In the other direction we explore the quartic K3 surfaces which contain lines. We list all subfamilies of K3 surfaces for which a very general member contains 8, 16, 24, 32 or 48 lines.

We combine the two directions, by using the lines and conics found, to explore the Picard group of the various families found. In particular, not only do we work out the Picard rank of a very general member of a family, but we also decompose the Picard lattice into known lattices.

This thesis has a secondary focus on hyperelliptic curves of genus two with complex multiplication (CM). At the end of the thesis, we design an algorithm to find CM curves of genus two which are defined over quadratic extensions of the rationals. To do so we also develop an algorithm which makes the coefficients of a curve smaller.

Chapter 1

Introduction

One of the motivating questions in number theory is trying to solve Diophantine equations, that is, to find integral or rational solutions to a system of integral polynomials. Solving a Diophantine equation is equivalent to finding the rational points of its associated variety. As such, much work has been done to understand algebraic curves of low genus such as elliptic curves. While many problems are still open, mathematicians have also began to look in detail at curves of higher genus or at varieties of higher dimension. In this thesis we will do both, first looking at K3 surfaces then studying genus two hyperelliptic curves.

Elliptic curves and K3 surfaces have trivial canonical divisors, hence K3 surfaces can be thought of as two-dimensional analogues of elliptic curves. While our understanding of K3 surfaces has grown, answering questions on the arithmetic of K3 surfaces is still difficult. So instead of looking at all K3 surfaces, we impose some restrictions on the ones we investigate. For example, in this thesis we look at a family of quartic K3 surfaces that admit a $(\mathbb{Z}/2\mathbb{Z})^4$-action. More specifically, consider the $(\mathbb{Z}/2\mathbb{Z})^4$ subgroup of $\mathrm{Aut}(\mathbb{P}^3_{\overline{\mathbb{Q}}})$ generated by

$$[x, y, z, w] \mapsto \begin{cases} [y, x, w, z] \\ [z, w, x, y] \\ [x, y, -z, -w] \\ [x, -y, z, -w] \end{cases}.$$

We consider the quartic K3 surfaces in $\mathbb{P}^3_{\overline{\mathbb{Q}}}$ which are invariant under these transformations. We call such quartic surfaces *invariant quartic,* and the family of all invariant quartics is known to be parametrised by $\mathbb{P}^4$.

Eklund [Ekl10] was motivated by the question “which invariant quartic contains a conic?”. He proved that a very general invariant quartic contains at least 320 conics. On the other hand, Barth and Nieto [BN94] studied the invariant quartics which contain a line. They found the locus of invariant quartics containing lines to be a quintic threefold $N_5 \subset \mathbb{P}^4$, plus the tangent cones of the 10 singular points of N_5.

The first motivating question of this thesis is: if an invariant quartic is defined over a number field, what is the smallest field extension over which the 320 conics are defined? This leads us to look at the Monodromy group of the 320 conics of a very general invariant quartic surface. We conclude that the moduli space of pairs (X, C), where X is an invariant quartic K3 surface and C one of the mentioned 320 conic on X, has 10 irreducible components. We contrast this with the following conjecture: the moduli space of pairs (X, C), where X is a primitive K3 surface of genus g and C an irreducible nodal curve on X, is irreducible. This conjecture is proven by Ciliberto and Dedieu for $3 \leq g \leq 11$, $g \neq 10$ in [CD12].

The second motivating question is: if X is an invariant quartic surface containing a line, what can we say about its Picard group? As Eklund studies the quartic surfaces parametrised by N_5 in [Ekl10], we look at the surfaces parametrised by the tangent cones. Using this, we find invariant quartic surfaces containing $8, 16, 24$ or 32 lines, which we group in various families. For each family we calculate the Picard group of a very general member, decompose it into known lattices and show that it is generated by the lines and conics lying on the surface. We notice that for a family of dimension r, the Picard rank of a very general member is $20 - r$. This fits nicely with the fact that certain moduli spaces of K3 surfaces whose Picard group contains a fixed lattice M has dimension $20 - \operatorname{rank}(M)$.

We also turn our attention to curves of higher genus, namely we look at genus two hyperelliptic curves with complex multiplication (CM). A curve C/k of genus g is said to have *complex multiplication* if the endomorphism ring of its Jacobian, over $\overline{k}$, contains an order in a number field K of degree $2g$. It is known that there exists exactly 13 j-invariants giving elliptic curves over $\mathbb{Q}$ with CM, see for example [Cox13, Thm 7.30ii]. Analogously, Van Wamelen [vW99a] gives a list of 19 curves of genus two over $\mathbb{Q}$ with CM by a maximal order. As this did not account for all quartic CM-fields, we implement an algorithm to find those not defined over $\mathbb{Q}$. More precisely we extend his list to include genus two hyperelliptic curves with CM defined over certain real quadratic extensions of $\mathbb{Q}$. A recent paper by Kilicer and Streng [KS15] proves that our list is complete.

The layout of the thesis is as follows: Chapter 2 deals with background knowledge that is assumed in the rest of the thesis. In particular the last section covers the main aspect of the family of invariant quartic K3 surfaces. Chapter 3 and Chapter 4 deal with the two motivating questions respectively, while Chapter 5 explains the work on genus two hyperelliptic curves.

Note. Many of the calculations done at various points throughout this thesis were done with the help of the computer algebra package Magma [BCP97] and Sage [S^+13]. We will explicitly mention where other packages were used.

Chapter 2

Background

In this chapter we review some material and notation that will be used throughout the thesis. The first section gives an overview of the theory of lattices that we need. In particular, we look at how given a lattice we can decompose it into root lattices and the hyperbolic lattice. For that purpose Table 2.1 gives the invariants of these lattices. The second section gives a quick recap of some basic algebraic geometry notions that we need. We provide some references where more details can be found.

The final section is the most important one, as we introduce the family of quartic K3 surfaces that is the main study of this thesis. We also recap what we already know about this family, with many of the theorems being used explicitly and implicitly throughout the thesis.

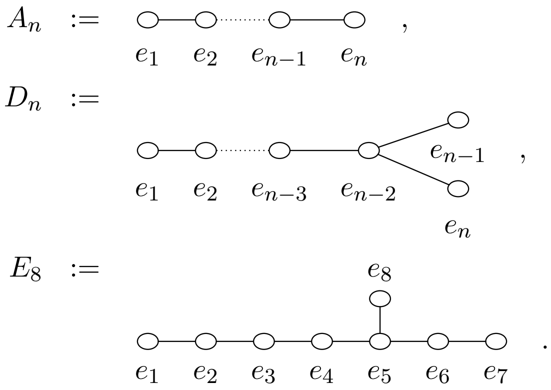
2.1 Lattices

In this thesis a *lattice*, L, is a free abelian group of finite rank equipped with a symmetric, non-degenerate, bilinear form $\langle\, ,\, \rangle : L \times L \to \mathbb{Z}$. We say it has *signature* (b_+, b_-) if for some basis $\{e_i\}$ of $L \otimes_{\mathbb{Z}} \mathbb{R}$ we have

$$\langle e_i, e_j \rangle = \begin{cases} 1 & i = j, i \in \{1, \dots, b_+\} \\ -1 & i = j, i \in \{b_+ + 1, \dots, b_+ + b_-\} \\ 0 & i \neq j \end{cases}.$$

A lattice is *positive definite* if it has signature $(b_+, 0)$, *negative definite* if it has signature $(0, b_-)$, and *indefinite* otherwise. A lattice, L, is *even* if $\langle x, x \rangle \in 2\mathbb{Z}$ for all $x \in L$. Let $\{e_i\}$ be a basis for L, then a *Gram matrix* of L (with respect to $\{e_i\}$) is the matrix $(\langle e_i, e_j \rangle)_{i,j}$. The *discriminant* of L, denoted $\mathrm{Disc}(L)$, is the determinant of a Gram matrix, which is invariant under change of basis. A lattice is *unimodular* if it has discriminant ± 1.

Example. Consider the following Dynkin diagrams:



Each diagram defines a (root) lattice, with basis $\{e_i\}$ and bilinear form

$$\langle e_i, e_j \rangle = \begin{cases} 2 & i = j \\ -1 & e_i \text{ --- } e_j \\ 0 & \text{otherwise} \end{cases} .$$

Another example of a lattice is the *hyperbolic plane lattice*, denoted U, which is the unique (up to isomorphism) rank 2 even indefinite unimodular lattice. For some basis, its Gram matrix is

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Given a lattice L with basis $\{e_i\}$ and $m \in \mathbb{Z}$, we denote by $L\langle m \rangle$ the lattice with basis $\{e_i\}$ and bilinear form $\langle e_i, e_j \rangle_{L\langle m \rangle} = m \langle e_i, e_j \rangle_L$. By abuse of notation, we denote the rank one lattice with bilinear form $\langle e_1, e_1 \rangle = m$ by $\langle m \rangle$. If L_1 and L_2 are two lattices with basis $\{e_i\}$, $\{f_i\}$ respectively, we denote by $L_1 \oplus L_2$ the lattice with basis $\{e_i\} \sqcup \{f_i\}$ and bilinear form given by $\langle e_i, f_j \rangle = 0$. We will say that a lattice L decomposes into $L_1, \ldots, L_n$ if $L \cong L_1 \oplus \cdots \oplus L_n$.

We say a lattice L_1 is a *sublattice* of a lattice L_2 if it is a subset of L_2 and if the bilinear form of L_2 restricted on L_1 agrees with the bilinear form of L_1. A sublattice is said to be *primitive* if L_2/L_1 is torsion free. If L_1 is a full-rank sublattice of L_2, i.e. $\mathrm{rank}(L_1) = \mathrm{rank}(L_2)$, then we call L_2 an *overlattice* of L_1. Note that in such case $\mathrm{Disc}(L_1)/\mathrm{Disc}(L_2) = [L_2 : L_1]^2$.

In Chapter 4 we try to find a decomposition of lattices into $A_n\langle m \rangle, D_n\langle m \rangle, E_8\langle m \rangle$ and $U\langle m \rangle$, to do so we will need some extra invariants. We may extend the bilinear form $\langle\, , \rangle$ on L $\mathbb{Q}$-linearly to $L \otimes \mathbb{Q}$ and define the *dual lattice* (which is often not a lattice with respect to our definition):

$$L^* := \mathrm{Hom}(L, \mathbb{Z}) \cong \{x \in L \otimes \mathbb{Q} : \langle x, y \rangle \in \mathbb{Z}\, \forall y \in L\}.$$

Definition 2.1.1. The *discriminant group* of a lattice L is the finite abelian group $A_L := L^*/L$. We denote by $\ell(A_L)$ the minimal number of generators of A_L.

The discriminant group comes with a bilinear form, $b_L : A_L \times A_L \to \mathbb{Q}/\mathbb{Z}$, defined by $b_L(x+L, y+L) \mapsto \langle x, y\rangle_{L^*} \mod \mathbb{Z}$.

For even lattices, we define the *discriminant form*, $q_L : A_L \to \mathbb{Q}/2\mathbb{Z}$, by $x + L \mapsto \langle x, x\rangle_{L^*} \mod 2\mathbb{Z}$.

The following theorem of Nikulin will help identify the lattices we will find:

Theorem 2.1.2 (Nikulin [Nik80, Cor. 1.13.3])**.** *If a lattice L is even, indefinite with $\operatorname{rank}(L) > \ell(A_L) + 2$, then L is determined up to isometry by its rank, signature and discriminant form.*

With that theorem in mind, we write down in Table 2.1 a summary of the rank, signature and discriminant form for the lattices U, E_8, $A_n\langle m\rangle$, $D_n\langle m\rangle$ and $\langle 2m\rangle$.

	Rank	Signature	Discriminant	A_L	q_L
U	2	$(1,1)$	1	$\{\mathrm{id}\}$	$\{0\}$
E_8	8	$(8,0)$	1	$\{\mathrm{id}\}$	$\{0\}$
$A_n\langle m\rangle$	n	$\begin{cases}(n,0) & m>0\\ (0,n) & m<0\end{cases}$	$(n+1)\cdot m^n$	$C_{(n+1)m}\times C_m^{n-1}$	$\left\{\frac{n}{(n+1)m}, \frac{2}{m}, \frac{n(n-1)}{m}\right\}$
$D_{2n}\langle m\rangle$	$2n$	$\begin{cases}(2n,0) & m>0\\ (0,2n) & m<0\end{cases}$	$4\cdot m^{2n}$	$C_{2m}^2\times C_m^{2n-2}$	$\left\{\frac{2}{2m}, \frac{n}{2m}, \frac{2}{m}\right\}$
$D_{2n+1}\langle m\rangle$	$2n+1$	$\begin{cases}(2n+1,0) & m>0\\ (0,2n+1) & m<0\end{cases}$	$4\cdot m^{2n+1}$	$C_{4m}\times C_m^{2n}$	$\left\{\frac{2n+1}{4m}, \frac{2}{m}, \frac{2n}{m}\right\}$
$\langle 2m\rangle$	1	$\begin{cases}(1,0) & m>0\\ (0,1) & m<0\end{cases}$	$2m$	C_{2m}	$\left\{\frac{1}{2m}\right\}$

Table 2.1: Invariant of Lattices

The row q_L lists the values of $q_L(x_i)$ where x_i are chosen generators of A_L, i.e., $A_L = \langle x_1\rangle \times \cdots \times \langle x_{\ell(A)}\rangle$. Therefore it only encodes partial information of the discriminant form and not the whole of it, but it encodes enough to rule out (in most cases) whether a summand occurs. As U and E_8 have trivial discriminant group, we use following theorem of Nikulin to identify copies of U and E_8 sitting inside a given lattice.

Theorem 2.1.3 (Nikulin [Nik80, Cor 1.13.15])**.** *Let L be an even lattice of signature (b_+, b_-).*

- *If $b_+ \geq 1, b_- \geq 1$ and $b_+ + b_- \geq 3 + \ell(A_L)$ then $L \cong U \oplus T$ for some T.*
- *If $b_+ \geq 1, b_- \geq 8$ and $b_+ + b_- \geq 9 + \ell(A_L)$ then $L \cong E_8\langle -1\rangle \oplus T$ for some T.*

We note that we can not always have a decomposition of lattices into $A_n\langle m\rangle, D_n\langle m\rangle, E_8\langle m\rangle$ and $U\langle m\rangle$. When this happens, we express our lattices as full rank sublattices of a lattice composed of $A_n\langle m\rangle, D_n\langle m\rangle, E_8\langle m\rangle$ and $U\langle m\rangle$. For this we will use:

Theorem 2.1.4. [Nik80, Prop 1.4.1] *Let L be an even lattice. Then there is a natural bijection between isotropic subgroups G of A_L (subgroups on which the discriminant form q_L satisfies $q_L(g) = 0$ for all $g \in G$) and overlattices L_G of L.*

Furthermore, the discriminant form q_{L_G} is given by the discriminant form q_L restricted to $G^{\perp}/G$, where orthogonality is with respect to b_L.

2.2 Basic Definitions

As this thesis deals with properties of general members of families, we start by reviewing some standard definitions. Let K be a field and X be a variety in $\mathbb{P}^n_K$.

Definition 2.2.1. A *general point* of X is a point $p \in X$ outside an (implied) Zariski closed proper subset of X.

Example. Let $X = \mathbb{P}^1_{\mathbb{Q}}$, then for a general point $p = [s, t] \in X$, we have that $t \neq 0$. (The implied Zariski closed proper subset being $\{t = 0\}$)

Definition 2.2.2. A *very general point* of X is a point $p \in X$ outside a countable union of Zariski closed proper subsets of X.

Example. Let $X = \mathbb{P}^1_{\mathbb{Q}}$, then for a very general point $p = [s, t] \in X$, we have that s/t is not a power of 2. This can be seen as s/t is a power of 2 if and only if there exists $d \in \mathbb{Z}$ such that $s - 2^d t = 0$. So p outside the countable union $\cup_{d\in\mathbb{Z}}\{s - 2^d t = 0\}$ is very general.

General and very general points have links with thin sets, which we review here. For the moment, suppose that K has characteristic 0.

Definition 2.2.3. A subset $\Omega \subseteq \mathbb{P}^n_K$ is said to be *thin* if there exists a variety X over K and a morphism $\pi : X \to \mathbb{P}^n_K$ such that

1. $\Omega \subset \pi(X(K))$,
2. The fibre of π over the generic point is finite, and π has no rational sections over K.

We differentiate between two types of thin sets.

Type 1 A thin set $\Omega \subseteq \mathbb{P}^n_K$ is said to be of type 1 if there exists a homogenous polynomial $F(T_1, \ldots, T_n) \in K[T]$ such that $F(t) = 0$ for all $t \in \Omega$. That is, Ω is a proper Zariski closed subset of $\mathbb{P}^n$.

Type 2 A thin set $\Omega \subseteq \mathbb{P}^n_K$ is said to be of type 2 if it is obtained as follows. Let $F(X, T_1, \ldots, T_n) \in K(T)[X]$ be absolutely irreducible (i.e., irreducible over $\overline{K}(T)[X]$), of degree ≥ 2 with respect to X, and such that the coefficients with respect to X are homogenous in T of the same degree m

(i.e., $F(X, \lambda T) = \lambda^m F(X, T)$). Then Ω is the set of all $t \in \mathbb{P}^n_K$ such that t is not a pole of the coefficients of F and $F(X, t)$ has a root in K.

Proposition 2.2.4 ([Ser97, Prop 9.1]). *Every thin subset of $\mathbb{P}^n_K$ is contained in a finite union of thin sets of type* 1 *and* 2.

Examples.

- The set $\Omega_1 = \{t : t = 0\} \subseteq \mathbb{P}^1_K$ is a thin set of type 1. Note that for $K = \mathbb{Q}$, a general point is outside this set.
- Consider the polynomial $S - TX^2 \in K(S, T)[X]$. Then $\Omega_2 = \{[s, t] \in \mathbb{P}^1_K : st \text{ is a square in } K, t \neq 0\}$ is a thin set of type 2.
- Fix $c \in K$ and let $F_c = S - cTX^2$, then each distinct class in $K^*/(K^2)^*$ gives rise to a different thin set Ω_c (of type 2). Furthermore, a finite union $\cup_{c \in K^*/(K^2)^*} \Omega_c$ is a thin set.

Definition 2.2.5. A field K, of characteristic 0, is called *Hilbertian* if for all $n \geq 1$, $\mathbb{P}^n_K$ is not thin.

Remark. For K to be Hilbertian it is enough to show that $\mathbb{P}^1_K$ is not thin (see [Ser97, Rmk 9.5.1]).

Examples.

- By the above example, if K is Hilbertian then $K^*/(K^2)^*$ is infinite, otherwise the finite union $\cup_{c \in K^*/(K^2)^*} \Omega_c = \mathbb{P}^1_K$ would be thin. In particular, local fields and algebraically closed fields are not Hilbertian.
- $\mathbb{Q}$ is Hilbertian (Hilbert's irreducibility theorem, see [Ser97, Thm 9.6] for a proof).
- If K is Hilbertian, any finite extension of K is also Hilbertian. In particular, any number field is Hilbertian.

We now turn our attention to reviewing some algebraic geometry notions we will need. We revert back to letting K be an arbitrary field.

Definition. A K3 *surface* is a smooth surface X over K with irregularity $q = \dim H^1(X, \mathcal{O}_X) = 0$ and trivial canonical sheaf $\omega_X \cong \mathcal{O}_X$.

Here a surface is a projective integral separated scheme of finite type and dimension 2 over K (see [Har77]).

Example. Smooth quartic surfaces in $\mathbb{P}^3$ are K3 surfaces.

Definition 2.2.6. Let X be a smooth surface over K:

- A *prime divisor* on X is a curve Y on X,
- A *(Weil) divisor* $D = \sum n_i Y_i$ is a finite formal sum of prime divisors,
- Let $\mathrm{Div}(X)$ be the set of Weil divisors on X,
- Let f be a non-zero rational function on X, we define the divisor of f as $(f) = \sum \nu_{Y_i}(f) Y_i$, where $\nu_{Y_i}(f)$ is the valuation of f in the discrete valuation ring associated to the generic point of Y.

We can define the *intersection pairing* $(\,,\,) : \mathrm{Div}(X) \times \mathrm{Div}(X) \to \mathbb{Z}$ which is a symmetric bilinear function on $\mathrm{Div}(X)$, refer to [Har77, Sec V.1] for more details. There are three basic equivalence relations we can put on $\mathrm{Div}(X)$:

Linear Two divisors $C, D \in \mathrm{Div}(X)$ are linearly equivalent if there exists a rational function f such that $C = D + (f)$

Algebraic Two divisors $C, D \in \mathrm{Div}(X)$ are algebraically equivalent if there exists a curve T, two closed points $0, 1 \in T$ and a divisor E in $X \times T$ such that $E_0 - E_1 = C - D$.

Numerical Two divisors $C, D \in \mathrm{Div}(X)$ are numerically equivalent if for all $E \in \mathrm{Div}(X)$, $(C, E) = (D, E)$.

We note that linear equivalence implies algebraic equivalence, which in turn implies numerical equivalence.

Definition 2.2.7. We define the *Picard group* of X to be $\mathrm{Pic}(X) := \mathrm{Div}(X)/\text{linear equivalence}$.

Let $\mathrm{Pic}^0(X)$ be the set of divisor classes algebraically equivalent to zero. We define the *Néron-Severi group* of X to be $\mathrm{NS}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^0(X)$.

Let $\mathrm{Pic}^\tau(X)$ be the set of divisor classes numerically equivalent to zero. We define $\mathrm{Num}(X) := \mathrm{Pic}(X)/\mathrm{Pic}^\tau(X)$.

Proposition 2.2.8. *Let X be a* K3 *surface over K, then* $\mathrm{Pic}(X) \cong \mathrm{NS}(X) \cong \mathrm{Num}(X)$.

See, for example, [VA15, Prop 1.8] for a proof.

Proposition 2.2.9. *[LN59] The Néron-Severi group of X is a finitely generated abelian group.*

Definition 2.2.10. The rank $\rho := \mathrm{rank}(\mathrm{NS}(X))$ is called the *Picard number* of X.

Remark. For a K3 surface X, we have that $\rho(X) \leq 22$ (see [BPVdV84, Prop VIII.8.3] and [BM77, Thm 5]). If X is defined over a field of characteristic 0, then $\rho(X) \leq 20$.

A K3 surface X with $\rho(X) = 20$ is called a *singular* K3 *surface.* A K3 surface X with $\rho(X) = 22$ is called a *supersingular* K3 *surface.*

Proposition 2.2.11. *Let X be a* K3 *surface. Then* $\mathrm{Pic}(X)$ *equipped with* $(\ ,\ ) : \mathrm{Pic}(X) \times \mathrm{Pic}(X) \to \mathbb{Z}$ *is an even lattice.*

Proof. Since for a K3 surface $\mathrm{Pic}(X) \cong \mathrm{Num}(X)$ and by definition $\mathrm{Num}(X)$ is torsion free, we have that $\mathrm{Pic}(X)$ is a lattice. Now using Riemann-Roch for surfaces, if C is a non-singular curve of genus g, then $(C, C) = (C, C + K) = 2g - 2$. Therefore, the Picard lattice is even. □

2.3 The Family $\mathcal{X}$

The first half of this thesis will concentrate on the variety $\mathcal{X} \subset \mathbb{P}^3_{[x,y,z,w]} \times \mathbb{P}^4_{[A,B,C,D,E]}$ defined by the following equation over $\overline{\mathbb{Q}}$

$$A(x^4+y^4+z^4+w^4)+Bxyzw+C(x^2y^2+z^2w^2)+D(x^2z^2+y^2w^2)+E(x^2w^2+y^2z^2) = 0.$$

We view $\mathcal{X}$ as a family of quartic surfaces over $\mathbb{P}^3$ parametrised by points $[A, B, C, D, E]$ in $\mathbb{P}^4$.

Notation. We will use X_p and $[A, B, C, D, E]$ to denote the quartic surface parametrised by the point $p = [A, B, C, D, E] \in \mathbb{P}^4$.

Note. If X_p is a smooth quartic surface, then it is a K3 surface.

Consider the group Ω acting on $\mathbb{P}^3 \times \mathbb{P}^4$ generated by the following five elements: the point $[x, y, z, w, A, B, C, D, E]$ is sent to

- $[x, y, z, -w, A, -B, C, D, E]$,
- $[x, y, w, z, A, B, C, E, D]$,
- $[x, z, y, w, A, B, D, C, E]$,
- $[x, y, iz, iw, A, -B, C, -D, -E]$,
- $[x-y, x+y, z-w, z+w, 2A+C, 8(D-E), 12A-2C, B+2D+2E, -B+2D+2E]$.

Denote these five elements by $\phi_1, \phi_2, \phi_3, \phi_4$ and ϕ_5 respectively. The group Ω fixes $\mathcal{X}$. While it is a rather large group with order $2^4 \cdot 6!$, we can pick out a normal subgroup Γ, which is generated by the following four elements

- $\gamma_1 := \phi_3\phi_4^2\phi_3\phi_5^2$,
- $\gamma_2 := \phi_4^2\phi_3\phi_5^2\phi_3$,
- $\gamma_3 := \phi_4^2$,
- $\gamma_4 := \phi_3\phi_4^2\phi_3$.

The group Γ consists of all elements of Ω which fix $\mathbb{P}^4_{[A,B,C,D,E]}$ in $\mathbb{P}^3 \times \mathbb{P}^4$. In particular upon picking a point $p \in \mathbb{P}^4$ we have that Γ is a subgroup of $\mathrm{Aut}(X_p)$ (when projecting the elements of Γ onto the $\mathbb{P}^3_{[x,y,z,w]}$ component). Explicitly, when regarding Γ as acting on $\mathbb{P}^3$, we have that its generators are

$$[x,y,z,w] \mapsto \begin{cases} [y,x,w,z] & \gamma_1 \\ [z,w,x,y] & \gamma_2 \\ [x,y,-z,-w] & \gamma_3 \\ [x,-y,z,-w] & \gamma_4 \end{cases}.$$

From this we know that $\Gamma \cong C_2^4$. We calculate that $\Omega/\Gamma \cong S_6$, but $\Omega \not\cong C_2^4 \times S_6$ because in particular Ω has trivial centre. For each $\gamma \in \Gamma \backslash \{\mathrm{id}\}$ we make a note of the points of $\mathbb{P}^3$ which are fixed by γ (restricted to $\mathbb{P}^3$). Each γ has two skew lines $L, \overline{L}$ of fixed points which are given by its $(+1)$ and (-1) eigenspaces, respectively its $(+i)$ and $(-i)$ eigenspaces. The lines are given in Table 2.2 (along with the Segre planes which we explain after Proposition 2.3.5). Note that every pair of lines is also fixed by any $\gamma \in \Gamma$ (on top of containing the fixed points of a particular γ).

	L_i	$\overline{L_i}$	Segre plane
γ_1	$[s,s,t,t]$	$[s,-s,t,-t]$	$q_{+C} = p_{+0} = p_{-1} = 0$
γ_2	$[s,t,s,t]$	$[s,t,-s,-t]$	$q_{+D} = p_{+0} = p_{-2} = 0$
$\gamma_1\gamma_2$	$[s,t,t,s]$	$[s,t,-t,-s]$	$q_{+E} = p_{+0} = p_{-3} = 0$
γ_3	$[s,t,0,0]$	$[0,0,s,t]$	$A = q_{+C} = q_{-C} = 0$
$\gamma_1\gamma_3$	$[s,-s,t,t]$	$[s,s,t,-t]$	$q_{-C} = p_{-0} = p_{+1} = 0$
$\gamma_2\gamma_3$	$[s,t,is,it]$	$[s,t,-is,-it]$	$q_{-D} = p_{-1} = p_{+3} = 0$
$\gamma_1\gamma_2\gamma_3$	$[s,t,it,is]$	$[s,t,-it,-is]$	$q_{-E} = p_{-1} = p_{+2} = 0$
γ_4	$[s,0,t,0]$	$[0,s,0,t]$	$A = q_{+D} = q_{-D} = 0$
$\gamma_1\gamma_4$	$[s,is,t,it]$	$[s,-is,t,-it]$	$q_{-C} = p_{-2} = p_{+3} = 0$
$\gamma_2\gamma_4$	$[s,t,-s,t]$	$[s,t,s,-t]$	$q_{+D} = p_{-0} = p_{+2} = 0$
$\gamma_1\gamma_2\gamma_4$	$[s,t,it,-is]$	$[s,t,-it,is]$	$q_{-E} = p_{+1} = p_{-2} = 0$
$\gamma_3\gamma_4$	$[s,0,0,t]$	$[0,s,t,0]$	$A = q_{+E} = q_{-E} = 0$
$\gamma_1\gamma_3\gamma_4$	$[s,-is,t,it]$	$[s,is,t,-it]$	$q_{-C} = p_{+2} = p_{-3} = 0$
$\gamma_2\gamma_3\gamma_4$	$[s,t,-is,it]$	$[s,t,is,-it]$	$q_{-D} = p_{+1} = p_{-3} = 0$
$\gamma_1\gamma_2\gamma_3\gamma_4$	$[s,t,t,-s]$	$[s,t,-t,s]$	$q_{+W} = p_{-0} = p_{+3} = 0$

Table 2.2: List of Invariant Lines

Notation. We shall denote by $\mathcal{L}$ the union of the 15 pairs of lines.

We now consider the cases when X_p is not a smooth surface using the following proposition taken from [Ekl10, Prop 2.1].

Proposition 2.3.1. *Let $p = [A,B,C,D,E] \in \mathbb{P}^4$. The surface X_p is singular if and* *only if*

$$\Delta \cdot A \cdot q_{+C} \cdot q_{-C} \cdot q_{+D} \cdot q_{-D} \cdot q_{+E} \cdot q_{-E} \cdot p_{+0} \cdot p_{+1} \cdot p_{+2} \cdot p_{+3} \cdot p_{-0} \cdot p_{-1} \cdot p_{-2} \cdot p_{-3} = 0,$$

where:

$$\Delta = 16A^3 + AB^2 - 4A(C^2 + D^2 + E^2) + 4CDE \tag{2.3.1}$$

$$\begin{aligned}
q_{+C} &= 2A + C & q_{+D} &= 2A + D & q_{+E} &= 2A + E \\
q_{-C} &= 2A - C & q_{-D} &= 2A - D & q_{-E} &= 2A - E \\
p_{+0} &= 4A + B + 2C + 2D + 2E & & & p_{-0} &= 4A - B + 2C + 2D + 2E \\
p_{+1} &= 4A + B + 2C - 2D - 2E & & & p_{-1} &= 4A - B + 2C - 2D - 2E \\
p_{+2} &= 4A + B - 2C + 2D - 2E & & & p_{-2} &= 4A - B - 2C + 2D - 2E \\
p_{+3} &= 4A + B - 2C - 2D + 2E & & & p_{-3} &= 4A - B - 2C - 2D + 2E.
\end{aligned}$$

Definition 2.3.2. The surface $S_3 = \{\Delta = 0\} \subset \mathbb{P}^4$ is the *Segre cubic*. We shall refer to the 15 hyperplanes in $\mathbb{P}^4$ defined by the 15 equations

$$\{A, p_{\pm j}, q_{\pm\alpha} : \alpha \in \{C, D, E\}, j \in \{0, 1, 2, 3\}\}$$

above as the *singular hyperplanes*.

The Segre cubic has 10 singular points, namely:

$$[1, 0, -2, -2, 2], [1, 0, -2, 2, -2], [1, 0, 2, -2, -2], [1, 0, 2, 2, 2],$$

$$[0, -2, 1, 0, 0], [0, 2, 1, 0, 0], [0, -2, 0, 1, 0], [0, 2, 0, 1, 0], [0, -2, 0, 0, 1], \text{ and } [0, 2, 0, 0, 1].$$

We shall denote these 10 points by q_i, $i \in [1, \dots, 10]$, as ordered above. These 10 points have associated quartics in $\mathbb{P}^3$, which turns out to be quadrics of multiplicity two. We denote the quadric associated to the point q_i by Q_i. Explicitly they are:

$$x^2 - y^2 - z^2 + w^2 = 0,\ x^2 - y^2 + z^2 - w^2 = 0,\ x^2 + y^2 - z^2 - w^2 = 0,\ x^2 + y^2 + z^2 + w^2 = 0,$$

$$xy - zw = 0,\ xy + zw = 0,\ xz - yw = 0,\ xz + yw = 0,\ xw - yz = 0, \text{ and } xw + yz = 0.$$

Remark 2.3.3. We note that the action Ω on $\mathbb{P}^3 \times \mathbb{P}^4$ induces an action on the set of the 15 singular hyperplanes, and a second action on the set of the 10 singular points q_i. The actions induced are as follows (using permutation notation):

- ϕ_1 acts as $(p_{+0}, p_{-0})(p_{+1}, p_{-1})(p_{+2}, p_{-2})(p_{+3}, p_{-3})$ and as $(q_5, q_6)(q_7, q_8)(q_9, q_{10})$,
- ϕ_2 acts as $(q_{+D}, q_{+E})(q_{-D}, q_{-E})(p_{+2}, p_{+3})(p_{-2}, p_{-3})$ and as $(q_1, q_2)(q_7, q_9)(q_8, q_{10})$,
- ϕ_3 acts as $(q_{+C}, q_{+D})(q_{-C}, q_{-D})(p_{+1}, p_{+2})(p_{-1}, p_{-2})$ and as $(q_2, q_3)(q_5, q_7)(q_6, q_8)$,

- ϕ_4 acts as $(q_{+D}, q_{-D})(q_{+E}, q_{-E})(p_{+0}, p_{-1})(p_{-0}, p_{+1})(p_{+2}, p_{-3})(p_{-2}, p_{+3})$ and as $(q_1, q_2)(q_3, q_4)(q_5, q_6)$,
- ϕ_5 acts as $(A, q_{+C})(q_{+D}, p_{+0})(q_{-D}, p_{-1})(q_{+E}, p_{-0})(q_{-E}, p_{+1})(p_{+2}, p_{-3})$ and as $(q_1, q_5)(q_2, q_6)(q_7, q_{10})$.

It is known that for a general point on S_3 the corresponding surface is a Kummer surface ([Ekl10, Prop 2.2]). The following two propositions link such Kummer surfaces with their singular points.

Proposition 2.3.4. *Let $p = [x, y, z, w]$ be a point in $\mathbb{P}^3 \setminus \mathcal{L}$ and let*

- $A = (yz + xw)(yz - xw)(xz + yw)(xz - yw)(zw + xy)(zw - xy)$,
- $B = 2xyzw(-x^2-y^2+z^2+w^2)(-x^2+y^2+z^2-w^2)(x^2-y^2+z^2-w^2)(x^2+y^2+z^2+w^2)$,
- $C = (yz + xw)(yz - xw)(xz + yw)(xz - yw)(x^4 + y^4 - z^4 - w^4)$,
- $D = (yz + xw)(yz - xw)(zw + xy)(zw - xy)(-x^4 + y^4 - z^4 + w^4)$,
- $E = (xz + yw)(xz - yw)(zw + xy)(zw - xy)(x^4 - y^4 - z^4 + w^4)$.

Then the point $[A, B, C, D, E]$ lies on the Segre cubic and the associated Kummer surface has the 16 singular points $\{\gamma(p) : \gamma \in \Gamma\}$.

Proof. Let $F = A(X^4 + Y^4 + Z^4 + W^4) + \cdots + E(X^2W^2 + Z^2Y^2)$. By algebraic manipulation, the system of linear equations

$$F(p) = \frac{\partial F}{\partial X}(p) = \frac{\partial F}{\partial Y}(p) = \frac{\partial F}{\partial Z}(p) = \frac{\partial F}{\partial W}(p) = 0$$

has a unique solution $[A, B, C, D, E] \in \mathbb{P}^4$ as given above. Hence, for the rest of the proof let A, B, C, D, E be as in the proposition. The quartic surface defined by $[A, B, C, D, E]$ has p as a singular point. Substituting $[A, B, C, D, E]$ into the equation Δ, we see that $[A, B, C, D, E]$ lies on on the Segre cubic. Finally, having found one singular point, we note that since Γ fixes the quartic surface defined by $[A, B, C, D, E]$, any point $\gamma(p)$ with $\gamma \in \Gamma$ must also be a point of singularity.

Note that in particular, any point $p \in \mathbb{P}^3 \setminus \mathcal{L}$ uniquely defines the Kummer surface of which it is a singular point. $\square$

Proposition 2.3.5. *Let $[A, B, C, D, E]$ be a point on the Segre cubic not lying on one of the* 15 *singular hyperplanes. Then the associated surface's 16 singular points are $[x, y, z, w]$ where x, y, z and w solve the following equations:*

- $az^8 + bz^6w^2 + cz^4w^4 + bz^2w^6 + aw^8 = 0$, *with* $a = -A^2B^2$, $b = 4(2AD - CE)(2AE - CD)$ *and* $c = 2(A^2B^2 - 2(E^2 + D^2)(4A^2 + C^2) + 16ACDE)$,

- $(4A^2 - C^2)(Ez^2 - Dw^2)y^2 + A\big((4A^2 - C^2)(z^4 - w^4) + (E^2 - D^2)(z^4 + w^4)\big) + C(E^2 - D^2)z^2w^2 = 0,$
- $2(C^2 - 4A^2)xyzw + BCz^2w^2 + AB\left(w^4 + z^4\right) = 0.$

Proof. Without loss of generality, using the action of Γ, we assume $w = 1$. Then the first equation can be considered as a symmetric quartic polynomial with the variable z^2, and hence z can be written as a radical function of A, B, C, D, E, i.e,

$$z = \pm\sqrt{\frac{u_\pm \pm \sqrt{u_\pm^2 - 4}}{2}}, \text{ where } u_\pm = \frac{-b \pm \sqrt{b^2 - 4a(c - 2a)}}{2a}.$$

Similarly, we can write x and y as radical functions of A, B, C, D, E. Substituting the point $[x, y, z, 1]$ (written in terms of A, B, C, D, E) into the equations of Proposition 2.3.4, we get an equality. Since a point uniquely defines the Kummer surface, we must have that the point $[x, y, z, 1]$ is a singular point of $[A, B, C, D, E]$. $\square$

We explain why we need the hypothesis in the two previous propositions, namely taking a point in $\mathbb{P}^3$ away from $\mathcal{L}$ and taking a point in $\mathbb{P}^4$ away from the singular hyperplanes. First we note that the intersection of one of the singular hyperplanes with the Segre cubic breaks down into 3 planes. For example

$$\begin{aligned} \{q_{+C} = 0\} \cap \{\Delta = 0\} &= \{q_{+C} = 0, q_{-C} = 0, A = 0\} \\ &\cup \{q_{+C} = 0, p_{-0} = 0, p_{+1} = 0\} \\ &\cup \{q_{+C} = 0, p_{+0} = 0, p_{-1} = 0\}. \end{aligned}$$

We check that we get 15 planes this way, which we shall refer to as the 15 *Segre planes.*

Suppose the surface X_p is represented by a point p lying on one of the 15 Segre planes, that is p does not satisfy the hypothesis of Proposition 2.3.5. Then, we calculate that X_p does not have only 16 singular points, but rather two skew singular lines. Namely one of the 15 pairs of lines in $\mathcal{L}$. On the other hand, consider the surface X_p with $p \in \mathbb{P}^4$, which has the singular point $q \in \mathcal{L}$. By Proposition 2.3.1 we know that either p lies on the Segre cubic or on one of the 15 singular hyperplanes. If p lies on the Segre cubic, then in fact p lies on one of the Segre planes. If p lies on a singular hyperplane and not on the Segre cubic, then q lies on 3 lines contained in $\mathcal{L}$ (c.f. A.1 in the Appendix).

Hence we have a one to one correspondence between the 15 pairs of skew lines of $\mathcal{L}$ and the 15 Segre planes. Table 2.2 shows which Segre plane corresponds to which pair of lines.

Definition 2.3.6. Let Y be a quartic surface in $\mathbb{P}^3$. We say that a plane T in $\mathbb{P}^3$ is a *trope* of Y if $Y \cap T$ is an irreducible conic counted with multiplicity two.

Lemma 2.3.7. *A quartic surface $Y \subset \mathbb{P}^3$ which has a trope T is singular.*

We now turn to the theorem from Eklund, [Ekl10, Thm 4.3], which motivated the questions of this thesis.

Theorem 2.3.8. *A general K3 surface X from the family $\mathcal{X}$ contains at least 320 smooth conics.*

Proof. The proof is adapted from [Ekl10, Thm 4.3], which we reproduce here as we will use some elements of the proof in the rest of the thesis. For this proof if Y is a hypersurface, fix $\widetilde{Y}$ to be an equation defining Y.

Pick $p \in \mathbb{P}^4$ general and let q_i be a singular point of the Segre cubic S_3 (in particular fix i). We have that the associated surface to q_i is Q_i^2, a quadric of multiplicity two. The line through p and q_i intersects S_3 in exactly one other point, call it p_i. Hence we have $\widetilde{X_p} = \alpha\left(\widetilde{Q_i}\right)^2 + \alpha'\widetilde{X_{p_i}}$ for some $\alpha, \alpha' \in \overline{\mathbb{Q}}$. As p is general by Proposition 2.3.5, we have that the associated surface X_{p_i} is Kummer. Pick a singular point on X_{p_i}, say $[a, b, c, d]$, and consider its dual $T := \{ax + by + cz + dw = 0\} \subset \mathbb{P}^3_{[x,y,z,w]}$. As T is a trope of X_{p_i} (see [Ekl10, pg 12] for more details) we have that $\widetilde{X_{p_i}} = \mu(Q')^2 + \lambda\widetilde{T}$, for some $\mu \in \overline{\mathbb{Q}}$, a cubic equation λ and a quadratic equation Q'. Hence, as an equation,

$$\begin{aligned}\widetilde{X_p} &= \alpha\left(\widetilde{Q_i}\right) + \alpha'\mu\left(Q'\right)^2 + \alpha'\lambda\widetilde{T} \\ &= (\sqrt{\alpha}\widetilde{Q_i} + \sqrt{-\alpha'\mu}Q')(\sqrt{\alpha}\widetilde{Q_i} - \sqrt{-\alpha'\mu}Q') + \alpha'\lambda\widetilde{T}. \end{aligned} \tag{2.3.2}$$

So $X_p \cap T$ is the union of two conics. As the general member of the family does not contain any line (see [Ekl10, Prop 2.3]), nor does it have a trope (Lemma 2.3.7), we have that the two conics, $\sqrt{\alpha}\widetilde{Q}_i \pm \sqrt{-\alpha'\mu}Q'$, are smooth (as they cannot be the union of two lines) and distinct.

Since a general Kummer surface of $\mathcal{X}$ is determined by any of its singular points (by Proposition 2.3.4) and hence by its tropes, all the tropes defined by using the 10 nodes q of S_3 are different. As two different planes cannot have a smooth conic in common, we conclude that we have constructed $10 \cdot 16 \cdot 2 = 320$ smooth conics on X_p. □

Chapter 3

The Galois and Monodromy Group

The motivating question for this chapter is "if X is an invariant quartic K3 surface defined over a number field K, what is the smallest field extension over which the 320 conics are defined?". The first section is dedicated to answering that question. After finding the field of definition of the 320 conics, in the second section, we are interested in studying the Monodromy group of the 320 conics. The main result of this chapter is that both the Galois group of the field of definition of the 320 conics, and the Monodromy group of the 320 conics are isomorphic to C_2^{10}. As a corollary we deduce that the moduli space of pairs (X, C), where X is an invariant quartic K3 surface and and C one of the 320 conic, has 10 irreducible components.

The results of this chapter have been put together into a preprint [Bou15a].

3.1 The Galois Group

In this section we are going to shift away from working over $\overline{\mathbb{Q}}$ to working over number fields. Let K be a number field, and let $p = [A, B, C, D, E]$ be a very general point in $\mathbb{P}^4_K$. Then the associated K3 surface, X_p, has 320 conics on it. So let L be the smallest number field containing K over which those conics are defined. Hence L is an extension of of K. We want to work out the Galois group of the field of definition of the 320 conics. That is, we are interested in $\mathrm{Gal}(L/K)$ so we first find L.

From Theorem 2.3.8 we have a constructive proof of the existence of the conics, so we want to construct equations defining the conics. In particular, we want to work out, for a fixed p and for each conic, the field of definition of α, α', μ and Q' (since Q_i is defined over $\mathbb{Q} \subseteq K$ for all i). Note that we have that α and α' depend only on Q_i (or more specifically on the point q_i), while μ and Q' depend both on Q_i and the trope T (of which there are 16 choices once Q_i is fixed). So let α_i and α'_i be associated to Q_i. Using the equations defining the line through the point p and the point q_i, with the cubic equation defining S_3 we can find the point p_i. Hence we write $[A_i, B_i, C_i, D_i, E_i] = X_{p_i}$ in terms of A, B, C, D and E. Since we know X_{q_i}, we can use simple algebra to calculate α_i and α_i'. We find that $\alpha_i = \Delta\beta_i^{-1}$ and $\alpha_i' = \begin{cases} \beta_i^{-1} & i \in [1,\dots,4] \\ (4\beta_i)^{-1} & i \in [5,\dots,10] \end{cases}$ where

$$\begin{aligned}
\beta_1 =& 12A^2 + \frac{1}{4}B^2 + 4A(C+D-E) - (C^2+D^2+E^2) + 2(CD-CE-DE)\\
\beta_2 =& 12A^2 + \frac{1}{4}B^2 + 4A(C-D+E) - (C^2+D^2+E^2) + 2(-CD+CE-DE)\\
\beta_3 =& 12A^2 + \frac{1}{4}B^2 + 4A(-C+D+E) - (C^2+D^2+E^2) + 2(-CD-CE+DE)\\
\beta_4 =& 12A^2 + \frac{1}{4}B^2 - 4A(C+D+E) + (C^2+D^2+E^2) + 2(CD+CE+DE)
\end{aligned}$$

$$\begin{aligned}
\beta_5 &= -(AB+2AC-DE) & \beta_6 &= AB-2AC+DE\\
\beta_7 &= -(AB+2AD-CE) & \beta_8 &= AB-2AD+CE\\
\beta_9 &= -(AB+2AE-CD) & \beta_{10} &= AB-2AE+CD
\end{aligned}$$

and Δ is defined by Equation (2.3.1). In particular, letting $\widetilde{Y}$ be the equation defining the hypersurface Y as in Theorem 2.3.8, we have $\widetilde{X_p} = \beta_i^{-1}\left(\Delta\left(\widetilde{Q_i}\right)^2 + \widetilde{X_{p_i}}\right)$, with the $\frac{1}{4}$ factor absorbed in the equation defining X_{p_i} when needed. Hence, using Equation (2.3.2) and by rescaling with β_i, we have, for a fixed Q_i and T,

$$X_p \cap T = \left\{\left(\widetilde{Q_i} + \sqrt{-\frac{\mu_i}{\Delta}}Q'\right)\left(\widetilde{Q_i} - \sqrt{-\frac{\mu_i}{\Delta}}Q'\right) = 0, \widetilde{X_p} = 0\right\}.$$

In particular, the field of definition of the pair of conics defined by T only depends on $\sqrt{-\frac{\mu_i}{\Delta}}$ and Q'. So let us fix an i, set $X_{p_i} = [A_i, B_i, C_i, D_i, E_i]$ and let us fix T by choosing the singular point $[r_{3,i}, r_{2,i}, r_{1,i}, 1]$ on X_{p_i} (using Proposition 2.3.5). That is, T is defined by $r_{3,i}x + r_{2,i}y + r_{1,i}z + w = 0$. As the singular point uniquely defines X_{p_i}, we use the equations in Proposition 2.3.4 to rewrite the equation defining X_{p_i} in terms of $r_{3,i}, r_{2,i}$ and $r_{1,i}$. Since $\widetilde{X_{p_i}} \cap \widetilde{T} = (Q')^2$ substituting $w = -(r_{3,i}x + r_{2,i}y + r_{1,i}z)$ into X_{p_i} we find that $(Q')^2 = (a_0x^2 + a_1y^2 + a_2z^2 + a_3xy + a_4xz + a_5yz)^2$ where

- $a_0 = (r_2r_3 - r_1)\cdot(r_2r_3 + r_1)\cdot(r_1r_3 - r_2)\cdot(r_1r_3 + r_2)$,
- $a_1 = (r_2r_3 - r_1)\cdot(r_2r_3 + r_1)\cdot(r_1r_2 - r_3)\cdot(r_1r_2 + r_3)$,
- $a_2 = (r_1r_3 - r_2)\cdot(r_1r_3 + r_2)\cdot(r_1r_2 - r_3)\cdot(r_1r_2 + r_3)$,
- $a_3 = r_3\cdot r_2\cdot(2r_1^2r_2^2r_3^2 - r_1^4 - r_2^4 - r_3^4 + 1)$,
- $a_4 = r_3\cdot r_1\cdot(2r_1^2r_2^2r_3^2 - r_1^4 - r_2^4 - r_3^4 + 1)$,
- $a_5 = r_2\cdot r_1\cdot(2r_1^2r_2^2r_3^2 - r_1^4 - r_2^4 - r_3^4 + 1)$.

Hence for a fixed i and trope T, the associated quadratic equation Q' is defined over the fixed field $K(r_{1,i}, r_{2,i})$ (recall that $r_{3,i}$ is a K-linear combination of $r_{1,i}, r_{2,i}$, see Proposition 2.3.5). Now each trope T, and hence each associated Q', of X_{p_i} is defined by Γ acting on the point $[r_{3,i}, r_{2,i}, r_{1,i}, 1]$. So we have that once i has been fixed, the 16 tropes and the 16 associated quadratic equations are all defined over $K(r_{1,i}, r_{2,i})$.

Next we work out μ_i (which also depends on the singular point $[r_{3,i}, r_{2,i}, r_{1,i}, 1]$). We use the fact that (as equations) $\widetilde{X_{p_i}} = \mu_i\,(Q')^2 + \lambda\widetilde{T}$ and that Q' has no w terms, to find that

$$\mu_i = (A_i r_{1,i}^4 + C_i r_{1,i}^2 + A_i)\cdot a_2^{-2}.$$

Again we see that the action of Γ on $[r_{3,i}, r_{2,i}, r_{1,i}, 1]$ will give us the other 15 μ's. In particular, as the 16 singular points have z-coordinates $\pm r_{1,i}, \pm\frac{1}{r_{1,i}}, \pm\frac{r_{2,i}}{r_{3,i}}, \pm\frac{r_{3,i}}{r_{2,i}}$, there are only four different μ's, namely μ, $\frac{1}{r_{1,i}^2}\mu$, $\overline{\mu} = (A_i\frac{r_{2,i}^4}{r_{3,i}^4} + C_i\frac{r_{2,i}^2}{r_{3,i}^2} + A_i)\cdot\overline{a_2}^{-2}$ and $\frac{r_{3,i}^2}{r_{2,i}^2}\overline{\mu}$ (where $\overline{a_2}$ can be calculated, but will not be needed). Putting all of this together we have proven the following proposition.

Proposition 3.1.1. *Let $p = [A, B, C, D, E] \in \mathbb{P}^4$ and fix $i \in [1, \ldots, 10]$. Let $p_i = [A_i, B_i, C_i, D_i, E_i] \in \mathbb{P}^4$ be the third point of intersection between the Segre cubic S_3, and the line joining q_i and p. Then the 32 conics lying on X_p and associated to the point q_i (as per the construction in Theorem 2.3.8) are defined over*

$$K_i = K(r_{1,i}, r_{2,i}, r_{\mu,i}, \overline{r_{\mu,i}}) \tag{3.1.1}$$

where

$$\begin{aligned}
r_{1,i} &\quad \textit{is a root of} \quad ax^8 + bx^6 + cx^4 + bx^2 + a, && (3.1.2)\\
r_{2,i} &\quad \textit{is a root of} \quad d(E_i r_{1,i}^2 - D_i)x^2 + A_i\left(d\left(r_{1,i}^4 - 1\right) + e\left(r_{1,i}^4 + 1\right)\right) + C_i e r_{1,i}^2, && (3.1.3)\\
r_{\mu,i} &\quad \textit{is a root of} \quad x^2 + \frac{1}{\Delta}(A_i r_{1,i}^4 + C_i r_{1,i}^2 + A_i), && (3.1.4)\\
\overline{r_{\mu,i}} &\quad \textit{is a root of} \quad x^2 + \frac{1}{\Delta}(A_i \overline{r}_{1,i}^4 + C_i \overline{r}_{1,i}^2 + A_i), && (3.1.5)
\end{aligned}$$

with $\overline{r}_{1,i} = \frac{r_{2,i}}{r_{3,i}}$ (which can be expressed in terms of $r_{1,i}$) and

$$a = -A_i^2 B_i^2,$$

$$b = 4(2A_iD_i - C_iE_i)(2A_iE_i - C_iD_i),$$

$$c = 2(A_i^2B_i^2 - 2(E_i^2 + D_i^2)(4A_i^2 + C_i^2) + 16A_iC_iD_iE_i),$$

$$d = 4A_i^2 - C_i^2, \qquad e = E_i^2 - D_i^2.$$

Remark. Magma [BCP97] was used for most of the computations leading to the above proposition. Part of the calculations are available on [Bou].

Proposition 3.1.2. *Let X_p be a K3 surface in the family $\mathcal{X}$. For each $i \in [1, \dots, 10]$ define K_i as in Proposition 3.1.1. Then $\mathrm{Gal}(K_i/K) \cong C_2^n$ for some $0 \leq n \leq 5$ (that is n copies of $\mathbb{Z}/2\mathbb{Z}$). Futhermore, $K_i = K(r_{1,i}, r_{2,i}, r_{\mu,i})$ (i.e., adjoining $\overline{r_{\mu,i}}$ is redundant).*

Proof. We show that if the polynomials (3.1.2) to (3.1.5) are irreducible then $\mathrm{Gal}(K_i/K) \cong C_2^5$. If any of the polynomials are not irreducible, then $\mathrm{Gal}(K_i/K)$ is a subgroup of C_2^5, and hence must be C_2^n for some $0 \leq n \leq 5$.

To do so we use the resolvent method. Consider the group

$$\langle (12)(34)(56)(78), (13)(24)(57)(68), (15)(37)(26)(48) \rangle \leq S_8.$$

Note that this is the group of translations of a fundamental cube inside $(\mathbb{Z}/2\mathbb{Z})^3$ (label the eight vertices of the cube 1 to 8), hence it is C_2^3. Let $x_1, \dots, x_8$ be indeterminate variables, then S_8 acts on them by $x_i \mapsto x_{\sigma(i)}$. Note that the monomial $x_1x_3 + x_2x_4 + x_5x_7 + x_6x_8$ is C_2^3-invariant, so we can construct the resolvent polynomial $R_{C_2^3} = \prod_{j=1}^{g}(X - P_j)$ where P_j are the elements in the S_8-orbit of $x_1x_3 + x_2x_4 + x_5x_7 + x_6x_8$.

We first consider the Galois group of $K(r_{1,i})$ over K, and call it G. As the polynomial (3.1.2) has as roots the eight different z coordinates of the 16 singular points, we have that the minimal polynomial of $r_{1,i}$ factorises as

$$(x - r_{1,i})(x + r_{1,i})\left(x - \frac{1}{r_{1,i}}\right)\left(x + \frac{1}{r_{1,i}}\right)(x - \overline{r}_{1,i})(x + \overline{r}_{1,i})\left(x - \frac{1}{\overline{r}_{1,i}}\right)\left(x + \frac{1}{\overline{r}_{1,i}}\right),$$

where, as above, $\overline{r}_{1,i} = \frac{r_{2,i}}{r_{3,i}}$. If we substitute the x_j with the jth root of the minimal polynomial of $r_{1,i}$ (as ordered above), we find that

$$x_1x_3 + x_2x_4 + x_5x_7 + x_6x_8 \quad = \quad 4.$$

Hence in this case $R_{C_2^3}$ has a K-rational non-repeated root, so $G \subseteq C_2^3$. But since the minimal polynomial of $r_{1,i}$ is already of degree 8, we must have $G \cong C_2^3$. In fact G is generated by $r_{1,i} \mapsto -r_{1,i}$, $r_{1,i} \mapsto \frac{1}{r_{1,i}}$ and $r_{1,i} \mapsto \overline{r}_{1,i}$; denote them by σ_2, σ_3 and σ_4 respectively.

Next, we consider the Galois group of $K(r_{1,i}, r_{2,i})$ over K. We have that the minimal polynomial of $r_{2,i}$ is of degree 8 (either through direct calculation, see [Bou], or the fact that $r_{2,i}$ solves a quadratic in $r_{1,i}^2$ which itself solves a quartic). We can find all the conjugates of $r_{2,i}$, by noting that if we let $\sigma_2, \sigma_3, \sigma_4$ act on the polynomial (3.1.3), we get with $\pm r_{2,i}$ a total of eight conjugates. Furthermore, we know that $\pm r_{2,i}$ corresponds to the y-coordinate of the singular points which have z-coordinate $r_{1,i}$. Similarly, $\sigma_j(\pm r_{2,i})$ corresponds to the y-coordinate of the singular points which have

z-coordinate $\sigma_j(r_{1,i})$. Hence we know that the minimal polynomial of $r_{2,i}$ factorises as

$$(x-r_{2,i})(x+r_{2,i})\left(x-\frac{1}{r_{2,i}}\right)\left(x+\frac{1}{r_{2,i}}\right)(x-\overline{r}_{2,i})(x+\overline{r}_{2,i})\left(x-\frac{1}{\overline{r}_{2,i}}\right)\left(x+\frac{1}{\overline{r}_{2,i}}\right),$$

where $\overline{r}_{2,i} = \frac{r_{1,i}}{\overline{r}_{1,i}} r_{2,i}$. As above, we can see that the Galois group of $K(r_{2,i})$ over K is C_2^3, and in particular, we now know that the field extension $K(r_{1,i}, r_{2,i})/K$ is Galois. After having made some choice of sign on $\sigma_j(r_{2,i})$ for $2 \leq j \leq 4$, it is not hard to see that in fact $\mathrm{Gal}(K(r_{1,i}, r_{2,i})/K) \cong C_2^4$ generated by $\sigma_1, \sigma_2, \sigma_3$ and σ_4, where $\sigma_1(r_{1,i}) = r_{1,i}$ and $\sigma_1(r_{2,i}) = -r_{2,i}$.

Finally we look at $K(r_{\mu,i}, \overline{r}_{\mu,i})$, and first note that $r_{\mu,i}$ and $\overline{r}_{\mu,i}$ have the same minimal polynomial over K. In fact, we have that the minimal polynomial of $r_{\mu,i}$ factorises as

$$(x-r_{\mu,i})(x+r_{\mu,i})\left(x-\frac{r_{\mu,i}}{r_{1,i}^2}\right)\left(x+\frac{r_{\mu,i}}{r_{1,i}^2}\right)(x-\overline{r}_{\mu,i})(x+\overline{r}_{\mu,i})\left(x-\frac{\overline{r}_{\mu,i}}{\overline{r}_{1,i}^2}\right)\left(x+\frac{\overline{r}_{\mu,i}}{\overline{r}_{1,i}^2}\right).$$

In this case, if we substitute the x_j with the jth root of the minimal polynomial of $r_{\mu,i}$ (as ordered above), we find that

$$\begin{aligned} x_1x_3 + x_2x_4 + x_5x_7 + x_6x_8 &= 2\left(\frac{r_{\mu,i}^2}{r_{1,i}^2} + \frac{\overline{r}_{\mu,i}^2}{\overline{r}_{1,i}^2}\right) \\ &= -\frac{2}{\Delta}\left(2C_1 + A_1\left(r_{1,i}^2 + \frac{1}{r_{1,i}^2} + \overline{r}_{1,i}^2 + \frac{1}{\overline{r}_{1,i}^2}\right)\right). \end{aligned}$$

Since $r_{1,i}^2$ solves a quartic polynomial whose other roots are $\frac{1}{r_{1,i}^2}, \overline{r}_{1,i}^2, \frac{1}{\overline{r}_{1,i}^2}$, we have that the above expression is in K. So $R_{C_2^3}$ has a K-rational non-repeated root, hence $\mathrm{Gal}(K(r_{\mu,i}, \overline{r}_{\mu,i})/K) \subseteq C_2^3$. But since the minimal polynomial of $r_{\mu,i}$ is already of degree 8, we must have $\mathrm{Gal}(K(r_{\mu,i}, \overline{r}_{\mu,i})/K) \cong C_2^3$, in particular $K(r_{\mu,i}, \overline{r}_{\mu,i}) \cong K(r_{\mu,i})$.

Hence we have $[K_i : K] = 2 \cdot 2 \cdot 8 = 32$, so we are looking for a group of order 32, which has $C_2^4 = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$ as a subgroup (after having extended σ_j properly on K_i by making some choice of the sign of $\sigma_j(r_{\mu,i})$). Let σ_5 be the element fixing $r_{1,i}, r_{2,i}$ and sending $r_{\mu,i} \mapsto \frac{1}{r_{\mu,i}}$, and note it has order 2 but is not in the subgroup $\langle \sigma_1, \sigma_2, \sigma_3, \sigma_4 \rangle$. Furthermore, one can check that σ_5 commutes with σ_j for $1 \leq j \leq 4$. Hence we have that $\mathrm{Gal}(K_i/K) \cong C_2^5$. □

The following lemma allows us to find another way of expressing K_i, which will help us find L. While this lemma is quite standard, the proof has been included as it details how we can construct a field isomorphic to K_i.

Lemma 3.1.3. *If $\mathrm{Gal}(L/K) \cong C_2^n$ for some n, then there exist $\Delta_1, \ldots, \Delta_n \in K$ whose images are linearly independent in the $\mathbb{F}_2$-vector space $K^*/(K^*)^2$, such that $L \cong K\left(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_n}\right)$.*

Proof. Let $\mathrm{Gal}(L/K) = \left\langle \sigma_1, \ldots, \sigma_n | \sigma_i^2 = (\sigma_i\sigma_j)^2 = 1 \right\rangle \cong C_2^n$ and for each $i \in \{1, \ldots, n\}$ let

$$\widetilde{\sigma_i} = \langle \sigma_1, \ldots, \sigma_{i-1}, \sigma_{i+1}, \ldots, \sigma_n \rangle \cong C_2^{n-1}.$$

For each i, we have $[\mathrm{Gal}(L/K) : \widetilde{\sigma_i}] = 2$, so $L^{\widetilde{\sigma_i}}$ the fixed field of $\widetilde{\sigma_i}$, is a degree 2 extension of K. Hence $L^{\widetilde{\sigma_i}} = K(\sqrt{\Delta_i})$ for some square free $\Delta_i \in K$.

We prove that $[K(\sqrt{\Delta_i})(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_{i-1}}) : K(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_{i-1}})] = 2$ by showing that $\sqrt{\Delta_i} \notin K(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_{i-1}})$. For a contradiction suppose that $\sqrt{\Delta_i} \in K(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_{i-1}})$, then by considering minimal polynomials, we can show that $\sqrt{\Delta_i} = \alpha\sqrt{\Delta_{i_1} \ldots \Delta_{i_s}}$ for some $\alpha \in K$ and subset $\{i_1, \ldots, i_s\}$ of $\{1, \ldots, i-1\}$, i.e., Δ_i is not linearly independent of $\Delta_1, \ldots, \Delta_{i-1}$ in $K^*/(K^*)^2$. Hence $K(\sqrt{\Delta_i}) \cong K(\sqrt{\Delta_{i_1} \ldots \Delta_{i_s}})$ and $\sigma_{i_1} \in \widetilde{\sigma_i}$ fixes $\sqrt{\Delta_{i_1} \ldots \Delta_{i_s}}$. But since $\sigma_{i_1} \in \widetilde{\sigma_j}$ for $j \in \{i_2, \ldots, i_s\}$, we also have that σ_{i_1} fixes $\sqrt{\Delta_j}$. So

$$\begin{aligned}
\sqrt{\Delta_{i_1} \ldots \Delta_{i_s}} &= \sigma_{i_1}(\sqrt{\Delta_{i_1} \ldots \Delta_{i_s}}) \\
&= \sigma_{i_1}(\sqrt{\Delta_{i_1}})\sqrt{\Delta_{i_2} \ldots \Delta_{i_s}},
\end{aligned}$$

hence σ_{i_1} fixes $\sqrt{\Delta_{i_1}}$. This is a contradiction, since then $K(\sqrt{\Delta_{i_1}})$ is the fixed field of $\widetilde{\sigma}_{i_1} \times \langle \sigma_{i_1} \rangle = \mathrm{Gal}(L/K)$.

As $L^{\widetilde{\sigma_i}} \subset L$, we have that $\sqrt{\Delta_i} \in L$. So $K(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_n}) \subset L$, but by the previous paragraph and the Tower Law, we also have $[K(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_n}) : K] = 2^n$. Hence $L \cong K(\sqrt{\Delta_1}, \ldots, \sqrt{\Delta_n})$. □

This means that for each of the fields K_i we can find an isomorphic field of the form $K(\sqrt{\Delta_{1,i}}, \ldots, \sqrt{\Delta_{5,i}})$. Then L, which is the compositium of the K_i, will be $K(\sqrt{\Delta_{1,1}}, \ldots, \sqrt{\Delta_{5,10}})$. Hence, our next step is to determine $\Delta_{i,j} \in K$. First, we will look at K_1.

Proposition 3.1.4. *Let $p = [A, B, C, D, E] \in \mathbb{P}^4$ be a very general point not lying on the Segre cubic nor on the 15 singular hyperplanes, then the 32 conics lying on X_p associated to the point q_1 are defined over the field*

$$K_1 \cong K\left(\sqrt{\Delta q_{+C} p_{-0} p_{+1}}, \sqrt{\Delta q_{+C} p_{+0} p_{-1}}, \sqrt{\Delta q_{+D} p_{+0} p_{-2}}, \sqrt{\Delta q_{+D} p_{-0} p_{+2}}, \sqrt{-\Delta q_{-E} p_{+1} p_{-2}}\right).$$

Proof. We use Lemma 3.1.3 to construct K_1. From Proposition 3.1.2, $\mathrm{Gal}(K_1/K) = \langle \sigma_1, \sigma_2, \sigma_3, \sigma_4, \sigma_5 \rangle$ where σ_j acts on $r_{1,1}, r_{2,1}, r_{\mu,1}$ according to the following table

	σ_1	σ_2	σ_3	σ_4	σ_5
$r_{1,1}$	$-r_{1,1}$	$\frac{1}{r_{1,1}}$	$\overline{r}_{1,1}$	$r_{1,1}$	$r_{1,1}$
$r_{2,1}$	$r_{2,1}$	$\overline{r}_{2,1}$	$\frac{1}{r_{2,1}}$	$-r_{2,1}$	$r_{2,1}$
$r_{\mu,1}$	$r_{\mu,1}$	$\frac{r_{\mu,1}}{r_{1,1}^2}$	$\overline{r}_{\mu,1}$	$r_{\mu,1}$	$-r_{\mu,1}$

(with $\overline{r}_{2,1} = \frac{r_{1,1}}{\overline{r}_{1,1}} r_{2,1}$). We calculate the fixed field of $\widetilde{\sigma_1} = \langle \sigma_2, \ldots, \sigma_5 \rangle$ by considering the expression $r_{1,1}+\frac{1}{r_{1,1}}+\overline{r}_{1,1}+\frac{1}{\overline{r}_{1,1}}$ which is fixed under σ_j for $j \in \{2,\dots,5\}$ but not under σ_1. Hence, upon calculating the discriminant of the minimal polynomial (after checking it is quadratic) of such an expression, we have that the fixed field of $\widetilde{\sigma}_1$ is $K\left(\sqrt{p_{-0}\,p_{+1}\,p_{+2}\,p_{-2}\,q_{+D}\,(-q_{-E})}\right)$ (where $p_{\pm i}, q_{\pm\alpha}$ are the equations in Proposition 2.3.1). Similarly we can use the following expressions to calculate the respective fixed fields:

- $r_{1,1}^2+\overline{r}_{1,1}^2$ for $\widetilde{\sigma}_2$ giving $K\left(\sqrt{p_{+1}\,p_{-1}\,p_{+2}\,p_{-2}}\right)$,
- $r_{1,1}^2+\frac{1}{r_{1,1}^2}$ for $\widetilde{\sigma}_3$ giving $K(\sqrt{p_{+0}\,p_{-0}\,p_{+1}\,p_{-1}})$,
- $r_{2,1}+\frac{1}{r_{2,1}}+\overline{r}_{2,1}+\frac{1}{\overline{r}_{2,1}}$ for $\widetilde{\sigma}_4$ giving $K\left(\sqrt{p_{+0}\,p_{+1}\,p_{-1}\,p_{-2}\,q_{+C}\,(-q_{-E})}\right)$,
- $r_{\mu,1}+\frac{r_{\mu,1}}{r_{1,1}^2}+\overline{r}_{\mu,1}+\frac{\overline{r}_{\mu,1}}{\overline{r}_{1,1}^2}$ for $\widetilde{\sigma}_5$ giving $K\left(\sqrt{\Delta\,p_{+0}\,p_{-0}\,q_{+C}\,q_{+D}\,(-q_{-E})}\right)$.

Putting all of this together and rearranging, we get the required result. $\square$

Theorem 3.1.5. *Let $p=[A,B,C,D,E]\in\mathbb{P}^4_K$ be a very general point not lying on the Segre cubic nor on the 15 singular hyperplanes and let L be the field where the 320 conics of X_p are defined. Then $\mathrm{Gal}(L/K)\cong C_2^{10}$.*

Proof. The first step is to calculate K_i for $i\in\{2,\dots,10\}$ in terms of square roots of elements in K. This is done by doing the same calculations as the above proposition with different q_i (and hence $r_{1,i}$, $r_{2,i}$, $r_{\mu,i}$). We find, up to rearrangements,

$$K_2\cong K\left(\sqrt{\Delta q_{+C}p_{-0}p_{+1}},\sqrt{\Delta q_{+C}p_{+0}p_{-1}},\sqrt{\Delta q_{+E}p_{+0}p_{-3}},\sqrt{\Delta q_{+E}p_{-0}p_{+3}},\sqrt{-\Delta q_{-D}p_{+1}p_{-3}}\right),$$

$$K_3\cong K\left(\sqrt{\Delta q_{+D}p_{-0}p_{+2}},\sqrt{\Delta q_{+D}p_{+0}p_{-2}},\sqrt{\Delta q_{+E}p_{+0}p_{-3}},\sqrt{\Delta q_{+E}p_{-0}p_{+3}},\sqrt{-\Delta q_{-C}p_{+2}p_{-3}}\right),$$

$$K_4\cong K\left(\sqrt{-\Delta q_{-D}p_{+1}p_{-3}},\sqrt{-\Delta q_{-D}p_{-1}p_{+3}},\sqrt{-\Delta q_{-E}p_{-1}p_{+2}},\sqrt{-\Delta q_{-E}p_{+1}p_{-2}},\sqrt{-\Delta q_{-C}p_{+2}p_{-3}}\right),$$

$$K_5\cong K\left(\sqrt{-\Delta A q_{+E}q_{-E}},\sqrt{-\Delta A q_{+D}q_{-D}},\sqrt{\Delta q_{+D}p_{+0}p_{-2}},\sqrt{\Delta q_{+E}p_{+0}p_{-3}},\sqrt{-\Delta q_{-E}p_{+1}p_{-2}}\right),$$

$$K_6\cong K\left(\sqrt{-\Delta A q_{+E}q_{-E}},\sqrt{-\Delta A q_{+D}q_{-D}},\sqrt{\Delta q_{+D}p_{-0}p_{+2}},\sqrt{\Delta q_{+E}p_{-0}p_{+3}},\sqrt{-\Delta q_{-E}p_{-1}p_{+2}}\right),$$

$$K_7\cong K\left(\sqrt{-\Delta A q_{+C}q_{-C}},\sqrt{-\Delta A q_{+E}q_{-E}},\sqrt{\Delta q_{+C}p_{+0}p_{-1}},\sqrt{\Delta q_{+E}p_{+0}p_{-3}},\sqrt{-\Delta q_{-E}p_{+1}p_{-2}}\right),$$

$$K_8\cong K\left(\sqrt{-\Delta A q_{+C}q_{-C}},\sqrt{-\Delta A q_{+E}q_{-E}},\sqrt{\Delta q_{+C}p_{-0}p_{+1}},\sqrt{\Delta q_{+E}p_{-0}p_{+3}},\sqrt{-\Delta q_{-E}p_{-1}p_{+2}}\right),$$

$$K_9\cong K\left(\sqrt{-\Delta A q_{+C}q_{-C}},\sqrt{-\Delta A q_{+D}q_{-D}},\sqrt{\Delta q_{+C}p_{+0}p_{-1}},\sqrt{\Delta q_{+D}p_{+0}p_{-2}},\sqrt{-\Delta q_{-D}p_{+1}p_{-3}}\right),$$

$$K_{10}\cong K\left(\sqrt{-\Delta A q_{+C}q_{-C}},\sqrt{-\Delta A q_{+D}q_{-D}},\sqrt{\Delta q_{+C}p_{-0}p_{+1}},\sqrt{\Delta q_{+D}p_{-0}p_{+2}},\sqrt{-\Delta q_{-D}p_{-1}p_{+3}}\right).$$

Then as the 320 conics of X_p are defined over the compositum of $K_1,\dots,K_{10}$, we see that L is the field extension

$$K\Bigg(\sqrt{-\Delta A q_{+C}q_{-C}},\sqrt{-\Delta A q_{+D}q_{-D}},\sqrt{-\Delta A q_{+E}q_{-E}},\sqrt{\Delta q_{+C}p_{+0}p_{-1}},\sqrt{\Delta q_{+C}p_{-0}p_{+1}},$$

$\sqrt{\Delta q_{+D} p_{+0} p_{-2}}, \sqrt{\Delta q_{+D} p_{-0} p_{+2}}, \sqrt{\Delta q_{+E} p_{+0} p_{-3}}, \sqrt{\Delta q_{+E} p_{-0} p_{+3}}, \sqrt{-\Delta q_{-C} p_{+2} p_{-3}}\Big).$

Let $\Omega \subseteq \mathbb{P}^4_K$ be the set of points where the image of the above ten expressions $-\Delta A q_{+C} q_{-C}, \ldots, -\Delta q_{-C} p_{+2} p_{-3}$ are linearly independent in the $\mathbb{F}_2$-vector space $K^*/(K^*)^2$. Then Ω is a thin set (it is a union of $2^{10}-1$ type 2 thin sets). As K is Hilbertian (it is a number field) the complement of Ω is non empty, so for a very general point $\mathrm{Gal}(L/K) \cong C_2^{10}$. □

Example. In the case $K = \mathbb{Q}$ and $[A, B, C, D, E] = [3, 3, 0, 3, 4]$ or $[1, 1, 0, 4, 8]$, we indeed have $\mathrm{Gal}(L/K) \cong C_2^{10}$.

Remark. While Magma [BCP97] could, in most cases, calculate the minimal polynomial of the expressions needed for Theorem 3.1.5 ($r_{1,i}^2 + \overline{r}_{1,i}^2$, $r_{1,i}^2 + \frac{1}{r_{1,i}^2}$ etc, see [Bou]) there were some difficult cases. Explicitly, for $i = 5, \ldots, 10$ Magma could not calculate the minimal polynomial of the expression $r_{\mu,i} + \frac{r_{\mu,i}}{r_{1,i}^2} + \overline{r}_{\mu,i} + \frac{\overline{r}_{\mu,i}}{\overline{r}_{1,i}^2}$. We dealt with this by considering

$$\left(r_{\mu,i} + \frac{r_{\mu,i}}{r_{1,i}^2} + \overline{r}_{\mu,i} + \frac{\overline{r}_{\mu,i}}{\overline{r}_{1,i}^2}\right)^2 =$$

$$\left(r_{\mu,i}^2 + \frac{r_{\mu,i}^2}{r_{1,i}^4} + \overline{r}_{\mu,i}^2 + \frac{\overline{r}_{\mu,i}^2}{\overline{r}_{1,i}^4} + 2\left(\frac{r_{\mu,i}^2}{r_{1,i}^2} + \frac{\overline{r}_{\mu,i}^2}{\overline{r}_{1,i}^2}\right) + 2r_{\mu,i}\overline{r}_{\mu,i}\left(1 + \frac{1}{r_{1,i}^2}\right)\left(1 + \frac{1}{\overline{r}_{1,i}^2}\right)\right).$$

We already know that

$$\left(\frac{r_{\mu,i}^2}{r_{1,i}^2} + \frac{\overline{r}_{\mu,i}^2}{\overline{r}_{1,i}^2}\right) \in K,$$

but we also have that

$$r_{\mu,i}^2 + \frac{r_{\mu,i}^2}{r_{1,i}^4} + \overline{r}_{\mu,i}^2 + \frac{\overline{r}_{\mu,i}^2}{\overline{r}_{1,i}^4}$$

$$= -\frac{1}{\Delta}\left(A_i\left(r_{1,i}^4 + 2 + \frac{1}{r_{1,i}^4} + \overline{r}_{1,i}^4 + 2 + \frac{1}{\overline{r}_{1,i}^4}\right) + C_i\left(r_{1,i}^2 + \frac{1}{r_{1,i}^2} + \overline{r}_{1,i}^2 + \frac{1}{\overline{r}_{1,i}^2}\right)\right)$$

$$= -\frac{1}{\Delta}\left(A_i\left(\left(r_{1,i}^2 + \frac{1}{r_{1,i}^2}\right)^2 + \left(\overline{r}_{1,i}^2 + \frac{1}{\overline{r}_{1,i}^2}\right)^2\right) + C_i\left(r_{1,i}^2 + \frac{1}{r_{1,i}^2} + \overline{r}_{1,i}^2 + \frac{1}{\overline{r}_{1,i}^2}\right)\right) \in K.$$

So, instead we ask Magma to calculate the minimal polynomial of

$$2r_{\mu,i}\overline{r}_{\mu,i}\left(1 + \frac{1}{r_{1,i}^2}\right)\left(1 + \frac{1}{\overline{r}_{1,i}^2}\right).$$

which turned out to be linear for $i = 5, \ldots, 10$, and hence in K. Therefore we explicitly calculate

$$\left(r_{\mu,i} + \frac{r_{\mu,i}}{r_{1,i}^2} + \overline{r}_{\mu,i} + \frac{\overline{r}_{\mu,i}}{\overline{r}_{1,i}^2}\right)^2 \in K$$

ourselves, and hence could implement the method of Proposition 3.1.4 and finish the proof of Theorem 3.1.5.

Remark. Out of interest, we list the 50 discriminants we got that way in Table A.2 in the Appendix.

3.2 Monodromy Group

We remark that the field of definition of the 320 conics involves the same equations which characterises when a surface X in $\mathcal{X}$ is singular (c.f. Theorem 3.1.5 and Proposition 2.3.1). We explain this by studying the Monodromy group of the conics over a general non-singular K3 surface in $\mathcal{X}$. First we briefly recall what this Monodromy group is.

Let Z be an algebraic variety with $\pi : Z \to X$ a surjective finite map of degree $d > 0$. Let $p \in X$ be a general point of X and $\pi^{-1}(p) = \{q_0, \ldots, q_{d-1}\}$ be its fibre. Let $U \subset X$ be a suitable small open subset of X and set $V = \pi^{-1}(U)$. Then for any loop $\lambda : [0,1] \to U$ based at p, and any point $q_i \in \pi^{-1}(p)$, there exists a unique path $\widetilde{\lambda}_i$ in V such that $\pi\left(\widetilde{\lambda}_i\right) = \lambda$ and $\widetilde{\lambda}_i(0) = q_i$. So we may define a permutation σ_λ of $\pi^{-1}(p)$ by sending each point q_i to $\widetilde{\lambda}_i(1) = q_j$ (for some j). Since σ_λ only depends on the homotopy class of λ, we have a homomorphism $\pi_1(U,p) \to S_d$. The image of this homomorphism is called the *Monodromy group* of the map π. Note that where π is smooth $\pi(U)$ is d disjoint subset of Z. Therefore any loop λ in U must lift to a loop $\widetilde{\lambda}$ contained in one component of $\pi^{-1}(U)$, and hence have $\widetilde{\lambda}(0) = \widetilde{\lambda}(1)$. Therefore, to study the Monodromy group of π, one needs to look at where π is not smooth.

In our case the variety Z parametrises the quartic surfaces of $\mathcal{X}$ with the 320 conics on them and the map $\pi : Z \to \mathbb{P}^4_{[A,B,C,D,E]}$ is the natural projection. We want to study the Monodromy group of π, that is, we want to look at the permutations of the conics as we draw loops on $\mathbb{P}^4$. First we study a simpler problem, namely we will look at the Monodromy group of the 16 planes associated to the point $q_1 = [1, 0, -2, -2, 2] \in \mathbb{P}^4$. We shall denote this set of 16 planes by T_1 (the planes are the tropes described in the proof of Theorem 2.3.8).

Lemma 3.2.1. *Let $p = [A,B,C,D,E] \in \mathbb{P}^4$ be a general point not lying on the Segre cubic nor on the* 15 *singular hyperplanes. The set T_1 is $\{\gamma(r_{0,1}x+r_{1,1}y+r_{2,1}z+r_{3,1}w) = 0 | \gamma \in \Gamma\}$ where*

- $r_{0,1} = 2^3 B\sqrt{-q_{+D}q_{+C}q_{-E}}$,
- $r_{1,1} = \sqrt{q_{+C}}\left(\sqrt{p_{-2}\,p_{-0}\,p_{+2}\,p_{+1}} + p_{+2}\sqrt{p_{+1}\,p_{+0}} + \sqrt{p_{-2}\,p_{-1}\,p_{+2}\,p_{+0}} + p_{-2}\sqrt{p_{-1}\,p_{-0}}\right)$,
- $r_{2,1} = \sqrt{q_{+D}}\left(\sqrt{p_{-1}\,p_{-0}\,p_{+1}\,p_{+2}} + p_{+1}\sqrt{p_{+2}\,p_{+0}} + \sqrt{p_{-1}\,p_{-2}\,p_{+1}\,p_{+0}} + p_{-1}\sqrt{p_{-2}\,p_{-0}}\right)$,
- $r_{3,1} = -\sqrt{-q_{-E}}\left(\sqrt{p_{-0}\,p_{-1}\,p_{+0}\,p_{+2}} + p_{+0}\sqrt{p_{+2}\,p_{+1}} + \sqrt{p_{-0}\,p_{-2}\,p_{+0}\,p_{+1}} + p_{-0}\sqrt{p_{-2}\,p_{-1}}\right)$,

where $p_{\pm i}, q_{\pm\alpha}$ are the equations in Proposition 2.3.1.

Proof. Theorem 3.1.1 already gives an expression for the planes, but we use the fact that $K(r_{1,1}, r_{2,1}, r_{\mu,1})$ is isomorphic to

$$K\left(\sqrt{\Delta q_{+C}p_{-0}p_{+1}}, \sqrt{\Delta q_{+C}p_{+0}p_{-1}}, \sqrt{\Delta q_{+D}p_{+0}p_{-2}}, \sqrt{\Delta q_{+D}p_{-0}p_{+2}}, \sqrt{-\Delta q_{-E}p_{+1}p_{-2}}\right)$$

and rewrite the singular point $[r_{3,1}, r_{2,1}, r_{1,1}, 1]$ in terms of linear combinations of square roots.

That is, let r be any of the coordinates, we know that r solves a degree 8 polynomial whose terms are all even, and $K(r) \cong K(\sqrt{\Delta_1}, \sqrt{\Delta_2}, \sqrt{\Delta_3})$ for some Δ_i's. So let $r = a_0 + a_1\sqrt{\Delta_1} + a_2\sqrt{\Delta_2} + \cdots + a_7\sqrt{\Delta_1\Delta_2\Delta_3}$. The Galois group of $K(\sqrt{\Delta_1}, \sqrt{\Delta_2}, \sqrt{\Delta_3})$ is naturally generated by $\sigma_1, \sigma_2, \sigma_3$ where $\sigma_i(\sqrt{\Delta_j}) = \begin{cases} -\sqrt{\Delta_j} & i = j \\ \sqrt{\Delta_j} & j \neq i \end{cases}$. On one hand we know that the minimal polynomial of r factorises as

$$(x-r)(x+r)\left(x-\frac{1}{r}\right)\left(x+\frac{1}{r}\right)(x-\overline{r})(x+\overline{r})\left(x-\frac{1}{\overline{r}}\right)\left(x+\frac{1}{\overline{r}}\right),$$

and on the other hand, it factorises as

$$\prod_{\sigma\in\mathrm{Gal}(K(r)/K)} \left(x - \sigma\left(a_0 + a_1\sqrt{\Delta_1} + \cdots + a_7\sqrt{\Delta_1\Delta_2\Delta_3}\right)\right).$$

Now comparing the x^7 coefficient of both factorisation we deduce $0 = r - r + \cdots + \frac{1}{\overline{r}} - \frac{1}{\overline{r}} = 8a_0$, hence $a_0 = 0$. Without loss of generality, suppose that $\sigma_1(r) = -r$, then $0 = r - r = a_2\sqrt{\Delta_2} + a_3\sqrt{\Delta_3} + a_6\sqrt{\Delta_2\Delta_3}$, so

$$r = \sqrt{\Delta_1}(a_1 + a_4\sqrt{\Delta_2} + a_5\sqrt{\Delta_3} + a_7\sqrt{\Delta_2\Delta_3}).$$

If we suppose $\sigma_2(r) = \frac{1}{r}$ and $\sigma_3(r) = \overline{r}$, then

$$\begin{aligned} r + \frac{1}{r} + \overline{r} + \frac{1}{\overline{r}} &= 4a_1\sqrt{\Delta_1}, \\ r - \frac{1}{r} + \overline{r} - \frac{1}{\overline{r}} &= 4a_4\sqrt{\Delta_1\Delta_2}, \\ r + \frac{1}{r} - \overline{r} - \frac{1}{\overline{r}} &= 4a_5\sqrt{\Delta_1\Delta_3}, \\ r - \frac{1}{r} - \overline{r} + \frac{1}{\overline{r}} &= 4a_7\sqrt{\Delta_1\Delta_2\Delta_3}. \end{aligned}$$

Hence we can easily work out a_1, a_4, a_5, a_7, and therefore r, by calculating the minimal polynomial of the above four expressions. We apply that theory to $r_{3,1}$, $r_{2,1}$ and $r_{1,1}$ in turn, and after some rearrangement we get the required result. $\square$

Remark. Similarly, for every $i \in \{1, \ldots, 10\}$, we can work out T_i, the set of the 16 planes associated to the point q_i. The list of the 10 sets T_i can be found in Appendix

A.3. Furthermore, the implementation of these calculations can be found [Bou].

Hence, to study the Monodromy group of the set T_1 on a K3 surface, we need the object $\mathcal{Z}$ defined by

$$\left\{([A,B,C,D,E],[a,b,c,d]) \;\middle|\; [a,b,c,d] \in \{\gamma\left([r_{0,1},r_{1,1},r_{2,1},r_{3,1}]\right) : \gamma \in \Gamma\}\right\} \subset \mathbb{P}^4 \times \mathbb{P}^3.$$

Note that we set up $\mathbb{P}^3_{[a,b,c,d]}$ to be the dual of $\mathbb{P}^3_{[x,y,z,w]}$, that is a point $[a,b,c,d] \in \mathbb{P}^3_{[a,b,c,d]}$ represents the plane $ax+by+cz+dw=0$ in $\mathbb{P}^3_{[x,y,z,w]}$. Now by the above lemma, $r_{i,1}$ involves square roots and hence $\mathcal{Z}$ is not a variety. So instead of looking at the planes defined by the point $[A,B,C,D,E]$, we look at the points $[A,B,C,D,E]$ that can be defined by a given plane $ax+by+cz+dw=0$. Pick a point $[a,b,c,d] \in \mathbb{P}^3$ and, following Proposition 2.3.4, let

- $A_1 = (bc+ad)(bc-ad)(ac+bd)(ac-bd)(cd+ab)(cd-ab)$,
- $B_1 = 2abcd(-a^2-b^2+c^2+d^2)(-a^2+b^2+c^2-d^2)(a^2-b^2+c^2-d^2)(a^2+b^2+c^2+d^2)$,
- $C_1 = (bc+ad)(bc-ad)(ac+db)(ac-bd)(a^4+b^4-c^4-d^4)$,
- $D_1 = (bc+ad)(bc-ad)(cd+ab)(cd-ab)(-a^4+b^4-c^4+d^4)$,
- $E_1 = (ac+db)(ac-bd)(cd+ab)(cd-ab)(a^4-b^4-c^4+d^4)$.

The point $[A,B,C,D,E]$ defines a quartic surface whose intersection with the plane $ax+by+cz+dy=0$ is two conics if and only if $[A,B,C,D,E]$ lies on the line joining $[A_1,B_1,C_1,D_1,E_1]$ and $q_1 = [1,0,-2,-2,2]$. The line in $\mathbb{P}^4_{[A,B,C,D,E]}$ joining these two points is defined by the equations

- $g_1 := 2(E_1+D_1)A - (2A_1-E_1)D - (2A_1+D_1)E = 0$
- $g_2 := 2(C_1-D_1)A - (2A_1+D_1)C + (2A_1+C_1)D = 0$
- $g_3 := 2B_1A - (2A_1+D_1)B + B_1D = 0$

So consider the variety $\mathcal{V}_1$ defined by

$$\{g_1 = g_2 = g_3 = 0\} \subseteq \mathbb{P}^4_{[A,B,C,D,E]} \times \mathbb{P}^3_{[a,b,c,d]}.$$

The variety $\mathcal{V}_1$ has the following properties:

- over any point of the dual of $\mathcal{L}$, we have a copy of $\mathbb{P}^4_{[A,B,C,D,E]}$,
- over the conic $a^2-b^2-c^2+d^2$, we have a copy of $\mathbb{P}^4_{[A,B,C,D,E]}$,
- over the point q_1, we have a copy of $\mathbb{P}^3_{[a:b:c:d]}$,
- everywhere else, the variety $\mathcal{V}_1$ coincides with $\mathcal{Z}$.

We note that the dual of $\mathcal{L} \subset \mathbb{P}^3_{[x,y,z,w]}$ is itself $\mathcal{L} \subset \mathbb{P}^3_{[a,b,c,d]}$ (by making the correspondence $x \leftrightarrow a, \ldots, w \leftrightarrow d$), as each L_i is dual to $\overline{L}_i$. Similarly, we have that the dual of each quadric $Q_i \subset \mathbb{P}^3_{[x,y,z,w]}$ can be identified with the quadric itself $Q_i \subset \mathbb{P}^3_{[a,b,c,d]}$.

Lemma 3.2.2. *Let $\mathcal{V}_1 \subset \mathbb{P}^4_{[A,B,C,D,E]} \times \mathbb{P}^3_{[a,b,c,d]}$ be as above and π_1, π_2 the projective maps $\mathcal{V}_1 \mapsto \mathbb{P}^3_{[a,b,c,d]}$ and $\mathcal{V}_1 \mapsto \mathbb{P}^4_{[A,B,C,D,E]}$ respectively. Then the projective map $\pi_2 : \mathcal{V}_1 \to \mathbb{P}^4_{[A,B,C,D,E]}$ is smooth away from the points p such that $\pi_1(p)$ lies on 10 quadrics Q_i, or $\pi_2(p)$ is the point $q_1 = [1, 0, -2, -2, 2]$.*

Proof. Note that the union of the 15 pairs of lines $\mathcal{L}$ are contained in the 10 quadrics Q_i. Away from these 15 pairs of lines, once we have fixed $[a, b, c, d]$, we have that the point $[A, B, C, D, E]$ is of the form

$$[\mu A_1 + (1-\mu), \mu B_1, \mu C_1 - 2(1-\mu), \mu D_1 - 2(1-\mu), \mu E_1 + 2(1-\mu)]$$

for some $\mu \in K$. We want to show that the Jacobian matrix

$$\left(\left.\frac{\partial g_i}{\partial x_j}\right|_{A=\mu A_1+(1-\mu),B=\mu B_1,C=\mu C_1-2(1-\mu),D=\mu D_1-2(1-\mu),E=\mu E_1+2(1-\mu)} \right)_{i,j}$$

has rank 3. This is equivalent to showing that the determinant of at least one of the four matrices obtained from deleting a row in the Jacobian is non-zero. We calculate ([Bou]) that the four determinants are $8\mu^3 aF$, $-8\mu^3 bF$, $8\mu^3 cF$, and $-8\mu^3 dF$ where

$$\begin{aligned} F &= (bc-ad)^4(bc+ad)^4(ac-bd)^2(ac+bd)^2(ab-cd)^4(ab+cd)^4 \\ &\quad \cdot(a^2-b^2-c^2+d^2)^6(a^2-b^2+c^2-d^2)^2(a^2+b^2-c^2-d^2)^4(a^2+b^2+c^2+d^2)^2. \end{aligned}$$

Note that F is a product of the 10 quadratics defining Q_i, and hence cannot be 0. While if $\mu = 0$, then the surface $[A, B, C, D, E]$ is $[1, 0, -2, -2, 2]$. If $F \neq 0$ and $\mu \neq 0$, then one of the four determinants must be non-zero, hence the projection map is smooth at that place. □

So we only need to worry about points lying on one of the 10 quadrics Q_i. From our construction (i.e., the object $\mathcal{Z}$ and not the variety $\mathcal{V}_1$), any such point either lies on $\mathcal{L}$ or gives rise to one of the 10 singular points on S_3, and hence corresponds to a quartic K3 surface lying on one of the 15 singular hyperplanes. Therefore to study the Monodromy group of T_1, we only need to look at the 15 singular hyperplanes. We will study this on the level of the object $\mathcal{Z}$ and not the variety $\mathcal{V}_1$.

Proposition 3.2.3. *Given a non-singular K3 surface defined by the point $p = [A, B, C, D, E] \in \mathbb{P}^4$, we can find a loop based at p that:*

1. *goes around the singular hyperplane $\{q_{+C} = 0\}$ and avoids the 15 singular hyperplanes,*

2. changes the sign of $\sqrt{q_{+C}}$ in the equations defining $r_{0,1}, r_{1,1}, r_{2,1}$ and $r_{3,1}$.

This loop sends the plane

$$r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w = 0$$

to the plane

$$\gamma_3 \left(r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w \right) = 0$$

(where $\gamma_3 \in \Gamma < \Omega$ as defined in Section 2.3).

Proof. Pick a point $[A, B, C, D, E] \in \mathbb{P}^4$ which does not represent a singular K3 surface, and note that $C \in K \subset \mathbb{C}$ can be written uniquely as $-2A + re^{i\phi}$ for some $r \in \mathbb{R}_{>0}$ and $\phi \in [0, 2\pi)$. Define a loop $\widetilde{\lambda}(t) = [\widetilde{\lambda}_A(t), \widetilde{\lambda}_B(t), \widetilde{\lambda}_C(t), \widetilde{\lambda}_D(t), \widetilde{\lambda}_E(t)]$ $(0 \leq t \leq 3)$ as $\widetilde{\lambda}_j = j$ for all $t \in [0, 3]$, $j \in \{A, B, D, E\}$ and $\widetilde{\lambda}_C = -2A + f(t)$, with f composed of the following three segments:

$$f(t) = \begin{cases} (\rho t + r(1-t))e^{i\phi} & t \in [0,1] \\ \rho e^{i\phi + i(t-1)2\pi} & t \in [1,2] \\ (\rho(3-t) + r(t-2))e^{i\phi} & t \in [2,3] \end{cases}$$

and where $\rho \in \mathbb{R}_{>0}$ satisfies

$$\rho < \min \left\{ |B + 2D + 2E| , |-B + 2D + 2E| , |8A + B + 2D - 2E| , |8A - B + 2D - 2E| \right\}.$$

Now consider how the point $[r_{0,1}, r_{1,1}, r_{2,1}, r_{3,1}]$ (where we have fixed a root for each square roots) changes as we travel along this loop. As t changes, the 10 equations occurring in $r_{0,1}, r_{1,1}, r_{2,1}$ and $r_{3,1}$ are affected in the following ways:

- $q_{+C} = f(t)$,
- q_{+D}, q_{-E}, and B all stay the same,
- $p_{+0} = -p_{-1} = B + 2D + 2E + 2f(t)$,
- $p_{-0} = -p_{+1} = -B + 2D + 2E + 2f(t)$,
- $p_{+2} = 8A + B + 2D - 2E - 2f(t)$,
- $p_{-2} = 8A - B + 2D - 2E - 2f(t)$.

For ease of argument we assume that for $0 \leq t \leq 1$, none of $\{p_{+0}, p_{-0}, p_{+1}, p_{-1}, p_{+2}, p_{-2}\}$ are 0 (if they are, the argument can be changed by slightly curving the first segment instead of using a straight line). So for the first segment, we see that nothing remarkable happens. During the second segment, we have chosen ρ small enough so that none of $p_{+0}, p_{-0}, p_{+1}, p_{-1}, p_{+2}$ and p_{-2} are 0, but we see that $\sqrt{q_{+C}}$ is affected. Indeed, if we choose the square root of $e^{i\phi}$ to be $e^{\frac{i\phi}{2}}$, we see that $\sqrt{q_{+C}} = \sqrt{\rho}e^{\frac{i\phi}{2}+i(t-1)\pi}$. Hence at $t=1$, $\sqrt{q_{+C}}$ is the chosen root, but at $t=2$ the sign has changed. Note that the third segment is the same as the first segment but backwards.

Finally, one can see that by changing the sign of $\sqrt{q_{+C}}$, we have $r_{0,1} \mapsto -r_{0,1}$, $r_{1,1} \mapsto -r_{1,1}$, $r_{2,1} \mapsto r_{2,1}$ and $r_{3,1} \mapsto r_{3,1}$. Hence the plane $r_{0,1}x+r_{1,1}y+r_{2,1}z+r_{3,1}w = 0$ gets mapped to the plane $r_{0,1}x+r_{1,1}y-r_{2,1}z-r_{3,1}w = 0 = \gamma_3(r_{0,1}x+r_{1,1}y+r_{2,1}z+r_{3,1}w)$. □

A very similar argument works for the singular hyperplanes defined by q_{+D}, q_{-E}, p_{+0}, p_{-0}, p_{+1}, p_{-1}, p_{+2} and p_{-2}. For the singular hyperplanes defined by A, q_{-C}, q_{-D}, q_{+E}, p_{+3} and p_{-3}, we note that either $[r_{0,1}, r_{1,1}, r_{2,1}, r_{3,1}]$ are completely unaffected, or see by direct calculations that we still have 16 different planes when substituting in $A=0$, or $q_{-C}=0, \ldots$ etc.

Notation. Out of the 15 singular hyperplanes, the point q_1 lies on nine of them, namely those defined by q_{+C}, q_{+D}, q_{-E}, p_{+0}, p_{-0}, p_{+1}, p_{-1}, p_{+2} and p_{-2}. We shall denote the set of these nine singular hyperplanes by Σ_{q_1}.

Proposition 3.2.4. *The Monodromy group of the set T_1 is isomorphic to Γ and hence C_2^4.*

Proof. By the above discussion, the permutations of the 16 planes in T_1 come from changing the sign of the square roots $\sqrt{\Delta_i}$ for $\Delta_i \in \Sigma_{q_1}$. By direct calculation, letting Π be the plane $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w = 0$, we have:

- $\sqrt{q_{+C}} \mapsto -\sqrt{q_{+C}}$ corresponds to $\Pi \mapsto \gamma_3(\Pi)$,
- $\sqrt{q_{+D}} \mapsto -\sqrt{q_{+D}}$ corresponds to $\Pi \mapsto \gamma_4(\Pi)$,
- $\sqrt{-q_{-E}} \mapsto -\sqrt{-q_{-E}}$ corresponds to $\Pi \mapsto \gamma_3\gamma_4(\Pi)$,
- $\sqrt{p_{+0}} \mapsto -\sqrt{p_{+0}}$ corresponds to $\Pi \mapsto \gamma_1\gamma_2(\Pi)$,
- $\sqrt{p_{-0}} \mapsto -\sqrt{p_{-0}}$ corresponds to $\Pi \mapsto \gamma_1\gamma_2\gamma_3\gamma_4(\Pi)$,
- $\sqrt{p_{+1}} \mapsto -\sqrt{p_{+1}}$ corresponds to $\Pi \mapsto \gamma_2\gamma_3\gamma_4(\Pi)$,
- $\sqrt{p_{-1}} \mapsto -\sqrt{p_{-1}}$ corresponds to $\Pi \mapsto \gamma_2\gamma_3(\Pi)$,
- $\sqrt{p_{+2}} \mapsto -\sqrt{p_{+2}}$ corresponds to $\Pi \mapsto \gamma_1\gamma_3\gamma_4(\Pi)$,
- $\sqrt{p_{-2}} \mapsto -\sqrt{p_{-2}}$ corresponds to $\Pi \mapsto \gamma_1\gamma_4(\Pi)$.

Hence, we see that the Monodromy group of T_1 is isomorphic to Γ. □

Next we want to calculate the Monodromy group of the 160 planes which intersect a surface in $\mathcal{X}$ to give the 320 conics. First, for each $i \in [2, \ldots, 10]$, we want to calculate the Monodromy group of the 16 planes in T_i. To do so, we use the action induced by the group Ω acting on $\mathbb{P}^3 \times \mathbb{P}^4$ as described in Remark 2.3.3 with the Monodromy group of T_1. The idea is as follows: if we pick an element $\phi \in \Omega$ which permutes q_1 and q_i, then the Monodromy group of T_1 on $\phi(X)$ is the same as the Monodromy group of T_i on X.

In more detail, pick an element $\phi \in \Omega$ which permutes q_1 and q_i. Let H be a singular hyperplane, then $\phi(H)$ is also a singular hyperplane as Ω acts on the 15 singular hyperplanes. Using the isomorphism between the Mondromy group of T_1 and Γ, let $\gamma_{\phi(H)} \in \Gamma$ be the element associated to $\phi(H)$. Then the element of the Monodromy group associated to H is $\phi^{-1} \cdot \gamma_{\phi(H)} \cdot \phi \in \Gamma$, as Γ is normal in Ω.

Example. We work out explicitly some of the cases for the point q_2. We use the element ϕ_2 which permutes the points q_1 and q_2.

Since $\phi_2(A) = A$, the element corresponding to the hyperplane A in the Monodromy group of T_2 is $\phi_2^{-1}\gamma_A\phi_2 = \phi_2^{-1} \cdot \mathrm{id} \cdot \phi_2 = \mathrm{id}$.

Since $\phi_2(q_{+E}) = q_{+D}$, the element corresponding to the hyperplane q_{+E} in the Monodromy group of T_2 is $\phi_2^{-1}\gamma_{q_{+D}}\phi_2 = \phi_2^{-1} \cdot \gamma_4 \cdot \phi_2 = \gamma_3\gamma_4$.

We summarise the information in Table 3.1 below, the row headings are the equations defining the 15 singular hyperplanes and the column headings are the 10 set T_i. Each entry is an element $\gamma \in \Gamma$ and represents how changing the sign of the square root of that equation permutes the 16 planes in T_i (which we know can be represented as an element of Γ). An empty box stands for the identity element in Γ.

Lemma 3.2.2 why we need to look at the 15 singular hyperplanes for the Monodromy group of the 16 planes. We argue here that for the 160 planes, it is sufficient to look at the 15 singular hyperplanes. Let $\mathcal{V}_1 \subset \mathbb{P}^4_{[A,B,C,D,E]} \times \mathbb{P}^3_{[a,b,c,d]}$ be as before, and let $\mathcal{V}_i \subset \mathbb{P}^4_{[A,B,C,D,E]} \times \mathbb{P}^3_{[a_i,b_i,c_i,d_i]}$ be the corresponding variety for T_i, $i \in [2, \ldots, 10]$. For $1 \leq i \leq 10$, let $\pi_i : \mathcal{V}_i \to \mathbb{P}^4_{[A,B,C,D,E]}$ be the natural projections, which we know are smooth away from the 15 singular hyperplanes. Embed each $\mathcal{V}_i$ in $\mathbb{P}^4_{[A,B,C,D,E]} \times \mathbb{P}^3_{[a,b,c,d]} \times \mathbb{P}^3_{[a_2,b_2,c_2,d_2]} \times \cdots \times \mathbb{P}^3_{[a_{10},b_{10},c_{10},d_{10}]}$, and let $\mathcal{V} = \cap \mathcal{V}_i$ be the variety for the 160 planes. Considering $\pi : \mathcal{V} \to \mathbb{P}^4_{[A,B,C,D,E]}$, we see that the Jacobian matrix of π is $J_\pi = \oplus_{i=1}^{10} J_{\pi_i}$. Hence, we conclude that π is smooth away from the 15 singular hyperplane.

Theorem 3.2.5. *The Monodromy group of the* 160 *planes is* C_2^9.

Proof. We use the information given in Table 3.1. The Monodromy group is a subgroup of S_{160}. After embedding in S_{160} the elements associated to the 15 singular planes, we check ([Bou]) that they generate a subgroup of order 2^9. From the table, we see that the elements associated to the 15 singular planes commute with each other and have order 2, hence we know that all non-trivial element of the Monodromy group has order 2. Since the only group of order 2^9 with every non-trivial elements being involutions is C_2^9, the Monodromy group of the 160 planes is C_2^9. $\square$

	T_1	T_2	T_3	T_4	T_5	T_6	T_7	T_8	T_9	T_{10}
A					γ_3	γ_3	γ_4	γ_4	$\gamma_3\gamma_4$	$\gamma_3\gamma_4$
q_{+C}	γ_3	γ_3					$\gamma_1\gamma_3$	γ_1	$\gamma_1\gamma_3$	γ_1
$-q_{-C}$			γ_3	γ_3			$\gamma_1\gamma_4$	$\gamma_1\gamma_3\gamma_4$	$\gamma_1\gamma_3\gamma_4$	$\gamma_1\gamma_4$
q_{+D}	γ_4		γ_4		$\gamma_2\gamma_4$	γ_2			$\gamma_2\gamma_4$	γ_2
$-q_{-D}$		γ_4		γ_4	$\gamma_2\gamma_3$	$\gamma_2\gamma_3\gamma_4$			$\gamma_2\gamma_3\gamma_4$	$\gamma_2\gamma_3$
q_{+E}		$\gamma_3\gamma_4$	$\gamma_3\gamma_4$		$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2$	$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2$		
$-q_{-E}$	$\gamma_3\gamma_4$			$\gamma_3\gamma_4$	$\gamma_1\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_4$	$\gamma_1\gamma_2\gamma_4$	$\gamma_1\gamma_2\gamma_3$		
p_{+0}	$\gamma_1\gamma_2$	γ_2	γ_1		γ_1		γ_2		$\gamma_1\gamma_2$	
p_{-0}	$\gamma_1\gamma_2\gamma_3\gamma_4$	$\gamma_2\gamma_4$	$\gamma_1\gamma_3$			$\gamma_1\gamma_3$		$\gamma_2\gamma_4$		$\gamma_1\gamma_2\gamma_3\gamma_4$
p_{+1}	$\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2\gamma_4$		$\gamma_1\gamma_3$	$\gamma_1\gamma_3$			$\gamma_2\gamma_3\gamma_4$		$\gamma_1\gamma_2\gamma_4$
p_{-1}	$\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3$		γ_1		γ_1	$\gamma_2\gamma_3$		$\gamma_1\gamma_2\gamma_3$	
p_{+2}	$\gamma_1\gamma_3\gamma_4$		$\gamma_1\gamma_2\gamma_3$	$\gamma_2\gamma_4$		$\gamma_1\gamma_3\gamma_4$	$\gamma_2\gamma_4$			$\gamma_1\gamma_2\gamma_3$
p_{-2}	$\gamma_1\gamma_4$		$\gamma_1\gamma_2\gamma_4$	γ_2	$\gamma_1\gamma_4$			γ_2	$\gamma_1\gamma_2\gamma_4$	
p_{+3}		$\gamma_1\gamma_4$	$\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3\gamma_4$		$\gamma_1\gamma_4$		$\gamma_2\gamma_3$	$\gamma_1\gamma_2\gamma_3\gamma_4$	
p_{-3}		$\gamma_1\gamma_3\gamma_4$	$\gamma_2\gamma_3\gamma_4$	$\gamma_1\gamma_2$	$\gamma_1\gamma_3\gamma_4$		$\gamma_2\gamma_3\gamma_4$			$\gamma_1\gamma_2$

Table 3.1: The Monodromy group of the 160 planes

Definition 3.2.6. Each conic comes in a natural pair, i.e., each of the 160 plane intersecting the K3 surface gives two conics. We shall call two such conics *conjugates* of each other.

Recall from Section 3.1 that given the K3 surface X_p and the plane T: $r_{0,1}x+r_{1,1}y+r_{2,1}z+r_{3,1}w=0$ in T_1, the two conics in $T \cap X_p$ are $Q_1+\sqrt{\frac{\mu_1}{\Delta}}Q'$ and $Q_1-\sqrt{\frac{\mu_1}{\Delta}}Q'$. Let $r_{\mu,1}=\sqrt{\frac{\mu_1}{\Delta}}$, so that the conics are expressed as $Q_1 \pm r_{\mu,1}Q'$. Using the same method as in the proof of Lemma 3.2.1, we express $r_{\mu,1}$ explicitly in terms of A, B, C, D, E and find that:

$$r_{\mu,1} = \frac{\sqrt{-q_{+C}q_{+D}q_{-E}}}{\sqrt{\Delta}a_2}\left(b_1\sqrt{p_{-0}p_{+0}}+b_2\sqrt{p_{+1}p_{-1}}+b_3\sqrt{p_{+2}p_{-2}}+b_4\sqrt{p_{+0}p_{-0}p_{+1}p_{-1}p_{+2}p_{-2}}\right),$$

$$\text{where } a_2 = b_5\sqrt{p_{+1}p_{-1}p_{+2}p_{-2}}+b_6\sqrt{p_{+0}p_{-0}p_{+2}p_{-2}}+b_7\sqrt{p_{+0}p_{-0}p_{+1}p_{-1}}+b_8$$

where $b_i \in \mathbb{Z}[A, B, C, D, E]$. From the equation of the conics $Q_1 \pm r_{\mu,1}Q'$ we get the equations of the other 30 conics associated to the point q_1 using the action of Γ. From $r_{\mu,1}$ we see that on top of looking at the loops going around the 15 singular hyperplanes, we need to look at loops going around the Segre cubic. For this, we need the following lemma.

Lemma 3.2.7. *Let* $|x| < \min\left\{1, \left|\frac{1}{a}\right|, \left|\frac{1}{b}\right|, \left|\frac{c}{2a}\right|, \left|\frac{c}{2b}\right|\right\}$, *then* x *satisfies* $\left|ax+bx^2\right| \leq |ax|+\left|bx^2\right| < |c|$. *In particular this implies that* $c+ax+bx^2 \neq 0$.

Proof. This is a simple case by case proof:

Case 1. $|x| < 1 = \min\left\{1, \left|\frac{1}{a}\right|, \left|\frac{1}{b}\right|, \left|\frac{c}{2a}\right|, \left|\frac{c}{2b}\right|\right\}$. Then $|ax|+\left|bx^2\right| < |a|+|b|$, since $1 \leq \left|\frac{c}{2a}\right|$ and $\left|\frac{c}{2b}\right|$, we know that $a, b \leq \left|\frac{c}{2}\right|$. Hence $|a|+|b| \leq |c|$.

Case 2. $|x| < \frac{1}{|a|} = \min\left\{1, \left|\frac{1}{a}\right|, \left|\frac{1}{b}\right|, \left|\frac{c}{2a}\right|, \left|\frac{c}{2b}\right|\right\}$. Then $|ax|+\left|bx^2\right| < 1+\left|\frac{b}{a^2}\right| \leq 1+\left|\frac{b}{a}\right|$. Since $\frac{1}{|a|} \leq \frac{1}{|b|}$ implies $\left|\frac{b}{a}\right| \leq 1$, and $\frac{1}{|a|} \leq \frac{|c|}{|2a|}$ implies $2 \leq |c|$, then $1+\left|\frac{b}{a}\right| \leq |c|$.

Case 3. $|x| < \frac{1}{|b|} = \min\left\{1, \left|\frac{1}{a}\right|, \left|\frac{1}{b}\right|, \left|\frac{c}{2a}\right|, \left|\frac{c}{2b}\right|\right\}$. Then $|ax|+\left|bx^2\right| < \left|\frac{a}{b}\right|+\left|\frac{1}{b}\right|$. As in case 2, we see that $\left|\frac{a}{b}\right| \leq 1$ and $2 \leq |c|$, hence $\left|\frac{a}{b}\right|+\left|\frac{1}{b}\right| \leq 2 \leq |c|$.

Case 4. $|x| < \left|\frac{c}{2a}\right| = \min\left\{1, \left|\frac{1}{a}\right|, \left|\frac{1}{b}\right|, \left|\frac{c}{2a}\right|, \left|\frac{c}{2b}\right|\right\}$. Then $|ax|+\left|bx^2\right| < \left|\frac{c}{2}\right|+\left|\frac{bc^2}{2a^2}\right| \leq \left|\frac{c}{2}\right|+\left|\frac{bc}{2a}\right|$. As $\left|\frac{c}{2a}\right| \leq \left|\frac{c}{2b}\right|$ implies $\left|\frac{b}{a}\right| \leq 1$, we have that $\left|\frac{c}{2}\right|+\left|\frac{b}{a}\right|\left|\frac{c}{2}\right| \leq |c|$.

Case 5. $|x| < \left|\frac{c}{2b}\right| = \min\left\{1, \left|\frac{1}{a}\right|, \left|\frac{1}{b}\right|, \left|\frac{c}{2a}\right|, \left|\frac{c}{2b}\right|\right\}$. Then $|ax|+\left|bx^2\right| < \left|\frac{a}{b}\right|\left|\frac{c}{2}\right|+\left|\frac{c}{2b}\right|\left|\frac{c}{2}\right| \leq \left|\frac{c}{2}\right|+\left|\frac{c}{2}\right| \leq |c|$.

$\square$

Proposition 3.2.8. *Given a non-singular K3 surface defined by the point $p = [A, B, C, D, E] \in \mathbb{P}^4$, we can find a loop based at p that:*

1. *goes around the singular hyperplane $\{q_{+C} = 0\}$ and avoids the 15 singular hyperplanes,*

2. *changes the sign of $\sqrt{-q}$ in the equations defining $r_{0,1}, r_{1,1}, r_{2,1}$, $r_{3,1}$ and $r_{\mu,1}$.*

This loop sends a conic on the plane $r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w = 0$ to a conic on the plane $\gamma_3 (r_{0,1}x + r_{1,1}y + r_{2,1}z + r_{3,1}w) = 0$.

Proof. We construct the same loop as in Theorem 3.2.3 with a slight modification. This time we put the constraint that $\rho \in \mathbb{R}_{>0}$ satisfies

$$\rho < \min \Big\{ |B + 2D + 2E| , |-B + 2D + 2E| , |8A + B + 2D - 2E| , |8A - B + 2D - 2E| , 1,$$
$$\frac{1}{|-4A^2 + 4DE|}, \frac{1}{|4A|}, \frac{|A^2B - 4A(D^2 + E^2) + 8ADE|}{2\,|-4A^2 + 4DE|}, \frac{|A^2B - 4A(D^2 + E^2) + 8ADE|}{2\,|4A|} \Big\}.$$

The extra conditions mean that, by Lemma 3.2.7, $\Delta = \big(A^2B - 4A(D^2 + E^2) + 8ADE\big) + (4CD - 4A^2)f(t) - 4Af(t)^2 \neq 0$ during the second segment of the loop. At the same time, we see that the sign of the square root cannot change. Finally, this extra condition on ρ does not effect the rest of the proof of Theorem 3.2.3. $\square$

As with Theorem 3.2.3, we can adapt the above proof for loops going around the 15 singular planes. The next proposition looks at loops going around the Segre cubic.

Proposition 3.2.9. *Given a non-singular K3 surface defined by the point $p = [A, B, C, D, E] \in \mathbb{P}^4$, we can find a loop based at p that:*

1. *goes around the Segre cubic $\{\Delta = 0\}$ and avoids the 15 singular hyperplanes,*

2. *changes the sign of $\sqrt{\Delta}$ in the equations defining $r_{0,1}, r_{1,1}, r_{2,1}$, $r_{3,1}$ and $r_{\mu,1}$.*

This loop permutes the two conjugate conics on the plane $r_{0,1}x+r_{1,1}y+r_{2,1}z+r_{3,1}w = 0$.

Proof. First we claim that given such a K3 surface there exists B' such that the surface $[A, B', C, D, E]$ lies on the Segre cubic but not on the 15 singular hyperplanes. This first part of the statement is easy to see, solve

$$16A^3 + A\left(B'\right)^2 - 4A(C^2 + D^2 + E^2) + 4CDE = 0$$

in terms on B'. Since $A \neq 0$ as our surface is non-singular and we are working over $\overline{\mathbb{Q}}$, this has a solution (in fact B' is at worse in a degree 2 extension of the field of definition of A, B, C, D, E). For the second part, recall that if a point lies on the Segre cubic and one of the 15 singular hyperplanes, then it lies on a Segre plane, i.e., it must lie on a further two singular hyperplanes. But note that any surface lying on a Segre plane must lie on one singular hyperplane which is defined with no B (and hence B')

term. Therefore if $[A, B', C, D, E]$ lied on such an singular hyperplane then so would $[A, B, C, D, E]$, contradicting our assumption that the surface is non-singular.

We construct a loop similar to the one in the proof of Theorem 3.2.3. Note that B can be written uniquely as $B' + re^{i\phi}$ for some $r \in \mathbb{R}_{>0}$ and $\phi \in [0, 2\pi)$. Define a loop $\widetilde{\gamma}(t) = [\widetilde{\gamma}_A(t), \widetilde{\gamma}_B(t), \widetilde{\gamma}_C(t), \widetilde{\gamma}_D(t), \widetilde{\gamma}_E(t)]$ $(0 \leq t \leq 3)$ as $\widetilde{\gamma}_i = i$ for all t and $i \in \{A, C, D, E\}$ and $\widetilde{\gamma}_B = B' + f(t)$, where f is composed of the following 3 segments:

$$f(t) = \begin{cases} (\rho t + r(1-t))e^{i\phi} & t \in [0,1] \\ \rho e^{i\phi + i(t-1)2\pi} & t \in [1,2] \\ (\rho(3-t) + r(t-2))e^{i\phi} & t \in [2,3] \end{cases}$$

and $\rho \in \mathbb{R}_{>0}$ satisfies

$$\begin{aligned} \rho \quad < \quad \min \Big\{ & \left|4A \pm B' + 2C + 2D + 2E\right|, \left|4A \pm B' + 2C - 2D - 2E\right|, \\ & \left|4A \pm B' - 2C + 2D - 2E\right|, \left|4A \pm B' - 2C - 2D + 2E\right|, \left|2B'\right| \Big\}. \end{aligned}$$

Note that with the conditions on ρ the loop γ never loops around the 15 singular hyperplanes, hence as we have seen before, the 15 square roots (of the defining equations of the 15 singular hyperplanes) do not have a sign change. As for $\sqrt{\Delta}$, note that the first and third segments leave it untouched, while for the second segment

$$\begin{aligned} \Delta &= 16A^3 - 4A(C^2 + D^2 + E^2) + 4CDE + A(B' + \rho e^{i\phi + i(t-1)2\pi})^2 \\ &= -AB'^2 + A(B' + \rho e^{i\phi + i(t-1)2\pi})^2 \\ &= A\left(B' + \rho e^{i\phi + i(t-1)2\pi} + B'\right)\left(B' + \rho e^{i\phi + i(t-1)2\pi} - B'\right) \\ &= A\rho e^{i\phi + i(t-1)2\pi}\left(2B' + \rho e^{i\phi + i(t-1)2\pi}\right). \end{aligned}$$

Hence as in the previous proof, we find that as we loop around the Segre cubic, $\Delta = 0$, the sign of $\sqrt{\Delta}$ changes. As only the sign of $\sqrt{\Delta}$ changes, we see $Q_1 + r_{\mu,1}Q'$ is sent to $Q_1 - r_{\mu,1}Q'$.

Note that the above only works under the assumption that $B' \neq 0$. In the case $B' = 0$, we first need to find a path from our point $[A, B, C, D, E]$ to the point $[A + \epsilon, B, C, D, E]$, where ϵ is small enough that we do not go near any singular hyperplane nor the Segre cubic. In that case, we use the point $[A + \epsilon, B, C, D, E]$ as our starting point. □

Hence we use the explicit equations of the conics to find the Monodromy group of the 32 conics defined by the point q_1. Then, as before, we use the group Ω acting on our set of points $\{q_i\}$ and 15 singular hyperplanes to find the Monodromy group of the 32 conics defined by each of the points q_i. We summarise the information in Table

3.2, where again the rows are the 15 singular hyperplanes or the Segre cubic and the columns are the points q_i. Entries are empty for the identity element or are elements of Γ with a $\pm$ sign. A -1 denotes the elements that conjugate conics, that is permute conics defined on the same plane, and $\gamma \in \Gamma$ corresponds to the permutation of the defining plane (as with Table 3.1).

Theorem 3.2.10. *The Monodromy group of the* 320 *conics is* C_2^{10}.

Proof. We use the information given in Table 3.2. The Monodromy group is a subgroup of S_{320}. After embedding in S_{320} the elements associated to the 15 singular planes and Segre cubic, we check ([Bou]) that they generate a subgroup of order 2^{10}. From the table, we see that the elements associated to the 15 singular planes and Segre cubic commute with each other and have order 2, hence we know that every non-trivial element of the Monodromy group has order 2. Since the only group of order 2^{10} with all non-trivial elements being involutions is C_2^{10}, the Monodromy group of the 320 conics is C_2^{10}. $\square$

Corollary 3.2.11. *The moduli space of pairs* (X, C)*, where* X *is a Heisenberg-invariant quartic K3 surface and* C *one of the* 320 *conic on* X*, has 10 irreducible components.*

Proof. Let Z be the moduli space of pairs (X, C) with X a surface in $\mathcal{X}$ and C a conic lying on X. We showed that the Monodromy group of π, $\pi : Z \to \mathbb{P}^4_{[A,B,C,D,E]}$, breaks the 320 conics on X in 10 orbits of size 32. Since calculating the Monodromy group involves lifting a path in $\mathbb{P}^4_{[A,B,C,D,E]}$ to a path in Z, any two elements in the same orbit represent two connected elements in Z. Finally, since the paths avoided where π was not smooth, the 10 orbits correspond to 10 smooth connected components of Z, i.e., 10 irreducible components. $\square$

3.3 K3 surfaces with many $\mathbb{Q}$-conics

We finish this chapter by using the field of definition of the conics, and the construction of the planes, to find rational invariant K3 surfaces with as many rational conics as possible.

Lemma 3.3.1. *Let* X *be an invariant* K3 *surface containing one rational conic. Then* X *contains* 32 *rational conics all associated to the same point* q_i *for some* $i \in [1, \ldots, 10]$.

Proof. Without loss of generality, assume the rational conic X contains is associated to the point q_1. Its conjugate (i.e., the other conic lying on the same plane) must also be defined over $\mathbb{Q}$ and is associated to the point q_1. Let Γ act on X. As Γ is defined over $\mathbb{Q}$, the orbit of the two conics, which are 32 conics, are defined over $\mathbb{Q}$. As Γ fixes X and q_1 in $\mathbb{P}^4$, the 32 conics are all associated to the point q_1. $\square$

	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	q_9	q_{10}
Δ	-1	-1	-1	-1	-1	-1	-1	-1	-1	-1
A					$-\gamma_3$	$-\gamma_3$	$-\gamma_4$	$-\gamma_4$	$-\gamma_3\gamma_4$	$-\gamma_3\gamma_4$
q_{+C}	$-\gamma_3$	$-\gamma_3$					$-\gamma_1\gamma_3$	$-\gamma_1$	$-\gamma_1\gamma_3$	$-\gamma_1$
$-q_{-C}$			$-\gamma_3$	$-\gamma_3$			$-\gamma_1\gamma_4$	$-\gamma_1\gamma_3\gamma_4$	$-\gamma_1\gamma_3\gamma_4$	$-\gamma_1\gamma_4$
q_{+D}	$-\gamma_4$		$-\gamma_4$		$-\gamma_2\gamma_4$	$-\gamma_2$			$-\gamma_2\gamma_4$	$-\gamma_2$
$-q_{-D}$		$-\gamma_4$		$-\gamma_4$	$-\gamma_2\gamma_3$	$-\gamma_2\gamma_3\gamma_4$			$-\gamma_2\gamma_3\gamma_4$	$-\gamma_2\gamma_3$
q_{+E}		$-\gamma_3\gamma_4$	$-\gamma_3\gamma_4$		$-\gamma_1\gamma_2\gamma_3\gamma_4$	$-\gamma_1\gamma_2$	$-\gamma_1\gamma_2\gamma_3\gamma_4$	$-\gamma_1\gamma_2$		
$-q_{-E}$	$-\gamma_3\gamma_4$			$-\gamma_3\gamma_4$	$-\gamma_1\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_4$	$-\gamma_1\gamma_2\gamma_4$	$-\gamma_1\gamma_2\gamma_3$		
p_{+0}	$-\gamma_1\gamma_2$	$-\gamma_2$	$-\gamma_1$		$-\gamma_1$		$-\gamma_2$		$-\gamma_1\gamma_2$	
p_{-0}	$-\gamma_1\gamma_2\gamma_3\gamma_4$	$-\gamma_2\gamma_4$	$-\gamma_1\gamma_3$			$-\gamma_1\gamma_3$		$-\gamma_2\gamma_4$		$-\gamma_1\gamma_2\gamma_3\gamma_4$
p_{+1}	$-\gamma_2\gamma_3\gamma_4$	$-\gamma_1\gamma_2\gamma_4$		$-\gamma_1\gamma_3$	$-\gamma_1\gamma_3$			$-\gamma_2\gamma_3\gamma_4$		$-\gamma_1\gamma_2\gamma_4$
p_{-1}	$-\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3$		$-\gamma_1$		$-\gamma_1$	$-\gamma_2\gamma_3$		$-\gamma_1\gamma_2\gamma_3$	
p_{+2}	$-\gamma_1\gamma_3\gamma_4$		$-\gamma_1\gamma_2\gamma_3$	$-\gamma_2\gamma_4$		$-\gamma_1\gamma_3\gamma_4$	$-\gamma_2\gamma_4$			$-\gamma_1\gamma_2\gamma_3$
p_{-2}	$-\gamma_1\gamma_4$		$-\gamma_1\gamma_2\gamma_4$	$-\gamma_2$	$-\gamma_1\gamma_4$			$-\gamma_2$	$-\gamma_1\gamma_2\gamma_4$	
p_{+3}		$-\gamma_1\gamma_4$	$-\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3\gamma_4$		$-\gamma_1\gamma_4$		$-\gamma_2\gamma_3$	$-\gamma_1\gamma_2\gamma_3\gamma_4$	
p_{-3}		$-\gamma_1\gamma_3\gamma_4$	$-\gamma_2\gamma_3\gamma_4$	$-\gamma_1\gamma_2$	$-\gamma_1\gamma_3\gamma_4$		$-\gamma_2\gamma_3\gamma_4$			$-\gamma_1\gamma_2$

Table 3.2: Monodromy of the 320 conics

Proposition 3.3.2. *Over $\mathbb{Q}$, there exists a four dimensional subfamily of $\mathcal{X}$, parametrised by $\mathbb{P}^3_{[a,b,c,d]} \times \mathbb{P}^1_{[m,n]}$, such that every rational member contains at least* 32 *conics defined over $\mathbb{Q}$.*

Proof. As in the previous section instead of starting with an invariant K3 surface and constructing a plane, we instead start with a plane and construct an invariant K3 surface. We start with a point $[a, b, c, d] \in \mathbb{P}^3$ defined over $\mathbb{Q}$ (i.e., there exists $\lambda \in \overline{\mathbb{Q}}^*$ such that $\lambda a, \lambda b, \lambda c, \lambda d \in \mathbb{Q}$). Following Proposition 2.3.4, let $[A_1, B_1, C_1, D_1, E_1] \in \mathbb{P}^4$ represent the Kummer surface which has $[a, b, c, d]$ as a singular point. Then any points on the line joining $[A_1, B_1, C_1, D_1, E_1]$ and $[1, 0, -2, -2, 2] = q_1$ represents an invariant K3 surface which intersects the plane $T : ax + by + cz + dw = 0$ in two conics. Parametrise the line joining $[A_1, B_1, C_1, D_1, E_1]$ and q_1 by the set of points $p_{m,n} = [A_1 m + n, B_1 m, C_1 m - 2n, D_1 m - 2n, E_1 m + 2n]$ with $[m, n] \in \mathbb{P}^1$. Let $X_{m,n}$ be the invariant K3 surface defined by the point $p_{m,n}$. Then, whenever $[m, n] \in \mathbb{P}^1$ is defined over $\mathbb{Q}$, $X_{m,n}$ is defined over $\mathbb{Q}$, and since T is defined over $\mathbb{Q}$, we have that the two conics on $X_{m,n} \cap T$ are at most on a degree 2 extension of $\mathbb{Q}$. We can use Proposition 3.1.1, substituting in $X_{m,n}$, to find that the field of definition of the 32 conics associated to the point q_1 is $K_1 = \mathbb{Q}(\sqrt{nm})$.

Hence the family of surfaces X_{m^2,n^2} parametrised by $\mathbb{P}^3_{[a,b,c,d]} \times \mathbb{P}^1_{[m,n]}$ contain at least 32 conics defined over $\mathbb{Q}$, namely the conics associated to the point $q_1 = [1, 0, -2, -2, 2]$. □

Remark. The above argument applies to any choice of q_i. Hence, given any rational plane and a point q_i, there exists infinitely many invariant K3 surfaces which contain two rational conics defined on that plane, and are associated to the point q_i.

Now that we can easily construct invariant K3 surfaces with 32 conics, we make the following remark. Let $p \in \mathbb{P}^4_{[A,B,C,D,E]}$ define an invariant K3 surface, with 32 conics defined over $\mathbb{Q}$ associated to the point q_i, and let p_i be the point of intersection between the Segre cubic and the line joining p and q_i. Consider the action on $\mathbb{P}^4_{[A,B,C,D,E]}$ induced by Ω and notice the Segre cubic is fixed by Ω. Let $\phi \in \Omega$ and notice that $\phi(p)$ defines an invariant K3 surface, with 32 conics defined over $\mathbb{Q}$ associated to the point $\phi(q_i)$, and $\phi(p_i)$ is the point of intersection between the Segre cubic and the line joining $\phi(p)$ and $\phi(q_i)$. Hence if p is such that $\phi(p) = p$, then it defines an invariant K3 surface with 64 conics defined over $\mathbb{Q}$.

While Ω has no fixed points in $\mathbb{P}^4$, finding a large subgroup of Ω which has some fixed points would give us good candidates for invariant K3 surfaces with many conics defined over $\mathbb{Q}$. Note for example the subfamily $[A, 0, C, C, C]$ of $\mathcal{X}$ is fixed by ϕ_1, ϕ_2 and ϕ_3. This leads us to the next proposition.

Proposition 3.3.3. *Over $\mathbb{Q}$, there exists a one dimensional subfamily of $\mathcal{X}$, parametrised by $\mathbb{P}^1_{[m,n]}$, such that for every member, X, there exists* 160 *$\mathbb{Q}$-planes intersecting X in two conics.*

Proof. Let $A, C \in \mathbb{Q}$ and consider the surface X defined by the point $[A, 0, C, C, C]$. Using the same method as in the proof of Theorem 3.1.5 (using the data of Appendix A.2, but not including $r_{\mu,i}$), we see that the field of definition of the 160 planes is

$$K_T = \mathbb{Q}\left(\sqrt{-(2A+C)(2A+3C)}, \sqrt{A(2A+C)}\right).$$

Without loss of generality, we can assume that $A \in \mathbb{Z}_{>0}$, $C \in \mathbb{Z}$ and $\gcd(A, C) = 1$. Hence $\gcd(A, 2A+C) = 1$, so $K_T \cong \mathbb{Q}$ if and only if there exists $r, s, t \in \mathbb{Q}$ such that $A = r^2, 2A + C = s^2$ and $2A + 3C = -t^2$. Hence consider the equation

$$\begin{aligned} 3(2A+C) - (2A+3C) &= 4A \\ 3s^2 + t^2 &= 4r^2. \end{aligned}$$

This describes a conic in $\mathbb{P}^2_{[r,s,t]}$ with a rational point $[1, 0, 2]$, hence its rational points can be parametrise by $\mathbb{P}^1_{[m,n]}$. We find the following parametrisation $[4m^2 + 3n^2, 8mn, 8m^2 - 6n^2]$.

Hence let $m, n \in \mathbb{Q}$, the surface X defined by the point

$$[(4m^2+3n^2)^2, 0, 2(8m^2n^2-16m^4-9n^4), 2(8m^2n^2-16m^4-9n^4), 2(8m^2n^2-16m^4-9n^4)]$$

has the 160 planes defined over $\mathbb{Q}$. □

Let $A, C \in \mathbb{Q}$ and consider the surface X defined by the point $[A, 0, C, C, C]$. If X has one conic defined over $\mathbb{Q}$, then as the conics come in pairs, X has two conics defined over $\mathbb{Q}$. Using the action of Γ, we see that X has 32 conics defined over $\mathbb{Q}$, all associated to one point q_i. So, let us use Theorem 3.1.5 and find the field of definition of each 32 conics. Note that since ϕ_1, ϕ_2 and ϕ_3 fix X, and the set $\{q_i : i \in [1, \ldots, 10]\}$ partitions into three orbits, $\{q_1, q_2, q_3\}$,$\{q_4\}$ and $\{q_5, q_6, q_7, q_8, q_9, q_{10}\}$, under ϕ_1, ϕ_2 and ϕ_3, we just need to calculate K_1, K_4 and K_5. We find that

$$\begin{aligned} K_1 &\cong \mathbb{Q}\left(\sqrt{-(A+C)(2A-C)}, \sqrt{-(2A+C)(2A+3C)}\right) \\ K_4 &\cong \mathbb{Q}\left(\sqrt{-(A+C)(2A-C)}\right) \\ K_5 &\cong \mathbb{Q}\left(\sqrt{-(A+C)(2A-C)}, \sqrt{-(2A+C)(2A+3C)}, \sqrt{A(2A+C)}\right). \end{aligned}$$

We see that there is a parametrisation (namely $A = m^2 - n^2, C = 2m^2 + n^2)$) of the invariant K3 surfaces of the form $[A, 0, C, C, C]$ with 32 conics defined over $\mathbb{Q}$ and associated to the point q_4. We use the next lemma to show that invariant K3 surfaces of the form $[A, 0, C, C, C]$ cannot have more than 32 conics defined over $\mathbb{Q}$.

Lemma 3.3.4. *Let $A, C \in \mathbb{Q}$, then $-(A+C)(2A-C)$ and $-(2A+C)(2A+3C)$ cannot both be non-zero squares.*

Proof. Without loss of generality assume that $A \in \mathbb{Z}_{>0}, C \in \mathbb{Z}$ and $\gcd(A, C) = 1$, then note that $\gcd(A+C, 2A-C)|3$ and $\gcd(2A+C, 2A+3C)|2$. If $-(2A+C)(2A+3C)$ is a square, we need $-2A < C < 0$, so we have the number line

$$\begin{array}{ccccccc} 2A+3C & & 2(A+C) & 0 & 2A+C & & 2A-C\,. \\ & C & & C & & 2C & \end{array}$$

Hence, the above two expression are both squares if and only if we are in one of the following cases:

1. There exists $r, s, t, u \in \mathbb{Q}$ such that $A + C = -r^2$, $2A - C = s^2$, $2A + C = t^2$ and $2A + 3C = -u^2$,
2. There exists $r, s, t, u \in \mathbb{Q}$ such that $A + C = -3r^2$, $2A - C = 3s^2$, $2A + C = t^2$ and $2A + 3C = -u^2$,
3. There exists $r, s, t, u \in \mathbb{Q}$ such that $A + C = -r^2$, $2A - C = s^2$, $2A + C = 2t^2$ and $2A + 3C = -2u^2$,
4. There exists $r, s, t, u \in \mathbb{Q}$ such that $A + C = -3r^2$, $2A - C = 3s^2$, $2A + C = 2t^2$ and $2A + 3C = -2u^2$.

To see if such $r, s, t, u \in \mathbb{Q}$ exists, we use a proof similar to the proof showing there exists no (non-trivial) arithmetic progression with four consecutive terms being squares.

1. We rearrange, using the number line above, to see we need r, s, t, u to satisfy the following two conditions

$$\begin{aligned} s^2 - t^2 &= 2(t^2 + 2r^2) \\ t^2 + 2r^2 &= -2r^2 + u^2. \end{aligned}$$

   This describes a genus one curve in $\mathbb{P}^3_{[r,s,t,u]}$, and as it has the rational point $[1, 2, 0, 2]$, it is an elliptic curve. Using Magma [BCP97], we find that it is isomorphic to the elliptic curve $E : y^2 = x^3 + x^2 - 2x$. This elliptic curve has $E(\mathbb{Q}) \cong (\mathbb{Z}/2\mathbb{Z})^2$, so we conclude that the only such r, s, t, u are $[1, 2, 0, 2]$, $[1, 2, 0, -2]$, $[1, -2, 0, 2]$ and $[1, -2, 0, -2]$. But as all four cases leads to $2A+C = 0$, hence contradicting the assumption of $-(2A+C)(2A+3C)$ being a non-zero squares.

2. Similarly, we rearrange to get the conditions

$$\begin{aligned} 3s^2 - t^2 &= 2(t^2 + 6r^2) \\ t^2 + 6r^2 &= -6r^2 + u^2. \end{aligned}$$

This has the rational point $[0,1,1,1]$ and hence is isomorphic to the elliptic curve $E : y^2 = x^3 + x^2 - 2x$. Like in case 1, we conclude the four different solutions lead to cases we do not want.

3. This time, we get the conditions

$$\begin{aligned} s^2 - 2t^2 &= 2(2t^2 + 2r^2) \\ 2t^2 + 2r^2 &= -2r^2 + 2u^2. \end{aligned}$$

We show that this has no rational points by showing, without loss of generality, it has no integral points with $\gcd(r,s,t,u) = 1$. Looking at the first equation $6t^2 + 4r^2 - s^2$ modulo 4, we conclude that $2|s$ and $2|t$. Using the second equation $t^2 + 2r^2 - u^2$, we conclude $2|u$, and hence $2|r$, contradicting that they are all coprime.

4. For this last case, we get the conditions

$$\begin{aligned} 3s^2 - 2t^2 &= 2(2t^2 + 6r^2) \\ 2t^2 + 6r^2 &= -6r^2 + 2u^2. \end{aligned}$$

Again, looking at the first equation modulo 4 and the second (once we have divided by 2), we reach the conclusion that this has no rational points.

□

Corollary 3.3.5. *Let $A, C \in \mathbb{Q}$, the surface defined by the point $[A, 0, C, C, C]$ has no more than 32 conics defined over $\mathbb{Q}$.*

Proof. Keeping with the notation above, Lemma 3.3.4 implies that $K_1 \not\cong \mathbb{Q}$ for any $A, C \in \mathbb{Q}$. Furthermore K_5 is an extension of K_1, hence $K_5 \not\cong \mathbb{Q}$. Therefore, using Lemma 3.3.1, none of the conics associated to q_1, q_2, q_3 (which has field of definition K_1) or $q_5, q_6, q_7, q_8, q_9, q_{10}$ (which has field of definition K_5) are defined over $\mathbb{Q}$. We conclude that only the conics associated to the point q_4 could be defined over $\mathbb{Q}$. □

Corollary 3.3.6. *Pick a point $[m, n] \in \mathbb{P}^1$ with $m, n \in \mathbb{Q}$, and let $A = (4m^2 + 3n^2)^2$, $C = 2(8m^2n^2 - 16m^4 - 9n^4)$. Then the surface X associated to the point $[A, 0, C, C, C]$ has none of its 320 conics defined over $\mathbb{Q}$. Instead, all 320 conics are defined over $\mathbb{Q}\left(\sqrt{-(A+C)(2A-C)}\right)$.*

Proof. By Proposition 3.3.3, X has all 160 planes defined over $\mathbb{Q}$, hence

$$\mathbb{Q}\left(\sqrt{-(2A+C)(2A+3C)}, \sqrt{A(2A+C)}\right) \cong \mathbb{Q}.$$

As $K_5 \cong K_4\left(\sqrt{-(2A+C)(2A+3C)}, \sqrt{A(2A+C)}\right)$ (keeping with the notation above), we have $K_5 \cong K_4 \cong K_1$. In particular, if $K_4 \cong \mathbb{Q}$ then X would have all 320 conics defined over $\mathbb{Q}$, contradicting the previous corollary.

We conclude that all 320 conics are defined over $K_4 \cong \mathbb{Q}\left(\sqrt{-(A+C)(2A-C)}\right) \not\cong \mathbb{Q}$ by Lemma 3.3.4. □

We know turn our attention to surfaces which are fixed under ϕ_1 and ϕ_3, i.e., surfaces defined by points of the form $[A, 0, C, C, E]$. This time $\{q_i : i \in [1, \dots, 10]\}$ is partitioned in 5, $\{q_1\}$, $\{q_2, q_3\}$, $\{q_4\}$, $\{q_5, q_6, q_7, q_8\}$ and $\{q_9, q_{10}\}$. Using the fact that for surfaces fixed under ϕ_1 and ϕ_2 we have $q_{+C} = q_{+D}$, $q_{-C} = q_{-D}$, $p_{-0} = p_{+0}$, $p_{-1} = p_{+1} = p_{-2} = p_{+2}$ and $p_{-3} = p_3$, we notice from Theorem 3.1.5 that

$$\begin{aligned}
K_1 &\cong \mathbb{Q}\left(\sqrt{\Delta q_{+C}p_{-0}p_{-1}}, \sqrt{-\Delta q_{-E}}\right),\\
K_2 &\cong \mathbb{Q}\left(\sqrt{\Delta q_{+C}p_{-0}p_{-1}}, \sqrt{-\Delta q_{-C}p_{-1}p_{-3}}, \sqrt{\Delta q_{+E}p_{-0}p_{-3}}\right),\\
K_4 &\cong \mathbb{Q}\left(\sqrt{-\Delta q_{-C}p_{-1}p_{-3}}, \sqrt{-\Delta q_{-E}}\right),\\
K_5 &\cong \mathbb{Q}\left(\sqrt{-\Delta A q_{+C}q_{-C}}, \sqrt{-\Delta A q_{+E}q_{-E}}, \sqrt{\Delta q_{+C}p_{-0}p_{-1}}, \sqrt{\Delta q_{+E}p_{-0}p_{-3}}, \sqrt{-\Delta q_{-E}}\right),\\
K_9 &\cong \mathbb{Q}\left(\sqrt{-\Delta A q_{+C}q_{-C}}, \sqrt{\Delta q_{+C}p_{-0}p_{-1}}, \sqrt{-\Delta q_{-C}p_{-1}p_{-3}}\right).
\end{aligned}$$

We see that $K_1 \subset K_5$ and $K_2, K_4, K_9 \subset K_5(\sqrt{-\Delta q_{-C}p_{-1}p_{-3}})$. Hence, we try to use Proposition 3.3.2 to construct an invariant K3 surface which contains a $\mathbb{Q}$-conic associated to q_5. By the above, such a K3 surface would have $5 \cdot 32 = 160$ conics defined over $\mathbb{Q}$, and the other 160 conics defined over $\mathbb{Q}\left(\sqrt{-\Delta q_{-C}p_{-1}p_{-3}}\right)$.

Proposition 3.3.7. *Let $a, b, c, d \in \mathbb{Z}$ satisfy*

1. $abcd(a^2 - b^2 - c^2 + d^2)(a^2 - b^2 + c^2 - d^2)(a^2 + b^2 - c^2 - d^2)(a^2 + b^2 + c^2 + d^2)$ *is a non-zero square, and*
2. $ad(a^2 - d^2) = bc(b^2 - c^2)$,.

Define

- $A = (bc + ad)(bc - ad)(ac + bd)(ac - bd)(cd + ab)(cd - ab)$,
- $C = (bc + ad)(bc - ad)(cd + ab)(cd - ab)(-a^4 + b^4 - c^4 + d^4)$,
- $E = (ac + db)(ac - bd)(cd + ab)(cd - ab)(a^4 - b^4 - c^4 + d^4)$.

Then the invariant K3 *surface X defined by the point $[A, 0, C, C, E]$ is non-singular, and has all* 320 *conics defined over $\mathbb{Q}$.*

Proof. As in proof of Proposition 3.3.2, we start from the point $[a, b, c, d] \times [m, n] \in \mathbb{P}^3 \times \mathbb{P}^1$ with $a, b, c, d, m, n \in \mathbb{Q}$, and construct the invariant K3 surfaces X defined by the point $[A_1m^2, B_1m^2 - 2n^2, C_1m^2 + n^2, D_1m^2, E_1m^2]$. Then X contains 32 $\mathbb{Q}$-conics associated to the point q_5.

As we want X to be invariant under ϕ_1, we need $B_1m^2 - 2n^2 = 0$. As

$$B_1 = 2abcd(a^2 - b^2 - c^2 + d^2)(a^2 - b^2 + c^2 - d^2)(a^2 + b^2 - c^2 - d^2)(a^2 + b^2 + c^2 + d^2)$$

then $B_1m^2 - 2n^2 = 0$ if and only if condition 1 is satisfied. In this case, without loss of generality, we assume $m^2 = 1$ and $n^2 = B_1/2$. If $B_1 = 0$, then $n^2 = 0$ and hence X is the Kummer surface associated to the point $[A_1, B_1, C_1, D_1, E_1]$, i.e., it is singular.

As we want X to be invariant under ϕ_3, we need $C_1m^2 + n^2 = D_1m^2$. As

$$C_1 + \frac{B_1}{2} - D_1 = (ab+cd)(ab-cd)(a^2-b^2-c^2+d^2)(a^2+b^2+c^2+d^2)(a^3d-ad^3-b^3c+bc^3)$$

and we want X to be non-singular, i.e., $A_1 \neq 0$ and as seen above $B_1 \neq 0$, then $C_1m^2 + n^2 = D_1m^2$ if and only if condition 2, is satisfied.

Hence, if both condition 1 and 2 are satisfied, X is defined by the point $[A_1, 0, D_1, D_1, E_1] = [A, 0, C, C, E]$ (with A, C, E as defined in the proposition). From the construction, X has the 32 conics associated to q_5 defined over $\mathbb{Q}$. As X is invariant under ϕ_1 and ϕ_3, by the above discussion X has 160 conics (associated respectively to q_1, q_5, q_6, q_7 and q_8) defined over $\mathbb{Q}$. The other 160 conics are defined over $\mathbb{Q}\left(\sqrt{-\Delta q_{-C}p_{-1}p_{-3}}\right)$.

To finish the proof we show that, if $a, b, c, d \in \mathbb{Z}$ satisfy conditions 1 and 2 then $\mathbb{Q}\left(\sqrt{-\Delta q_{-C}p_{-1}p_{-3}}\right) \cong \mathbb{Q}$. We have $-\Delta q_{-C}p_{-1}p_{-3}$ is

$$\begin{aligned}2^4 \cdot (bc - ad)^2 \cdot (bc + ad)^2 \cdot (ac - bd)^2 \cdot (ac + bd)^2 \cdot (-ab + cd)^6 \cdot (ab + cd)^6 \\ \cdot(-a^2 + b^2 + c^2 - d^2)^3 \cdot (a^2 + b^2 + c^2 + d^2)^5 \cdot (a^2 - b^2 + c^2 - d^2) \\ \cdot(-b^2c^2 + c^4 + a^2d^2 - d^4) \cdot (a^4 - b^4 + b^2c^2 - a^2d^2) \cdot f\end{aligned}$$

where

$$\begin{aligned} f &= a^4c^2 + a^2b^2c^2 - 2b^4c^2 - a^2c^4 + 2b^2c^4 - 2a^4d^2 \\ &\quad +a^2b^2d^2 + b^4d^2 - a^2c^2d^2 - b^2c^2d^2 + 2a^2d^4 - b^2d^4 \end{aligned}$$

Using condition 1, up to squares we have

$$\begin{aligned} & (-a^2 + b^2 + c^2 - d^2)^3 \cdot (a^2 + b^2 + c^2 + d^2)^5 \cdot (a^2 - b^2 + c^2 - d^2) \\ = \; & -abcd(a^2 + b^2 - c^2 - d^2). \end{aligned}$$

So consider

$$\begin{aligned} & -abcd(-b^2c^2 + c^4 + a^2d^2 - d^4)(a^4 - b^4 + b^2c^2 - a^2d^2) \\ = \; & -bc\left(ad^2(a^2 - d^2) - ac^2(b^2 - c^2)\right)\left(a^2d(a^2 - d^2) - b^2d(b^2 - c^2)\right) \\ = \; & -bc\left(c(bd - ac)(b^2 - c^2)\right)\left(b(ac - bd)(b^2 - c^2)\right) \quad \text{by condition 2} \\ = \; & b^2c^2(bd - ac)^2(b^2 - c^2)^2. \end{aligned}$$

At this stage, we have that up to squares $-\Delta q_{-C} p_{-1} p_{-3}$ is $(a^2+b^2-c^2-d^2)\cdot f$. But note that $f = (ac-bd)^2(a^2+b^2-c^2-d^2) - 2(ad-bc)\left[ad(a^2-d^2) - bc(b^2-c^2)\right]$. Hence by condition 2, $(a^2+b^2-c^2-d^2)\cdot f = (ac-bd)^2(a^2+b^2-c^2-d^2)^2$, and $-\Delta q_{-C} p_{-1} p_{-3}$ is a square, finishing the proof. $\square$

Corollary 3.3.8. *The invariant* K3 *surface X defined by the point*

$$[425, 0, -1025, -1025, 1207]$$

has 320 *conics defined over* $\mathbb{Q}$.

Proof. Note that $a = 5$, $b = 7$, $c = 15$ and $d = 16$ satisfy conditions 1 and 2 of Proposition 3.3.7. Hence the surface, X, defined by $[425, 0, -1025, -1025, 1207]$, (dividing by $\gcd(A, C, E)$), has all 320 conics defined over $\mathbb{Q}$. $\square$

Remark. As all the conics of the surface X defined by the point $[425, 0, -1025, -1025, 1207]$ are defined over $\mathbb{Q}$, we can easily calculate explicit equations for them using the formulas for the planes listed in Appendix A.3. Doing so enables us to find that 128 conics have a rational point (and hence infinitely many) and the other 192 have no rational points.

These calculations, and other complementary calculations made in this section can be found online [Bou].

Chapter 4

Picard Lattices of Subfamilies

This chapter was motivated by the question: “if X is an invariant quartic surface containing a line, what can we say about its Picard Group?”. As Eklund [Ekl10] studied the quartic surfaces parametrised by the Niento quintic, N_5, we look at the surfaces parametrised by the tangent cones of the 10 singular points of N_5.

In particular we consider :

- a four dimensional family $\mathcal{X}$ (as described in Section 2.3),
- a three dimensional family $\mathcal{X}_{C,D,E}$,
- a two dimensional family $\mathcal{X}_{C,D}$,
- a one dimensional family $\mathcal{X}_B$,
- a one dimensional family $\mathcal{X}_C$,
- a specific quartic K3 surface Y,
- and the Fermat quartic, F_4.

For each of these families, we look at the lines a very general member contains. We use these and the 320 conics that Eklund found to calculate the Picard group of a very general member. Our main result (Theorem 4.2.8) can be summarised as follows:

Theorem (Summarised Theorem 4.2.8)**.**

- *A very general member of $\mathcal{X}$ contains no lines, and has Picard rank* 16*,*
- *A very general member of $\mathcal{X}_{C,D,E}$ contains exactly* 8 *lines, and has Picard rank* 17*,*
- *A very general member of $\mathcal{X}_{C,D}$ contains exactly* 16 *lines, and has Picard rank* 18*,*
- *A very general member of $\mathcal{X}_B$ contains exactly* 24 *lines, and has Picard rank* 19*,*

- *A very general member of $\mathcal{X}_C$ contains exactly* 32 *lines, and has Picard rank* 19*,*
- *The surface Y contains exactly* 32 *lines, and has Picard rank* 20*,*
- *The Fermat quartic, F_4, contains exactly* 48 *lines, and has Picard rank* 20*.*

Possibly except for the surface Y, the Picard group is generated by the lines and conics lying on the surface. In all cases, we decompose the Picard group into known lattices.

Remark. The result about a very general member of $\mathcal{X}$ having Picard rank 16, with the Picard group generated by the conics, was already proven by Eklund [Ekl10, Thm 3.5, Cor 7.4] but in this chapter we prove this using a different method.

The fact that the Fermat quartic has 48 lines, who generate the Picard group of rank 20, is a classical result. We will use that result in our proof of Theorem 4.2.8.

We note that Theorem 4.2.8 fits nicely with the fact that certain moduli spaces of K3 surfaces whose Picard group contains a fixed lattice M have dimension $20 - \text{rank}(M)$. That is, in each of the above, a Picard group of rank r, fits nicely with a $20 - r$ dimensional family.

The results of this chapter have been put together into a preprint [Bou15b].

4.1 The Families and Lines

Definition 4.1.1. We define the *Nieto quintic,* $N_5 \subseteq \mathbb{P}^4_{[A,B,C,D,E]}$, by the equation

$$4A^3(48A^2 - B^2) - A(32A^2 - B^2)(C^2 + D^2 + E^2) + B^2CDE$$

$$+4A(C + D + E)(C + D - E)(C - D + E)(-C + D + E) = 0$$

The Nieto quintic was studied by Barth and Nieto when they were looking at K3 surfaces in $\mathcal{X}$ containing lines. In particular, they proved in [BN94, Section 7 and 8] the following proposition.

Proposition 4.1.2. *Let $p \in \mathbb{P}^4$, then the surface X_p contains a line, L, if and only if p is in N_5 or in one of the* 10 *tangent cones to the isolated singular points of N_5 (i.e., the* 10 *nodes of S_3).*

In the case where p lies on the tangent cone of q_i, then L lies on Q_i.

As Eklund studies in detail the K3 surfaces defined by a point lying on the Nieto quintic [Ekl10], we study here those surfaces defined by a point lying on the 10 tangent cones. We first make the following remark:

Remark. The four roots of the equation $f = x^4 + cx^2 + 1$ are of the form

$$\alpha = \frac{1}{2}\left(\sqrt{-c+2} + \sqrt{-c-2}\right).$$

To see this, note that $\alpha^2 = \frac{1}{2}(-c + \sqrt{c^2 - 4})$ which solves $y^2 + cy + 1$.

Proposition 4.1.3. *Let $p \in \mathbb{P}^4$ lie on one of the* 10 *tangent cones to the isolated singular points of N_5, away from N_5 and the* 15 *singular planes. Then the surface X_p contains eight lines. In the case where p lies on a unique tangent cone, X_p contains exactly eight lines.*

Proof. If $p \in \mathbb{P}^4 \setminus N_5$ lies on a unique tangent cone, say q_i, then by Proposition 4.1.2 all the lines lying on X_p must be lines lying on Q_i.

We first prove that when $p = [A, B, C, D, E] \in \mathbb{P}^4$ lies on the tangent cone of the point q_6, there are exactly eight lines lying on $Q_6 \cap X_p \subseteq \mathbb{P}^3$. By the work done in Section 3.1, we have that p satisfies the equation $AB - 2AC + DE = 0$. The quadric $Q_6 : xy + wz = 0$ has the following lines (for any $\alpha \in K^*$)

- $x + \alpha z = y - \alpha^{-1}w = 0$,
- $x + \alpha w = y - \alpha^{-1}z = 0$,
- $x = z = 0$,
- $x = w = 0$,
- $y = z = 0$,
- $y = w = 0$.

Note that the last four lines can not lie on X_p, as p does not lie on the 15 singular planes (hence $A \neq 0$). Now $X_p \cap \{x + \alpha z = y - \alpha^{-1}w = 0\}$ is defined by the equations:

$$\begin{aligned} x + \alpha z &= 0, \\ y - \alpha^{-1}w &= 0, \\ (A\alpha^4 + D\alpha^2 + A)\left(z^4 + \frac{w^4}{\alpha^4}\right) + \left(E\alpha^4 + (2C - B)\alpha^2 + E\right)\frac{z^2w^2}{\alpha^2} &= 0. \end{aligned}$$

As $AB - 2AC + DE = 0$ implies

$$\begin{aligned} E\alpha^4 + (2C - B)\alpha^2 + E &= E\alpha^4 + \frac{DE}{A}\alpha^2 + E \\ &= \frac{E}{A}(A\alpha^4 + D\alpha^2 + A), \end{aligned}$$

we have that the last equation becomes

$$\begin{aligned} & (A\alpha^4 + D\alpha^2 + A)\left(z^4 + \frac{w^4}{\alpha^4}\right) + (A\alpha^4 + D\alpha^2 + A)\frac{Ez^2w^2}{A\alpha^2} \\ = \;& (A\alpha^4 + D\alpha^2 + A)(z^4 + \frac{Ez^2w^2}{A\alpha^2} + \frac{w^4}{\alpha^4}). \end{aligned}$$

This is identically zero if and only $A\alpha^4 + D\alpha^2 + A = 0$. Hence there are exactly four lines of the form $x + \alpha z = y - \alpha^{-1}w = 0$ on X_p, corresponding to the four roots of

$A\alpha^4+D\alpha^2+A=0$. We can run through exactly the same process for lines of the form $x+\alpha w=y-\alpha^{-1}z=0$ and find that this time α needs to solve $A\alpha^4+E\alpha^2+A=0$. Hence by letting

$$\alpha = \frac{\sqrt{A}}{2A}\left(\sqrt{q_{-D}}+\sqrt{-q_{+D}}\right)$$
$$\beta = \frac{\sqrt{A}}{2A}\left(\sqrt{q_{-E}}+\sqrt{-q_{+E}}\right)$$

we have the eight lines

- $x+\alpha z=y-\alpha^{-1}w=0$,
- $x+\alpha^{-1}z=y-\alpha w=0$,
- $x-\alpha z=y+\alpha^{-1}w=0$,
- $x-\alpha^{-1}z=y+\alpha w=0$,
- $x+\beta w=y-\beta^{-1}z=0$,
- $x+\beta^{-1}w=y-\beta z=0$,
- $x-\beta w=y+\beta^{-1}z=0$,
- $x-\beta^{-1}w=y+\beta z=0$,

which lie on our surface X_p, and these are the only lines on $X_p\cap Q_6$.

To finish the proof, we use the action induced by the group Ω acting on $\mathbb{P}^3\times\mathbb{P}^4$ as described in Remark 2.3.3. That is by applying the appropriate element $\phi\in\Omega$ on the above eight lines, we get the equations of the eight lines lying on the surface $X_{\phi(p)}$. We have listed the equations of the lines in Table A.4 in the Appendix. □

Using the fact that the eight lines come from the two different rulings of the quadric (one set using α, the other β), it is not hard to see that the lines come in two sets of four skew lines. Furthermore each line from one set intersects each of the four lines in the other set.

Finally, using the explicit equations, we note that given two (not necessarily distinct) lines in one set, L_1 and L_2, and two in the other set M_1 and M_2, there exists a unique $\gamma\in\Gamma$ interchanging L_1 with L_2 and M_1 with M_2.

We can use Proposition 4.1.3 to find various families containing $8, 16, 24, 32$ and 48 lines.

Lemma 4.1.4.

- *A very general surface in the family* $[A,(DE-2AC)/A,C,D,E]$ *contains exactly* 8 *lines. We denote this family by* $\mathcal{X}_{C,D,E}$,
- *A very general surface in the family* $[A,0,C,D,2AC/D]$ *contains exactly* 16 *lines. We denote this family by* $\mathcal{X}_{C,D}$,
- *A very general surface in the family* $[A,B(2A-B)/A,B,B,B]$ *contains exactly* 24 *lines. We denote this family by* $\mathcal{X}_{B}$,
- *A very general surface in the family* $[A,0,C,0,0]$ *contains exactly* 32 *lines. We denote this family by* $\mathcal{X}_{C}$,

- *The surface* $[\sqrt{-3}, 12(\sqrt{-3}-1), 6, 6, -6]$ *contains exactly* 32 *lines. We denote this surface by* Y*,*

- *The Fermat quartic* $[1, 0, 0, 0, 0]$ *contains exactly* 48 *lines. We denote this surface by* F_4*.*

Up to an action of Ω*, there are no other families whose very general member is smooth and lies on the tangent cones to one of the points* q_i*.*

Proof. Note that for each family, a very general point will not be on N_5, hence if for each family a very general member lie on m distinct tangent cones, then it will contain $8m$ lines as claimed.

Recall that Ω acts on the 10 points q_i (as described in Remark 2.3.3), and hence on the 10 tangent cones. For each $m \in \{1, \ldots, 10\}$, we find representatives of the action of Ω on sets of size m. For example, when $m = 2$, as Ω is two-transitive, we have the representative $\{q_1, q_2\}$, for $m = 3$, we have two representative $\{q_1, q_2, q_3\}$ and $\{q_2, q_4, q_5\}$. Starting from $m = 10$ to 1, for each representative we intersect the corresponding tangent cones. We look at its irreducible components and discard any that is a subset of $\mathcal{L}$ (the union of the 15 singular hyperplanes), any component remaining give us a family that we list. This also verifies that our list is complete. This calculation is available online [Bou]. ☐

We illustrate how the families fit together with Figure 4.1.1. The lines show which family is a subfamily of another family.

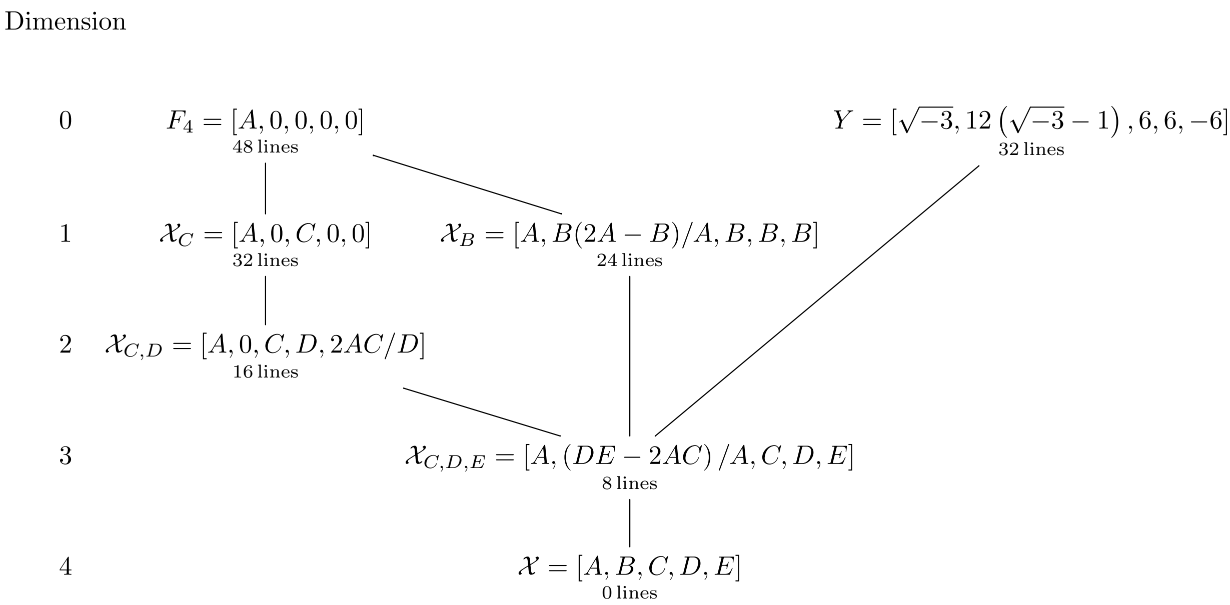


Figure 4.1.1: The various families

4.2 The Picard Group

We now turn to proving that the Picard rank of the families given above are those claimed by (the summarised) Theorem 4.2.8. Note that we already know this to be true for the Fermat quartic, $x^4 + y^4 + z^4 + w^4$ (see for example [AS83]) and the family, $\mathcal{X}$, parameterised by $\mathbb{P}^4$ ([Ekl10]). To achieve this, for each family we will bound the rank from below and above. To bound the Picard rank from below, we use Theorem 2.3.8, that is a very general invariant quartic K3 surface contains 320 conics.

The equations of the conics can be listed explicitly in terms of the point $[A, B, C, D, E] \in \mathbb{P}^4$ associated to the surface X, as explained in Lemma 3.2.1. As the lines and conics are elements of $\mathrm{Pic}(X)$, they form a sublattice of it. Hence by using the explicit equations of the 320 conics and $8m$ lines, we can calculate their intersection matrix. The rank of said matrix, which is the rank of the sublattice generated by the lines and conics, is a lower bound to the rank of the Picard group.

To calculate an upper bound, we use three main ideas:

4.2.1 Reduction at a good prime

The first idea is due to Van Luijk [vL07], which we briefly recap here.

Theorem 4.2.1. *Let X be a* K3 *surface defined over a number field K. Choose a finite prime $\mathfrak{p} \subseteq \mathcal{O}_K$ of good reduction for X. Let $R = (\mathcal{O}_K)_{\mathfrak{p}}$ and k its residue field. Fix an algebraic closure $\overline{K}$ of K, $\overline{R}$ the integral closure of R in $\overline{K}$, and let $\overline{k} = \overline{R}/\mathfrak{p}$ be the algebraic closure of k. There are natural injective homomorphisms*

$$\mathrm{NS}(X_{\overline{K}}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{NS}(X_{\overline{k}}) \otimes \mathbb{Q}_\ell \hookrightarrow H^2_{\text{ét}}(X_{\overline{k}}, \mathbb{Q}_\ell(1))$$

of finite dimensional vector space over $\mathbb{Q}_\ell$. The second injection respects the Galois action $\mathrm{Gal}(\overline{k}/k)$.

Proposition 4.2.2. *Let X be a* K3 *surface defined over a finite field $\mathbb{F}_q$ with $q = p^r$. Let $F_q : X \to X$ be the* absolute Frobenius map *of X, which acts on the identity on points, and by $x \mapsto x^p$ on the structure sheaf. Set $\Phi_q = F_q^r$ and let Φ_q^* denote the automorphism on $H^2_{\text{ét}}(\overline{X}, \mathbb{Q}_\ell)$ induced by $\Phi_q \times 1$ acting on $X_{\overline{\mathbb{F}}_q}$. Then the rank of $NS(X_{\overline{\mathbb{F}}_q})$ is bounded above by the number of eigenvalues λ of Φ_q^* for which λ/q is a root of unity (counted with multiplicity).*

Hence, given a K3 surface over a number field K, its Picard rank, $\rho(X_{\overline{K}})$, is bounded above by eigenvalues in a certain form of Φ_q^*. Such eigenvalues can be read off from the characteristic polynomial, $f_q(x)$, of Φ_q^*. To calculate the characteristic polynomial we use the Lefschetz formula:

$$\mathrm{Tr}\left(\left(\Phi_q^*\right)^i\right) = \#X_k(\mathbb{F}_{q^i}) - 1 - q^{2i},$$

and the following lemma:

Lemma 4.2.3 (Newton's Identity). *Let V be a vector space of dimension n and T a linear operator on V. Let t_i denote the trace of T^i. Then the characteristic polynomial of T is equal to*

$$f_T(x) = \det(x \cdot \mathrm{id} - T) = x^n + c_1 x^{n-1} + c_2 x^{n-2} + \cdots + c_n$$

where the c_i are given recursively by $c_1 = -t_1$ and

$$-kc_k = t_k + \sum_{i=1}^{k-1} c_i t_{k-i}.$$

So in theory, since $n = 22$ as X is a K3 surface, we can calculate the characteristic polynomial by counting points over $\mathbb{F}_{q^i}$ for $i = 1, \ldots, 22$. But this is computationally infeasible. To make the computation more feasible we use the fact that from the Weil conjectures we have the functional equation

$$p^{22} f_q(x) = \pm x^{22} f_q(p^2/x).$$

Second of all, in our cases we have an explicit submodule $M \subseteq \mathrm{NS}X_{\overline{k}}$ of rank r, namely the one generated by the lines and conics lying on X. Hence we can calculate the characteristic polynomial $f_M(x)$ of Frobenius acting on M. Since $f_M(x)|f_q(x)$, we can compute two possible polynomials $f_{q,+}(x)$ and $f_{q,-}(x)$ (one for each possible sign in the functional equation) by counting points on $X_k(\mathbb{F}_{q^i})$ for $i = 1, \ldots (22-r)/2$. Explicitly, suppose $f_M(x) = \prod_j g_j(x)^{e_j}$ with $\deg(g_j) = d_j$, hence $\sum d_j e_j = r$. Note that $f_q'(x) = f_M(x)h'(x) + f_M'(x)h(x)$, hence if $e_j > 1$ then $g_j(x)|f_q'(x)$, and in general $g_j(x)$ divides the $(e_j - 1)$th derivative of $f_q(x)$. Therefore, we can use the roots of M to construct $r/2$ linear equations in the 11 coefficients of $f_q(x)$ (by assuming $f_q(x)$ satisfies one of the functional equation). Hence we just need to count points on $X_k(\mathbb{F}_{q^i})$ for $i = 1, \ldots (22-r)/2$ to be able to use linear algebra and find the 11 coefficients of $f_q(x)$. Note that when we assume the negative functional equations, we have in fact only 10 coefficients of $f_q(x)$, as $c_{11} = 0$. Hence, we end up not using all the information from $f_M(x)$, therefore it is possible to construct $f_q(x)$ such that $f_M(x) \nmid f_q(x)$. This is a contradiction, meaning that $f_q(x)$ satisfies the positive functional equation and not the negative.

Finally, note that by rescaling $f_q(x)$ by $f_q(x/p)$, we just need to count the roots which are also roots of unity.

4.2.2 Artin-Tate conjecture

Unfortunately, as the roots come in conjugate pairs, the above method can only ever give an even upper bound. The following proposition can potentially reduce the upper bound by one more than the above bound.

Proposition 4.2.4. *Let X be a* K3 *surface defined over a number field K and let $\mathfrak{p}$ and $\mathfrak{p}'$ be two primes of good reductions. Suppose that $\rho\left(\overline{X}_{\mathfrak{p}}\right) = \rho\left(\overline{X}_{\mathfrak{p}'}\right) = n$ but the discriminants* $\mathrm{Disc}(\mathrm{NS}\left(\overline{X}_{\mathfrak{p}}\right))$ *and* $\mathrm{Disc}(\mathrm{NS}\left(\overline{X}_{\mathfrak{p}'}\right))$ *are different in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$. Then $\rho\left(\overline{X}\right) < n$.*

Proof. By the above, we know that $\rho\left(\overline{X}\right) \leq n$. If $\rho\left(\overline{X}\right) = n$, then $\mathrm{NS}\left(\overline{X}\right)$ is a full rank sublattice of $\mathrm{NS}\left(\overline{X}_{\mathfrak{p}}\right)$ and $\mathrm{NS}\left(\overline{X}_{\mathfrak{p}'}\right)$. But in that case, as elements of $\mathbb{Q}^*/(\mathbb{Q}^*)^2$, all three discriminants should be equal, which is a contradiction to the hypothesis. □

As the proposition requires us to calculate the discriminants of $\mathrm{NS}(\overline{X}_{\mathfrak{p}})$ and $\mathrm{NS}(\overline{X}_{\mathfrak{p}'})$ we use the following conjecture:

Conjecture 4.2.5 (Artin - Tate)**.** *Let X be a* K3 *surface over a finite field $\mathbb{F}_q$. Let ρ and* Disc *denote respectively the rank and discriminant of the Picard group defined over $\mathbb{F}_q$. Then*

$$|\mathrm{Disc}| = \frac{\lim_{T\to q} \frac{\Phi(T)}{(T-q)^{\rho}}}{q^{21-\rho}\#\mathrm{Br}(X)}.$$

Here Φ is the characteristic polynomial of Frob *on $H^2_{\text{ét}}(X_{\overline{\mathbb{F}}_q}, \mathbb{Q}_l)$. Finally,* $\mathrm{Br}(X)$ *is the Brauer group of X.*

In the case when q is odd, then the above conjecture has been proven to be true (using the fact that it follows from the Tate Conjecture [Mil75] which has been proven for K3 surfaces [Nyg83, NO85, Cha13, Mau14, MP15]). Furthermore in the case the conjecture is true we have that $\#\,\mathrm{Br}(X)$ is a square. Hence, by picking q large enough so that $\rho(X_q) = \rho(\overline{X}_q)$, we can find $|\mathrm{Disc}|$ as an element of $\mathbb{Q}^*/\left(\mathbb{Q}^*\right)^2$.

4.2.3 Only finitely many singular K3 surfaces

Suppose that a general member of the family $\mathcal{Y}$ has Picard rank at least 19 and the family $\mathcal{Y}$ is parameterised by a one dimensional curve. The third idea uses the fact that, up to $\overline{\mathbb{Q}}$-isomorphism, there only finitely many K3 surfaces over $\mathbb{Q}$ which are singular, i.e., with Picard rank 20. Hence if a very general member of $\mathcal{Y}$ has Picard rank 20, then every member of $\mathcal{Y}$ is singular. Therefore $\mathcal{Y}$ parametrises a set of isomorphic surfaces. If we can show that there are two $\mathbb{Q}$-surfaces in $\mathcal{Y}$ which are not isomorphic, then a very general member of the family $\mathcal{Y}$ has Picard rank at most 19 (as it can not be 20).

We implement this by noting that in each of the cases we are interested in, the Fermat quartic, F_4, belongs to our family $\mathcal{Y}$. Furthermore the Fermat quartic is supersingular over algebraically closed fields of characteristic 3 mod 4, i.e., $\rho\left(\overline{F_{4,p}}\right) = 22$ for $p \equiv 3 \mod 4$ [Tat65]. Hence if there is another surface in $\mathcal{Y}$ with $\rho\left(\overline{X}_p\right) = 20$ over a prime $p \equiv 3 \mod 4$, then F_4 and X are not isomorphic (since their specialisations to the field $\mathbb{F}_p$ are not isomorphic, as they have different Picard rank).

With all these tools we tackle the following proposition:

Proposition 4.2.6.

- *A very general surface in the family $\mathcal{X}$ has Picard rank* 16*,*
- *A very general surface in the family $\mathcal{X}_{C,D,E}$ has Picard rank* 17*,*
- *A very general surface in the family $\mathcal{X}_{C,D}$ has Picard rank* 18*,*
- *A very general surface in the family $\mathcal{X}_B$ has Picard rank* 19*,*
- *A very general surface in the family $\mathcal{X}_C$ has Picard rank* 19*,*
- *The surface Y is singular.*

Proof. To get the lower bound we want to calculate the intersection matrix of the conics and lines lying on a very general member of each family. The lines and conics are defined over a degree 2^{10} field extension, hence calculating the intersection matrix is computationally infeasible. Instead we do the calculations over finite fields. Pick X in one of the families (call it $\mathcal{X}_*$) and let p be a prime of good reduction. Then we know that the conics and lines of $X_{\mathbb{F}_p}$ are defined over $\mathbb{F}_{p^2}$ (due to having explicit equations and there are only two square classes in $\mathbb{F}_p$) and so we calculate with ease the intersection matrix. By Theorem 4.2.1 $\mathrm{NS}(X_{\overline{\mathbb{Q}}}) \otimes \mathbb{Q}_\ell \hookrightarrow \mathrm{NS}(X_{\overline{\mathbb{F}_p}}) \otimes \mathbb{Q}_\ell$ is injective, so the intersection matrix of the lines and conics over $\mathbb{F}_{p^2}$ is the same as the intersection matrix of the lines and conics over $\overline{\mathbb{Q}}$. Furthermore, as the set of surfaces in $\mathcal{X}_*$ which reduce to $X_{\mathbb{F}_p}$ is Zariski open, the intersection matrix calculated is the same as the intersection matrix of a very general member of $\mathcal{X}_*$.

As the intersection matrix is a large matrix, we have included in Appendix A.5 a full rank minor of the matrix for each family (in particular, the lower bound is the dimension of said minor). We work (see [Bou]) through the families in reverse order from the list above.

- As M_Y has rank 20, we know that $\rho(Y) = 20$ and hence the surface Y is singular.
- As M_C has rank 19, we know that a very general surface X_C of $\mathcal{X}_C$ has $\rho(X_C) \geq 19$. Using the idea in Subsection 4.2.3 we see that the surface X_0, associated to the point $[1, 0, 0, 0, 0]$, is the Fermat quartic so it is supersingular over $\overline{\mathbb{F}}_{19}$. On the other hand consider the surface X_5, associated to the point $[1, 0, 5, 0, 0]$, over $\mathbb{F}_{19}$. The characteristic polynomial of Frobenius acting on conics and lines on X_5 is $f(x) = (x-1)^{10}(x+1)^9$. Hence we just need to count points over $\mathbb{F}_{19}$ and $\mathbb{F}_{19^2}$ to find the two possible characteristic polynomials for Φ_{19}^*. We find, after rescaling, $f_{19,+}(x) = \frac{1}{19}(x-1)^{10}(x+1)^{10}(19x^2 - 22x + 19)$ and a contradiction

for $f_{19,-}(x) = \frac{1}{19}(x-1)^9(x+1)^9(19x^4 - 22x^3 - 22x + 19)$ as $f(x) \nmid f_{19,-}(x)$. As X_2 is not supersingular, X_0 and X_2 are not isomorphic over $\overline{\mathbb{F}}_{19}$. Therefore a very general surface in $\mathcal{X}_C$ has Picard number 19.

- As M_B has rank 19, we know that a very general surface X_B of $\mathcal{X}_B$ has $\rho(X_B) \geq 19$. Using the idea in Subsection 4.2.3 we see that X_2, associated to the point $[1,0,0,0,0]$, is the Fermat quartic so is supersingular over $\overline{\mathbb{F}}_{19}$. On the other hand consider the surface X_1, associated to the point $[1,1,1,1,1]$, over $\mathbb{F}_{19}$. The characteristic polynomial of Frobenius acting on conics and lines on X_1 is $f(x) = (x-1)^{16}(x+1)^3$. After point counting over $\mathbb{F}_{19}$ and $\mathbb{F}_{19^2}$ we find the possible two characteristic polynomials for Φ_{19}^*, namely $f_{19,+}(x) = \frac{1}{19}(x-1)^{16}(x+1)^4(19x^2 - 18x + 19)$ and a contradiction for $f_{19,-}(x) = \frac{1}{19}(x-1)^{15}(x+1)^3(19x^4 - 18x^3 - 18x + 19)$. As X_1 is not supersingular, X_2 and X_1 are not isomorphic over $\overline{\mathbb{F}}_{19}$. Therefore a very general surface in $\mathcal{X}_B$ has Picard number 19.

- As $M_{C,D}$ has rank 18, we know that a very general surface $X_{C,D}$ of $\mathcal{X}_{C,D}$ has $\rho(X_{C,D}) \geq 18$. We use the idea in Subsection 4.2.1 and find a surface whose reduction at a prime p gives an upper bound of 18. To make point counting easier, we will work over $\mathbb{F}_{13}$ and the surface $X_{4,1}$, associated to the point $[1,0,4,1,8]$. Our first step is to calculate the characteristic polynomial of Frobenius acting on conics and lines, which is $f(x) = (x-1)^{10}(x+1)^8$. After point counting over $\mathbb{F}_{13}$ and $\mathbb{F}_{13^2}$ we find the two possible characteristic polynomials for Φ_{13}^*, namely $f_{13,+}(x) = \frac{1}{13}(x-1)^{10}(x+1)^8(13t^4 + 12t^3 + 14t^2 + 12t + 13)$ and a contradiction for $f_{13,-}(x) = \frac{1}{13}(x-1)^9(x+1)^9(13t^4 - 14t^3 + 16t^2 - 14t + 13)$ (since $f(x) \nmid f_{13,-}(x)$). Hence $\rho(\overline{X}_{4,1}) \leq 18$, so a very general surface in $\mathcal{X}_{C,D}$ has Picard number 18.

- As $M_{C,D,E}$ has rank 17, we know that a very general surface $X_{C,D,E}$ of $\mathcal{X}_{C,D,E}$ has $\rho(X_{C,D,E}) \geq 17$. We use the idea in Subsection 4.2.1 and find a surface whose reduction at two primes p and p' gives an upper bound of 18. We work with the surface $X_{3,5,7}$, associated to the point $[1,29,3,5,7]$, over the fields $\mathbb{F}_{13}$ and $\mathbb{F}_{19}$. The characteristic polynomial of Frobenius acting on conics and lines over $\mathbb{F}_{13}$ is $f_{13}(x) = (x-1)^8(x+1)^9$ and over $\mathbb{F}_{19}$ is $f_{19}(x) = (x-1)^9(x+1)^8$. We find the following possible characteristic polynomials (after rescaling):

	f_+	f_-
$\mathbb{F}_{13}$	$\frac{1}{13}(x-1)^8(x+1)^{10}(13x^4 + 22x^2 + 13)$	$\frac{1}{13}(x-1)^9(x+1)^9(13x^4 + 26x^3 + 48x^2 + 26x + 13)$
$\mathbb{F}_{19}$	$\frac{1}{19}(x-1)^{10}(x+1)^8(19x^4 + 32x^3 + 42x^2 + 32x + 19)$	$\frac{1}{19}(x-1)^9(x+1)^9(19x^4 - 6x^3 + 16x^2 - 6x + 19)$

We then apply the idea in Subsection 4.2.2, by working over $\mathbb{F}_{13^2}$ and $\mathbb{F}_{19^2}$. We find that, up to squares, the discriminants are as follow:

	$\lvert\mathrm{Disc}_+\rvert$	$\lvert\mathrm{Disc}_-\rvert$
$\mathbb{F}_{13^2}$	13	$13 \cdot 17 \cdot 61$
$\mathbb{F}_{19^2}$	18691	75011

As these four discriminants are all different elements in $\mathbb{Q}^*/(\mathbb{Q}^*)^2$ we have $\mathrm{Disc}\left(\mathrm{NS}(X_{3,5,7,\mathbb{F}_{13^2}})\right) \neq \mathrm{Disc}\left(\mathrm{NS}(X_{3,5,7,\mathbb{F}_{19^2}})\right)$ and so a very general surface in $\mathcal{X}_{C,D,E}$ has Picard number 17.

- As M has rank 16, we know a very general surface X of $\mathcal{X}$ has $\rho(X) \geq 16$. We use the idea in Subsection 4.2.1 and find a surface whose reduction at a prime p gives an upper bound of 16. We work over $\mathbb{F}_{19}$ and let X be the surface defined by the point $[1, 2, 7, 11, 13]$. We calculate that the characteristic polynomial of Frobenius acting on conics and lines is $f(x) = (x-1)^8(x+1)^8$, hence we need to count points over $\mathbb{F}_{19}$, $\mathbb{F}_{19^2}$ and $\mathbb{F}_{19^3}$ to find the two possible characteristic polynomials for Φ_{19}^*. We find that, after rescaling

$$f_{19,+}(x) = \frac{1}{19}(x-1)^8(x+1)^8(19x^6 + 10x^5 + 29x^4 + 12x^3 + 29x^2 + 10x + 19),$$

and a contradiction for

$$f_{19,-}(x) = \frac{1}{19}(x-1)^7(x+1)^9(19x^6 - 28x^5 + 47x^4 - 64x^3 + 47x^2 - 28x + 19),$$

as $f(x) \nmid f_{19,-}(x)$. Hence a very general surface in $\mathcal{X}$ has Picard number 16.

$\square$

Now that we know the rank of the Picard group of a very general member of each family, we can prove the following proposition:

Proposition 4.2.7. *For a very general member of the families $\mathcal{X}, \mathcal{X}_{C,D,E}, \mathcal{X}_{C,D}, \mathcal{X}_B$ and $\mathcal{X}_C$, as well as the Fermat quartic, F_4, the Picard groups are generated by lines and conics.*

In particular the matrices $M, M_{C,D,E}, M_{C,D}, M_B$ and M_C as defined in Appendix A.5 define the Picard group of a very general member of the families $\mathcal{X}, \mathcal{X}_{C,D,E}, \mathcal{X}_{C,D}, \mathcal{X}_B$ and $\mathcal{X}_C$ respectively.

Proof. First note that if $L_1 \hookrightarrow L_2$ is primitive, then no overlattice L' of L_1 can be a sublattice of L_2. Let $\mathcal{X}$ and $\mathcal{Y}$ are two families of K3 surfaces, with $\mathcal{Y}$ a subfamily of $\mathcal{X}$. If X and Y denote a very general member of $\mathcal{X}$ and $\mathcal{Y}$, then $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(Y)$ as the elements of $\mathrm{Pic}(X)$ must specialise to elements of $\mathrm{Pic}(Y)$.

With this in mind we start with the proven fact (see for example [SSvL10]) that the Picard group of the Fermat quartic, denoted by $\mathrm{Pic}(F_4)$, is generated by lines. Upon calculating the Gram matrix of the 48 lines on F_4, we find that the Picard group has discriminant -64. On the other hand we calculate the following Gram matrix, which we denote M_{F_4}, generated by 16 conics and four lines (each line coming from a different set of eight lines associated to a point q_i):

$$\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 1 & 0 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 0 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 0 & 1 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 & 0 & 1 & 1 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 & 1 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -2 & 0 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -2 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & -2
\end{pmatrix}.$$

M_{F_4} has determinant -64 and hence does represent $\mathrm{Pic}(F_4)$.

Let X_C be a very general surface in $\mathcal{X}_C$. We extracted the matrix M_C, from the intersection matrix of the lines and conics on X_C, by looking at the lines and conics which specialise to a subset of the 16 conics and four lines that lie on the Fermat quartic (which makes sense since $X_0 \in \mathcal{X}_C$ is the Fermat quartic). We ended up with 16 conics and three lines (which must come from three different sets of eight lines) and hence we have a rank 19, i.e. full rank, sublattice of $\mathrm{Pic}(X_C)$. Notice that M_C is a minor of M_{F_4} (just remove the last row and column), and the lines and conics generating M_C specialise to those generating the corresponding minor of M_{F_4}. Hence the lattice defined by M_C is a primitive sublattice of $\mathrm{Pic}(F_4)$. If M_C did not define $\mathrm{Pic}(X_C)$ then $\mathrm{Pic}(X_C)$ would be an overlattice of M_C. Furthermore by the remark at the beginning of the proof $\mathrm{Pic}(X_C)$ would be a sublattice of $\mathrm{Pic}(F_4)$. This is a contradiction to the fact that M_C is already a primitive sublattice of $\mathrm{Pic}(F_4)$. Hence the lattice defined by M_C, which is generated by lines and conics, is $\mathrm{Pic}(X_C)$.

Similarly we extracted $M_{C,D}$ from the intersection matrix using M_C (and note it is a minor of M_C by removing the last row and column), $M_{C,D,E}$ using $M_{C,D}$ (a minor of

$M_{C,D}$ by removing the last row and column) and M using $M_{C,D,E}$. Hence by the same argument, they represent respectively $\mathrm{Pic}(X_{C,D})$, $\mathrm{Pic}(X_{C,D,E})$ and $\mathrm{Pic}(X)$. Finally, we extracted M_B from F_4 using the same process (and notice it is a minor of F_4 by removing column and row 18), finishing the proof. □

Notation. Let M, N be matrices, then we use M^N to mean $N^T \cdot M \cdot N$.

We now have all the tools to prove our main result.

Theorem 4.2.8. *Let* $p = [A, B, C, D, E] \in \mathbb{P}^4$ *define the quartic* $X_p : A(x^4 + y^4 + z^4 + w^4) + Bxyzw + C(x^2y^2 + z^2w^2) + D(x^2z^2 + y^2w^2) + E(x^2w^2 + y^2z^2) \subset \mathbb{P}^3$. *Then*

- *A very general member of family parameterised by* $\mathbb{P}^4$ *contains no lines, has Picard rank* 16 *and Picard group isomorphic to* $E_8\langle -1\rangle \oplus U \oplus D_5\langle -2\rangle \oplus \langle -4\rangle$,
- *A very general member of the family parameterised by* $[A, (DE-2AC)/A, C, D, E]$ *contains exactly eight lines, has rank* 17 *and Picard group isomorphic to* $E_8\langle -1\rangle \oplus U \oplus A_2\langle -2\rangle \oplus (D_4\langle -1\rangle \oplus \langle -2\rangle)^N$, *with*

$$N = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{pmatrix},$$

- *A very general member of the family parameterised by* $[A, 0, C, D, 2AC/D]$ *contains exactly* 16 *lines, has rank* 18 *and Picard group isomorphic to* $E_8\langle -1\rangle \oplus U \oplus A_7\langle -1\rangle^{I_{4,2}} \oplus \langle -8\rangle$, *with* $I_{4,2} = \mathrm{Diag}([1,1,1,2,1,1,1])$,
- *A very general member of the family parameterised by* $[A, B(2A-B)/A, B, B, B]$ *contains exactly* 24 *lines, has rank* 19 *and Picard group isomorphic to* $E_8\langle -1\rangle \oplus U \oplus D_8\langle -1\rangle \oplus \langle -40\rangle$,
- *A very general member of the family parameterised by* $[A, 0, C, 0, 0]$ *contains exactly* 32 *lines, has rank* 19 *and Picard group isomorphic to* $E_8\langle -1\rangle \oplus U \oplus D_8\langle -1\rangle^{I_{8,2}} \oplus \langle -8\rangle$ *with* $I_{8,2} = \mathrm{Diag}([1,1,1,1,1,1,1,2])$,
- *The surface defined by the point* $[\sqrt{-3}, 12(\sqrt{-3}-1), 6, 6, -6]$ *contains exactly* 32 *lines, has rank* 20 *and Picard group isomorphic either to* $E_8\langle -1\rangle^{\oplus 2} \oplus U \oplus \langle -4\rangle \oplus \langle -24\rangle$ *or to* $E_8\langle -1\rangle^{\oplus 2} \oplus U \oplus \langle -4\rangle \oplus \langle -6\rangle$ *(but not both),*
- *The Fermat quartic defined by the point* $[1, 0, 0, 0, 0]$ *contains exactly* 48 *lines, has rank* 20 *and Picard group isomorphic to* $E_8\langle -1\rangle^{\oplus 2} \oplus U \oplus \langle -8\rangle \oplus \langle -8\rangle$.

Possibly except for the point $[\sqrt{-3}, 12(\sqrt{-3}-1), 6, 6, -6]$, *the Picard group is generated by the lines and conics lying on the surface.*

Proof. The claim about the number of lines each very general member contains is proven in Lemma 4.1.4 while the rank is proven in Proposition 4.2.6. Apart from the surface defined by $[\sqrt{-3}, 12(\sqrt{-3}-1), 6, 6, -6]$, the claim that the Picard group of a very general member is defined by lines and conics is proven by Proposition 4.2.7.

As all the Picard groups are even and indefinite, and in each case the rank is large enough, we can apply Theorem 2.1.2 to each of our Picard groups. Specifically one can check (see [Bou]) that the discriminant form, rank and signature of the lattices defined by $M, M_{C,D,E}, M_{C,D}, M_B, M_C$ and M_{F_4} are the same as the discriminant form, rank and signature of the lattice

- $E_8\langle -1\rangle \oplus U \oplus D_5\langle -2\rangle \oplus \langle -4\rangle$,
- $E_8\langle -1\rangle \oplus U \oplus A_2\langle -2\rangle \oplus (D_4\langle -1\rangle \oplus \langle -2\rangle)^N$,
- $E_8\langle -1\rangle \oplus U \oplus A_7\langle -1\rangle^{I_{4,2}} \oplus \langle -8\rangle$,
- $E_8\langle -1\rangle \oplus U \oplus D_8\langle -1\rangle \oplus \langle -40\rangle$,
- $E_8\langle -1\rangle \oplus U \oplus D_8\langle -1\rangle^{I_{8,2}} \oplus \langle -8\rangle$,
- and $E_8\langle -1\rangle^{\oplus 2} \oplus U \oplus \langle -8\rangle \oplus \langle -8\rangle$ respectively.

For the surface Y, defined by $[\sqrt{-3}, 12(\sqrt{-3}-1), 6, 6, -6]$, the lattice defined by M_Y is isomorphic to $E_8\langle -1\rangle^{\oplus 2} \oplus U \oplus \langle -4\rangle \oplus \langle -24\rangle$. While we don't know that the lattice defined by M_Y is the Picard group of Y, we know that it is a full rank sublattice of it. One can then use Theorem 2.1.4 to find all overlattices of it, of which there is only one, and use Theorem 2.1.2 to identify said lattice using its discriminant form, rank and signature. $\square$

Recall that we had a diagram, Figure 4.1.1, illustrating the various subfamilies of $\mathcal{X}$ containing lines and how they fitted together. In Figure 4.2.1 we reproduce the same diagram, where instead of the families, we put together the Picard group of the generic member of each family (except for the surface Y, where we put the two possible Picard groups), and instead of the dimension of each family we put the rank of the Picard group.

4.2.4 Method

We include here two examples of how the isomorphic lattices were found for Theorem 4.2.8, which the reader might find useful. Those two examples illustrate the two different approaches we took in identifying the lattices.

We start with the lattice defined by M, i.e. the Picard group of a very general member X of $\mathcal{X}$. We know that M has signature $(1, 15)$ and rank 16. We calculate its discriminant group to be $C_2^4 \times C_4 \times C_8$, and $\mathrm{Pic}(X)$ has discriminant -512 (this concurs

$$
\begin{tikzcd}[column sep=small]
\text{rank} & & & \\
20 & \substack{E_8\langle -1\rangle^{\oplus 2}\oplus U\oplus\langle -8\rangle\oplus\langle -8\rangle \\ \text{Discriminant } -2^6} & \substack{E_8\langle -1\rangle^{\oplus 2}\oplus U\oplus\langle -4\rangle\oplus\langle -6\rangle \\ \text{Discriminant } -2^3\cdot 3} & \substack{E_8\langle -1\rangle^{\oplus 2}\oplus U\oplus\langle -4\rangle\oplus\langle -24\rangle \\ \text{Discriminant } -2^5\cdot 3} \arrow[l, hook', "\textbf{index 2}"'] \\
19 & \substack{E_8\langle -1\rangle\oplus U\oplus D_8\langle -1\rangle^{I_{8,2}}\oplus\langle -8\rangle \\ \text{Discriminant } 2^7} \arrow[u, hook, "\text{primitive}"'] & \substack{E_8\langle -1\rangle\oplus U\oplus D_8\langle -1\rangle\oplus\langle -40\rangle \\ \text{Discriminant } 2^5\cdot 5} \arrow[ul, hook, "\text{primitive}"] & \\
18 & \substack{E_8\langle -1\rangle\oplus U\oplus A_7\langle -1\rangle^{I_{4,2}}\oplus\langle -8\rangle \\ \text{Discriminant } -2^8} \arrow[u, hook, "\text{primitive}"'] & & \\
17 & & \substack{E_8\langle -1\rangle\oplus U\oplus A_2\langle -2\rangle\oplus(D_4\langle -1\rangle\oplus\langle -2\rangle)^N \\ \text{Discriminant } 2^7\cdot 3} \arrow[ul, hook, "\text{primitive}"] \arrow[uu, hook, "\text{primitive}"'] \arrow[uuur, hook, "\text{primitive}"'] & \\
16 & & \substack{E_8\langle -1\rangle\oplus U\oplus D_5\langle -2\rangle\oplus\langle -4\rangle \\ \text{Discriminant } -2^9} \arrow[u, hook, "\text{primitive}"'] &
\end{tikzcd}
$$

Figure 4.2.1: Lattices of the families

with the proof of [Ekl10, Thm 7.3, Cor 7.4]). By Theorem 2.1.3, we see that we can fit in one copy of $E_8(-1)$ and one copy of U in Pic(X), i.e., $\mathrm{Pic}(X) = U \oplus E_8\langle -1\rangle \oplus T$ where T is a lattice with the same discriminant group and discriminant form as Pic(X), but with signature $(0,6)$.

Recall that A_L denotes the discriminant group of a lattice L, and q_L its discriminant form. We calculate the discriminant form and find that:

- If $x \in A_{\mathrm{Pic}(X)}$ has order 2 then $q_{\mathrm{Pic}(X)}(x) \in \{0,1\}$,
- If $x \in A_{\mathrm{Pic}(X)}$ has order 4 then $q_{\mathrm{Pic}(X)}(x) \in \{-\frac{3}{4}, -\frac{2}{4}, -\frac{1}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\}$,
- If $x \in A_{\mathrm{Pic}(X)}$ has order 8 then $q_{\mathrm{Pic}(X)}(x) \in \{-\frac{7}{8}, -\frac{5}{8}, \frac{1}{8}, \frac{3}{8}\}$.

As the lattice $\langle -4\rangle$ has discriminant form $-\frac{1}{4}$ and discriminant group C_4, we guess that it appears as one of the summands of Pic(X). Using Table 2.1 and the fact we need negative definite lattices, we see that the C_8 factor could arise from $A_7\langle -1\rangle$, $A_3\langle -2\rangle$, $\langle -8\rangle$ or $D_{2n+1}\langle -2\rangle$. As $A_7\langle -1\rangle$ has too large of a rank (greater than six), and both $A_3\langle -2\rangle$ and $\langle -8\rangle$ have an element of order 8 with discriminant form $-\frac{3}{8}$ and $-\frac{1}{8}$ respectively, they can not be a factor of Pic(X). On the other hand, $D_5\langle -2\rangle$ does not give any obvious contradiction while having discriminant group $C_2^4 \times C_8$. We guess that it is a factor of Pic(X). Hence putting everything together we check that $\mathrm{Pic}(X) \cong U \otimes E_8\langle -1\rangle \otimes D_5\langle -2\rangle \otimes \langle -4\rangle$. It is easy to see they have the same rank and signature; and a calculation checks they have the same discriminant form, namely both discriminant group have a basis $\{g_1, \dots, g_6\}$ such that the discriminant form is given by

$$
\begin{aligned}
M_{q_L}(a_{ij}) &= \begin{cases} q_L(g_i+g_j) & i \neq j \\ q_L(g_i) & i = j \end{cases} \\
&= \begin{pmatrix} 0 & 1 & 1 & 1 & -\frac{1}{4} & -\frac{5}{8} \\ 1 & 1 & 1 & 0 & \frac{3}{4} & -\frac{5}{8} \\ 1 & 1 & 1 & 0 & \frac{3}{4} & -\frac{5}{8} \\ 1 & 0 & 0 & 0 & -\frac{1}{4} & \frac{3}{8} \\ -\frac{1}{4} & \frac{3}{4} & \frac{3}{4} & -\frac{1}{4} & -\frac{1}{4} & -\frac{7}{8} \\ -\frac{5}{8} & -\frac{5}{8} & -\frac{5}{8} & \frac{3}{8} & -\frac{7}{8} & -\frac{5}{8} \end{pmatrix}.
\end{aligned}
$$

Our second example is with the lattice defined by M_C, i.e. the Picard group of a very general member X_C of $\mathcal{X}_C$. We know that M_C has signature $(1,18)$ and rank 19. We calculate that Pic(X_C) has discriminant 128 and discriminant group $C_4^2 \times C_8$. By Theorem 2.1.3, we know that $\mathrm{Pic}(X_C) \cong E_8\langle -1\rangle \oplus U \oplus T$, where T is a lattice of signature $(0,9)$ with discriminant group $C_4^2 \times C_8$ and discriminant form as:

- If $x \in A_{\mathrm{Pic}(X_C)}$ has order 2 then $q_{\mathrm{Pic}(X_C)}(x) \in \{0\}$,

- If $x \in A_{\mathrm{Pic}(X_C)}$ has order 4 then $q_{\mathrm{Pic}(X_C)}(x) \in \{-\frac{1}{2}, 0, \frac{1}{2}, 1\}$,
- If $x \in A_{\mathrm{Pic}(X_C)}$ has order 8 then $q_{\mathrm{Pic}(X_C)}(x) \in \{-\frac{5}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{7}{8}\}$.

As there is no negative definite lattice in Table 2.1 which gives a copy of C_4 without giving an element of discriminant form $\frac{2n+1}{4}$ for some n, we deduce that T can not be written simply in terms of scaled root lattices. Instead we use Theorem 2.1.4 to find an overlattice of $\mathrm{Pic}(X_C)$ that we can identify. In particular, if we let $\{e_i\}$ be the basis given by M_C, then $\frac{1}{2}(e_4 + e_5 + e_{10} + e_{11} + e_{13} + e_{14}) \in A_{\mathrm{Pic}(X_C)}$ has order two and discriminant form zero. This generates an isotropic subgroup of $A_{\mathrm{Pic}(X_C)}$ and gives a corresponding index two overlattice. This overlattice, L, has discriminant group $C_2^2 \times C_8$ and discriminant form given as:

- If $x \in A_L$ has order 2 then $q_L(x) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$,
- If $x \in A_L$ has order 4 then $q_L(x) \in \{-\frac{1}{2}, 0, \frac{1}{2}\}$,
- If $x \in A_L$ has order 8 then $q_L(x) \in \{-\frac{5}{8}, -\frac{1}{8}, \frac{3}{8}, \frac{7}{8}\}$.

Following our first example this allows us to guess that $L \cong E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle \oplus \langle -8 \rangle$. We check that is the case, as they both have rank 19, signature $(1, 18)$ and discriminant form given by

$$
\begin{aligned}
M_{q_L}(a_{ij}) &= \begin{cases} q_L(g_i + g_j) & i \neq j \\ q_L(g_i) & i = j \end{cases} \\
&= \begin{pmatrix} 0 & 1 & -\frac{1}{8} \\ 1 & 0 & \frac{7}{8} \\ -\frac{1}{8} & \frac{7}{8} & \frac{7}{8} \end{pmatrix}.
\end{aligned}
$$

Knowing that $\mathrm{Pic}(X)$ is an index two full rank sublattice of $E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle \oplus \langle -8 \rangle$, we enumerate the index two full rank sublattices of $E_8 \langle -1 \rangle \oplus U \oplus D_8 \langle -1 \rangle \oplus \langle -8 \rangle$ until we find one that has the same discriminant form as $\mathrm{Pic}(X)$.

Chapter 5

Examples of CM Curves of Genus two Defined over the Reflex Field

In this chapter, based on the collaborative paper [BS15], we assume that the field k has characteristic different from $2, 3, 5$.

We say that a curve C/k of genus g has *complex multiplication* (CM) if the endomorphism ring of its Jacobian over $\overline{k}$ contains an order in a number field K of degree $2g$. A *CM-field* is a totally imaginary quadratic extension K of a totally real number field K_0. If C has CM with $K \cong \mathrm{End}(\mathrm{Jac}(C)_{\overline{k}})$ then K is a CM-field.

It is known that there exists exactly 13 j-invariants giving elliptic curves over $\mathbb{Q}$ with CM, see for example [Cox13, Thm 7.30ii]. Analogously, Van Wamelen [vW99a] gives a list of 19 curves of genus two over $\mathbb{Q}$ with CM by a maximal order. We extend Van Wamelen's list by finding genus two hyperelliptic curves defined over certain real quadratic extensions of $\mathbb{Q}$. Our main result is:

Theorem. *For every row of the Tables 1a, 1b, and 2b on pages 79–89, let* $K = \mathbb{Q}[X]/(X^4 + AX^2 + B)$, *where* $[D, A, B]$ *is as in the first column of the table. Then the curves* $C : y^2 = f(x)$ *where* f *is as in the last column are exactly all curves with complex multiplication by the maximal order of* K*, up to isomorphism over* $\overline{\mathbb{Q}}$ *and up to automorphism of* $\overline{\mathbb{Q}}$.

The number a *that may appear in the coefficients of* f *is as follows. In table 1b, let* $D' = D$*, and in table 2b, let* $[D', A', B']$ *be as in the second column. Let* $\epsilon \in \{0, 1\}$ *be* D' *modulo 4. Then* a *is a root of* $x^2 + \epsilon x + (\epsilon - D')/4 = 0$.

Section 5.3 contains the more detailed Theorem 5.3.2, including an explanation of the other columns. Recently Pinar Kilicer and Marco Streng proved the completeness of the list [KS15]. That is, they proved that the first columns of Table 1a, 1b, and 2b contain exactly the quartic fields K for which there exists a curve C of genus two with $\mathrm{End}(\mathrm{Jac}(C)) = \mathcal{O}_K$ such that C is defined over the real quadratic subfield of the reflex field.

The first section briefly recaps the Igusa invariants of genus two hyperelliptic curves and Mestre's algorithm. The second section is dedicated to the theory of reducing the coefficients of the models we found. This is needed since Mestre's algorithm gives models for genus two hyperelliptic curves with coefficients of hundreds of digits. This is unpractical, so we use a reduction algorithm to reduce the coefficient size. The final section gives a more precise statement of the above theorem, as well as the curves found by applying our algorithm to the fields in the Echidna database [K$^+$07].

5.1 Invariants

In the same way that the j-invariant uniquely specifies isomorphism classes of elliptic curves over $\overline{k}$, the Igusa Invariants classify genus two hyperelliptic curves. These were introduced by Igusa [Igu60].

Let C/k be a hyperellitic curve of genus two, that is $C : y^2 = f(x)$ where $f \in k[x]$ has degree 6 and disctinct roots $\alpha_1, \dots, \alpha_6$. Let a_6 be the leading coefficient of f. For any $\sigma \in S_6$ let (ij) denote the difference $(\alpha_{\sigma(i)} - \alpha_{\sigma(j)})$. Then we can define the *Igusa Invariants* as the following, where the sums are taken over the distinct expressions as σ ranges in S_6:

$$\begin{aligned}
I_2 &= a_6^2 \sum_{15\,\text{terms}} (12)^2(34)^2(56)^2, \\
I_4 &= a_6^4 \sum_{10\,\text{terms}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2, \\
I_6 &= a_6^6 \sum_{60\,\text{terms}} (12)^2(23)^2(31)^2(45)^2(56)^2(64)^2(14)^2(25)^2(36)^2, \\
I_{10} &= a_6^{10} \prod_{i<j} (\alpha_i - \alpha_j)^2.
\end{aligned}$$

For two curves C and C', we have that $C_{\overline{k}} \cong C'_{\overline{k}}$ if and only if there exists $\lambda \in \overline{k}^*$ such that $I_j(C) = \lambda^j I_j(C')$ for $j = 2, 4, 6, 10$. Note that the Igusa Invariants are available in the software packages Magma ([BCP97]) and Sage ([S$^+$13]).

Given a quartic CM-field K there exists polynomials, called Igusa class polynomials, which allows us to recover $I_j(C)$ for all C with CM by $\mathcal{O}_K$. Details on how to compute these can be found in [Str10] and [BS15]. For now we assume that for each quartic CM-field, we have the list of Igusa Invariants defined over a number k. We want to decide whether there is a curve of genus two over k with those Igusa Invariants, and if so compute a model of the form $y^2 = f(x)$, $f \in k[x]$, for that curve.

Let k be any field of characteristic not 2, 3, or 5. Let

$$\mathcal{M}_2(\overline{k}) = \{(x_2, x_4, x_6, x_{10}) \in \overline{k}^4 \mid x_{10} \neq 0\}/\overline{k^*},$$

where $\lambda \in \overline{k^*}$ acts by a weighted scaling $\lambda(x_2, x_4, x_6, x_{10}) = (\lambda^2 x_2, \lambda^4 x_4, \lambda^6 x_6, \lambda^{10} x_{10})$.

We say that a point $x \in \mathcal{M}_2(\overline{k})$ is defined over k if $x \in \mathcal{M}_2(\overline{k})$ is stable under the action of $\mathrm{Gal}(\overline{k}/k)$. One can show (using Hilbert's Theorem 90) that this condition is satisfied if and only if x is the equivalence class of a quadruple with $x_n \in k$ for all $n \in [2, 4, 6, 10]$. The *field of moduli* k_0 of $C/\overline{k}$ is the smallest field over which the point $x = (I_n(C))_n \in \mathcal{M}_2(\overline{k})$ is defined. We say that a field $l \subset \overline{k}$ is a *field of definition* for C if there exists a curve D/l with $D_{\overline{k}} \cong C$.

Unlike the elliptic case, there is no simple formula for C given $(I_n(C))_n$, and C cannot always be defined over its field of moduli. Mestre [Mes91] designed an algorithm which finds a model for C given x, but it involves solving a conic, which is not always possible without extending the field. When it is possible to solve a conic over the base field, then it usually introduces large numbers, so that the output polynomial may have coefficients that are much too large to be practical.

In more detail, Mestre's algorithm works as follows. First of all, assume that the curve C with $x = (I_n(C))_n$ does not have any automorphisms other than the hyperelliptic involution $\iota : (x, y) \mapsto (x, -y)$. (If it does, then use the construction of Cardona and Quer [CQ05] instead of Mestre's.) From the coordinates x_n in the field of moduli k_0, one constructs homogeneous ternary forms $Q = Q_x$ and $T = T_x \in k_0[U, V, W]$ of degrees 2 and 3 (for equations, see [Mes91] or [SBa]). Let $M_x \subset \mathbb{P}^2$ be the conic defined by Q. If M_x has a point over a field $k \supset k_0$, then this gives rise to a parametrisation $\varphi : \mathbb{P}^1 \to M_x$ over k. Let $\varphi^* : k[U, V, W] \to k[X, Z]$ be the ring homomorphism inducing this parametrisation. We get a hyperelliptic curve $C_\varphi : Y^2 = \varphi^*(T)$, i.e., $C_\varphi : y^2 = T(\varphi(x : 1))$. The curve C_φ is a double cover of $\mathbb{P}^1$, ramified at the six points of $\mathbb{P}^1$ that map (under φ) to the six zeroes of T_x on M_x.

Theorem 5.1.1. *Given $x \in \mathcal{M}_2(k)$, assume that the curve $C/\overline{k}$ with $x = (I_n(C))_n$ satisfies* $\mathrm{Aut}(C) = \{1, \iota\}$.

- *If $M_x(k) = \emptyset$, then C has no model over k,*
- *If $M_x(k) \neq \emptyset$, then C_φ/k as above is a model of C.*

We use Magma [BCP97] to solve conics over number fields and we contributed our Sage implementation of Mestre's algorithm to Sage [S$^+$13], where it is available (as of version 5.13) through the command HyperellipticCurve_from_Invariants.

5.2 Reduction

In the previous section we described Mestre's algorithm for finding models of genus two curves over a number field k. However, these hyperelliptic models in practice have coefficients of hundreds of digits. In this section we describe how we make hyperelliptic curve equations over k smaller. We start by explaining the relation between twists of hyperelliptic curves and an action of $\mathrm{GL}_2(k) \times k^*$ on binary forms. The rest of the section is then about $(\mathrm{GL}_2(k) \times k^*)$-reduction of binary forms, and our algorithm consists of two parts:

1. Making a binary form integral with discriminant of small norm (Section 5.2.2),
2. Making the heights of the coefficients small by $(\mathrm{GL}_2(\mathcal{O}_k) \times \mathcal{O}_k^*)$-transformations, which preserve integrality and affect the discriminant only by units (Section 5.2.3).

We give the reduction algorithm for binary forms of general degree n, though it only applies to hyperelliptic curves in the case that n is even and greater than six.

5.2.1 Isomorphisms and twists

Fix an integer $n \geq 3$ and a field k, and let $H_n(k)$ be the set of separable binary forms of degree n in $k[X, Z]$. We interpret $F(X, Z) \in H_n(k)$ also as the pair $(n, f(x))$, where $f(x) = F(x, 1) \in k[x]$ is a polynomial of degree n or $n-1$. In the case where n is even and greater than six, let $g = (n-2)/2$ and interpret F as the hyperelliptic curve $C = C_f = C_F$ of genus g given by the affine equation $y^2 = f(x)$. We can also write C as the smooth curve given by $Y^2 = F(X, Z)$ in weighted projective space $\mathbb{P}^{(1,g+1,1)}$.

Given any element of $H_{2g+2}(k)$, we would like to find an isomorphic hyperelliptic curve with coefficients of small height, so first we determine when two hyperelliptic curves are isomorphic.

Note the natural right group actions of scaling and substitution for any n,

$$\begin{array}{lll} H_n(k) \circlearrowright k^* & : & (F(X,Z), u) \mapsto uF(X,Z), \quad \text{and} \\ H_n(k) \circlearrowright \mathrm{GL}_2(k) & : & (F(X,Z), A) \mapsto F(A \cdot (X,Z)), \end{array}$$

which together induce an action of $\mathrm{GL}_2(k) \times k^*$ on $H_n(k)$.

In terms of the polynomial $f(x) = F(x, 1) \in k[x]$, the action is

$$f(x) \cdot \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}, u \right] = u\ (cx+d)^n f\left(\frac{ax+b}{cx+d}\right).$$

Note that a hyperelliptic curve C always has the identity automorphism and the *hyperelliptic involution* $\iota : C \to C : (x, y) \mapsto (x, -y)$. We will often assume that these are the only automorphisms.

Proposition 5.2.1. *Given any two F, $F^\dagger \in H_{2g+2}(k)$, assume* $\mathrm{Aut}((C_F)_{\overline{k}}) = \{1, \iota\}$. *Then C_F and $C_{F^\dagger}$ are isomorphic over $\overline{k}$ if and only if F and $F^\dagger$ are in the same orbit under* $\mathrm{GL}_2(k) \times k^*$.

Proof. It is a standard result (see e.g. [CF96, p. 1] for the case of genus two) that two hyperelliptic curves C_F and $C_{F^\dagger}$ in $H_n(k)$ are isomorphic over k if and only if they are in the same orbit under $\mathrm{GL}_2(k) \times (k^*)^2$. Using $\mathrm{Aut}(C_{\overline{k}}) = \{1, \iota\}$, we get (see e.g. [HS00, Example C.5.1]) that all twists, up to isomorphisms over k, are given by the action of $H^1(k,\{1,\iota\}) = k^*/k^{*2} = \{1\} \times (k^*/k^{*2})$. $\square$

Remark. If $F^\dagger = F \cdot [(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}), v^2]$, then an isomorphism $C_{F^\dagger} \to C_F$ is given by $(x,y) \to (\frac{ax+b}{cx+d}, v^{-1}(cx+d)^{-g-1}y)$.

By Proposition 5.2.1, finding small-height models over k of hyperelliptic curves C/k with $\mathrm{Aut}(C_{\overline{k}}) = \{1,\iota\}$ is equivalent to finding small elements of $\mathrm{GL}_2(k) \times k^*$-orbits of binary forms of even degree ≥ 6. By [BS15, Lem 5.6] the hypothesis $\mathrm{Aut}(C_{\overline{k}}) = \{1,\iota\}$ is satisfied for the curves we deal with, except for one curve for which we do not need a reduction algorithm. If $\mathrm{Aut}(C_{\overline{k}}) \neq \{1,\iota\}$, then $\mathrm{GL}_2(k) \times k^*$-actions may be too restrictive, but by Remark 5.2.1, they do always give valid twists.

Our goal for the remainder of Section 5.2 is, given a binary form $F \in H_n(k)$, to find a $\mathrm{GL}_2(k) \times k^*$-equivalent form with small coefficients. We start by computing a discriminant-minimal form in Section 5.2.2, followed by discriminant-preserving $\mathrm{GL}_2(\mathcal{O}_k) \times \mathcal{O}_k^*$-reduction in Section 5.2.3.

5.2.2 Reduction of the discriminant

Given a binary form $F(X,Z) \in k[X,Z]$ of any degree $n \geq 3$, we wish to find a $\mathrm{GL}_2(k) \times k^*$-equivalent form with minimal discriminant. First we recall that the discriminant of a separable binary form

$$F(X,Z) = \prod_{i=1}^{n} (\gamma_i X - \alpha_i Z) \in k[X,Z]$$

with $\alpha_i, \gamma_i \in \overline{k}$ is

$$\Delta(F) = \prod_{i<j} (\gamma_j \alpha_i - \gamma_i \alpha_j)^2 \in k^*.$$

In terms of the polynomial $f = F(x,1)$ of degree n or $n-1$ with leading coefficient c, this is

$$\Delta(F) = \begin{cases} \Delta(f) & \text{if } \deg f = n, \\ c^2 \Delta(f) & \text{if } \deg f = n-1. \end{cases}$$

Let $g \in \mathbb{Z}$ be given by $n = 2g+2$ if n is even and $n = 2g+3$ if n is odd. If n is even and ≥ 6, then F corresponds to a hyperelliptic curve C_F of genus g with

$$\Delta(C_F) = 2^{4g} \Delta(F).$$

If n is odd, then there is no interpretation in terms of hyperelliptic curves and the number g is simply a convenient number in the algorithms and proofs.

The discriminant changes under the action of the group $\mathrm{GL}_2(k) \times k^*$ via

$$\Delta(F \cdot [A,u]) = u^{2(n-1)} \det(A)^{n(n-1)} \Delta(F). \tag{5.2.1}$$

Remark. In case $n = 6$, the Igusa Invariants of Section 5.1 satisfy $I_{10}(C) = 2^{12}\Delta(C) = 2^{20}\Delta(F)$ and

$$I_j(C_{F\cdot[A,u]}) = u^j \det(A)^{3j} I_j(C_F).$$

Before we describe how to reduce the discriminant globally over a number field, we first outline how to reduce the discriminant at just one prime.

5.2.2.1 Local reduction of the discriminant

Assume for now that k is the field of fractions of a discrete valuation ring R with valuation v. Let π be a uniformiser of v and $\mathfrak{m} = \pi R$ the maximal ideal.

We call F *minimal at* v if $v(\Delta(F))$ is minimal among all $\mathrm{GL}_2(k) \times k^*$-equivalent forms with v-integral coefficients.

Proposition 5.2.2. *Suppose $F \in H_n(k)$ has coefficients in R. Let $g = \lfloor n/2 \rfloor - 1$ be the largest integer smaller than or equal to $(n-2)/2$, so $n \in \{2g+2, 2g+3\}$. Then F is* non-*minimal at v if and only if we are in one of the following three cases:*

1. *The polynomial F is not primitive, so $F^\dagger = F \cdot [\mathrm{id}_2, \pi^{-1}]$ is integral and satisfies $v(\Delta(F^\dagger)) < v(\Delta(F))$,*

2. *The polynomial $(F(x,1) \bmod \mathfrak{m})$ has a $(g+2)$-fold root $\bar{t}$ in the residue field. Moreover, for* some *(equivalently* every*) lift $t \in R$ of $\bar{t}$, the form $F^\dagger = F \cdot [(\begin{smallmatrix} \pi & t \\ 0 & 1 \end{smallmatrix}), \pi^{-(g+2)}] = F(\pi X + tZ, Z)\pi^{-(g+2)}$ is integral and satisfies $v(\Delta(F^\dagger)) < v(\Delta(F))$,*

3. *The polynomial $(F(x,1) \bmod \mathfrak{m})$ has degree $\leq n - (g+2)$. Moreover, the form $F^\dagger = F \cdot [(\begin{smallmatrix} 1 & 0 \\ 0 & \pi \end{smallmatrix}), \pi^{-(g+2)}] = \pi^{-(g+2)} F(X, \pi Z)$ is integral and satisfies $v(\Delta(F^\dagger)) < v(\Delta(F))$.*

Proof. For the "if" part, note that in each of the three cases, the proposition gives an explicit equivalent form that proves that F is not minimal.

Conversely, suppose that F is non-minimal. Then there exists $[A, u] \in \mathrm{GL}_2(k) \times k^*$ with $F \cdot [A, u]$ integral of smaller discriminant. Write

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Let T be the subgroup $T = \{[\mu \mathrm{id}_2, \mu^{-n}] : \mu \in k^*\}$ of the centre of $\mathrm{GL}_2(k) \times k^*$, and note that T acts trivially on $H_n(k)$, so without loss of generality A has coprime coefficients in R. So either

(i) $c \in R^*$ or $d \in R^*$, or

(ii) $c \equiv d \equiv 0 \bmod \pi$ and either a or b is in R^*.

Note also that $\mathrm{GL}(R) \times R^*$ preserves integrality and the discriminant, so we use multiplication by $\mathrm{GL}(R)$ on the right to perform elementary column operations over R on A. Hence we get that without loss of generality either

(i) $d = 1$, $c = 0$, or

(ii) $a = 1$, $b = 0$, $c \equiv d \equiv 0 \bmod \pi$.

Note that in both cases $a \neq 0$ and $d \neq 0$, so with more $\mathrm{GL}(R) \times R^*$-multiplication, we get $a = \pi^k$, $d = \pi^l$, $u = \pi^{-m}$ with k, l, $m \in \mathbb{Z}$, k, $l \geq 0$, and by equation (5.2.1) also

$$2m > n(k + l). \tag{5.2.2}$$

We start with case (i).

Let $H(X, Z) = F(X+bZ, Z)$ and write $H(X, Z) = \sum_i h_i X^i Z^{n-i}$. Then $F\cdot[A, u] = \pi^{-m}H(\pi^k X, Z)$ is integral, so $v(h_i) \geq m - ki$. Together with (5.2.2), this gives

$$v(h_i) > \left(\frac{n}{2} - i\right) k.$$

In particular, if $k = 0$, then H is integral and non-primitive, hence so is $F(X, Z) = H(X - bZ, Z)$ and we are in case 1.

If $k \geq 1$, then for all i, we have $v(h_i) > \frac{n}{2} - i$, hence $v(h_i) > \lfloor n/2 \rfloor - i = g + 1 - i$, so $v(h_i) \geq g + 2 - i$. In particular, the form

$$F \cdot \left[\begin{pmatrix} \pi & b \\ 0 & 1 \end{pmatrix}, \pi^{-(g+2)}\right] = H \cdot \left[\begin{pmatrix} \pi & b \\ 0 & 1 \end{pmatrix}, \pi^{-(g+2)}\right]$$

is integral, and of strictly smaller discriminant than F. This proves that we are in case 2 for *some* lift $t = b$ of a $(g + 2)$-fold root $\overline{t} = \overline{b}$. To finish the proof of case 2, we need to prove that for *every* t' satisfying $\overline{t'} = \overline{b}$, the transformation $\left[\begin{pmatrix} \pi & b \\ 0 & 1 \end{pmatrix}, \pi^{-(g+2)}\right]$ also gives an integral equation. Let $y = (t' - b)/\pi \in \mathcal{O}_k$ and note

$$\begin{pmatrix} \pi & t' \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \pi & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \in \begin{pmatrix} \pi & b \\ 0 & 1 \end{pmatrix} \mathrm{GL}_2(R),$$

which proves that we are in case 2 for *every* lift t. This finishes case (i).

Now assume that we are in case (ii). Equation (5.2.2) gives $m > \frac{n}{2} \geq g + 1$. Write $F = \sum_{i=0}^n f_i X^i Z^{n-i}$. We will prove by induction that $v(f_j) \geq j + g + 2 - n$ holds for all j, which implies that $F(X, \pi Z)\pi^{-(g+2)}$ is integral, so we are in case 3. Note that the assertion is trivial for $j \leq n - g - 2$. Now suppose that it is true for all $j < J$. Then $F \cdot [A, u] = \pi^{-m}F(X, cX + dZ)$ is integral, so modulo π^{g+2}, we get $0 \equiv \sum_{i=0}^n f_i X^i (cX + dZ)^{n-i} \equiv \sum_{i=J}^n f_i X^i (cX + dZ)^{n-i}$. Looking at the coefficient of $X^J Z^{n-J}$, we get $f_J d^{n-J} \equiv 0 \bmod \pi^{g+2}$, so $\pi^{g+2-n+J} \mid f_J$. This finishes the proof. $\square$

We use Proposition 5.2.2 to create the following reduction algorithm.

Algorithm 5.2.3.
Input: *A binary form* $F \in H_n(k) \cap R[X,Y]$ *and a prime element* $\pi \in R$.
Output: *A binary form* $F^\dagger$ *that is* $\mathrm{GL}_2(k) \times k^*$*-equivalent to* F *and minimal at* ord_π.
First let $g = \lfloor n/2 \rfloor - 1$.

1. *If* F *mod* πR *is zero, then repeat the algorithm with* $F^\dagger = F \cdot [\mathrm{id}_2, \pi^{-1}]$. *(This corresponds to case 1. of 5.2.2.)*

2. *If* $F(x,1)$ *mod* πR *has degree* $\leq n-(g+2)$, *then let* $F^\dagger = F(X, \pi Z)\pi^{-(g+2)}$. *If* $F^\dagger$ *is integral, then repeat the algorithm with* $F^\dagger$. *(This corresponds to case 3. of 5.2.2.)*

3. *Factor* $\overline{f} = (f \bmod \pi)$ *over the finite* $R/\pi R$. *If* $\overline{f}$ *has a root* $\overline{t}$ *of multiplicity* $\geq g+2$, *then let* t *be a lift of* $\overline{t}$ *to* R. *If* $F^\dagger = F(\pi X + tZ, Z)\pi^{-(g+2)}$ *is integral, then repeat the algorithm with* $F^\dagger$. *(This corresponds to case 2. of 5.2.2.)*

4. *Return* F.

Proof of correctness of Algorithm 5.2.3. Every step of the algorithm leaves the model integral, and every iteration reduces $v(\Delta(F))$, so the algorithm terminates. It therefore suffices to prove that the output is not in any of the three cases of Proposition 5.2.2.

In case 1, the algorithm reduces the discriminant in step 1 and starts over. In case 3, the same happens with step 2, and in case 2, it happens with step 3 because a polynomial of degree $\leq 2g+3$ has at most one $(g+2)$-fold root $\overline{t}$. □

In many cases, we can do step 3 as follows without having to think about factoring of polynomials.

Lemma 5.2.4. *If* π *is coprime to* $n!$, *then step 3 can be replaced by the following.*

3'. *Let* $f = F(x,1)$, *calculate* $\gcd(f, f', f'', \ldots, f^{(g+1)})$ *over the finite field* $R/\pi R$, *and write it as* $\sum_{i=0}^{s} a_i x^s$ *with* $a_s \neq 0$. *If* $s > 0$, *then let* t *be such that* $t \equiv -a_{s-1}/(s a_s)$ *mod* πR. *If* $F^\dagger = F(\pi X + tZ, Z)\pi^{-(g+2)}$ *is integral, then repeat the algorithm with* $F^\dagger$.

Proof. It suffices to show that if $\overline{f}$ has a root $\overline{t}$ of multiplicity $\geq g+2$, then it is equal to $(-a_{s-1}/(s a_s) \bmod \pi R)$.

Let a be a root of exact multiplicity m of $\overline{f}$ over the algebraic closure of $R/\pi R$, that is, we have $\overline{f} = (x-a)^m g(x)$ with $g(a) \neq 0$. Then the i-th derivative $\overline{f}^{(i)}$ for $i \leq m$ is

$$\frac{m!}{(m-i)!}(x-a)^{m-i} g(x) \mod (x-a)^{m-i+1}.$$

In particular, $(x-a)$ is a factor of $\gcd(\overline{f}, \overline{f}', \ldots, \overline{f}^{(m-1)})$, but not of $\overline{f}^{(m)}$. Here we use that $m!$ is coprime to π.

It follows that only the (unique) root of multiplicity $\geq g+2$ appears in $\gcd(\overline{f}, \overline{f}', \ldots, \overline{f}^{(g+1)})$, that is, we get $\gcd = a_s(x - \overline{t})^s$, hence $a_{s-1} = -s\overline{t}a_s$, so $\overline{t} = -a_{s-1}/(sa_s)$. □

5.2.2.2 Global reduction of the discriminant

Now let us get back to the case where k is a number field with ring of integers $\mathcal{O}_k$. We prefer to have a binary form F where $v(\Delta(F))$ is minimal for all discrete valuations v of k.

If k has class number one, then such a form exists. Indeed, if we take π in Algorithm 5.2.3 to be a generator of the prime ideal corresponding to v, then this affects only v and no other valuations, so we can do this for each v separately. See Section 5.2.2.3 for what to do if the class group is non-trivial.

To be able to use our local reduction algorithm one prime at a time, we need to know the valuations v for which $v(\Delta(F))$ is non-minimal. The most straightforward method is to factor $\Delta(F)$. However, factorisation is computationally hard, so we will give some tricks for trying to avoid factorisation below. We needed to use a combination of sophisticated factorisation software and the tricks below for creating our tables. Indeed, on the one hand, without the tricks below, even the state-of-the-art factorisation software left us unable to reduce a couple of the curves. On the other hand, when just using the tricks below and the built-in factorisation functionality of pari-gp [PAR11] (through Sage [S$^+$13]), there are some curves that we were still unable to reduce. Only the combination of factoring software and the tricks below allowed us to complete the table.

For serious factoring, we combined the built-in implementation of Pollard's rho method and the elliptic curve method of Magma [BCP97], the GMP-ECM implementation of the elliptic curve method [Z$^+$12], and the CADO-NFS implementation of the number field sieve [GKM$^+$].

The method for avoiding factorisation is based on the following proposition.

Proposition 5.2.5. *Let $\mathfrak{a} = \pi\mathcal{O}_k$ be any (possibly non-prime) principal ideal in a number field k. Modify Algorithm 5.2.3 as follows.*

1. *Whenever testing whether an element b of $\mathcal{O}_k$ is zero modulo $\pi^j\mathcal{O}_k = \mathfrak{a}^j$ or whether an element $b/\pi^j \in k$ is integral (in steps 1, 2, and 3), compute $\mathfrak{d}_i = \gcd(b\mathcal{O}_k, \mathfrak{a}^i)$ for $i = 1, \ldots j-1$. If there exists an i with $\mathfrak{d}_i \notin \{\mathfrak{a}^{i-1}, \mathfrak{a}^i\}$, then for the smallest such i output the non-trivial factor $\mathfrak{d}_i/\mathfrak{a}^{i-1}$ of $\mathfrak{a}$.*

2. *Replace step 3 with step 3' of Lemma 5.2.4 regardless of whether π is coprime to $n!$. Compute* gcd*s of polynomials in $\mathcal{O}_k/\mathfrak{a}$ using Euclid's algorithm. For each division with remainder by a polynomial g, first compute the gcd of the leading coefficient of g with $\mathfrak{a}$ as in item 1.*

Then all steps of Algorithm 5.2.3 are polynomial-time computable and the output is either a polynomial $F^\dagger$ equivalent to F with $\Delta(F^\dagger) \mid \Delta(F)$ or a non-trivial factor of $\mathfrak{a}$. Moreover, if $\mathfrak{a}$ is square-free and coprime to $n!$ and the algorithm runs without returning a factor of $\mathfrak{a}$, then the output polynomial $F^\dagger$ is minimal at all primes dividing $\mathfrak{a}$.

Proof. Since the leading coefficient of a polynomial over $\mathcal{O}_k$ is either invertible modulo $\mathfrak{a}$ or has a non-trivial factor in common with $\mathfrak{a}$, division with remainder either works or provides such a non-trivial factor. This proves the first assertion in Proposition 5.2.5.

Next, suppose that $\mathfrak{a}$ is square-free and coprime to $n!$ and let F be as in Algorithm 5.2.3. If F is minimal at all primes dividing $\mathfrak{a}$, then we are done. If there is an $i \in \{1, 2, 3\}$ such that all primes dividing $\mathfrak{a}$ are as in case i. of Proposition 5.2.2, then the corresponding step (1, 3' or 2) in Algorithm 5.2.3 reduces the discriminant of F and we start over with a new F.

So without loss of generality, there are $i \in \{1, 2, 3\}$ and primes $\mathfrak{p}, \mathfrak{q} \mid \mathfrak{a}$ such that $\mathfrak{p}$ is as in case i. of Proposition 5.2.2 and $\mathfrak{q}$ is not. But then the corresponding step (1, 3' or 2) in Algorithm 5.2.3 returns a non-trivial factor of $\mathfrak{a}$. □

Based on Proposition 5.2.5, we get the following algorithm that tries to minimise the amount of factoring.

Algorithm 5.2.6.
Input: *A binary form $F \in H_n(k)$ for a number field k of class number one.*
Output: *A binary form $F^\dagger$ that is integral, is $\mathrm{GL}_2(k) \times k^*$-equivalent to F, and has minimal discriminant.*

1. *Let $\mathfrak{a} = \Delta(F)\mathcal{O}_k$ and $A = \{\mathfrak{a}\}$.*

2. *If the unit ideal is in A, remove it from A. If A is empty, return F.*

3. *For each $\mathfrak{a} \in A$, test if $\mathfrak{a}$ is a perfect power and replace it by its highest-power root.*

4. *Fix $B \in \mathbb{Z}$ with $B \geq n$ and apply trial division up to B to each element of A to find a small prime factor $\mathfrak{p} = (\pi)$. If no prime is found, go to Step 5. If a prime is found, then reduce the form locally using Algorithm 5.2.3 on $\mathfrak{p}$, remove all factors $\mathfrak{p}$ from all elements of A, and go to step 2.*

5. *For each $\mathfrak{a} \in A$, run Algorithm 5.2.3 on $\mathfrak{a}$ with the modifications of Proposition 5.2.5.*

    (a) *If it returns a non-trivial factor $\mathfrak{b}$ of $\mathfrak{a}$, then replace $\mathfrak{a}$ in A by $\mathfrak{b}$ and $\mathfrak{a}/\mathfrak{b}$ and go to step 3.*

    (b) *If it returns a binary form $F^\dagger \neq F$, then replace all $\mathfrak{a} \in A$ by $\mathfrak{a} + \Delta(F^\dagger)\mathcal{O}_k$, replace F by $F^\dagger$, and go to step 2.*

(c) If it returns F, then go to the next $\mathfrak{a}$ in A.

6. *Go to step 4 with a strictly larger trial division bound B (or more sophisticated factoring methods).*

Let us first show that this algorithm terminates in finite time and returns a minimal form. For minimality of the form, note that at every step in the algorithm, all primes at which F is non-minimal divide some element of A, and the algorithm terminates only if A is empty. To see that the algorithm ends, note that the norm $N = N_{k/\mathbb{Q}}(\Delta(F))$ never increases, while at every iteration either $N \in \mathbb{Z}$ decreases or $B \in \mathbb{Z}$ increases, so at some point we have $B > N$ after which a repeated application of step 4 finishes the algorithm.

Remark. There is no way to completely avoid factoring. Indeed, let p, q be large primes and consider the hyperelliptic curve

$$y^2 = f(x) = N^2x^6 + x + 1 \quad \text{where } N = pq^2,$$

which has discriminant $\Delta(f) = (5^2 - 6^6n^2)n^8$. Then for most p, q the twist-reduced hyperelliptic curve is

$$y^2 = f^\dagger(x) = f(x/p)p^2 = q^2x^6 + px + p^2$$

which has discriminant $\Delta(f^\dagger) = (5^5 - 6^6n^2)p^6q^8 = \Delta(f)/p^{10}$. Hence, reducing f is equivalent to finding $p = \sqrt[10]{\Delta(f)/\Delta(f^\dagger)}$ and factorising N.

Remark. In the genus two case (that is, $n = 6$) we can replace $\Delta(F)\mathcal{O}_k$ in the algorithm by the ideal $\gcd(I_2(C_F), I_4(C_F), I_6(C_F), \Delta(F))$, where I_2, I_4, I_6 are the Igusa-Clebsch invariants from Section 5.1. Indeed, we have that I_2, I_4, and I_6 satisfy the transformation formula of the remark after Equation 5.2.1, so all primes at which the model is non-minimal divide this gcd. The advantage is that this ideal is smaller than $\Delta(F)$, which speeds up the algorithm.

Remark. All of the above works if one wants a hyperelliptic curve model that is isomorphic over $\overline{k}$, but not necessarily over k. To get a minimal model of C_F that is isomorphic over k, one could do the following. First reduce F as above, and do some bookkeeping to find not only a twist-reduced model $C_{F^\dagger}/k$, but also $[A, u] \in \mathrm{GL}_2(k) \times k^*$ with $F^\dagger = F \cdot [A, u]$ and some information on the factorisation of u. Then all one needs is a minimal element $v \in u(k^*)^2 \cap \mathcal{O}_k$, because $C_{vF^\dagger}$ is then a minimal model. Such an element v exists if k has class number one, and can then be easily found if one is able to factor $u\mathcal{O}_k$.

5.2.2.3 Class number > 1

Everything in Section 5.2.2.2 is under the assumption that k had class number one, and hence a global minimal form exist. If k does not have class number one, then this is not always possible. Indeed, let F_v be a $\mathrm{GL}_2(k) \times k^*$-equivalent binary form with $v(\Delta(F_v))$ minimal, and let $\Delta_{\min}$ be the ideal with $v(\Delta_{\min}) = v(\Delta(F_v))$ for all v. If $\Delta_{\min}$ is not principal, then there is no form with that discriminant. In fact, if F is any form, and there exists a globally minimal equivalent form $F_{\min}$ with $\Delta(F_{\min}) = \Delta_{\min}$, then the ideal $\sqrt[\gcd(n,2)(n-1)]{\Delta(F)/\Delta_{\min}}$ is a principal ideal.

So instead of a globally reduced form, we look for an almost reduced form. Let S be a (small) set of (small) prime ideals that generate the class group. It is easy to change the methods above into an algorithm that finds a form that is reduced outside S. We now give the details of the algorithm that we used for this, which also makes the form reasonably simple at the primes of S.

Let T be any set of prime ideals that generate the class group and $\mathfrak{a}$ an ideal supported outside T. In Algorithm 5.2.3, to reduce at $\mathfrak{a}$ and stay reduced outside of T, we do the following. Take $\pi_u \in \mathfrak{a}$ and $\pi_l^{-1} \in \mathfrak{a}^{-1}$ such that $\pi_u/\mathfrak{a}$ and $\mathfrak{a}/\pi_l$ are supported on T. Then in that algorithm replace the formulas for $F^\dagger$ in cases 1, 2, 3 with

$$\pi_l^{-1} F(X,Z), \quad F(X/\pi_l, Z)\pi_u^{n-(g+2)}, \quad \text{and} \quad F(\pi_u X + tZ, Z)\pi_l^{-(g+2)} \tag{5.2.3}$$

respectively, where we make sure that t is divisible by $\pi_u/\mathfrak{a}$. Note that this gives integral forms, and worsens the discriminant only at T.

Our algorithm starts by taking T disjoint from S. First reduce at all primes of S, possibly worsening at T. Then take $T = S$ and reduce outside of S, possibly worsening at S.

Since we had a minimal form at the primes of S, the only non-minimality of the form at this stage is what was introduced by (5.2.3). In particular, it can be removed by transformations of the form $a^{-1}b^g F(b^{-1}X, Z)$. So we take $a, b \in \mathcal{O}_k$ with $a^2 b^{n-2g}$ of maximal norm such that $a^{-1}b^g F(b^{-1}X, Z)$ is integral. Note that no hard factoring is required in finding a and b since they are supported on the set of primes S.

We did the above for the field $K = \mathbb{Q}[X]/(X^4+46X^2+257)$ (denoted by $[17, 46, 257]$ in [K+07]). We used $S = \{\mathfrak{p}\}$ for a (non-principal) prime $\mathfrak{p}$ of norm 2 in the quadratic field $K_0^r = \mathbb{Q}(\sqrt{257})$, which has class group of order 3.

5.2.3 Reduction of coefficients: Stoll-Cremona reduction

At this point, we have an integral form $F \in H_n(k)$ where the norm $N(\Delta(F))$ is small. Next, we try to make the coefficients small. As we do not want to break integrality nor disturb the discriminant, we take transformations in $(\mathrm{GL}_2(\mathcal{O}_k) \times \mathcal{O}_k^*)$.

We use a notion of 'reduced' based on Stoll and Cremona [SC03]. We do not prove that this notion of 'reduced' yields small coefficients, but in practice it does.

5.2.3.1 The case $k = \mathbb{Q}$

Stoll and Cremona [SC03, Definition 4.3] give a definition of *reduced* for binary forms of degree ≥ 3 over $\mathbb{Q}$ under the action of $\mathrm{SL}_2(\mathbb{Z}) \times 1$, which we will summarise here.

Recall that $H_n(k)$ is the set of separable binary forms $F(X,Y)$ of degree n. Let $\mathcal{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$ be the complex upper half plane. We turn the standard left $\mathrm{GL}_2^+(\mathbb{R})$-action on $\mathcal{H}$ into a right action by

$$z \cdot A = A^{-1}(z) = \frac{dz - b}{-cz + a}$$

for $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$.

The idea behind [SC03] is to use an $\mathrm{SL}_2(\mathbb{R})$-covariant map $z : H_n(\mathbb{R}) \to \mathcal{H}$, which we do not explain here. In $\mathcal{H}$, there is a notion of $\mathrm{SL}_2(\mathbb{Z})$-reduction, and we just pull back that notion to $H_n(\mathbb{Q})$ via z. In other words, we have the following definition.

Definition 5.2.7. We call $F \in H_n(\mathbb{Q})$ *reduced for* $\mathrm{SL}_2(\mathbb{Z})$ if $z(F) = z = x+iy$ satisfies

(R) $|x| \leq \frac{1}{2}$, and

(M) $|z| \geq 1$.

This gives rise to the following algorithm.

Algorithm 5.2.8 (Stoll-Cremona reduction).
Input: $F \in H_n(\mathbb{Q})$
Output: *an* $\mathrm{SL}_2(\mathbb{Z})$*-reduced element of the orbit* $F \cdot (\mathrm{SL}_2(\mathbb{Z}) \times 1)$.

1. *Let m be the integer nearest to $x = \mathrm{Re}(z(F))$ and let $F \leftarrow F \cdot \left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right) = F(X + mZ, Z)$.*
   This replaces $z(F)$ with $\left(\begin{smallmatrix} 1 & -m \\ 0 & 1 \end{smallmatrix}\right) z(F) = z(F) - m$, which satisfies (R) above.

2. *If $|z(F)| < 1$, then let $F \leftarrow F \cdot \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) = F(Z, -X)$ and go back to step 1. This replaces $z(F)$ with $\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right) z(F) = -1/z(F)$, which satisfies (M) above.*

Stoll and Cremona [SC03, after Proposition 4.4] outline how one could extend the definition of reduced to binary forms over any number field k under the action of $\mathrm{SL}_2(\mathcal{O}_k) \times 1$. We work out the details in the case of a totally real field, and give an implementation and an improvement.

To generalise the algorithm, we need two ingredients: a covariant map, and a reduction algorithm on the codomain of that map.

5.2.3.2 The covariant for totally real fields

Let k be a totally real number field of degree d and let $\phi_1, \ldots, \phi_d$ be the d embeddings $k \to \mathbb{R}$. This induces embeddings $k \to \mathbb{R}^d$, $H_n(k) \to H_n(\mathbb{R})^d$ and $\mathrm{SL}_2(k) \to \mathrm{SL}_2(\mathbb{R})^d$, which we will use implicitly. Composing with the covariant map z on every component, we get a map $H_n(k) \to \mathbb{H}^d$, which is $\mathrm{SL}_2(k)$-covariant and which we also denote by z.

In fact, we can do slightly better. We identify $\mathcal{H}$ with $(\mathbb{C} \setminus \mathbb{R})$ modulo complex conjugation, that is, we identify $z \in -\mathcal{H}$ with $\overline{z} \in \mathcal{H}$. Then the $\mathrm{SL}_2(\mathbb{R})$-action on $\mathcal{H}$ extends to a $\mathrm{GL}_2(\mathbb{R})$-action also given by $z \cdot A = A^{-1}(z) = (dz - b)/(-cz + a)$ (up to complex conjugation). The covariant z as defned in [SC03] then turns out to also be $\mathrm{GL}_2(\mathbb{R})$-covariant. In particular, we get a map $z : H_n(k) \to \mathbb{H}^d$, which is $\mathrm{GL}_2(k)$-covariant.

5.2.3.3 Reduction for $\mathrm{GL}_2(\mathcal{O}_k)$ in $\mathbb{H}^d$

Let $N : \mathbb{R}^d \to \mathbb{R} : (x_m)_m \mapsto \prod_m x_m$, define $\mathrm{Re}, \mathrm{Im}, |\cdot| : \mathbb{C}^d \to \mathbb{R}^d$ component-wise and let $\log : \mathbb{R}^d \to \mathbb{R}^d : (x_m)_m \mapsto (\log |x_m|)_m$.

Definition 5.2.9. We call $z \in \mathbb{H}^d$ *reduced* for $\mathrm{GL}_2(\mathcal{O}_k)$ if it satisfies the following conditions:

(R) the point $\mathrm{Re}(z) \in \mathbb{R}^d$ is in some fixed chosen fundamental hyper-parallelogram for addition by $\mathcal{O}_k$,

(I) the point $\log(\mathrm{Im}(z)) \in \mathbb{R}^d$ is in some fixed chosen fundamental domain for addition by $\log(\mathcal{O}_k^*)$, and

(M) the norm $N(\mathrm{Im}(z))$ is maximal for the $\mathrm{GL}_2(\mathcal{O}_k)$-orbit $\mathrm{GL}_2(\mathcal{O}_k)z$.

Let us first see how this is an analogue of Definition 5.2.7. Note that in the case $k = \mathbb{Q}$, we can choose the hyper-parallelogram $[-\frac{1}{2}, \frac{1}{2}]$, and then conditions 5.2.7(R) and 5.2.9(R) coincide and condition 5.2.9(I) is empty. It is well-known that under condition 5.2.7(R), we have 5.2.9(M) if and only if 5.2.7(M).

The above gives rise to a notion of reduction for $\mathrm{GL}_2(\mathcal{O}_k) \times 1$ on $H_n(k)$. We then get the following sketch of a reduction algorithm.

Algorithm 5.2.10 (Reduction for $\mathrm{GL}_2(\mathcal{O}_k) \times \mathcal{O}_k^*$**).**
Input: *$F \in H_n(k)$.*
Output: *$F^\dagger \in H_n(k)$ that is $\mathrm{GL}_2(\mathcal{O}_k) \times \mathcal{O}_k^*$-equivalent to F and $\mathrm{GL}_2(\mathcal{O}_k)$-reduced.*

1. *Compute a fundamental domain $\mathcal{F}$ for addition by $\mathcal{O}_k$ in $\mathbb{R}^d$.*

2. *Compute a fundamental domain $\mathcal{G}$ for addition by $\log(\mathcal{O}_k^*)$ in $\mathbb{R}^d$.*

3. *Take* $u \in \mathcal{O}_k^*$ *such that* $\log \mathrm{Im}(z(F)) - (\log|\phi_m(u)|)_m \in \mathcal{G}$ *and replace* F *by*

$$F \cdot \left[\begin{pmatrix} u & 0 \\ 0 & 1 \end{pmatrix}, 1 \right] = F(uX, Z).$$

This replaces $z(F)$ *by* $u^{-1}z(F)$*, hence makes sure* F *satisfies (I) and preserves* $N(\mathrm{Im}(z))$.

4. *Take* $b \in \mathcal{O}_k$ *such that* $\mathrm{Re}(z(F)) - b \in \mathcal{F}$ *and replace* F *by*

$$F \cdot \left[\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1 \right] = F(X + bZ, Z).$$

This replaces $z(F)$ *by* $z(F) - b$*, hence makes sure* F *satisfies (R) and preserves (I) and* $N(\mathrm{Im}(z))$.

5. *Try to find a matrix* M *such that* $N(\mathrm{Im}(Mz)) > N(\mathrm{Im}(z))$*. If no such matrix exists, go to step 6. If such a matrix exists, replace* F *by* $F \cdot [M^{-1}, 1]$ *and go to step 3.*

6. *Try to find* $u \in \mathcal{O}_k^*$ *such that the maximum of the heights of the coefficients of* uF *is minimal and return* $F[1_2, u] = uF$.

Details on how we implemented this algorithm and on its correctness can be found in [BS15]. Furthermore we implemented this algorithm in Sage and it is available online at [SBb].

5.3 Results

To be able to read the tables presented in this section, first we need to explain what a CM-type is.

5.3.1 CM-types and reflex fields

We let k be a field of characteristic 0. Let K be a quartic CM field, $K = K_0(\sqrt{r})$, then K has a unique *complex conjugation* automorphism, which is the generator $\rho = \overline{\cdot}$ of $\mathrm{Gal}(K/K_0)$. For an embedding $\phi : K \to \overline{k}$, we write $\overline{\phi} : \phi \circ \rho$. Hence K has four embeddings into $\overline{k}$: $\{\phi_1, \phi_2, \overline{\phi_1}, \overline{\phi_2}\}$.

Definition 5.3.1. A *CM-type,* Φ*,* of a quartic CM field K, is a set of two embeddings $K \to \overline{k}$, such that $\Phi \cup \overline{\Phi}$ is the set of all four embeddings.

Let A be an abelian variety of dimension two with CM by K. The tangent space $T_0(A_{\overline{k}})$ of A over $\overline{k}$ at 0 is a two-dimensional $\overline{k}$-vector space. Differentiation gives a map $\mathrm{End}(A_{\overline{k}}) \to \mathrm{End}(T_0(A_{\overline{k}}))$. Choose an isomorphism $i : K \to \mathrm{End}(A_{\overline{k}})$, this induces a map $K \to \mathrm{End}(T_0(A_{\overline{k}}))$, i.e., a two-dimensional k-linear representation of K. By CM theory ([ST61]) there is a CM-type Φ, such that this representation is isomorphic to a direct sum of the two elements of Φ. We say that (A, i) is of *type* Φ and that Φ is *the CM-type of* (A, i). The *type norm* of Φ is the multiplicative map $N_\Phi : K \to \overline{k}$ defined by $\alpha \to \prod_{\phi \in \Phi} \phi(\alpha)$. The *reflex field* K^r is the field generated over $\mathbb{Q}$ by the set $\{N_\Phi(\alpha) : \alpha \in K\}$. The reflex field is again a quartic CM-field, hence it is a totally imaginary quadratic extension of a quadratic totally real number field K_0^r.

Note that the reflex field of the CM-type of (A, i) depends only on A, since composition of Φ with elements of $\mathrm{Aut}(K)$ does not change N_Φ.

5.3.2 The case distinctions

There are three possibilities for the Galois group of a quartic CM-field ([ST61, Example 8.4(2)]):

1. $K/\mathbb{Q}$ is Galois with cyclic Galois group C_4 of order 4,
2. $K/\mathbb{Q}$ is not normal, and its normal closure has dihedral Galois group D_4 of order 8,
3. $K/\mathbb{Q}$ is Galois over $\mathbb{Q}$ with Galois group $V_4 = C_2 \times C_2$.

It is known that case 3 of a *biquadratic* CM-field contradicts our assumption that $A = J(C)$ is simple over $\overline{k}$, that is A is not isogenious to a product of elliptic curves. So following the Echidna database [K+07], our tables will be partitioned into cases 1 and 2.

Recall that we are interested in curves with CM by the maximal order of a quartic CM-field K, which are defined over the reflex field K^r. We distinguish whether the curves are defined over:

a. $\mathbb{Q}$,

b. K_0^{r}, but not $\mathbb{Q}$,

c. K^{r}, but not K_0^{r}.

The motivation for this chapter was that case 2a is not possible, and during the construction of our list we found no example for case 1c. Hence we conjecture that case 1c is empty and we construct four tables corresponding to the four cases 1a, 1b, 2b, and 2c. Case 1a corresponds to Van Wamelen [vW99a].

5.3.3 Legend for the tables

In case 1, we have $K^r \cong K$ and $\mathrm{Aut}(K) = C_4$, so every abelian variety with CM by $\mathcal{O}_K$ is of all four CM-types, we therefore give K and f, but not Φ or K^r.

In case 2, we have two $\mathrm{Aut}(K)$-orbits of CM-types, and, given A, only one of these orbits correspond to A. We specify the correct CM-type orbit by specifying its reflex field K^r as an extension of the quadratic field $K_0^r = \mathbb{Q}(a)$.

A quartic CM-field K is given up to isomorphism by a unique triplet $[D, A, B]$ as follows, following the Echidna database [K+07]. Write $K = K_0(\sqrt{r})$ for some real quadratic field K_0 and some totally negative $r \in K_0$. Without loss of generality, we take $r \in \mathcal{O}_{K_0}$ with $A = -\mathrm{Tr}_{K_0/\mathbb{Q}}(r) \in \mathbb{Z}_{>0}$ *minimal*. Then let $B = N_{K_0/\mathbb{Q}}(r) \in \mathbb{Z}_{>0}$ and assume B is minimal for this A. Finally, let $D = \Delta_{K_0/\mathbb{Q}}$. We use the triplet $[D, A, B]$ to represent the isomorphism class of K, and note $K \cong \mathbb{Q}[X]/(X^4 + AX^2 + B)$.

Let us briefly state what the notation in the table means.

DAB With $[D, A, B]$ as in the first column, let $K = \mathbb{Q}(\beta)$, where β is a root of $X^4 + AX^2 + B$.

DAB^r In Tables 2b and 2c, let $[D^r, A^r, B^r]$ be as in the column DAB^r. Then let $K^r = \mathbb{Q}(\alpha)$, where α is a root of $X^4 + A^r X^2 + B^r$. In Tables 1a and 1b, we have $K^r \cong K$ and $[D^r, A^r, B^r] = [D, A, B]$.

a A root of $X^2 + \epsilon X + (D^r - \epsilon)/4$ with $\epsilon \in \{0, 1\}$ congruent to D^r modulo 4. We have $\mathbb{Z}[a] = \mathcal{O}_{K_0^r}$. In case 1, the field K^r is uniquely determined as a subset of $\overline{k}$ by $K^r \cong K$. In case 2, there are two quadratic extensions $K^r/\mathbb{Q}(a)$ that satisfy $K^r \cong \mathbb{Q}[X]/(X^4 + AX^2 + B)$, and they are conjugate over $\mathbb{Q}$. The expression of a in terms of α (in the column "a") tells us which of these extensions is $K^r = \mathbb{Q}(\alpha)$.

f, C The polynomial $f \in \mathbb{Z}[a][x]$ given in the final column defines a hyperelliptic curve $C : y^2 = f(x)$ of genus two.

$\Delta(C)$ The discriminant of the given model $y^2 = f(x)$ of C.

Δ_{stable} The minimal discriminant of all models of C over $\overline{\mathbb{Q}}$ of the form $y^2 + h(x)y = g(x)$ with coefficients in $\overline{\mathbb{Z}}$.

Φ One fixed CM-type of K with reflex field K^r, uniquely determined up to right-composition with $\mathrm{Aut}(K)$ by the following recipe. In case 1, we have $\mathrm{Aut}(K) = C_4$ and we fix an arbitrary CM-type. In case 2, the type Φ is unique up to complex conjugation and given as follows: Φ is a CM-type of K with values in a normal closure of K^r and reflex field K^r.

$(xa+y)_n^e$ The eth power of the principal $\mathbb{Z}[a]$-ideal of norm n generated by $xa+y$. This notation is used in the discriminant and obstruction columns.

5.3.4 Statement and proof of results regarding the table

Theorem 5.3.2. *With the notation as in the legend above, we have the following:*

1. *For every row of Tables 1a, 1b, and 2b, let K be as specified in that row (see "DAB" in the legend), and consider the curves C given in that row. Then the following holds.*

   (a) *In Table 1a, the given curves are exactly all $\overline{\mathbb{Q}}$-isomorphism classes of curves satisfying $\mathrm{End}(J(C)_{\overline{\mathbb{Q}}}) \cong \mathcal{O}_K$.*

   (b) *In Tables 1b and 2b, the given curves and their quadratic conjugates over $\mathbb{Q}$ are exactly all $\overline{\mathbb{Q}}$-isomorphism classes of curves satisfying $\mathrm{End}(J(C)_{\overline{\mathbb{Q}}}) \cong \mathcal{O}_K$.*

   (c) *In Tables 1a and 1b, the curves have CM-type Φ for every CM-type Φ of K.*

   (d) *In Table 2b, the given curves has the given CM-type Φ, and its quadratic conjugate has CM-type Φ' where $\Phi' \notin \{\Phi, \overline{\Phi}\}$.*

2. *The curves in Tables 1a, 1b, and 2b are all defined over K_0^r, and the entries $\Delta(C)/\Delta_{\mathrm{stable}}$ and Δ_{stable} are as explained in the legend above.*

3. *In Tables 1b and 2b, the discriminant $\Delta(C)$ is minimal (as defined in Section 5.2.2.1) among all $\overline{\mathbb{Q}}$-isomorphic models of the form $y^2 = g(x)$ with $g(x) \in \mathcal{O}_{K_0^r}[x]$, except for the case of the field $[17, 46, 257]$ in Table 2b, where a global minimal model does not exist, and the given model is minimal outside $(2, a+1)$. In Table 1a, the discriminant is minimal among such models with $g(x) \in \mathbb{Z}[x]$.*

4. *The curves in Tables 1b and 2b have Igusa Invariants that do not lie in $\mathbb{Q}$. In particular, they have no model over $\mathbb{Q}$.*

5. *For every row of Table 2c, the number in the final column is the number of curves over $\overline{\mathbb{Q}}$ with $\mathrm{End}(J(C)_{\overline{\mathbb{Q}}}) \cong \mathcal{O}_K$* of type Φ *up to isomorphism over $\overline{\mathbb{Q}}$. These curves all have Igusa Invariants in K_0^r but no model over K_0^r. They do have a model over K^r. The column "obstructions" gives exactly the set of places of K_0^r at which Mestre's conic locally has no point.*

Remark. Note that the curves in 1(a) and Table 1a were already given by Van Wamelen [vW99a] and proven correct by Van Wamelen [vW99b] and Bisson and Streng [BS13].

This list has been proven to be complete recently by Kilicer and Streng [KS15].

Proof. We compute the isomorphism class of the reflex field as follows. The reflex field is again a non-biquadratic quartic CM-field. In fact, one can compute that it is isomorphic to $\mathbb{Q}[X]/(X^4 + 2AX^2 + (A^2 - 4B))$. Let $[D', A', B']$ be the triplet that represents K^r as before. We do not necessarily have $A' = 2A$ and $B' = A^2 - 4B$, because those values are not always minimal. Note that we do have $K_0^r \cong \mathbb{Q}(\sqrt{D'}) \cong \mathbb{Q}(\sqrt{B})$.

Our computation of Igusa class polynomials, as explained in [BS15, Sec 2], shows that we have the correct number of curves for each field. Since we use interval arithmetic and the denominator formulas of Lauter and Viray [LV15], these computations even prove that the Igusa Invariants themselves are correct, including the ones for Table 2c, which are not listed. We used the Igusa Invariants to compute the curves and obstructions with Mestre's algorithm, which proves that the curves and obstructions are correct. In case 1, all CM-types are in the same orbit for $\mathrm{Aut}(K)$, so they are all correct. In cases 2b and 2c, the correct CM-type is determined using reduction modulo a suitable prime and the Shimura-Taniyama formula [ST61, Theorem 1(ii) in Section 13.1]. Proposition 5.2.2 and our reduction algorithm prove that the discriminant is minimal. The stable discriminant is computed directly from Igusa's arithmetic invariants [Igu60]. The set of obstructions in Table 2c is non-empty, hence there is no model over K_0^r. It remains to prove that there is a model over K^r, which can be verified by checking that the obstructions are inert or ramified in K^r/K_0^r, but which also follows from [BS15, Thm 5.3]. $\square$

Table 1a

DAB	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[5, 5, 5]$	1	$2^8 \cdot 5^5$	$x^5 - 1$
$[5, 10, 20]$	2^{12}	$2^{10} \cdot 5^5$	$4x^5 - 30x^3 + 45x - 22$
	$2^{12} \cdot 11^{12}$	$2^{10} \cdot 5^5$	$8x^6 + 52x^5 - 250x^3 + 321x - 131$
$[5, 65, 845]$	11^{12}	$2^{20} \cdot 5^5 \cdot 13^{10}$	$8x^6 - 112x^5 - 680x^4 + 8440x^3 + 28160x^2 - 55781x + 111804$
	$31^{12} \cdot 41^{12}$	$2^{20} \cdot 5^5 \cdot 13^{10}$	$-9986x^6 + 73293x^5 - 348400x^3 - 118976x - 826072$
$[5, 85, 1445]$	71^{12}	$2^{20} \cdot 5^5 \cdot 17^{10}$	$-73x^6 + 1005x^5 + 14430x^4 - 130240x^3 - 1029840x^2 + 760976x - 2315640$
	$11^{12} \cdot 41^{12} \cdot 61^{12}$	$2^{20} \cdot 5^5 \cdot 17^{10}$	$2160600x^6 - 8866880x^5 + 2656360x^4 - 582800x^3 + 44310170x^2 + 6986711x - 444408$
$[8, 4, 2]$	2^6	2^{15}	$x^5 - 3x^4 - 2x^3 + 6x^2 + 3x - 1$
$[8, 20, 50]$	$2^6 \cdot 7^{12} \cdot 23^{12}$	$2^{15} \cdot 5^{10}$	$-8x^6 - 530x^5 + 160x^4 + 64300x^3 - 265420x^2 - 529x$
	$2^6 \cdot 7^{12} \cdot 17^{12} \cdot 23^{12}$	$2^{15} \cdot 5^{10}$	$4116x^6 + 64582x^5 + 139790x^4 - 923200x^3 + 490750x^2 + 233309x - 9347$
$[13, 13, 13]$	1	$2^{20} \cdot 13^5$	$x^6 - 8x^4 - 8x^3 + 8x^2 + 12x - 8$
$[13, 26, 52]$	$2^{12} \cdot 3^{12} \cdot 23^{12}$	$2^{10} \cdot 13^5$	$-243x^6 - 2223x^5 - 1566x^4 + 19012x^3 + 903x^2 - 19041x - 5882$
	$2^{12} \cdot 3^{12} \cdot 23^{12} \cdot 131^{12}$	$2^{10} \cdot 13^5$	$59499x^6 - 125705x^5 - 801098x^4 + 1067988x^3 + 2452361x^2 + 707297x - 145830$
$[13, 65, 325]$	3^{12}	$2^{20} \cdot 5^{10} \cdot 13^5$	$36x^5 - 1040x^3 + 1560x^2 + 1560x + 1183$
	$3^{12} \cdot 53^{12}$	$2^{20} \cdot 5^{10} \cdot 13^5$	$-1323x^6 - 1161x^5 + 9360x^4 + 9590x^3 - 34755x^2 + 1091x + 32182$
$[29, 29, 29]$	5^{12}	$2^{20} \cdot 29^5$	$43x^6 - 216x^5 + 348x^4 - 348x^2 - 116x$
$[37, 37, 333]$	$3^{12} \cdot 11^{12}$	$2^{20} \cdot 37^5$	$-68x^6 + 57x^5 + 84x^4 - 680x^3 + 72x^2 - 1584x - 4536$
$[53, 53, 53]$	$17^{12} \cdot 29^{12}$	$2^{20} \cdot 53^5$	$-3800x^6 + 15337x^5 + 160303x^4 - 875462x^3 + 896582x^2 - 355411x + 50091$
$[61, 61, 549]$	$3^{24} \cdot 5^{12} \cdot 41^{12}$	$2^{20} \cdot 61^5$	$40824x^6 + 103680x^5 - 67608x^4 - 197944x^3 - 17574x^2 + 41271x + 103615$

Table 1b

DAB	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[5, 15, 45]$	$(2)^{12} \cdot (3)^6$	$(2a+1)_5^{10}$	$-x^6 + (-3a-3)\,x^5 + (5a+15)\,x^3 + (-15a-3)\,x - 4a + 1$
	$(2)^{12} \cdot (3)^6 \cdot (5a+2)_{31}^{12}$	$(2a+1)_5^{10}$	$(-2a+3)\,x^6 + (-9a+18)\,x^5 + (15a-70)\,x^3 + (39a+54)\,x - 52a - 1$
$[5, 30, 180]$	$(3a+2)_{11}^{12} \cdot (2)^{18} \cdot (3)^6 \cdot (5a+2)_{31}^{12}$	$(2a+1)_5^{10}$	$684x^6 + (390a+90)\,x^5 + (24a-3138)\,x^4 + (217a+401)\,x^3 + (96a+3918)\,x^2 + (-2112a-1698)\,x + 284a + 432$
	$(3a+1)_{11}^{12} \cdot (2a-11)_{139}^{12} \cdot (4a+3)_{19}^{12} \cdot (2)^{18} \cdot (3)^6 \cdot (5a+2)_{31}^{12}$	$(2a+1)_5^{10}$	$(927a+2906)\,x^6 + (5541a+18822)\,x^5 + (-33535a-124380)\,x^3 + (33417a+183726)\,x + 12641a - 31928$
$[5, 35, 245]$	$(3a+2)_{11}^{12} \cdot (2)^{12} \cdot (a+6)_{29}^{12} \cdot (7)^6 \cdot (a+9)_{71}^{12}$	$(2a+1)_5^{10}$	$(-4527a-783)\,x^6 + (6392a+7811)\,x^5 + (-4500a-17085)\,x^3 + (-6948a+9783)\,x - 1687a + 39$
	$(3a+1)_{11}^{12} \cdot (11a+5)_{151}^{12} \cdot (2a+15)_{191}^{12} \cdot (2)^{12} \cdot (a-5)_{29}^{12} \cdot (7)^6$	$(2a+1)_5^{10}$	$(-435a-521)\,x^6 + (353a+110)\,x^5 + (131927a+189531)\,x^4 + (-696187a-952511)\,x^3 + (-10094248a-15393369)\,x^2 + (94869598a+145990333)\,x - 210533420a - 329328479$
$[5, 105, 2205]$	$(3a+1)_{11}^{12} \cdot (3)^6 \cdot (7)^6$	$(2)^{20} \cdot (2a+1)_5^{10}$	$(-5a+4)\,x^6 + (-81a+30)\,x^5 + (-135a+210)\,x^4 + (450a-210)\,x^3 + (360a-1785)\,x^2 + (600a+15)\,x - 950a + 5625$
	$(a+11)_{109}^{12} \cdot (3a+2)_{11}^{12} \cdot (3)^6 \cdot (7)^6 \cdot (8a+3)_{79}^{12}$	$(2)^{20} \cdot (2a+1)_5^{10}$	$(-3a-260)\,x^6 + (1032a+1389)\,x^5 + (19160a+8760)\,x^3 + (-16224a+163200)\,x + 162976a + 114632$
$[8, 12, 18]$	$(a)_2^{12} \cdot (3)^6 \cdot (2a-1)_7^{12} \cdot (2a+1)_7^{12}$	$(a)_2^{30}$	$(24a-54)\,x^5 + (-66a+96)\,x^4 + (-32a+220)\,x^3 + (12a-312)\,x^2 + (96a+21)\,x - 5a - 16$

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Table 1b: *Continued from previous page*

DAB	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[17, 119, 3332]$	$(2a+15)_{179}^{12} \cdot (a+2)_2^{36} \cdot (a-1)_2^{12} \cdot (4a+7)_{43}^{12} \cdot (7)^6$	$(2a+1)_{17}^{10}$	$(213a+1875)\,x^6 + (8071a+4059)\,x^5 + (-1045a+58039)\,x^4 + (32898a+26657)\,x^3 + (-12585a+3550)\,x^2 + (-46889a-136176)\,x - 42057a - 104692$
$[17, 255, 15300]$	$(2a-5)_{19}^{12} \cdot (a+2)_2^{24} \cdot (a-1)_2^{24} \cdot (3)^6 \cdot (2a+31)_{883}^{12}$	$(2a+1)_{17}^{10} \cdot (5)^{10}$	$(-4264a-13208)\,x^6 + (9516a-94116)\,x^5 + (331770a-503670)\,x^4 + (-1195640a+1593625)\,x^3 + (1141785a-2476410)\,x^2 + (-69927a+2540472)\,x - 301251a - 1280828$
	$(2a+3)_{13}^{12} \cdot (4a+17)_{157}^{12} \cdot (2a+7)_{19}^{12} \cdot (a+2)_2^{12} \cdot (a-1)_2^{12} \cdot (3)^6 \cdot (4a+3)_{67}^{12} \cdot (2a-9)_{83}^{12} \cdot (2a+11)_{83}^{12}$	$(2a+1)_{17}^{10} \cdot (5)^{10}$	$(3703196a+9037010)\,x^6 + (12666396a+36366348)\,x^5 + (33133830a+56148570)\,x^4 + (35333760a+111063545)\,x^3 + (71845845a+45282705)\,x^2 + (154100103a-105860229)\,x + 81081415a - 36366223$

Table 2b

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[5, 11, 29]$	$[29, 7, 5]$	α^2+3	$(2)^{12} \cdot (a-1)_5^{12} \cdot (a+1)_7^{12}$	$(a+2)_5^{10}$	$(18a+60)\,x^6 + (-76a-246)\,x^5 + (127a+329)\,x^4 + (-77a-209)\,x^3 + (-30a+155)\,x^2 + (29a-69)\,x + 71a - 156$
			$(2)^{12} \cdot (a+6)_{23}^{12} \cdot (a-1)_5^{12}$	$(a+2)_5^{10}$	$(2a+1)\,x^6 + (-a-26)\,x^5 + (9a+38)\,x^4 + (-40a-25)\,x^3 + (-21a-37)\,x^2 + (100a+218)\,x + 102a + 268$
$[5, 13, 41]$	$[41, 11, 20]$	α^2+5	$(a-3)_2^{12}$	$(a+4)_2^{20} \cdot (2a-5)_5^{10}$	$(-a+3)\,x^6 + (4a-8)\,x^5 + 10x^4 + (-a+20)\,x^3 + (4a+5)\,x^2 + (a+4)\,x + 1$

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Table 2b: *Continued from previous page*

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[5, 17, 61]$	$[61, 9, 5]$	$\alpha^2 + 4$	$(a-3)_3^{12}$	$(2)^{20} \cdot (a-4)_5^{10}$	$(a+4)\,x^6 + (-8a-42)\,x^5 + (37a+117)\,x^4 + (-20a-240)\,x^3 + (56a-9)\,x^2 + (22a-114)\,x + 9a - 28$
$[5, 21, 109]$	$[109, 17, 45]$	$\alpha^2 + 8$	$(a-5)_3^{12} \cdot (3a+17)_5^{12}$	$(2)^{20} \cdot (3a-14)_5^{10}$	$(-28a+53)\,x^6 + (-113a+913)\,x^5 + (-495a+1890)\,x^4 + (-746a+3308)\,x^3 + (-563a+3574)\,x^2 + (-378a+1069)\,x - 151a - 227$
$[5, 26, 149]$	$[149, 13, 5]$	$\alpha^2 + 6$	$(a+7)_5^{12} \cdot (a-5)_7^{12}$	$(2)^{20} \cdot (a-6)_5^{10}$	$(-125a-875)\,x^6 + (-1375a-8575)\,x^5 + (-9090a-62160)\,x^4 + (-38862a-251798)\,x^3 + (-73257a-489843)\,x^2 + (-53235a-347403)\,x - 12896a - 86314$
$[5, 33, 261]$	$[29, 21, 45]$	$\frac{1}{3}\alpha^2 + 3$	$(a+5)_{13}^{12} \cdot (3)^6$	$(2)^{20} \cdot (a+2)_5^{10}$	$(-27a-96)\,x^5 + (-18a-51)\,x^4 + (-34a-58)\,x^3 + (-18a-36)\,x^2 - 15x - 9a - 27$
			$(3)^6 \cdot (a)_7^{12}$	$(2)^{20} \cdot (a+2)_5^{10}$	$(-3a+6)\,x^5 - 90x^4 + (-128a-136)\,x^3 + (-72a-744)\,x^2 + (-240a-240)\,x - 216$
$[5, 34, 269]$	$[269, 17, 5]$	$\alpha^2 + 8$	$(a-7)_{11}^{12} \cdot (2a-15)_{13}^{12} \cdot (a+9)_5^{12}$	$(2)^{20} \cdot (a-8)_5^{10}$	$(-283a+2246)\,x^6 + (-4563a+33800)\,x^5 + (-11932a+103166)\,x^4 + (127408a-1032304)\,x^3 + (998576a-7558008)\,x^2 + (2439792a-18969664)\,x + 2110776a - 16149072$
$[5, 41, 389]$	$[389, 37, 245]$	$\alpha^2 + 18$	$(2a+21)_{11}^{12} \cdot (8a+83)_{17}^{12} \cdot (5a+52)_{19}^{12} \cdot (3a-28)_5^{12}$	$(2)^{20} \cdot (3a+31)_5^{10}$	$(1248a-11685)\,x^6 + (-16097a+150611)\,x^5 + (37185a-349530)\,x^4 + (250806a-2359968)\,x^3 + (-972081a+9046728)\,x^2 + (-942318a+8701533)\,x + 4994791a - 46866753$

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Table 2b: *Continued from previous page*

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[5, 66, 909]$	$[101, 33, 45]$	$\frac{1}{3}\alpha^2 + 5$	$(a-2)_{19}^{12} \cdot (3)^6 \cdot (2a+13)_{43}^{12} \cdot (a-4)_5^{12}$	$(2)^{20} \cdot (a+5)_5^{10}$	$(-340a - 1674)\, x^6 + (-4179a - 26820)\, x^5 + (-26433a - 118800)\, x^4 + (-38358a - 315240)\, x^3 + (-46686a - 41130)\, x^2 + (40761a - 15348)\, x - 13013a + 39100$
			$(3)^6 \cdot (a+8)_{31}^{12} \cdot (2a-7)_{37}^{12} \cdot (a-4)_5^{12}$	$(2)^{20} \cdot (a+5)_5^{10}$	$(-6120a - 36189)\, x^6 + (-22143a - 102375)\, x^5 + (-21378a - 184140)\, x^4 + (-31356a - 65810)\, x^3 + (765a - 81765)\, x^2 + (-3783a + 6192)\, x$
$[8, 10, 17]$	$[17, 5, 2]$	$\alpha^2 + 2$	$(a+2)_2^6$	$(a+2)_2^{45} \cdot (a-1)_2^{20}$	$x^6 + (2a+4)\, x^5 + (3a+14)\, x^4 + (10a+8)\, x^3 + (-9a+32)\, x^2 + (16a-16)\, x - 4a + 8$
$[8, 18, 73]$	$[73, 9, 2]$	$\alpha^2 + 4$	$(a-4)_2^6 \cdot (a+5)_2^{12} \cdot (4a-15)_3^{12}$	$(a-4)_2^{45}$	$(a+5)\, x^6 + (28a+132)\, x^5 + (214a+1026)\, x^4 + (349a+1658)\, x^3 + (259a+1242)\, x^2 + (47a+222)\, x - 3a - 14$
$[8, 22, 89]$	$[89, 11, 8]$	$\alpha^2 + 5$	$(a-4)_2^{12} \cdot (a+5)_2^6 \cdot (4a-17)_5^{12}$	$(a+5)_2^{45}$	$(a-4)\, x^6 + (8a-36)\, x^5 + (16a-62)\, x^4 + (-13a+57)\, x^3 + (-17a+73)\, x^2 + (13a-57)\, x - a + 5$
$[8, 34, 281]$	$[281, 17, 2]$	$\alpha^2 + 8$	$(42a-331)_{17}^{12} \cdot (a-8)_2^6 \cdot (a+9)_2^{24} \cdot (76a+675)_5^{12} \cdot (8a-63)_7^{12}$	$(a-8)_2^{45}$	$(-15024a + 118185)\, x^6 + (310153a - 2435026)\, x^5 + (-2658057a + 20990488)\, x^4 + (12047831a - 97400942)\, x^3 + (-33280854a + 231380920)\, x^2 + (34989188a - 413796872)\, x - 37610304a + 81055944$
$[8, 38, 233]$	$[233, 19, 32]$	$\alpha^2 + 9$	$(38a-271)_{13}^{12} \cdot (a+8)_2^{12} \cdot (a-7)_2^6 \cdot (8a+65)_7^{12} \cdot (8a-57)_7^{12}$	$(a-8)_2^{45}$	$(-166628a - 1355047)\, x^6 + (-354121a - 2879769)\, x^5 + (-318274a - 2588269)\, x^4 + (-153661a - 1249743)\, x^3 + (-41827a - 339754)\, x^2 + (-6158a - 48444)\, x - 441a - 2400$
$[8, 50, 425]$	$[17, 25, 50]$	$\frac{1}{5}\alpha^2 + 2$	$(a+2)_2^6 \cdot (a-1)_2^{12} \cdot (5)^6$	$(a+2)_2^{45} \cdot (5)^{15}$	$(34a+80)\, x^6 + (140a+224)\, x^5 + (110a-220)\, x^4 + (-455a+220)\, x^3 + (-5a+190)\, x^2 + (91a-104)\, x + 254a - 395$

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Table 2b: *Continued from previous page*

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[8, 50, 425]$	$[17, 25, 50]$	$\frac{1}{5}\alpha^2 + 2$	$(2a+3)_{13}^{12} \cdot (2a-5)_{19}^{12} \cdot (a+2)_2^6 \cdot (a-1)_2^{24} \cdot (5)^6$	$(a+2)_2^{45} \cdot (5)^{15}$	$(-1455a + 1511)\, x^6 + (-1004a - 2656)\, x^5 + (-19100a + 20290)\, x^4 + (-3805a - 4380)\, x^3 + (-72745a + 108600)\, x^2 + (-7451a + 10748)\, x - 99295a + 155108$
$[8, 66, 1017]$	$[113, 33, 18]$	$\frac{1}{3}\alpha^2 + 5$	$(4a-19)_{11}^{12} \cdot (a+6)_2^{12} \cdot (a-5)_2^6 \cdot (3)^6 \cdot (8a+47)_{41}^{12} \cdot (6a+35)_7^{12} \cdot (6a-29)_7^{12}$	$(a-5)_2^{45}$	$(-4215a - 14698)\, x^6 + (30036a + 338652)\, x^5 + (-549576a - 134610)\, x^4 + (-2945519a + 22716733)\, x^3 + (12849441a - 76601511)\, x^2 + (234523575a - 1115687637)\, x - 843111919a + 4054444133$
			$(a+6)_2^{12} \cdot (a-5)_2^6 \cdot (3)^6 \cdot (2a+13)_{31}^{12} \cdot (28a + 163)_{53}^{12} \cdot (6a+35)_7^{12}$	$(a-5)_2^{45}$	$(-27a - 2538)\, x^6 + (7230a + 8412)\, x^5 + (-3867a - 272622)\, x^4 + (121693a + 458725)\, x^3 + (-1686144a + 6014715)\, x^2 + (-5324007a + 27892107)\, x + 110392412a - 532554277$
$[13, 9, 17]$	$[17, 15, 52]$	$\alpha^2 + 7$	$(a+2)_2^{12}$	$(2a-1)_{13}^{10} \cdot (a-1)_2^{20}$	$(a-2)\, x^6 + (-8a+8)\, x^5 + (14a - 32)\, x^4 + (-19a + 27)\, x^3 + (6a - 21)\, x^2 + (3a + 9)\, x - 4a - 7$
$[13, 18, 29]$	$[29, 9, 13]$	$\alpha^2 + 4$	$(a-1)_5^{12}$	$(a-4)_{13}^{10} \cdot (2)^{20}$	$(9a - 22)\, x^6 + (-19a + 21)\, x^5 + (8a - 95)\, x^4 + (-70a - 6)\, x^3 + (-23a - 148)\, x^2 + (-7a - 127)\, x - 18a - 7$
$[13, 29, 181]$	$[181, 41, 13]$	$\frac{1}{3}\alpha^2 + \frac{19}{3}$	$(6a-37)_{29}^{12} \cdot (a-6)_3^{12} \cdot (a+7)_3^{12} \cdot (4a+29)_5^{12}$	$(3a-19)_{13}^{10} \cdot (2)^{20}$	$(-16581a - 119826)\, x^6 + (-52472a - 379062)\, x^5 + (-67729a - 508419)\, x^4 + (-78876a - 162464)\, x^3 + (-44960a + 21657)\, x^2 + (14402a - 144114)\, x - 21885a + 131494$
$[13, 41, 157]$	$[157, 25, 117]$	$\alpha^2 + 12$	$(3a+20)_{11}^{12} \cdot (a-7)_{17}^{12} \cdot (a-6)_3^{12} \cdot (a+7)_3^{12}$	$(2a-11)_{13}^{10} \cdot (2)^{20}$	$(-1181a + 7035)\, x^6 + (18395a - 104353)\, x^5 + (-116071a + 664673)\, x^4 + (386042a - 2282384)\, x^3 + (-742970a + 4253365)\, x^2 + (784564a - 4063679)\, x - 253294a + 2224205$

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Table 2b: *Continued from previous page*

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[17, 5, 2]$	$[8, 10, 17]$	$\frac{1}{2}\alpha^2 + \frac{5}{2}$	1	$(3a+1)_{17}^{10} \cdot (a)_2^{30}$	$(-3a+4)\,x^5 - x^4 + (6a-2)\,x^3 + (9a-5)\,x^2 + (-3a+8)\,x - 3a + 6$
$[17, 15, 52]$	$[13, 9, 17]$	$\alpha^2 + 4$	$(a)_3^{12}$	$(a-4)_{17}^{10} \cdot (2)^{20}$	$-x^6 - 2ax^5 + (3a-3)\,x^4 + (8a+4)\,x^3 + (-19a+39)\,x^2 + (16a-30)\,x + 3a - 36$
$[17, 25, 50]$	$[8, 50, 425]$	$\frac{1}{10}\alpha^2 + \frac{5}{2}$	$(a)_2^{24} \cdot (2a+1)_7^{12}$	$(3a+1)_{17}^{10} \cdot (5)^{10}$	$(6a-2)\,x^6 + (-50a-64)\,x^5 + (285a+485)\,x^4 + (-485a-435)\,x^3 + (-70a+90)\,x^2 + (244a+92)\,x + 70a - 166$
			$(a)_2^{36} \cdot (a+7)_{47}^{12} \cdot (2a+1)_7^{12}$	$(3a+1)_{17}^{10} \cdot (5)^{10}$	$(315a+422)\,x^6 + (1212a+1757)\,x^5 + (-2605a-3240)\,x^4 + (-50a-625)\,x^3 + (1730a-570)\,x^2 + (864a-212)\,x + 72a + 456$
$[17, 46, 257]$ Class number not one	$[257, 23, 68]$	$\alpha^2 + 11$	$(11, a+5)^{12} \cdot (13, a+10)^{12} \cdot (2, a)^{12} \cdot (2, a+1)^{24} \cdot (59, a+14)^{12}$	$(17, a+6)^{10} \cdot (2, a+1)$	$(-22a-1802)\,x^6 + (3596a+11488)\,x^5 + (-30700a-354072)\,x^4 + (243927a+1843299)\,x^3 + (-616892a-5576996)\,x^2 + (647768a+5283496)\,x - 198146a - 1755298$
$[17, 47, 548]$	$[137, 35, 272]$	$\alpha^2 + 17$	$(14a-75)_{11}^{12} \cdot (4a+25)_{19}^{12} \cdot (3a-16)_2^{12} \cdot (3a+19)_2^{24}$	$(8a+51)_{17}^{10}$	$(285a+1620)\,x^6 + (-2683a-19110)\,x^5 + (13341a+76698)\,x^4 + (-28642a-195577)\,x^3 + (40284a+245904)\,x^2 + (-27600a-177408)\,x + 8154a + 51670$
$[29, 7, 5]$	$[5, 11, 29]$	$\alpha^2 + 5$	$(2)^{12} \cdot (2a+1)_5^{12}$	$(a-5)_{29}^{10}$	$(-4a-5)\,x^6 + (11a+37)\,x^5 + (-65a-62)\,x^4 + (111a+104)\,x^3 + (-28a-189)\,x^2 + (-28a+157)\,x - 19a - 76$
			$(2)^{12} \cdot (5a+3)_{31}^{12} \cdot (2a+1)_5^{12}$	$(a-5)_{29}^{10}$	$(18a+42)\,x^6 + (62a+194)\,x^5 + (-209a+31)\,x^4 + (-648a-471)\,x^3 + (116a+338)\,x^2 + (244a+259)\,x - 65a - 159$
$[29, 9, 13]$	$[13, 18, 29]$	$\frac{1}{4}\alpha^2 + \frac{7}{4}$	$(a)_3^{12}$	$(2)^{20} \cdot (3a+2)_{29}^{10}$	$(-25a+56)\,x^6 + (172a-39)\,x^5 + (-39a+561)\,x^4 + (312a+234)\,x^3 + (73a+354)\,x^2 + (76a+141)\,x + 15a + 37$

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Table 2b: *Continued from previous page*

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[29, 21, 45]$	$[5, 33, 261]$	$\frac{1}{3}\alpha^2 + 5$	$(4a+1)_{19}^{12} \cdot (3)^6$	$(2)^{20} \cdot (a-5)_{29}^{10}$	$(-a+20)\,x^6 + (-87a-18)\,x^5 + (-48a+198)\,x^4 + (-8a-296)\,x^3 + (384a+360)\,x^2 + (-384a-480)\,x + 144a + 216$
$[29, 21, 45]$	$[5, 33, 261]$	$\frac{1}{3}\alpha^2 + 5$	$(3)^6$	$(2)^{20} \cdot (a-5)_{29}^{10}$	$(-102a-165)\,x^5 + (45a+72)\,x^4 + (-174a-262)\,x^3 + (36a-66)\,x^2 + (69a-144)\,x + 5a - 107$
$[29, 26, 53]$	$[53, 13, 29]$	$\alpha^2 + 6$	$(a-1)_{11}^{12} \cdot (a+1)_{13}^{12} \cdot (a+6)_{17}^{12}$	$(2)^{20} \cdot (a-6)_{29}^{10}$	$(-790a+1564)\,x^6 + (241a-12431)\,x^5 + (-15139a-14345)\,x^4 + (-2950a-165614)\,x^3 + (-51588a-116086)\,x^2 + (-58139a-53507)\,x + 12653a - 123381$
$[41, 11, 20]$	$[5, 13, 41]$	$\alpha^2 + 6$	1	$(2)^{20} \cdot (a-6)_{29}^{10}$	$(a+4)\,x^6 + (6a-2)\,x^5 + 17x^4 + (-12a-16)\,x^3 + (24a-5)\,x^2 + (-54a-16)\,x + 33a + 9$
$[53, 13, 29]$	$[29, 26, 53]$	$\frac{1}{4}\alpha^2 + \frac{11}{4}$	$(a+6)_{23}^{12} \cdot (a-1)_5^{12} \cdot (a)_7^{12}$	$(2)^{20} \cdot (3a+5)_{53}^{10}$	$(-31a+70)\,x^6 + (151a-322)\,x^5 + (-405a+658)\,x^4 + (238a-846)\,x^3 + (3288a+2437)\,x^2 + (-3262a+12157)\,x - 27420a - 58255$
$[61, 9, 5]$	$[5, 17, 61]$	$\frac{1}{3}\alpha^2 + \frac{7}{3}$	1	$(2)^{20} \cdot (7a+4)_{61}^{10}$	$(a+2)\,x^6 + (-2a-15)\,x^5 + (36a-4)\,x^4 + (72a+24)\,x^3 + (8a-24)\,x^2 + (-48a-80)\,x - 24a - 40$
$[73, 9, 2]$	$[8, 18, 73]$	$\frac{1}{2}\alpha^2 + \frac{9}{2}$	$(a)_2^{24} \cdot (2a-1)_7^{12}$	$(2a-9)_{73}^{10}$	$(-12a-6)\,x^6 + (8a+82)\,x^5 + (-51a+92)\,x^4 + (-126a-1)\,x^3 + (-36a+35)\,x^2 + (32a+50)\,x + 10a + 8$
$[73, 47, 388]$	$[97, 94, 657]$	$\frac{1}{8}\alpha^2 + \frac{43}{8}$	$(20a+109)_{101}^{12} \cdot (7a+38)_2^{24} \cdot (7a-31)_2^{12} \cdot (2a-9)_3^{12} \cdot (2a+11)_3^{12} \cdot (30a+163)_{79}^{12}$	$(22a+119)_{73}^{10}$	$(23a-43)\,x^6 + (-149a-1221)\,x^5 + (8675a+44883)\,x^4 + (-128038a-698079)\,x^3 + (928849a+5037588)\,x^2 + (123515a+671208)\,x + 4023a + 21640$
$[89, 11, 8]$	$[8, 22, 89]$	$\frac{1}{4}\alpha^2 + \frac{11}{4}$	$(a)_2^{24}$	$(7a+3)_{89}^{10}$	$-x^5 + (-4a+2)\,x^4 + 21x^3 + (-16a+64)\,x^2 - 160x + 142a - 190$

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Table 2b: *Continued from previous page*

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[97, 94, 657]$	$[73, 47, 388]$	$\frac{1}{3}\alpha^2 + \frac{22}{3}$	$(a-4)_2^{12} \cdot (a+5)_2^{12} \cdot (14a-53)_{23}^{12} \cdot (4a-15)_3^{12} \cdot (4a+19)_3^{12} \cdot (30a+143)_{41}^{12} \cdot (10a+47)_{61}^{12}$	$(24a+115)_{97}^{10}$	$(-128252a-611298)\,x^6 + (-984572a-4709700)\,x^5 + (-3071730a-15394554)\,x^4 + (-6889006a-20077475)\,x^3 + (-39650571a+105355350)\,x^2 + (174191751a-679664106)\,x + 256866525a-973717416$
$[101, 33, 45]$	$[5, 66, 909]$	$\frac{1}{12}\alpha^2 + \frac{9}{4}$	$(3)^6 \cdot (2a+1)_5^{12} \cdot (7a+3)_{61}^{12}$	$(9a+5)_{101}^{10} \cdot (2)^{20}$	$(-216a+464)\,x^6 + (-2304a-48)\,x^5 + (-3984a-960)\,x^4 + (-864a+3088)\,x^3 + (-720a+1422)\,x^2 + (-4047a-5322)\,x - 818a-2423$
			$(4a+3)_{19}^{12} \cdot (4a+1)_{19}^{12} \cdot (3)^6 \cdot (5a+3)_{31}^{12} \cdot (2a+1)_5^{12}$	$(9a+5)_{101}^{10} \cdot (2)^{20}$	$(-5229a+4019)\,x^6 + (-6132a-6909)\,x^5 + (44637a-2364)\,x^4 + (53094a+58660)\,x^3 + (-39159a+19266)\,x^2 + (-30363a-55761)\,x - 16848a-16911$
$[109, 17, 45]$	$[5, 21, 109]$	$\alpha^2 + 10$	$(2a+1)_5^{12}$	$(a-10)_{109}^{10} \cdot (2)^{20}$	$(-8a-8)\,x^6 - 16x^5 + (8a+72)\,x^4 + (152a+184)\,x^3 + (6a+84)\,x^2 + (-255a-339)\,x - 319a-524$
$[113, 33, 18]$	$[8, 66, 1017]$	$\frac{1}{6}\alpha^2 + \frac{11}{2}$	$(3a+11)_{103}^{12} \cdot (a)_2^{24} \cdot (3)^6 \cdot (4a-1)_{31}^{12} \cdot (2a-1)_7^{12} \cdot (2a+1)_7^{12}$	$(2a-11)_{113}^{10}$	$(122a+800)\,x^6 + (-1509a-909)\,x^5 + (36762a-85470)\,x^4 + (-116871a+265713)\,x^3 + (-467682a+704460)\,x^2 + (-480528a+365352)\,x - 7616a+226442$
			$(a)_2^{24} \cdot (3)^6 \cdot (4a+1)_{31}^{12} \cdot (2a+1)_7^{12}$	$(2a-11)_{113}^{10}$	$(-418a-190)\,x^6 + (1476a-660)\,x^5 + (1146a+6810)\,x^4 + (2145a+2175)\,x^3 + (-1437a-3489)\,x^2 + (-42a-2736)\,x + 830a+394$
$[137, 35, 272]$	$[17, 47, 548]$	$\alpha^2 + 23$	$(2a-5)_{19}^{12} \cdot (a+2)_2^{12} \cdot (a-1)_2^{12}$	$(6a-1)_{137}^{10}$	$(4a+6)\,x^6 + (8a+36)\,x^5 + (-4a+42)\,x^4 + (586a+1289)\,x^3 + (1066a+2808)\,x^2 + 4ax + 25596a+65566$

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Table 2b: *Continued from previous page*

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[149, 13, 5]$	$[5, 26, 149]$	$\frac{1}{4}\alpha^2 + \frac{11}{4}$	$(3a+1)_{11}^{12}$	$(11a+7)_{149}^{10} \cdot (2)^{20}$	$8x^6 + 96x^5 + (-24a+168)\,x^4 + (-576a-808)\,x^3 + (66a-132)\,x^2 + (292a+47)\,x + 86a - 87$
$[157, 25, 117]$	$[13, 41, 157]$	$\frac{1}{9}\alpha^2 + \frac{16}{9}$	$(a-4)_{17}^{12} \cdot (3a-1)_{23}^{12} \cdot (a)_3^{24} \cdot (a+1)_3^{12}$	$(7a+5)_{157}^{10} \cdot (2)^{20}$	$(-3328a-7633)\,x^6 + (-17510a-39323)\,x^5 + (-32518a-68044)\,x^4 + (-17960a-66720)\,x^3 + (256a-51704)\,x^2 + (5184a-22864)\,x + 1432a - 5264$
$[181, 41, 13]$	$[13, 29, 181]$	$\frac{1}{3}\alpha^2 + \frac{13}{3}$	$(a+5)_{17}^{12} \cdot (3a+2)_{29}^{12} \cdot (a)_3^{24} \cdot (a+1)_3^{12}$	$(3a-13)_{181}^{10} \cdot (2)^{20}$	$(330a+1417)\,x^6 + (11102a+1701)\,x^5 + (1396a+59742)\,x^4 + (24016a+92792)\,x^3 + (74408a+38064)\,x^2 + (35248a+26160)\,x - 5784a + 21888$
$[233, 19, 32]$	$[8, 38, 233]$	$\frac{1}{8}\alpha^2 + \frac{19}{8}$	$(a)_2^{24} \cdot (a-5)_{23}^{12} \cdot (a+5)_{23}^{12} \cdot (2a+1)_7^{12}$	$(11a+3)_{233}^{10}$	$(2348a-3554)\,x^6 + (11828a-12348)\,x^5 + (4498a-23598)\,x^4 + (12704a+9133)\,x^3 + (-3151a-14433)\,x^2 + (5344a-1974)\,x + 18a - 604$
$[257, 23, 68]$	$[17, 46, 257]$	$\frac{1}{8}\alpha^2 + \frac{19}{8}$	$(2a+3)_{13}^{12} \cdot (a+2)_2^{12} \cdot (a-1)_2^{24} \cdot (4a-3)_{43}^{12} \cdot (2a+9)_{47}^{12} \cdot (4a+13)_{53}^{12}$	$(8a-19)_{257}^{10}$	$(-2809a-7326)\,x^6 + (5069a+3572)\,x^5 + (52427a-51416)\,x^4 + (249518a+105951)\,x^3 + (-311115a-180355)\,x^2 + (156533a-20215)\,x - 34657a + 19003$
$[269, 17, 5]$	$[5, 34, 269]$	$\frac{1}{4}\alpha^2 + \frac{15}{4}$	$(3a+1)_{11}^{12} \cdot (2a+1)_5^{12}$	$(2)^{20} \cdot (15a+11)_{269}^{10}$	$(-168a-272)\,x^6 + (960a+1696)\,x^5 + (472a-1008)\,x^4 + (-4448a-1552)\,x^3 + (358a+904)\,x^2 + (945a+1690)\,x$
$[281, 17, 2]$	$[8, 34, 281]$	$\frac{1}{2}\alpha^2 + \frac{17}{2}$	$(a)_2^{36} \cdot (4a+1)_{31}^{12} \cdot (2a-1)_7^{12} \cdot (2a+1)_7^{12}$	$(2a-17)_{281}^{10}$	$(-835a+1960)\,x^6 + (1343a+7589)\,x^5 + (19630a+6428)\,x^4 + (26923a+13601)\,x^3 + (-6743a+44228)\,x^2 + (-5762a+18262)\,x + 17138a - 23184$

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Table 2b: *Continued from previous page*

DAB	**DAB**^{r}	a	Δ_{stable}	$\Delta(C)/\Delta_{\text{stable}}$	f, where $C : y^2 = f$
$[389, 37, 245]$	$[5, 41, 389]$	$\frac{1}{5}\alpha^2 + \frac{18}{5}$	$(3a+1)_{11}^{12} \cdot (3a+2)_{11}^{12} \cdot$ $(4a+3)_{19}^{12} \cdot (4a+1)_{19}^{12} \cdot$ $(a+6)_{29}^{12} \cdot (2a+1)_{5}^{12}$	$(2)^{20} \cdot$ $(18a+13)_{389}^{10}$	$(-22952a - 6848)\,x^6 + (162272a - 61136)\,x^5 +$ $(296568a + 208208)\,x^4 + (-212600a - 959344)\,x^3 +$ $(89874a + 1610270)\,x^2 + (-428348a - 1023457)\,x + 315516a + 343397$

Table 2c

DAB	**DAB reflex**	a	**Obstructions**	**Curves**
$[8, 14, 41]$	$[41, 7, 2]$	$\alpha^2 + 3$	$(a+4)_2, (a-3)_2$	2
$[8, 26, 137]$	$[137, 13, 8]$	$\alpha^2 + 6$	$(3a-16)_2, (3a+19)_2$	2
$[8, 30, 153]$	$[17, 15, 18]$	$\frac{1}{3}\alpha^2 + 2$	$(a+2)_2, (a-1)_2$	4
$[12, 8, 13]$	$[13, 10, 12]$	$\frac{1}{2}\alpha^2 + 2$	$(a+1)_3, (2)$	2
$[12, 10, 13]$	$[13, 5, 3]$	$\alpha^2 + 2$	$(a+1)_3, (2)$	2
$[12, 14, 37]$	$[37, 7, 3]$	$\alpha^2 + 3$	$(a+3)_3, (2)$	2
$[12, 26, 61]$	$[61, 13, 27]$	$\alpha^2 + 6$	$(a-3)_3, (2)$	2
$[12, 26, 157]$	$[157, 13, 3]$	$\alpha^2 + 6$	$(a-6)_3, (2)$	2
$[12, 50, 325]$	$[13, 25, 75]$	$\frac{1}{5}\alpha^2 + 2$	$(a+1)_3, (2)$	4
$[44, 8, 5]$	$[5, 14, 44]$	$\frac{1}{2}\alpha^2 + 3$	$(2), (3a+2)_{11}$	2
$[44, 14, 5]$	$[5, 7, 11]$	$\alpha^2 + 3$	$(2), (3a+2)_{11}$	2
$[44, 42, 45]$	$[5, 21, 99]$	$\frac{1}{3}\alpha^2 + 3$	$(2), (3a+2)_{11}$	4
$[76, 18, 5]$	$[5, 9, 19]$	$\alpha^2 + 4$	$(2), (4a+3)_{19}$	2
$[172, 34, 117]$	$[13, 17, 43]$	$\frac{1}{3}\alpha^2 + \frac{7}{3}$	$(2), (4a+5)_{43}$	2
$[236, 32, 20]$	$[5, 16, 59]$	$\frac{1}{2}\alpha^2 + \frac{7}{2}$	$(2), (7a+5)_{59}$	2

Appendix A

A.1 List of Singular Points

The following table gives the list of the four singular points for the surface X_p, where $p \in \mathbb{P}^4_{[A,B,C,D,E]}$ lies on one of the 15 singular hyperplane.

Hyperplane	**List of Singular Points** ($i = \sqrt{-1}$)
$A = 0$	$[1:0:0:0], [0:1:0:0], [0:0:1:0], [0:0:0:1]$
$q_{+C} = 0$	$[1:1:0:0], [1:-1:0:0], [0:0:1:1], [0:0:1:-1]$
$q_{-C} = 0$	$[1:i:0:0], [1:-i:0:0], [0:0:1:i], [0:0:1:-i]$
$q_{+D} = 0$	$[1:0:1:0], [1:0:-1:0], [0:1:0:1], [0:1:0:-1]$
$q_{-D} = 0$	$[1:0:i:0], [1:0:-i:0], [0:1:0:i], [0:1:0:-i]$
$q_{+E} = 0$	$[1:0:0:1], [1:0:0:-1], [0:1:1:0], [0:1:-1:0]$
$q_{-E} = 0$	$[1:i:0:0], [1:-i:0:0], [0:1:i:0], [0:1:-i:0]$
$p_{+0} = 0$	$[1:1:1:1], [1:-1:1:-1], [1:1:-1:-1,], [1:-1:-1:1]$
$p_{-0} = 0$	$[1:1:1:-1], [1:1:-1:1], [1:-1:1:1], [1:-1:-1:-1]$
$p_{+1} = 0$	$[1:1:i:-i], [1:1:-i:i], [1:-1:-i:-i], [1:-1:i:i]$
$p_{-1} = 0$	$[1:1:i:i], [1:-1:i:-i], [1:-1:-i:i], [1:1:-i:-i]$
$p_{+2} = 0$	$[1:i:1:-i], [1:-i:1:i], [1:-i:-1:-i], [1:i:-1:i]$
$p_{-2} = 0$	$[1:i:1:i], [1:i:-1:-i], [1:-i:-1:i], [1:-i:1:-i]$
$p_{+3} = 0$	$[1:i:-i:1], [1:-i:i:1], [1:-i:-i:-1], [1:i:i:-1]$
$p_{-3} = 0$	$[1:i:i:1], [1:i:-i:-1], [1:-i:i:-1], [1:-i:-i:1]$

Table A.1: Table of singular surfaces and their singular points

A.2 List of Discriminants

The table below lists the 50 discriminants (up to squares) calculated for Theorem 3.1.5.

Field	**Discriminant**
$K\left(r_{1,1}^2 + \frac{1}{r_{1,1}^2}\right)$	$p_{+1} \cdot p_{-1} \cdot p_{-0} \cdot p_{+0}$
$K\left(r_{1,1}^2 + \overline{r}_{1,1}^2\right)$	$p_{+1} \cdot p_{-1} \cdot p_{+2} \cdot p_{-2}$
$K\left(r_{1,1} + \frac{1}{r_{1,1}} + \overline{r}_{1,1} + \frac{1}{\overline{r}_{1,1}}\right)$	$-1 \cdot p_{+1} \cdot p_{-0} \cdot p_{+2} \cdot p_{-2} \cdot q_{-E} \cdot q_{+D}$

Table A.2 List Of Discriminants

Field	**Discriminant**
$K\left(r_{2,1}+\frac{1}{r_{2,1}}+\overline{r}_{2,1}+\frac{1}{\overline{r}_{2,1}}\right)$	$-1\cdot p_{+1}\cdot p_{-1}\cdot p_{+0}\cdot p_{-2}\cdot q_{-E}\cdot q_{+C}$
$K\left(r_{\mu,1}+\frac{r_{\mu,1}}{r_{1,1}^2}+\overline{r}_{\mu,1}+\frac{\overline{r}_{\mu,1}}{\overline{r}_{1,1}^2}\right)$	$-1\cdot p_{+0}\cdot p_{-0}\cdot q_{+C}\cdot q_{-E}\cdot q_{+D}\cdot\Delta$
$K\left(r_{1,2}^2+\frac{1}{r_{1,2}^2}\right)$	$p_{+1}\cdot p_{-1}\cdot p_{-0}\cdot p_{+0}$
$K\left(r_{1,2}^2+\overline{r}_{1,2}^2\right)$	$p_{+3}\cdot p_{-3}\cdot p_{-0}\cdot p_{+0}$
$K\left(r_{1,2}+\frac{1}{r_{1,2}}+\overline{r}_{1,2}+\frac{1}{\overline{r}_{1,2}}\right)$	$-1\cdot p_{-1}\cdot p_{+0}\cdot p_{-3}\cdot p_{+3}\cdot q_{-D}\cdot q_{+E}$
$K\left(r_{2,2}+\frac{1}{r_{2,2}}+\overline{r}_{2,2}+\frac{1}{\overline{r}_{2,2}}\right)$	$p_{+1}\cdot p_{+3}\cdot q_{+E}\cdot q_{+C}$
$K\left(r_{\mu,2}+\frac{r_{\mu,1}}{r_{1,2}^2}+\overline{r}_{\mu,2}+\frac{\overline{r}_{\mu,1}}{\overline{r}_{1,2}^2}\right)$	$-1\cdot p_{+1}\cdot p_{-1}\cdot q_{+C}\cdot q_{-D}\cdot q_{+E}\cdot\Delta$
$K\left(r_{1,3}^2+\frac{1}{r_{1,3}^2}\right)$	$p_{+3}\cdot p_{-3}\cdot p_{+2}\cdot p_{-2}$
$K\left(r_{1,3}^2+\overline{r}_{1,3}^2\right)$	$p_{+3}\cdot p_{-3}\cdot p_{+0}\cdot p_{-0}$
$K\left(r_{1,3}+\frac{1}{r_{1,3}}+\overline{r}_{1,3}+\frac{1}{\overline{r}_{1,3}}\right)$	$p_{-3}\cdot p_{-2}\cdot q_{+E}\cdot q_{+D}$
$K\left(r_{2,3}+\frac{1}{r_{2,3}}+\overline{r}_{2,3}+\frac{1}{\overline{r}_{2,3}}\right)$	$-1\cdot p_{+3}\cdot p_{-3}\cdot p_{-2}\cdot p_{+0}\cdot q_{-C}\cdot q_{+E}$
$K\left(r_{\mu,3}+\frac{r_{\mu,3}}{r_{1,3}^2}+\overline{r}_{\mu,3}+\frac{\overline{r}_{\mu,3}}{\overline{r}_{1,3}^2}\right)$	$-1\cdot p_{+3}\cdot p_{-3}\cdot p_{+2}\cdot p_{-2}\cdot p_{+0}\cdot p_{-0}\cdot q_{-C}\cdot q_{+D}\cdot q_{+E}\cdot\Delta$
$K\left(r_{1,4}^2+\frac{1}{r_{1,4}^2}\right)$	$p_{+3}\cdot p_{-3}\cdot p_{+2}\cdot p_{-2}$
$K\left(r_{1,4}^2+\overline{r}_{1,4}^2\right)$	$p_{+1}\cdot p_{-1}\cdot p_{+2}\cdot p_{-2}$
$K\left(r_{1,4}+\frac{1}{r_{1,4}}+\overline{r}_{1,4}+\frac{1}{\overline{r}_{1,4}}\right)$	$p_{-3}\cdot p_{-2}\cdot q_{-D}\cdot q_{-E}$
$K\left(r_{2,4}+\frac{1}{r_{2,4}}+\overline{r}_{2,4}+\frac{1}{\overline{r}_{2,4}}\right)$	$p_{-1}\cdot p_{-3}\cdot q_{-C}\cdot q_{-E}$
$K\left(r_{\mu,4}+\frac{r_{\mu,4}}{r_{1,4}^2}+\overline{r}_{\mu,4}+\frac{\overline{r}_{\mu,4}}{\overline{r}_{1,4}^2}\right)$	$-1\cdot p_{+1}\cdot p_{-1}\cdot p_{+2}\cdot p_{-2}\cdot p_{+3}\cdot p_{-3}\cdot q_{-C}\cdot q_{-D}\cdot q_{-E}\cdot\Delta$
$K\left(r_{1,5}^2+\frac{1}{r_{1,5}^2}\right)$	$q_{+D}\cdot q_{+E}\cdot q_{-D}\cdot q_{-E}$
$K\left(r_{1,5}^2+\overline{r}_{1,5}^2\right)$	$-1\cdot p_{+0}\cdot p_{+1}\cdot p_{-2}\cdot p_{-3}\cdot q_{+D}\cdot q_{-D}$
$K\left(r_{1,5}+\frac{1}{r_{1,5}}+\overline{r}_{1,5}+\frac{1}{\overline{r}_{1,5}}\right)$	$p_{-2}\cdot p_{-3}\cdot q_{-D}\cdot q_{-E}$
$K\left(r_{2,5}+\frac{1}{r_{2,5}}+\overline{r}_{2,5}+\frac{1}{\overline{r}_{2,5}}\right)$	$A\cdot p_{+1}\cdot p_{-3}\cdot q_{-D}\cdot q_{+E}\cdot q_{-E}$
$K\left(r_{\mu,5}+\frac{r_{\mu,5}}{r_{1,5}^2}+\overline{r}_{\mu,5}+\frac{\overline{r}_{\mu,5}}{\overline{r}_{1,5}^2}\right)$	$-1\cdot A\cdot p_{+1}\cdot p_{+0}\cdot q_{-D}\cdot q_{-E}\cdot\Delta$
$K\left(r_{1,6}^2+\frac{1}{r_{1,6}^2}\right)$	$q_{+D}\cdot q_{+E}\cdot q_{-D}\cdot q_{-E}$
$K\left(r_{1,6}^2+\overline{r}_{1,6}^2\right)$	$-1\cdot p_{-0}\cdot p_{-1}\cdot p_{+2}\cdot p_{+3}\cdot q_{+E}\cdot q_{-E}$
$K\left(r_{1,6}+\frac{1}{r_{1,6}}+\overline{r}_{1,6}+\frac{1}{\overline{r}_{1,6}}\right)$	$p_{+2}\cdot p_{+3}\cdot q_{+D}\cdot q_{+E}$
$K\left(r_{2,6}+\frac{1}{r_{2,6}}+\overline{r}_{2,6}+\frac{1}{\overline{r}_{2,6}}\right)$	$A\cdot p_{-1}\cdot p_{+3}\cdot q_{-D}\cdot q_{+E}\cdot q_{-E}$
$K\left(r_{\mu,6}+\frac{r_{\mu,6}}{r_{1,6}^2}+\overline{r}_{\mu,6}+\frac{\overline{r}_{\mu,6}}{\overline{r}_{1,6}^2}\right)$	$A\cdot p_{-0}\cdot p_{-1}\cdot q_{+D}\cdot q_{+E}\cdot\Delta$
$K\left(r_{1,7}^2+\frac{1}{r_{1,7}^2}\right)$	$-1\cdot p_{+0}\cdot p_{-1}\cdot p_{+2}\cdot p_{-3}\cdot q_{-E}\cdot q_{+E}$

Table A.2 List Of Discriminants

Field	Discriminant
$K\left(r_{1,7}^2+\overline{r}_{1,7}^2\right)$	$q_{+C}\cdot q_{-C}\cdot q_{+E}\cdot q_{-E}$
$K\left(r_{1,7}+\frac{1}{r_{1,7}}+\overline{r}_{1,7}+\frac{1}{\overline{r}_{1,7}}\right)$	$-1\cdot A\cdot p_{+0}\cdot p_{-1}\cdot q_{-C}$
$K\left(r_{2,7}+\frac{1}{r_{2,7}}+\overline{r}_{2,7}+\frac{1}{\overline{r}_{2,7}}\right)$	$p_{-1}\cdot p_{-3}\cdot q_{-C}\cdot q_{-E}$
$K\left(r_{\mu,7}+\frac{r_{\mu,7}}{r_{1,7}^2}+\overline{r}_{\mu,7}+\frac{\overline{r}_{\mu,7}}{\overline{r}_{1,7}^2}\right)$	$A\cdot p_{-1}\cdot p_{-3}\cdot q_{+C}\cdot q_{-E}\cdot\Delta$
$K\left(r_{1,8}^2+\frac{1}{r_{1,8}^2}\right)$	$-1\cdot p_{-0}\cdot p_{+1}\cdot p_{-2}\cdot p_{+3}\cdot q_{+E}\cdot q_{-E}$
$K\left(r_{1,8}^2+\overline{r}_{1,8}^2\right)$	$q_{-C}\cdot q_{+C}\cdot q_{+E}\cdot q_{-E}$
$K\left(r_{1,8}+\frac{1}{r_{1,8}}+\overline{r}_{1,8}+\frac{1}{\overline{r}_{1,8}}\right)$	$A\cdot p_{-2}\cdot p_{+3}\cdot q_{-C}\cdot q_{+E}\cdot q_{-E}$
$K\left(r_{2,8}+\frac{1}{r_{2,8}}+\overline{r}_{2,8}+\frac{1}{\overline{r}_{2,8}}\right)$	$p_{+1}\cdot p_{+3}\cdot q_{+C}\cdot q_{+E}$
$K\left(r_{\mu,8}+\frac{r_{\mu,8}}{r_{1,8}^2}+\overline{r}_{\mu,8}+\frac{\overline{r}_{\mu,8}}{\overline{r}_{1,8}^2}\right)$	$A\cdot p_{-0}\cdot p_{-2}\cdot q_{+C}\cdot q_{+E}\cdot\Delta$
$K\left(r_{1,9}^2+\frac{1}{r_{1,9}^2}\right)$	$-1\cdot p_{+0}\cdot p_{-1}\cdot p_{-2}\cdot p_{+3}\cdot q_{+D}\cdot q_{-D}$
$K\left(r_{1,9}^2+\overline{r}_{1,9}^2\right)$	$q_{+C}\cdot q_{-C}\cdot q_{+D}\cdot q_{-D}$
$K\left(r_{1,9}+\frac{1}{r_{1,9}}+\overline{r}_{1,9}+\frac{1}{\overline{r}_{1,9}}\right)$	$A\cdot p_{-2}\cdot p_{+3}\cdot q_{-C}\cdot q_{+D}\cdot q_{-D}$
$K\left(r_{2,9}+\frac{1}{r_{2,9}}+\overline{r}_{2,9}+\frac{1}{\overline{r}_{2,9}}\right)$	$A\cdot p_{-1}\cdot p_{+3}\cdot q_{+C}\cdot q_{-C}\cdot q_{-D}$
$K\left(r_{\mu,9}+\frac{r_{\mu,9}}{r_{1,9}^2}+\overline{r}_{\mu,9}+\frac{\overline{r}_{\mu,9}}{\overline{r}_{1,9}^2}\right)$	$-1\cdot A\cdot p_{+0}\cdot p_{+3}\cdot q_{+C}\cdot q_{+D}\cdot\Delta$
$K\left(r_{1,10}^2+\frac{1}{r_{1,10}^2}\right)$	$-1\cdot p_{-0}\cdot p_{+1}\cdot p_{+2}\cdot p_{-3}\cdot q_{-D}\cdot q_{+D}$
$K\left(r_{1,10}^2+\overline{r}_{1,10}^2\right)$	$-1\cdot p_{-0}\cdot p_{+1}\cdot p_{+2}\cdot p_{-3}\cdot q_{+C}\cdot q_{-C}$
$K\left(r_{1,10}+\frac{1}{r_{1,10}}+\overline{r}_{1,10}+\frac{1}{\overline{r}_{1,10}}\right)$	$-1\cdot A\cdot p_{+1}\cdot p_{-0}\cdot q_{-C}$
$K\left(r_{2,10}+\frac{1}{r_{2,10}}+\overline{r}_{2,10}+\frac{1}{\overline{r}_{2,10}}\right)$	$A\cdot p_{+1}\cdot p_{-3}\cdot q_{+C}\cdot q_{-C}\cdot q_{-D}$
$K\left(r_{\mu,10}+\frac{r_{\mu,10}}{r_{1,10}^2}+\overline{r}_{\mu,10}+\frac{\overline{r}_{\mu,10}}{\overline{r}_{1,10}^2}\right)$	$A\cdot p_{-0}\cdot p_{-3}\cdot q_{+C}\cdot q_{+D}\cdot\Delta$

Table A.2 List Of Discriminants

A.3 List of Planes

Let $p=[A,B,C,D,E]\in\mathbb{P}^4$ be a general point not lying on the Segre cubic nor the 15 singular hyperplanes. There exists $10\cdot 16$ planes intersecting the quartic K3 surface X_p into $2\cdot 10\cdot 16$ conics (see Theorem 2.3.8). Each plane is associated to a singular point, q_i, of the Segre cubic. So let T_i be the set of the 16 planes associated to the point q_i. Then $T_i=\{\gamma\left(r_{0,i}x+r_{1,i}y+r_{2,i}z+r_{3,i}w\right)=0|\gamma\in\Gamma)\}$, where $r_{j,i}$ is listed below (c.f. Lemma 3.2.1)

The point $q_1=[1,0,-2,-2,2]$:

- $r_{1,0}=2^3B\sqrt{-q_{+D}q_{+C}q_{-E}}$

- $r_{1,1} = \sqrt{q_{+C}}\left(\sqrt{p_{-2}p_{-0}p_{+2}p_{+1}} + p_{+2}\sqrt{p_{+1}p_{+0}} + \sqrt{p_{-2}p_{-1}p_{+2}p_{+0}} + p_{-2}\sqrt{p_{-1}p_{-0}}\right)$
- $r_{1,2} = \sqrt{q_{+D}}\left(\sqrt{p_{-1}p_{-0}p_{+1}p_{+2}} + p_{+1}\sqrt{p_{+2}p_{+0}} + \sqrt{p_{-1}p_{-2}p_{+1}p_{+0}} + p_{-1}\sqrt{p_{-2}p_{-0}}\right)$
- $r_{1,3} = -\sqrt{-q_{-E}}\left(\sqrt{p_{-0}p_{-1}p_{+0}p_{+2}} + p_{+0}\sqrt{p_{+2}p_{+1}} + \sqrt{p_{-0}p_{-2}p_{+0}p_{+1}} + p_{-0}\sqrt{p_{-2}p_{-1}}\right)$

The point $q_2 = [1, 0, -2, 2, -2]$:

- $r_{2,0} = 2^3 B\sqrt{-q_{+C}q_{-D}q_{+E}}$
- $r_{2,1} = \sqrt{q_{+C}}\left(\sqrt{p_{-3}p_{-0}p_{+3}p_{+1}} + p_{+3}\sqrt{p_{+1}p_{+0}} + \sqrt{p_{-3}p_{-1}p_{+3}p_{+0}} + p_{-3}\sqrt{p_{-1}p_{-0}}\right)$
- $r_{2,2} = -\sqrt{-q_{-D}}\left(p_{-0}\sqrt{p_{-1}p_{-3}} + p_{+0}\sqrt{p_{+1}p_{+3}} + \sqrt{p_{-1}p_{-0}p_{+0}p_{+3}} + \sqrt{p_{-3}p_{-0}p_{+0}p_{+1}}\right)$
- $r_{2,3} = \sqrt{q_{+E}}\left(p_{-1}\sqrt{p_{-0}p_{-3}} + p_{+1}\sqrt{p_{+0}p_{+3}} + \sqrt{p_{-3}p_{-1}p_{+1}p_{+0}} + \sqrt{p_{-0}p_{-1}p_{+1}p_{+3}}\right)$

The point $q_3 = [1, 0, 2, -2, -2]$:

- $r_{3,0} = 2^3 B\sqrt{-q_{-C}q_{+D}q_{+E}}$
- $r_{3,1} = -\sqrt{-q_{-C}}\left(\sqrt{p_{-2}p_{-0}p_{+3}p_{+0}} + p_{+0}\sqrt{p_{+3}p_{+2}} + \sqrt{p_{-3}p_{-0}p_{+2}p_{+0}} + p_{-0}\sqrt{p_{-3}p_{-2}}\right)$
- $r_{3,2} = \sqrt{q_{+D}}\left(\sqrt{p_{-0}p_{-3}p_{+2}p_{+3}} + p_{+3}\sqrt{p_{+0}p_{+2}} + \sqrt{p_{-2}p_{-3}p_{+0}p_{+3}} + p_{-3}\sqrt{p_{-0}p_{-2}}\right)$
- $r_{3,3} = \sqrt{q_{+E}}(\sqrt{p_{-0}p_{-2}p_{+3}p_{+2}} + p_{+2}\sqrt{p_{+0}p_{+3}} + \sqrt{p_{-3}p_{-2}p_{+0}p_{+2}} + p_{-2}\sqrt{p_{-0}p_{-3}})$

The point $q_4 = [1, 0, 2, 2, 2]$:

- $r_{4,0} = 2^3 B\sqrt{-q_{-C}q_{-D}q_{-E}}$
- $r_{4,1} = \sqrt{-q_{-C}}\left(\sqrt{p_{-3}p_{-1}p_{+2}p_{+1}} + p_{+1}\sqrt{p_{+2}p_{+3}} + \sqrt{p_{-2}p_{-1}p_{+3}p_{+1}} + p_{-1}\sqrt{p_{-2}p_{-3}}\right)$
- $r_{4,2} = \sqrt{-q_{-D}}\left(\sqrt{p_{-1}p_{-2}p_{+3}p_{+2}} + p_{+2}\sqrt{p_{+1}p_{+3}} + \sqrt{p_{-3}p_{-2}p_{+1}p_{+2}} + p_{-2}\sqrt{p_{-1}p_{-3}}\right)$
- $r_{4,3} = \sqrt{-q_{-E}}\left(\sqrt{p_{-2}p_{-3}p_{+1}p_{+3}} + p_{+3}\sqrt{p_{+1}p_{+2}} + \sqrt{p_{-1}p_{-3}p_{+2}p_{+3}} + p_{-3}\sqrt{p_{-1}p_{-2}}\right)$

The point $q_5 = [0, 2, -1, 0, 0]$:

- $r_{5,0} = 4\sqrt{A}\left(\Delta - 2B(AB + 2AC - DE)\right)$
- $$r_{5,1} = \sqrt{A}\cdot\Big(p_{+1}\sqrt{p_{-3}p_{-2}q_{+E}q_{+D}} - p_{+0}\sqrt{p_{-3}p_{-2}q_{-E}q_{-D}} + p_{-2}\sqrt{-p_{+1}p_{+0}q_{+E}q_{-D}} + p_{-3}\sqrt{-p_{+1}p_{+0}q_{-E}q_{+D}}\Big)$$
- $$r_{5,2} = 2\cdot\Big(q_{+D}\sqrt{q_{-E}q_{+E}q_{-D}p_{-3}p_{+1}} - q_{-D}\sqrt{-q_{-E}q_{+E}q_{+D}p_{-2}p_{+0}} + (AB + 2AC - DE)\cdot\left(\sqrt{-q_{-D}p_{-2}p_{+0}} + \sqrt{q_{+D}p_{-3}p_{+1}}\right)\Big)$$
- $$r_{5,3} = 2\cdot\Big(q_{+E}\sqrt{q_{-D}q_{+D}q_{-E}p_{-2}p_{+1}} - q_{-E}\sqrt{-q_{-D}q_{+D}q_{+E}p_{-3}p_{+0}} + (AB + 2AC - DE)\cdot\left(\sqrt{-q_{-E}p_{-3}p_{+0}} + \sqrt{q_{+E}p_{-2}p_{+1}}\right)\Big)$$

The point $q_6 = [0, 2, 1, 0, 0]$:

- $r_{6,0} = 4\sqrt{A}\,(\Delta - 2B(AB - 2AC + DE))$
- $r_{6,1} = \sqrt{A} \cdot \Big(p_{-1}\sqrt{p_{+3}p_{+2}q_{+D}q_{+E}} - p_{-0}\sqrt{p_{+3}p_{+2}q_{-D}q_{-E}}$
  $+p_{+2}\sqrt{-p_{-0}p_{-1}q_{-D}q_{+E}} + p_{+3}\sqrt{-p_{-0}p_{-1}q_{+D}q_{-E}}\Big)$
- $r_{6,2} = -2 \cdot \Big(- q_{+D}\sqrt{q_{-D}q_{+E}q_{-E}p_{-1}p_{+3}} + q_{-D}\sqrt{-q_{+D}q_{+E}q_{-E}p_{-0}p_{+2}}$
  $+(AB - 2AC + DE) \cdot \big(\sqrt{q_{+D}p_{-1}p_{+3}} + \sqrt{-q_{-D}p_{-0}p_{+2}}\big)\Big)$
- $r_{6,3} = 2 \cdot \Big(- q_{+E}\sqrt{q_{-D}q_{+D}q_{-E}p_{-1}p_{+2}} + q_{-E}\sqrt{-q_{+D}q_{-D}q_{+E}p_{-0}p_{+3}}$
  $+(AB - 2AC + DE) \cdot \big(\sqrt{q_{+E}p_{-1}p_{+2}} + \sqrt{-q_{-E}p_{-0}p_{+3}}\big)\Big)$

The point $q_7 = [0, 2, 0, -1, 0]$:

- $r_{7,0} = 4\sqrt{A}\,(\Delta - 2B(AB + 2AD - CE))$
- $r_{7,1} = 2 \cdot \Big(q_{+C}\sqrt{q_{-C}q_{+E}q_{-E}p_{+2}p_{-3}} - q_{-C}\sqrt{-q_{+C}q_{+E}q_{-E}p_{+0}p_{-1}}$
  $+(AB + 2AD - CE) \cdot \big(\sqrt{q_{+C}p_{+2}p_{-3}} + \sqrt{-q_{-C}p_{+0}p_{-1}}\big)\Big)$
- $r_{7,2} = \sqrt{A} \cdot \Big(p_{-1}\sqrt{-q_{+E}q_{-C}p_{+2}p_{+0}} - p_{+0}\sqrt{q_{-E}q_{-C}p_{-1}p_{-3}}$
  $+p_{+2}\sqrt{q_{+C}q_{+E}p_{-1}p_{p-3}} + p_{-3}\sqrt{-q_{+C}q_{-E}p_{+0}p_{+2}}\Big)$
- $r_{7,3} = 2 \cdot \Big(q_{+E}\sqrt{q_{-C}q_{+C}q_{-E}p_{-1}p_{+2}} - q_{-E}\sqrt{-q_{-C}q_{+C}q_{+E}p_{+0}p_{-3}}$
  $+(AB + 2AD - CE) \cdot \big(\sqrt{q_{+E}p_{-1}p_{+2}} + \sqrt{-q_{-E}p_{+0}p_{-3}}\big)\Big)$

The point $q_8 = [0, 2, 0, 1, 0]$:

- $r_{8,0} = 4\sqrt{A}\,(\Delta - 2B(AB - 2AD + CE))$
- $r_{8,1} = 2 \cdot \Big(- q_{+C}\sqrt{q_{-C}q_{+E}q_{-E}p_{-2}p_{+3}} + q_{-C}\sqrt{-q_{+C}q_{+E}q_{-E}p_{-0}p_{+1}}$
  $+(AB - 2AD + CE) \cdot \big(\sqrt{q_{+C}p_{-2}p_{+3}} + \sqrt{-q_{-C}p_{-0}p_{+1}}\big)\Big)$
- $r_{8,2} = \sqrt{A} \cdot \Big(p_{+1}\sqrt{-q_{-C}q_{+E}p_{-0}p_{-2}} - p_{-0}\sqrt{q_{-C}q_{-E}p_{+1}p_{+3}}$
  $+p_{-2}\sqrt{q_{+C}q_{+E}p_{+1}p_{+3}} + p_{+3}\sqrt{-q_{+C}q_{-E}p_{-0}p_{-2}}\Big)$
- $r_{8,3} = -2 \cdot \Big(- q_{+E}\sqrt{q_{-C}q_{+C}q_{-E}p_{+1}p_{-2}} + q_{-E}\sqrt{-q_{+C}q_{-C}q_{+E}p_{-0}p_{+3}}$
  $+(AB - 2AD + CE) \cdot \big(\sqrt{q_{+E}p_{+1}p_{-2}} + \sqrt{-q_{-E}p_{-0}p_{+3}}\big)\Big)$

The point $q_9 = [0, 2, 0, 0, -1]$:

- $r_{9,0} = 4\sqrt{A}\,(\Delta - 2B(AB + 2AE - CD))$
- $r_{9,1} = 2 \cdot \Big(q_{+C}\sqrt{q_{-C}q_{-D}q_{+D}p_{-2}p_{+3}} - q_{-C}\sqrt{-q_{+C}q_{-D}q_{+D}p_{+0}p_{-1}}$
$$+(AB + 2AE - CD) \cdot \big(\sqrt{q_{+C}p_{-2}p_{+3}} + \sqrt{-q_{-C}p_{+0}p_{-1}}\big)\Big)$$
- $r_{9,2} = 2 \cdot \Big(q_{+D}\sqrt{q_{+C}q_{-C}q_{+D}p_{-1}p_{+3}} - q_{-D}\sqrt{-q_{+C}q_{-C}q_{+D}p_{+0}p_{-2}}$
$$+(AB + 2AE - CD) \cdot \big(\sqrt{q_{+D}p_{-1}p_{+3}} + \sqrt{-q_{-D}p_{+0}p_{-2}}\big)\Big)$$
- $r_{9,3} = \sqrt{A} \cdot \Big(p_{-1}\sqrt{-q_{-C}q_{+D}p_{+0}p_{+3}} - p_{+0}\sqrt{q_{-C}q_{-D}p_{-1}p_{-2}}$
$$+p_{-2}\sqrt{-q_{+C}q_{-D}p_{+0}p_{+3}} + p_{+3}\sqrt{q_{+C}q_{+D}p_{-1}p_{-2}}\Big)$$

The point $q_{10} = [0, 2, 0, 0, 1]$:

- $r_{10,0} = 4\sqrt{A}\,(\Delta - 2B(AB - 2AE + CD))$
- $r_{10,1} = -2 \cdot \Big(- q_{+C}\sqrt{q_{-C}q_{+D}q_{-D}p_{+2}p_{-3}} + q_{-C}\sqrt{-q_{+C}q_{-D}q_{+D}p_{-0}p_{+1}}$
$$+(AB - 2AE + CD) \cdot \big(\sqrt{q_{+C}p_{+2}p_{-3}} + \sqrt{-q_{-C}p_{-0}p_{+1}}\big)\Big)$$
- $r_{10,2} = 2 \cdot \Big(- q_{+D}\sqrt{q_{-C}q_{+C}q_{-D}p_{+1}p_{-3}} + q_{-D}\sqrt{-q_{-C}q_{+C}q_{+D}p_{-0}p_{+2}}$
$$+(AB - 2AE + CD) \cdot \big(\sqrt{q_{+C}p_{+2}p_{-3}} + \sqrt{-q_{-C}p_{-0}p_{+1}}\big)\Big)$$
- $r_{9,3} = \sqrt{A} \cdot \Big(p_{+1}\sqrt{-q_{-C}q_{+D}p_{-0}p_{-3}} - p_{-0}\sqrt{q_{-C}q_{-D}p_{+1}p_{+2}}$
$$+p_{+2}\sqrt{-q_{+C}q_{-D}p_{-0}p_{-3}} + p_{-3}\sqrt{q_{+C}q_{+D}p_{+1}p_{+2}}\Big)$$

A.4 The Equations of the Lines

In the table below (Table A.4) let p lie on the tangent cone to q_i on N_5, the Neron surface. Then X_p has eight lines (see Proposition 4.1.3), of which we list two. The other lines can be obtained through the action of Γ on X. Note that the pair of lines come from two different ruling of the associated quadric.

Point	**Associated Quadric**	**Lines**
q_1	$x^2-y^2-z^2+w^2$	$2\sqrt{q_{+C}}x+\sqrt{p_{-1}}z+\sqrt{-p_{+0}}w=2\sqrt{q_{+C}}y+\sqrt{-p_{+0}}z+\sqrt{p_{-1}}w=0$ $2\sqrt{q_{+C}}x+\sqrt{p_{+1}}z+\sqrt{-p_{-0}}w=2\sqrt{q_{+C}}y-\sqrt{-p_{-0}}z-\sqrt{p_{+1}}w=0$
q_2	$x^2-y^2+z^2-w^2$	$2\sqrt{q_{+C}}x+\sqrt{-p_{+0}}z+\sqrt{p_{-1}}w=2\sqrt{q_{+C}}y+\sqrt{p_{-1}}z+\sqrt{-p_{+0}}w=0$ $2\sqrt{q_{+C}}x+\sqrt{-p_{-0}}z+\sqrt{p_{+1}}w=2\sqrt{q_{+C}}y-\sqrt{p_{+1}}z-\sqrt{-p_{-0}}w=0$
q_3	$x^2+y^2-z^2-w^2$	$2\sqrt{q_{-C}}x+\sqrt{p_{+3}}z+\sqrt{p_{-2}}w=2\sqrt{q_{-C}}y-\sqrt{p_{-2}}z+\sqrt{p_{+3}}w=0$ $2\sqrt{q_{-C}}x+\sqrt{p_{-3}}z+\sqrt{p_{+2}}w=2\sqrt{q_{-C}}y+\sqrt{p_{+2}}z-\sqrt{p_{-3}}w=0$
q_4	$x^2+y^2+z^2+w^2$	$2\sqrt{q_{-C}}x+\sqrt{-p_{-2}}z+\sqrt{-p_{+3}}w=2\sqrt{q_{-C}}y-\sqrt{-p_{+3}}z+\sqrt{-p_{-2}}w=0$ $2\sqrt{q_{-C}}x+\sqrt{-p_{+2}}z+\sqrt{-p_{-3}}w=2\sqrt{q_{-C}}y+\sqrt{-p_{-3}}z-\sqrt{-p_{+2}}w=0$
q_5	$xy-zw$	$2\sqrt{A}x+\left(\sqrt{q_{-D}}+\sqrt{-q_{+D}}\right)z=2\sqrt{A}y+\left(\sqrt{q_{-D}}-\sqrt{-q_{+D}}\right)w=0$ $2\sqrt{A}x+\left(\sqrt{q_{-E}}+\sqrt{-q_{+E}}\right)w=2\sqrt{A}y+\left(\sqrt{q_{-E}}-\sqrt{-q_{+E}}\right)z=0$
q_6	$xy+zw$	$2\sqrt{A}x+\left(\sqrt{q_{-D}}+\sqrt{-q_{+D}}\right)z=2\sqrt{A}y-\left(\sqrt{q_{-D}}-\sqrt{-q_{+D}}\right)w=0$ $2\sqrt{A}x+\left(\sqrt{q_{-E}}+\sqrt{-q_{+E}}\right)w=2\sqrt{A}y-\left(\sqrt{q_{-E}}-\sqrt{-q_{+E}}\right)z=0$
q_7	$xz-yw$	$2\sqrt{A}x+\left(\sqrt{q_{-C}}+\sqrt{-q_{+C}}\right)y=2\sqrt{A}z+\left(\sqrt{q_{-C}}-\sqrt{-q_{+C}}\right)w=0$ $2\sqrt{A}x+\left(\sqrt{q_{-E}}+\sqrt{-q_{+E}}\right)w=2\sqrt{A}z+\left(\sqrt{q_{-E}}-\sqrt{-q_{+E}}\right)zy=0$
q_8	$xz+yw$	$2\sqrt{A}x+\left(\sqrt{q_{-C}}+\sqrt{-q_{+C}}\right)y=2\sqrt{A}z-\left(\sqrt{q_{-C}}-\sqrt{-q_{+C}}\right)w=0$ $2\sqrt{A}x+\left(\sqrt{q_{-E}}+\sqrt{-q_{+E}}\right)w=2\sqrt{A}z-\left(\sqrt{q_{-E}}-\sqrt{-q_{+E}}\right)y=0$
q_9	$xw-yz$	$2\sqrt{A}x+\left(\sqrt{q_{-C}}+\sqrt{-q_{+C}}\right)y=2\sqrt{A}w+\left(\sqrt{q_{-C}}-\sqrt{-q_{+C}}\right)z=0$ $2\sqrt{A}x+\left(\sqrt{q_{-D}}+\sqrt{-q_{+D}}\right)z=2\sqrt{A}w+\left(\sqrt{q_{-D}}-\sqrt{-q_{+D}}\right)y=0$
q_{10}	$xw+yz$	$2\sqrt{A}x+\left(\sqrt{q_{-C}}+\sqrt{-q_{+C}}\right)y=2\sqrt{A}w-\left(\sqrt{q_{-C}}-\sqrt{-q_{+C}}\right)z=0$ $2\sqrt{A}x+\left(\sqrt{q_{-D}}+\sqrt{-q_{+D}}\right)z=2\sqrt{A}w-\left(\sqrt{q_{-D}}-\sqrt{-q_{+D}}\right)y=0$

Table A.4: Equations of Lines

A.5 List of Gram Matrices

For Proposition 4.2.6, for each of the families $\mathcal{X}, \mathcal{X}_{C,D,E}, \mathcal{X}_{C,D}, \mathcal{X}_B$ and $\mathcal{X}_C$, we calculated the intersection matrix of the lines and conics of a general member of that family. As discussed in the proof of Proposition 4.2.7, for each intersection matrix we extracted a full rank minor which we list below. For the surface Y, by seeing the intersection matrix as a lattice, we calculate an integral basis to extract a minor with minimal discriminant.

For a very general member of the family $\mathcal{X}$ a full rank minor, denoted by M, is:

$$\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2
\end{pmatrix}$$

For a very general member of the family $\mathcal{X}_{C,D,E}$, a full rank minor, denoted by $M_{C,D,E}$, is:

$$\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & 1 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2
\end{pmatrix}$$

For a very general member of the family $\mathcal{X}_{C,D}$, a full rank minor, denoted $M_{C,D}$, is:

$$\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 0 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 & 0 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -2
\end{pmatrix}$$

For a very general member of the family $\mathcal{X}_B$, a full rank minor, denoted M_B, is:

$$\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 1 & 0 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 1 & 0 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 0 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 1 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 & 1 & 1 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & -2 & 1 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & -2
\end{pmatrix}$$

For a very general member of the family $\mathcal{X}_C$, a full rank minor, denoted M_C, is:

$$\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 0 & 1 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 1 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 0 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 & 0 & 1 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 & 1 & 1 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & 1 & 0 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & -2 & 0 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & -2
\end{pmatrix}$$

For the surface Y, a full rank minor of minimal discriminant, denoted M_Y, is:

$$\begin{pmatrix}
-2 & 0 & 0 & 2 & 2 & 0 & 0 & 1 & 2 & 1 & 0 & 0 & 0 & 1 & 2 & 0 & 1 & 2 & 1 & 0 \\
0 & -2 & 2 & 2 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 2 & 1 & 0 \\
0 & 2 & -2 & 0 & 2 & 2 & 1 & 2 & 1 & 1 & 0 & 0 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 \\
2 & 2 & 0 & -2 & 2 & 2 & 2 & 1 & 0 & 0 & 1 & 1 & 1 & 2 & 1 & 1 & 1 & 0 & 0 & 1 \\
2 & 0 & 2 & 2 & -2 & 2 & 0 & 2 & 1 & 2 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 2 & 2 & 2 & -2 & 1 & 0 & 2 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 2 & 0 & 0 \\
0 & 0 & 1 & 2 & 0 & 1 & -2 & 1 & 2 & 1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 1 & 1 & 0 \\
1 & 1 & 2 & 1 & 2 & 0 & 1 & -2 & 1 & 1 & 2 & 2 & 0 & 0 & 2 & 1 & 1 & 1 & 0 & 0 \\
2 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 2 & 1 & 2 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0 & 2 & 0 & 1 & 1 & 2 & -2 & 0 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 1 & 0 & -2 & 0 & 2 & 2 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 2 & 2 & 0 & 0 & -2 & 1 & 1 & 0 & 2 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 2 & 2 & 1 & -2 & 0 & 2 & 0 & 1 & 1 & 0 & 0 \\
1 & 0 & 2 & 2 & 0 & 0 & 0 & 0 & 1 & 2 & 2 & 1 & 0 & -2 & 2 & 2 & 0 & 1 & 0 & 0 \\
2 & 1 & 1 & 1 & 1 & 1 & 2 & 2 & 1 & 1 & 1 & 0 & 2 & 2 & -2 & 2 & 1 & 0 & 1 & 2 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 0 & 2 & 2 & -2 & 1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & -2 & 0 & 0 & 0 \\
2 & 2 & 0 & 0 & 0 & 2 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & -2 & 0 & 1 \\
1 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & -2 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & -2
\end{pmatrix}$$

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