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Convergence Analysis of the Straightforward Expansion Perturbation Method for Weakly Nonlinear Vibrations

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Abstract: There are typically several *perturbation methods* for approaching the solution of *weakly nonlinear vibrations* (where the nonlinear terms are "small" compared to the linear ones): the *Method of Strained Parameters*, the *Naive Singular Perturbation Method*, the *Method of Multiple Scales*, the *Method of Harmonic Balance* and the *Method of Averaging*. The *Straightforward Expansion Perturbation Method* (SEPM) applied to weakly nonlinear vibrations does not usually yield to correct solutions. In this manuscript, we provide mathematical proof of the inaccuracy of the SEPM in general cases. Nevertheless, we also provide a sufficient condition for the SEPM to be successfully applied to weakly nonlinear vibrations. This mathematical formalism is written in the syntax of the first-order formal language of Set Theory under the methodology framework provided by the Category Theory.

Keywords: numerical analysis; approximation theory; nonlinear vibration; perturbation method; Banach space; unitary algebra; sup norm

MSC: 47L05; 47L90; 49J30; 90B50

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1. Introduction

This paper originates from and is motivated by a recently published manuscript [1], where the influence of different support types in the nonlinear vibrations of beams is analyzed. During the development of this latter paper, the authors searched for a mathematical proof of the Straightforward Expansion Perturbation Method (SEPM), not reaching any manuscript containing a proper and rigorous definition and proof of the method. The main objective of this paper is to provide a mathematical formalism of SEPM.

Currently, a large number of nonlinear vibration problems in Engineering are solved by the *Nonlinear Finite Element Method*. However, in many cases, it is necessary to find an analytical solution in order to better understand the contribution of forces, masses or geometries. In the process of searching for an analytical solution, hypotheses, simplifications and linearizations are raised, which usually lead to approximations of the exact analytical solutions. Traditionally, nonlinear problems have been solved by perturbations methods in order to eliminate the generated secular terms. According to these techniques, the solution is represented by a few terms of an expansion, usually no more than two or three terms. Therefore, the deviation between the approximate analytical solution and the exact analytical solution depends on the number of selected expansion terms and the amplitude of the vibration [2–6].

In accordance with ([7], Subsection 3.5.1), there are typically three *perturbation methods* for approaching the solution of weakly nonlinear vibrations: the *Method of Multiple Scales*, the *Method of Harmonic Balance* and the *Method of Averaging*. There is a fourth method, simpler to apply than the previous three but much more imprecise, called the *Straightforward Expansion Perturbation Method* (SEPM). This perturbation method applied to weakly nonlinear vibrations does not usually yield to correct solutions, as shown in ([7], Subsection 3.5.2). The appearance of secular terms, i.e., terms of the form $t\sin(t+\varphi_0)$, leads to incorrect solutions due to, among other physical reasons, the unbounded growth of the term with time t. In Theorem 3 and Corollary 4, we prove that, if the terms of the expansion are uniformly bounded, then the SEPM leads to a correct solution of a weakly nonlinear vibration.

There are other perturbation methods such as the *Method of Strained Parameters* (also known as *Lindstedt-Poincaré Method*) ([6], Section 3.1) and the *Naive Singular Perturbation Method* [8]. Both methods can also be consider *expansion methods*, in the sense that the solution of the nonlinear equation is calculated by means of an expansion series, like it occurs with SEPM. The main difference between the SEPM and the previous two methods relies on the way that the expansion terms are computed. On the other hand, as mentioned above, we provide a mathematical proof that assures that, under uniform boundedness of the expansion terms, the expansion series in the SEPM converges to the solution. We have conveyed a search and we have found no mathematical theorems or formalism for the *Method of Strained Parameters* or the *Naive Singular Perturbation Method*. Both methods are described by means of examples and successfully applied in many situations. In this direction, we refer the reader to several works by Van Groesen and his students Karjanto and Cahyono [9–12], who accomplished more successful applications of perturbartion methods in different settings, such as wave modelling.

In 1998, Liao [13] provided a new analytic technique, called Homotopy Analysis Method (HAM), which differs from perturbation methods in the essential fact that the validity of HAM is independent on whether or not there exist small parameters in considered nonlinear equations. Therefore, HAM is a powerful tool to deal with strongly nonlinear problems. As we will show throughout this manuscript, the existence of small parameters is crucial towards the convergence of SEPM (see Theorem 3 and Section 4). In fact, SEPM is a powerful tool for weakly nonlinear vibrations in which the norm of the expansion terms $(\theta_i)_{i\geq 0}$ can be uniformly controlled. It is worth mentioning that we provide a mathematical proof for the validity of SEPM in the previously mentioned weakly nonlinear vibrations, whereas no mathematical proof is given for a general validation of HAM. However, Liao applied HAM successfully in several situations such as boundary element methods [14], the laminar viscous flow over a semi-infinite flat plate [15], and nonlinear oscillations [16]. In [17], HAM is compared with Euler transformations.

Many authors have investigated the SEPM and developed formulations. However, a rigorous mathematical treatment has not been carried out. Therefore, it is necessary to define the limit of the SEPM in order to provide mathematical formalism and contribute to the advancement of this method. A Banach algebra of differentiable functions endowed with an extended supremum norm has been used in this work. Many theorems, corollaries and lemmas have been formulated with the purpose to assure the stability of the solutions found with this method. In addition, all the methodology used has been applied to the example of the famous pendulum [18].

2. Materials and Methods

The form of the motion equation of *nonlinear vibrations* is generally expressed as [7]:

$$\begin{cases} \ddot{\theta}(t)+c\dot{\theta}(t)+k\theta(t)=g\big(t,\theta(t),\dot{\theta}(t)\big), \\ \text{with initial conditions } \theta(0) \text{ and } \dot{\theta}(0), \\ \theta\in\mathcal{C}^2([a,b],\mathbb{R}), \\ k,c\in\mathbb{R},\ c^2-4k<0, \\ g\in\mathcal{C}^3\big(\mathbb{R}^3,\mathbb{R}\big). \end{cases} \tag{1}$$

Notice that the homogeneous equation $\ddot{\theta}+c\dot{\theta}+k\theta=0$ has harmonic solutions if and only if $c^2-4k<0$. This is why this condition is imposed in Equation (1). On the other hand, $\theta_0 \in \mathbb{R}$ is said to be an equilibrium point of Equation (1) provided that the constant function $\theta(t) := \theta_0$ is a solution of Equation (1). It is trivial to check that $\theta_0 \in \mathbb{R}$ is an equilibrium point of Equation (1) if and only if θ_0 is a solution of $g(t,\theta,0) - k\theta = 0$ for all $t \in [a,b]$. When the "size" of the nonlinear terms of Equation (1), $g\big(t,\theta(t),\dot{\theta}(t)\big)$, is small compared to the one of the linear terms, then we call Equation (1) a *weakly nonlinear vibration*. Later on, in the next section, we will rigorously define the "size" of a function by means of a norm (see Equation (2)).

2.1. The Banach Space Unitary Algebra $\mathcal{C}^m([a,b],\mathbb{R})$

We will deal with the unitary algebra $\mathcal{C}^m([a,b],\mathbb{R})$ of $\mathcal{C}^m$-differentiable real-valued functions defined on the real interval $[a,b]$ with $a < b$, i.e., real-valued functions which are m times continuously differentiable on $[a,b]$. We will endowed it with the norm

$$\|f\|_{(m)} := \max\left\{\|f\|_\infty, \|f'\|_\infty, \dots, \left\|f^{(m)}\right\|_\infty\right\}, \tag{2}$$

where $\|\cdot\|_\infty$ stands for the sup norm on $\mathcal{C}([a,b],\mathbb{R})$, i.e.,

$$\|f\|_\infty := \max_{a\le t\le b} |f(t)|,$$

for each $f \in \mathcal{C}([a,b],\mathbb{R})$. Recall that $\mathcal{C}([a,b],\mathbb{R})$ is a Banach algebra endowed with the sup norm, i.e., $\|fg\|_\infty \le \|f\|_\infty\|g\|_\infty$ for every $f,g \in \mathcal{C}([a,b],\mathbb{R})$. Note also that the uniform convergence of functions in $\mathcal{C}([a,b],\mathbb{R})$ is precisely the $\|\cdot\|_\infty$-convergence (see [19] for a complete study on Banach algebras). It is trivial to check that $\|f\|_{(p)} \le \|f\|_{(q)}$ whenever $p \le q$ and $f \in \mathcal{C}^q([a,b],\mathbb{R})$.

For the sake of completeness and to make this manuscript as self-contained as possible, we will prove that $\mathcal{C}^m([a,b],\mathbb{R})$ is a Banach space when endowed with the norm (2). We first need a technical lemma.

Lemma 1. *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{C}^1([a,b],\mathbb{R})$. Assume that the following conditions hold:*

1. *There exists $t_0 \in [a,b]$ such that $(f_n(t_0))$ is convergent.*
2. *There exists $g \in \mathcal{C}([a,b],\mathbb{R})$ such that $(f'_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_\infty$-convergent to g.*

Then there exists $f \in \mathcal{C}^1([a,b],\mathbb{R})$ such that $f' = g$ and $(f_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_\infty$-convergent to f.

Proof. Let $L := \lim_{n\to\infty} f_n(t_0)$ and define

$$F_n(t) := \int_a^t f'_n(s)ds + f_n(t_0) - \int_a^{t_0} f'(s)ds, \quad n \in \mathbb{N},$$

and

$$f(t) := \int_a^t g(s)ds + L - \int_a^{t_0} g(s)ds$$

for every $t \in [a,b]$. Observe that $f_n(t) = F_n(t)$ for every $t \in [a,b]$ and every $n \in \mathbb{N}$. Thus, it only remains to prove that $(F_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_\infty$-convergent to f. For every $t \in [a,b]$ and every $n \in \mathbb{N}$, we have that

$$\begin{aligned}
|F_n(t) - f(t)| &= \left|\int_a^t f'_n(s)ds + f_n(t_0) - \int_a^{t_0} f'(s)ds - \int_a^t g(s)ds - L + \int_a^{t_0} g(s)ds\right| \\
&\le \int_a^t |f'_n(s) - g(s)|ds + |f_n(t_0) - L| + \int_a^{t_0} |f'_n(s) - g(s)|ds \\
&\le \|f'_n - g\|_\infty (t-a) + |f_n(t_0) - L| + \|f'_n - g\|_\infty (t_0 - a) \\
&\le 2\|f'_n - g\|_\infty (b-a) + |f_n(t_0) - L|.
\end{aligned}$$

This shows that, for all $n \in \mathbb{N}$,

$$\|F_n - f\|_\infty \leq 2\|f'_n - g\|_\infty (b-a) + |f_n(t_0) - L|.$$

Since $(f'_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_\infty$-convergent to g and $(f_n(t_0))_{n\in\mathbb{N}}$ is convergent to L by hypothesis, we deduce that $(F_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_\infty$-convergent to f. $\square$

Now, we are in the right position to prove that $\mathcal{C}^m([a,b],\mathbb{R})$ is a Banach space. In the following theorem and throughout the rest of this manuscript, f^{-1} stands for the inverse of f with respect to the multiplication operation in the algebra $\mathcal{C}^m([a,b],\mathbb{R})$.

Theorem 1. *The unitary algebra $\mathcal{C}^m([a,b],\mathbb{R})$ becomes a complete normed space endowed with the norm (2) and satisfies, for all $f,g \in \mathcal{C}^m([a,b],\mathbb{R})$, that:*

1. $\|1\|_{(m)} = 1$.
2. $\|fg\|_{(m)} \leq 2^m \|f\|_{(m)} \|g\|_{(m)}$.
3. *If f is invertible, then $\|f^{-1}\|_{(m)} \geq 2^{-m}\|f\|_{(m)}^{-1}$.*

Proof. Let $(f_n)_{n\in\mathbb{N}} \subseteq \mathcal{C}^m([a,b],\mathbb{R})$ be a Cauchy sequence for the norm given in (2). Since $\left\|f_p^{(j)} - f_q^{(j)}\right\|_\infty \leq \|f_p - f_q\|_{(m)}$ for all $p,q \in \mathbb{N}$ and all $j \in \{0,\ldots,m\}$, we conclude that $\left(f_n^{(j)}\right)_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathcal{C}([a,b],\mathbb{R})$ for the sup norm. The completeness of $\mathcal{C}([a,b],\mathbb{R})$ endowed with the sup norm assures the existence of $g_j \in \mathcal{C}([a,b],\mathbb{R})$ such that $\left(f_n^{(j)}\right)_{n\in\mathbb{N}}$ is $\|\cdot\|_\infty$-convergent to g_j for all $j \in \{0,\ldots,m\}$. At this stage, we only need to call on Lemma 1 to deduce that, if $f := g_0$, then $f \in \mathcal{C}^m([a,b],\mathbb{R})$ and $f^{(j)} = g_j$ for all $j \in \{0,\ldots,m\}$. Then $\left(f_n^{(j)}\right)_{n\in\mathbb{N}}$ is $\|\cdot\|_\infty$-convergent to $f^{(j)}$ for all $j \in \{0,\ldots,m\}$. Since j ranges a finite set, we can conclude that $(f_n)_{n\in\mathbb{N}}$ converges to f in $\mathcal{C}^m([a,b],\mathbb{R})$. Next, we will prove the three items of the statement of the theorem:

1. It is clear $\|1\|_{(m)} = 1$ since
$$\|1\|_{(m)} = \max\{\|1\|_\infty, \|0\|_\infty, \ldots, \|0\|_\infty\} = 1.$$
2. We will prove, by induction on $m \in \mathbb{N} \cup \{0\}$, that
$$(fg)^{(m)} = \sum_{i=0}^{m} \binom{m}{i} f^{(m-i)} g^{(i)} \tag{3}$$
and
$$\left\|(fg)^{(m)}\right\|_{(m)} \leq 2^m \|f\|_{(m)} \|g\|_{(m)}. \tag{4}$$
For $m = 0$. It is clear that (3) holds if $m = 0$. Since $\mathcal{C}([a,b],\mathbb{R})$ is a Banach algebra endowed with the sup norm, we have that
$$\|fg\|_{(0)} = \|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty = \|f\|_{(0)} \|g\|_{(0)} = 2^0 \|f\|_{(0)} \|g\|_{(0)}.$$
For $m = 1$. By using the product rule, we have that $(fg)' = f'g + fg'$, therefore
$$\|(fg)'\|_\infty \leq \|f'\|_\infty \|g\|_\infty + \|f\|_\infty \|g'\|_\infty \leq 2\|f\|_{(1)} \|g\|_{(1)}.$$
By induction hypothesis, $\|fg\|_\infty \leq \|fg\|_{(0)} \leq \|f\|_{(0)} \|g\|_{(0)} \leq 2\|f\|_{(1)} \|g\|_{(1)}$, so we conclude that
$$\|fg\|_{(1)} = \max\{\|fg\|_\infty, \|(fg)'\|_\infty\} \leq 2^1 \|f\|_{(1)} \|g\|_{(1)}.$$

For $m = 2$. For the second derivative, we have $(fg)'' = f''g + 2f'g' + fg''$, so

$$\|(fg)''\|_\infty \leq \|f''\|_\infty \|g\|_\infty + 2\|f'\|_\infty \|g'\|_\infty + \|f\|_\infty \|g''\|_\infty \leq 4\|f\|_{(2)}\|g\|_{(2)}.$$

By induction hypothesis, $\|fg\|_\infty \leq \|fg\|_{(0)} \leq \|f\|_{(0)}\|g\|_{(0)} \leq 2^2\|f\|_{(2)}\|g\|_{(2)}$ and $\|(fg)'\|_\infty \leq \|(fg)'\|_{(1)} \leq 2\|f\|_{(1)}\|g\|_{(1)} \leq 2^2\|f\|_{(2)}\|g\|_{(2)}$, thus we obtain that

$$\|fg\|_{(2)} = \max\{\|fg\|_\infty, \|(fg)'\|_\infty, \|(fg)''\|_\infty\} \leq 2^2\|f\|_{(2)}\|g\|_{(2)}.$$

Suppose that (3) and (4) hold for $m \geq 2$. Let us prove it for $m+1$. By relying again on the product rule,

$$\begin{aligned}
(fg)^{(m+1)} &= \left((fg)^{(m)}\right)' \\
&= \left(\sum_{i=0}^{m} \binom{m}{i} f^{(m-i)} g^{(i)}\right)' \\
&= \sum_{i=0}^{m} \binom{m}{i} \left(f^{(m-i)} g^{(i)}\right)' \\
&= \sum_{i=0}^{m} \binom{m}{i} f^{(m+1-i)} g^{(i)} + \sum_{i=0}^{m} \binom{m}{i} f^{(m-i)} g^{(i+1)} \\
&= f^{(m+1)}g + \sum_{i=1}^{m} \binom{m}{i} f^{(m+1-i)} g^{(i)} + \sum_{i=0}^{m-1} \binom{m}{i} f^{(m-i)} g^{(i+1)} + fg^{(m+1)} \\
&= f^{(m+1)}g + \sum_{i=1}^{m} \binom{m}{i} f^{(m+1-i)} g^{(i)} + \sum_{i=1}^{m} \binom{m}{i-1} f^{(m-(i-1))} g^{(i)} + fg^{(m+1)} \\
&= f^{(m+1)}g + \sum_{i=1}^{m} \binom{m}{i-1} f^{(m+1-i)} g^{(i)} + \sum_{i=1}^{m} \binom{m}{i} f^{(m+1-i)} g^{(i)} + fg^{(m+1)} \\
&= f^{(m+1)}g + \sum_{i=1}^{m} \binom{m+1}{i} f^{(m+1-i)} g^{(i)} + fg^{(m+1)} \\
&= \sum_{i=0}^{m+1} \binom{m+1}{i} f^{(m+1-i)} g^{(i)}.
\end{aligned}$$

Next, we will prove that $\|fg\|_{(m+1)} \leq 2^{m+1}\|f\|_{(m+1)}\|g\|_{(m+1)}$ by relying on our induction Hypothesis (4). Notice that it is sufficient to show that $\left\|(fg)^{(k)}\right\|_\infty \leq 2^{m+1}\|f\|_{(m+1)}\|g\|_{(m+1)}$ for all $k = 0, \ldots, m+1$. If $k \in \{0, \ldots, m\}$, by (4) we know that

$$\left\|(fg)^{(k)}\right\|_\infty \leq \|fg\|_{(m)} \leq 2^m\|f\|_{(m)}\|g\|_{(m)} \leq 2^{m+1}\|f\|_{(m+1)}\|g\|_{(m+1)}.$$

It only remains to show that $\left\|(fg)^{(m+1)}\right\|_\infty \leq 2^{m+1}\|f\|_{(m+1)}\|g\|_{(m+1)}$. Observe that

$$\begin{aligned}
&\left\|(fg)^{(m+1)}\right\|_\infty = \left\|\sum_{i=0}^{m+1} \binom{m+1}{i} f^{(m+1-i)} g^{(i)}\right\|_\infty \\
\leq\ & \sum_{i=0}^{m+1} \binom{m+1}{i} \left\|f^{(m+1-i)} g^{(i)}\right\|_\infty \leq \sum_{i=0}^{m+1} \binom{m+1}{i} \left\|f^{(m+1-i)}\right\|_\infty \left\|g^{(i)}\right\|_\infty \\
\leq\ & \|f\|_{(m+1)}\|g\|_{(m+1)} \sum_{i=0}^{m+1} \binom{m+1}{i} \leq 2^{m+1}\|f\|_{(m+1)}\|g\|_{(m+1)}.
\end{aligned}$$

3. Finally, if f is invertible, by applying Theorem 1(2),
$$1 = \|1\|_{(m)} = \left\|ff^{-1}\right\|_{(m)} \leq 2^m \|f\|_{(m)} \left\|f^{-1}\right\|_{(m)}$$
so
$$\left\|f^{-1}\right\|_{(m)} \geq 2^{-m} \|f\|_{(m)}^{-1}.$$

□

Remark 1. *If $f, g, h \in \mathcal{C}^m([a,b],\mathbb{R})$, then in view of Theorem 1(2),*
$$\|fgh\|_{(m)} \leq 2^m \|fg\|_{(m)} \|h\|_{(m)} \leq 4^m \|f\|_{(m)} \|g\|_{(m)} \|h\|_{(m)}.$$

Remark 2. *The inequality $\|fg\|_{(m)} \leq 2^m \|f\|_{(m)} \|g\|_{(m)}$ proved in Theorem 1(2) for the Banach space unitary algebra $\mathcal{C}^m([a,b],\mathbb{R})$, does not turn $\mathcal{C}^m([a,b],\mathbb{R})$ into a Banach algebra, unless $m = 0$. However, a very simple renorming does:*
$$|||f|||_{(m)} := 2^m \|f\|_{(m)}.$$
With this new equivalent norm on $\mathcal{C}^m([a,b],\mathbb{R})$, we can trivially obtained that
$$|||fg|||_{(m)} \leq |||f|||_{(m)} |||g|||_{(m)}$$
for all $fg, \in \mathcal{C}^m([a,b],\mathbb{R})$, turning $\mathcal{C}^m([a,b],\mathbb{R})$ into a Banach algebra. However, we will keep working with the norm given by (2).

Corollary 1. *A sequence $(f_n)_{n\in\mathbb{N}}$ is convergent in $\mathcal{C}^m([a,b],\mathbb{R})$ to $f \in \mathcal{C}^m([a,b],\mathbb{R})$ if and only if $\left(f_n^{(p)}\right)_{n\in\mathbb{N}}$ is convergent to $f^{(p)}$ in $\mathcal{C}([a,b],\mathbb{R})$ for every $0 \leq p \leq m$.*

The following corollary will be crucial towards accomplishing our results.

Corollary 2. *Let $\varepsilon \in (0,1)$ and let $(\theta_i)_{i\geq 0} \subseteq \mathcal{C}^2([a,b],\mathbb{R})$ be a bounded sequence. Then $\theta := \sum_{i=0}^{\infty} \varepsilon^i \theta_i \in \mathcal{C}^2([a,b],\mathbb{R})$. Furthermore, $\dot{\theta} = \sum_{i=0}^{\infty} \varepsilon^i \dot{\theta}_i$ in $\mathcal{C}^1([a,b],\mathbb{R})$ and $\ddot{\theta} = \sum_{i=0}^{\infty} \varepsilon^i \ddot{\theta}_i$ in $\mathcal{C}([a,b],\mathbb{R})$.*

Proof. We will prove first that $\left(\sum_{i=0}^{n} \varepsilon^i \theta_i\right)_{n\geq 0}$ is convergent in $\mathcal{C}^2([a,b],\mathbb{R})$. Indeed, take $M := \sup_{i\geq 0} \|\theta_i\|_{(2)}$. Notice that $\sum_{i=0}^{n} \left\|\varepsilon^i \theta_i\right\|_{(2)} \leq M \sum_{i=0}^{n} \varepsilon^i$ for every $n \geq 0$, so $\left(\sum_{i=0}^{n} \varepsilon^i \theta_i\right)_{n\geq 0}$ is absolutely convergent in $\mathcal{C}^2([a,b],\mathbb{R})$ because $0 < \varepsilon < 1$. Since $\mathcal{C}^2([a,b],\mathbb{R})$ is a Banach space, we conclude that $\left(\sum_{i=0}^{n} \varepsilon^i \theta_i\right)_{n\geq 0}$ is convergent in $\mathcal{C}^2([a,b],\mathbb{R})$. Finally, Corollary 1 assures that $\dot{\theta} = \sum_{i=0}^{\infty} \varepsilon^i \dot{\theta}_i$ in $\mathcal{C}^1([a,b],\mathbb{R})$ and $\ddot{\theta} = \sum_{i=0}^{\infty} \varepsilon^i \ddot{\theta}_i$ in $\mathcal{C}([a,b],\mathbb{R})$. □

We finalize this section with the following technical lemma of great importance towards the development of the upcoming sections.

Lemma 2. *If $h \in \mathcal{C}^2([a,b],\mathbb{R})$, then $H \in \mathcal{C}^2([a,b],\mathbb{R})$ and $\|H\|_{(2)} \leq \|h\|_{(2)} \max\{b-a, 1\}$, where $H(t) := \int_a^t h(s)ds$.*

Proof. In the first place,
$$|H(t)| = \left|\int_a^t h(s)ds\right| \leq \int_a^t |h(s)|ds \leq \|h\|_\infty (t-a) \leq \|h\|_\infty (b-a) \leq \|h\|_{(2)} (b-a)$$

for all $t \in [a,b]$, therefore $\|H\|_\infty \le \|h\|_\infty(b-a) \le \|h\|_{(2)}(b-a)$. Next, $\dot{H} = h$, thus $\|\dot{H}\|_\infty = \|h\|_\infty \le \|h\|_{(2)}$. Finally, $\ddot{H} = \dot{h}$, thus $\|\ddot{H}\|_\infty = \|\dot{h}\|_\infty \le \|h\|_{(2)}$. As a consequence,

$$\|H\|_{(2)} = \max\{\|H\|_\infty, \|\dot{H}\|_\infty, \|\ddot{H}\|_\infty\} \le \|h\|_{(2)} \max\{b-a, 1\}.$$

□

Observe that, in the settings of Lemma 2, H is, in fact, in $\mathcal{C}^3([a,b],\mathbb{R})$.

2.2. Perturbation to Second-Order Linear Ordinary Differential Equations with Constant Coefficients

We will deal with the solutions of a second-order linear ordinary differential equation (ODE) with constant coefficients whose independent term has been applied a perturbation to. Consider the following Initial Value Problem (IVP):

$$\begin{cases} \ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = g(t), \\ \text{with initial conditions } \theta(0) \text{ and } \dot{\theta}(0), \\ \theta \in \mathcal{C}^2([a,b],\mathbb{R}), \\ k, c \in \mathbb{R},\ c^2 - 4k < 0, \\ g \in \mathcal{C}^2([a,b],\mathbb{R}). \end{cases} \tag{5}$$

An example of perturbation of the IVP (5) is the following IVP:

$$\begin{cases} \ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = h(t), \\ \text{with initial conditions } \theta(0) \text{ and } \dot{\theta}(0), \end{cases} \tag{6}$$

where $h \in \mathcal{C}^2([a,b],\mathbb{R})$ and $\|g-h\|_{(2)} < \varepsilon$ for a certain $0 < \varepsilon < 1$.

Our objective is to estimate $\|\theta - \vartheta\|_{(2)}$, where θ and ϑ are the unique solutions of (5) and (6), respectively. Notice that $\theta - \vartheta$ is the unique solution of the IVP:

$$\begin{cases} \ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = g(t) - h(t), \\ \theta(0) = 0, \\ \dot{\theta}(0) = 0. \end{cases} \tag{7}$$

Therefore, everything is reduced to estimate the norm of the unique solution of the IVP:

$$\begin{cases} \ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = r(t), \\ \theta(0) = 0, \\ \dot{\theta}(0) = 0. \end{cases} \tag{8}$$

where $r \in \mathcal{C}^2([a,b],\mathbb{R})$.

Suppose now that $\{\theta_{\mathbf{c}}, \theta_{\mathbf{s}}\}$ is a basis of the vector space of solutions of the homogeneous equation $\ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = 0$. Let $W(\theta_{\mathbf{c}}(t), \theta_{\mathbf{s}}(t))$ be the Wronskian of $\{\theta_{\mathbf{c}}, \theta_{\mathbf{s}}\}$, i.e.,

$$W(\theta_{\mathbf{c}}(t), \theta_{\mathbf{s}}(t)) := \begin{vmatrix} \theta_{\mathbf{c}}(t) & \theta_{\mathbf{s}}(t) \\ \dot{\theta}_{\mathbf{c}}(t) & \dot{\theta}_{\mathbf{s}}(t) \end{vmatrix} = \theta_{\mathbf{c}}(t)\dot{\theta}_{\mathbf{s}}(t) - \dot{\theta}_{\mathbf{c}}(t)\theta_{\mathbf{s}}(t).$$

Recall that if $c_1, c_2 \in \mathcal{C}^2([a,b],\mathbb{R})$ satisfy

$$\dot{c}_1(t) = \frac{-\theta_{\mathbf{s}}(t)r(t)}{W(\theta_{\mathbf{c}}(t), \theta_{\mathbf{s}}(t))} \text{ and } \dot{c}_2(t) = \frac{\theta_{\mathbf{c}}(t)r(t)}{W(\theta_{\mathbf{c}}(t), \theta_{\mathbf{s}}(t))}, \tag{9}$$

then it is trivial that

$$\begin{cases} \dot{c}_1\theta_{\mathbf{c}} + \dot{c}_2\theta_{\mathbf{s}} = 0, \\ \dot{c}_1\dot{\theta}_{\mathbf{c}} + \dot{c}_2\dot{\theta}_{\mathbf{s}} = r. \end{cases}$$

Hence, $c_1\theta_{\mathbf{c}} + c_2\theta_{\mathbf{s}}$ is a particular solution of $\ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = r(t)$. Thus, the general solution of $\ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = r(t)$ is given by

$$\theta(t) := C_1\theta_{\mathbf{c}}(t) + C_2\theta_{\mathbf{s}}(t) + c_1(t)\theta_{\mathbf{c}}(t) + c_2(t)\theta_{\mathbf{s}}(t), \tag{10}$$

where $C_1, C_2 \in \mathbb{R}$ are constants. Let us find the unique solution of (8). Notice that

$$c_1(t) := \int_a^t \frac{-\theta_{\mathbf{s}}(s)r(s)}{W(\theta_{\mathbf{c}}(s), \theta_{\mathbf{s}}(s))} ds \text{ and } c_2(t) := \int_a^t \frac{\theta_{\mathbf{c}}(s)r(s)}{W(\theta_{\mathbf{c}}(s), \theta_{\mathbf{s}}(s))} ds$$

both satisfy that $c_1, c_2 \in \mathcal{C}^2([a,b], \mathbb{R})$ and verify Equation (9). Next, since $c^2 - 4k < 0$ by initial assumption, we can choose

$$\theta_{\mathbf{c}}(t) := e^{-\frac{c}{2}t} \cos\left(t\sqrt{k - \frac{c^2}{4}}\right) \text{ and } \theta_{\mathbf{s}}(t) := e^{-\frac{c}{2}t} \sin\left(t\sqrt{k - \frac{c^2}{4}}\right).$$

It is not hard to check that

$$\begin{cases} \theta_{\mathbf{c}}(0) = 1, \\ \dot{\theta}_{\mathbf{c}}(0) = -\frac{c}{2}, \end{cases} \text{ and } \begin{cases} \theta_{\mathbf{s}}(0) = 0, \\ \dot{\theta}_{\mathbf{s}}(0) = \sqrt{k - \frac{c^2}{4}}. \end{cases}$$

Then

$$\begin{cases} 0 = \theta(0) = C_1 + c_1(0), \\ 0 = \dot{\theta}(0) = C_1\left(-\frac{c}{2}\right) + C_2\sqrt{k - \frac{c^2}{4}} + \dot{c}_1(0) + c_1(0)\left(-\frac{c}{2}\right) + c_2(0)\sqrt{k - \frac{c^2}{4}}. \end{cases}$$

Observe that

$$C_1\left(-\frac{c}{2}\right) + c_1(0)\left(-\frac{c}{2}\right) = 0,$$

therefore, we obtain that

$$\begin{cases} C_1 = -c_1(0), \\ C_2 = \dfrac{-\dot{c}_1(0) - c_2(0)\sqrt{k - \frac{c^2}{4}}}{\sqrt{k - \frac{c^2}{4}}}. \end{cases}$$

Now, we are in the right position to estimate the norm of the unique solution of (8).

Theorem 2. *If $\theta \in \mathcal{C}^2([a,b], \mathbb{R})$ is the unique solution of the IVP (8), then*

$$\|\theta\|_{(2)} \leq \left(160\|\theta_{\mathbf{c}}\|_{(2)} \max\{b-a, 1\} + \frac{16}{\sqrt{k - \frac{c^2}{4}}}\|\theta_{\mathbf{s}}\|_{(2)}\right)\|\theta_{\mathbf{s}}\|_{(2)}\left\|\left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1}\right\|_{(2)}\|r\|_{(2)}.$$

Proof. Following Equation (10), we have that

$$\theta(t) = C_1\theta_{\mathbf{c}}(t) + C_2\theta_{\mathbf{s}}(t) + c_1(t)\theta_{\mathbf{c}}(t) + c_2(t)\theta_{\mathbf{s}}(t),$$

where

$$\begin{cases} C_1 = -c_1(0), \\ C_2 = \dfrac{-\dot{c}_1(0) - c_2(0)\sqrt{k - \frac{c^2}{4}}}{\sqrt{k - \frac{c^2}{4}}}, \end{cases}$$

and

$$c_1(t) := \int_a^t \frac{-\theta_{\mathbf{s}}(s)r(s)}{W(\theta_{\mathbf{c}}(s), \theta_{\mathbf{s}}(s))} ds \text{ and } c_2(t) := \int_a^t \frac{\theta_{\mathbf{c}}(s)r(s)}{W(\theta_{\mathbf{c}}(s), \theta_{\mathbf{s}}(s))} ds.$$

In view of Lemma 2 and by taking into consideration Theorem 1 together with Remark 1, we have that

$$\begin{aligned} |C_1| &= |c_1(0)| \leq \|c_1\|_\infty \leq \|c_1\|_{(2)} \leq \left\| \frac{-\theta_{\mathbf{s}} r}{W(\theta_{\mathbf{c}}, \theta_{\mathbf{s}})} \right\|_{(2)} \max\{b-a, 1\} \\ &\leq 16\|\theta_{\mathbf{s}}\|_{(2)} \|r\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \max\{b-a, 1\} \end{aligned}$$

and

$$\begin{aligned} |C_2| &\leq \frac{|\dot{c}_1(0)|}{\sqrt{k - \frac{c^2}{4}}} + |c_2(0)| \leq \frac{\|\dot{c}_1\|_\infty}{\sqrt{k - \frac{c^2}{4}}} + \|c_2\|_\infty \leq \frac{\|\dot{c}_1\|_{(2)}}{\sqrt{k - \frac{c^2}{4}}} + \|c_2\|_{(2)} \\ &\leq \frac{\left\| \frac{-\theta_{\mathbf{s}} r}{W(\theta_{\mathbf{c}}, \theta_{\mathbf{s}})} \right\|_{(2)}}{\sqrt{k - \frac{c^2}{4}}} + \left\| \frac{\theta_{\mathbf{c}} r}{W(\theta_{\mathbf{c}}, \theta_{\mathbf{s}})} \right\|_{(2)} \max\{b-a, 1\} \\ &\leq \frac{16}{\sqrt{k - \frac{c^2}{4}}} \|\theta_{\mathbf{s}}\|_{(2)} \|r\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \\ &+ 16\|\theta_{\mathbf{c}}\|_{(2)} \|r\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \max\{b-a, 1\}. \end{aligned}$$

Finally, by using again Lemma 2 and Theorem 1, we have that

$$\begin{aligned} \|\theta\|_{(2)} &\leq |C_1| \|\theta_{\mathbf{c}}\|_{(2)} + |C_2| \|\theta_{\mathbf{s}}\|_{(2)} + \|c_1\theta_{\mathbf{c}}\|_{(2)} + \|c_2\theta_{\mathbf{s}}\|_{(2)} \\ &\leq |C_1| \|\theta_{\mathbf{c}}\|_{(2)} + |C_2| \|\theta_{\mathbf{s}}\|_{(2)} + 4\|c_1\|_{(2)} \|\theta_{\mathbf{c}}\|_{(2)} + 4\|c_2\|_{(2)} \|\theta_{\mathbf{s}}\|_{(2)} \\ &\leq 16\|\theta_{\mathbf{s}}\|_{(2)} \|r\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \max\{b-a, 1\} \|\theta_{\mathbf{c}}\|_{(2)} \\ &+ \frac{16}{\sqrt{k - \frac{c^2}{4}}} \|\theta_{\mathbf{s}}\|_{(2)}^2 \|r\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \\ &+ 16\|\theta_{\mathbf{c}}\|_{(2)} \|r\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \max\{b-a, 1\} \|\theta_{\mathbf{s}}\|_{(2)} \\ &+ 64\|\theta_{\mathbf{s}}\|_{(2)} \|r\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \max\{b-a, 1\} \|\theta_{\mathbf{c}}\|_{(2)} \\ &+ 64\|\theta_{\mathbf{c}}\|_{(2)} \|r\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \max\{b-a, 1\} \|\theta_{\mathbf{s}}\|_{(2)} \\ &= \left(160\|\theta_{\mathbf{c}}\|_{(2)} \max\{b-a, 1\} + \frac{16}{\sqrt{k - \frac{c^2}{4}}} \|\theta_{\mathbf{s}}\|_{(2)}\right) \|\theta_{\mathbf{s}}\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)} \|r\|_{(2)}. \end{aligned}$$

□

Corollary 3. *If $\theta, \vartheta \in \mathcal{C}^2([a,b], \mathbb{R})$ are the unique solutions of the IVPs (5) and (6), respectively, then*

$$\|\theta - \vartheta\|_{(2)} \leq L\|g - h\|_{(2)},$$

where

$$L := \left(160\|\theta_{\mathbf{c}}\|_{(2)} \max\{b-a, 1\} + \frac{16}{\sqrt{k - \frac{c^2}{4}}} \|\theta_{\mathbf{s}}\|_{(2)}\right) \|\theta_{\mathbf{s}}\|_{(2)} \left\| \left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1} \right\|_{(2)}.$$

Proof. Like we previously mentioned, $\theta - \vartheta$ is the unique solution of the IVP (7). Thus, we only need to apply Theorem 2 □

Observe that $L \geq 0$ and, since L only depends on the homogeneous equation $\ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) = 0$, i.e., on the real numbers a, b, c, k and the fundamental set of solutions $\{\theta_{\mathbf{c}}, \theta_{\mathbf{s}}\}$, we can state that, under the settings of Corollary 3, if h approaches g, then ϑ approaches θ.

2.3. Functions of Polynomial Behavior

To accomplish our goals, we will strongly rely on the algebra of polynomials with several variables, $\mathbb{R}[x_1, \ldots, x_n]$, and on the algebra of formal series $\mathbb{R}[[x]]$ (see [20]). We recall the reader that

$$\begin{aligned}\mathbb{R}[x_1, \ldots, x_n] \quad := \quad & \Bigg\{ \sum_{k_1+\cdots+k_n=k,\ k=0}^{p} a_{k_1 \ldots k_n} x_1^{k_1} \cdots x_n^{k_n} : p \in \mathbb{N} \cup \{0\},\ \forall k \in \{0, \ldots, p\} \\ & \forall i \in \{1, \ldots, n\}\ k_i \in \mathbb{N} \cup \{0\},\ k_1 + \cdots + k_n = k,\ a_{k_1 \ldots k_n} \in \mathbb{R} \Bigg\}\end{aligned}$$

and

$$\mathbb{R}[[x]] := \left\{ \sum_{k=0}^{\infty} a_k x^k : \forall k \in \mathbb{N} \cup \{0\}\ a_k \in \mathbb{R} \right\}.$$

Notice that $\mathbb{R}[[x]]$ is a purely algebraic object, so no convergence is required for the series (in fact, no topology is given).

Definition 1. *A function $g \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R})$ is said to have polynomial behavior provided that there exists a sequence $(g_i)_{i\geq 0}$ of functions $g_i \in \mathcal{C}^2(\mathbb{R}^{i+1} \times \mathbb{R}^{i+1}, \mathbb{R})$ such that*

$$g\left(\sum_{i=0}^{\infty} \varepsilon^i a_i, \sum_{i=0}^{\infty} \varepsilon^i b_i\right) = \sum_{i=0}^{\infty} \varepsilon^i g_i(a_0, \ldots, a_i, b_0, \ldots, b_i)$$

for every bounded sequences $(a_i)_{i\geq 0}, (b_i)_{i\geq 0}$ of real numbers and every $0 < \varepsilon < 1$.

As expected, polynomials in two variables have polynomial behavior.

Proposition 1. *Every $p(x, y) \in \mathbb{R}[x, y]$ has polynomial behavior.*

Proof. Let

$$p(x, y) = \sum_{i+j=n,\ n=0}^{k} c_{ij} x^i y^j.$$

If $\alpha(z) = \sum_{i=0}^{\infty} \alpha_i z^i \in \mathbb{R}[[z]]$ and $\beta(z) = \sum_{i=0}^{\infty} \beta_i z^i \in \mathbb{R}[[z]]$ are formal series, then $\delta(z) := p(\alpha(z), \beta(z)) \in \mathbb{R}[[z]]$ is another formal series because $\mathbb{R}[[z]]$ is an algebra. Then we can write $\delta(z) = \sum_{i=0}^{\infty} \delta_i z^i$ and, since the coefficients c_{ij} of $p(x, y)$ are fixed, we have that δ_i is a function of $\alpha_0, \ldots, \alpha_i, \beta_0, \ldots, \beta_i$, i.e.,

$$\delta_i = \delta_i(\alpha_0, \ldots, \alpha_i, \beta_0, \ldots, \beta_i)$$

for every $i \geq 0$. Finally, it only suffices to define

$$\begin{aligned} g_i(x_0, \ldots, x_i, y_0, \ldots, y_i) \quad & := \quad \delta_i(x_0, \ldots, x_i, y_0, \ldots, y_i) \\ & \in \quad \mathbb{R}[x_0, \ldots, x_i, y_0, \ldots, y_i] \\ & \subseteq \quad \mathcal{C}^2\left(\mathbb{R}^{i+1} \times \mathbb{R}^{i+1}, \mathbb{R}\right) \end{aligned}$$

for every $i \geq 0$. Indeed, if $0 < \varepsilon < 1$ and $(a_i)_{i\geq 0}, (b_i)_{i\geq 0}$ are bounded sequences of real numbers, then

$$p\left(\sum_{i=0}^{\infty}\varepsilon^i a_i, \sum_{i=0}^{\infty}\varepsilon^i b_i\right) = \sum_{i=0}^{\infty}\delta_i(a_0,\ldots,a_i,b_0,\ldots,b_i)\varepsilon^i = \sum_{i=0}^{\infty}\varepsilon^i g_i(a_0,\ldots,a_i,b_0,\ldots,b_i).$$

□

3. Results

After the application of the proposed methodology in this work, we obtain as a result the mathematical formalism which yields to the validity of the SEPM. In particular, this method will be applied to a simplified version of Equation (1) by introducing a perturbation ($0 < \varepsilon < 1$) and a function with polynomial behavior (g). We will deal with Equation (11):

$$\begin{cases} \ddot{\theta} + c\dot{\theta} + k\theta = \varepsilon g(\theta, \dot{\theta}), \\ \text{with initial conditions } \theta(0) \text{ and } \dot{\theta}(0), \\ \theta \in \mathcal{C}^2([a,b],\mathbb{R}), \\ \varepsilon \in (0,1), \\ k, c \in \mathbb{R},\ c^2 - 4k < 0, \\ g \in \mathcal{C}^2(\mathbb{R}^2,\mathbb{R}) \text{ with polynomial behavior.} \end{cases} \tag{11}$$

Algorithm of Application for the Straightforward Expansion Perturbation Method

Since $g \in \mathcal{C}^2(\mathbb{R}^2,\mathbb{R})$ is of polynomial behavior, in view of Definition 1, there exists a sequence $(g_i)_{i\geq 0}$ of functions $g_i \in \mathcal{C}^2(\mathbb{R}^{i+1}\times\mathbb{R}^{i+1},\mathbb{R})$ such that

$$g\left(\sum_{i=0}^{\infty}\varepsilon^i a_i, \sum_{i=0}^{\infty}\varepsilon^i b_i\right) = \sum_{i=0}^{\infty}\varepsilon^i g_i(a_0,\ldots,a_i,b_0,\ldots,b_i)$$

for every bounded sequences $(a_i)_{i\geq 0}, (b_i)_{i\geq 0}$ of real numbers and every $0 < \varepsilon < 1$.

By relying on the sequence $(g_i)_{i\in\mathbb{N}}$, we will follow an inductive process:

1. θ_0 is the unique solution of the second-order linear IVP
$$\begin{cases} \ddot{\theta}_0 + c\dot{\theta}_0 + k\theta_0 = 0, \\ \theta_0(0) = \theta(0), \\ \dot{\theta}_0(0) = \dot{\theta}(0). \end{cases}$$
2. θ_1 is the unique solution of the second-order linear IVP
$$\begin{cases} \ddot{\theta}_1 + c\dot{\theta}_1 + k\theta_1 = g_0(\theta_0,\dot{\theta}_0), \\ \theta_1(0) = 0, \\ \dot{\theta}_1(0) = 0. \end{cases}$$
3. θ_2 is the unique solution of the second-order linear IVP
$$\begin{cases} \ddot{\theta}_2 + c\dot{\theta}_2 + k\theta_2 = g_1(\theta_0,\theta_1,\dot{\theta}_0,\dot{\theta}_1), \\ \theta_2(0) = 0, \\ \dot{\theta}_2(0) = 0. \end{cases}$$
4. And so on.

We can summarize this inductive process as follows: For $i \geq 0$, θ_i is the unique solution of the IVP

$$i = 0 \Rightarrow \begin{cases} \ddot{\theta}_0 + c\dot{\theta}_0 + k\theta_0 = 0, \\ \theta_0(0) = \theta(0), \\ \dot{\theta}_0(0) = \dot{\theta}(0), \end{cases} \qquad i \geq 1 \Rightarrow \begin{cases} \ddot{\theta}_i + c\dot{\theta}_i + k\theta_i = g_{i-1}(\theta_0,\ldots,\theta_{i-1},\dot{\theta}_0,\ldots,\dot{\theta}_{i-1}), \\ \theta_i(0) = 0, \\ \dot{\theta}_i(0) = 0. \end{cases} \tag{12}$$

Notice that

$$\theta_0(t) = \theta(0)e^{-\frac{c}{2}t}\cos\left(t\sqrt{k-\frac{c^2}{4}}\right) + \frac{\dot{\theta}(0)+\theta(0)\frac{c}{2}}{\sqrt{k-\frac{c^2}{4}}}e^{-\frac{c}{2}t}\sin\left(t\sqrt{k-\frac{c^2}{4}}\right)$$

for all $t \in [a,b]$ and it is infinitely differentiable on $[a,b]$. Therefore,

$$g_{i-1}(\theta_0,\ldots,\theta_{i-1},\dot{\theta}_0,\ldots,\dot{\theta}_{i-1}) \in \mathcal{C}^2([a,b],\mathbb{R})$$

and, according to Theorem 2,

$$\|\theta_i\|_{(2)} \leq L\big\|g_{i-1}(\theta_0,\ldots,\theta_{i-1},\dot{\theta}_0,\ldots,\dot{\theta}_{i-1})\big\|_{(2)} \tag{13}$$

for all $i \geq 1$.

Theorem 3. *For every $i \geq 0$, let $\theta_i \in \mathcal{C}^2([a,b],\mathbb{R})$ be a solution of (12). If $(\theta_i)_{i\geq 0}$ is bounded in $\mathcal{C}^2([a,b],\mathbb{R})$, then $\theta := \sum_{i=0}^{\infty} \varepsilon^i \theta_i \in \mathcal{C}^2([a,b],\mathbb{R})$ is the unique solution of (11).*

Proof. By bearing in mind Corollary 2, $\theta := \sum_{i=0}^{\infty} \varepsilon^i \theta_i \in \mathcal{C}^2([a,b],\mathbb{R})$. Furthermore, $\dot{\theta} = \sum_{i=0}^{\infty} \varepsilon^i \dot{\theta}_i$ in $\mathcal{C}^1([a,b],\mathbb{R})$ and $\ddot{\theta} = \sum_{i=0}^{\infty} \varepsilon^i \ddot{\theta}_i$ in $\mathcal{C}([a,b],\mathbb{R})$. Finally, if we take into consideration that, for every $t \in [a,b]$, $(\theta_i(t))_{i\geq 0}$ and $\big(\dot{\theta}_i(t)\big)_{i\geq 0}$ are bounded sequences of real numbers since

$$\begin{aligned}
\sup\{|\theta_i(t)|, |\dot{\theta}_i(t)| : i \geq 0, t \in [a,b]\} &\leq \sup\{\|\theta_i\|_\infty, \|\dot{\theta}_i\|_\infty : i \geq 0\} \\
&\leq \sup\Big\{\|\theta_i\|_{(2)} : i \geq 0\Big\} \\
&< \infty,
\end{aligned}$$

then we have that

$$\begin{aligned}
\ddot{\theta}(t) + c\dot{\theta}(t) + k\theta(t) &= \sum_{i=0}^{\infty} \varepsilon^i \ddot{\theta}_i(t) + c\sum_{i=0}^{\infty} \varepsilon^i \dot{\theta}_i(t) + k\sum_{i=0}^{\infty} \varepsilon^i \theta_i(t) \\
&= \sum_{i=0}^{\infty} \varepsilon^i \big(\ddot{\theta}_i(t) + c\dot{\theta}_i(t) + k\theta_i(t)\big) = \sum_{i=1}^{\infty} \varepsilon^i \big(\ddot{\theta}_i(t) + c\dot{\theta}_i(t) + k\theta_i(t)\big) \\
&= \sum_{i=1}^{\infty} \varepsilon^i g_{i-1}\big(\theta_0(t),\ldots,\theta_{i-1}(t),\dot{\theta}_0(t),\ldots,\dot{\theta}_{i-1}(t)\big) \\
&= \varepsilon\sum_{i=1}^{\infty} \varepsilon^{i-1} g_{i-1}\big(\theta_0(t),\ldots,\theta_{i-1}(t),\dot{\theta}_0(t),\ldots,\dot{\theta}_{i-1}(t)\big) \\
&= \varepsilon\sum_{i=0}^{\infty} \varepsilon^{i} g_{i}\big(\theta_0(t),\ldots,\theta_{i}(t),\dot{\theta}_0(t),\ldots,\dot{\theta}_{i}(t)\big) \\
&= \varepsilon g\left(\sum_{i=0}^{\infty} \varepsilon^i \theta_i(t), \sum_{i=0}^{\infty} \varepsilon^i \dot{\theta}_i(t)\right) = \varepsilon g\big(\theta(t),\dot{\theta}(t)\big).
\end{aligned}$$

□

Please note that by bearing in mind Equation (13), if $\big(g_i(\theta_0,\ldots,\theta_i,\dot{\theta}_0,\ldots,\dot{\theta}_i)\big)_{i\geq 0}$ is a bounded sequence in $\mathcal{C}^2([a,b],\mathbb{R})$, then so is $(\theta_i)_{i\geq 0}$.

Sometimes, for computation reasons, (12) is perturbated to obtain an easier IVP, such as

$$i = 0 \Rightarrow \begin{cases} \ddot{\theta}_0 + c\dot{\theta}_0 + k\theta_0 = 0, \\ \theta_0(0) = \theta(0), \\ \dot{\theta}_0(0) = \dot{\theta}(0), \end{cases} ; \quad i \geq 1 \Rightarrow \begin{cases} \ddot{\theta}_i + c\dot{\theta}_i + k\theta_i = h_{i-1}, \\ \theta_i(0) = 0, \\ \dot{\theta}_i(0) = 0. \end{cases} \tag{14}$$

where $h_i \in \mathcal{C}^2([a,b],\mathbb{R})$ is closed to $g_i(\theta_0,\ldots,\theta_i,\dot{\theta}_0,\ldots,\dot{\theta}_i)$ in the norm of $\mathcal{C}^2([a,b],\mathbb{R})$. In other words, we require that

$$\big\|h_i - g_i(\theta_0,\ldots,\theta_i,\dot{\theta}_0,\ldots,\dot{\theta}_i)\big\|_{(2)}$$

be sufficiently small for all $i \geq 0$ in such a way that

$$\sup\left\{\left\|h_i - g_i(\theta_0, \ldots, \theta_i, \dot{\theta}_0, \ldots, \dot{\theta}_i)\right\|_{(2)} : i \geq 0\right\}$$

will also be small.

In this situation, under certain circumstances, we can assure that the solutions of the perturbated IVP approach the solution of the original IVP (11).

Corollary 4. *For every $i \geq 0$, let $\vartheta_i \in \mathcal{C}^2([a,b],\mathbb{R})$ be a solution of (14). If $(\vartheta_i)_{i\geq 0}$ is bounded in $\mathcal{C}^2([a,b],\mathbb{R})$, then $\vartheta := \sum_{i=0}^{\infty} \varepsilon^i \theta_i \in \mathcal{C}^2([a,b],\mathbb{R})$ satisfies that*

$$\|\vartheta - \theta\|_{(2)} \leq \frac{\varepsilon}{1-\varepsilon} L \sup\left\{\left\|h_i - g_i(\theta_0, \ldots, \theta_i, \dot{\theta}_0, \ldots, \dot{\theta}_i)\right\|_{(2)} : i \geq 0\right\},$$

where $\theta \in \mathcal{C}^2([a,b],\mathbb{R})$ is the unique solution of (11).

Proof. By relying on Theorem 2 and on the fact that $\theta_0 = \vartheta_0$, it only suffices to observe that

$$\begin{aligned}
\|\vartheta - \theta\|_{(2)} &= \left\|\sum_{i=0}^{\infty} \varepsilon^i \vartheta_i - \sum_{i=0}^{\infty} \varepsilon^i \theta_i\right\|_{(2)} \leq \sum_{i=0}^{\infty} \varepsilon^i \|\vartheta_i - \theta_i\|_{(2)} = \sum_{i=1}^{\infty} \varepsilon^i \|\vartheta_i - \theta_i\|_{(2)} \\
&\leq \sum_{i=1}^{\infty} \varepsilon^i L \left\|h_{i-1} - g_{i-1}(\theta_0, \ldots, \theta_{i-1}, \dot{\theta}_0, \ldots, \dot{\theta}_{i-1})\right\|_{(2)} \\
&= \varepsilon \sum_{i=1}^{\infty} \varepsilon^{i-1} L \left\|h_{i-1} - g_{i-1}(\theta_0, \ldots, \theta_{i-1}, \dot{\theta}_0, \ldots, \dot{\theta}_{i-1})\right\|_{(2)} \\
&= \varepsilon \sum_{i=0}^{\infty} \varepsilon^{i} L \left\|h_{i} - g_{i}(\theta_0, \ldots, \theta_{i}, \dot{\theta}_0, \ldots, \dot{\theta}_{i})\right\|_{(2)} \\
&\leq \varepsilon L \sup\left\{\left\|h_i - g_i(\theta_0, \ldots, \theta_i, \dot{\theta}_0, \ldots, \dot{\theta}_i)\right\|_{(2)} : i \geq 0\right\} \sum_{i=0}^{\infty} \varepsilon^i \\
&= \frac{\varepsilon}{1-\varepsilon} L \sup\left\{\left\|h_i - g_i(\theta_0, \ldots, \theta_i, \dot{\theta}_0, \ldots, \dot{\theta}_i)\right\|_{(2)} : i \geq 0\right\}.
\end{aligned}$$

□

4. Discussion

In this section, we will validate the formalism proposed in the previous section by means of the pendulum example for a fixed bounded time interval.

Validation Example (Pendulum) for a Fixed Bounded Time Interval

In this subsection, we will apply our theorems to the weakly nonlinear vibration example of the famous pendulum. This problem can be expressed by Equation (15) by using Taylor approximations and can be found in [7] (see also [6,18]):

$$\begin{cases} \ddot{\theta} + \theta = \varepsilon \frac{1}{6}\theta^3, \\ \text{with initial conditions } \theta(0) \text{ and } \dot{\theta}(0), \\ \theta \in \mathcal{C}^2([a,b],\mathbb{R}), \\ \varepsilon \in (0,1). \end{cases} \tag{15}$$

In this situation, $c = 0$, $k = 1$ and $g(x,y) = \frac{1}{6}x^3 \in \mathbb{R}[x,y] \subseteq \mathcal{C}^2(\mathbb{R}^2,\mathbb{R})$ is a polynomial in two variables so it is of polynomial behavior in view of Proposition 1. In fact:

$$g\left(\sum_{i=0} \varepsilon^i a_i, \sum_{i=0} \varepsilon^i b_i\right) = \frac{1}{6}\left(\sum_{i=0} \varepsilon^i a_i\right)^3 = \frac{1}{6}\left(a_0 + \varepsilon a_1 + \varepsilon^2 a_2 + \cdots\right)^3 = \frac{1}{6}a_0^3 + \frac{1}{2}a_0^2 a_1 \varepsilon + \cdots,$$

therefore

$$\begin{cases} g_0(x,y) = \dfrac{1}{6}x^3, \\ \\ g_1(x,y) = \dfrac{1}{2}x^2 y, \\ \vdots \end{cases}$$

The SEPM applied to (15) follows the next steps:

1. θ_0 is the unique solution of the IVP $\begin{cases} \ddot{\theta}_0 + \theta_0 = 0, \\ \theta_0(0) = \theta(0), \\ \dot{\theta}_0(0) = \dot{\theta}(0). \end{cases}$
2. θ_1 is the unique solution of the IVP $\begin{cases} \ddot{\theta}_1 + \theta_1 = \frac{1}{6}\theta_0^3, \\ \theta_1(0) = 0, \\ \dot{\theta}_1(0) = 0. \end{cases}$
3. θ_2 is the unique solution of the IVP $\begin{cases} \ddot{\theta}_2 + \theta_2 = \frac{1}{2}\theta_0^2\theta_1, \\ \theta_2(0) = 0, \\ \dot{\theta}_2(0) = 0. \end{cases}$
4. And so on.

Notice that θ_0 can be easily computed, in fact,

$$\theta_0(t) = \theta(0)\cos(t) + \dot{\theta}(0)\sin(t)$$

for all $t \in [a,b]$. As a consequence, by relying on the Hölder's inequality (see [21]),

$$\|\theta_0\|_{(2)} \leq \sqrt{|\theta(0)|^2 + \left|\dot{\theta}(0)\right|^2}. \tag{16}$$

Since, in the case of (15), $\theta_{\mathbf{c}}(t) := \cos(t)$ and $\theta_{\mathbf{s}}(t) := \sin(t)$, the constant L can also be bounded above by

$$\begin{aligned} L \quad &:= \quad \left(160\|\theta_{\mathbf{c}}\|_{(2)} \max\{b-a,1\} + \frac{16}{\sqrt{k - \frac{c^2}{4}}}\|\theta_{\mathbf{s}}\|_{(2)}\right)\|\theta_{\mathbf{s}}\|_{(2)}\left\|\left(\theta_{\mathbf{c}}\dot{\theta}_{\mathbf{s}} - \theta_{\mathbf{s}}\dot{\theta}_{\mathbf{c}}\right)^{-1}\right\|_{(2)} \\ &\leq \quad 160\max\{b-a,1\} + 16. \end{aligned}$$

In view of Theorem 3, the SEPM leads to a correct solution of (15) if we can control the norms $\|\theta_1\|_{(2)}, \|\theta_2\|_{(2)}, \ldots$

Next, keep in mind that Theorem 2 together with Theorem 1(2) applied to $\frac{1}{6}\theta_0^3$ and $\frac{1}{2}\theta_0^2\theta_1$ yield

$$\begin{cases} \|\theta_1\|_{(2)} \leq L\left\|\dfrac{1}{6}\theta_0^3\right\|_{(2)} \leq \dfrac{8}{3}L\|\theta_0\|_{(2)}^3, \\ \\ \|\theta_2\|_{(2)} \leq L\left\|\dfrac{1}{2}\theta_0^2\theta_1\right\|_{(2)} \leq L8\|\theta_0\|_{(2)}^2\|\theta_1\|_{(2)} \leq \dfrac{64}{3}L^2\|\theta_0\|_{(2)}^5, \\ \vdots \end{cases}$$

At this stage, in order to control the norms $\|\theta_1\|_{(2)}, \|\theta_2\|_{(2)}, \ldots$, we need $\|\theta_0\|_{(2)}$ to be sufficiently small in such a way that $\|\theta_0\|_{(2)}^3, \|\theta_0\|_{(2)}^5, \ldots$ can counteract the growth of the constants $\frac{8}{3}L, \frac{64}{3}L^2, \ldots$. In view of Equation (16), by choosing the initial conditions, $\theta(0)$ and $\dot{\theta}(0)$, sufficiently small, the SEPM leads to a correct solution of (15) by bearing in mind our Theorem 3.

5. Conclusions

The main conclusion that we infer from our work is that the SEPM works in certain cases (see Theorem 3 and Corollary 4), contrary to what is stated in ([7], Subsection 3.5.2). In ([7], Subsection 3.5.2), the SEPM is neglected as a perturbation method simply because it does not yield to correct solutions in only one example. No mathematical proof is provided to show that the SEPM never works. Therefore, the SEPM should have never been disregarded as a perturbation method, at least, until a mathematical proof states so. In fact, here in this work, we demonstrate that the SEPM cannot be rejected as a perturbation method because we provide a rigorous mathematical proof that shows the validity of the SEPM by leading to a correct solution of the nonlinear vibration in the case where the sequence of expansion terms $(\theta_i)_{i\geq 0}$ is uniformly bounded. As an application of our theorems, if we look at Section 4, then we can see that if we choose the initial conditions, $\theta(0)$ and $\dot{\theta}(0)$, sufficiently small as well as the length of the interval $[a,b]$ around 0, then the SEPM successfully converges to the correct solution of (15).

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Abbreviations

The following abbreviations are used in this manuscript:

SEPM	Straightforward Expansion Perturbation Method
HAM	Homotopy Analysis Method
ODE	Ordinary Differential Equation
IVP	Initial Value Problem

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