Some properties for probabilistic and multinomial (probabilistic) values on cooperative games *

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Abstract

We investigate the conditions for the coefficients of probabilistic and multinomial values of cooperative games necessary and / or sufficient in order to satisfy some properties, including marginal contributions, balanced contributions, desirability relation and null player exclusion property. Moreover, a similar analysis is conduced for transfer property of probabilistic power indices on the domain of simple games.

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1 Introduction

Weber's general model for assessing cooperative games [27] is based on *probabilistic values*, a family of values axiomatically characterized by means of linearity, positivity, and the dummy player property. Every probabilistic value allocates, to each player in each game of its domain, a weighted (convex) sum of the marginal contributions of the player in the game. These allocations can be interpreted as a measure of players' bargaining relative strength. The most conspicuous member of this family (in fact, the inspiring one) is the *Shapley value* [26]. In the present paper we also focus on a subfamily of probabilistic values called *multinomial (probabilistic) values*. These values were introduced in reliability by Puente [24] (see also [15]) with the name of "multibinary probabilistic values." They were independently defined by Carreras [2], for simple games only —i.e. as power indices—, in a work on decisiveness (see also [3]) where they were called "Banzhaf α–indices." As it is shown in Carreras and

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Puente [8], multinomial values (n parameters, n being the number of players) offer a deal of flexibility clearly greater than *binomial semivalues* (one parameter) [1, 24], and hence many more possibilities to introduce additional information when evaluating a game. Technically, their main characteristic is the *systematic generation of the weighting coefficients in terms of a few parameters (one parameter per player).* In [17] the multinomial values are used to study the effects of the partnership formation in cooperative games, comparing the joint effect on the involved players with the alternative alliance formation. Recently, Carreras and Puente [9] introduced the *coalitional multinomial probabilistic values*, a new family of coalitional values designed to take into account players' attitudes with regard cooperation. This new family applies to cooperative games with a coalition structure by combining the Shapley value and the multinomial probabilistic values.

For more than a decade, our research group has been studying *semivalues*, a subfamily of probabilistic values introduced by Dubey et al. [12], characterized by anonymity, and including the Shapley value as the only *efficient* member. In the analysis of certain cooperative problems we have successfully used *binomial semivalues*, a single parametric subfamily defined by Puente that includes the Banzhaf value introduced by Owen [22].[1]

Fig. 1 describes the relationships between the above values and families of values and the main characteristics of each one of them.

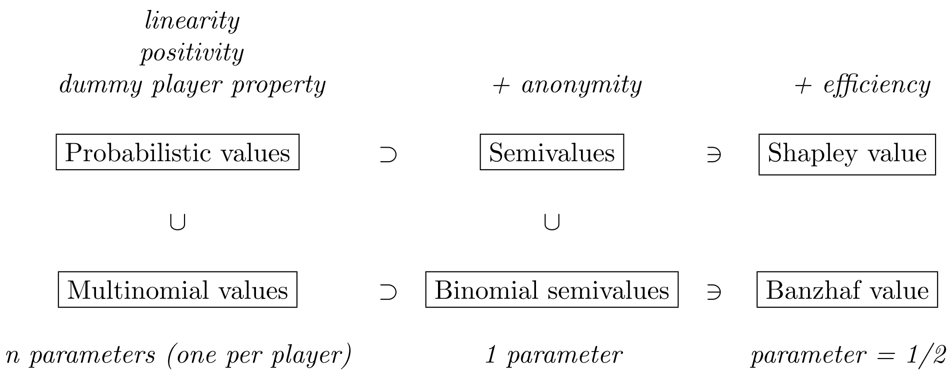


Fig. 1: Inclusion relationships between values and families of values

Indeed, parameters defining probabilistic values (Weber [27]) and semivalues (Dubey et al. [12]), and in particular multinomial values and binomial semivalues [24], introduce in the evaluation of games additional information not stored in the characteristic function. These parameters will be addressed here to cope with different attitudes that the players may hold when playing a given game, even if they are not individuals but countries, enterprizes, parties, trade unions, or collectivities of any other kind. We will attach to parameter p_i the meaning of *generical tendency of player i to form coalitions*, assuming p_i and p_j independent of each other if $i \neq j$.

[1] [5], [6] and [7] are samples of our work in this line.

Summing up, the paper tries to contribute to a better understanding of multinomial values as a consistent alternative or complement to classical values (Shapley and Banzhaf). The fact that they are based on tendency profiles provides new tools to encompass a wide variety of situations arising from players' personality when playing a given game. In this sense, multinomial values represent a wide generalization of binomial semivalues, whose single parametric condition implies a quite limited capability of analysis of such situations. Of course, these situations can neither be analyzed, without modifying the game, by means of the classical values, which can be concerned only with the structure of the game.

The organization of the paper is then as follows. Section 2 includes a minimum of preliminaries. In Section 3, we specialize to multinomial values as a natural generalization of the binomial semivalues. In Section 4 we compare the behavior of probabilistic values and multinomial values with respect to several standard properties for values and power indices, concerning null and nonnull players, balanced contributions, dominance, monotonicity, sensitivity and donation. The notions of *regularity* and *hereditary value* arise in a natural way as a convenient condition to guarantee the validity of some of them. Particularly, we characterize the class of regular values within the class of probabilistic values and, the class of solutions satisfying the balanced contributions property within the class of regular and hereditary regular probabilistic values. Section 5 focuses on multinomial values and provides a measure of the effect of a variation of p_j $(j \neq i)$ on the payoff to i by investigating second partial derivatives of the multilinear extension of the games. Section 6 contains an example of application of the multinomial values. Finally, Section 7 provides a summary of the paper and concluding remarks. Proofs of the statements in Sections 4 and 5 will be found in Appendices A and B, respectively.

2 Preliminaries

Let N be a finite set of *players*, where N is any set of natural numbers, and 2^N be the set of its *coalitions* (subsets of N). A (TU) *cooperative game* on N is a function $v : 2^N \to \mathbb{R}$ that assigns a real number $v(S)$ to each coalition $S \subseteq N$, with $v(\emptyset) = 0$. This number is usually understood as the utility that coalition S can obtain by itself, that is, independently of the actions of the remaining players.

A game v is *monotonic* iff $v(S) \leq v(T)$ whenever $S \subset T \subseteq N$. A player $i \in N$ is a *dummy* in v iff $v(S \cup \{i\}) = v(S) + v(\{i\})$ for all $S \subseteq N \backslash \{i\}$, and *null* iff, moreover, $v(\{i\}) = 0$. Two players $i, j \in N$ are *symmetric* in v iff $v(S \cup \{i\}) = v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i, j\}$.

Endowed with the natural operations for real–valued functions, i.e. $v + v'$ and λv for all $\lambda \in \mathbb{R}$, the set of all cooperative games on N is a vector space $\mathcal{G}_N$. For every nonempty coalition $T \subseteq N$, the *unanimity game* u_T is defined by $u_T(S) = 1$ if $T \subseteq S$ and $u_T(S) = 0$ otherwise, and it is easily checked that the set of all unanimity games is a basis for $\mathcal{G}_N$, so that $\dim \mathcal{G}_N = 2^n - 1$. Finally, every permutation θ on N induces a linear automorphism of $\mathcal{G}_N$ given by $(\theta v)(S) = v(\theta^{-1} S)$ for all $S \subseteq N$ and all v.

By a *value* on $\mathcal{G}_N$ we will mean a map $g : \mathcal{G}_N \to \mathbb{R}^n$, $n = |N|$, which assigns to every game v a vector $g[v]$ with components $g_i[v]$ for all $i \in N$.

Following Weber's [27] axiomatic definition, $\phi : \mathcal{G}_N \to \mathbb{R}^n$ is a (group) *probabilistic value* iff it satisfies the following properties:

(i) *linearity*: $\phi[v+v'] = \phi[v]+\phi[v']$ and $\phi[\lambda v] = \lambda\phi[v]$ for all $v, v' \in \mathcal{G}_N$ and $\lambda \in \mathbb{R}$;
(ii) *positivity*[2]: if v is monotonic, then $\phi[v] \geq 0$;
(iii) *dummy player property*: if $i \in N$ is a dummy in game v, then $\phi_i[v] = v(\{i\})$.

There is an interesting characterization of the probabilistic values, also in [27]: (a) given a set of $n2^{n-1}$ *weighting coefficients* $\{p^i_S : i \in N,\ S \subseteq N\backslash\{i\}\}$ such that $\sum_{S\subseteq N\backslash\{i\}} p^i_S = 1$ for each $i \in N$ and $p^i_S \geq 0$ for all $i \in N$ and $S \subseteq N\backslash\{i\}$, the expression

$$\phi_i[v] = \sum_{S\subseteq N\backslash\{i\}} p^i_S[v(S\cup\{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N \tag{1}$$

defines a probabilistic value ϕ on N; (b) conversely, every probabilistic value can be obtained in this way; (c) the correspondence given by $\{p^i_S : i \in N,\ S \subseteq N\backslash\{i\}\} \mapsto \phi$ is one–to–one.

Thus, the payoff that a probabilistic value allocates to every player in any game is a weighted sum of his marginal contributions in the game. We quote from [27]:

> "Let player i view his participation in a game v as consisting merely of joining some coalition S and then receiving as a reward his marginal contribution to the coalition. If p^i_S is the probability that he joins coalition S, then $\phi_i[v]$ is his expected payoff from the game."

All probabilistic values are linear, so that it is interesting to know their action on unanimity games, which form a basis of the space of games. It is as follows:

$$\phi_i[u_T] = \sum_{\substack{S\subseteq N\backslash\{i\}:\\ T\backslash\{i\}\subseteq S}} p^i_S \quad \text{if } i \in T \qquad \text{and} \qquad \phi_i[u_T] = 0 \quad \text{otherwise.} \tag{2}$$

Among the probabilistic values, *semivalues*, introduced by Dubey et al. [12], are characterized by the *anonymity* property: $\phi_{\theta i}[\theta v] = \phi_i[v]$ for all $i \in N$, $v \in \mathcal{G}_N$ and for every permutation θ on N. Alternatively, this is equivalent to saying that, if $n = |N|$, there is a vector $\{p_s\}_{s=0}^{n-1}$ such that $p^i_S = p_s$ for all $i \in N$ and all $S \subseteq N\backslash\{i\}$, where $s = |S|$, so that all coalitions of a given size share a common weight that applies to all (external) players, and hence Eq. (1) reduces to

$$\phi_i[v] = \sum_{S\subseteq N\backslash\{i\}} p_s[v(S\cup\{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N.$$

The weighting coefficients $\{p_s\}_{s=0}^{n-1}$ of any semivalue ϕ satisfy therefore two characteristic conditions:

$$\text{each } p_s \geq 0 \qquad \text{and} \qquad \sum_{s=0}^{n-1} \binom{n-1}{s} p_s = 1.$$

[2] In [27] this property is called *monotonicity*, but we prefer to call to it *positivity* as in [12].

Among semivalues, the *Shapley value* (Shapley [26]), denoted here by φ and defined by $p_s = 1/n\binom{n-1}{s}$ for all s, is the only *efficient* semivalue, in the sense that $\sum_{i \in N} \varphi_i[v] = v(N)$ for every $v \in \mathcal{G}_N$. The *Banzhaf value* (Owen [21, 23]), denoted here by β and defined by $p_s = 1/2^{n-1}$ for all s, is the only semivalue with constant (i.e. independent of s) weighting coefficients.

Finally, the *multilinear extension* (Owen [21]) of a game $v \in \mathcal{G}_N$ is the real–valued function defined on $\mathbb{R}^n$ by

$$f(x_1, x_2, \ldots, x_n) = \sum_{S \subseteq N} \prod_{i \in S} x_i \prod_{j \in N \setminus S} (1 - x_j) v(S). \tag{3}$$

As is well known, both the Shapley and Banzhaf values of any game v can be obtained from its multilinear extension. Indeed, $\varphi[v]$ can be calculated by integrating the partial derivatives of the multilinear extension of the game along the main diagonal $x_1 = x_2 = \cdots = x_n$ of the cube $[0, 1]^n$ [21], while the partial derivatives of that multilinear extension evaluated at point $(1/2, 1/2, \ldots, 1/2)$ give $\beta[v]$ [22].

3 Multinomial (probabilistic) values

The multinomial (probabilistic) values were introduced in reliability by Puente [24] (see also [15]) with the name of "multibinary probabilistic values" as follows:

Definition 3.1 Set $N = \{1, 2, \ldots, n\}$ and let $\mathbf{p} \in [0, 1]^n$, that is, $\mathbf{p} = (p_1, p_2, \ldots, p_n)$ with $0 \leq p_i \leq 1$ for $i = 1, 2, \ldots, n$, be given. Then the coefficients

$$p_S^i = \prod_{j \in S} p_j \prod_{\substack{k \in N \setminus S \\ k \neq i}} (1 - p_k) \quad \text{for all } i \in N \text{ and } S \subseteq N \setminus \{i\} \tag{4}$$

(where the empty product, arising if $S = \emptyset$ or $S = N \setminus \{i\}$, is taken to be 1) define a probabilistic value on $\mathcal{G}_N$ that will be called the $\mathbf{p}$–*multinomial probabilistic value* and denoted as $\lambda^{\mathbf{p}}$. Thus,

$$\lambda_i^{\mathbf{p}}[v] = \sum_{S \subseteq N \setminus \{i\}} \prod_{j \in S} p_j \prod_{\substack{k \in N \setminus S \\ k \neq i}} (1 - p_k)[v(S \cup \{i\}) - v(S)] \quad \text{for all } i \in N \text{ and } v \in \mathcal{G}_N.$$

We will attach to p_i the meaning of *generical tendency of player i to form coalitions*, and thus we will say that $\mathbf{p}$ is a *(tendency) profile* on N. According to Eq. (4), this implies that coefficient p_S^i, the probability of i to join S, will depend on the positive tendencies of the members of S to form coalitions and also on the negative tendencies in this sense of the outside players, i.e. the members of $N \setminus (S \cup \{i\})$.

Remarks 3.2 (a) For example, for $n = 2$ we have $\mathbf{p} = (p_1, p_2)$ and, if $i \neq j$,

$$\lambda_i^{\mathbf{p}}[v] = (1 - p_j)[v(\{i\} - v(\emptyset)] + p_j[v(N) - v(\{j\})].$$

Thus, the payoff allocated by $\lambda^{\mathbf{p}}$ to player i does not depend on p_i but only on p_j. If player j is not greatly interested in cooperating, and hence p_j is small, player i mainly receives his individual utility whereas, otherwise, if player j is interested in cooperating, and hence p_j is great, player i mainly receives his marginal contribution to the grand coalition.

(b) It is easy to check that the action of $\lambda^{\mathbf{p}}$ on a unanimity game u_T is given by:

$$\lambda_i^{\mathbf{p}}[u_T] = \prod_{\substack{j \in T \\ j \neq i}} p_j \quad \text{if } i \in T \qquad \text{and} \qquad \lambda_i^{\mathbf{p}}[u_T] = 0 \quad \text{otherwise.} \tag{5}$$

(c) Whenever, in particular, $p_1 = p_2 = \cdots = p_n = q$ for some $q \in [0,1]$, coefficients p_S^i reduce, for all $i \in N$, to

$$p_S^i = p_s = q^s(1-q)^{n-s-1} \quad \text{for } s = 0, 1, \ldots, n-1,$$

where $s = |S|$ and $0^0 = 1$ by convention in cases $q = 0$ and $q = 1$. These coefficients $\{p_s\}_{s=0}^{n-1}$ define the *q–binomial semivalue* ψ^q introduced by Puente [24] and, obviously, $\lambda^{\mathbf{p}} = \psi^q$. If, moreover, $q = 1/2$ then we obtain $\psi^{1/2} = \beta$, the Banzhaf value.

(d) As it is shown in [24, 1], the multilinear extension representation of the Banzhaf value extends well to all binomial semivalues. In [24, 15], the method is also extended to any multinomial probabilistic value: if $\lambda^{\mathbf{p}}$ is such a value and f is the multilinear extension of game $v \in \mathcal{G}_N$ then

$$\lambda_i^{\mathbf{p}}[v] = \frac{\partial f}{\partial x_i}(p_1, p_2, \ldots, p_n) \quad \text{for all } i \in N.$$

4 Regularity and other properties

In this section first we study for probabilistic values a series of standard properties considered in the literature on value theory. Most of them hold for semivalues but, as we will see, things are not so simple when we use probabilistic and multinomial values, where the tendency profile plays an important role. Next, we restrict our analysis to simple games and *probabilistic power indices.* In both cases some of these properties only hold for a special subclass of probabilistic values (indices) that we will call *regular.*

4.1 Dominance, monotonicity and sensitivity properties

The first property concerns the *desirability* and *indifference* relations. Let us considerer $v \in \mathcal{G}_N$ and $i, j \in N$. Following Isbell [18], we set

$$iDj \text{ in } v \qquad \text{iff} \qquad v(S \cup \{i\}) \geq v(S \cup \{j\}) \quad \text{for all } \; S \subseteq N \backslash \{i,j\}.$$

Thus, iDj in v means that player i *dominates* (i.e., is "at least as desirable as") j as a coalition partner in v. We also set iIj in v iff iDj and jDi in v, which means that players i, j are symmetric, that is, *indifferent* (perfect substitutes of each other) as coalition partners. If iDj but $j \not{D} i$ in v then we say that i dominates j *strictly*. It is not difficult to verify that D is a preordering and I is an equivalence relation (both on N). Notice that these relations depend only on the game structure. As it is shown in [4], g being a semivalue on N, iDj in v implies $g_i[v] \geq g_j[v]$, and hence iIj in v implies $g_i[v] = g_j[v]$, although not always iDj and $j \not{D} i$ together in v imply $g_i[v] > g_j[v]$. However, thinks are not so simple when probabilistic values are used.

Next example refers to the *desirability* and *indifference* relations and it can help us to a better understanding of the multinomial values.

Example 4.1 Let $n = 4$, $\mathbf{p} = (p_1, p_2, p_3, p_4)$, and v be the game given by

$$\begin{array}{lll} v(\emptyset) = 0, & v(\{1\}) = v(\{2\}) = 1, & v(\{1,2\}) = 4, \\ v(\{1,3\}) = v(\{2,3\}) = 3, & v(\{1,4\}) = v(\{2,4\}) = 2, & v(\{3,4\}) = 1, \\ v(\{1,2,3\}) = 4, & v(\{1,2,4\}) = 5, & v(\{1,3,4\}) = v(\{2,3,4\}) = 4, \\ v(N) = 6, & v(S) = 0 \ \text{otherwise.} & \end{array}$$

Equivalently,

$$v = [u_{\{1\}} + u_{\{2\}}] + 2u_{\{1,2\}} + 2[u_{\{1,3\}} + u_{\{2,3\}}] + [u_{\{1,4\}} + u_{\{2,4\}}] + u_{\{3,4\}} - \\ 4u_{\{1,2,3\}} - u_{\{1,2,4\}} - [u_{\{1,3,4\}} + u_{\{2,3,4\}}] + 2u_N$$

and hence the multilinear extension of this game is

$$f(x_1, x_2, x_3, x_4) = x_1 + x_2 + 2x_1x_2 + 2x_1x_3 + 2x_2x_3 + x_1x_4 + x_2x_4 + x_3x_4 - \\ 4x_1x_2x_3 - x_1x_2x_4 - x_1x_3x_4 - x_2x_3x_4 + 2x_1x_2x_3x_4.$$

Here we have, for instance, $1I2$, $1D3$ but $3\not{D}1$. If g is a semivalue on N, implies $g_1[v] = g_2[v]$, although it does not always imply $g_1[v] > g_3[v]$. What happens when multinomial values are used?

From Remark 3.2(d) we can obtain $\lambda_i^{\mathbf{p}}[v]$ for i=1, 2 and 3 from the multilinear extension of the game as follows:

$$\begin{aligned} \lambda_1^{\mathbf{p}}[v] &= 1 + 2p_2 + 2p_3 + p_4 - 4p_2p_3 - p_2p_4 - p_3p_4 + 2p_2p_3p_4, \\ \lambda_2^{\mathbf{p}}[v] &= 1 + 2p_1 + 2p_3 + p_4 - 4p_1p_3 - p_1p_4 - p_3p_4 + 2p_1p_3p_4, \\ \lambda_3^{\mathbf{p}}[v] &= 2p_1 + 2p_2 + p_4 - 4p_1p_2 - p_1p_4 - p_2p_4 + 2p_1p_2p_4. \end{aligned}$$

Notice that the introduction of tendency profiles breaks the symmetry between players 1 and 2. Nevertheless, a "structural" symmetry still exists, between $\lambda_1^{\mathbf{p}}[v]$ and $\lambda_2^{\mathbf{p}}[v]$ since $\lambda_2^{\mathbf{p}}[v]$ is obtained from $\lambda_1^{\mathbf{p}}[v]$ by replacing p_2 with p_1. This is due to the symmetrical positions of each pair of players in the game, which translates to the multilinear extension.

Next we consider some particular samples of profiles that show the different relations between $\lambda_1^{\mathbf{p}}[v]$ and $\lambda_2^{\mathbf{p}}[v]$ and $\lambda_1^{\mathbf{p}}[v]$ and $\lambda_3^{\mathbf{p}}[v]$.

- For players 1 and 2 ($1I2$):

If $\mathbf{p} = (0.6, 0.6, 0.8, 0.7)$ we have $p_1 = p_2$ and $\lambda_1^{\mathbf{p}}[v] = \lambda_2^{\mathbf{p}}[v] = 2.2720$.

If $\mathbf{p} = (0, 0.7, 0.5, 0.2)$ we have $p_1 < p_2$ and $\lambda_1^{\mathbf{p}}[v] = \lambda_2^{\mathbf{p}}[v] = 2.1000$.
If $\mathbf{p} = (0.7, 0.2, 0.5, 0)$ we have $p_1 > p_2$ and $\lambda_1^{\mathbf{p}}[v] = \lambda_2^{\mathbf{p}}[v] = 2.0000$.
If $\mathbf{p} = (0.4, 0.5, 0.1, 0.2)$ we have $p_1 < p_2$ and $\lambda_1^{\mathbf{p}}[v] = 2.1000 > 1.9560 = \lambda_2^{\mathbf{p}}[v]$.

• For players 1 and 3 ($1D3$ but $3\not{D}1$):

If $\mathbf{p} = (0.3, 0.7, 0.3, 0.9)$ we have $p_1 = p_3$ and $\lambda_1^{\mathbf{p}}[v] = 2.5380 > \lambda_3^{\mathbf{p}}[v] = 1.5380$
If $\mathbf{p} = (0.8, 0.4, 0.2, 0.7)$ we have $p_1 > p_3$ and $\lambda_1^{\mathbf{p}}[v] = 2.2720 > \lambda_3^{\mathbf{p}}[v] = 1.4280$.
If $\mathbf{p} = (0.9, 0.1, 0.2, 0.1)$ we have $p_1 > p_3$ and $\lambda_1^{\mathbf{p}}[v] = 1.5940 < \lambda_3^{\mathbf{p}}[v] = 1.6580$.
If $\mathbf{p} = (0.3, 0.6, 0.4, 0.7)$ we have $p_1 < p_3$ and $\lambda_1^{\mathbf{p}}[v] = 2.3760 > \lambda_2^{\mathbf{p}}[v] = 1.4020$.

Table 1 in Example 5.6 yields more particular samples of profiles that cover, more or less, all the cases of Corollaries 5.4 and 5.5 that will be studied in Section 5.

Now, we introduce *regular* probabilistic values.

Definition 4.2 A probabilistic value ϕ on $\mathcal{G}_N$ is *regular* iff $p_S^i > 0$ for all $i \in N$ and for all $S \subseteq N \backslash \{i\}$.

Notice that a multinomial value $\lambda^{\mathbf{p}}$ is regular iff $0 < p_i < 1$ for all $i = 1, 2, ..., n$ and, in particular, a regular multinomial value is defined by a positive profile.

Definition 4.3 A probabilistic value ϕ on $\mathcal{G}_N$ is *regular* for player i iff $p_S^i > 0$ for all $S \subseteq N \backslash \{i\}$.

Particularly, a multinomial value $\lambda^{\mathbf{p}}$ is regular for player i iff $0 < p_k < 1$ for all $k \neq i$.

Definition 4.4 Let $i, j \in N$ be two distinct players and let ϕ be a probabilistic value on $\mathcal{G}_N$.
ϕ is (i, j)–symmetric iff for all $S \subseteq N \backslash \{i, j\}$ $p_S^i = p_S^j$ and $p_{S \cup \{j\}}^i = p_{S \cup \{i\}}^j$.

Next proposition characterizes (i, j)–symmetric values within the class of probabilistic values.

Proposition 4.5 *(Dominance property) Let $i, j \in N$ be two distinct players and let ϕ be a probabilistic value on $\mathcal{G}_N$. The following properties are equivalent:*
(1) ϕ is (i, j)–symmetric,
(2) for all $v \in \mathcal{G}_N$, if iIj in v then $\phi_i[v] = \phi_j[v]$,
(3) for all $v \in \mathcal{G}_N$, if iDj in v then $\phi_i[v] \geq \phi_j[v]$.
If additionally for all $S \subseteq N \backslash \{i, j\}$, $p_S^i + p_{S \cup \{j\}}^i > 0$ then (1) is equivalent to
(4) for all $v \in \mathcal{G}_N$, if iDj and $j\not{D}i$ in v then $\phi_i[v] > \phi_j[v]$.

Carreras and Freixas [4], prove that all semivalues satisfy (2) and (3) in Proposition 4.5, and a semivalue satisfies (4) if additionally $p_s + p_{s+1} > 0$ for all $s = 0, 1, ..., n-2$.

Corollary 4.6 *Let $i, j \in N$ be two distinct players and let $\lambda^{\mathbf{p}}$ be a regular multinomial probabilistic value on $\mathcal{G}_N$. The following properties are equivalent:*
(1) $p_i = p_j$,
(2) for all $v \in \mathcal{G}_N$, if iIj in v then $\lambda_i^{\mathbf{p}}[v] = \lambda_j^{\mathbf{p}}[v]$,
(3) for all $v \in \mathcal{G}_N$, if iDj in v then $\lambda_i^{\mathbf{p}}[v] \geq \lambda_j^{\mathbf{p}}[v]$.
(4) for all $v \in \mathcal{G}_N$, if iDj and $j\not{D}i$ in v then $\lambda_i^{\mathbf{p}}[v] > \lambda_j^{\mathbf{p}}[v]$.

From now on we will focus on the monotonicity conditions considered by Young [28], when providing an axiomatic characterization of the Shapley value without using additivity, and extended to semivalues by Carreras and Freixas [4]. Let $v, w \in \mathcal{G}_N$ be and $i \in N$. Following [4], we set

$$v\,B\,w \text{ for } i \qquad \text{iff} \qquad v(S \cup \{i\}) - v(S) \geq w(S \cup \{i\}) - w(S) \quad \text{for all } \; S \subseteq N \backslash \{i\},$$

i.e., iff i's marginal contributions are better (not smaller) in v than in w.

If g is a value on $\mathcal{G}_N$, we shall say that g satisfies the *monotonicity property* iff $v\,B\,w$ for i implies $g_i[v] \geq g_i[w]$, and that g satisfies the *sensitivity property* iff $v\,B\,w$ and $w\,\not{B}\,v$ together for i imply $g_i[v] > g_i[w]$.

In [4] it is shown that all semivalues satisfy the monotonicity property, and also that a semivalue satisfies the sensitivity property iff it is regular. It is not difficult to extend these results to probabilistic values.

Proposition 4.7 *(Monotonicity and sensitivity properties) Let ϕ a probabilistic value on $\mathcal{G}_N$ and $v, w \in \mathcal{G}_N$ be distinct games. Then, for each $i \in N$:*
(a) $v\,B\,w$ for i implies $\phi_i[v] \geq \phi_i[w]$.
(b) $v\,B\,w$ and $w\,B\,v$ for i implies $\phi_i[v] = \phi_i[w]$.
(c) $v\,B\,w$ and $w\,\not{B}\,v$ for i implies $\phi_i[v] > \phi_i[w]$ iff ϕ is regular for i.

4.2 Nonnull player, null player exclusion and balanced contributions properties

The first property of this section refers to nonnull players. Usually, if g is a value on $\mathcal{G}_N$, a nonnull player $i \in N$ in a *monotonic* game v gets a payoff $g_i[v] > 0$. This property holds for —and in fact characterizes for $n \geq 2$— all regular probabilistic values within the class of probabilistic values.

Proposition 4.8 *(Nonnull player property) A probabilistic value ϕ allocates a positive payoff to every nonnull player in any monotonic game $v \in \mathcal{G}_N$ if, and only if whenever $n \geq 2$, ϕ is regular.*

Before studying the following properties, we need the notions of subgame and subprofile, both with regard to a nonempty $T \subseteq N$. If $v \in \mathcal{G}_N$, the game $v_T \in \mathcal{G}_T$, defined by $v_T(S) = v(S)$ for all $S \subseteq T$, is a *subgame* of v. Analogously, if $\mathbf{p} \in [0,1]^n$ is a profile on N, we will say that $\mathbf{p}^T \in [0,1]^t$ (where $t = |T|$), defined by $p_i^T = p_i$ for all $i \in T$, is a *subprofile* of $\mathbf{p}$. Thus, any multinomial probabilistic value $\lambda^{\mathbf{p}}$ on $\mathcal{G}_N$ induces a multinomial probabilistic value $\lambda^{\mathbf{p}^T}$ on $\mathcal{G}_T$ for each nonempty $T \subseteq N$.

Notice that, in general, such a procedure does not work for an arbitrary probabilistic value on N.

In the particular case where $T = N\backslash\{i\}$ for some $i \in N$ we will prefer to write $v_{-\{i\}}$ and $\mathbf{p}^{-i}$ instead of $v_{N\backslash\{i\}}$ and $\mathbf{p}^{N\backslash\{i\}}$, respectively. That is, if $i \in N$ then any multinomial probabilistic value $\lambda^{\mathbf{p}}$ on $\mathcal{G}_N$ induces a multinomial probabilistic value $\lambda^{\mathbf{p}^{-i}}$ on $\mathcal{G}_{N\backslash\{i\}}$ with profile $\mathbf{p}^{-i}$, whose weighting coefficients will be denoted as $(p^{-i})^j_S$ for each $j \in N\backslash\{i\}$ and each $S \subseteq N\backslash\{i, j\}$.

The following property refers to the effect of a null player leaving the game. It is desirable that the payoffs given by a value to the remaining players are not affected by this exclusion. As we will see, this property holds for any multinomial probabilistic value.

Definition 4.9 (*Null player exclusion property*) A value g on $\mathcal{G}_N$ satisfies the null player exclusion property if for all $v \in \mathcal{G}_N$

$$g_j[v] = g_j[v_{-\{i\}}],$$

for all $i, j \in N$ such that i is a null player in v.

The *null player exclusion property* or *null players out property* studies the consequences of excluding a null player on the payoff of the remaining players. For more details about it we refer the reader to Derks and Haller [10].

Definition 4.10 (*Balanced contributions property*) A value g on $\mathcal{G}_N$ satisfies the property of balanced contributions if for all $v \in \mathcal{G}_N$ and all $i, j \in N$

$$g_i[v] - g_i[v_{-\{j\}}] = g_j[v] - g_j[v_{-\{i\}}].$$

As we will see, these two properties have not sense for probabilistic values in general and we need to introduce a definition that will be also useful later.

Definition 4.11 A probabilistic value ϕ on $\mathcal{G}_N$ is *hereditary* iff its weighted coefficients on $N\backslash\{i\}$ satisfy

$$(p^{-i})^j_S = p^j_S + p^j_{S\cup\{i\}} \quad \text{for all} \;\; S \subseteq N\backslash\{i, j\}.$$

Proposition 4.12 *Every multinomial probabilistic value on $\mathcal{G}_N$ is hereditary.*

Remark 4.13 Taking into account the last proposition, a multinomial value induces multinomial values on lower cardinalities. We can say that the multinomial value $\lambda^{\mathbf{p}}$ is *hereditary* in the sense of that for each player set N, all its induced values in $T \subseteq N$ are also multinomial values. However, this is not true for all probabilistic values. For instance, the probabilistic value defined by

$$p^i_S = \begin{cases} 1 \text{ if either } n \text{ is even and } |S| = n-1 \text{ or } n \text{ is odd and } |S| = 0, \\ 0 \text{ otherwise,} \end{cases}$$

is not hereditary.

Next proposition shows that within the class of probabilistic values, ϕ satisfying the null player property is equivalent to ϕ being hereditary.

Proposition 4.14 *(Null player exclusion property) Let ϕ be a probabilistic value on $\mathcal{G}_N$. Then*

$$\phi \text{ satisfies the null player exclusion property iff } \phi \text{ is hereditary.}$$

Corollary 4.15 *Any multinomial probabilistic value on $\mathcal{G}_N$ satisfies the null player exclusion property.*

The following property refers to the effect of excluding a player on the payoff to any other player and characterizes the class of solutions satisfying the balanced contributions property within the class of regular and hereditary regular probabilistic values.

Proposition 4.16 *(Balanced contributions property) Given $v \in \mathcal{G}_N$, $i, j \in N$ and ϕ a regular and hereditary probabilistic value, then*

$$\phi \text{ satisfies the balanced contributions property iff } p^j_{S\cup\{i\}} = p^i_{S\cup\{j\}} \text{ for all } S \subseteq N\backslash\{i,j\}.$$

Proposition 4.17 *Let us assume $v \in \mathcal{G}_N$.*

(a) (Weighted balanced contributions property)[3] *Any multinomial probabilistic value $\lambda^{\mathbf{p}}$ on $\mathcal{G}_N$ satisfies*

$$p_i(\lambda^{\mathbf{p}}_i[v] - \lambda^{\mathbf{p}_{-j}}_i[v_{-\{j\}}]) = p_j(\lambda^{\mathbf{p}}_j[v] - \lambda^{\mathbf{p}_{-i}}_j[v_{-\{i\}}]).$$

(b) (Balanced contributions property) If $\lambda^{\mathbf{p}}$ is a regular multinomial probabilistic value then

$$\lambda^{\mathbf{p}} \text{ satisfies the balanced contributions property iff } p_i = p_j.$$

Remark 4.18 Particularly, Proposition 4.16 proves that all semivalue satisfies balanced contributions property, generalizing the results obtained in [13]. Myerson [20] proved that the balanced contributions property and efficiency characterize the Shapley value and, in a more general context, Dragan [11] shows that the balanced contributions property is equivalent to the existence of a potential function.

[3] Notice that Corollary 4.15 follows from this weighted balanced contributions property but only for players j with $p_j > 0$.

4.3 Simple games and donation property

Simple games constitute very often a test bed for many cooperative concepts. In particular, they have been intensively applied to describe and analyze collective decision and the notion of voting power has been closely attached to them.

We recall that v is a *simple game* if it is monotonic and $v(S) = 0$ or 1 for all $S \subseteq N$. It is determined by the set $W = \{S \subseteq N : v(S) = 1\}$ of *winning coalitions* and even by the subset $W^m = \{S \in W : T \notin W \text{ if } T \subset S\}$ of *minimal winning coalitions.* In particular, v is a *weighted majority game* if there exist a *quota* $q > 0$ and *weights* $w_1, w_2, \ldots, w_n \geq 0$ such that $S \in W$ if and only if $\sum_{i \in S} w_i \geq q$. We denote this fact by $v \equiv [q; w_1, w_2, \ldots, w_n]$, although this *representation* of v is never unique. All unanimity games are simple (and weighted majority games).

If (N, v) is a simple game, a player $i \in N$ is said to be *crucial* for a coalition $S \subseteq N \backslash \{i\}$ if $S \notin W$ but $S \cup \{i\} \in W$; we then write $S \in \mathcal{C}(i, v)$. It seems desirable that a measure of power takes account of the times that each player is crucial in a game.

Particularly, all properties stated for probabilistic values in this paper make sense for probabilistic (power) indices and the restriction to $\mathcal{SG}_N$ of any probabilistic value on $\mathcal{G}_N$ is, of course, a power index, which will be denoted by the same symbol as follows: if $v \in \mathcal{SG}_N$ and $i \in N$ then,

$$\phi_i[v] = \sum_{S \in \mathcal{C}(i,v)} p_S^i. \tag{6}$$

Following Remark 2.3(c) in [2], an alternative interpretation in simple games of the profile that defines a multinomial value is as follows: there is a status quo Q and a proposal P to modify it. The action of the parliamentary members reduces to vote for or against P. Then *each p_i can be viewed as the probability that player i votes for P.* Since the result of a voting is essentially equivalent to forming a coalition (the coalition of players that vote for P), this interpretation of p_i agrees with that of "tendency to form a coalition" that we are using in this paper.

Now we consider probabilistic indices on weighted majority games. The situation that we analyze is as follows. Two weighted majority games $(N, v) \equiv [q; w_1, w_2, ..., w_n]$ and $(N, v') \equiv [q; w_1', w_2', ..., w_n']$ are given with common player set, quota and total weight $\sum_{i \in N} w_i = \sum_{i \in N} w_i'$. A player i for which $w_i > w_i'$ is called a *donor* and a player j for which $w_j < w_j'$ is called a *recipient.* Intuitively, the idea is that (N, v) represents the initial distribution of weights and (N, v') arises from it by redistribution, the donor(s) "donating" some weight to the recipient(s), so each donor loses weight and each recipient gains some, while the total weight remains unchanged.

Proposition 4.19 *(Donation property). If, in the preceding situation, there is just a donor i and a recipient j, then, for every probabilistic index ϕ, $\phi_i[v] \geq \phi_i[v']$.*

Proposition 4.20 *(Donation strict property). Assume, moreover, that $v \neq v'$. A probabilistic index ϕ satisfies $\phi_i[v] > \phi_i[v']$ iff ϕ is regular for i.*

Remark 4.21 Schotter [25] presents and discusses cases in which is possible to decrease a voter's weight and, at the same time, increase his power. This phenomenon has been regarded as paradoxical; indeed, Fisher and Schotter [14] termed it the *paradox of redistribution*. As we have proved on Proposition 4.19 and Proposition 4.20, things are easier when we restrict our analysis to probabilistic indices. In addition, Proposition 4.20 generalizes a result on the Shapley-Shubik index given by Gambarelli [16]. For more information about results related to both propositions, we refer the reader to Felsenthal and Machover [13].

The donation strict property was generalized by Felsenthal and Machover [13] to all simple games by the *transfer property* as follows. Let (N, v) and (N, v') be *different* simple games. Assume that there are players $i, j \in N$ such that, for any $S \subseteq N$,

(a) if either $i, j \in S$ or $i, j \notin S$, then $S \in W$ iff $S \in W'$;

(b) if $i \in S$, $j \notin S$ and $S \in W'$ then $S \in W$;

(c) if $i \notin S$, $j \in S$ and $S \in W$ then $S \in W'$.

We say that there is a *donation* from (N, v) to (N, v'). Player i can lose in v' some crucial positions he had in v, whereas player j can gain crucial positions in v' he had in v. Hence some transfer of power should be expected from i to j when passing from v to v'.

Proposition 4.22 *(Generalized donation strict property). A probabilistic index ϕ satisfies $\phi_i[v] > \phi_j[v]$ in a donation from (N, v) to (N, v') iff ϕ is regular for i.*

5 Second partial derivatives to compute multinomial values

One of the distinguished features of multinomial probabilistic values is that the payoff obtained by any player i does not depend on his tendency p_i to cooperate, but only on the tendencies of some or all the remaining players. Then it is interesting to measure the effect of a variation of p_j $(j \neq i)$ on the payoff to i. Since this payoff is given by the first partial derivative of f, the multilinear extension of the game, with respect to variable x_i at point $\mathbf{p}$, this leads us to investigate second partial derivatives of the form $\dfrac{\partial^2 f}{\partial x_j \partial x_i}$ with $j \neq i$. Notice that, as stated in Corollary 4.6, when $p_i = p_j$ the payoffs agree with the dominance (strict or not) and indifference relations iDj, iIj, or iDj but $j \not{D} i$, which depend on the game structure only.

What happens when $p_i \neq p_j$? For example, in the unanimity game $u_{\{1,2\}}$ (for any $n \geq 2$), we have $1I2$ but, for any profile $\mathbf{p}$, $\lambda_1^{\mathbf{p}}[u_{\{1,2\}}] - \lambda_2^{\mathbf{p}}[u_{\{1,2\}}] = p_2 - p_1 \neq 0$ if $p_1 \neq p_2$. We seek an answer to the question, which is important for discarding any sensation that the multinomial values are "perverse" in the sense that the more is the tendency of a player to cooperate, the less is the payoff he gets...

We begin by introducing a bit more of notation. If $v \in \mathcal{G}_N$, its multilinear extension f is, in principle, defined on the whole Euclidean space $\mathbb{R}^n$ although, generally, we are only interested in its behavior in the n–cube $[0,1]^n$. The set 2^N of coalitions of N can be identified with $\{0,1\}^n$, the set of vertices of the cube, through the map $S \mapsto \mathbf{x}^S = (x_1^S, x_2^S, \ldots, x_n^S)$ given by $x_i^S = 1$ if $i \in S$ or else $x_i^S = 0$. Then, $v(S) = f(\mathbf{x}^S)$ for all $S \subseteq N$.

We shall use in the sequel, for any $\mathbf{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, notation like e.g.

$$f(1_i, \mathbf{x}) = f(x_1, \ldots, \underbrace{1}_{i}, \ldots, x_n) \quad \text{or} \quad f(0_i, 1_j, \mathbf{x}) = f(x_1, \ldots, \underbrace{0}_{i}, \ldots, \underbrace{1}_{j}, \ldots, x_n).$$

Notice that, for each $i \in N$, f is a linear function of x_i, that is, $f(\mathbf{x}) = a + bx_i$ where $a = f(0_i, \mathbf{x})$ and $b = f(1_i, \mathbf{x}) - f(0_i, \mathbf{x})$ for all $\mathbf{x}$. Then

$$f(\mathbf{x}) = x_i f(1_i, \mathbf{x}) + (1 - x_i) f(0_i, \mathbf{x}) \quad \text{for all } \mathbf{x}. \tag{7}$$

Lemma 5.1 *Let $\mathbf{p}$ be a profile on N. For all distinct $i, j \in N$ and $v \in \mathcal{G}_N$, if f is the multilinear extension of v then:*

(a) $\lambda_i^{\mathbf{p}}[v] = \dfrac{\partial f}{\partial x_i}(\mathbf{p}) = f(1_i, \mathbf{p}) - f(0_i, \mathbf{p})$.

(b) $\dfrac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{p}) = f(1_i, 1_j, \mathbf{p}) - f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p}) + f(0_i, 0_j, \mathbf{p})$.

Theorem 5.2 *Let $i, j \in N$ be distinct players, $v \in \mathcal{G}_N$, and f be the multilinear extension of v. Then, for any profile $\mathbf{p}$ on N,*

$$\lambda_i^{\mathbf{p}}[v] - \lambda_j^{\mathbf{p}}[v] = (p_j - p_i) \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{p}) + f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p}).$$

Now we are ready to provide some answers to the question stated above as immediate consequences of Theorem 5.2. To ease the notation, let us set

$$d(i, j, \mathbf{p}) = \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{p}) \qquad \text{and} \qquad \Delta(i, j, \mathbf{p}) = f(1_i, 0_j, \mathbf{p}) - f(0_i, 1_j, \mathbf{p}),$$

so that

$$\lambda_i^{\mathbf{p}}[v] - \lambda_j^{\mathbf{p}}[v] = (p_j - p_i) d(i, j, \mathbf{p}) + \Delta(i, j, \mathbf{p}).$$

Notice that, by the multilinearity of f,

$$v(S \cup \{i\}) \geq v(S \cup \{j\}) \text{ for all } S \subseteq N \backslash \{i, j\} \quad \text{iff} \quad \Delta(i, j, \mathbf{p}) \geq 0 \text{ for all } \mathbf{p},$$

but even if, moreover, $v(S \cup \{i\}) > v(S \cup \{j\})$ for some such S, we cannot conclude that $\Delta(i, j, \mathbf{p}) > 0$ for all $\mathbf{p}$ (see the case of players 4 and 3 and coalition $S = \{1, 2\}$ in Example 5.6).

Corollary 5.3 *If* iDj *in* v *then* $\Delta(i,j,\mathbf{p}) \geq 0$ *for all* $\mathbf{p}$. *If, moreover,*

(a) $d(i,j,\mathbf{p}) = 0$, *then* $\lambda_i^{\mathbf{p}}[v] \geq \lambda_j^{\mathbf{p}}[v]$;

(b) $d(i,j,\mathbf{p}) > 0$ *and* $p_i < p_j$, *then* $\lambda_i^{\mathbf{p}}[v] > \lambda_j^{\mathbf{p}}[v]$;

(c) $d(i,j,\mathbf{p}) < 0$ *and* $p_i > p_j$, *then* $\lambda_i^{\mathbf{p}}[v] > \lambda_j^{\mathbf{p}}[v]$. $\square$

Corollary 5.4 *If* iIj *in* v *then* $\Delta(i,j,\mathbf{p}) = 0$ *for all* $\mathbf{p}$. *If, moreover,*

(a) $d(i,j,\mathbf{p}) = 0$, *then* $\lambda_i^{\mathbf{p}}[v] = \lambda_j^{\mathbf{p}}[v]$;

(b) $d(i,j,\mathbf{p}) > 0$ *and* $p_i < p_j$, *then* $\lambda_i^{\mathbf{p}}[v] > \lambda_j^{\mathbf{p}}[v]$;

(c) $d(i,j,\mathbf{p}) > 0$ *and* $p_i > p_j$, *then* $\lambda_i^{\mathbf{p}}[v] < \lambda_j^{\mathbf{p}}[v]$;

(d) $d(i,j,\mathbf{p}) < 0$ *and* $p_i < p_j$, *then* $\lambda_i^{\mathbf{p}}[v] < \lambda_j^{\mathbf{p}}[v]$;

(e) $d(i,j,\mathbf{p}) < 0$ *and* $p_i > p_j$, *then* $\lambda_i^{\mathbf{p}}[v] > \lambda_j^{\mathbf{p}}[v]$. $\square$

Corollary 5.5 *If* iDj *but* $j \not\!\!D\, i$ *in* v *then* $\Delta(i,j,\mathbf{p}) \geq 0$ *for all* $\mathbf{p}$ *and* $\Delta(i,j,\mathbf{p}) > 0$ *for some* $\mathbf{p}$. *If, moreover,*

(a) $d(i,j,\mathbf{p}) = 0$, *then* $\lambda_i^{\mathbf{p}}[v] > \lambda_j^{\mathbf{p}}[v]$ *for any such* $\mathbf{p}$;

(b) $d(i,j,\mathbf{p}) > 0$ *and* $p_i < p_j$, *then* $\lambda_i^{\mathbf{p}}[v] > \lambda_j^{\mathbf{p}}[v]$ *for any such* $\mathbf{p}$;

(c) $d(i,j,\mathbf{p}) < 0$ *and* $p_i > p_j$, *then* $\lambda_i^{\mathbf{p}}[v] > \lambda_j^{\mathbf{p}}[v]$ *for any such* $\mathbf{p}$. $\square$

We have not detailed, in Corollaries 5.3 and 5.5, the cases where no conclusion can be established due to the influence of $\Delta(i,j,\mathbf{p})$.

Now we consider the previous Example 4.1 that will illustrate some of these results.

Example 5.6 For $n = 4$, $\mathbf{p} = (p_1, p_2, p_3, p_4)$, and v the game given by

$$\begin{array}{lll} v(\emptyset) = 0, & v(\{1\}) = v(\{2\}) = 1, & v(\{1,2\}) = 4, \\ v(\{1,3\}) = v(\{2,3\}) = 3, & v(\{1,4\}) = v(\{2,4\}) = 2, & v(\{3,4\}) = 1, \\ v(\{1,2,3\}) = 4, & v(\{1,2,4\}) = 5, & v(\{1,3,4\}) = v(\{2,3,4\}) = 4, \\ v(N) = 6, & v(S) = 0 \text{ otherwise.} & \end{array}$$

Here we have $1I2$, $1D3$ but $3\not\!\!D\,1$, $1D4$ but $4\not\!\!D\,1$, and $3\not\!\!D\,4$ and $4\not\!\!D\,3$. Table 1 yields some particular samples of profiles that cover, more or less, all cases of Corollaries 5.4 and 5.5. We add that, for all $\mathbf{p}$,

$$\Delta(1,2,\mathbf{p}) = 0, \quad \Delta(1,3,\mathbf{p}) = 1 \quad \text{and} \quad \Delta(1,4,\mathbf{p}) = 1 + p_2 + p_3 - 3p_2p_3,$$

and also that $d(1,4,\mathbf{p})$ is never negative and vanishes only at two points.

6 An example of application: multinomial values among players connected by a network

In [19], Myerson used graph theory to analyze cooperation structure in games. His main idea is that players might cooperate in a game by forming agreements among themselves. These cooperative agreements can be represented by links between the agreeing players. Then any cooperation structure can be represented by a set of links.

case for i, j	p_1	p_2	p_3	p_4	$d(i,j,p)$	p_i ? p_j	$\lambda_i^{\mathbf{p}}[v]$	sign	$\lambda_j^{\mathbf{p}}[v]$
	0.0	0.7	0.5	0.2	0.0000	$p_i < p_j$	2.1000	=	2.1000
	0.7	0.2	0.5	0.0	0.0000	$p_i > p_j$	2.0000	=	2.0000
$1I2$	0.4	0.5	0.1	0.2	1.4400	$p_i < p_j$	2.1000	>	1.9560
($1D2$ and $2D1$)	0.6	0.5	0.1	0.2	1.4400	$p_i > p_j$	2.1000	<	2.2440
	0.4	0.5	0.8	0.7	−0.7800	$p_i < p_j$	2.3500	<	2.4280
	0.6	0.5	0.8	0.7	−0.7800	$p_i > p_j$	2.3500	>	2.2720
	0.1	0.5	0.8	0.1	0.0000	$p_i < p_j$	2.0500	>	1.0500
	0.8	0.5	0.1	0.7	0.0000	$p_i > p_j$	2.3500	>	1.3500
$1D3$ and	0.4	0.3	0.5	0.7	0.5200	$p_i < p_j$	2.3500	>	1.2980
$3\not{D}1$	0.8	0.4	0.2	0.7	0.2600	$p_i > p_j$	2.2720	>	1.4280
	0.3	0.6	0.4	0.7	−0.2600	$p_i < p_j$	2.3760	>	1.4020
	0.3	0.7	0.2	0.1	−0.7600	$p_i > p_j$	2.2780	>	1.2020
	0.1	0.0	1.0	0.8	0.0000	$p_i < p_j$	3.0000	>	1.0000
$1D4$ and	0.8	0.0	1.0	0.1	0.0000	$p_i > p_j$	3.0000	>	1.0000
$4\not{D}1$	0.6	0.3	0.4	0.8	0.5400	$p_i < p_j$	2.3520	>	0.9040
	0.2	0.3	0.4	0.6	0.5400	$p_i > p_j$	2.2440	>	0.6880

Table 1: Behavior of multinomial values as to indifference and strict dominance

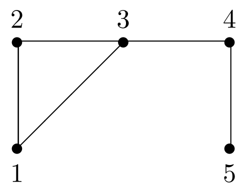
Once the graph (or network) is fixed, it may be viewed as a cooperative game where the role of the network is to define which coalitions can work. Coalition's worth is obtained by adding the utilities achieved by those members who are communicated by the network. These situations holds particularly when players may, due to affinity, consanguinity or other factors, have clear preferences for joining certain coalitions as opposed to others.

Cases in which many theoretically possible coalitions will not realistically be formed are not limited to social situations alone. If one is studying cooperative coalitions among players connected by supply routes, computer networks or web links, there are clear structural reason for entirely excluding some coalitions and including in consideration instead only coalitions that are connected along the network.

As we will see in the following example, the way in which the players are connected to each other is important and the tendency profile defining multinomial values provides new tools to study these situations.

Example 6.1 Consider a situation where 5 players are connected in some network Γ, like friendships and social relationships, communication lines or alliances given by

Γ :



Let v be the cooperative game defined by $v(S) = \dfrac{s(s-1)}{2}$ where $s = |S|$. The multilinear extension of the game v_Γ associated to the graph Γ in Myerson's sense is

$$f(x_1, x_2, x_3, x_4, x_5) = x_1x_2 + x_1x_3 + x_2x_3 + x_3x_4 + x_4x_5 + x_1x_3x_4 + x_2x_3x_4 + x_3x_4x_5 + x_1x_3x_4x_5 + x_2x_3x_4x_5$$

Let $\mathbf{p} = (p_1, p_2, p_3, p_4, p_5)$ a tendency profile.

The calculation of $\lambda_i^{\mathbf{p}}[v_\Gamma]$ for i=1, 2, 3, 4 and 5 derives from Remark 3.2(d):

$$\begin{aligned}
\lambda_1^{\mathbf{p}}[v_\Gamma] &= p_2 + p_3 + p_3p_4(1 + p_5),\\
\lambda_2^{\mathbf{p}}[v_\Gamma] &= p_1 + p_3 + p_3p_4(1 + p_5),\\
\lambda_3^{\mathbf{p}}[v_\Gamma] &= p_1 + p_2 + p_4(1 + p_5)(1 + p_1 + p_2),\\
\lambda_4^{\mathbf{p}}[v_\Gamma] &= p_5 + p_3(1 + p_5)(1 + p_1 + p_2),\\
\lambda_5^{\mathbf{p}}[v_\Gamma] &= p_4[1 + p_3(1 + p_1 + p_2)].
\end{aligned}$$

Notice that the introduction of tendency profiles breaks the symmetry between players 1 and 2 in the game v_Γ. Nevertheless, a "structural" symmetry still exists, between $\lambda_1^{\mathbf{p}}[v_\Gamma]$ and $\lambda_2^{\mathbf{p}}[v_\Gamma]$ since $\lambda_2^{\mathbf{p}}[v_\Gamma]$ is obtained from $\lambda_1^{\mathbf{p}}[v_\Gamma]$ by replacing p_2 with p_1.

These allocations reflect the a priori power distribution. The possibilities of player 3 depend on p_4 as well as p_1 and p_2 as a consequence of the central position of the player 3 in the network. The allocations for player 4 are strongly influenced by p_3. Moreover, the possibilities of player 5 clearly rest upon the interest of player 4 to form coalitions.

	$\mathbf{p} = (0.5, 0.5, 0.5, 0.5, 0.5)$	$\mathbf{p} = (0.1, 0.8, 0.2, 0.5, 0.9)$	$\mathbf{p} = (0.1, 0.8, 0.2, 0.9, 0.9)$
$\lambda_1^{\mathbf{p}}[v_\Gamma]$	1.375 (16.67 %)	1.190 (17.77 %)	1.342 (14.92 %)
$\lambda_2^{\mathbf{p}}[v_\Gamma]$	1.375 (16.67 %)	0.490 (7.32 %)	0.642 (7.14 %)
$\lambda_3^{\mathbf{p}}[v_\Gamma]$	2.500 (30.30 %)	2.705 (40.39 %)	4.149 (46.12 %)
$\lambda_4^{\mathbf{p}}[v_\Gamma]$	2.000 (24.24 %)	1.622 (24.22 %)	1.622 (18.03 %)
$\lambda_5^{\mathbf{p}}[v_\Gamma]$	1.000 (12.12 %)	0.690 (10.30 %)	1.242 (13.80 %)

Table 2. Multinomial values for several values of $\mathbf{p}$

7 Concluding remarks

As we have said before, in this paper we have been concerned to the analysis of multinomial values and probabilistic values within the framework of cooperative games. The aim of the work is that providing a self–contained theory for multinomial values. We investigate the conditions for the coefficients of these values necessary and / or sufficient in order to satisfy some properties including marginal contributions, balanced contributions, desirability relation and null player exclusion property. The notions of regularity and hereditary value guaranty the validity of some of them. Some of these properties were studied by Felshenthal and Machover [13] for several power indices and by Carreras and Freixas [4] for semivalues.

Two kind of results have been established in Section 4. On one hand, general results for all probabilistic values like Propositions 4.7(a) and (b) and 4.19. On the other hand, results not so general but interesting in the sense that they evidence the great influence that the weighting coefficients and the tendency profiles may exert on the validity of classical properties in this setup, as is the case of Propositions 4.5, 4.7(c), 4.8, 4.17(b) and 4.20. Moreover, Propositions 4.14 and 4.16, where the notion of hereditary value it is necessary, have not sense for probabilistic values in general.

In this work we also characterize: (i) the class of regular values within the class of probabilistic values; (ii) the (i,j)–symmetric values within the class of probabilistic values; and (iii) the class of solutions satisfying the balanced contributions property within the class of regular and hereditary regular probabilistic values. Moreover, a discussion based on the second partial derivatives of the multilinear extension of the game provides additional insight on the meaning of multinomial values.

Even a potential theory can be developed when the profile is positive. However, for not to enlarge this paper too much, we prefer to leave all this material for a future article.

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Appendix A: proofs of Section 4

Proof of Proposition 4.5: The proof goes as follows: (1) ⇒ (3) ⇒ (2) ⇒ (1) ⇒ (4) ⇒ (1).

(1) ⇒ (3) Assume iDj in v. Starting at Eq. (1) and splitting the sum into two parts, we have

$$\phi_i[v] = \sum_{S \subseteq N \backslash \{i\}} p_S^i [v(S \cup \{i\}) - v(S)] =$$

$$\sum_{S \subseteq N \backslash \{i,j\}} \left[p_S^i [v(S \cup \{i\}) - v(S)] + p_{S \cup \{j\}}^i [v(S \cup \{i\} \cup \{j\}) - v(S \cup \{j\})] \right].$$

Now, by comparing this expression with the analogous expression for $\phi_j[v]$, it follows that

$$\phi_i[v] - \phi_j[v] = \sum_{S \subseteq N \backslash \{i,j\}} (p_S^i + p_{S \cup \{j\}}^i)[v(S \cup \{i\}) - v(S \cup \{j\})] \geq 0$$

because $p_S^i = p_S^j$, $p_{S \cup \{j\}}^i = p_{S \cup \{i\}}^j$ and $v(S \cup \{i\}) \geq v(S \cup \{j\})$ for all $S \subseteq N \backslash \{i,j\}$, since iDj in v.

(3) $\Rightarrow$ (2) It suffices to apply the previous result twice.

(2) $\Rightarrow$ (1) Let us first assume that $p^i_S \neq p^j_S$ for some $S \subseteq N\backslash\{i,j\}$. Let $w \in \mathcal{G}_N$ be defined by

$$w(T) = \begin{cases} 1 & \text{if } T \supseteq S \cup \{k\} \text{ for any } k \in N\backslash S \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

Since, in fact, iIj in w, without loss of generality we may assume that $p^i_S < p^j_S$. But then $\phi_i[w] = p^i_S < p^j_S = \phi_j[w]$, a contradiction.

Now we assume that $p^i_{S\cup\{j\}} \neq p^j_{S\cup\{i\}}$ for some $S \subseteq N\backslash\{i,j\}$. Consider $w^* \in \mathcal{G}_N$ defined by

$$w^*(T) = \begin{cases} 1 & \text{if } T \supseteq S \cup \{k\}, \text{ for any k} \neq i,j \textit{ or } T \supseteq S \cup \{i\} \cup \{j\} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

Since, in fact, iIj in w^*, without loss of generality we may assume that $p^i_{S\cup\{j\}} < p^j_{S\cup\{i\}}$. But then $\phi_i[w^*] = p^i_{S\cup\{j\}} < p^j_{S\cup\{i\}} = \phi_j[w^*]$, a contradiction.

(1) $\Rightarrow$ (4) If iDj and $j \not\!D\, i$ in v then, for some $S \subseteq N\backslash\{i,j\}$ we have $v(S\cup\{i\}) > v(S\cup\{j\})$. As $p^i_S = p^j_S$, $p^i_{S\cup\{j\}} = p^j_{S\cup\{i\}}$ and $p^i_S + p^i_{S\cup\{j\}} > 0$ for all S$\subseteq N\backslash\{i,j\}$, it follows from the proof of (1) $\Rightarrow$ (3) that $\phi_i[v] > \phi_j[v]$.

(4) $\Rightarrow$ (1) First, let us assume that $p^i_S \neq p^j_S$ for some $S \subseteq N\backslash\{i,j\}$ and e.g. $p^j_S > p^i_S$. Consider $v = 2u_{\{i\}} + u_{\{j\}} + cw$, with $c > \dfrac{1}{p^j_S - p^i_S}$ and $w \in \mathcal{G}_N$ as defined in (8).

Then it is clear that iDj and $j \not\!D\, i$ in v. However,

$$\phi_i[v] = 2 + cp^i_S \qquad \text{and} \qquad \phi_j[v] = 1 + cp^j_S,$$

and so $\phi_i[v] - \phi_j[v] = 1 - c(p^j_S - p^i_S) < 0$, a contradiction.[4]

Now, we assume that $p^i_{S\cup\{j\}} \neq p^j_{S\cup\{i\}}$ for some $S \subseteq N\backslash\{i,j\}$ and e.g. $p^j_{S\cup\{i\}} > p^i_{S\cup\{j\}}$. Consider $v = 2u_{\{i\}} + u_{\{j\}} + cw^*$, with $c > \dfrac{1}{p^j_{S\cup\{i\}} - p^i_{S\cup\{j\}}}$ and $w^* \in \mathcal{G}_N$ as defined in (9).

Then it is clear that iDj and $j \not\!D\, i$ in v. However,

$$\phi_i[v] = 2 + cp^i_{S\cup\{j\}} \qquad \text{and} \qquad \phi_j[v] = 1 + cp^j_{S\cup\{i\}},$$

and so $\phi_i[v] - \phi_j[v] = 1 - c(p^j_{S\cup\{i\}} - p^i_{S\cup\{j\}}) < 0$, a contradiction.[5] $\square$

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[4]If $p^j_S < p^i_S$ the argument is similar, with $c < \dfrac{1}{p^j_S - p^i_S}$.

[5]If $p^j_{S\cup\{i\}} < p^i_{S\cup\{j\}}$ the argument is similar, with $c < \dfrac{1}{p^i_{S\cup\{i\}} - p^j_{S\cup\{j\}}}$.

Proof of Corollary 4.6: It follows from the fact that $p_i = p_j$ and $\lambda^{\mathbf{p}}$ regular is equivalent to (i,j)–symmetry of $\lambda^{\mathbf{p}}$ and for regular multinomial values the condition $p_S + p_{S\cup\{j\}} > 0$ holds. $\square$

Proof of Proposition 4.7: (a) By a mere inspection of

$$\phi_i[v] = \sum_{S\subseteq N\backslash\{i\}} p_S^i[v(S\cup\{i\}) - v(S)] \quad \text{and} \quad \phi_i[w] = \sum_{S\subseteq N\backslash\{i\}} p_S^i[w(S\cup\{i\}) - w(S)],$$

it follows that $\phi_i[v] \geq \phi_i[w]$, because $v(S\cup\{i\}) - v(S) \geq w(S\cup\{i\}) - w(S)$ and $p_S^i \geq 0$ for all $S \subseteq N\backslash\{i\}$.

(b) It suffices to apply (a) twice.

(c) ($\Leftarrow$) If $v\,B\,w$ and $w\,\not\!B\,v$ for i, then we have $v(S\cup\{i\})-v(S) > w(S\cup\{i\})-w(S)$ for some $S \subseteq N\backslash\{i\}$. If ϕ is regular for player i, $\phi_i[v] > \phi_i[w]$ follows from the proof of (a).

($\Rightarrow$) If $n = 1$, the statement holds trivially because any probabilistic value is regular. If $n \geq 2$, let us assume that ϕ is not regular for player i. Then there is some $S \subseteq N\backslash\{i\}$ such that $p_S^i = 0$. Take $w, w^{**} \in \mathcal{G}_N$; w as defined in (8) and

$$w^{**}(T) = \begin{cases} 1 & \text{if } T \supseteq S\cup\{k\}, \text{ for any } k \in N\backslash S,\ k \neq i \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Again, it is clear that $w\,B\,w^{**}$ and $w^{**}\,\not\!B\,w$ for i but $\phi_i[w] = p_S^i = \phi_i[w^{**}] = 0$, a contradiction. $\square$

Proof of Proposition 4.8: ($\Leftarrow$) Assume that ϕ is regular and let $i \in N$ be a nonnull player in a monotonic game $v \in \mathcal{G}_N$. Then, by monotonicity, $v(S\cup\{i\}) \geq v(S)$ for all $S \subseteq N\backslash\{i\}$, and $v(S\cup\{i\}) > v(S)$ for some such S since i is nonnull. Moreover, from regularity it follows that $p_S^i > 0$ for all $S \subseteq N\backslash\{i\}$. Hence

$$\phi_i[v] = \sum_{S\subseteq N\backslash\{i\}} p_S^i[v(S\cup\{i\}) - v(S)] > 0.$$

($\Rightarrow$) If $n = 1$, v is monotonic, and 1 is nonnull in v, then $v(\{1\}) > 0$, $p_\emptyset^1 = 1$ for any profile $\mathbf{p}$ and $\phi_1[v] = v(\{1\}) > 0$ for all ϕ, regular or not. Now, consider $n \geq 2$ and assume that ϕ is not regular. Then there is some $S \subseteq N\backslash\{i\}$ such that $p_S^i = 0$.

Let $w \in \mathcal{G}_N$ be as defined in (8).

Then w is monotonic and i is a nonnull player in w since $w(S\cup\{i\}) - w(S) = 1$. However, $\phi_i[w] = p_S^i = 0$. $\square$

Proof of Proposition 4.12: For all $S \subseteq N\backslash\{i,j\}$ we have

$$p_S^j = (1-p_i)\prod_{k\in S} p_k \prod_{\substack{h\in N\backslash S\\ h\neq i,j}} (1-p_h) \qquad \text{and} \qquad p_{S\cup\{i\}}^j = p_i \prod_{k\in S} p_k \prod_{\substack{h\in N\backslash S\\ h\neq i,j}} (1-p_h)$$

so that

$$p_S^j + p_{S\cup\{i\}}^j = \prod_{k\in S} p_k \prod_{\substack{h\in N\setminus S\\ h\neq i,j}} (1-p_h) = (p^{-i})_S^j. \quad \square$$

Proof of Proposition 4.14: ($\Leftarrow$) Following the initial argument of the proof of Proposition 4.5 we obtain

$$\phi_j[v] = \sum_{S\subseteq N\setminus\{j\}} p_S^j[v(S\cup\{j\}) - v(S)] =$$

$$\sum_{S\subseteq N\setminus\{i,j\}} \left[p_S^j[v(S\cup\{j\}) - v(S)] + p_{S\cup\{i\}}^j[v(S\cup\{i\}\cup\{j\}) - v(S\cup\{i\})]\right].$$

As i is a null player in v, we have $v(S\cup\{i\}\cup\{j\}) = v(S\cup\{j\})$ and $v(S\cup\{i\}) = v(S)$. Thus, using that ϕ is hereditary,

$$\phi_j[v] = \sum_{S\subseteq N\setminus\{i,j\}} (p_S^j + p_{S\cup\{i\}}^j)[v(S\cup\{j\}) - v(S)] =$$

$$\sum_{S\subseteq(N\setminus\{i\})\setminus\{j\}} (p^{-i})_S^j[v_{-\{i\}}(S\cup\{j\}) - v_{-\{i\}}(S)] = \phi_j[v_{-\{i\}}].$$

($\Rightarrow$) Assume that ϕ is not an hereditary probabilistic value. That is, there is some $S\subseteq N\setminus\{i,j\}$ such that $(p^{-i})_S^j \neq p_S^j + p_{S\cup\{i\}}^j$. Let $w^{**}\in\mathcal{G}_N$ be as defined in (10).

Notice that player i is null in w^{**} and $\phi_j[w^{**}] = p_S^j + p_{s\cup\{i\}}^j \neq (p^{-i})_S^j = \phi_j[w_{-\{i\}}^{**}]$. $\square$

Proof of Corollary 4.15: It is straightforward to verify with Proposition 4.12. $\square$

Proof of Proposition 4.16: ($\Leftarrow$) Using the fact that ϕ is hereditary we have

$\phi_i[v] - \phi_i[v_{-\{j\}}] = \sum_{S\subseteq N\setminus\{i,j\}} p_S^i[v(S\cup\{i\}) - v(S)] + \sum_{S\subseteq N\setminus\{i,j\}} p_{S\cup\{j\}}^i[v(S\cup\{i\}\cup\{j\}) - v(S\cup\{j\})] - \sum_{S\subseteq N\setminus\{i,j\}} (p^{-j})_S^i[v((S\cup\{i\}) - v(S)] = \sum_{S\subseteq N\setminus\{i,j\}} p_{S\cup\{j\}}^i[v(S\cup\{i\}\cup\{j\}) - v(S\cup\{i\}) - v(S\cup\{j\}) + v(S)]$ and analogously for

$$\phi_j[v] - \phi_i[v_{-\{i\}}] = \sum_{S\subseteq N\setminus\{i,j\}} p_{S\cup\{i\}}^j[v(S\cup\{i\}\cup\{j\}) - v(S\cup\{i\}) - v(S\cup\{j\}) + v(S)].$$

($\Rightarrow$) Assume that $p_{S\cup\{j\}}^i \neq p_{S\cup\{i\}}^j$ for some $S\subseteq \setminus N\{i,j\}$. Let $w\in\mathcal{G}_N$ be as defined in (8).

Then $\phi_i[w] = p_S^i$, $\phi_i[w_{-\{j\}}] = (p^{-j})_S^i$, $\phi_j[w] = p_S^j$ and $\phi_j[w_{-\{i\}}] = (p^{-i})_S^j$. Using that ϕ is hereditary we find, because of the regularity of ϕ,

$$\phi_i[w] - \phi_i[w_{-\{j\}}] = -p_{S\cup\{j\}}^i \neq -p_{S\cup\{i\}}^j = \phi_j[w] - \phi_j[w_{-\{i\}}]. \quad \square$$

Proof of Proposition 4.17: (a)

$$p_i(\lambda_i^{\mathbf{p}}[v] - \lambda_i^{\mathbf{p}_{-j}}[v_{-\{j\}}]) = f(\mathbf{p}) - f(0_i, \mathbf{p}) - f(0_j, \mathbf{p}) - f(0_i, 0_j, \mathbf{p})$$

The symmetrical appearance of i and j in this expression shows that the claimed equality holds.

(b) ($\Leftarrow$) It follows at once from Proposition 4.16. $\square$

Proof of Proposition 4.19: It suffices to prove that $\mathcal{C}(i, v') \subseteq \mathcal{C}(i, v)$.

Given $S \in \mathcal{C}(i, v')$, then $w'(S) < q$ and $w'(S) + w'_i \geq q$, where $w'(S) = \sum_{i \in S} w'_i$. We have to distinguish two cases:

(i) If $j \in S$, $w(S) = w_j + w(S \backslash \{j\}) = w_j + w'(S \backslash \{j\}) < w'_j + w'(S \backslash \{j\}) = w'(S) < q$, $w(S) + w_i = w_j + w(S \backslash \{j\}) + w_i = w'_j + w'(S \backslash \{j\}) + w'_i = w'(S) + w'_i \geq q$ and then $S \in \mathcal{C}(i, v)$.

(ii) If $j \notin S$, $w(S) = w'(S) < q$, $w(S) + w_i = w'(S) + w_i > w'(S) + w'_i \geq q$ and then $S \in \mathcal{C}(i, v)$. $\square$

Proof of Proposition 4.20: ($\Leftarrow$) As ϕ is regular it suffices to prove that there is some $S \in \mathcal{C}(i, v) \backslash \mathcal{C}(i, v')$. From condition $v \neq v'$ it follows that either (a) there is S such that $S \in W$ and $S \notin W'$ (and hence $i \in S$ and $j \notin S$) or (b) there is some S such that $S \notin W$ and $S \in W'$ (and hence $i \notin S$ and $j \in S$). In case (a), $S \backslash \{i\} \in \mathcal{C}(i, v) \backslash \mathcal{C}(i, v')$; in case (b), $S \in \mathcal{C}(i, v) \backslash \mathcal{C}(i, v')$.

($\Rightarrow$) Assume that ϕ is not regular for player i and $n > 2$. Then it follows that there is some $S \subseteq N \backslash \{i\}$ such that $p^i_S = 0$. If $j \notin S$, let v be the majority game defined by

$$w_k = \begin{cases} 1 & \text{if } k \in S, \\ s + 2 & \text{if } k = i, \\ s & \text{if } k = j, \\ 2s + 2 & \text{otherwise,} \end{cases}$$

$q = 2s + 2$, and v' be the game obtained from v after player i gives one unit of weight to player j. Then it is easily seen that $\mathcal{C}(i, v) \backslash \mathcal{C}(i, v') = S$, so that $v \neq v'$ but $\phi_i[v] = \phi_i[v']$.

If $j \in S$ let $v \in \mathcal{G}_N$ be defined by

$$w_k = \begin{cases} s + 1 & \text{if } k = i, \\ s + 3 & \text{if } k = j, \\ 1 & \text{if } k \in S \backslash \{j\}, \\ 2s + 3 & \text{otherwise,} \end{cases}$$

$q = 2s + 3$ and v' be the game obtained from v after player i gives one unit of weight to player j. Then it is easily seen that $\mathcal{C}(i, v) \backslash \mathcal{C}(i, v') = S$, so that $v \neq v'$ but $\phi_i[v] = \phi_i[v']$.

Proof of Proposition 4.22: The proof is analogously to Proposition 4.20. $\square$

Appendix B: proofs of Section 5

Proof of Lemma 5.1: (a) By the linearity of f with respect to variable x_i, we have

$$\lambda_i^{\mathbf{p}}[v] = \frac{\partial f}{\partial x_i}(\mathbf{p}) = f(1_i, \mathbf{p}) - f(0_i, \mathbf{p}).$$

(b) The second equality in (a) holds[6] not only for any profile but for all $\mathbf{x} \in \mathbb{R}^n$:

$$\frac{\partial f}{\partial x_i}(\mathbf{x}) = f(1_i, \mathbf{x}) - f(0_i, \mathbf{x}).$$

Then, by deriving this equation and evaluating at $\mathbf{x} = \mathbf{p}$ we get the result. □

Proof of Theorem 5.2: From Lemma 5.1 and Eq. (7) we find

$$\lambda_i^{\mathbf{p}}[v] = f(1_i, \mathbf{p}) - f(0_i, \mathbf{p}) =$$

$$p_j f(1_i, 1_j, \mathbf{p}) + (1 - p_j) f(1_i, 0_j, \mathbf{p}) - p_j f(0_i, 1_j, \mathbf{p}) - (1 - p_j) f(0_i, 0_j, \mathbf{p})$$

and a similar expresion for $\lambda_j^{\mathbf{p}}[v]$. Hence

$$\lambda_i^{\mathbf{p}}[v] = p_j \frac{\partial^2 f}{\partial x_j \partial x_i}(\mathbf{p}) + f(1_i, 0_j, \mathbf{p}) - f(0_i, 0_j, \mathbf{p})$$

$$\lambda_j^{\mathbf{p}}[v] = p_i \frac{\partial^2 f}{\partial x_i \partial x_j}(\mathbf{p}) + f(0_i, 1_j, \mathbf{p}) - f(0_i, 0_j, \mathbf{p})$$

and, f being of class C^∞ and satisfying therefore Schwartz's theorem on coincidence of crossed partial derivatives, by subtracting the above expressions and cancelling the common term the result follows at once. □

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