Qualitative Theory of Dynamical Systems (2021) 20:66
https://doi.org/10.1007/s12346-021-00505-6

Compression–Expansion Fixed Point Theorems for Decomposable Maps and Applications to Discontinuous ϕ-Laplacian problems

Jorge Rodríguez–López[1]

Received: 10 May 2021 / Accepted: 8 July 2021

Abstract
In this paper, we prove new compression–expansion type fixed point theorems in cones for the so-called decomposable maps, that is, compositions of two upper semicontinuous multivalued maps. As an application, we obtain existence and localization of positive solutions for a differential equation with ϕ-Laplacian and discontinuous nonlinearity subject to multi-point boundary conditions. As far as we are aware, the existence results are new even in the classical case of continuous nonlinearities.

Keywords Compression–expansion fixed point theorem · Discontinuous differential equation · Positive solution · ϕ-Laplacian equation · Differential inclusion

Mathematics Subject Classification 47H10 · 34A36 · 47H04 · 34A60

1 Introduction

We are concerned with the existence of positive solutions to the following multi–point boundary value problem involving the ϕ-Laplacian

$$- (\phi(u'))' = f(t, u), \quad u'(0) = 0, \ u(1) = \sum_{i=1}^{n} \alpha_i \, u(\eta_i), \tag{1.1}$$

where $\alpha_i \in [0, 1)$, $\eta_i \in (0, 1)$ $(i = 1, 2, \dots, n)$, $\phi : (-a, a) \to (-b, b)$ $(0 < a, b \leq +\infty)$ is an odd increasing homeomorphism and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ may be discontinuous with respect to both variables.

✉ Jorge Rodríguez–López
jorgerodriguez.lopez@usc.es

1 Departamento de Estatística, Análise Matemática e Optimización, Instituto de Matemáticas, Universidade de Santiago de Compostela, 15782 Facultade de Matemáticas, Campus Vida, Santiago, Spain

Published online: 14 July 2021

In recent years, a lot of attention have been paid to the existence of solutions to boundary value problems with ϕ-Laplacian (see, for instance, [5,9,16,18,19,27] and the references therein). Here, we study in a unified way the *classical* homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$, the *singular* homeomorphism $\phi : (-a, a) \to \mathbb{R}$ and the *bounded* one $\phi : \mathbb{R} \to (-b, b)$.

As usual in the related literature when f is discontinuous (see [18,19]), we consider the following regularized problem

$$- (\phi(u'))' \in F(t, u), \quad u'(0) = 0, \ u(1) = \sum_{i=1}^{n} \alpha_i \, u(\eta_i), \tag{1.2}$$

where F is an usc multivalued map with closed convex values. In order to prove the existence of positive solutions to (1.2), by means of a fixed point approach, it appears the fixed point problem

$$x \in (\Psi \circ \Phi)(x),$$

where Ψ is a nonlinear continuous map and Φ is an usc multivalued map with closed convex values. Since the values of the composition $\Psi \circ \Phi$ can be non-convex, the classical generalization of Krasnosel'skiĭ's fixed point theorem in cones for usc multivalued maps [13] is not applicable here.

To overcome this difficulty we prove a new version of the compression–expansion fixed point theorem in cones for *decomposable maps*, that is, compositions of two usc multivalued maps which cover the fixed point problem above. Our approach is based on a suitable computation of the fixed point index for this class of maps, which was developed in [27]. Some previous results on fixed point theory for decomposable maps can be found in the pioneering paper [24] and in [11].

Once the existence of positive solutions for the regularized problem (1.2) is established, we wonder whether such solutions are also Carathéodory type solutions for (1.1). This question is not new and was studied, for instance, in case of systems of first-order equations in [10], second-order BVPs in [6] or reaction-diffusion equations in [20]. Here a transversality condition on the discontinuities of f is imposed in order to prove that all the solutions of (1.2) are in fact solutions of (1.1). Roughly speaking, the function f may be discontinuous over the graphs of a countable number of the so-called *admissible* curves.

To our best knowledge, the main existence result for problem (1.1) is new even in the classical case of a continuous nonlinearity. It is based on a new Harnack type inequality for supersolutions of (1.1), what enables us to include the boundary value problem (1.1) between the class of problems which can be studied by means of the general technique developed in [14].

Finally, we shall discuss the existence of positive solutions for problem (1.1) under asymptotic conditions on f at 0 and/or infinity, which were inspired by those in [4]. If ϕ is singular, we prove that (1.1) has at least one positive solution provided that f is superlinear at 0 with respect to ϕ, that is,

$$\lim_{x\to 0^+} \frac{f(t,x)}{\phi(x)} = +\infty \quad \text{uniformly with } t \in [0,1]. \tag{1.3}$$

In the case of a classical ϕ, if (1.3) holds and, moreover, f is sublinear at infinity with respect to ϕ, i.e.,

$$\lim_{x\to \infty} \frac{f(t,x)}{\phi(x)} = 0 \quad \text{uniformly with } t \in [0,1],$$

then the same conclusion is obtained. Note that, unlike [9] or [15], no monotonicity assumptions on f are required.

2 Compression–Expansion Fixed Point Theorem for Decomposable Maps

In the recent paper [27], the authors define a fixed point index theory for the composition of two multivalued maps. We recall here its definition and its main properties.

Let X and Y be Banach spaces and $K_X \subset X$, $K_Y \subset Y$ be closed convex sets. We are interested into the class of decomposable maps, i.e., multivalued maps $T : \overline{\Omega} \to 2^{K_X}$, where $\Omega \subset K_X$ is open in K_X, which can be represented as a composition $T = \Psi \circ \Phi$ of two multivalued maps Φ and Ψ with the following properties:

- (i) $\Phi : \overline{\Omega} \to 2^{K_Y}$ is upper semicontinuous (usc, for short) with closed convex values and relatively compact range;
- (ii) $\Psi : K_Y \to 2^{K_X}$ is usc with compact convex values.

Definition 2.1 Let Φ and Ψ be two multivalued maps satisfying conditions (i), (ii) and such that $x \notin (\Psi \circ \Phi)(x)$ for all $x \in \partial_{K_X} \Omega$.

The fixed point index for the pair of maps (Φ, Ψ) over Ω with respect to $K := K_X \times K_Y$ is defined as

$$ind_K(\Phi, \Psi, \Omega) := i_K(\Pi, \Omega \times K_Y),$$

where $\Pi : \overline{\Omega} \times K_Y \to 2^{K_X \times K_Y}$ is the multivalued map associated to the pair (Φ, Ψ) given by $\Pi(x, y) = \Psi y \times \Phi x$.

Note that the fixed point index over $\Omega \times K_Y$ with respect to K for the map Π, $i_K(\Pi, \Omega \times K_Y)$, is well–defined according to the Fitzpatrick–Petryshyn degree theory for multivalued maps [13].

Theorem 2.1 *The fixed point index $ind_K(\Phi, \Psi, \Omega)$ has the following properties:*

(1) (Additivity) If $\Omega_1, \Omega_2 \subset K_X$ are disjoint open in K_X and $\Psi \circ \Phi$ is fixed point free on $\partial_{K_X}\Omega_1 \cup \partial_{K_X}\Omega_2$, then $ind_K(\Phi, \Psi, \Omega_1 \cup \Omega_2) = ind_K(\Phi, \Psi, \Omega_1) + ind_K(\Phi, \Psi, \Omega_2)$.

(2) (Existence) If $ind_K(\Phi, \Psi, \Omega) \neq 0$, then there exist $x \in \Omega$ and $y \in K_Y$ with $x \in \Psi y$ and $y \in \Phi x$, consequently $x \in (\Psi \circ \Phi)(x)$ and $y \in (\Phi \circ \Psi)(y)$.

(3) (Normalization) For every $x_0 \in \Omega$*, one has* $ind_K(\Phi, x_0, \Omega) = 1$*.*

(4) (Homotopy) If $\phi : \overline{\Omega} \times [0,1] \to 2^{K_Y}$ *is usc with closed convex values and relatively compact range,* $\psi : K_Y \times [0,1] \to 2^{K_X}$ *is usc with compact convex values, and* $x \notin \psi(\phi(x,\lambda), \lambda)$ *for all* $x \in \partial_{K_X}\Omega$ *and* $\lambda \in [0,1]$*, then the index* $ind_K(\phi(\cdot,\lambda), \psi(\cdot,\lambda), \Omega)$ *does not depend on* λ*.*

In the sequel we need the following definition. A closed convex subset K of a Banach space X is a cone if it satisfies that $K \cap (-K) = \{0\}$ and $\lambda x \in K$ for every $x \in K$ and for all $\lambda \geq 0$.

Now we prove some sufficient conditions for guaranteeing that the fixed point index for decomposable maps defined above is 0 or 1 in certain open subsets of a cone.

Proposition 2.1 *Let X and Y be Banach spaces and* $K_X \subset X$*,* $K_Y \subset Y$ *cones in X and Y, respectively. Let* Ω *be a relatively open subset of* K_X *and let* $T = \Psi \circ \Phi : \overline{\Omega} \longrightarrow 2^{K_X}$ *be a multivalued operator such that* Φ *and* Ψ *fulfill conditions* (i) *and* (ii)*.*

If $0 \in \Omega$ *and* $\lambda x \notin Tx$ *for all* $x \in \partial_{K_X}\Omega$ *and all* $\lambda \geq 1$*, then* $ind_K(\Phi, \Psi, \Omega) = 1$*.*

Proof Let $\phi : \overline{\Omega} \times [0,1] \to 2^{K_Y}$ and $\psi : K_Y \times [0,1] \to 2^{K_X}$ be defined by

$$\phi(x,\lambda) = \Phi x, \quad \psi(y,\lambda) = \lambda \Psi y.$$

The fact that $\lambda x \notin (\Psi \circ \Phi)(x)$ for all $x \in \partial_{K_X}\Omega$ and $\lambda \geq 1$ implies that the homotopy given by ϕ and ψ is *admissible* and then $ind_K(\Phi, 0, \Omega) = ind_K(\Phi, \Psi, \Omega)$. Since $0 \in \Omega$, the normalization property ensures that $ind_K(\Phi, 0, \Omega) = 1$ and so $ind_K(\Phi, \Psi, \Omega) = 1$. □

Proposition 2.2 *Let X and Y be Banach spaces and* $K_X \subset X$*,* $K_Y \subset Y$ *cones in X and Y, respectively. Let* Ω *be a relatively open and bounded subset of* K_X *and let* $T = \Psi \circ \Phi : \overline{\Omega} \longrightarrow 2^{K_X}$ *be a multivalued operator such that* Φ *and* Ψ *fulfill conditions* (i) *and* (ii)*.*

If there exists $w \in K_X \backslash \{0\}$ *such that* $x \notin Tx + \mu w$ *for all* $x \in \partial_{K_X}\Omega$ *and all* $\mu \geq 0$*, then* $ind_K(\Phi, \Psi, \Omega) = 0$*.*

Proof Assume that $ind_K(\Phi, \Psi, \Omega) \neq 0$. Since $\Psi(\Phi(\overline{\Omega}))$ is relatively compact (as the image of a relatively compact set by an usc map with compact values), there exists $\mu_0 > 0$ such that $\|y\| < \mu_0$ for all $y \in T(\overline{\Omega})$. Define $R := \sup\{\|x\| : x \in \overline{\Omega}\}$, choose $\mu > (R + \mu_0)/\|w\|$ and consider the homotopy given by

$$\phi(x,\lambda) = \Phi x, \quad \psi(y,\lambda) = \Psi y + \lambda \mu w.$$

This homotopy is *admissible* and thus $ind_K(\Phi, \Psi, \Omega) = ind_K(\Phi, \Psi + \mu w, \Omega)$. Hence, by the existence property, there exists $x \in \Omega$ such that $x \in (\Psi \circ \Phi)(x) + \mu w$. Then,

$$\|x - \mu w\| \geq \mu \|w\| - \|x\| > (R + \mu_0) - \|x\| \geq \mu_0 > \|y\|$$

for all $y \in (\Psi \circ \Phi)(x)$, a contradiction. Therefore, $ind_K(\Phi, \Psi, \Omega) = 0$. □

As a consequence of the previous theory, we generalize to this context the Krasnosel'skiĭ's compression–expansion fixed point theorem in cones, see [1,17].

Theorem 2.2 *Let X and Y be Banach spaces and $K_X \subset X$, $K_Y \subset Y$ cones in X and Y, respectively. Let Ω_1 and Ω_2 be two relatively open and bounded subsets of K_X, with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T = \Psi \circ \Phi : \overline{\Omega}_2 \longrightarrow 2^{K_X}$ be a multivalued operator such that*

(i) $\Phi : \overline{\Omega}_2 \to 2^{K_Y}$ is usc with closed convex values and relatively compact range;
(ii) $\Psi : K_Y \to 2^{K_X}$ is usc with compact convex values.

Assume that T satisfies either of the following two conditions:

(1) $\lambda x \notin Tx$ for all $x \in \partial_{K_X}\Omega_1$ and $\lambda > 1$; and there exists $w \in K_X\backslash\{0\}$ such that $x \notin Tx + \mu w$ for all $x \in \partial_{K_X}\Omega_2$ and $\mu > 0$.
(2) $\lambda x \notin Tx$ for all $x \in \partial_{K_X}\Omega_2$ and $\lambda > 1$; and there exists $w \in K_X\backslash\{0\}$ such that $x \notin Tx + \mu w$ for all $x \in \partial_{K_X}\Omega_1$ and $\mu > 0$.

Then T has at least a fixed point in $\overline{\Omega}_2\backslash\Omega_1$.

Proof Suppose that condition (1) holds. If $x \in Tx$ for some $x \in \partial_{K_X}\Omega_1 \cup \partial_{K_X}\Omega_2$, then we are done.

Otherwise, we have that $\lambda x \notin Tx$ for all $x \in \partial_{K_X}\Omega_1$ and $\lambda \geq 1$, which based on Proposition 2.1 implies $ind_K(\Phi, \Psi, \Omega_1) = 1$, and that $x \notin Tx+\mu w$ for all $x \in \partial_{K_X}\Omega_2$ and $\mu \geq 0$, and thus by Proposition 2.2 we obtain that $ind_K(\Phi, \Psi, \Omega_2) = 0$.

Therefore, as a consequence of the additivity property, $ind_K(\Phi, \Psi, \Omega_2\backslash\overline{\Omega}_1) = -1$ and so the conclusion is deduced from the existence property of the fixed point index. The reasoning is similar in case that (2) holds. □

We note that Ω_1 and Ω_2 are arbitrarily open bounded subsets of a cone instead of just intersections of open balls and cones, which enlarges the applicability of the previous result, cf. [11].

Remark 2.1 In particular, if $X = Y$, $K_X = K_Y$ and $\Psi = I$, where I is the identity map in K_X, then Theorem 2.2 is the well–known compression–expansion fixed point theorem in cones for usc maps, see [13].

Remark 2.2 In Theorem 2.2 we present an extension of the well–known Krasnosel'skiĭ's fixed point theorem in cones, but clearly we can adapt much more fixed point theorems (all those whose proof is just based on fixed point index properties) to the context of decomposable maps. This is the case, for instance, of Leggett–Williams' fixed point theorem [22] and its generalizations [2,3].

3 Positive Solutions to a Multi–point BVP Involving the ϕ–Laplacian

We consider the following boundary value problem with ϕ–Laplacian subject to multi–point boundary conditions

$$(\phi(u'))' + f(t,u) = 0, \quad u'(0) = 0,\ u(1) = \sum_{i=1}^{n} \alpha_i\, u(\eta_i), \tag{3.1}$$

where $n \in \mathbb{N}, \alpha_i \geq 0\,(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i < 1, 0 < \eta_1 < \eta_2 < \cdots < \eta_n < 1$, $\phi : (-a, a) \to (-b, b)$ $(0 < a, b \leq +\infty)$ is an odd increasing homeomorphism and the function $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is non necessarily continuous.

By a *Carathéodory solution* of (3.1) we mean a function $u \in \mathcal{C}^1[0, 1]$ with $u'(0) = 0 = u(1) - \sum_{i=1}^{n} \alpha_i\, u(\eta_i)$ such that $u'(t) \in (-a, a)$ for all $t \in I := [0, 1]$, $\phi(u') \in W^{1,1}(I)$ and which satisfies the differential equation in (3.1) for a.a. $t \in I$.

In the sequel, the space of continuous functions $\mathcal{C}(I)$ will be endowed with the usual supremum norm $\|\cdot\|_\infty$ and the cone of nonnegative continuous functions will be denoted by P.

Let us assume that the function $f : I \times [0, \infty) \to [0, \infty)$ satisfies the following basic conditions:

$(H1)$ Compositions $f(\cdot, u(\cdot))$ are measurable whenever $u \in P$;

$(H2)$ If $b < +\infty$, for each $r > 0$ there exists $c_r < b$ such that $f(t, u) \leq c_r$ on $I \times [0, r]$, and if $b = \infty$, there exist $c_1, c_2 \geq 0$ and $p \geq 1$ such that $f(t, u) \leq c_1 u^p + c_2$ for a.a. $t \in I$ and all $u \in [0, \infty)$.

Now we present the integral formulation of problem (3.1). By integration of the differential equation, $u'(0) = 0$ and $\phi(0) = 0$ we have

$$\phi(u'(t)) = -\int_0^t f(s, u(s))\, ds.$$

Then, integrating from t to 1, we obtain

$$u(t) = u(1) + \int_t^1 \phi^{-1}\left(\int_0^s f(r, u(r))\, dr\right) ds.$$

In particular, evaluating at $t = \eta_i$,

$$u(\eta_i) = u(1) + \int_{\eta_i}^1 \phi^{-1}\left(\int_0^s f(r, u(r))\, dr\right) ds,$$

and so, taking into account the condition $u(1) = \sum_{i=1}^{n} \alpha_i\, u(\eta_i)$,

$$u(1) = \frac{1}{1 - \sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{n} \alpha_i \int_{\eta_i}^1 \phi^{-1}\left(\int_0^s f(r, u(r))\, dr\right) ds.$$

Hence, we define the operator $N : P \longrightarrow P$ by

$$Nu(t) = \frac{1}{1 - \sum_{i=1}^{n} \alpha_i} \sum_{i=1}^{n} \alpha_i \int_{\eta_i}^1 \phi^{-1}\left(\int_0^s f(r, u(r))\, dr\right) ds + \int_t^1 \phi^{-1}\left(\int_0^s f(r, u(r))\, dr\right) ds,$$

whose fixed points correspond with solutions to (3.1).

Since f is discontinuous in both variables, the operator N is not continuous and thus we transform (3.1) into a multi-point boundary value problem with a nonlinear differential inclusion

$$-(\phi(u'))' \in F(t,u) \ \text{ for a.a. } t \in I,\ u'(0) = 0 = u(1) - \sum_{i=1}^{n} \alpha_i\, u(\eta_i), \tag{3.2}$$

where $F : I \times [0,\infty) \to 2^{[0,\infty)}$ is the multivalued map given by

$$F(t,x) = \bigcap_{\varepsilon>0} \overline{\text{co}} f\left(t, \overline{B}_\varepsilon(x)\right),$$

with $\overline{B}_\varepsilon(x) := [x-\varepsilon, x+\varepsilon]$. Equivalently, F can be rewritten as

$$F(t,x) = \left[\min\left\{f(t,x), \liminf_{y\to x} f(t,y)\right\}, \max\left\{f(t,x), \limsup_{y\to x} f(t,y)\right\}\right].$$

Observe that $F(t,x) = \{f(t,x)\}$ whenever $f(t,\cdot)$ is continuous at x.

Note that the solutions of (3.2) are usually called *Krasovskii solutions* of (3.1).

To solve (3.2), it suffices to consider the operator inclusion $u \in (\Psi \circ \Phi)(u)$, with

$$\Psi v(t) = \frac{1}{1-\sum_{i=1}^{n}\alpha_i} \sum_{i=1}^{n} \alpha_i \int_{\eta_i}^{1} \phi^{-1}(v(s))\, ds + \int_t^1 \phi^{-1}(v(s))\, ds$$

and $\Phi(u) = \Gamma \circ \mathcal{N}_F(u)$, where

$$\Gamma v(s) = \int_0^s v(r)\, dr$$

and $\mathcal{N}_F$ denotes the Nemytskii operator

$$\mathcal{N}_F(u) = \{v \in L^1(I) : v(t) \in F(t,u(t)) \ \text{ for a.a. } t \in I\}.$$

We shall need the following result concerning the upper semicontinuity of the Nemytskii operator (see [8,25], for details).

Lemma 3.1 *Assume that the function $f : I \times [0,\infty) \to [0,\infty)$ satisfies conditions $(H1)$ and $(H2)$.*

Then the Nemytskii operator $\mathcal{N}_F : P \to 2^{L^1(I)}$ is usc.

In order to apply the compression–expansion fixed point theorem in cones, we need the following Harnack type inequality. Its proof is adapted from the ideas in [9,15].

Lemma 3.2 *For every function $u \in \mathcal{C}^1(I)$ with $u'(0) = 0 = u(1) - \sum \alpha_i\, u(\eta_i)$, $u'(t) \in (-a,a)$ for every $t \in I$, $\phi(u') \in W^{1,1}(I)$ and $(\phi(u'))' \le 0$ on I, one has the following inequality:*

$$\min_{t\in[0,\eta_i]} u(t) \ge \frac{1-\eta_i}{1-\alpha_i\,\eta_i}\, \|u\|_\infty, \tag{3.3}$$

for each $i \in \{1, 2, \ldots, n\}$.

Proof From $(\phi(u'))' \leq 0$, we have that $\phi(u')$ is nonincreasing and so is $u' = \phi^{-1}(\phi(u'))$. Thus u is concave on I. On the other hand, since $\phi(u')$ is nonincreasing and vanishes at $t = 0$, we have that $\phi(u') \leq 0$, and then $u' \leq 0$. Hence, u is nonincreasing on I.

Now, to see that $u \geq 0$, it suffices to show that $u(1) \geq 0$. Since u is nonincreasing on I, we have that $u(\eta_i) \geq u(1)$ for all $i \in \{1, 2, \ldots, n\}$ and thus condition $u(1) = \sum \alpha_i \, u(\eta_i)$ implies that $(1 - \sum \alpha_i) u(1) \geq 0$. Then, by assumption $\sum \alpha_i < 1$, we obtain that $u(1) \geq 0$. Therefore, u is a nonnegative, nonincreasing, concave function and $\|u\|_\infty = u(0)$.

Let us fix an arbitrary $i \in \{1, 2, \ldots, n\}$. By its concavity, the graph of u restricted to $[0, \eta_i]$ lies under the line containing the points $(1, u(1))$ and $(\eta_i, u(\eta_i))$. Thus,

$$u(0) \leq u(1) + \frac{u(\eta_i) - u(1)}{1 - \eta_i}.$$

Since $u(1) \geq \alpha_i \, u(\eta_i)$, we have

$$u(0) \leq \frac{1 - \alpha_i \, \eta_i}{1 - \eta_i} u(\eta_i).$$

Finally, (3.3) follows immediately from $u(\eta_i) = \min_{t \in [0, \eta_i]} u(t)$. □

In the sequel, let us denote $\alpha := \alpha_n$ and $\eta := \eta_n$. We shall prove the existence of positive solutions to (3.1) in the following subcone of P:

$$K = \left\{ u \in P \ : \ \min_{t \in [0, \eta]} u(t) \geq c \, \|u\|_\infty \right\},$$

with $c := (1 - \eta)/(1 - \alpha \, \eta)$.

In order to apply Theorem 2.2, we let $X = Y = \mathcal{C}(I)$, $K_X = K$ and

$$K_Y = \{u \in P \ : \ u \text{ is nondecreasing}, u(0) = 0\}.$$

In the following result, we show that the operator $T = \Psi \circ \Phi$ is a multivalued decomposable map, what justifies the application of the theory developed in Sect. 2.

Theorem 3.1 *Assume that* f *satisfies conditions* $(H1)$ *and* $(H2)$.
Then the operators

$$\Phi : K \to 2^{K_Y} \quad \textit{and} \quad \Psi : K_Y \to K$$

are well–defined; Φ *is usc with closed convex values and maps bounded sets into relatively compact sets; and* Ψ *is a single–valued continuous operator.*

Proof It is clear that $\Phi(K) \subset K_Y$ since $\Phi = \Gamma \circ \mathcal{N}_F$, f is nonnegative and $\Gamma v(t) = \int_0^t v(r)\,dr$. Let us show that $\Psi(K_Y) \subset K$. Take an arbitrary function $v \in K_Y$ and let $u := \Psi v$, then $u \in P$. Moreover, $\phi(u') = -v$, and thus $\phi(u')$ is nonincreasing. Furthermore, $u'(0) = 0 = u(1) - \sum \alpha_i\, u(\eta_i)$. By Lemma 3.2, we have

$$\min_{t\in[0,\eta]} u(t) \geq \frac{1-\eta}{1-\alpha\,\eta}\, \|u\|_\infty$$

and, therefore, $u \in K$.

In addition, Γ as a linear operator from $L^1(I)$ to $\mathcal{C}(I)$ is compact and, by Lemma 3.1, $\mathcal{N}_F$ is usc from the topology of $\mathcal{C}(I)$ to that of $L^1(I)$. Hence, Φ is usc and maps bounded sets into relatively compact sets. Also, Φ has closed and convex values, see [26, Theorem 2].

Finally, the continuity and the compactness of the operator Ψ are standard consequences of Lebesgue's dominated convergence theorem and Ascoli-Arzela's theorem. □

Now we present some sufficient conditions about f for guaranteeing the cone–compression or cone–expansion conditions for the operator $T = \Psi \circ \Phi$ on the boundary of two nested neighborhoods of the origin.

Let us introduce some notations. For $r > 0$ and $\varepsilon \in (0, r)$, denote

$$m_{r,\varepsilon} := \inf_{t\in[0,\eta],\, x\in[r-\varepsilon,(r+\varepsilon)/c]} f(t,x) \qquad \text{and} \qquad M_{r,\varepsilon} := \sup_{t\in I,\, x\in[0,r+\varepsilon]} f(t,x).$$

We also make use of the following relatively open set of the cone K,

$$V_r = \left\{ u \in K \ :\ \min_{t\in[0,\eta]} u(t) < r \right\},$$

which is similar to that introduced in [21].

Notice that $B_r \subset V_r \subset B_{r/c}$, where $B_\rho := \{u \in K \ :\ \|u\|_\infty < \rho\}$.

Lemma 3.3 *Assume that f satisfies conditions $(H1)$ and $(H2)$. If there exist $r > 0$ and $\varepsilon > 0$ such that*

$$\phi^{-1}\left(M_{r,\varepsilon}\right) \leq \left(1 - \sum_{i=1}^{n} \alpha_i\right) r, \tag{3.4}$$

then we have that $\lambda u \notin Tu$ for all $u \in K$, $\|u\|_\infty = r$, and all $\lambda > 1$.

Proof Let us show that

$$\|v\|_\infty \leq r \quad \text{for all } v \in Tu \text{ and } u \in K \text{ with } \|u\|_\infty = r,$$

which implies that

$$\lambda u \notin Tu \quad \text{for all } \lambda > 1 \text{ and } u \in K \text{ with } \|u\|_\infty = r.$$

Assume to the contrary that there exist $v \in Tu$ and $u \in K$ with $\|u\|_\infty = r$ such that $r < \|v\|_\infty$. Observe that if $w \in \mathcal{N}_F(u)$ and $\|u\|_\infty = r$, by the definition of the regularized multivalued map F, we have

$$w(t) \leq \sup_{t\in I,\, x\in[0,r+\varepsilon]} f(t,x) =: M_{r,\varepsilon} \quad \text{for a.a. } t \in I.$$

Hence, since $v \in Tu = \Psi \circ \Gamma \circ \mathcal{N}_F(u)$, we obtain

$$\begin{aligned}\|v\|_\infty &\leq \frac{1}{1-\sum_{i=1}^n \alpha_i} \sum_{i=1}^n \alpha_i \int_{\eta_i}^1 \phi^{-1}\left(\int_0^s M_{r,\varepsilon}\, dr\right) ds + \int_0^1 \phi^{-1}\left(\int_0^s M_{r,\varepsilon}\, dr\right) ds \\ &= \frac{1}{1-\sum_{i=1}^n \alpha_i} \sum_{i=1}^n \alpha_i \int_{\eta_i}^1 \phi^{-1}\left(M_{r,\varepsilon}\, s\right) ds + \int_0^1 \phi^{-1}\left(M_{r,\varepsilon}\, s\right) ds\end{aligned}$$

and so, from $\eta_i \in (0,1)$ for all $i \in \{1,\ldots,n\}$, it follows

$$\|v\|_\infty \leq \frac{1}{1-\sum_{i=1}^n \alpha_i} \int_0^1 \phi^{-1}\left(M_{r,\varepsilon}\, s\right) ds \leq \frac{1}{1-\sum_{i=1}^n \alpha_i} \phi^{-1}\left(M_{r,\varepsilon}\right).$$

Finally, inequality (3.4) yields the contradiction $r < \|v\|_\infty \leq r$. □

Remark 3.1 Of course, condition (3.4) holds for r large enough if ϕ is singular. Therefore, in this case, problem (3.2) is always solvable under the basic assumptions $(H1)$ and $(H2)$. Moreover, it is worth mentioning that if ϕ is singular and f is continuous, then problem (3.1) has a nonnegative (maybe trivial) solution without additional hypotheses, as a consequence of Proposition 2.1. This is similar to what happens for the Dirichlet problem, see [5].

Lemma 3.4 *Assume that f satisfies conditions $(H1)$ and $(H2)$. If there exist $r > 0$ and $\varepsilon \in (0,r)$ such that*

$$\phi^{-1}\left(\eta\, m_{r,\varepsilon}\right) \geq \frac{1-\sum_{i=1}^n \alpha_i}{1-\eta} r, \tag{3.5}$$

then $u \notin Tu + \mu w$ for all $u \in \partial_K V_r$ and all $\mu > 0$ with $w \equiv 1$.

Proof Suppose that there exist $u \in \partial_K V_r$ and $\mu > 0$ such that $u \in Tu + \mu w$. Since by definition $T = \Psi \circ \Gamma \circ \mathcal{N}_F$, then there exists $v \in \mathcal{N}_F(u)$ such that

$$u(t) = \frac{1}{1-\sum_{i=1}^n \alpha_i} \sum_{i=1}^n \alpha_i \int_{\eta_i}^1 \phi^{-1}\left(\int_0^s v(r)\, dr\right) ds + \int_t^1 \phi^{-1}\left(\int_0^s v(r)\, dr\right) ds + \mu.$$

Notice that if $v \in \mathcal{N}_F(u)$ and $u \in \partial_K V_r$, by the definition of the regularized multivalued map F, we have

$$v(t) \geq \inf_{t\in[0,\eta],\, x\in[r-\varepsilon,(r+\varepsilon)/c]} f(t,x) =: m_{r,\varepsilon} \quad \text{for a.a. } t \in [0,\eta].$$

Therefore, since $0 < \eta_i \leq \eta < 1$ for every $i \in \{1, \ldots, n-1\}$, we have for $t \in [0, \eta]$,

$$\begin{aligned} u(t) &\geq \frac{1}{1-\sum_{i=1}^n \alpha_i} \sum_{i=1}^n \alpha_i \int_\eta^1 \phi^{-1}\left(\int_0^s v(r)\,dr\right) ds + \int_\eta^1 \phi^{-1}\left(\int_0^s v(r)\,dr\right) ds + \mu \\ &\geq \frac{1}{1-\sum_{i=1}^n \alpha_i} \int_\eta^1 \phi^{-1}\left(\int_0^\eta m_{r,\varepsilon}\,dr\right) ds + \mu \\ &\geq \frac{1}{1-\sum_{i=1}^n \alpha_i} \int_\eta^1 \phi^{-1}\left(\eta\, m_{r,\varepsilon}\right) ds + \mu \\ &= \frac{1-\eta}{1-\sum_{i=1}^n \alpha_i} \phi^{-1}\left(\eta\, m_{r,\varepsilon}\right) + \mu. \end{aligned}$$

Hence, condition (3.5) implies that $u(t) \geq r + \mu$ for $t \in [0, \eta]$. Taking the infimum in $[0, \eta]$, we get the contradiction $r \geq r + \mu$. ◻

Now we are in a position to prove the existence of positive solutions to the inclusion problem (3.2) based on the previous lemmas and the compression-expansion fixed point theorem in Sect. 2.

Theorem 3.2 *Assume that f satisfies conditions $(H1)$ and $(H2)$. Moreover, assume that there exist $r_1, r_2 > 0$ and $\varepsilon \in (0, r_2)$ such that*

$$\phi^{-1}\left(M_{r_1,\varepsilon}\right) \leq \left(1 - \sum \alpha_i\right) r_1 \quad \textit{and} \quad \phi^{-1}\left(\eta\, m_{r_2,\varepsilon}\right) \geq \frac{1-\sum \alpha_i}{1-\eta} r_2. \tag{3.6}$$

- *If $r_1 < r_2$, then problem (3.2) has one positive solution u such that $r_1 \leq \|u\|_\infty \leq r_2/c$.*
- *If $r_2/c < r_1$, then problem (3.2) has one positive solution u such that $r_2 \leq \|u\|_\infty \leq r_1$.*

Proof By Lemmas 3.3 and 3.4 we obtain, thanks to condition (3.6), that

$$\lambda u \notin Tu \text{ for all } u \in \partial_K B_{r_1} \text{ and all } \lambda > 1,$$

and

$$u \notin Tu + \mu \text{ for all } u \in \partial_K V_{r_2} \text{ and all } \mu > 0.$$

First, if $r_1 < r_2$, it follows that $0 \in B_{r_1} \subset \overline{B}_{r_1} \subset V_{r_2}$ and thus Theorem 2.2 ensures that the operator T has at least one fixed point in $\overline{V}_{r_2} \backslash B_{r_1}$. Now, the fact that $V_{r_2} \subset B_{r_2/c}$ gives the conclusion.

On the other hand, in case that $r_2/c < r_1$, we have that $0 \in V_{r_2} \subset \overline{V}_{r_2} \subset B_{r_1}$ and as a consequence of Theorem 2.2 we obtain that T has at least one fixed point in $\overline{B}_{r_1} \backslash V_{r_2} \subset \overline{B}_{r_1} \backslash B_{r_2}$. ◻

Remark 3.2 We use the open set V_r in Lemma 3.4 to prove the index zero result in Proposition 2.2. As explained in [17], the requirement about the growth condition of f is less stringent than if the set $B_{r/c}$ is employed instead.

Remark 3.3 Observe that condition (3.6) holds if there exists two constants $M_1, M_2 \in (0, b)$ such that $M_1 \leq f(t, x) \leq M_2$ for all $(t, x) \in I \times [0, \infty)$, as required in [16].

The existence of positive solutions for the differential inclusion (3.2) is meaningful itself when f is a discontinuous function, see [7,18,19]. However, we are concerned with the existence of positive Carathéodory type solutions for (3.1) and so we need some additional condition about f which implies that all the solutions of (3.2) are in fact Carathéodory solutions of (3.1). It is given by the notion of *admissible discontinuity curves* presented in the following definition, which can be traced back to [23] (see also [12,25]).

Definition 3.1 A function $\gamma \in \mathcal{C}^1(I)$ with $\phi \circ \gamma' \in W^{1,1}(I)$ is called an *admissible discontinuity curve* for the differential equation $-(\phi(u'))' = f(t, u)$ on the subinterval $J \subset I$ if

$$\{-(\phi(\gamma'(t)))'\} \cap F(t, \gamma(t)) \subset \{f(t, \gamma(t))\} \quad \text{for a.a. } t \in J. \tag{3.7}$$

Note that a sufficient condition for a curve $\gamma : I \to \mathbb{R}$ to be *admissible* on a subinterval $J \subset I$ can be stated as follows: there exist $\delta, \epsilon > 0$ such that

$$-(\phi(\gamma'(t)))' + \delta \leq f(t, y) \quad \text{for a.a. } t \in J \text{ and all } y \in \left[\gamma(t) - \epsilon, \gamma(t) + \epsilon\right], \tag{3.8}$$

or

$$-(\phi(\gamma'(t)))' - \delta \geq f(t, y) \quad \text{for a.a. } t \in J \text{ and all } y \in \left[\gamma(t) - \epsilon, \gamma(t) + \epsilon\right]. \tag{3.9}$$

Obviously, any Carathéodory solution for $-(\phi(u'))' = f(t, u)$ on J is also an admissible discontinuity curve.

The *transversality* condition (3.7) is the key ingredient to show that all solutions of (3.2) are Carathéodory solutions of (3.1). The proof of this result can be looked up in [27, Lemma 3.3].

Lemma 3.5 *Assume that* $f : I \times [0, \infty) \to [0, \infty)$ *satisfies conditions* $(H1)$, $(H2)$ *and*

$(H3)$ *There is a countable number of admissible discontinuity curves* γ_n *on the subintervals* $J_n \subset I$ *such that*

$$f(t, \cdot) \text{ is continuous on } [0, \infty) \setminus \bigcup_{\{n:\, t \in J_n\}} \{\gamma_n(t)\} \quad \text{for a.a. } t \in I.$$

Then the set of solutions to problem (3.2) *coincides with the set of solutions to* (3.1).

Now, as a straightforward consequence of Theorem 3.2 and Lemma 3.5, we derive the existence of positive solutions for problem (3.1).

Theorem 3.3 *Assume that f satisfies conditions $(H1)$, $(H2)$ and $(H3)$. Moreover, assume that there exist $r_1, r_2 > 0$ and $\varepsilon \in (0, r_2)$ such that*

$$\phi^{-1}\left(M_{r_1,\varepsilon}\right) \leq \left(1 - \sum \alpha_i\right) r_1 \quad \text{and} \quad \phi^{-1}\left(\eta\, m_{r_2,\varepsilon}\right) \geq \frac{1 - \sum \alpha_i}{1 - \eta} r_2.$$

- $(S1)$ *If $r_1 < r_2$, then problem (3.1) has one positive solution u such that $r_1 \leq \|u\|_\infty \leq r_2/c$.*
- $(S2)$ *If $r_2/c < r_1$, then problem (3.1) has one positive solution u such that $r_2 \leq \|u\|_\infty \leq r_1$.*

We illustrate the applicability of our result with the following example.

Example 3.1 Consider the problem

$$-\left(\frac{u'}{\sqrt{1 + |u'|^2}}\right)' = f(t, u) \quad \text{for a.a. } t \in I, \quad u'(0) = 0, \; u(1) = \frac{1}{5} u\left(\frac{1}{2}\right), \tag{3.10}$$

with

$$f(t, x) = \frac{1}{4}\left(1 + \sin^2\left(\left[\frac{1}{x + 1 - t}\right]\right)\right) \quad \text{for all } x \in [0, 1],$$

where $[x]$ denotes the integer part of x.

Note that the function f satisfies that

$$\frac{1}{4} \leq f(t, x) \leq \frac{1}{2} \quad \text{for } t, x \in [0, 1].$$

Hence inequalities (3.6) hold, for instance, with $r_1 = 4/5$, $r_2 = 1/20$ and ε small enough. Because of this localization, it suffices to define f in $I \times [0, 1]$.

Moreover, the function f is discontinuous over the graphs of a countable number of lines, which are those given by

$$\gamma_n(t) = t - 1 + \frac{1}{n}, \quad t \in \left[1 - \frac{1}{n}, 1\right], \quad n \in \mathbb{N}.$$

They are admissible discontinuity curves in the sense of Definition 3.1 since $(\phi(\gamma_n'))' \equiv 0$ for all $n \in \mathbb{N}$ and thus inequality (3.8) is trivially satisfied for any $\delta \in (0, 1/4)$ and any $\epsilon > 0$. This implies that f satisfies condition $(H3)$.

Then Theorem 3.3 ensures that problem (3.10) has a positive solution u such that

$$\frac{1}{20} \leq \|u\|_\infty \leq \frac{4}{5}.$$

We highlight that, as far as we are aware, the previous existence result is new even in the case of continuous nonlinearities. Note that in this case, since $F(t, x) = \{f(t, x)\}$, the positive number ε can be omitted in conditions (3.4) and (3.5).

Corollary 3.1 *Assume that $f : I \times [0,\infty) \to [0,\infty)$ is continuous. Moreover, assume that there exist $r_1, r_2 > 0$ such that*

$$\phi^{-1}\left(\max_{t\in I,\, x\in[0,r_1]} f(t,x)\right) \le \left(1-\sum \alpha_i\right) r_1 \text{ and } \phi^{-1}\left(\eta \min_{t\in[0,\eta],\, x\in[r_2,r_2/c]} f(t,x)\right) \ge \frac{1-\sum\alpha_i}{1-\eta} r_2.$$

- $(S1)$ *If $r_1 < r_2$, then problem (3.1) has one positive solution u such that $r_1 \le \|u\|_\infty \le r_2/c$.*
- $(S2)$ *If $r_2/c < r_1$, then problem (3.1) has one positive solution u such that $r_2 \le \|u\|_\infty \le r_1$.*

Remark 3.4 Obviously, from Theorem 3.3, one can obtain multiplicity results for problem (3.1) provided that there exist several pairs of numbers (r_1, r_2) satisfying conditions (3.4) and (3.5), respectively.

Finally, we show that problem (3.1) has at least one positive solution under suitable asymptotic conditions. Next results are inspired by those in [4], but our arguments are slightly different since they rely on Krasnosel'skiĭ's type fixed point theorem in cones.

Proposition 3.1 *Assume that f satisfies conditions $(H1)$, $(H2)$ and that*

$$\lim_{x\to 0^+} \frac{f(t,x)}{\phi(x)} = +\infty \quad \text{uniformly with } t \in I \tag{3.11}$$

and

$$\limsup_{x\to 0} \frac{\phi(\tau x)}{\phi(x)} < +\infty \quad \text{for all } \tau > 0. \tag{3.12}$$

Then there exists $r > 0$ such that $u \notin Tu + \mu w$ for all $u \in \partial_K V_r$ and all $\mu > 0$ with $w \equiv 1$.

Proof By (3.12), with $\tau = 2\left(1-\sum\alpha_i\right)/(1-\eta)$, there exists $L > 0$ so that

$$\eta L > \limsup_{x\to 0} \frac{\phi\left(\frac{1-\sum\alpha_i}{1-\eta}x\right)}{\phi\left(\frac{x}{2}\right)},$$

and thus there exists $\rho > 0$ ($\rho < a$) such that

$$\eta L \phi\left(\frac{x}{2}\right) \ge \phi\left(\frac{1-\sum\alpha_i}{1-\eta}x\right) \quad \text{for all } x \in (0,\rho). \tag{3.13}$$

On the other hand, by (3.11), there is $r > 0$ (we may suppose $r < 2c\rho/3$) such that

$$f(t,x) \ge L\phi(x) \quad \text{for all } t \in I \text{ and all } x \in \left(0, \frac{3r}{2c}\right],$$

so it follows that

$$\inf_{t\in[0,\eta],\, x\in[r/2,3r/(2c)]} f(t,x) \geq L\phi\left(\frac{r}{2}\right).$$

Now, inequality (3.13) implies that

$$\inf_{t\in[0,\eta],\, x\in[r/2,3r/(2c)]} f(t,x) \geq \frac{1}{\eta}\phi\left(\frac{1-\sum\alpha_i}{1-\eta}r\right).$$

Finally, the conclusion is derived from Lemma 3.4 with $\varepsilon = r/2$. □

If ϕ is singular (i.e., $a < +\infty$, $b = +\infty$), then problem (3.1) has at least one positive solution provided that f is superlinear at 0 with respect to ϕ.

Theorem 3.4 *Assume that conditions* $(H1)-(H3)$*,* (3.11) *and* (3.12) *are fulfilled. Then problem* (3.1) *has at least one positive solution if* ϕ *is singular.*

Proof By Remark 3.1 and Proposition 3.1, problem (3.2) has at least one positive solution. Finally, Lemma 3.5 implies that it is also a Carathéodory solution for (3.1). □

A remarkable particular problem is that given by the *relativistic operator*, namely,

$$-\left(\frac{u'}{\sqrt{1-|u'|^2}}\right)' = f(t,u) \quad (t\in I), \quad u'(0)=0,\ u(1)=\sum_{i=1}^{n}\alpha_i\, u(\eta_i). \tag{3.14}$$

Corollary 3.2 *Assume that conditions* $(H1)-(H3)$ *hold and*

$$\lim_{x\to 0^+}\frac{f(t,x)}{x} = +\infty \quad \textit{uniformly with } t\in I. \tag{3.15}$$

Then problem (3.14) *has at least one positive solution.*

Proof The conclusion follows from Theorem 3.4 with the homeomorphism $\phi : (-1,1)\to\mathbb{R}$ given by $\phi(x) = x/\sqrt{1-x^2}$. Note that

$$\lim_{x\to 0}\frac{\phi(\tau x)}{\phi(x)} = \tau \quad \text{for all } \tau > 0,$$

and that, in this case, condition (3.15) implies (3.11). □

Unlike the singular case, if ϕ is a classical homeomorphism (i.e., $a = b = +\infty$), then some additional condition to the asymptotic behavior of f at 0 is needed to ensure the existence of a positive solution for (3.1). To this end, it is enough to assume that f is sublinear at infinity with respect to ϕ.

Proposition 3.2 *Assume that f satisfies conditions $(H1)$, $(H2)$ and that*

$$\lim_{x\to\infty} \frac{f(t,x)}{\phi(x)} = 0 \quad \textit{uniformly with } t \in I \tag{3.16}$$

and ϕ is a classical homeomorphism such that

$$\limsup_{x\to\infty} \frac{\phi(x)}{\phi(\tau x)} < +\infty \quad \textit{for all } \tau \in (0,1). \tag{3.17}$$

Then there exists $R > 0$ such that $\lambda u \notin Tu$ for all $u \in K$, $\|u\|_\infty = R$, and all $\lambda > 1$.

Proof First, note that (3.17), with $\tau = 2\left(1 - \sum \alpha_i\right)/3$, implies that there exist positive numbers ξ and ρ such that

$$\phi\left(\frac{3}{2}x\right) \le \xi \phi\left(\left(1 - \sum \alpha_i\right)x\right) \quad \text{for all } x > \rho. \tag{3.18}$$

Next, by (3.16), for each $L > 0$ there exists $m > 0$ such that

$$f(t,x) \le m + L\phi(x) \quad \text{for all } t \in I \text{ and all } x \ge 0.$$

We can choose $L > 0$ small enough such that $2L\xi \le 1$. Since $\phi : \mathbb{R} \to \mathbb{R}$ is an increasing unbounded homeomorphism, we can also take $R > 0$ large enough (suppose $R > \rho$) so that $m \le L\phi(3R/2)$. Hence, we have that

$$f(t,x) \le 2L\phi\left(\frac{3}{2}R\right) \quad \text{for all } t \in I \text{ and all } x \in \left[0, \frac{3}{2}R\right].$$

From the choice of L and inequality (3.18), we obtain that

$$f(t,x) \le \phi\left(\left(1 - \sum \alpha_i\right)R\right) \quad \text{for all } t \in I \text{ and all } x \in \left[0, \frac{3}{2}R\right].$$

Therefore, condition (3.4) holds with $r = R$ and $\varepsilon = R/2$, so the conclusion is obtained by application of Lemma 3.3. □

Theorem 3.5 *Assume that conditions $(H1)-(H3)$, (3.11), (3.12), (3.16) and (3.17) hold. Then problem (3.1) has at least one positive solution provided that ϕ is classical.*

Proof Propositions 3.1 and 3.2 allow us to deduce, by application of Theorem 2.2, that problem (3.2) has at least one positive solution which belongs to $\overline{B}_R \backslash V_r$. Finally, Lemma 3.5 implies that it is also a Carathéodory solution for (3.1). □

As a consequence, in the classical p-Laplacian case,

$$-\left(\left|u'\right|^{p-2}u'\right)' = f(t,u) \quad (t \in I), \quad u'(0) = 0, \; u(1) = \sum_{i=1}^{n} \alpha_i\, u(\eta_i), \tag{3.19}$$

the existence of one positive solution is ensured provided that f is superlinear at 0 and sublinear at infinity with respect to $\phi_p(x) = |x|^{p-2}x$, $p > 1$.

Corollary 3.3 *Assume that conditions* $(H1) - (H3)$ *hold,*

$$\lim_{x\to 0^+} \frac{f(t,x)}{x^{p-1}} = +\infty \ \textit{and} \ \lim_{x\to\infty} \frac{f(t,x)}{x^{p-1}} = 0 \ \textit{uniformly with } t \in I.$$

Then problem (3.19) *has at least one positive solution.*

Proof It suffices to apply Theorem 3.5 with the homeomorphism $\phi(x) = \phi_p(x) = |x|^{p-2}x$. Note that ϕ_p clearly satisfies conditions (3.12) and (3.17). □

Funding Jorge Rodríguez-López was partially supported by Xunta de Galicia ED431C 2019/02.

Availability of data and material Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

References

1. Amann, H.: Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. SIAM Rev. **18**(4), 620–709 (1976)
2. Avery, R.I.: A generalization of the Leggett–Williams fixed point theorem. Math. Sci. Res. Hot-Line **3**, 9–14 (1999)
3. Avery, R.I., Peterson, A.C.: Three positive fixed points of nonlinear operators on ordered Banach spaces. Comput. Math. Appl. **42**, 313–322 (2001)
4. Bereanu, C., Jebelean, P., Torres, P.J.: Positive radial solutions for Dirichlet problems with mean curvature operators in Minkowski space. J. Funct. Anal. **264**, 270–287 (2013)
5. Bereanu, C., Mawhin, J.: Existence and multiplicity results for some nonlinear problems with singular ϕ-Laplacian. J. Differ. Equ. **243**(2), 536–557 (2007)
6. Bonanno, G., Iannizzotto, A., Marras, M.: On ordinary differential inclusions with mixed boundary conditions. Differ. Integral Equ. **30**, 273–288 (2017)
7. Bonanno, G., Jebelean, P., Şerban, C.: Three periodic solutions for discontinuous perturbations of the vector p-Laplacian operator. Proc. R. Soc. Edinb. Sect. A **147**, 673–681 (2017)
8. Cellina, A., Fryszkowski, A., Rzezuchowski, T.: Upper semicontinuity of Nemytskij operators. Ann. Mat. Pura Appl. **160**(4), 321–330 (1991)
9. Chinní, A., Di Bella, B., Jebelean, P., Precup, R.: A four-point boundary value problem with singular ϕ-Laplacian. J. Fixed Point Theory Appl. **21**(66), 1–16 (2019)
10. Cid, J. Á., Pouso, R. L.: Ordinary differential equations and systems with time-dependent discontinuity sets. Proc. R. Soc. Edinb. Sect. A, **134**, 617–637 (2004)
11. Couchouron, J.-F., Precup, R.: Homotopy method for positive solutions of p-Laplace inclusions. Topol. Methods Nonlinear Anal. **30**, 157–169 (2007)
12. Figueroa, R., Infante, G.: A Schauder-type theorem for discontinuous operators with applications to second-order BVPs. Fixed Point Theory Appl. **2016**, 53 (2016)

13. Fitzpatrick, P.M., Petryshyn, W.V.: Fixed point theorems and the fixed point index for multivalued mappings in cones. J. Lond. Math. Soc. **2**(11), 75–85 (1975)
14. Herlea, D.-R., O'Regan, D., Precup, R.: Harnack type inequalities and multiple solutions in cones of nonlinear problems. Z. Anal. Anwend. **39**, 151–170 (2020)
15. Herlea, D.-R., Precup, R.: Existence, localization and multiplicity of positive solutions to ϕ-Laplace equations and systems. Taiwan. J. Math. **20**, 77–89 (2016)
16. Hu, L.-G., Xu, J.: Positive solutions of nonhomogeneous boundary value problems for some nonlinear equation with ϕ-Laplacian. Bound. Value Probl. **2012**, 130 (2012)
17. Infante, G.: A short course on positive solutions of systems of ODEs via fixed point index. Lect. Notes Nonlinear Anal. **16**, 93–140 (2017)
18. Jebelean, P., Mawhin, J., Şerban, C.: Periodic solutions for discontinuous perturbations of the relativistic operator. Bull. Sci. Math. **140**, 99–117 (2016)
19. Jebelean, P., Şerban, C.: Boundary value problems for discontinuous perturbations of singular ϕ-Laplacian operator. J. Math. Anal. Appl. **431**, 662–681 (2015)
20. Kryszewski, W., Maciejewski, M.: Degree for weakly upper semicontinuous perturbations of quasi-m-accretive operators. Phil. Trans. R. Soc. A **379**, 20190377 (2021)
21. Lan, K.Q.: Multiple positive solutions of semilinear differential equations with singularities. J. Lond. Math. Soc. **63**, 690–704 (2001)
22. Leggett, R.W., Williams, L.R.: Multiple positive fixed points of nonlinear operators on ordered Banach spaces. Indiana Univ. Math. J. **28**(4), 673–688 (1979)
23. Pouso, R.L.: Schauder's fixed-point theorem: new applications and a new version for discontinuous operators. Bound. Value Probl. **2012**, 92 (2012)
24. Precup, R.: Fixed point theorems for decomposable multi-valued maps and applications. Z. Anal. Anwend. **22**, 843–861 (2003)
25. Precup, R., Rodríguez-López, J.: Positive solutions for discontinuous problems with applications to ϕ-Laplacian equations. J. Fixed Point Theory Appl. **20**(156), 1–17 (2018)
26. Precup, R., Rodríguez-López, J.: Positive solutions for ϕ-Laplace equations with discontinuous state-dependent forcing terms. Nonlinear Anal. Model. Control **24**, 447–461 (2019)
27. Precup, R., Rodríguez-López, J.: Fixed point index theory for decomposable multivalued maps and applications to discontinuous ϕ-Laplacian problems. Nonlinear Anal. **199**, 111958 (2020)