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Abstract. Five types of constructions are introduced for non-prioritized belief revision, i.e., belief revision in which the input sentence is not always accepted. These constructions include generalizations of entrenchment-based and sphere-based revision. Axiomatic characterizations are provided, and close interconnections are shown to hold between the different constructions.

§1. Introduction. In conventional, AGM-style belief revision [1] the input sentence is always accepted. In non-prioritized belief revision, this requirement is relaxed. Most models of non-prioritized revision operate on belief bases [4, 9, 11, 16, 17, 19]. A couple of proposals for belief sets (theories) have been put forward [3, 15], but a full treatment including axiomatic characterizations of alternative constructions has been lacking. The purpose of the present paper is to fill this gap in the literature by presenting a wide array of alternative constructions and providing axiomatic characterizations that clarify their properties and their interconnections. The intuitive importance of this exercise should be evident; actual agents differ from the idealized agents of conventional belief change theory in often rejecting inputs that are offerred for revision.

We will assume a language $\mathscr{L}$ that is closed under truth-functional operations and a consequence operator Cn for $\mathscr{L}$. Cn satisfies the standard Tarskian properties, namely inclusion ($A \subseteq \text{Cn}(A)$), monotony (if $A \subseteq B$, then $\text{Cn}(A) \subseteq \text{Cn}(B)$), and iteration ($\text{Cn}(A) = \text{Cn}(\text{Cn}(A))$). It is supraclassical and compact, and satisfies deduction (if $\beta \in \text{Cn}(A \cup \{\alpha\})$, then $(\alpha \rightarrow \beta) \in \text{Cn}(A)$). $A \vdash \alpha$ will be used as an alternative notation for $\alpha \in \text{Cn}(A)$ and $\text{Cn}(\alpha)$ for $\text{Cn}(\{\alpha\})$. Upper-case letters denote subsets of $\mathscr{L}$. K denotes a set such that $K = \text{Cn}(K) \subseteq \mathscr{L}$. Lower-case Greek letters denote elements of $\mathscr{L}$. $\top$ is an arbitrary tautology and $\bot$ an arbitrary contradiction. $\|\alpha\|$ is the set of inclusion-maximal consistent subsets of $\mathscr{L}$ that have α as an element, and $\|A\|$ the set of inclusion-maximal consistent subsets of $\mathscr{L}$ that include A. Furthermore, for any maximal consistent subset Y of $\mathscr{L}$, $\text{Th}(Y) = \{\alpha : Y \subseteq \|\alpha\|\}$. We use $\circ$ to denote (conventional or non-prioritized) revision operators in general, and $*$ to

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denote AGM revision. Furthermore, $+$ denotes AGM expansion, i.e., $K+\alpha = \mathrm{Cn}(K \cup \{\alpha\})$.

In Section 2, postulates for non-prioritized belief revision are proposed. In Section 3, five types of constructions for non-prioritized revision are proposed, and in Section 4 they are axiomatically characterized.

§2. Postulates and their interrelations. The following postulates are used in AGM theory to characterize the operators of revision [1]:

If $K = \mathrm{Cn}(K)$, then $K \circ \alpha = \mathrm{Cn}(K \circ \alpha)$ (*closure*).
$\alpha \in K \circ \alpha$ (*success*).
$K \circ \alpha \subseteq K+\alpha$ (*inclusion*).
If $\neg\alpha \notin K$, then $K \circ \alpha = K+\alpha$ (*vacuity*).
If α is consistent, then so is $K \circ \alpha$ (*consistency*).
If $\alpha \leftrightarrow \beta \in \mathrm{Cn}(\emptyset)$, then $K \circ \alpha = K \circ \beta$ (*extensionality*).
$K \circ (\alpha \wedge \beta) \subseteq (K \circ \alpha)+\beta$ (*superexpansion*).
If $K \circ \alpha \nvdash \neg\beta$, then $(K \circ \alpha)+\beta \subseteq K \circ (\alpha \wedge \beta)$ (*subexpansion*).

The first six of these are the basic Gärdenfors postulates and the last two the supplementary Gärdenfors postulates. The latter are closely connected with the following three postulates:

$(K \circ \alpha) \cap (K \circ \beta) \subseteq K \circ (\alpha \vee \beta)$ (*disjunctive overlap*).
If $K \circ (\alpha \vee \beta) \nvdash \neg\alpha$, then $K \circ (\alpha \vee \beta) \subseteq K \circ \alpha$ (*disjunctive inclusion*).
Either $K \circ (\alpha \vee \beta) = K \circ \alpha$, $K \circ (\alpha \vee \beta) = K \circ \beta$, or $K \circ (\alpha \vee \beta) = (K \circ \alpha) \cap (K \circ \beta)$ (*disjunctive factoring*).

In the presence of the six basic postulates, superexpansion is equivalent with disjunctive overlap, subexpansion is equivalent with disjunctive inclusion, and disjunctive factoring holds if and only if both disjunctive overlap and disjunctive inclusion hold [1, 5].

Our general approach will be to give up the success postulate while retaining as much as possible of the other Gärdenfors postulates. The following are useful weakenings of the success postulate:

Either $K \circ \alpha = K$ or $\alpha \in K \circ \alpha$ (*relative success*).
Either $\alpha \in K \circ \alpha$ or $\neg\alpha \in K \circ \neg\alpha$ (*disjunctive success*).
If $\alpha \in K \circ \alpha$ and $\vdash \alpha \rightarrow \beta$, then $\beta \in K \circ \beta$ (*strict improvement*).
If $\beta \in K \circ \alpha$, then $\beta \in K \circ \beta$ (*regularity*).
If $\neg\beta \notin K \circ \alpha$, then $\beta \in K \circ \beta$ (*strong regularity*).
If $\alpha \vee \beta \in K \circ (\alpha \vee \beta)$, then either $\alpha \in K \circ \alpha$ or $\beta \in K \circ \beta$ (*disjunctive distribution*).

Intuitively, we may call a sentence α credible, relative to a belief set K and a revision operator $\circ$ for K, if and only if $\alpha \in K \circ \alpha$. Under this interpretation, strict improvement says that credibility is inherited by logically weaker sentences, regularity that the resulting new belief state contains only credible sentences and strong regularity that it contains all sentences with incredible negations.

The consistency postulate of AGM requires $K \circ \alpha$ to be consistent only when α is consistent. This is because the success postulate ($\alpha \in K \circ \alpha$) is given higher priority than consistency. In non-prioritized belief revision, success is relaxed, and it is therefore natural to consider the following stronger consistency postulate:

$K\circ\alpha$ is consistent (*strong consistency*) [10].

We will also have use for some other properties related to consistency:

If K is consistent, then $K\circ\alpha$ is consistent (*consistency preservation*) [15].
If both K and α are consistent, then so is $K\circ\alpha$ (*weak consistency preservation*) [13].
If $K \nsubseteq K\circ\alpha$, then $K \cup (K\circ\alpha) \vdash \bot$ (*consistent expansion*) [3].

Consistent expansion follows from vacuity and relative success.

Subexpansion is a fairly plausible property for conventional (prioritized) belief revision, but it is much less so for non-prioritized revision. This can be seen from examples such that $\beta \notin K$, $\neg\beta \notin K$, $\alpha \notin K\circ\alpha$, and $\alpha \wedge \beta \notin K\circ(\alpha \wedge \beta)$ (for instance, let α denote that there is a living dinosaur in Australia and β that there is a living tree in Australia of a species that existed at the time of the dinosaurs). Such a pattern cannot simultaneously satisfy subexpansion, relative success, and closure (from $\alpha \notin K\circ\alpha$ and relative success follows $K\circ\alpha = K$. Since $\neg\beta \notin K$ we then have $\neg\beta \notin K\circ\alpha$, and due to closure, $K\circ\alpha \nvdash \neg\beta$, so that by subexpansion $\beta \in K\circ(\alpha \wedge \beta)$. Since $\beta \notin K$ it follows from relative success that $\alpha \wedge \beta \in K\circ(\alpha \wedge \beta)$, contrary to the conditions). This problem can be avoided if we replace subexpansion by the following variant, that is equivalent with subexpansion whenever success holds.

If $\alpha \in K\circ\alpha$ and $K\circ\alpha \nvdash \neg\beta$, then $(K\circ\alpha)+\beta \subseteq K\circ(\alpha \wedge \beta)$ (*guarded subexpansion*).

§3. **Constructions.** In this section, we are going to introduce five constructions of non-prioritized revision on belief sets. The first of these is the most general one. Its basic assumption is simply that some inputs are accepted, whereas others are not. Those that are accepted form the set $\mathscr{C}$ of credible sentences. This will be called credibility-limited revision since the acceptance of inputs is limited to sentences with sufficient credibility.

DEFINITION 1. *Let K be a logically closed set of sentences. The operation $\circ$ on K is a* credibility-limited revision *on K if and only if there is an AGM revision $*$ on K (satisfying the six basic postulates) and a set $\mathscr{C}$ of sentences such that for all sentences α:*

$$K\circ\alpha = \begin{cases} K*\alpha & \text{if } \alpha \in \mathscr{C} \\ K & \text{otherwise} \end{cases}$$

The following are some plausible conditions on $\mathscr{C}$.

If $\vdash \alpha \leftrightarrow \beta$ and $\alpha \in \mathscr{C}$, then $\beta \in \mathscr{C}$ (*closure under logical equivalence*).
If $\alpha \in \mathscr{C}$ then $\mathrm{Cn}(\{\alpha\}) \subseteq \mathscr{C}$ (*single sentence closure*) Section 1.10 of [12].
If $\alpha \vee \beta \in \mathscr{C}$, then either $\alpha \in \mathscr{C}$ or $\beta \in \mathscr{C}$ (*disjunctive completeness*).
$\alpha \in \mathscr{C}$ or $\neg\alpha \in \mathscr{C}$ (*negation completeness*).
If $\alpha \in \mathscr{C}$, then $\alpha \nvdash \bot$ (*element consistency*).
If $K \nvdash \alpha$, then $\neg\alpha \in \mathscr{C}$ (*expansive credibility*).
If $\alpha \in \mathscr{C}$, then $K\circ\alpha \subseteq \mathscr{C}$.

The generalization of single sentence closure to full logical closure ($\mathrm{Cn}(\mathscr{C}) \subseteq \mathscr{C}$) is patently unreasonable; each of α and β may be credible without $\alpha \wedge \beta$ being so (for an obvious example, let $\beta = \neg\alpha$).

Our second construction is a modified version of David Makinson's screened revision [15]. Makinson made use of a set A of potential core beliefs that are immune to revision. The belief set K should be revised by the input sentence α if α is consistent with the set $A \cap K$ of actual core beliefs, otherwise not. In our version, we have replaced $A \cap K$ by a set A of core beliefs. For expository convenience we will present this construction as a special case of Definition 1, with the set A of core beliefs as the determinant of whether or not a sentence α is a member of the set $\mathscr{C}$ of credible sentences.

Definition 2. *Let $\circ$ be a credibility-limited revision operator for K, based on $*$ and $\mathscr{C}$. Then it is*

1. *an operator of* core beliefs revision *if and only if there is a set $A \subseteq \mathscr{L}$ such that $\alpha \in \mathscr{C}$ iff $A \nvdash \neg\alpha$.*
2. *an operator of* consistent core beliefs revision *if and only if there is a consistent set $A \subseteq \mathscr{L}$ such that $\alpha \in \mathscr{C}$ iff $A \nvdash \neg\alpha$.*
3. *an operator of* endorsed core beliefs revision *if and only if there is a set $A \subseteq K$ such that $\alpha \in \mathscr{C}$ iff $A \nvdash \neg\alpha$.*

If K is consistent, then all endorsed core beliefs revisions are also consistent core beliefs revisions.

Our third construction is a modification of epistemic entrenchment. In the standard account of entrenchment, due to Peter Gärdenfors [5, 6], revision is based on a relation $\leq$ that corresponds to usefulness in inquiry or deliberation, or to the amount of epistemic value.

Definition 3. [5, 6] *A standard entrenchment ordering for a belief set K is a relation $\leq$ on $\mathscr{L}$ that satisfies:*

(EE1) *If $\alpha \leq \beta$ and $\beta \leq \delta$, then $\alpha \leq \delta$* (transitivity).
(EE2) *If $\alpha \vdash \beta$, then $\alpha \leq \beta$* (dominance).
(EE3) *Either $\alpha \leq \alpha \wedge \beta$ or $\beta \leq \alpha \wedge \beta$* (conjunctiveness).
(EE4) *If the belief set K is consistent, then $\alpha \notin K$ if and only if $\alpha \leq \beta$ for all β* (minimality).
(EE5) *If $\beta \leq \alpha$ for all β, then $\vdash \alpha$* (maximality).

$<$ is the strict and $\equiv$ the symmetric part of $\leq$. A sentence α is maximally entrenched according to $\leq$ if and only if $\delta \leq \alpha$ for all sentences δ.

Entrenchment-based revision is usually defined via entrenchment-based contraction [5, 6]. However, it is also possible to obtain revision directly from entrenchment, as follows [14, 18]:

($\leq *1$) $\beta \in K{*}\alpha$ if and only if either $(\alpha \to \neg\beta) < (\alpha \to \beta)$ or $\vdash \neg\alpha$.

Given the standard properties of the entrenchment relation, this is equivalent with:

($\leq *2$) $\beta \in K{*}\alpha$ if and only if either $(\alpha \to \neg\beta) < (\alpha \to \beta)$ or $\neg\alpha$ is maximally entrenched.

To construct non-prioritized entrenchment-based revision, we can make use of EE1-EE4 but give up EE5 (maximality). Furthermore, we can use the following variant of ($\leq *2$).

Definition 4. *$\circ$ is an* entrenchment-based non-prioritized revision operator *based on $\leq$ if and only if:*

($\leq \circ$) *$\beta \in K\circ\alpha$ if and only if either $(\alpha \rightarrow \neg\beta) < (\alpha \rightarrow \beta)$, or $\beta \in K$ and $\neg\alpha$ is maximally entrenched.*

The added condition $\beta \in K$ is needed to assure that (strong) consistency is given priority over success.

Our fourth construction makes use of the one-to-one correspondence that persists between propositions (sets of possible worlds) and belief sets. In a propositional approach, operations of belief change are performed on the set $\|K\|$ of possible worlds. Indeed, the standard AGM revision operator (partial meet revision) of K by a corresponds to the selection of a subset of $\|\alpha\|$ that is non-empty if $\|\alpha\|$ is non-empty and equal to $\|K\| \cap \|\alpha\|$ if $\|K\| \cap \|\alpha\|$ is non-empty [8, 12]. We propose to distinguish between credible and incredible worlds, and to require that the latter never be included in an outcome proposition. Again, it is convenient to introduce the new construction as a special case of credibility-limited revision.

Definition 5. *Let $\circ$ be a credibility-limited revision operator for the belief set K, based on $\mathscr{C}$. Then $\circ$ is*

1. *an operator of* credible worlds revision *if and only if there is a set $\mathscr{W}_{\mathscr{C}}$ of possible worlds such that: $\alpha \in \mathscr{C}$ if and only if there is some $w \in \mathscr{W}_{\mathscr{C}}$ such that $\alpha \in w$.*
2. *an operator of* non-empty credible worlds revision *if and only if this holds for a set $\mathscr{W}_{\mathscr{C}} \neq \emptyset$ of possible worlds.*
3. *an operator of* endorsed credible worlds revision *if and only if this holds for a set $\mathscr{W}_{\mathscr{C}}$ such that $\|K\| \subseteq \mathscr{W}_{\mathscr{C}}$.*

If K is consistent, then all endorsed credible worlds revisions are non-empty credible worlds revisions. Two plausible additional conditions should be mentioned that relate $\mathscr{W}_{\mathscr{C}}$ to the outcome of the operation:

$\|K\circ\alpha\| \cap \mathscr{W}_{\mathscr{C}} \neq \emptyset$ (*outcome credibility*).
$\|K\circ\alpha\| \subseteq \mathscr{W}_{\mathscr{C}}$ (*strong outcome credibility*).

Our fifth and last model is a variant of the previous one. Grove's sphere-based operations make use of the simple intuition that the outcome of revising $\|K\|$ by $\|\alpha\|$ consists of those elements of $\|\alpha\|$ that are as close as possible to $\|K\|$. For that purpose, $\|K\|$ can be thought of as surrounded by a system of concentric spheres [8]. Each sphere represents a degree of closeness or similarity to $\|K\|$. The outcome of revising $\|K\|$ by $\|\alpha\|$ should be the intersection of $\|\alpha\|$ with the narrowest sphere around $\|K\|$ that has a non-empty intersection with $\|\alpha\|$. The equivalence of this construction with the full set of (basic and supplementary) Gärdenfors postulates is a fundamental result in AGM theory [5].

Our modification consists in relaxing the standard requirements on sphere systems, so that not all possible worlds are elements of any sphere.

Definition 6. *$\mathscr{S}$ is a* system of spheres *around* $\mathrm{Th}(\cap\mathscr{S})$ *if and only if it satisfies:*

(S1) $\emptyset \neq \mathscr{S} \subseteq \mathscr{P}(\mathscr{L}\perp\!\!\perp)$,
(S2) $\cap\mathscr{S} \in \mathscr{S}$,
(S3) *If $G, G' \in \mathscr{S}$, then either $G \subseteq G'$ or $G' \subseteq G$,*
(S4) $\cup\mathscr{S} \in \mathscr{S}$, *and*
(S5) *If $\|\alpha\| \cap (\cup\mathscr{S}) \neq \emptyset$, then $S_\alpha \in \mathscr{S}$ and $S_\alpha \cap \|\alpha\| \neq \emptyset$, where $S_\alpha = \cap\{G \in \mathscr{S} : G \cap \|\alpha\| \neq \emptyset\}$.*

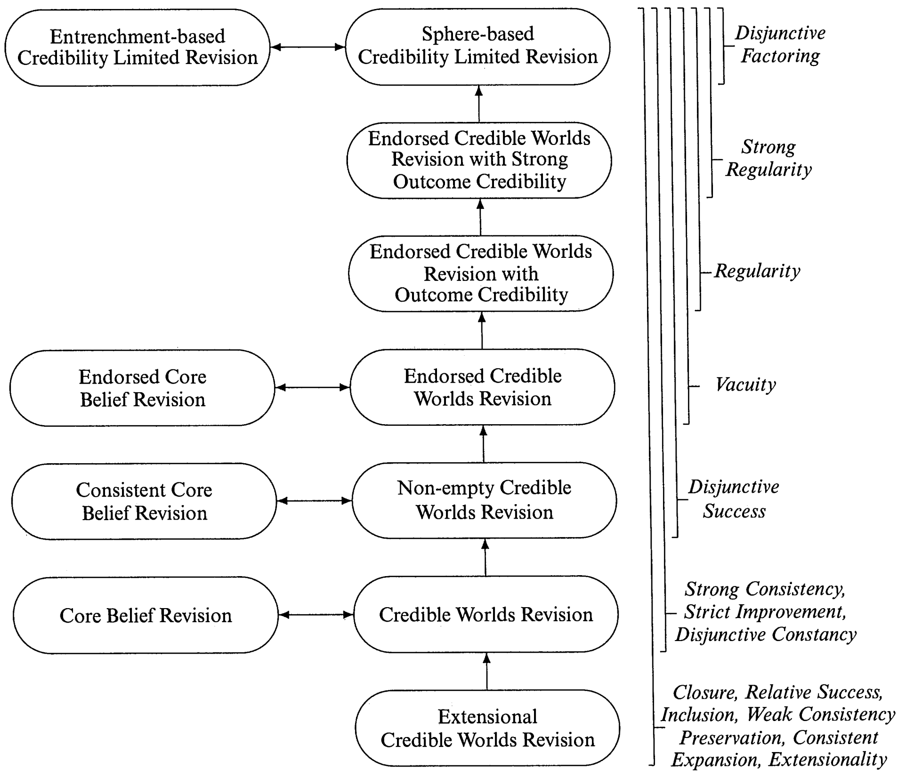

FIGURE 1. Results from Theorems 11-13.

For any set A of sentences and sentence α, $A \perp \alpha$ is the set of maximal consistent subsets of A not implying α. Intuitively, S_α is the smallest sphere that contains some α-world.

DEFINITION 7. *Let $\mathcal{S}$ be a system of spheres around K. The operator $\circ$ is a* non-prioritized sphere-based revision operator *for $\mathcal{S}$ if and only if it satisfies*:

$$K \circ \alpha = \begin{cases} \mathrm{Th}(\|\alpha\| \cap S_\alpha) & \text{if } \|\alpha\| \cap (\cup \mathcal{S}) \neq \emptyset \\ K & \text{otherwise} \end{cases}$$

§4. **Representation theorems.** This section reports a series of representation results through which the postulates of Section 2 and the constructions of Section 3 are closely knit together. Theorem 8 provides the starting-point, characterizing essentially those credibility-limited revisions that are available within an extensional framework. Theorem 10 exhibits some one-to-one correspondences between additional revision postulates and additional properties of the set $\mathscr{C}$ of credible sentences. Theorems 11-13 provide us with a series of axiomatically characterized constructions of increasing strength. The major results of this section are summarized in Figure 1.

THEOREM 8. *Let K be a consistent and logically closed set and* $\circ$ *an operation on K. Then the following three conditions are equivalent*:

1. $\circ$ *satisfies closure, relative success, inclusion, weak consistency preservation, consistent expansion, and extensionality.*
2. *There is an AGM revision operator* $*$ *for K and a set* $\mathscr{C} \subseteq \mathscr{L}$ *that is closed under logical equivalence, and such that* $\circ$ *is the credibility-limited revision induced by* $*$ *and* $\mathscr{C}$.
3. *There is an AGM revision operator* $*$ *for K and a set* $\mathscr{C} \subseteq \mathscr{L}$ *that satisfies* $K \subseteq \mathscr{C}$ *and is closed under logical equivalence, and such that* $\circ$ *is the credibility-limited revision induced by* $*$ *and* $\mathscr{C}$.

PROOF. **(2)-to-(1)**: It follows directly from the construction that closure, relative success, inclusion, weak consistency preservation, and extensionality are satisfied. For consistent expansion, let $K \nsubseteq K{\circ}\alpha$. Then $K{\circ}\alpha = K{*}\alpha$ and $\alpha \in \mathscr{C}$. It follows from the vacuity and success postulates satisfied by $*$ that $*$ satisfies consistent expansion.

(1)-to-(3): Let $*$ be the operation such that

(i) if $\alpha \in K{\circ}\alpha$, then $K{*}\alpha = K{\circ}\alpha$,
(ii) if $\alpha \notin K{\circ}\alpha$, then $K{*}\alpha = K{*}'\alpha$ for some AGM revision operator $*'$.

Furthermore, let $\mathscr{C} = \{\alpha : \alpha \in K{\circ}\alpha\}$. We need to show: (A1) that $\mathscr{C}$ is closed under logical equivalence, (A2) that $K \subseteq \mathscr{C}$, (B) that $*$ is an AGM revision operator, and (C) that $\circ$ is induced by $*$ and $\mathscr{C}$.

Part A1: To show that $\mathscr{C}$ is closed under logical equivalence, let $\alpha \in \mathscr{C}$ and let $\vdash \alpha \leftrightarrow \beta$. Then $\alpha \in K{\circ}\alpha$. It follows from $\circ$-closure that $\beta \in K{\circ}\alpha$ and from $\circ$-extensionality that $K{\circ}\alpha = K{\circ}\beta$. Then $\beta \in K{\circ}\beta$, hence $\beta \in \mathscr{C}$.

Part A2: Let $\alpha \in K$. It follows from relative success that $\alpha \in K{\circ}\alpha$, hence $\alpha \in \mathscr{C}$.

Part B: This can be proved by verifying that $*$ satisfies the six basic AGM postulates.

Part C: There are two cases. (1) If $\alpha \in \mathscr{C}$, then $K{\circ}\alpha = K{*}\alpha$. (2) If $\alpha \notin \mathscr{C}$, then $\alpha \notin K{\circ}\alpha$. It follows from $\circ$-relative success that $K{\circ}\alpha = K$.

(3)-to-(2): Obvious. ⊣

It follows from Theorem 8 that the condition $K \subseteq \mathscr{C}$ has no effects on the properties of the operator $\circ$. The reason for this should be clear from the following observation.

OBSERVATION 9. *Let K be a consistent and logically closed set of sentences and* $*$ *an AGM revision on K. Let* $\mathscr{C}_1$ *and* $\mathscr{C}_2$ *be two sets of sentences. Let* $\circ_1$ *be the credibility-limited revision based on* $\mathscr{C}_1$ *and* $*$, *and* $\circ_2$ *that based on* $\mathscr{C}_2$ *and* $*$. *Then*:

$$\text{If } \mathscr{C}_1 \setminus K = \mathscr{C}_2 \setminus K, \text{ then } K{\circ_1}\alpha = K{\circ_2}\alpha \text{ for all } \alpha.$$

The proof of this observation is left to the reader.

THEOREM 10. *Let K be a consistent and logically closed set and* $\circ$ *an operation on K. Then the following pairs of conditions are equivalent*:

1. $\circ$ *satisfies closure, relative success, inclusion, weak consistency preservation, consistent expansion, extensionality and*
   (a) *If* $\alpha \in K{\circ}\alpha$ *and* $\alpha \vdash \beta$, *then* $\beta \in K{\circ}\beta$ (strict improvement).
   (b) *If* $\alpha \vee \beta \in K{\circ}(\alpha \vee \beta)$, *then either* $\alpha \in K{\circ}\alpha$ *or* $\beta \in K{\circ}\beta$ (disjunctive distribution).

(c) *Either* $\alpha \in K\circ\alpha$ *or* $\neg\alpha \in K\circ\neg\alpha$ (disjunctive success).
(d) $K\circ\alpha \nvdash \bot$ (strong consistency).
(e) *If* $\neg\alpha \notin K$, *then* $K\circ\alpha = K+\alpha$ (vacuity).

2. *There is an AGM revision operator* $*$ *for* K *and a set* $\mathscr{C} \subseteq \mathscr{L}$ *that is closed under logical equivalence and satisfies*
   (a) *If* $\alpha \in \mathscr{C}$ *then* $\mathrm{Cn}(\{\alpha\}) \subseteq \mathscr{C}$ (single sentence closure).
   (b) *If* $\alpha \vee \beta \in \mathscr{C}$, *then either* $\alpha \in \mathscr{C}$ *or* $\beta \in \mathscr{C}$ (disjunctive completeness).
   (c) *Either* $\alpha \in \mathscr{C}$ *or* $\neg\alpha \in \mathscr{C}$ (negation-completeness).
   (d) *If* $\alpha \in \mathscr{C}$, *then* $\alpha \nvdash \bot$ (element consistency).
   (e) *If* $K \nvdash \alpha$, *then* $\neg\alpha \in \mathscr{C}$ (expansive credibility).
   and such that $\circ$ *is the credibility-limited revision induced by* $*$ *and* $\mathscr{C}$.

The proofs of the various parts of Theorem 10 follow directly by adding the respective condition to the construction introduced in the proof of Theorem 8.

THEOREM 11. *Let K be a consistent and logically closed set and* $\circ$ *an operation on K. Then the following four conditions are equivalent*:

1. $\circ$ *satisfies closure, relative success, inclusion,* strong consistency, *consistent expansion, extensionality,* strict improvement, *and* disjunctive distribution.
2. *There is an AGM revision operator* $*$ *for* K *and a set* $\mathscr{C} \subseteq \mathscr{L}$ *that is closed under logical equivalence, and satisfies* single sentence closure, disjunctive completeness *and* element consistency, *and such that* $\circ$ *is the credibility-limited revision induced by* $*$ *and* $\mathscr{C}$.
3. *It is a core beliefs revision.*
4. *It is a credible worlds revision.*

The postulates listed in Theorem 11 will be referred to as the *core postulates*. Weak consistency preservation could be redundantly added to the core postulates, since it follows from strong consistency.

PROOF OF THEOREM 11. **(1)-to-(2)** and **(2)-to-(1)**: Directly from Theorems 8 and 10.

(2)-to-(3): Let the three conditions hold. Let $A = \{\alpha : \neg\alpha \notin \mathscr{C}\}$. Then $\alpha \in \mathscr{C}$ iff $\neg\alpha \notin A$. In order to prove that $\alpha \in \mathscr{C}$ iff $A \nvdash \neg\alpha$ it is sufficient to show that $A = \mathrm{Cn}(A)$. By element consistency, $\bot \notin \mathscr{C}$, so that $\top \in A$ and hence $A \neq \emptyset$. Let $\alpha \in \mathrm{Cn}(A)$. We assume compactness. Since A is non-empty, there are $\beta_1, \dots, \beta_n \in A$ such that $\{\beta_1, \dots, \beta_n\} \vdash \alpha$. We need to show that $\alpha \in A$.

It follows from $\beta_1, \dots, \beta_n \in A$ that $\neg\beta_1, \dots, \neg\beta_n \notin \mathscr{C}$. It follows from repeated use of disjunctive completeness that $\neg\beta_1 \vee \dots \vee \neg\beta_n \notin \mathscr{C}$.

Suppose that $\neg\alpha \in \mathscr{C}$. Then, since $\neg\alpha \vdash \neg\beta_1 \vee \dots \vee \neg\beta_n$, single sentence closure yields $\neg\beta_1 \vee \dots \vee \neg\beta_n \in \mathscr{C}$, contrary to what was just shown. We may conclude that $\neg\alpha \notin \mathscr{C}$, hence $\alpha \in A$. This finishes the proof.

(3)-to(2): We need to show that all core beliefs revisions satisfy the three postulates. Let the operator be one of core beliefs revision, i.e., let there be some A such that $\alpha \in \mathscr{C}$ iff $A \nvdash \neg\alpha$.

To show that single sentence closure is satisfied, let $\alpha \in \mathscr{C}$ and $\vdash \alpha \rightarrow \beta$. Then $\alpha \in \mathscr{C}$ yields $A \nvdash \neg\alpha$, and $\vdash \alpha \rightarrow \beta$ yields $\vdash \neg\beta \rightarrow \neg\alpha$. Hence $A \nvdash \neg\beta$, so that $\beta \in \mathscr{C}$.

To show that disjunctive completeness is satisfied, let $\alpha \vee \beta \in \mathscr{C}$. It follows from the definition of core beliefs revisions that $A \nvdash \neg(\alpha \vee \beta)$, hence $A \nvdash \neg\alpha \wedge \neg\beta$, hence either $A \nvdash \neg\alpha$ or $A \nvdash \neg\beta$. In the former case, $\alpha \in \mathscr{C}$, in the latter $\beta \in \mathscr{C}$.

To show that element consistency is satisfied, let $\alpha \vdash \bot$. Then $A \vdash \neg\alpha$, hence $\alpha \notin \mathscr{C}$.

(3)-to-(4): Let the operator be a core beliefs revision. Let $\mathscr{W}_{\mathscr{C}} = \|A\|$. We then have $\alpha \in \mathscr{C}$ iff $A \nvdash \neg\alpha$, iff there is some $w \in \mathscr{W}_{\mathscr{C}}$ such that $w \nvdash \neg\alpha$, iff there is some $w \in \mathscr{W}_{\mathscr{C}}$ such that $\alpha \in w$.

(4)-to-(3): Let the operator be a credible worlds revision. Let $A = \mathrm{Th}(\mathscr{W}_{\mathscr{C}})$. Then $\alpha \in \mathscr{C}$ iff there is some $w \in \mathscr{W}_{\mathscr{C}}$ such that $\alpha \in w$, iff there is some $w \in \mathscr{W}_{\mathscr{C}}$ such that $w \nvdash \neg\alpha$, iff $\mathrm{Th}(\mathscr{W}_{\mathscr{C}}) \nvdash \neg\alpha$, iff $A \nvdash \neg\alpha$. ⊣

COROLLARY 1. *Let K be a consistent and logically closed set and* $\circ$ *an operation on K. Then the following four conditions are equivalent*:

1. $\circ$ *satisfies the core postulates and* disjunctive success.
2. *There is an AGM revision operator* $*$ *for K and a set* $\mathscr{C} \subseteq \mathscr{L}$ *that is closed under logical equivalence, and satisfies* single sentence closure, disjunctive completeness, element consistency *and* negation completeness, *and such that* $\circ$ *is the credibility-limited revision induced by* $*$ *and* $\mathscr{C}$.
3. *It is a consistent core beliefs revision.*
4. *It is a non-empty credible worlds revision.*

COROLLARY 2. *Let K be a consistent and logically closed set and* $\circ$ *an operation on K. Then the following four conditions are equivalent*:

1. $\circ$ *satisfies the core postulates and* vacuity.
2. *There is an AGM revision operator* $*$ *for K and a set* $\mathscr{C} \subseteq \mathscr{L}$ *that is closed under logical equivalence, and satisfies* single sentence closure, disjunctive completeness, element consistency, *and* expansive credibility, *and such that* $\circ$ *is the credibility-limited revision induced by* $*$ *and* $\mathscr{C}$.
3. *It is an endorsed core beliefs revision.*
4. *It is an endorsed credible worlds revision.*

It should be noted that disjunctive success follows from vacuity.

THEOREM 12. *Let K be a consistent and logically closed set and* $\circ$ *an endorsed credible worlds revision on K. Then*:

1. $\circ$ *satisfies* regularity *if and only if it* $\circ$ *is an endorsed credible worlds revision satisfying outcome credibility.*
2. $\circ$ *satisfies* strong regularity *if and only if it is an endorsed core beliefs revision that satisfies strong outcome credibility.*

PROOF. *Part 1, from the postulates to the construction:* We will use the same construction as in Theorem 8, but with the further specification that $*'$ (the revision operator for residual cases) is defined so that $K *' \alpha = \mathrm{Cn}(\{\alpha\})$ if $\neg\alpha \in K$ (it can easily be checked that this is an AGM operator). It remains to be shown that outcome credibility holds, i.e., that $\|K \circ \alpha\| \cap \mathscr{W}_{\mathscr{C}} \neq \emptyset$. We can prove that $K \circ \alpha \subseteq \mathscr{C}$ by assuming $\beta \notin \mathscr{C}$ and showing $\beta \notin K \circ \alpha$. Let $\beta \notin \mathscr{C}$. Then, $K \circ \beta = K$. It follows from $K \subseteq \mathscr{C}$ that $\beta \notin K$, hence $\beta \notin K \circ \beta$, and by regularity $\beta \notin K \circ \alpha$. It follows that $K \circ \alpha \subseteq \mathscr{C}$, and thus $\|K \circ \alpha\| \cap \mathscr{W}_{\mathscr{C}} \neq \emptyset$.

Part 1, from the construction to the postulates: It follows from the construction that for all α, if $\alpha \in \mathscr{C}$, then $K \circ \alpha \subseteq \mathscr{C}$. Let $\beta \in K \circ \alpha$. There are two cases:

Case 1, $\alpha \notin \mathscr{C}$. Then $K \circ \alpha = K$ and $\beta \in K$. Since this is an endorsed core beliefs revision, it is also an endorsed credible worlds revision (see Corollary 2 of Theorem 11), i.e., $\|K\| \subseteq \mathscr{W}_{\mathscr{C}}$. Since K is consistent, it follows that $\|K\| \cap \mathscr{W}_{\mathscr{C}} \neq \emptyset$, and since $\|K\| \subseteq \|\beta\|$ it follows from this that $\|\beta\| \cap \mathscr{W}_{\mathscr{C}} \neq \emptyset$ or equivalently $\beta \in \mathscr{C}$, from which follows $\beta \in K \circ \beta$.

Case 2, $\alpha \in \mathscr{C}$. Then $K \circ \alpha = K * \alpha$. It follows that $K * \alpha \subseteq \mathscr{C}$, hence $K \circ \alpha \subseteq \mathscr{C}$. Since $\beta \in K \circ \alpha$ it follows that $\alpha \in \mathscr{C}$, hence $\beta \in K \circ \beta$.

Part 2, from the postulates to the construction: Let strong regularity be satisfied. We can, without loss of generality, assume that $\circ$ is an endorsed credible worlds revision based on a set $\mathscr{W}_{\mathscr{C}}$ of credible worlds such that $\mathscr{W}_{\mathscr{C}} = \|\operatorname{Th}(\mathscr{W}_{\mathscr{C}})\|$ (to see this, note that there is a w such that $\alpha \in w \in \mathscr{W}_{\mathscr{C}}$ iff there is a w such that $\alpha \in w \in \|\operatorname{Th}(\mathscr{W}_{\mathscr{C}})\|$).

Let $\beta \in \operatorname{Th}(\mathscr{W}_{\mathscr{C}})$. Then $\|\operatorname{Th}(\mathscr{W}_{\mathscr{C}})\| \subseteq \|\beta\|$, consequently $\mathscr{W}_{\mathscr{C}} \subseteq \|\beta\|$, from which it follows that $\mathscr{W}_{\mathscr{C}} \cap \|\neg\beta\| = \emptyset$. It follows from the definition of $\mathscr{W}_{\mathscr{C}}$ that $\neg\beta \notin K \circ \neg\beta$. Applying strong regularity to this we obtain $\beta \in K \circ \alpha$. Hence we have proved that $\operatorname{Th}(\mathscr{W}_{\mathscr{C}}) \subseteq K \circ \alpha$. From this follows $\|K \circ \alpha\| \subseteq \mathscr{W}_{\mathscr{C}}$.

Part 2, from the construction to the postulates: Let $\neg\beta \notin K \circ \alpha$. From $\neg\beta \notin K \circ \alpha$ follows $\|K \circ \alpha\| \nsubseteq \|\neg\beta\|$, and then from strong outcome credibility ($\|K \circ \alpha\| \subseteq \mathscr{W}_{\mathscr{C}}$) that $\mathscr{W}_{\mathscr{C}} \nsubseteq \|\neg\beta\|$, hence $\mathscr{W}_{\mathscr{C}} \cap \|\beta\| \neq \emptyset$, equivalently $\beta \in \mathscr{C}$, from which follows $\beta \in K \circ \beta$. ⊣

Results from Theorems 11 and 12 are diagrammatically summarized in Figure 2. In endorsed credible worlds revision, the set of credible worlds is a superset of the set $\|K\|$ of worlds compatible with the belief set. If $\|\alpha\|$ intersects with $\|K\|$, then the outcome of revision is equal to the belief set corresponding to $\|K\| \cap \|\alpha\|$, see (1) in the figure. If $\|\alpha\|$ does not intersect with $\mathscr{W}_{\mathscr{C}}$, as in (3), then the outcome is $\|K\|$. In the intermediate case, when $\|\alpha\|$ intersects with $\mathscr{W}_{\mathscr{C}}$ but not with $\|K\|$, the outcome may be a proposition that either (2a) consists only of credible worlds, (2b) consists in part of credible and in part of incredible worlds, or (2c) consists only of incredible worlds. A good case can be made that (2c), and perhaps also (2b), should be excluded. Regularity corresponds exactly to the exclusion of case (2c) and strong regularity to the exclusion of both cases (2b) and case (2c).

THEOREM 13. *Let K be a consistent and logically closed set and $\circ$ an operator on K. Then the following three conditions are equivalent*:

1. *$\circ$ satisfies the core postulates and* vacuity, strong regularity, *and* disjunctive factoring.
2. *$\circ$ is an entrenchment-based non-prioritized revision in the sense of Definition* 4, *based on an entrenchment relation $\leq$ on K that satisfies properties* EE1-EE4.
3. *$\circ$ is a sphere-based revision operator around K in the sense of Definitions* 6-7.

The following lemmas are needed for the proof of the theorem. The proofs of the lemmas are left to the reader.

LEMMA 14. *Let $\leq$ be a relation on $\mathscr{L}$ that satisfies transitivity, dominance and conjunctiveness. Then*:

1. *It satisfies intersubstitutivity* (*If $\alpha \leftrightarrow \alpha' \in \operatorname{Cn}(\emptyset)$ and $\beta \leftrightarrow \beta' \in \operatorname{Cn}(\emptyset)$, then $\alpha \leq \beta$ if and only if $\alpha' \leq \beta'$*) [7].

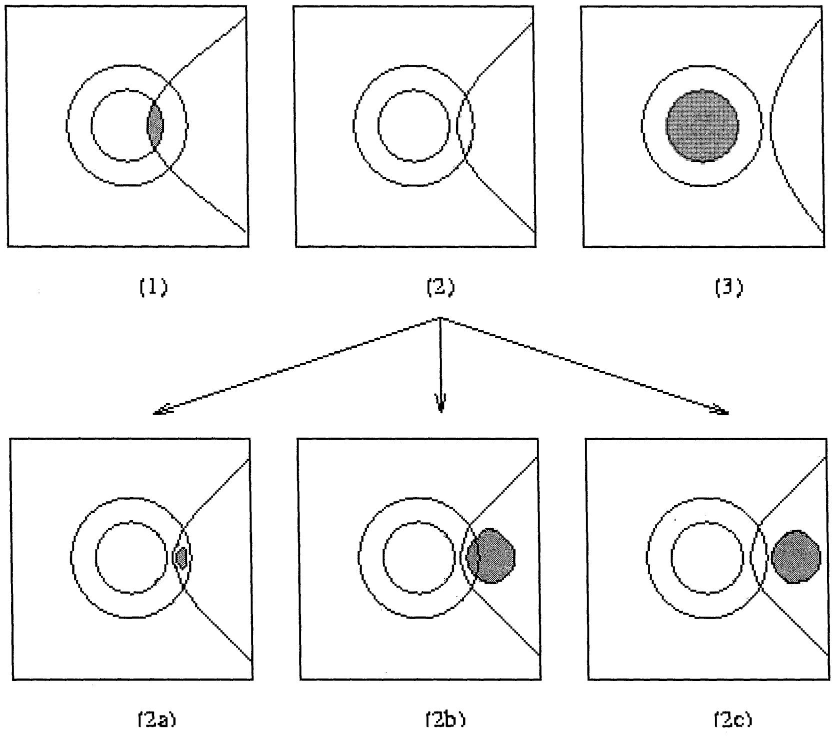

FIGURE 2. Results from Theorems 11 and 12.

2. *It satisfies connectedness* (*Either* $\alpha \leq \beta$ *or* $\beta \leq \alpha$) [6].
3. $\neg\alpha < (\alpha \rightarrow \beta)$ *holds if and only if* $(\alpha \rightarrow \neg\beta) < (\alpha \rightarrow \beta)$ [12].
4. $\top \leq \alpha$ *holds if and only if there is no* δ *such that* $\alpha < \delta$.
5. *If* $\alpha < \beta$, *then* $\alpha \equiv \alpha \wedge \beta$.
6. *If* $\alpha < \beta$ *and* $\alpha < \delta$, *then* $\alpha < \beta \wedge \delta$ [7].
7. $\|\{\delta : \alpha \leq \delta\}\| \subseteq \|\beta\|$ *if and only if* $\alpha \leq \beta$.
8. *If* $\leq$ *satisfies minimality* (EE4) *with respect to a consistent belief set K, then* $K = \{\beta : \bot < \beta\}$.

LEMMA 15. *Let* $\circ$ *satisfy vacuity, relative success, strict improvement and strong consistency. Then it satisfies*: *If* $\alpha \wedge \beta \in K\circ\neg(\alpha \wedge \beta)$ *then* $\beta \in K\circ\neg\beta$.

LEMMA 16. *Let K be a consistent belief set, and let* $\circ$ *satisfy closure, vacuity, relative success, extensionality, disjunctive inclusion, and strong consistency. Then it satisfies*: *If* $\neg\beta \in K\circ\beta$, *then* $\neg\beta \in K\circ(\alpha \vee \beta)$.

LEMMA 17. *Let* $\circ$ *satisfy vacuity. Then it satisfies*: *If* $\alpha \in K\circ\neg(\alpha \wedge \beta)$, *then* $\alpha \in K$.

LEMMA 18. (Modified from [2]) *Let D be a non-empty subset of* $\mathscr{P}(\mathscr{L})$ *such that* (1) *for all X in D*, $X = \|\mathrm{Th}(X)\|$, *and* (2) *for all elements X and Y of D, either* $X \subseteq Y$ *or* $Y \subseteq X$. *Furthermore, let* $\alpha \in \mathscr{L}$. *Then*: *If* $\cap D \subseteq \|\alpha\|$ *then there is some element X of D such that* $X \subseteq \|\alpha\|$.

PROOF OF THEOREM 13. **(1)-to-(2)** *From postulates to entrenchment:* We assume that the postulates given in (1) hold for a given operator $\circ$, and we let $\leq$ be defined as follows:

$$\alpha \leq \beta \text{ iff: If } \alpha \in K\circ\neg(\alpha \wedge \beta), \text{ then } \beta \in K\circ\neg(\alpha \wedge \beta)$$

We need to show that EE1-EE4 hold and that $\circ$ is entrenchment-based on $\leq$ in the sense of Definition 4. This is straightforward except for EE1. To show that EE1 holds, let $\alpha \leq \beta$, $\beta \leq \delta$ and $\alpha \in K\circ\neg(\alpha \wedge \delta)$. We need to prove that $\delta \in K\circ\neg(\alpha \wedge \delta)$. There are two cases.

Case 1, $\alpha \in K\circ\neg(\alpha \wedge \beta)$: Since $\alpha \leq \beta$ we then have $\beta \in K\circ\neg(\alpha \wedge \beta)$. By closure, $\alpha \wedge \beta \in K\circ\neg(\alpha \wedge \beta)$. Lemma 15 yields $\beta \in K\circ\neg\beta$ and Lemma 16 yields $\beta \in K\circ\neg(\beta \wedge \delta)$. Since $\beta \leq \delta$, it follows that $\delta \in K\circ\neg(\beta \wedge \delta)$. By closure, $\beta \wedge \delta \in K\circ\neg(\beta \wedge \delta)$. Lemma 15 then yields $\delta \in K\circ\neg\delta$, and Lemma 16 yields $\delta \in K\circ\neg(\alpha \wedge \delta)$.

Case 2, $\alpha \notin K\circ\neg(\alpha \wedge \beta)$: It then follows from Lemma 16 that $\alpha \notin K\circ\neg\alpha$. We are going to assume for *reductio* that $\delta \notin K\circ\neg(\alpha \wedge \delta)$. It then follows directly by Lemma 16 that $\delta \notin K\circ\neg\delta$. Lemma 15 then yields $\beta \wedge \delta \notin K\circ\neg(\beta \wedge \delta)$. Since $\beta \leq \delta$, we can conclude from this (using closure) that $\beta \notin K\circ\neg(\beta \wedge \delta)$. We are going to show (1) that $\alpha \in K\circ\neg(\alpha \wedge \beta \wedge \delta)$ and (2) that $\alpha \notin K\circ\neg(\alpha \wedge \beta \wedge \delta)$.

Ad 1: Since $\alpha \in K\circ\neg(\alpha \wedge \delta)$, Lemma 17 yields $\alpha \in K$. Hence by relative success and closure, $\alpha \in K\circ(\alpha \wedge \neg\beta)$. By assumption, $\alpha \in K\circ\neg(\alpha \wedge \delta)$. Due to disjunctive overlap (that follows from disjunctive factoring), $K\circ\neg(\alpha \wedge \delta) \cap K\circ(\alpha \wedge \neg\beta) \subseteq K\circ\neg(\alpha \wedge \beta \wedge \delta)$, hence $\alpha \in K\circ\neg(\alpha \wedge \beta \wedge \delta)$.

Ad 2: It follows from disjunctive factoring and extensionality that $K\circ\neg(\alpha \wedge \beta \wedge \delta)$ is equal to one of $K\circ\neg(\alpha \wedge \beta) \cap K\circ\neg\delta$, $K\circ\neg(\alpha \wedge \beta)$ and $K\circ\neg\delta$. Since, as we have just seen, $\alpha \notin K\circ\neg(\alpha \wedge \beta)$ and $\delta \notin K\circ\neg\delta$, it follows that either $\alpha \notin K\circ\neg(\alpha \wedge \beta \wedge \delta)$ or $\delta \notin K\circ\neg(\alpha \wedge \beta \wedge \delta)$. In the former case we are done. In the latter case, we also have $\beta \wedge \delta \notin K\circ\neg(\alpha \wedge \beta \wedge \delta)$, and it follows by disjunctive inclusion (that follows from disjunctive factoring and vacuity in the presence of the core postulates) and extensionality that $K\circ\neg(\alpha \wedge \beta \wedge \delta) \subseteq K\circ\neg(\beta \wedge \delta)$. Since $\beta \notin K\circ\neg(\beta \wedge \delta)$ as shown above, it follows that $\beta \notin K\circ\neg(\alpha \wedge \beta \wedge \delta)$, hence due to closure and extensionality $\alpha \wedge \beta \notin K\circ(\neg(\alpha \wedge \beta) \vee \neg\delta)$. It follows from disjunctive inclusion that $K\circ(\neg(\alpha \wedge \beta) \vee \neg\delta) \subseteq K\circ\neg(\alpha \wedge \beta)$. Extensionality yields $K\circ\neg(\alpha \wedge \beta \wedge \delta) \subseteq K\circ\neg(\alpha \wedge \beta)$. Since in this case $\alpha \notin K\circ\neg(\alpha \wedge \beta)$, it follows that $\alpha \notin K\circ\neg(\alpha \wedge \beta \wedge \delta)$. This is the contradiction we needed.

(2)-to-(1) *From entrenchment to postulates:* Let $\leq$ be an entrenchment relation satisfying EE1-EE4 with respect to K, and let $\circ$ be the operator that is based on $\leq$ in the manner of Definition 4. We need to show that the listed postulates hold. This is straightforward except for closure and disjunctive factoring.

For closure, let $\varphi \in \text{Cn}(K\circ\alpha)$. Then there is, by compactness of the underlying logic, a finite subset $\{\beta_1, \ldots, \beta_n\}$ of $K\circ\alpha$ such that $\{\beta_1, \ldots, \beta_n\} \vdash \varphi$. Using the above definition of our entrenchment-based revision $\circ$, we can show that if $\beta_1 \in K\circ\alpha$ and $\beta_2 \in K\circ\alpha$, then $\beta_1 \wedge \beta_2 \in K\circ\alpha$. By iteration, $\beta_1 \wedge \ldots \wedge \beta_n \in K\circ\alpha$, from which it can be shown that $\varphi \in K\circ\alpha$.

As for disjunctive factoring, there are three cases:

Case 1, $\alpha \notin K\circ\alpha$ and $\beta \notin K\circ\beta$. From $\alpha \notin K\circ\alpha$ follows, via the definition of $\circ$, that $\alpha \rightarrow \neg\alpha \not< \alpha \rightarrow \alpha$, hence by connectedness and intersubstitutivity $\top \leq \neg\alpha$, hence by dominance $\neg\alpha$ is maximal. It follows from Definition 4 that $\alpha \notin K$. Hence, according to the same definition, for all δ, $\delta \in K \circ \alpha$ iff $\delta \in K$, hence $K \circ \alpha = K$. It follows in the same way that $K \circ \beta = K$. We need to show that $K \circ (\alpha \vee \beta) = K$.

Case 1a, let both $\neg\alpha$ and $\neg\beta$ be maximally entrenched. Then it follows by conjunctiveness that $\neg\alpha \wedge \neg\beta$ is maximally entrenched. Since $\neg\alpha \wedge \neg\beta$ is maximally entrenched, it follows from dominance that so is $(\alpha \vee \beta) \rightarrow \neg\delta$ for all δ. It then follows from Definition 4 that $K \circ (\alpha \vee \beta) = K$.

Case 1b, $\neg\alpha$ is not maximally entrenched. We then have $\neg\alpha < \top$, hence $\neg\alpha < \alpha \rightarrow \alpha$, hence by the definition of $\circ$, $\alpha \in K \circ \alpha = K$. Since K is logically closed it follows that $\alpha \vee \beta \in K$. Since K is consistent, vacuity yields $K \circ (\alpha \vee \beta) = K + (\alpha \vee \beta) = K$.

Case 2, $\alpha \notin K \circ \alpha$ and $\beta \in K \circ \beta$. We are first going to show that it is not the case that $\beta \in K$ and $\neg\beta$ is maximally entrenched. Suppose to the contrary that this is the case. Since K is consistent, it then follows from minimality that $\neg\beta$ is not minimal. Then according to minimality $\neg\beta \in K$, contrary to the consistency of K. We may conclude from this contradiction that it is not the case that $\beta \in K$ and $\neg\beta$ is maximal.

Hence, since $\beta \in K \circ \beta$, we can conclude from the definition of $\circ$ that $\beta \rightarrow \neg\beta < \beta \rightarrow \beta$, hence by intersubstitutivity $\neg\beta < \top$. By dominance, $\neg\alpha \wedge \neg\beta \leq \neg\beta$, hence by transitivity $\neg\alpha \wedge \neg\beta < \top$. Since $\top \leq \neg\alpha$, it follows from conjunctiveness and dominance that $\neg\alpha \wedge \neg\beta \equiv \neg\beta$. By intersubstitutivity, $\neg(\alpha \vee \beta) \equiv \neg\beta$.

Let δ be any sentence. By dominance, since $\neg\alpha$ is maximal, so is $\alpha \rightarrow \delta$. Hence by conjunctiveness and dominance, $(\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta) \equiv (\beta \rightarrow \delta)$. By intersubstitutivity, $(\alpha \vee \beta \rightarrow \delta) \equiv (\beta \rightarrow \delta)$. Hence for all δ, $\neg\beta < \beta \rightarrow \delta$ if and only if $\neg(\alpha \vee \beta) < (\alpha \vee \beta) \rightarrow \delta$. Since neither $\neg\alpha \wedge \neg\beta$ nor $\neg\beta$ is maximal, it follows from the definition of $\circ$ that for all δ, $\delta \in K \circ (\alpha \vee \beta)$ iff $\delta \in K \circ \beta$.

Case 3, $\alpha \in K \circ \alpha$ and $\beta \in K \circ \beta$: Using the symmetry of this case, we have two subcases.

Case 3a, $\neg\alpha < \neg\beta$: For one direction, let $\delta \in K \circ \alpha$. Then, since $\neg\alpha$ is not maximal, according to the definition of $\circ$ we have $\neg\alpha < \alpha \rightarrow \delta$. It also follows from $\neg\alpha < \neg\beta$, by part 5 of Lemma 14 that $\neg\alpha \equiv \neg\alpha \wedge \neg\beta$. Since dominance yields $\neg\beta \leq \beta \rightarrow \delta$, we can use transitivity to obtain both $\neg\alpha \wedge \neg\beta < \beta \rightarrow \delta$ and $\neg\alpha \wedge \neg\beta < \alpha \rightarrow \delta$. Conjunctiveness yields $\neg\alpha \wedge \neg\beta < (\alpha \rightarrow \delta) \wedge (\beta \rightarrow \delta)$, hence by intersubstitutivity $\neg\alpha \wedge \neg\beta < (\alpha \vee \beta \rightarrow \delta)$, hence $\delta \in K \circ (\alpha \vee \beta)$.

For the other direction, let $\delta \in K \circ (\alpha \vee \beta)$. It follows from $\neg\alpha < \neg\beta$ that $\neg\alpha \wedge \neg\beta \equiv \neg\alpha$, hence $\neg\alpha \wedge \neg\beta$ is not maximal, hence it follows from $\delta \in K \circ (\alpha \vee \beta)$ that $\neg\alpha \wedge \neg\beta < \alpha \vee \beta \rightarrow \delta$. By dominance, $\alpha \vee \beta \rightarrow \delta \leq \alpha \rightarrow \delta$. Transitivity yields $\neg\alpha < \alpha \rightarrow \delta$, hence $\delta \in K \circ \alpha$.

Case 3b, $\neg\alpha \equiv \neg\beta$. Then $\neg\alpha \equiv \neg\beta \equiv \neg\alpha \wedge \neg\beta$. For one direction, let $\delta \in K \circ \alpha \cap K \circ \beta$. Then $\neg\alpha < \alpha \rightarrow \delta$ and $\neg\beta < \beta \rightarrow \delta$. Then by dominance and transitivity $\neg\alpha \wedge \neg\beta < \alpha \rightarrow \delta$ and $\neg\alpha \wedge \neg\beta < \beta \rightarrow \delta$. Conjunctiveness and dominance yield $\neg\alpha \wedge \neg\beta < \alpha \vee \beta \rightarrow \delta$. Hence $\delta \in K \circ (\alpha \vee \beta)$.

For the other direction, let $\delta \in K \circ (\alpha \vee \beta)$. Then we have $\neg\alpha \wedge \neg\beta < \alpha \vee \beta \rightarrow \delta$. We already know that $\neg\beta \equiv \neg\alpha \wedge \neg\beta$, and dominance yields $\alpha \vee \beta \rightarrow \delta \leq \beta \rightarrow \delta$. Using transitivity to combine this, we obtain $\neg\beta < \beta \rightarrow \delta$, hence $\delta \in K \circ \beta$.

(2)-to-(3) *From entrenchment to spheres:* Let $\leq$ be a relation on $\mathscr{L}$ that satisfies EE1-EE4 with respect to the consistent belief set K. Furthermore, let $\mathcal{S}_{\leq}$ be the set such that $X \in \mathcal{S}_{\leq}$ iff it satisfies the following four conditions:

($S_{\leq}1$) $\|K\| \subseteq X$.
($S_{\leq}2$) $X = \cap\{\|\alpha\| : X \subseteq \|\alpha\|\}$.
($S_{\leq}3$) $X \subseteq \mathscr{P}(\mathscr{L}\perp\perp)$.
($S_{\leq}4$) For all $\alpha, \beta \in \mathscr{L}$, if $X \subseteq \|\alpha\|$ and $\alpha \leq \beta$, then $X \subseteq \|\beta\|$.

Let $S_\alpha = \cap\{G \in \mathcal{S}_{\leq} : G \cap \|\alpha\| \neq \emptyset\}$, and let $\text{Th}(\cap\mathcal{S}_{\leq}) = K$. We need to prove that $\mathcal{S}_{\leq}$ is a sphere system around K, i.e., that it satisfies S1-S5, and that the revision operator based on it, for all inputs, yields the same outcome as the entrenchment-based revision operator based on $\leq$ in the manner of Definition 4 (note in what follows that $\|\{\delta : \alpha \leq \delta\}\| = \cap\{\|\delta\| : \alpha \leq \delta\}$).

In order to show S4, first show that: If $\perp < \alpha$, then $\|\{\delta : \alpha \leq \delta\}\| \in \mathcal{S}_{\leq}$.

Given S4, S1 and S5 follow directly. The proof of S3 is also straightforward. The same applies to S2 if we first use Lemma 18 to prove that for any non-empty subset D of $\mathcal{S}_{\leq}$, it holds that $\cap D \in \mathcal{S}_{\leq}$. Given S2, we can prove that $\text{Th}(\cap\mathcal{S}_{\leq}) = K$. It remains to prove revision-equivalence. For that purpose, let $\circ_{\leq}$ be the credibility-limited revision operator based on $\leq$ in the manner of Definition 4, i.e.,

$$K\circ_{\leq}\alpha = \begin{cases} \{\beta : \neg\alpha < \alpha \rightarrow \beta\} & \text{if } \neg\alpha < \top \\ K & \text{otherwise} \end{cases}$$

Let $\circ_{\mathcal{S}_{\leq}}$ be the operator based on $\mathcal{S}_{\leq}$ in the manner for Definition 7, i.e., let

$$K\circ_{\mathcal{S}_{\leq}}\alpha = \begin{cases} \text{Th}(\|\alpha\| \cap S_\alpha) & \text{if } \|\alpha\| \cap (\cup\mathcal{S}_{\leq}) \neq \emptyset \\ K & \text{otherwise} \end{cases}$$

We need to show that for all α, $K\circ_{\leq}\alpha = K\circ_{\mathcal{S}_{\leq}}\alpha$. This can be done with the help of the following two intermediate results: (1) $\alpha < \top$ iff $\cup\mathcal{S}_{\leq} \nsubseteq \|\alpha\|$, (2) If $\neg\alpha < \top$, then $S_\alpha = \cap\{\|\beta\| : \neg\alpha < \beta\}$.

(3)-to-(2) *From spheres to entrenchment:* Given the sphere system $\mathcal{S}$ around K, we define the following entrenchment relation for K:

$$\alpha \leq \beta \text{ iff it holds for all } S \in \mathcal{S} \text{ that if } S \subseteq \|\alpha\| \text{ then } S \subseteq \|\beta\|.$$

It is easy to verify that $\leq$ satisfies EE1-EE4. It remains to show that the entrenchment-based operator $\circ_{\leq}$ that it gives rise to is identical with the sphere-based operator $\circ_{\mathcal{S}}$ that is based on $\mathcal{S}$.

For one direction of that proof, let $\beta \in K\circ_{\mathcal{S}}\alpha$. We have two cases.

First case, $\cup\mathcal{S} \nsubseteq \|\neg\alpha\|$. Then $\beta \in \text{Th}(S_\alpha \cap \|\alpha\|)$, hence $S_\alpha \cap \|\alpha\| \subseteq \|\beta\|$, so that $S_\alpha \subseteq \|\neg\alpha \vee \beta\|$. By definition, $S_\alpha \nsubseteq \|\neg\alpha\|$.

It follows from EE2 (dominance) that $\neg\alpha \leq \neg\alpha \vee \beta$. From $S_\alpha \subseteq \|\neg\alpha \vee \beta\|$ and $S_\alpha \nsubseteq \|\neg\alpha\|$ follows that $\neg\alpha \vee \beta \leq \neg\alpha$ does *not* hold, hence $\neg\alpha < \neg\alpha \vee \beta$. It follows from the definition of $\circ_{\leq}$ that $\beta \in K\circ_{\leq}\alpha$.

Second case, $\cup\mathcal{S} \subseteq \|\neg\alpha\|$. Then $K\circ_{\mathcal{S}}\alpha = K$ by the definition of $\circ_{\mathcal{S}}$, hence $\beta \in K$. Next, let S be any sphere such that $S \subseteq \|\top\|$. Then $S \subseteq \cup\mathcal{S} \subseteq \|\neg\alpha\|$. Hence $\top \leq \neg\alpha$.

For the other direction, let $\beta \in K\circ_{\leq}\alpha$. According to the definition of $\circ_{\leq}$ there are two cases.

First case, $\neg\alpha < \alpha \rightarrow \beta$. Rewriting this condition, using the definition above of $\leq$, it holds (a) that for all $G \in \mathcal{S}$, if $G \subseteq \|\neg\alpha\|$ then $G \subseteq \|\neg\alpha \vee \beta\|$, and (b) that there is some $G' \in \mathcal{S}$ such that $G' \subseteq \|\neg\alpha \vee \beta\|$ and $G' \nsubseteq \|\neg\alpha\|$.

It follows from $G' \nsubseteq \|\neg\alpha\|$ that $G' \cap \|\alpha\| \neq \emptyset$, so that $S_\alpha \subseteq G'$. Hence $S_\alpha \subseteq \|\neg\alpha \vee \beta\| = \|\neg\alpha\| \cup \|\beta\|$, hence $S_\alpha \cap \|\alpha\| \subseteq \|\beta\|$, from which follows that $\beta \in \mathrm{Th}(S_\alpha \cap \|\alpha\|)$. We also know from $G' \nsubseteq \|\neg\alpha\|$ that $G' \cap \|\alpha\| \neq \emptyset$, hence $(\cup\mathcal{S}) \cap \|\alpha\| \neq \emptyset$. It follows from this and $\beta \in Th(S_\alpha \cap \|\alpha\|)$ that $\beta \in K\circ_{\mathcal{S}}\alpha$.

Second case, $\top \leq \neg\alpha$ and $\beta \in K$. Let $G \in \mathcal{S}$. Then it follows from $\top \leq \neg\alpha$ and the definition of $\leq$ that if $G \subseteq \|\top\|$ then $G \subseteq \|\neg\alpha\|$. Since $G \subseteq \|\top\|$ is true for all $G \in \mathcal{S}$, it follows that $G \subseteq \|\neg\alpha\|$ for all $G \in \mathcal{S}$, hence $\cup\mathcal{S} \subseteq \|\neg\alpha\|$, so that $(\cup\mathcal{S}) \cap \|\alpha\| = \emptyset$.

It follows by EE4 (minimality) from $\beta \in K$, i.e., (by part 8 of Lemma 14) $\perp < \beta$, by the definition of $\leq$, that there is some $G \in \mathcal{S}$ such that $G \subseteq \|\beta\|$. Hence $\cap\mathcal{S} \subseteq \|\beta\|$, so that $\beta \in \mathrm{Th}(\cap\mathcal{S}) = K$. It follows from this and $(\cup\mathcal{S}) \cap \|\alpha\| = \emptyset$ that $\beta \in K\circ_{\mathcal{S}}\alpha$. ⊣

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