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New geometries for the characterization of dark matter phenomena

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Abstract. We present some new geometries with spheroidal symmetry, with and without mass, that involve new kind of energy momentum tensors, and which are suitable for the description of dark matter phenomena.

1. Introduction

The standard characterization of dark matter phenomena is through models that assume the generally accepted cold dark matter model. However, when studying dark matter phenomena with different techniques one often finds non-trivial disagreement among the measurements.

Notably, when estimating the matter content in a region using gravitational weak lensing effects and dynamical studies, the different techniques do not coincideSerra and Romero(2011) in the estimated value.

These problems might be related to the way in which one normally deals with inhomogeneities in cosmology. We will comment briefly on the inherent problems involved in the notion of averaging of tensors; that contribute to unexpected terms in the energy momentum tensor.

In a previous study of weak lensing we have noticed that a spacelike contribution of the energy-momentum tensor has been neglectedGallo and Moreschi(2011) in previous works. This is the source of inspiration for the suggestion of a family of solutions with a nontrivial contribution to the geometry but with less requirement of mass content. In the past we have presented static spherically symmetric solutionsGallo and Moreschi(2012); in this case we generalize to spheroidal symmetry,

We present some new geometries that involve new kind of energy momentum tensors which are suitable for the description of dark matter phenomena.

1.1. What could be missing from the standard picture?

The problem with implicit averages: In a simple cosmological model one can consider a Universe made out of small pieces of matter distributed in corresponding islands. If a photon would reach us from one of those bodies it would feel: a vanishing Ricci tensor and a non-vanishing Weyl tensor, namely:

$$R_{ab} = 0 \qquad , \qquad W_{abc}{}^{d} \neq 0.$$

While in a smooth averaged description, one would have the contrary, that is: a non-vanishing Ricci tensor and a vanishing Weyl tensor:

$$R_{ab} \neq 0 \qquad , \qquad W_{abc}{}^{d} = 0;$$

as is the case in the Robertson-Walker spacetimes.

One normally thinks that the Robertson-Walker spacetimes are a good model for the large scale structure of the Universe in which the small scales inhomogeneities are smooth out in some kind of averaging process. However, there is no notion of average that coming from a zero tensor would produce a non-zero average. As is the case with the Ricci tensor as mention above.

The standard approach to weak lenses: In standard textbooks, such as: *Gravitational lenses*, P. Schneider, J. Ehlers and E.E. Falco (1992)Schneider et al.(1992)Schneider, Ehlers, and Falco, one finds that the deflection angle is expressed by:

$$\hat{\alpha}(\vec{\xi}) = \frac{4G}{c^2} \int_{\mathbb{R}^2} \Sigma(\vec{\xi}') \frac{\vec{\xi} - \vec{\xi}'}{|\vec{\xi} - \vec{\xi}'|^2} d^2\xi', \tag{1}$$

where $\Sigma(\vec{\xi})$ is the mass density projected onto a plane perpendicular to the light path, $\vec{\xi}$ describes the position of the light ray in the lens plane.

Instead we have shown in [GM11]Gallo and Moreschi(2011) the following expressions for the bending angle in terms of energy-momentum components and the mass content $M(r)$, of a spherically symmetric stationary spacetime

$$\boxed{\alpha(J) = J \int_{-d_l}^{d_{ls}} \left[\frac{3J^2}{r^2} \left(\frac{M(r)}{r^3} - \frac{4\pi}{3} \varrho(r) \right) + 4\pi \left(\varrho(r) + P_r(r) \right) \right] dy}; \tag{2}$$

where $J = |\vec{\xi}|$ is the impact parameter and $r = \sqrt{J^2 + y^2}$.

Let us observe the appearance of a term proportional to the radial component of the energy-momentum tensor; namely P_r.

This suggested us to consider a simple model with $P_r \neq 0$ and $M(r) = 0$ (zero mass), $\rho(r) = 0$ (zero mass density); which describes fairly well dark matter phenomena; id.est. rotation curves, weak lens, scape velocities; as we have shown in previous works[GM12]Gallo and Moreschi(2012).

Here we present a new exact solution of Einstein equations with prolate and oblate spheroidal symmetry and zero mass, which is the natural generalization of our previous construction with spherical symmetry. We also present a family of solutions with mass resembling well known profiles.

2. A spacetime with prolate spheroidal symmetry and zero mass

2.1. Using the hyperbolic coordinate

The metric

We will consider spacetimes with spheroidal symmetry of the form

$$\begin{aligned} ds^2 =& a(\xi,t)dt^2 - b(\xi,t) r_\mu^2 (\sinh^2(\xi) + \sin^2(\theta)) d\xi^2 \\ & - r_\mu^2 \left((\sinh^2(\xi) + \sin^2(\theta)) d\theta^2 + \sinh^2(\xi) \sin^2(\theta) d\phi^2 \right); \end{aligned} \tag{3}$$

where r_μ characterizes the position of the focus, for the spheroidal geometry, as it will become more clear later when we relate the geometric coordinate ξ with the radial coordinate r.

In particular we present the static solution given by

$$a = a_0(\xi + C)^2, \tag{4}$$

and

$$b = 1; \tag{5}$$

so that the whole geometry is characterized by the two constants a_0 and C.

The Einstein tensor

The corresponding components of the Einstein tensor which are different from zero are:

$$G_{\xi\xi} = -\frac{(2\cosh^2(\xi) - 2 + \sin^2(\theta))\cosh(\xi)\sinh(\xi)}{(\cosh^2(\xi) + \sin^2(\theta) - 1)(\cosh^2(\xi) - 1)(\xi + C)}, \tag{6}$$

$$G_{\xi\theta} = -\frac{\cos(\theta)\sin(\theta)}{(\cosh^2(\xi) - 1 + \sin^2(\theta))(\xi + C)}, \tag{7}$$

$$G_{\theta\theta} = -\frac{\cosh(\xi)\sin^2(\theta)\sinh(\xi)}{(\cosh^2(\xi) - 1 + \sin^2(\theta))(\cosh^2(\xi) - 1)(\xi + C)}. \tag{8}$$

2.2. Using the radial coordinate

The metric

From the relation

$$\xi = \operatorname{arcsinh}\left(\frac{r}{r_\mu}\right) = \ln\left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1}\right); \tag{9}$$

or alternatively $r = r_\mu \sinh(\xi)$; one can express the metric as:

$$ds^2 = a(r)dt^2 - \left((r^2 + r_\mu^2\sin^2(\theta))\left(\frac{dr^2}{r^2 + r_\mu^2} + d\theta^2\right) + r^2\sin^2(\theta)d\phi^2\right), \tag{10}$$

and the timelike component of the metric is

$$a = a_0\left(\ln\left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1}\right) + C\right)^2. \tag{11}$$

The Einstein tensor

The corresponding components of the Einstein tensor which are different from zero are:

$$G_{rr} = -\frac{(2r^2 + r_\mu^2\sin^2(\theta))}{\sqrt{r^2 + r_\mu^2}(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right) + C)(r_\mu^2\sin^2(\theta) + r^2)r}, \tag{12}$$

$$G_{r\theta} = -\frac{r_\mu^2 \cos(\theta)\sin(\theta)}{\sqrt{r^2+r_\mu^2}(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right)+C)(r_\mu^2\sin^2(\theta)+r^2)}, \tag{13}$$

$$G_{\theta\theta} = -\frac{(r^2+r_\mu^2)r_\mu^2 \sin^2(\theta)}{\sqrt{r^2+r_\mu^2}(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right)+C)(r_\mu^2\sin^2(\theta)+r^2)r}. \tag{14}$$

The Riemann tensor

The components of the Riemann tensor which are different from zero are:

$$R_{trtr} = \frac{-(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right)+C)a_0 r}{\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2+r^2)}, \tag{15}$$

$$R_{trt\theta} = \frac{-(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right)+C)\cos(\theta)\sin(\theta)a_0 r_\mu^2}{\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2+r^2)}, \tag{16}$$

$$R_{t\theta t\theta} = \frac{(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right)+C)(r^2+r_\mu^2)a_0 r}{\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2+r^2)}, \tag{17}$$

$$R_{t\phi t\phi} = \frac{(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right)+C)(r^2+r_\mu^2)\sin(\theta)^2 a_0 r}{\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2+r^2)}. \tag{18}$$

3. A prolate spheroidal distribution as a gravitational lens

3.1. The adapted coordinate system

We have in mind a gravitational lens configuration in which the source is located far away close to the y axis, the lens is near the origin of the frame, and the observer along negative values of the y axis; as depicted in figure 1. We use coordinates (x, z) for the plane of the lens.

3.2. A rotated spheroid

The spheroidal distribution is assumed to be at an angle ι from the z axis in the direction of y.

3.3. Gravitational lens geometry for prolate spheroidal distributions

In the calculation of gravitational lens, one needs to calculate the spinor components of the Ricci tensor Φ_{00} and the Weyl component Ψ_0, with respect to a null tetrad adapted to the null geodesic congruence of the photons.

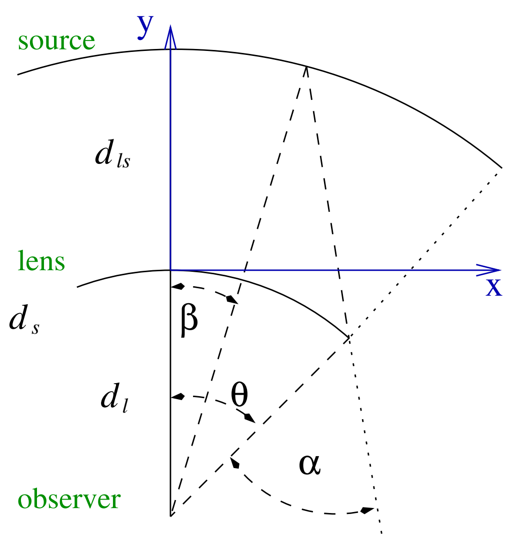

Figure 1. Standard notation for deviation angles and background coordinate system. d_s denotes the distance to the source of the image; d_l to the lens and d_{ls} the lens-source distance.

We choose the null tetrad in the flat background as in our previous article; so that in the (t, x, y, z) frame, one has

$$l^{\underline{a}} = (-1, 0, 1, 0), \tag{19}$$

$$m^{\underline{a}} = \frac{1}{\sqrt{2}}(0, i, 0, 1), \tag{20}$$

$$\bar{m}^{\underline{a}} = \frac{1}{\sqrt{2}}(0, -i, 0, 1), \tag{21}$$

$$n^{\underline{a}} = \frac{1}{2}(-1, 0, -1, 0). \tag{22}$$

Let us note that the Ricci component is:

$$\Phi_{00} = -\frac{1}{2}R_{ab}l^a l^b = -\frac{1}{2}G_{ab}l^a l^b. \tag{23}$$

The Weyl component is given by:

$$\Psi_0 = C_{abcd}l^a m^b l^c m^d. \tag{24}$$

We finally obtain the expression

$$
\begin{aligned}
\Phi_{00} =& -\frac{1}{2}\left(G_{rr}l^r l^r + 2G_{r\theta}l^r l^\theta + G_{\theta\theta}l^\theta l^\theta\right) \\
=& \frac{1}{2\sqrt{r^2+r_\mu^2}(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right)+C)(r_\mu^2\sin^2(\theta)+r^2)r} \\
& \Big((2r^2+r_\mu^2\sin^2(\theta))l^r l^r + 2\,r\,r_\mu^2\cos(\theta)\sin(\theta)l^r l^\theta \\
& + (r^2+r_\mu^2)r_\mu^2\sin^2(\theta)l^\theta l^\theta\Big);
\end{aligned}
\tag{25}
$$

while the Weyl component is given by:

$$
\begin{aligned}
\Psi_0 =& \frac{1}{g_{tt}}R_{tbtd}\,m^b\,m^d \\
=& \frac{1}{g_{tt}}R_{trtr}(m^r)^2 + \frac{2}{g_{tt}}R_{trt\theta}\,m^r\,m^\theta \\
& + \frac{1}{g_{tt}}R_{t\theta t\theta}(m^\theta)^2 + \frac{1}{g_{tt}}R_{t\phi t\phi}(m^\phi)^2 \\
=& \frac{1}{(\ln\left(\frac{\sqrt{r^2+r_\mu^2}+r}{r_\mu}\right)+C)\sqrt{r^2+r_\mu^2}(\sin(\theta)^2 r_\mu^2+r^2)} \\
& \Big(-r(m^r)^2 - 2\cos(\theta)\sin(\theta)\,r_\mu^2\,m^r\,m^\theta \\
& + (r^2+r_\mu^2)r(m^\theta)^2 + (r^2+r_\mu^2)r\sin(\theta)^2(m^\phi)^2\Big).
\end{aligned}
\tag{26}
$$

3.4. The optical scalars

Let us recall from [GM11]Gallo and Moreschi(2011) that the optical scalars, namely, the expansion κ and the shear components γ_1 and γ_2, in the thin lens approximation, are given by:

$$\kappa = \frac{d_l d_{ls}}{d_s}\hat{\Phi}_{00}, \tag{27}$$

$$\gamma_1 + i\gamma_2 = \frac{d_l d_{ls}}{d_s}\hat{\Psi}_0, \tag{28}$$

where

$$
\begin{aligned}
\hat{\Phi}_{00} &= \int_0^{d_s} \Phi_{00}\,d\lambda, \\
\hat{\Psi}_0 &= \int_0^{d_s} \Psi_0\,d\lambda,
\end{aligned}
\tag{29}
$$

are the projected curvature scalars along the line of sight.

4. Numeric calculation of the optical scalars

4.1. The expansion

For the numerical calculation we have taken the following values: The parameter C was taken as $-\ln(\mu)$, from reference [GM12]Gallo and Moreschi(2012) which it was adjusted to the observations of weak lens in the Coma cluster. The radius r_μ was arbitrarily taken to have the value 3Mpc. The rotation angle ι was chosen to be $\frac{\pi}{4}$. The lens distances were taken as: $d_l = 97.10$Mpc, $d_s = 1068.03$Mpc, $d_{ls} = 970.92$Mpc; which are values from the Coma cluster used in our previous work. The integration was carried out using Chebyshev-Gauss techniques. The number of points evaluated was automatically adjusted to a chosen tolerance. The results are presented in the graphics of Figures 2, 3 and 4.

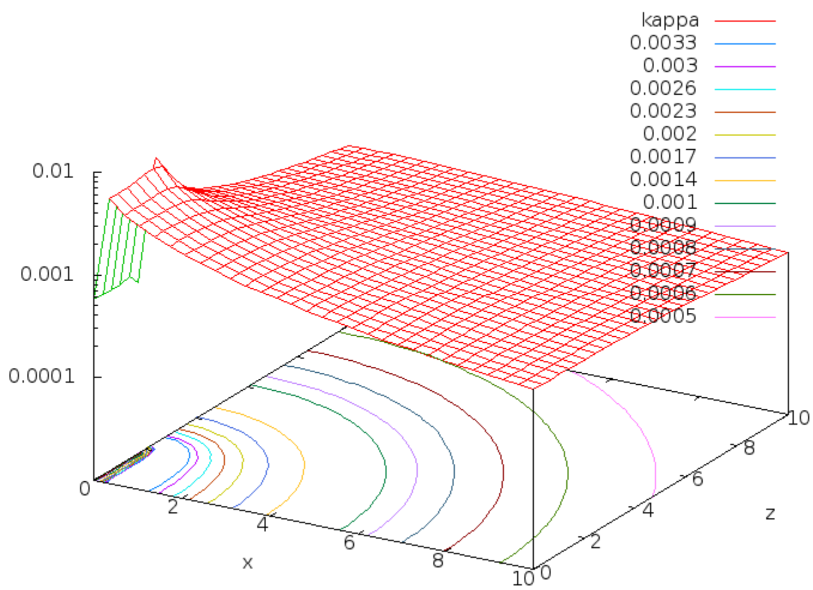

Figure 2. The expansion κ plotted in a log scale. One can see that it copies the geometry of the projected spheroids.

In figure 2 we plot the expansion optical scalar κ, with the contour level at the bottom. One can see that the contours copy very well the projection of the spheroidal geometry, to the (xz) plane.

Figure 3 shows the plot of the modulus of shear optical scalar γ, with the contour level at the bottom. It is observed that in this case the structure is much more complicated, and that in the inner region the behaviour of the modulus does not follow the projection of the spheroidal geometry. However, in figure 4; where the shear is represented by small segments, it is easier to follow and understand

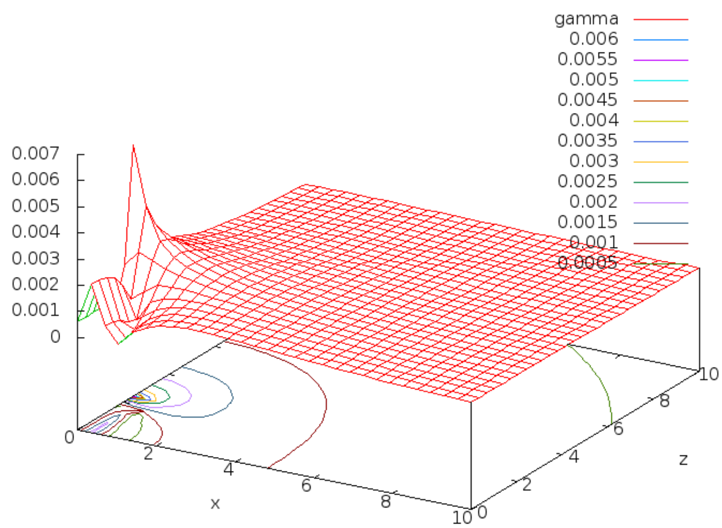

Figure 3. The modulus γ of the shear expansion. The contour curves are more complicated in this case.

the effects of the spheroidal geometry on the gravitational lens. The segments represent the direction of the maximum shear deformation.

5. Spacetimes with prolate spheroidal symmetry and mass

The zero mass spacetime just presented can be generalized to spacetimes with mass content; as we do next.

The metric

Here we present a new stationary solution with mass content, spheroidal symmetry and a non-trivial spacelike component of the energy momentum tensor whose metric is:

$$ds^2 = a(r)dt^2 - \left((r^2 + r_\mu^2 \sin^2(\theta)) \left(\frac{dr^2}{r^2 - 2M(r)r + r_\mu^2} + d\theta^2 \right) + r^2 \sin^2(\theta) d\phi^2 \right), \tag{30}$$

and the timelike component of the metric is:

$$a = a_0 \left(\ln \left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1} \right) + C \right)^2, \tag{31}$$

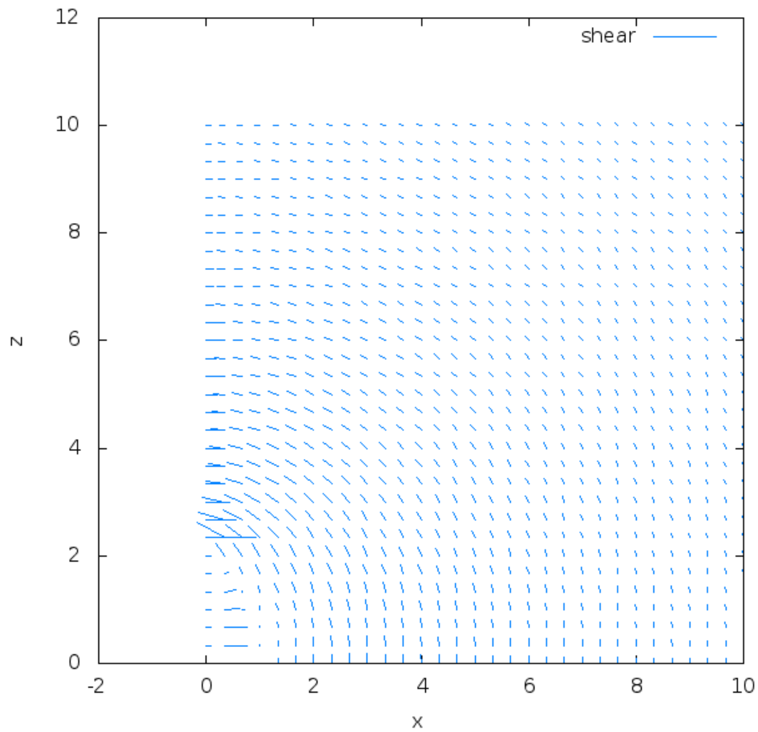

Figure 4. The shear plotted as segments in the plane of the lens.

and where $M(r)$ is:

$$M(r) = \frac{M_*}{r_*} r \quad \text{for} \quad r \leqslant r_* \text{ and } M(r) = M_* \quad \text{for} \quad r > r_* \text{ (isothermal) or} \tag{32}$$

$$M(r) = 4\pi \rho_* r_*^3 \left(\ln(1 + \frac{r}{r_*}) - \frac{\frac{r}{r_*}}{1 + \frac{r}{r_*}} \right) \qquad \text{(NFW);} \tag{33}$$

where the constant M_* is the mass of the generalized isothermal distribution, r_* denotes the maximum radius for the isothermal distribution, or the characteristic radius for the generalized Navarro-Frenk-White (NFW) distribution, and ρ_* the density parameter. These two mass distributions, considered in these solutions, are the natural generalization of the isothermal mass density and of the NFW profile to the spheroidal geometry.

6. A spacetime with oblate spheroidal symmetry and zero mass

We generalize here the previous discussion to oblate spheroidal symmetry.

6.1. Using the radial coordinate

The metric

From the relation $r = r_\mu \sinh(\xi)$; one can express the metric as:

$$ds^2 = a(r)dt^2 - \left((r^2 + r_\mu^2\cos^2(\theta))\left(\frac{dr^2}{r^2 + r_\mu^2} + d\theta^2\right) + (r^2 + r_\mu^2)\sin^2(\theta)d\phi^2\right), \tag{34}$$

and the timelike component of the metric is

$$a = a_0\left(\ln\left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1}\right) + C\right)^2. \tag{35}$$

7. Spacetimes with oblate spheroidal symmetry and mass

The zero mass spacetime just presented can be generalized to spacetimes with mass content; as we do next.

The metric

Here we present a new stationary solution with mass content, spheroidal symmetry and a non-trivial spacelike component of the energy momentum tensor. The metric is:

$$\begin{aligned} ds^2 = a(r)dt^2 &- (r^2 + r_\mu^2\cos^2(\theta))\frac{dr^2}{r^2 - 2M(r)r + r_\mu^2} \\ &- \left((r^2 + r_\mu^2\cos^2(\theta))d\theta^2 + (r^2 + r_\mu^2)\sin^2(\theta)d\phi^2\right), \end{aligned} \tag{36}$$

and the timelike component of the metric is

$$a = a_0\left(\ln\left(\frac{r}{r_\mu} + \sqrt{(\frac{r}{r_\mu})^2 + 1}\right) + C\right)^2, \tag{37}$$

and where $M(r)$ is:

$$M(r) = \frac{M_*}{r_*}r \quad \text{for} \quad r \leqslant r_* \text{ and } M(r) = M_* \quad \text{for} \quad r > r_* \text{ (isothermal) or} \tag{38}$$

$$M(r) = 4\pi\rho_* r_*^3\left(\ln(1 + \frac{r}{r_*}) - \frac{\frac{r}{r_*}}{1 + \frac{r}{r_*}}\right) \qquad \text{(NFW)}. \tag{39}$$

These mass distributions are the natural generalization of the isothermal mass density to the spheroidal geometry and of the NFW distribution to spheroidal geometry.

8. Final comments

We have presented several new static exact solutions of the Einstein equations, with spheroidal symmetry. Some of them have $T_{tt} = 0$, and therefore they have *zero mass*, although with a non-trivial geometry whose gravitational effects are of interest for the explanation of dark matter phenomena.

They are the natural generalization of a previous geometry we presented before[GM12]Gallo and Moreschi(2012), with spherical symmetry; that adequately represents *dark mater observations*.

The behaviour of the shear in the weak lens calculation, for the prolate zero mass case, is not yet well understood; but it might indicate a non-trivial behaviour of the spin 2 nature of the Weyl Ψ_0 component.

These geometries have the property that they can naturally be generalize to other matter distributions with spheroidal symmetry; using the same form of the metric. That is they represent a family of solutions with multiple possibilities.

We wish to develop these techniques for applications to typical non-spheric systems as binary systems, irregular clusters, galaxies, and others.

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