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**Proceedings Paper:**

Glass, AMW and Macpherson, HD [orcid.org/0000-0003-0277-7561](https://orcid.org/0000-0003-0277-7561) (2017) Permutation groups without irreducible elements. In: Droste, M, Fuchs, L, Goldsmith, B and Strümgmann, L, (eds.) Groups, Modules, and Model Theory - Surveys and Recent Developments: In Memory of Rüdiger Göbel. New Pathways between Group Theory and Model Theory, 01-04 Feb 2016, Mülheim an der Ruhr, Germany. Springer , pp. 331-332. ISBN 978-3-319-51717-9

[https://doi.org/10.1007/978-3-319-51718-6\\_17](https://doi.org/10.1007/978-3-319-51718-6_17)

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# Permutation groups without irreducible elements

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May 10, 2016

Dedicated to the memory of Rüdiger Göbel

## Abstract

We call a non-identity element of a permutation group irreducible if it cannot be written as a product of non-identity elements of disjoint support. We show that it is indeed possible for a sublattice subgroup of  $\text{Aut}(\mathbb{R}, \leq)$  to have no irreducible elements and still be transitive on the set of pairs  $\alpha < \beta$  in  $\mathbb{R}$ . This answers a question raised in “The first-order theory of  $\ell$ -permutation groups”, a Conference talk by the first author.

$G \upharpoonright H$

Let  $(\Omega, \leq)$  be a totally ordered set and  $G$  be a subgroup of  $\text{Aut}(\Omega, \leq)$ . Let  $1$  be the identity element of  $\text{Aut}(\Omega, \leq)$  and  $g \in G \setminus \{1\}$ . Then  $g$  is said to be *irreducible* if  $g = g_1 g_2$  with  $g_1, g_2 \in G$  and  $\text{supp}(g_1) \cap \text{supp}(g_2) = \emptyset$  implies  $g_1 = 1$  or  $g_2 = 1$ . Note that if  $G = \text{Aut}(\Omega, \leq)$ , then  $g \in G$  is irreducible if and only if  $g$  has a single supporting interval; *i. e.*, there is  $\sigma \in \text{supp}(g)$  such that the convexification in  $\Omega$  of  $\{\sigma g^n \mid n \in \mathbb{Z}\}$  is  $\text{supp}(g)$ . We prove:

**Theorem A.** *There is an  $\ell$ -subgroup of  $\text{Aut}(\mathbb{R}, \leq)$  that is transitive on ordered pairs  $\alpha < \beta$  and has no irreducible elements.*

Here, an  $\ell$ -subgroup of  $\text{Aut}(\mathbb{R}, \leq)$  is a subgroup  $G$  of  $\text{Aut}(\mathbb{R}, \leq)$  such that  $g_+ \in G$  whenever  $g \in G$ , where  $\alpha g_+ := \alpha g$  if  $\alpha g \geq \alpha$  and  $\alpha g_+ = \alpha$  if  $\alpha g \leq \alpha$

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2010 AMS Classification: 20B22, 06F15.

Keywords: order-preserving permutation,  $\ell$ -permutation group.

( $\alpha \in \mathbb{R}$ ). In particular,  $G$  is a lattice-ordered group where  $f \vee g = (fg^{-1} \vee 1)g$  and  $f \wedge g = (f^{-1} \vee g^{-1})^{-1}$ . For background on ordered permutation groups and  $\ell$ -groups see [1].

*Proof of Theorem A.* Let  $g \in \text{Aut}(\mathbb{R}, \leq)$ . We say that  $g$  has period  $n \in \mathbb{Z}_+$  if  $(\alpha + n)g = \alpha g + n$  for all  $\alpha \in \mathbb{R}$ . Let

$$G := \{g \in \text{Aut}(\mathbb{R}, \leq) \mid (\exists m \in \mathbb{Z}_+)(g \text{ has period } m)\}.$$

Then  $G$  is transitive on ordered pairs  $\alpha < \beta$  in  $\mathbb{R}$  and it is easily checked that  $(G, \mathbb{R})$  is an  $\ell$ -permutation group. Obviously, if  $f \in G$  fixes no point in  $\mathbb{R}$ , then it must be irreducible. So  $G$  has irreducible elements. On the other hand, if  $g \in G$  has period  $m$  and is not the identity but fixes  $\alpha_0 \in \mathbb{R}$  (and so fixes  $\alpha_0 + km$  for all  $k \in \mathbb{Z}$ ), define  $g_1, g_2 \in G$ , each with periods  $2m$ , as follows:

$$g_1(x) = \begin{cases} g(x) & \text{if } x \in [\alpha_0 + 2km, \alpha_0 + (2k+1)m), \quad k \in \mathbb{Z} \\ x & \text{if } x \in [\alpha_0 + (2k+1)m, \alpha_0 + (2k+2)m), \quad k \in \mathbb{Z} \end{cases}$$

$$g_2(x) = \begin{cases} g(x) & \text{if } x \in [\alpha_0 + (2k+1)m, \alpha_0 + (2k+2)m), \quad k \in \mathbb{Z} \\ x & \text{if } x \in [\alpha_0 + 2km, \alpha_0 + (2k+1)m), \quad k \in \mathbb{Z} \end{cases}.$$

Then  $g_1$  and  $g_2$  have disjoint supports and  $g = g_1 g_2$ , so  $g$  is reducible. Thus if  $H := \{g \in G \mid 0g = 0\}$ , then  $H$  has no irreducible elements. Now  $H$  acts faithfully on  $\mathbb{R}_+$  and  $(H \upharpoonright \mathbb{R}_+, \mathbb{R}_+)$  (the permutation group induced by  $H$  on  $\mathbb{R}_+$ ) is an  $\ell$ -permutation group that is transitive on ordered pairs  $\alpha < \beta$  in  $\mathbb{R}_+$ . Consequently we obtain an  $\ell$ -permutation group  $(H^*, \mathbb{R})$  that is transitive on pairs  $\alpha < \beta$  in  $\mathbb{R}$  and has no irreducible elements. For let  $\varphi : \mathbb{R} \rightarrow \mathbb{R}_+$  be an order-preserving bijection between  $\mathbb{R}$  and  $\mathbb{R}_+$  and  $h^* \in \text{Aut}(\mathbb{R}, \leq)$  be given by  $\alpha h^* = (\alpha \varphi) h \varphi^{-1}$  ( $\alpha \in \mathbb{R}, h \in H$ ). Then the desired properties transfer from  $H$  (acting on  $\mathbb{R}_+$ ) to  $H^* = \{h^* : h \in H\}$  (acting on  $\mathbb{R}$ ).  $\square$

The above proof can similarly be adapted to  $\ell$ -permutation groups  $(L, \mathbb{Q})$  that are transitive on pairs  $\alpha < \beta$  in  $\mathbb{Q}$  and have no irreducible elements.

**Acknowledgment.** We are most grateful to Queens' College, Cambridge and the Engineering and Physical Sciences Research Council (grant EP/K020692/1) for funding to attend the Conference in memory of Rüdiger Göbel.

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