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Pareto Optimality for Multioptimization of Continuous Linear Operators

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Abstract: This manuscript determines the set of Pareto optimal solutions of certain multiobjective-optimization problems involving continuous linear operators defined on Banach spaces and Hilbert spaces. These multioptimization problems typically arise in engineering. In order to accomplish our goals, we first characterize, in an abstract setting, the set of Pareto optimal solutions of any multiobjective optimization problem. We then provide sufficient topological conditions to ensure the existence of Pareto optimal solutions. Next, we determine the Pareto optimal solutions of convex max–min problems involving continuous linear operators defined on Banach spaces. We prove that the set of Pareto optimal solutions of a convex max–min of form max $\|T(x)\|$, min $\|x\|$ coincides with the set of multiples of supporting vectors of T. Lastly, we apply this result to convex max–min problems in the Hilbert space setting, which also applies to convex max–min problems that arise in the design of truly optimal coils in engineering.

Keywords: multioptimization; Pareto optimality; linear operators; adjoint operators; normed spaces; matrix norms

MSC: 47L05; 47L90; 49J30; 90B50

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1. Introduction

Multiobjective optimization problems (MOPs) appear quite often in all areas of pure and applied mathematics, for instance, in the geometry of Banach spaces [1–3], in operator theory [4–7], in lineability theory [8–10], in differential geometry [11–14], and in all areas of Experimental, Medical and Social Sciences [15–20]. By means of MOPs, many real-life situations can be modeled accurately. However, the existence of a global solution that optimizes all the objective functions of an MOP at once is very unlikely. This is were Pareto optimal solutions (POS) come into play. Informally speaking, a POS is a feasible solution such that, if any other feasible solution is more optimal at one objective function, then it is less optimal at another objective function. Pareto optimal solutions are sometimes graphically displayed in Pareto charts (PC). In this manuscript, we prove a characterization of POS by relying on orderings and equivalence relations. We also provide a sufficient topological condition to guarantee the existence of Pareto optimal solutions.

This work is mainly motivated by certain MOPs appearing in engineering, such as the design of truly optimal transcranial magnetic stimulation (TMS) coils [18–23]. The main goal of this manuscript is to characterize (Theorem 6) the set of Pareto optimal solutions of the MOPs that appear in the design of coils, such as (3). In the case of MOPs in which operators are defined on Hilbert spaces, this characterization is improved (Corollary 1). Under this Hilbert space setting, we also study the relationships between different MOPs involving different operators, but which are defined on the same Hilbert space. These operators can be naturally combined to obtain a new MOP. The set of Pareto optimal solutions of this new MOP is compared (Corollary 2) to the set of Pareto optimal solutions of the initial MOPs.

2. Materials and Methods

In this section, we compile all necessary tools to accomplish our results. We also develop new and original tools, such as Theorem 1 and Corollary 2, which contribute to enriching the literature on optimization theory.

2.1. Formal Description of MOPs

A generic multiobjective optimization problem (MOP) has the following form:

$$M := \begin{cases} \max f_i(x) & i = 1, \ldots, p, \\ \min g_j(x) & j = 1, \ldots, q, \\ x \in \mathcal{R}, \end{cases} \tag{1}$$

where $f_i, g_j : X \to \mathbb{R}$ are called objective functions, defined on a nonempty set X, and $\mathcal{R}$ is a nonempty subset of X called the feasible region or region of constraints/restrictions. The set of general solutions of the above MOP is denoted by $\operatorname{sol}(M)$. In fact,

$$\operatorname{sol}(M) := \{x \in \mathcal{R} : \forall y \in \mathcal{R} \forall i \in \{1, \ldots, p\} \forall j \in \{1, \ldots, q\} f_i(x) \geq f_i(y), g_j(x) \leq g_j(y)\}.$$

It is obvious that

$$\operatorname{sol}(M) = \operatorname{sol}(P_1) \cap \cdots \cap \operatorname{sol}(P_p) \cap \operatorname{sol}(Q_1) \cap \cdots \cap \operatorname{sol}(Q_q) \tag{2}$$

where

$$P_i := \begin{cases} \max f_i(x), \\ x \in \mathcal{R}, \end{cases} \quad \text{and} \quad Q_j := \begin{cases} \min g_j(x), \\ x \in \mathcal{R}, \end{cases}$$

are single-objective optimization problems (SOPs) and $\operatorname{sol}(P_i)$, $\operatorname{sol}(Q_j)$ denote the set of general solutions of P_i, Q_j for $i = 1, \ldots, p$ and $j = 1, \ldots, q$, respectively. The set of Pareto optimal solutions of MOP M is defined as

$$\begin{aligned} \operatorname{Pos}(M) := \{x \in \mathcal{R} \quad : \quad & \forall y \in \mathcal{R}, \text{ if there exists } i \in \{1, \ldots, p\} \text{ with } f_i(y) > f_i(x) \\ & \text{or } j \in \{1, \ldots, q\} \text{ with } g_j(y) < g_j(x), \text{ then there exists} \\ & i' \in \{1, \ldots, p\} \text{ with } f_{i'}(y) < f_{i'}(x) \text{ or } j' \in \{1, \ldots, q\} \\ & \text{ with } g_{j'}(x) < g_{j'}(y)\}. \end{aligned}$$

To guarantee the existence of general solutions, it is usually asked for X to be a Hausdorff topological space, $\mathcal{R}$ is a compact subset of X, f_is are upper semicontinuous, and g_js are lower semicontinuous. This way, at least we make sure that the SOPs P_is and Q_js have at least one solution (Weierstrass extreme value theorem). Even more, solution sets $\operatorname{sol}(P_i)$ and $\operatorname{sol}(Q_j)$ are closed and thus compact, which makes $\operatorname{sol}(M)$ also compact. Nevertheless, even under these conditions, $\operatorname{sol}(M)$ might still be empty, as we can easily infer from Equation (2).

2.2. Characterizing Pareto Optimal Solutions

A more abstract way to construct the set of Pareto optimal solutions follows. Let X be a nonempty set, $f_i, g_j : X \to \mathbb{R}$ functions and $\mathcal{R}$ a nonempty subset of X. In $\mathcal{R}$, consider the equivalence relation given by

$$\mathcal{S} := \left\{(x, y) \in \mathcal{R}^2 : \forall i = 1, \ldots, p \ \forall j = 1, \ldots, q \ f_i(x) = f_i(y), g_j(x) = g_j(y)\right\}.$$

Next, in the quotient set of $\mathcal{R}$ by $\mathcal{S}$, $\frac{\mathcal{R}}{\mathcal{S}}$, consider the order relation given by

$$[x]_{\mathcal{S}} \leq [y]_{\mathcal{S}} \Leftrightarrow \forall i = 1, \ldots, p \; \forall j = 1, \ldots, q \; f_i(x) \leq f_i(y), g_j(y) \leq g_j(x).$$

Theorem 1. *Consider MOP* (1). *Then,*

$$\mathrm{Pos}(M) = \left\{ x \in \mathcal{R} : [x]_{\mathcal{S}} \textit{ is a maximal element of } \frac{\mathcal{R}}{\mathcal{S}} \textit{ endowed with } \leq \right\}$$

and

$$\mathrm{sol}(M) := \left\{ x \in \mathcal{R} : [x]_{\mathcal{S}} \textit{ is the maximum of } \frac{\mathcal{R}}{\mathcal{S}} \textit{ endowed with } \leq \right\}.$$

As a consequence, $\mathrm{sol}(M) \subseteq \mathrm{Pos}(M)$. *If there exists* $i_1 \in \{1, \ldots, p\}$ *or* $j_1 \in \{1, \ldots, q\}$ *such that* $\mathrm{sol}(P_{i_1})$ *or* $\mathrm{sol}(Q_{j_1})$ *is a singleton, respectively, then* $\mathrm{sol}(P_{i_1}) \subseteq \mathrm{Pos}(M)$ *or* $\mathrm{sol}(Q_{j_1}) \subseteq \mathrm{Pos}(M)$, *respectively.*

Proof. Fix an arbitrary $x_0 \in \mathrm{Pos}(M)$. Let us assume that there is $y \in \mathcal{R}$, so that $[x_0]_{\mathcal{S}} < [y]_{\mathcal{S}}$. Then, $f_i(x_0) \leq f_i(y)$ for all $i = 1, \ldots, p$ and $g_j(x_0) \geq g_j(y)$ for all $j = 1, \ldots, q$. However, $[x_0]_{\mathcal{S}} \neq [y]_{\mathcal{S}}$; therefore, there exists $i_0 \in \{1, \ldots, p\}$ or $j_0 \in \{1, \ldots, q\}$ such that $f_{i_0}(x_0) < f_{i_0}(y)$ or $g_{j_0}(x_0) < g_{j_0}(y)$, respectively. Since $x_0 \in \mathrm{Pos}(M)$ by assumption, there exists $i_1 \in \{1, \ldots, p\}$ or $j_1 \in \{1, \ldots, q\}$, such that $f_{i_1}(x_0) > f_{i_1}(y)$ or $g_{j_1}(x_0) < g_{j_1}(y)$, respectively, which is a contradiction. Therefore, $[x_0]_{\mathcal{S}}$ is a maximal element of $\frac{\mathcal{R}}{\mathcal{S}}$ endowed with $\leq$. The arbitrariness of $x_0 \in \mathrm{Pos}(M)$ shows that

$$\mathrm{Pos}(M) \subseteq \left\{ x \in \mathcal{R} : [x]_{\mathcal{S}} \text{ is a maximal element of } \frac{\mathcal{R}}{\mathcal{S}} \text{ endowed with } \leq \right\}.$$

Conversely, fix an arbitrary $x_0 \in \mathcal{R}$, such that $[x_0]_{\mathcal{S}}$ is a maximal element of $\frac{\mathcal{R}}{\mathcal{S}}$ endowed with $\leq$. Take $y \in \mathcal{R}$ satisfying that there exists $i_0 \in \{1, \ldots, p\}$ or $j_0 \in \{1, \ldots, q\}$ with $f_{i_0}(y) > f_{i_0}(x_0)$ or $g_{j_0}(y) < g_{j_0}(x_0)$, respectively. If $f_i(x_0) \leq f_i(y)$ for all $i \in \{1, \ldots, p\} \setminus \{i_0\}$ and $g_j(x_0) \geq g_j(y)$ for all $j \in \{1, \ldots, q\} \setminus \{j_0\}$, then $[x_0]_{\mathcal{S}} < [y]_{\mathcal{S}}$, reaching a contradiction with the maximality of $[x_0]_{\mathcal{S}}$ in $\frac{\mathcal{R}}{\mathcal{S}}$ endowed with $\leq$. This shows that

$$\mathrm{Pos}(M) = \left\{ x \in \mathcal{R} : [x]_{\mathcal{S}} \text{ is a maximal element of } \frac{\mathcal{R}}{\mathcal{S}} \text{ endowed with } \leq \right\}.$$

Next, fix an arbitrary $x_0 \in \mathrm{sol}(M)$. For every $y \in \mathcal{R}$, $f_i(x_0) \geq f_i(y)$ and $g_j(x_0) \leq g_j(y)$ for all $i = 1, \ldots, p$ and all $j = 1, \ldots, q$. Then, $[x_0]_{\mathcal{S}} \geq [y]_{\mathcal{S}}$. The arbitrariness of $y \in \mathcal{R}$ ensures that $[x_0]_{\mathcal{S}}$ is a maximal element of $\frac{\mathcal{R}}{\mathcal{S}}$ endowed with $\leq$. Conversely, fix an arbitrary $x_0 \in \mathcal{R}$, such that $[x_0]_{\mathcal{S}}$ is a maximal element of $\frac{\mathcal{R}}{\mathcal{S}}$ endowed with $\leq$. For every $y \in \mathcal{R}$, $[y]_{\mathcal{S}} \geq [x_0]_{\mathcal{S}}$; therefore, $f_i(x_0) \geq f_i(y)$ and $g_j(x_0) \leq g_j(y)$ for all $i = 1, \ldots, p$ and all $j = 1, \ldots, q$. The arbitrariness of $y \in \mathcal{R}$ proves that $x_0 \in \mathrm{sol}(M)$. We proved that

$$\mathrm{sol}(M) := \left\{ x \in \mathcal{R} : [x]_{\mathcal{S}} \text{ is the maximum of } \frac{\mathcal{R}}{\mathcal{S}} \text{ endowed with } \leq \right\}.$$

Lastly, suppose that $\mathrm{sol}(P_{i_1})$ is a singleton for some $i_1 \in \{1, \ldots, p\}$, and write $\mathrm{sol}(P_{i_1}) = \{x_0\}$. Take $y \in \mathcal{R}$ satisfying that there exists $i_0 \in \{1, \ldots, p\}$ or $j_0 \in \{1, \ldots, q\}$ with $f_{i_0}(y) > f_{i_0}(x_0)$ or $g_{j_0}(y) < g_{j_0}(x_0)$, respectively. If such i_0 exists, then $i_0 \neq i_1$. By hypothesis, $f_{i_1}(x_0) > f_{i_1}(y)$ since $y \notin \mathrm{sol}(P_{i_1})$. This shows that $x_0 \in \mathrm{Pos}(M)$. Likewise, $\mathrm{sol}(Q_{j_1}) \subseteq \mathrm{Pos}(M)$ provides that $\mathrm{sol}(Q_{j_1})$ is a singleton. $\square$

Lemma 1. *Consider MOP (1). Let* $i_0 \in \{1, \dots, p\}, j_0 \in \{1, \dots, q\}$*. Then,*

1. *If there is* $x_{i_0} \in \mathcal{R}$ *so that* $\left[x_{i_0}\right]_{\mathcal{S}}$ *is a maximal element of* $\left\{[x]_{\mathcal{S}} : x \in \arg\max_{\mathcal{R}} f_{i_0}\right\}$*, then* $\left[x_{i_0}\right]_{\mathcal{S}}$ *is a maximal element of* $\mathcal{R}/\mathcal{S}$*. Hence,* $x_{i_0} \in \operatorname{Pos}(M)$*.*
2. *If there is* $x_{j_0} \in \mathcal{R}$ *so that* $\left[x_{j_0}\right]_{\mathcal{S}}$ *is a maximal element of* $\left\{[x]_{\mathcal{S}} : x \in \arg\min_{\mathcal{R}} g_{j_0}\right\}$*, then* $\left[x_{j_0}\right]_{\mathcal{S}}$ *is a maximal element of* $\mathcal{R}/\mathcal{S}$*. Hence,* $x_{j_0} \in \operatorname{Pos}(M)$

Proof. We only prove the first item since the other follows a dual proof. Assume that $\left[x_{i_0}\right]_{\mathcal{S}}$ is not a maximal element of $\mathcal{R}/\mathcal{S}$. Then, we can find $y \in \mathcal{R}$ in such a way that $\left[x_{i_0}\right]_{\mathcal{S}} < [y]_{\mathcal{S}}$. In particular, $f_{i_0}\left(x_{i_0}\right) \leq f_{i_0}(y)$; therefore, $f_{i_0}(y) = \max_{\mathcal{R}} f_{i_0}$; hence, $y \in \arg\max_{\mathcal{R}} f_{i_0}$. As a consequence, $[y]_{\mathcal{S}} \in \left\{[x]_{\mathcal{S}} : x \in \arg\max_{\mathcal{R}} f_{i_0}\right\}$, contradicting that $\left[x_{i_0}\right]_{\mathcal{S}}$ be a maximal element of $\left\{[x]_{\mathcal{S}} : x \in \arg\max_{\mathcal{R}} f_{i_0}\right\}$. $\square$

Theorem 2. *Consider MOP (1). If X is a topological space,* $\mathcal{R}$ *is a compact Hausdorff subset of X and all the objective functions are continuous, then* $\operatorname{Pos}(M) \neq \varnothing$*.*

Proof. Fix $i_0 \in \{1, \dots, p\}$. In accordance with Lemma 1, it is only sufficient to find a maximal element of $A := \left\{[x]_{\mathcal{S}} : x \in \arg\max_{\mathcal{R}} f_{i_0}\right\}$. We rely on Zorn's lemma. Consider a chain in A, that is, a totally ordered subset of elements $[x_k]_{\mathcal{S}}$, with k ranging a totally ordered set K in such a way that $k_1 < k_2$ if and only if $\left[x_{k_1}\right]_{\mathcal{S}} < \left[x_{k_2}\right]_{\mathcal{S}}$. Since K is totally ordered, we have that $(x_k)_{k \in K}$ is a net in $\mathcal{R}$. The compactness of $\mathcal{R}$ allows for extracting a subnet $(y_h)_{h \in H}$ of $(x_k)_{k \in K}$ convergent to some $x_0 \in \mathcal{R}$. Let us first show that $x_0 \in \arg\max_{\mathcal{R}} f_{i_0}$. The continuity of f_{i_0} implies that $\left(f_{i_0}(y_h)\right)_{h \in H}$ converges to $f_{i_0}(x_0)$. Fix any $\varepsilon > 0$. There is $h_\varepsilon \in H$ satisfying that, if $h \geq h_\varepsilon$, then $\left|f_{i_0}(y_h) - f_{i_0}(x_0)\right| < \varepsilon$. Fix any $k_0 \in K$. There is $h_0 \in H$, so that $\{y_h : h \geq h_0\} \subseteq \{x_k : k \geq k_0\}$. Since H is a directed set, we can find $h_1 \in H$ with $h_1 \geq h_\varepsilon$ and $h_1 \geq h_0$. There exists $k_1 \in K$ with $k_1 \geq k_0$ such that $y_{h_1} = x_{k_1}$. Next, $f_{i_0}\left(y_{h_1}\right) = f_{i_0}\left(x_{k_1}\right) = \max_{\mathcal{R}} f_{i_0}$. As a consequence,

$$\max_{\mathcal{R}} f_{i_0} - f(x_0) = f_{i_0}\left(y_{h_1}\right) - f(x_0) < \varepsilon.$$

The arbitrariness of ε shows that $\max_{\mathcal{R}} f_{i_0} = f(x_0)$. Lastly, we prove that $[x_0]_{\mathcal{S}}$ is an upper bound for chain $\{[x_k]_{\mathcal{S}} : k \in K\}$. Fix an arbitrary $k_0 \in K$. In order to prove that $\left[x_{k_0}\right]_{\mathcal{S}} \leq [x_0]_{\mathcal{S}}$, we have to check that $f_i\left(x_{k_0}\right) \leq f_i(x_0)$ for all $i \in \{1, \dots, p\}$ and $g_j\left(x_{k_0}\right) \geq g_j(x_0)$ for all $j \in \{1, \dots, q\}$. Indeed, fix $i \in \{1, \dots, p\}$ and suppose to the contrary that $f_i\left(x_{k_0}\right) > f_i(x_0)$. Let $0 < \varepsilon < f_i\left(x_{k_0}\right) - f_i(x_0)$. There exists $h_\varepsilon \in H$ such that, if $h \geq h_\varepsilon$, then $|f_i(y_h) - f_i(x_0)| < \varepsilon$. We can find $h_0 \in H$, such that $\{y_h : h \geq h_0\} \subseteq \{x_k : k \geq k_0\}$. Since H is a directed set, we can find $h_1 \in H$ with $h_1 \geq h_\varepsilon$ and $h_1 \geq h_0$. There exists $k_1 \in K$ with $k_1 \geq k_0$ such that $y_{h_1} = x_{k_1}$. Since $k_0 \leq k_1$, we have that $\left[x_{k_0}\right]_{\mathcal{S}} \leq \left[x_{k_1}\right]_{\mathcal{S}}$. Thus,

$$f_i\left(y_{h_1}\right) = f_i\left(x_{k_1}\right) \geq f_i\left(x_{k_0}\right) > f_i(x_0),$$

contradicting that $\left|f_i\left(y_{h_1}\right) - f_i(x_0)\right| < \varepsilon$. In a similar way, it can be shown that $g_j\left(x_{k_0}\right) \geq g_j(x_0)$ for all $j \in \{1, \dots, q\}$. As a consequence, $\left[x_{k_0}\right]_{\mathcal{S}} \leq [x_0]_{\mathcal{S}}$. In other words, $[x_0]_{\mathcal{S}}$ is an upper bound for the chain $\{[x_k]_{\mathcal{S}} : k \in K\}$. Since every chain of A has an upper bound, Zorn's lemma ensures the existence of maximal elements in A. $\square$

2.3. MOPs in a Functional-Analysis Context

A large number of objective functions in an MOP may cause a lack of general solutions, that is, $\operatorname{sol}(M) = \varnothing$. This happens quite often with MOPs involving matrices. Even if the number of objective functions is short, we might still have $\operatorname{sol}(M) = \varnothing$. The following theorem [20], Theorem 2, is a very representative example of this situation of lack of general solutions.

Theorem 3. *Let $T : X \to Y$ be a nonzero continuous linear operator, where X, Y are normed spaces; then, the following max–min problem is free of general solutions:*

$$\begin{cases} \max \|T(x)\|, \\ \min \|x\|, \\ x \in X. \end{cases} \tag{3}$$

Equation (3) describes an MOP that appears in bioengineering quite often after the linearization of forces or fields [18].

3. Results

We focus on MOPs similar to (3). In fact, we find Pos(3) (Theorem 6 and Corollary 1). If X, Y are Hilbert spaces, say H, K, and $T_1, \dots, T_k \in \mathcal{B}(H, K)$ are continuous linear operators, then the sets of Pareto optimal solutions of the MOPs

$$\begin{cases} \max \|T_i(x)\|, \\ \min \|x\|, \\ x \in H, \end{cases} \tag{4}$$

for $i = 1, \dots, k$ are compared (Corollary 2) with the set of Pareto optimal solutions of MOP

$$\begin{cases} \max \|T(x)\|, \\ \min \|x\|, \\ x \in H, \end{cases} \tag{5}$$

where

$$\begin{array}{rccl} T: & H & \to & K \oplus_2 \overset{k}{\cdots} \oplus_2 K \\ & x & \mapsto & T(x) = (T_1(x), \dots, T_k(x)). \end{array}$$

3.1. Formatting of Mathematical Components

Let X, Y be normed spaces. Consider a nonzero continuous linear operator $T : X \to Y$. Then

$$\|T\| := \sup\{\|T(x)\| : x \in \mathsf{B}_X\}$$

is the norm of T. On the other hand,

$$\operatorname{suppv}(T) := \{x \in \mathsf{S}_X : \|T(x)\| = \|T\|\},$$

stands for the set of supporting vectors of T, where $\mathsf{B}_X := \{x \in X : \|x\| \leq 1\}$ is a (closed) unit ball, and $\mathsf{S}_X := \{x \in X : \|x\| = 1\}$ is the unit sphere. Continuous linear operators are also called bounded because they are bounded on the unit ball. The space of bounded linear operators from X to Y is denoted as $\mathcal{B}(X, Y)$.

Let H be a Hilbert space, and consider the dual map of H:

$$\begin{array}{rccl} J_H: & H & \to & H^* \\ & k & \mapsto & J_H(k) := k^* = (\bullet | k). \end{array}$$

J_H is a surjective linear isometry between H and H^* (Riesz representation theorem). In the frame of the geometry of Banach spaces, J_H is called duality mapping.

Consider H, K Hilbert spaces, and let $T \in \mathcal{B}(H, K)$ be a bounded linear operator. We define the adjoint operator of T as $T' := (J_H)^{-1} \circ T^* \circ J_K \in \mathcal{B}(K, H)$, with $T^* : K^* \to H^*$ as the dual operator of T. The most representative property of the adjoint operator is that it is the unique operator in $\mathcal{B}(K, H)$ satisfying $(T(x)|y) = (x|T'(y))$ for all $x \in H$ and all $y \in K$. It holds that $\|T'\| = \|T\|$, $(T')' = T$, $(T + S)' = T' + S'$ and $(\lambda T)' = \overline{\lambda} T'$.

If $T \in \mathcal{B}(H)$ verifies $T = T'$, then T is self-adjoint. This is equivalent to equality $(T(x)|y) = (x|T(y))$ held for every $x, y \in H$. If T satisfies $(T(x)|x) \geq 0$ for each $x \in H$, then T is called positive. If H is complex, then $T \in \mathcal{B}(H)$ is self-adjoint if and only if $(T(x)|x) \in \mathbb{R}$ for each $x \in H$. Thus, in complex Hilbert spaces, positive operators are self-adjoint. T is strongly positive if there exists $S \in \mathcal{B}(H, K)$ with $T = S' \circ S$. Typical examples of self-adjoint positive operators are strongly positive operators.

For each $T \in \mathcal{B}(H)$, the following set is the spectrum of T

$$\sigma(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin \mathcal{U}(\mathcal{B}(H))\},$$

where $\mathcal{U}(\mathcal{B}(H))$ is the multiplicative group of invertible operators on H. Among spectral properties, it is compact, nonempty, and $\|T\| \geq \max|\sigma(T)|$. We work with a special subset of the spectrum:

$$\sigma_p(T) := \{\lambda \in \mathbb{C} : \ker(\lambda I - T) \neq \{0\}\},$$

called the point spectrum, whose elements are eigenvalues of T. It is clear that $\sigma_p(T) \subseteq \sigma(T)$. In addition, if $\lambda \in \sigma_p(T)$, the subspace of associated eigenvectors to λ is

$$V(\lambda) := \{x \in H : T(x) = \lambda x\}.$$

If $\|T\|$ is an eigenvalue of T or, in other words, $\|T\| \in \sigma_p(T)$, then $\|T\|$ is the maximal element of $|\sigma(T)|$, i.e., $\|T\| = \max|\sigma(T)|$. In this situation, we also write $\|T\| = \lambda_{\max}(T)$.

Example 1. *Let $T : H \to K$ be a continuous linear operator where H, K are Hilbert spaces, such that $\|T\| \in \sigma_p(T)$; then, $V(\|T\|) \cap \mathsf{S}_X \subseteq \operatorname{suppv}(T)$. If $x \in V(\|T\|) \cap \mathsf{S}_X$, then $T(x) = \|T\|x$; therefore, $\|T(x)\| = \|T\|$ and hence $x \in \operatorname{suppv}(T)$.*

In general, $\|T\| \notin \sigma_p(T)$, unless, for instance, T is compact, self-adjoint, and positive. This is why we have to rely on adjoint T' and strongly positive operator $T' \circ T$. It is straightforward to verify that the eigenvalues of a positive operator are positive, and in the case of a self-adjoint operator, the eigenvalues are real. When T is compact, it holds that $T' \circ T$ is compact, self-adjoint, and positive.

The next result was obtained by refining ([10] [Theorem 9]). In particular, we obtained the same conclusions with fewer hypotheses.

Theorem 4. *Consider H, K Hilbert spaces, and $T \in \mathcal{B}(H, K)$. Then,*

1. $\|T\|^2 = \|T' \circ T\|$.
2. $\operatorname{suppv}(T) \subseteq \operatorname{suppv}(T' \circ T)$.
3. $\operatorname{supp}(T) \neq \varnothing$ *if and only if* $\|T' \circ T\| \in \sigma_p(T' \circ T)$.

In this situation, $\|T\| = \sqrt{\lambda_{\max}(T' \circ T)}$ and $\operatorname{suppv}(T) = V(\lambda_{\max}(T' \circ T)) \cap \mathsf{S}_H$.

Proof.

1. Fix an element $x \in H$, and the associated mapping $x^* := (\bullet|x)$. Then,

$$\begin{aligned} \|T(x)\|^2 &= (T(x)|T(x)) = \big(T'(T(x))|x\big) = x^*\big((T' \circ T)(x)\big) && (6)\\ &\leq \|x^*\|\big\|T'(T(x))\big\| \leq \|x^*\|\big\|T' \circ T\big\|\|x\| = \big\|T' \circ T\big\|\|x\|^2 && (7)\\ &\leq \big\|T'\big\|\|T\|\|x\|^2 = \|T\|^2\|x\|^2. && (8) \end{aligned}$$

If element x is taken in the unit sphere, i.e., $x \in \mathsf{S}_X$, and considering the previous inequalities, we concluded that $\|T\|^2 = \|T' \circ T\|$.

2. Let $x \in \operatorname{suppv}(T)$ be an arbitrary element; then, Equation (6) implies that

$$\|T'(T(x))\| = \|T\|^2 = \|T\|\|T(x)\| = \|T'\|\|T(x)\|.$$

Then, $x \in \operatorname{suppv}(T' \circ T)$.

3. Take $v \in \operatorname{suppv}(T)$. Before anything else, since $\operatorname{suppv}(T) \subseteq \operatorname{suppv}(T' \circ T)$, we have that

$$\left\| \frac{(T' \circ T)(v)}{\|T' \circ T\|} \right\| = \frac{\|(T' \circ T)(v)\|}{\|T' \circ T\|} = \frac{\|T' \circ T\|}{\|T' \circ T\|} = 1. \tag{9}$$

Following chain of equalities (6),

$$v^* \left(\frac{(T' \circ T)(v)}{\|T' \circ T\|} \right) = \frac{\|T(v)\|^2}{\|T' \circ T\|} = \frac{\|T\|^2\|v\|^2}{\|T' \circ T\|} = 1. \tag{10}$$

Thanks to the strict convexity of space H,

$$\frac{(T' \circ T)(v)}{\|T' \circ T\|} = v,$$

that is,

$$(T' \circ T)(v) = \|T' \circ T\| v$$

and so $\|T' \circ T\| \in \sigma_p(T' \circ T)$. We implicitly proved that $\operatorname{suppv}(T) \subseteq V(\|T' \circ T\|) \cap \mathsf{S}_H$. Conversely, let us suppose that $\|T' \circ T\| \in \sigma_p(T' \circ T)$. As we remarked before, $T' \circ T$ is a strongly positive operator, so the eigenvalues of that operator are real and positive. Therefore, equality $\lambda_{\max}(T' \circ T) = \|T' \circ T\|$ holds, which implies that

$$\|T\| = \sqrt{\|T\|^2} = \sqrt{\|T' \circ T\|} = \sqrt{\lambda_{\max}(T' \circ T)}.$$

Take $w \in V(\lambda_{\max}(T' \circ T)) \cap \mathsf{S}_H$. Then

$$\begin{aligned} \|T(w)\|^2 &= w^*\big((T' \circ T)(w)\big) \\ &= w^*\big(\lambda_{\max}(T' \circ T)w\big) \\ &= \lambda_{\max}\big(T' \circ T\big) \\ &= \|T' \circ T\| \\ &= \|T\|^2. \end{aligned}$$

This chain of equalities proves that $w \in \operatorname{suppv}(T)$. Consequently,

$$\mathsf{V}\big(\lambda_{\max}(T' \circ T)\big) \cap \mathsf{S}_H \subseteq \operatorname{suppv}(T).$$

□

The following technical lemma establishes the behavior of the point spectrum of a linear combination of operators. However, we first introduce some notation. Considering bounder linear operator $T \in \mathcal{B}(H, K)$ defined between H and K, Hilbert spaces, then

$$V(T) := \bigcup_{\lambda \in \sigma_p(T)} V(\lambda).$$

Lemma 2. *If we consider Hilbert spaces, H, K, and $T_1, \ldots, T_k \in \mathcal{B}(H, K)$, then, for every $\alpha_1, \ldots, \alpha_k \in \mathbb{C}$,*

$$\bigcap_{i=1}^{k} V(T_i) \subseteq V\left(\sum_{i=1}^{k} \alpha_i T_i \right).$$

Proof. Take any $x \in \bigcap_{i=1}^{k} V(T_i)$. If $x = 0$, there is nothing to prove, x is actually in $V\left(\sum_{i=1}^{k} \alpha_i T_i\right)$. So, assume that $x \neq 0$. For every $i \in \{1, \dots, k\}$, there exists $\lambda_i \in \sigma(T_i)$, such that $x \in V(\lambda_i)$, that is, $T_i(x) = \lambda_i x$. Then

$$\left(\sum_{i=1}^{k} \alpha_i T_i\right)(x) = \sum_{i=1}^{k} \alpha_i T_i(x) = \sum_{i=1}^{k} \alpha_i \lambda_i x = \left(\sum_{i=1}^{k} \alpha_i \lambda_i\right) x.$$

This shows that

$$x \in V\left(\sum_{i=1}^{k} \alpha_i \lambda_i\right) \subseteq V\left(\sum_{i=1}^{k} \alpha_i T_i\right).$$

□

The hypothesis in Lemma 3 is, in fact, very restrictive.

Lemma 3. *If H, K are Hilbert spaces, and $T_1, \dots, T_k \in \mathcal{B}(H, K)$, such that $V(T_i) \setminus \ker(T_i) \subseteq \ker(T_j)$ for all $i, j \in \{1, \dots, k\}$ with $i \neq j$. For every $i \in \{1, \dots, k\}$ and every $x_i \in V(T_i) \setminus \ker(T_i)$, there are $\alpha_1, \dots, \alpha_k \in \mathbb{C}$, such that*

$$\sum_{i=1}^{k} \beta_i x_i \in V\left(\sum_{i=1}^{k} \alpha_i T_i\right)$$

for every $\beta_1, \dots, \beta_k \in \mathbb{C}$.

Proof. For every $i \in \{1, \dots, k\}$, there exists $\lambda_i \in \sigma(T_i) \setminus \{0\}$, such that $x_i \in V(\lambda_i)$, that is, $T_i(x_i) = \lambda_i x_i$. Define $\alpha_i := \lambda_i^{-1}$ for every $i \in \{1, \dots, k\}$. Then

$$\left(\sum_{i=1}^{k} \alpha_i T_i\right)\left(\sum_{i=1}^{k} \beta_i x_i\right) = \sum_{i=1}^{k} \alpha_i \beta_i T_i(x_i) = \sum_{i=1}^{k} \alpha_i \beta_i \lambda_i x_i = \sum_{i=1}^{k} \beta_i x_i.$$

□

If $H_1, \dots, H_p$ are Hilbert spaces, then $\left(\bigoplus_{i=1}^{p} H_i\right)_2$ is a Hilbert space, considering the following scalar product and norm

$$\left((h_i)_{i=1}^{p} | (k_i)_{i=1}^{p}\right) := \sum_{i=1}^{p} (h_i | k_i), \quad \left\|(h_i)_{i=1}^{p}\right\| := \sqrt{\sum_{i=1}^{p} \|h_i\|^2},$$

for all $(h_i)_{i=1}^{p}, (k_i)_{i=1}^{p} \in \left(\bigoplus_{i=1}^{p} H_i\right)_2$. If H is another Hilbert space and $T_i : H \to H_i$ is a continuous linear operator for each $i = 1, \dots, p$, then the direct sum of $T_1, \dots, T_p$ is defined as

$$\begin{array}{rrcl} \left(\bigoplus_{i=1}^{p} T_i\right)_2 : & H & \to & \left(\bigoplus_{i=1}^{p} H_i\right)_2 \\ & x & \mapsto & \left(\bigoplus_{i=1}^{p} T_i\right)_2 (x) := (T_i(x))_{i=1}^{p}. \end{array}$$

If $S_i : H_i \to H$ is a continuous linear operator for each $i = 1, \dots, p$, then the direct sum of $S_1, \dots, S_p$ is now defined as

$$\begin{array}{rrcl} \left(\bigoplus_{i=1}^{p} S_i\right)_2 : & \left(\bigoplus_{i=1}^{p} H_i\right)_2 & \to & H \\ & (h_i)_{i=1}^{p} & \mapsto & \left(\bigoplus_{i=1}^{p} S_i\right)_2 \left((h_i)_{i=1}^{p}\right) := \sum_{i=1}^{p} S_i(h_i). \end{array}$$

Theorem 5. *Suppose that $H, H_1, \ldots, H_p$ are Hilbert spaces, and let $T_i : H \to H_i$ be a continuous linear operator for each $i = 1, \ldots, p$. Then, $\left(\bigoplus_{i=1}^{p} T_i\right)_2' = \left(\bigoplus_{i=1}^{p} T_i'\right)_2$ and*

$$\left(\bigoplus_{i=1}^{p} T_i\right)_2' \circ \left(\bigoplus_{i=1}^{p} T_i\right)_2 = \sum_{i=1}^{p} T_i' \circ T_i.$$

Proof. Fix arbitrary elements $x \in H$ and $(h_i)_{i=1}^p \in \left(\bigoplus_{i=1}^{p} H_i\right)_2$. Then,

$$\begin{aligned}
\left(x \middle| \left(\bigoplus_{i=1}^{p} T_i'\right)_2 \left((h_i)_{i=1}^p\right)\right) &= \left(x \middle| \sum_{i=1}^{p} T_i'(h_i)\right) \\
&= \sum_{i=1}^{p} \left(x \middle| T_i'(h_i)\right) \\
&= \sum_{i=1}^{p} \left(T_i(x) \middle| h_i\right) \\
&= \left((T_i(x))_{i=1}^p \middle| (h_i)_{i=1}^p\right) \\
&= \left(\left(\bigoplus_{i=1}^{p} T_i\right)_2 (x) \middle| (h_i)_{i=1}^p\right).
\end{aligned}$$

Lastly, for each $x \in H$,

$$\begin{aligned}
\left(\left(\bigoplus_{i=1}^{p} T_i\right)_2' \circ \left(\bigoplus_{i=1}^{p} T_i\right)_2\right)(x) &= \left(\bigoplus_{i=1}^{p} T_i\right)_2' \left(\left(\bigoplus_{i=1}^{p} T_i\right)_2 (x)\right) \\
&= \left(\bigoplus_{i=1}^{p} T_i'\right)_2 \left((T_i(x))_{i=1}^p\right) \\
&= \sum_{i=1}^{p} T_i'(T_i(x)) \\
&= \sum_{i=1}^{p} (T_i' \circ T_i)(x).
\end{aligned}$$

□

3.2. *Pareto Optimal Solutions of the MOP* max $\|T(x)\|$, min $\|x\|$, $x \in X$

Under the settings of Theorem 3, $\arg\min_{x \in X} \|x\| = \{0\}$; therefore, in view of Theorem 1, $0 \in \mathrm{Pos}(3)$. This Pareto optimal solution is usually disregarded when it comes to a real-life problem.

Theorem 6. *Let X, Y be normed spaces, and $T : X \to Y$ be a nonzero continuous linear operator. Then,* $\mathrm{Pos}(3) = \mathbb{R}\mathrm{suppv}(T)$.

Proof. Fix an arbitrary $x_0 \in \mathrm{Pos}(3)$. Since $x_0 = \|x_0\| \frac{x_0}{\|x_0\|}$, it is sufficient if we show that $\frac{x_0}{\|x_0\|} \in \mathrm{suppv}(T)$. Therefore, we may assume that $\|x_0\| = 1$, so our aim was summed up to prove that $\|T(x_0)\| = \|T\|$. Since $x_0 \in \mathsf{S}_X \subseteq \mathsf{B}_X$, $\|T(x_0)\| \leq \|T\|$. Suppose that $\|T(x_0)\| < \|T\|$. By the definition of sup,there exists $y \in \mathsf{B}_X$, such that $\|T(x_0)\| < \|T(y)\| \leq \|T\|$. $\|y\| \leq 1 = \|x_0\|$ and $\|T(x_0)\| < \|T(y)\|$, which contradicts that $x_0 \in \mathrm{Pos}(3)$. As a consequence, $\|T(x_0)\| = \|T\|$; hence, $x_0 \in \mathrm{suppv}(T)$. The arbitrariness of $x_0 \in \mathrm{Pos}(3)$ shows that $\mathrm{Pos}(3) \subseteq \mathbb{R}\mathrm{suppv}(T)$. Conversely, fix an arbitrary $x_0 \in \mathbb{R}\mathrm{suppv}(T)$. There exists $y_0 \in \mathrm{suppv}(T)$ and $\alpha \in \mathbb{R}$, such that $x_0 = \alpha y_0$. Observe that $\|x_0\| = |\alpha|\|y_0\| = |\alpha|$. We prove that $x_0 \in \mathrm{Pos}(3)$. Let us consider an element $y \in X$ satisfying that $\|y\| < \|x_0\| = |\alpha|$, and we distinguish cases: if $y = 0$, then $\|T(x_0)\| = |\alpha|\|T(y_0)\| = |\alpha|\|T\| > 0 = \|T(y)\|$. If $y \neq 0$, then

$$\|T(x_0)\| = |\alpha|\|T(y_0)\| = |\alpha|\|T\| \geq |\alpha|\left\|T\left(\frac{y}{\|y\|}\right)\right\| > \|T(y)\|.$$

Lastly, if there exists $y \in X$, such that $\|T(y)\| > \|T(x_0)\|$, then

$$|\alpha|\|T\| = \|T(x_0)\| < \|T(y)\| \leq \|T\|\|y\|,$$

which means that $\|x_0\| = |\alpha| < \|y\|$. □

When X, Y are Hilbert spaces, the Pareto optimal solutions of (3) are directly obtained via combining Theorems 4 and 6.

Corollary 1. *Let $T : H \to K$ be a continuous linear operator with H, K Hilbert spaces. Then,* $\mathrm{Pos}(3) = V(\lambda_{\max}(T' \circ T))$.

This last result allows for solving the following MOP (motivated in Section 4), given by

$$\begin{cases} \max \|T_1(x)\|^2 + \cdots + \|T_k(x)\|^2, \\ \min \|x\|, \\ x \in H. \end{cases} \tag{11}$$

The Pareto optimal solutions of (11) are related to those of

$$\begin{cases} \max \|T_i(x)\|, \\ \min \|x\|, \\ x \in H, \end{cases} \tag{12}$$

for $i = 1, \ldots, k$.

Corollary 2. *If $T_1, \ldots, T_k \in \mathcal{B}(H, K)$ are continuous linear operators between Hilbert spaces H and K, then:*

1. $\mathrm{Pos}(11) = V\left(\lambda_{\max}\left(\sum_{i=1}^k T_i' \circ T_i\right)\right)$.
2. $\bigcap_{i=1}^k \mathrm{Pos}(12) \subseteq \mathrm{Pos}(11)$.

Proof. Consider bounded linear operator

$$\begin{array}{rccl} T_1 \oplus_2 \overset{k}{\cdots} \oplus_2 T_k : & H & \to & K \oplus_2 \overset{k}{\cdots} \oplus_2 K \\ & x & \mapsto & \left(T_1 \oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right)(x) := (T_1(x), \ldots, T_k(x)). \end{array}$$

The next equality trivially holds for every $x \in H$,

$$\|(T_1(x), \ldots, T_k(x))\|^2 = \|T_1(x)\|^2 + \cdots + \|T_k(x)\|^2.$$

Since the square root is strictly increasing, (11) is equivalent to

$$\begin{cases} \max \left\|\left(T_1 \oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right)(x)\right\|, \\ \min \|x\|, \\ x \in H, \end{cases} \tag{13}$$

which is an MOP of form (3).

1. According to Corollary 1 and Theorem 5,

$$\begin{aligned}\operatorname{Pos}(11) &= \operatorname{Pos}(13)\\ &= V\left(\lambda_{\max}\left(\left(T_1\oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right)' \circ \left(T_1\oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right)\right)\right)\\ &= V\left(\lambda_{\max}\left(\sum_{i=1}^{k} T_i' \circ T_i\right)\right).\end{aligned}$$

2. We rely on Theorem 6 and Corollary 1. Fix an arbitrary $x \in \bigcap_{i=1}^{k} \operatorname{Pos}(12)$. If $x = 0$, then $x \in \mathbb{R}\operatorname{suppv}\left(T_1\oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right) = \operatorname{Pos}(13) = \operatorname{Pos}(11)$. Suppose that $x \neq 0$. In view of Theorem 6, $\frac{x}{\|x\|} \in \bigcap_{i=1}^{k} \operatorname{suppv}(T_i)$. We prove that $\frac{x}{\|x\|} \in \operatorname{suppv}\left(T_1\oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right)$. Take any $y \in \mathsf{B}_H$. Since $\frac{x}{\|x\|} \in \bigcap_{i=1}^{k} \operatorname{suppv}(T_i)$, for every $i \in \{1, \ldots, k\}$,

$$\left\| T_i\left(\frac{x}{\|x\|}\right)\right\| \geq \|T_i(y)\|.$$

As a consequence,

$$\begin{aligned}\left\|\left(T_1\oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right)\left(\frac{x}{\|x\|}\right)\right\| &= \sqrt{\left\|T_1\left(\frac{x}{\|x\|}\right)\right\|^2 + \cdots + \left\|T_k\left(\frac{x}{\|x\|}\right)\right\|^2}\\ &\geq \sqrt{\|T_1(y)\|^2 + \cdots + \|T_k(y)\|^2}\\ &= \left\|\left(T_1\oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right)(y)\right\|.\end{aligned}$$

This means that $\frac{x}{\|x\|} \in \operatorname{suppv}\left(T_1\oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right)$. In accordance with Theorem 6,

$$x = \|x\|\frac{x}{\|x\|} \in \mathbb{R}\operatorname{suppv}\left(T_1\oplus_2 \overset{k}{\cdots} \oplus_2 T_k\right) = \operatorname{Pos}(13) = \operatorname{Pos}(11).$$

$\square$

4. Discussion

In order to design truly optimal TMS coils, and depending on the nature and characteristics of the coil that we want to maximize or minimize, a linearization technique is applied to the electromagnetic field [18,23–25]; then, MOPs like (3) come out:

$$\left\{\begin{array}{l}\max \|E_x\psi\|_2,\\ \min \psi^T L\psi,\\ \psi \in \mathbb{R}^n,\end{array}\right. \quad \left\{\begin{array}{l}\max \|E_y\psi\|_2,\\ \min \psi^T L\psi,\\ \psi \in \mathbb{R}^n,\end{array}\right. \quad \left\{\begin{array}{l}\max \|E_z\psi\|_2,\\ \min \psi^T L\psi,\\ \psi \in \mathbb{R}^n,\end{array}\right. \tag{14}$$

where E is a matrix representing the electromagnetic field, E_x, E_y, E_z are the components of E, and L represents inductance with a positive definite symmetric matrix. Using Cholesky decomposition, as L is positive definite and symmetric, the existence of an invertible matrix C, such that $L = C^T C$, is guaranteed. Then,

$$\psi^T L\psi = \psi^T C^T C\psi = (C\psi)^T(C\psi) = \|C\psi\|_2^2.$$

Next, we apply the following change of variables: $\varphi := C\psi$. Then, the previous problems can be rewritten as follows:

$$\left\{\begin{array}{l}\max \left\| \left(E_x C^{-1} \right) \varphi \right\|_2, \\ \min \|\varphi\|_2^2, \\ \varphi \in \mathbb{R}^n,\end{array}\right. \quad \left\{\begin{array}{l}\max \left\| \left(E_y C^{-1} \right) \varphi \right\|_2, \\ \min \|\varphi\|_2^2, \\ \varphi \in \mathbb{R}^n,\end{array}\right. \quad \left\{\begin{array}{l}\max \left\| \left(E_z C^{-1} \right) \varphi \right\|_2, \\ \min \|\varphi\|_2^2, \\ \varphi \in \mathbb{R}^n.\end{array}\right. \tag{15}$$

Since the square root is strictly increasing, the previous MOPs are equivalent to the following (in the sense that they have the same set of global solutions and the same set of Pareto optimal solutions):

$$\left\{\begin{array}{l}\max \left\| \left(E_x C^{-1} \right) \varphi \right\|_2, \\ \min \|\varphi\|_2, \\ \varphi \in \mathbb{R}^n,\end{array}\right. \quad \left\{\begin{array}{l}\max \left\| \left(E_y C^{-1} \right) \varphi \right\|_2, \\ \min \|\varphi\|_2, \\ \varphi \in \mathbb{R}^n,\end{array}\right. \quad \left\{\begin{array}{l}\max \left\| \left(E_z C^{-1} \right) \varphi \right\|_2, \\ \min \|\varphi\|_2, \\ \varphi \in \mathbb{R}^n.\end{array}\right. \tag{16}$$

The three MOPs above are of the form (3). Therefore, in view of Corollary 1, the Pareto optimal solutions of each of them is determined by

$$\begin{array}{c} V\left(\lambda_{\max}\left(\left(E_x C^{-1}\right)^T \left(E_x C^{-1}\right)\right)\right), \\ V\left(\lambda_{\max}\left(\left(E_y C^{-1}\right)^T \left(E_y C^{-1}\right)\right)\right), \\ V\left(\lambda_{\max}\left(\left(E_z C^{-1}\right)^T \left(E_z C^{-1}\right)\right)\right), \end{array}$$

respectively. On the other hand, we can consider the combined MOP, as in (11):

$$\left\{\begin{array}{l}\max \left\| \left(E_x C^{-1} \right) \varphi \right\|_2^2 + \left\| \left(E_y C^{-1} \right) \varphi \right\|_2^2 + \left\| \left(E_z C^{-1} \right) \varphi \right\|_2^2, \\ \min \|\varphi\|_2, \\ \varphi \in \mathbb{R}^n.\end{array}\right. \tag{17}$$

Let us define the following linear operator:

$$\begin{array}{rccl} T: & \ell_2^n & \to & \ell_2^n \oplus_2 \ell_2^n \oplus_2 \ell_2^n \\ & \varphi & \mapsto & T(\varphi) = \left(\left(E_x C^{-1} \right) \varphi, \left(E_y C^{-1} \right) \varphi, \left(E_z C^{-1} \right) \varphi \right). \end{array}$$

The corresponding matrix to T is precisely

$$A := \begin{pmatrix} E_x C^{-1} \\ E_y C^{-1} \\ E_z C^{-1} \end{pmatrix}.$$

For every $\varphi \in \mathbb{R}^n$,

$$\|T(\varphi)\|_2 = \left(\left\| \left(E_x C^{-1} \right) \varphi \right\|_2^2 + \left\| \left(E_y C^{-1} \right) \varphi \right\|_2^2 + \left\| \left(E_z C^{-1} \right) \varphi \right\|_2^2 \right)^{\frac{1}{2}}.$$

Then (17) is the same as

$$\left\{\begin{array}{l}\max \|T(\varphi)\|_2, \\ \min \|\varphi\|_2, \\ \varphi \in \mathbb{R}^n.\end{array}\right. \tag{18}$$

According to Corollary 1,

$$\operatorname{Pos}(18) = \mathbb{R}\operatorname{suppv}(T) = V\left(\lambda_{\max}\left(A^T A\right)\right).$$

Equivalently, according to Corollary 2,

$$\begin{aligned} &\mathrm{Pos}(18) = \mathrm{Pos}(17) \\ = \;& V\left(\lambda_{\max}\left(\left(E_xC^{-1}\right)^T\left(E_xC^{-1}\right) + \left(E_yC^{-1}\right)^T\left(E_yC^{-1}\right) + \left(E_zC^{-1}\right)^T\left(E_zC^{-1}\right)\right)\right). \end{aligned}$$

A very illustrative example of this situation is displayed in the Appendix A

5. Conclusions

This section deals with linear combinations of MOPs of the form given in (3). Let H, K be Hilbert spaces. Consider continuous linear operators $T_1, \ldots, T_k \in \mathcal{B}(H, K)$ between H and K. Let $\alpha_1, \ldots, \alpha_k > 0$. In bioengineering, it is common to assign weights α_i to different operators T_i depending on the relevance of each T_i. Then, the following MOP comes into play:

$$\begin{cases} \max \alpha_1\|T_1(x)\|^2 + \cdots + \alpha_k\|T_k(x)\|^2, \\ \min \|x\|, \\ x \in H, \end{cases} \tag{19}$$

Nevertheless, the above MOP is, in fact, the same as the following:

$$\begin{cases} \max \|S_1(x)\|^2 + \cdots + \|S_k(x)\|^2, \\ \min \|x\|, \\ x \in H, \end{cases} \tag{20}$$

where $S_i := \sqrt{\alpha_i}T_i$ for each $i = 1, \ldots, k$. $\mathrm{suppv}(T_i) = \mathrm{suppv}\left(\sqrt{\alpha_i}T_i\right) = \mathrm{suppv}(S_i)$ for every $i = 1, \ldots, k$. By relying on Corollary 2, at least we can ensure that

$$\bigcap_{i=1}^{k} \mathrm{Pos}(12) \subseteq \mathrm{Pos}(20) = \mathrm{Pos}(19).$$

However, it is very unlikely that $\bigcap_{i=1}^{k} \mathrm{Pos}(12) \neq \{0\}$. Unless hypotheses similar to the ones employed in Lemma 2 or Lemma 3 are used, we cannot conclude any other relation between Pos(19) and Pos(12).

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Abbreviations

The following abbreviations are used in this manuscript:

MDPI	Multidisciplinary Digital Publishing Institute
DOAJ	Directory of open access journals
MOP	Multiobjective optimization problem
SOP	Single-objective optimization problem
POS	Pareto optimal solution
PC	Pareto chart
TMS	Transcranial magnetic stimulation
MRI	Magnetic resonance imaging
ROI	Region of interest

Appendix A. Illustrative Example on Coil Design

Appendix A.1. Coil Design in Engineering

The use of coils that optimize one or more components of an electromagnetic field while minimizing power dissipation or stored magnetic energy is often required in bioengineering applications such as TMS [21,23] and magnetic resonance imaging (MRI) [24,25] or in high-precision magnetic measurement systems in space missions such as eLISA [26–29].

All these applications are characterized by the need of generating a prescribed and localized electromagnetic field in a specific region, and are subject to other performance requirements such as the minimization of stored magnetic energy or dissipated power. Therefore, the design of electromagnetic coils for these applications can be considered to be an MOP. MOPs from coil design are frequently expressed as a convex optimization and formulated in terms of the stream function of a quasistatic current [18].

Appendix A.2. Design of Maximal B_x and B_y Coil for Magnetic Measurement Systems in a Space Missions

In the following, for the purpose of illustrating an application of the obtained theoretical results in this manuscript, we present the design of a planar coil over a (34 × 17 mm) PCBfor magnetic measurement systems in space missions. This coil was constructed with the aim of maximizing the magnetic field in a small and near region where a magnetic sensor capable of measuring the X and Y components of the B field is located. At the same time, resistance was minimized in order to avoid power dissipation.

Hence, the initial requirement that the coil had to satisfy was that it had to produce a maximal magnetic field in a region of interest (ROI) that was located in the same position as that of the sensor ($x = 22$ mm$, y = 6.5$ mm$, z = 1.25$ mm), with its same dimensions (5.8 × 3.5 mm), formed by 200 points. Figure A1 illustrates the available surface for the coil design along with the ROI.

In order to obtain stream function φ, which simultaneously maximizes B_x and B_y while minimizing power dissipation at the ROI, previously presented MOPs (16) and (17) were applied. Consequently, the current coil-design problem can be expressed as the following MOPs:

$$\left\{\begin{array}{l}\max\left\|\left(B_xC^{-1}\right)\varphi\right\|_2,\\ \min\|\varphi\|_2,\\ \varphi\in\mathbb{R}^n,\end{array}\right.\quad\left\{\begin{array}{l}\max\left\|\left(B_yC^{-1}\right)\varphi\right\|_2,\\ \min\|\varphi\|_2,\\ \varphi\in\mathbb{R}^n,\end{array}\right.\quad\left\{\begin{array}{l}\max\left\|\left(B_zC^{-1}\right)\varphi\right\|_2,\\ \min\|\varphi\|_2,\\ \varphi\in\mathbb{R}^n.\end{array}\right.\tag{A1}$$

$$\left\{\begin{array}{l}\max\alpha\left\|\left(B_xC^{-1}\right)\varphi\right\|_2^2+\beta\left\|\left(B_yC^{-1}\right)\varphi\right\|_2^2+\gamma\left\|\left(B_zC^{-1}\right)\varphi\right\|_2^2,\\ \min\|\varphi\|_2,\\ \varphi\in\mathbb{R}^n.\end{array}\right.\tag{A2}$$

where $B_i \in \mathbb{R}^{m\times n}$ stands for the matrix of the magnetic field in the i-th direction ($i = x, y, z$); $R \in \mathbb{R}^{n\times n}$ is the resistance matrix; n is the number of mesh points ($n = 2000$); m is the number of ROI points ($m = 200$); and α, β, and γ are constants that provide specific weights for maximizing each component of the field (B_x, B_y, B_z). Due to the fact that it is only necessary to maximize the B_x and B_y components in the current case, weights were chosen such that $\alpha = \beta < \gamma$ (in concrete $\alpha = \beta = 1$ and $\gamma = 10^{-2}$).

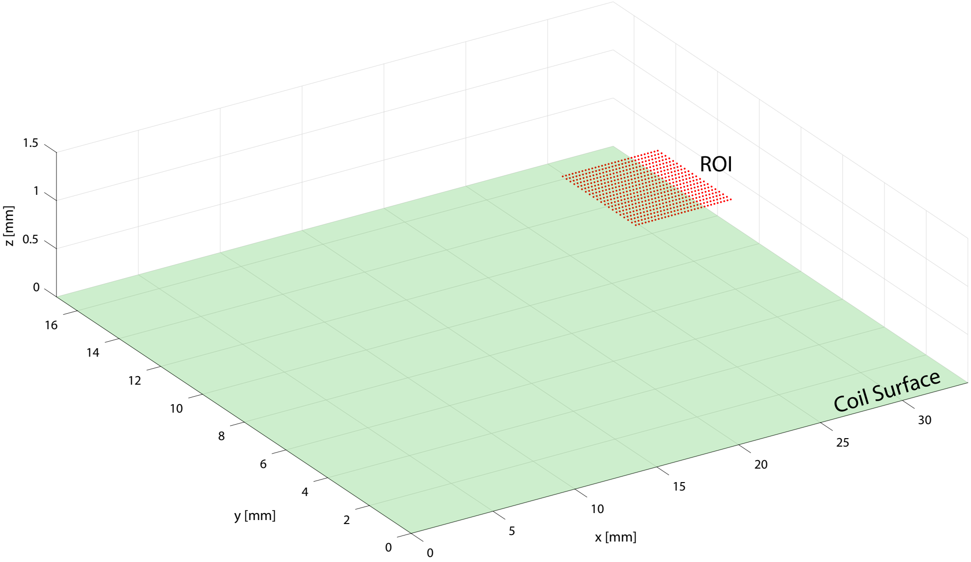

Figure A1. Representation of planar coil surface and region of interest (ROI) where optimal stream function φ is calculated.

Figure A2 shows the stream function solution from the (A1) and (A2) MOPs (red and blue functions, respectively) computed by using the theoretical model developed in [4,18,19]. Three different optimal stream functions were obtained from (A1) MOP (φ_x, φ_y, φ_z). Consequently, the final φ_1 solution was calculated as linear combination $\varphi_1 = \alpha\varphi_x + \beta\varphi_y + \gamma\varphi_z$. However, stream function φ_2 is the final solution obtained from the (A2) MOP. As expected from the conclusions of the manuscript, the stream functions were not equal.

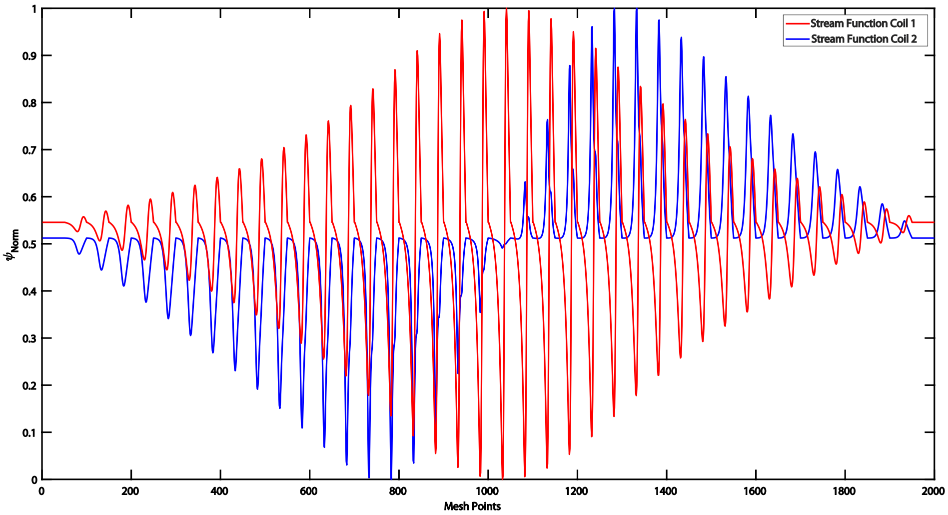

Figure A2. Stream functions obtained from Problems (A1) (Coil 1) and (A2) (Coil 2).

Furthermore, stream function contours over the coil surface can be considered to be the current wire path [30,31]. Accordingly, coil wires were designed as the stream function contour, as is depicted in Figures A3 and A4, where the designed coils are different depending on the MOP.

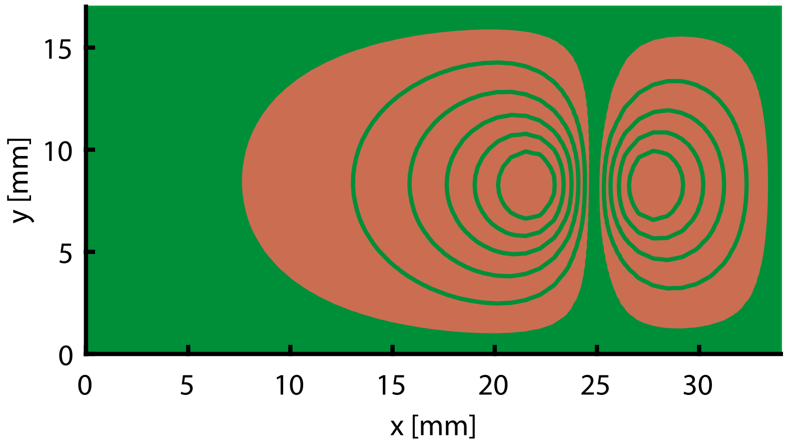

Figure A3. Obtained wire paths from Problem (A1) over PCBsurface.

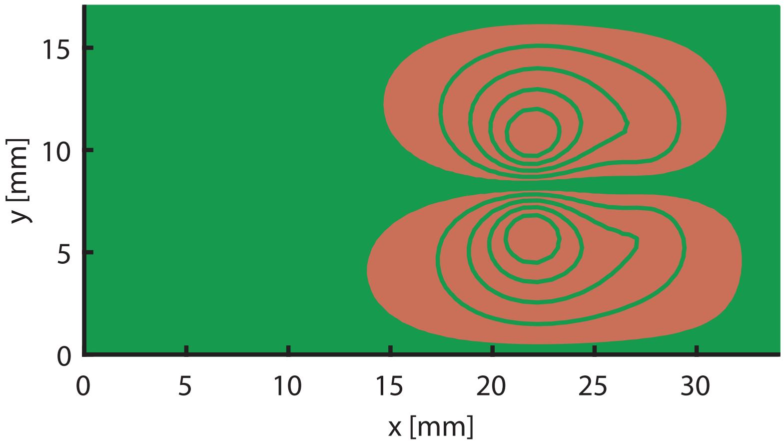

Figure A4. Obtained wire paths from Problem (A2) over the PCB surface.

In conclusion, as expected from the proposed theoretical model in this manuscript, the solutions of (A1) and (A2) were different; consequently, we could not conclude any relation between the two MOPs.

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