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Multiple Kernel Contraction

Abstract. This paper focuses on the extension of AGM that allows change for a belief base by a set of sentences instead of a single sentence. In [FH94], Fuhrmann and Hansson presented an axiomatic for *Multiple Contraction* and a construction based on the AGM *Partial Meet Contraction*. We propose for their model another way to construct functions: *Multiple Kernel Contraction*, that is a modification of *Kernel Contraction*, proposed by Hansson [Han94] to construct classical AGM contractions and belief base contractions. This construction works out the unsolved problem pointed out by Hansson in [Han99, pp. 369].

Keywords: Logic of Theory Change, Belief Bases, Kernel Contraction, Multiple Contraction

Introduction

In logic of theory change, the standard model is AGM, proposed by Alchourrón, Gärdenfors and Makinson [AGM85]. During the '90 the AGM model was extended in several ways, among them: models of belief change in which one or more of the postulates of the original AGM model are not satisfied; models that allow for sequences of operations (iterated or global functions); models that extend the language or the representation of the belief state; models that modify the representation of a belief state by introducing belief bases, etc.

When a new model is proposed, not only are the intuitions of it important, but also its axiomatic (which determines the behaviour of the functions) and the ways to construct functions.

This paper focuses on the extension of AGM that allows change for a belief base by a set of sentences instead of a single sentence. In [FH94], Fuhrmann and Hansson presented an axiomatic for *Multiple Contraction* and

[†] Personal note from Fermé: This paper pays a double tribute to Sven Ove Hansson. First, it combines his work on two topics: kernel contraction and multiple contraction. Second, the research reported here is the result of work done by Saez and Sanz in connection with their degree theses, and I have personally advised them following the example of Sven Ove, who was my own PhD advisor.

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a construction based on the AGM *Partial Meet Contraction*. We propose for their model another way to construct functions: *Multiple Kernel Contraction*, that is a modification of *Kernel Contraction*, proposed by Hansson [Han94] to construct classical AGM contractions and belief base contractions.

In section **Background** we introduce all the background needed to develop our method: the AGM model, belief base functions, kernel contraction and multiple contraction. In section **Multiple Kernel Contraction** we present our proposal and its axiomatic characterization. Proofs are deferred to an Appendix.

Background

Formal preliminaries: We will assume a language $\mathcal{L}$ that is closed under truth-functional operations and a consequence operator Cn for $\mathcal{L}$. Cn satisfies the standard Tarskian properties, namely inclusion ($A \subseteq Cn(A)$), monotony (if $A \subseteq B$, then $Cn(A) \subseteq Cn(B)$), and iteration ($Cn(A) = Cn(Cn(A))$). It is supraclassical and compact, and satisfies deduction (if $\beta \in Cn(A \cup \{\alpha\})$, then $(\alpha \to \beta) \in Cn(A)$). $A \vdash \alpha$ will be used as an alternative notation for $\alpha \in Cn(A)$, $\vdash \alpha$ for $\alpha \in Cn(\emptyset)$ and $Cn(\alpha)$ for $Cn(\{\alpha\})$. Upper-case letters denote subsets of $\mathcal{L}$. Lower-case Greek letters denote elements of $\mathcal{L}$. $\top$ is an arbitrary tautology and $\bot$ an arbitrary contradiction.

Belief Sets

In the AGM model of theory change [AGM85] belief states are represented by *belief sets*, i.e., set of sentences closed under Cn. Changes in beliefs are represented by operations on such sets. Among these operations, the *contraction* of a belief set A by a sentence α should be a new belief set $A' \subseteq A$ that does not contain α.

The most widely known method of contracting a sentence from a belief set is *partial meet contraction*, introduced by Alchourrón, Gärdenfors and Makinson [AGM85] in their seminal paper. For any set A of propositions, $A \bot \alpha$ be the set of all maximal subsets of A that do not imply α. Let γ be any function such that for any proposition α,$\gamma(A \bot \alpha)$ is a nonempty subset of A if the latter is non-empty, and $\gamma(A \bot \alpha) = \{A\}$ in the limiting case that $A \bot \alpha$ is empty.
The *partial meet contraction* on A that is generated by γ is the operation $\sim_\gamma$ such that for all sentence α:

$$A \sim_\gamma \alpha = \cap \gamma(A \bot \alpha)$$

One of the major achievements of AGM theory is the characterization of partial meet contraction for belief sets in terms of a set of postulates:

THEOREM 1 ([AGM85]). *Let A be a belief set. An operator $-$ on A is a partial meet contraction function if and only if $-$ satisfies:*

Closure *$A-\alpha$ is a belief set whenever A is a belief set.*

Success *If $\nvdash \alpha$, then $A-\alpha \nvdash \alpha$.*

Inclusion *$A-\alpha \subseteq A$.*

Vacuity *If $A \nvdash \alpha$, then $A \subseteq A-\alpha$.*

Extensionality *If $\vdash \alpha \leftrightarrow \beta$ then $A-\alpha = A-\beta$.*

Recovery *$A \subseteq Cn((A-\alpha) \cup \{\alpha\})$.*

Other methods to construct contraction functions are Safe Contraction [AM85], its generalization Kernel Contraction [Han94]; and contraction functions based on Epistemic Entrenchment [GM88].

Belief Bases

In recent years, alternative models have been presented in which the belief states are represented by *belief bases*, sets of sentences that are not necessarily closed under logical consequence. Hansson characterizes partial meet contraction for belief bases in terms of a set of postulates:

THEOREM 2 ([Han92]). *Let A be a belief set. An operator $-$ on A is a partial meet contraction function for A if and only if $-$ satisfies* success, inclusion*,*

Uniformity: *If it holds for all subsets A' of A that $\alpha \in Cn(A')$ if and only if $\beta \in Cn(A')$, then $A - \alpha = A - \beta$.*

Relevance *If $\beta \in A$ and $\beta \notin A-\alpha$ then there is some set A' such that $A-\alpha \subseteq A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A \cup \{\beta\})$.*

Uniformity is a strong version of extensionality, that is extended to sentences that have the same "behaviour" in a belief base and *relevance* is the postulate that represents the rationality criteria of *minimal change*.

Kernel Contraction

In [Han94] Hansson introduced *Kernel Contraction*, a generalization of Safe Contraction. It is based on a selection among the sentences of a set A that contribute effectively to imply α; and how to use this selection in contracting by α. Formally:

DEFINITION 1 ([Han94]). Let A be a set in $\mathcal{L}$ and α a sentence. Then $A \perp\!\!\!\perp \alpha$ is the set such that $B \in A \perp\!\!\!\perp \alpha$ if and only if:

$$\begin{cases} B \subseteq A \\ B \vdash \alpha \\ \text{If } B' \subset B \text{ then } B' \nvdash \alpha \end{cases}$$

$A \perp\!\!\!\perp \alpha$ is called the *kernel set of A with respect to α* and its elements are the *α-kernels* of A.

DEFINITION 2 ([Han94]). Let A be a set of sentences. Let $A \perp\!\!\!\perp \alpha$ be the kernel set of A respect to α. An *incision function* σ for A is a function such that for all sentences α:

$$\begin{cases} \sigma(A \perp\!\!\!\perp \alpha) \subseteq \bigcup(A \perp\!\!\!\perp \alpha) \\ \emptyset \neq B \in A \perp\!\!\!\perp \alpha, \text{ then } B \cap \sigma(A \perp\!\!\!\perp \alpha) \neq \emptyset \end{cases}$$

DEFINITION 3 ([Han94]). Let A be a set of sentences and σ an incision function for A. The *kernel contraction* $-_\sigma$ for A is defined as follows:

$$A -_\sigma \alpha = A \setminus \sigma(A \perp\!\!\!\perp \alpha).$$

An operator $-$ for a set A is a kernel contraction if and only if there is an incision function σ for A such that $A \sim \alpha = A -_\sigma \alpha$ for all sentences α.

Hansson also provided an axiomatic characterization for kernel contraction.

THEOREM 3 ([Han94]). *The operator $-$ for a set of sentences A is a kernel contraction if and only if it satisfies* success, inclusion, uniformity *and*

Core-retainment *If $\beta \in A$ and $\beta \notin A - \alpha$ then there is some set A' such that $A' \subseteq A$ and $\alpha \notin Cn(A')$ but $\alpha \in Cn(A' \cup \{\beta\})$.*

Since *core-retainment* is weaker than relevance, it follows for belief bases that all partial meet contraction are kernel contraction[1].

Multiple Contraction

Another extension of the AGM contraction consists in extending the model to contracting by sets of sentences [Fuh88, Nie91, FH94]. This extension allows two variants: *package contraction* where all the sentences must be

[1] For belief sets, partial meet contraction corresponds exactly to a special case called *smooth kernel contraction*.

removed from the belief base and *choice contraction*, where it is sufficient to remove at least one of the sentences.
Fuhrmann and Hansson [FH94] propose the *partial meet multiple contraction* for choice and package. For package contraction we will redefine the remainder operator "$\perp$" for sets by sets, whereas for choice contraction we will introduce the operator "$\angle$".

DEFINITION 4 (Package remainders [FH94]). $X \in A \perp B$ if and only if:

$$\begin{cases} X \subseteq A \\ B \cap Cn(X) = \emptyset \\ \text{If } X \subset Y \subseteq A \text{ then } B \cap Cn(Y) \neq \emptyset \end{cases}$$

$A \perp B$ is the set of all maximal subsets of A that do not overlap with B.

DEFINITION 5 ([FH94]). $X \in A \angle B$ if and only if:

$$\begin{cases} X \subseteq A \\ B \not\subseteq Cn(X) \\ \text{If } X \subset Y \subseteq A \text{ then } B \subseteq Cn(Y) \end{cases}$$

$A \angle B$ is the set of all maximal subsets of A that do not imply the whole elements of B.

It is important to note that $B \cap Cn(X) = \emptyset$ means that X does not imply any element of B (closely related with package contraction) whereas $B \not\subseteq Cn(X)$ means that there X does not imply all elements of B (closely related with choice contraction).

Partial meet package and choice contraction are defined based based on a selection from $A \perp B$ and $A \angle B$ respectively:

DEFINITION 6 ([FH94]). γ is a package selection function for A if and only if for all sets B:

1. If $A \perp B$ is non-empty, then $\gamma(A \perp B)$ is a non-empty subset of $A \perp B$.

2. If $A \perp B$ is empty, then $\gamma(A \perp B) = A$.

An operator $\div$ for A is an operator of *partial meet package contraction* if and only if there is some package selection function γ such that $A \div B = \bigcap \gamma(A \perp B)$ for all set B.

DEFINITION 7 ([FH94]). γ is a choice selection function for A if and only if for all sets B:

1. If $A \angle B$ is non-empty, then $\gamma(A \angle B)$ is a non-empty subset of $A \angle B$.

2. If $A\angle B$ is empty, then $\gamma(A\angle B) = A$.

An operator $\div$ for A is an operator of *partial meet choice contraction* if and only if there is some choice selection function γ such that $A \div B = \bigcap \gamma(A\angle B)$ for all sets B.

Fuhrmann and Hansson axiomatically characterized choice and package contraction as follows:

THEOREM 4 ([FH94]). *An operator $\div$ for a set A is an operator of* partial meet package contraction *if and only if it satisfies the following conditions:*

P-success *If $B \cap Cn(\emptyset) = \emptyset$ then $B \cap Cn(A \div B) = \emptyset$.*

P-inclusion $A \div B \subseteq A$.

P-relevance *If $\beta \in A$ and $\beta \notin A \div B$, then there exists a set C, such that $A \div B \subseteq C \subseteq A$ and $B \cap Cn(C) = \emptyset$ but $B \cap Cn(C \cup \{\beta\}) \neq \emptyset$.*

P-uniformity *If every subset A' of A implies some element of B if and only if A' implies some element of C, then $A \div B = A \div C$.*

THEOREM 5 ([FH94]). *An operator $\div$ for a set A is an operator of* partial meet choice contraction *if and only if it satisfies the following conditions:*

C-success *If $B \not\subseteq Cn(\emptyset)$ then $B \not\subseteq Cn(A \div B)$.*

C-inclusion $A \div B \subseteq A$.

C-relevance *If $\beta \in A$ and $\beta \notin A \div B$, then there exists a set C, such that $A \div B \subseteq C \subseteq A$, and $B \not\subseteq Cn(C)$ but $B \subseteq Cn(C \cup \{\beta\})$.*

C-uniformity *If for every subset A' of A it follows that $B \subseteq Cn(A')$ if and only if $C \subseteq Cn(A')$, then $A \div B = A \div C$.*

Multiple Kernel Contraction

The idea of *Multiple Kernel Contraction* is to define a constructive method to contract a belief base by a set of sentences. Multiple partial meet contraction is based on the maximal subsets of A that do not imply B (in its both means of implication, package and choice). Multiple Kernel Contraction is based on a selection among the elements of A that effectively contribute to imply B. The method to construct multiple kernel contraction is rather similar to kernel contraction for single sentences.

We must to define at first the *kernel set of A with respect to B*. We distinguish between package and choice.

DEFINITION 8 (Package Kernel Set). Let A, B be sets in $\mathcal{L}$. Then $A \perp\!\!\!\perp_P B$ is the set such that $X \in A \perp\!\!\!\perp_P B$ if and only if:

$$\begin{cases} X \subseteq A \\ B \cap Cn(X) \neq \emptyset \\ \text{If } Y \subset X \text{ then } B \cap Cn(Y) = \emptyset \end{cases}$$

It is important to note that $A \perp\!\!\!\perp_P B$ is different from $\bigcup_{\alpha \in B} A \perp\!\!\!\perp \alpha$. For example let $\beta_1,\ \beta_2 \in B$, $X \in A \perp\!\!\!\perp \beta_1$, $Y \in A \perp\!\!\!\perp \beta_2$ and $X \subset Y$. then $Y \in \bigcup_{\alpha \in B} A \perp\!\!\!\perp \alpha$, however by Definition 8, $Y \notin A \perp\!\!\!\perp_P B$.

The relationship among the elements from $A \perp\!\!\!\perp_P B$ and from $\bigcup_{\alpha \in B} A \perp\!\!\!\perp \alpha$, $\alpha \in B$ can be seen in the following observation:

Observation 1. *If $\alpha \in B$, $B \cap Cn(\emptyset) = \emptyset$ and $A \vdash \alpha$ then for all $X \in A \perp\!\!\!\perp \alpha$, there exists $\emptyset \neq Y \in A \perp\!\!\!\perp_P B$ such that $Y \subseteq X$.*

For Choice, the kernel set is defined as follows:

Definition 9 (Choice Kernel Set). Let A, B be sets in $\mathcal{L}$. Then $A \perp\!\!\!\perp_C B$ is the set such that $X \in A \perp\!\!\!\perp_P B$ if and only if:

$$\begin{cases} X \subseteq A \\ B \subseteq Cn(X) \\ \text{If } Y \subset X \text{ then } B \not\subseteq Cn(Y) \end{cases}$$

The following observation, inspired by the case of single sentences [Han94], will be useful in the characterization of multiple kernel contraction:

Observation 2. *The following conditions are equivalent:*

1. *For all subsets A' of A: $B \cap Cn(A') \neq \emptyset$ iff $C \cap Cn(A') \neq \emptyset$*
2. $A \perp\!\!\!\perp_P B = A \perp\!\!\!\perp_P C$
3. $A \perp\!\!\!\perp_C B = A \perp\!\!\!\perp_C C$

As in Kernel Contraction we must define an *incision function* to determine which sentences would be removed.

Definition 10. A function σ is a *incision function for A* if and only if it satisfies for all B:

1. $\sigma(A \perp\!\!\!\perp_i B) \subseteq \bigcup(A \perp\!\!\!\perp_i B)$
2. If $\emptyset \neq X \in A \perp\!\!\!\perp_i B$, then $X \bigcap \sigma(A \perp\!\!\!\perp_i B) \neq \emptyset$

   where $\perp\!\!\!\perp_i$ is $\perp\!\!\!\perp_P$ or $\perp\!\!\!\perp_C$.

The incision function gives rise to the *multiple kernel contraction* function.

DEFINITION 11. Let σ be an incision function for A. The *Multiple Kernel Contraction* $\approx_\sigma$ for A is defined as follows:

$$A \approx_\sigma B = A \backslash \sigma(A \perp\!\!\!\perp_i B)$$

where $\perp\!\!\!\perp_i$ is $\perp\!\!\!\perp_P$ or $\perp\!\!\!\perp_C$.

- An operator $\div$ for A is a multiple package contraction if and only if there is some package incision function σ for A such that $A \div B = A \approx_\sigma B$ for all the sets B.
- An operator $\div$ for A is a multiple choice contraction if and only if there is some choice incision function σ for A such that $A \div B = A \approx_\sigma B$ for all the sets B.

Axiomatic Characterization of Multiple Kernel Contraction

THEOREM 6. *An operator $\div$ for a set A is an operator of* kernel package contraction *if and only if it satisfies the following conditions:*

P-success *If $B \cap Cn(\emptyset) = \emptyset$ then $B \cap Cn(A \div B) = \emptyset$.*

P-inclusion *$A \div B \subseteq A$.*

P-core-retainment *If $\beta \in A$ and $\beta \notin A \div B$, then there exists a set C, such that $C \subseteq A$ and $B \cap Cn(C) = \emptyset$ but $B \cap Cn(C \cup \{\beta\}) \neq \emptyset$.*

P-uniformity *If every subset A' of A implies some element of B if and only if A' implies some element of C, then $A \div B = A \div C$.*

THEOREM 7. *An operator $\div$ for a set A is an operator of* kernel choice contraction *if and only if it satisfies the following conditions:*

C-success *If $B \not\subseteq Cn(\emptyset)$ then $B \not\subseteq Cn(A \div B)$.*

C-inclusion *$A \div B \subseteq A$.*

C-core-retainment *If $\beta \in A$ and $\beta \notin A \div B$, then there exists a set C, such that $C \subseteq A$, $B \not\subseteq Cn(C)$ and $B \subseteq Cn(C \cup \{\beta\})$.*

C-uniformity *If for every subset A' of A $B \subseteq Cn(A')$ if and only if $C \subseteq Cn(A')$, then $A \div B = A \div C$.*

Inclusion and *success* are the two basic postulates for a contraction function. *Uniformity* is a strong version of *extensionality*, that is extended to sentences that have the same "behavior" in a belief base, and *core-retainment* is the postulate that represent the rationality criteria of *minimal change* and is a postulate closely relate to the idea of "kernel-set", i.e., $C \cup \{\beta\}$ is a B-kernel element of A.

Conclusions and Remarks

We have defined multiple kernel contraction. This work can be considered as an extension of previous models in different ways:

- We extend the model of kernel contraction to multiple contraction.
- We define a new constructive method for multiple contraction.
- Since, as in the singleton case, all the multiple partial meet contractions are multiple kernel contractions, our model is a generalization of multiple partial meet contraction.

The connection with the supplementary postulates and its (probable) relationship with *multiple safe contraction* still awaits research.

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Appendix: Proofs

Proof of Observation 1.

Let $\alpha \in B$, $B \cap Cn(\emptyset) = \emptyset$ and $A \vdash \alpha$. Let $X \in A \perp\!\!\!\perp \alpha$: then it follows that $X \subseteq A$ and (since $\alpha \in B$) $B \cap Cn(X) \neq \emptyset$.
If $X \in A \perp\!\!\!\perp_P B$, let $Y = X$ and we done. Let $X \notin A \perp\!\!\!\perp_P B$, then (due the compactness of the underlying logic) there exists a subset of X, X' such that $B \cap Cn(X') \neq \emptyset$ and for all X'' such that $X'' \subset X'$, it follows $B \cap Cn(X'') = \emptyset$.
Hence $X' \in A \perp\!\!\!\perp_P B$. ■

Proof of Observation 2.

(1. $\Rightarrow$ 2.) Let $A \perp\!\!\!\perp_P B \neq A \perp\!\!\!\perp_P C$. We can, without loss of generality, assume that there exists X, such that $X \in A \perp\!\!\!\perp_P B$ and $X \notin A \perp\!\!\!\perp_P C$. If $C \cap Cn(X) = \emptyset$, we done, since $B \cap Cn(X) \neq \emptyset$. If $C \cap Cn(X) \neq \emptyset$, then by definition 8 (since $X \notin A \perp\!\!\!\perp_P C$), there exists $Y \subset X$, such that $C \cap Cn(Y) \neq \emptyset$. We done, since $B \cap Cn(Y) = \emptyset$.

(2. $\Rightarrow$ 1.) Let $X \subseteq A$ such that $B \cap Cn(X) \neq \emptyset$ and $C \cap Cn(X) = \emptyset$. Then (due the compactness of the underlying logic) there exists $X' \subseteq X$ such that $X' \in A \perp\!\!\!\perp_P B$. Since $X' \subseteq X$ and $C \cap Cn(X) = \emptyset$ it follows by definition 8 that $X' \notin A \perp\!\!\!\perp_P C$.

(1. $\Rightarrow$ 3.) Let $A \perp\!\!\!\perp_C B \neq A \perp\!\!\!\perp_C C$. We can, without loss of generality, assume that there exists X, such that $X \in A \perp\!\!\!\perp_C B$ and $X \notin A \perp\!\!\!\perp_C C$. If $C \nsubseteq Cn(X)$, we done, since $B \subseteq Cn(X)$. If $C \subseteq Cn(X)$, then there exists by definition 9 (since $X \notin A \perp\!\!\!\perp_C C$) $Y \subset X$, such that $C \subseteq Cn(Y)$. But $B \nsubseteq Cn(Y)$ and we done.

(3. $\Rightarrow$ 1.) Let $X \subseteq A$ such that $B \subseteq Cn(X)$ and $C \nsubseteq Cn(X)$. Then (by definition 9 and due the compactness of the underlying logic) there exists $X' \subseteq X$ such that $X' \in A \perp\!\!\!\perp_C B$. Since $X' \subseteq X$ y $C \nsubseteq Cn(X)$ then $X' \notin A \perp\!\!\!\perp_C C$. ■

Proof of Theorem 6.

Construction to Postulates: Let $\approx_\sigma$ be a Multiple Package Kernel Contraction operator for A.

P-success: Suppose by *reductio* that $B \cap Cn(\emptyset) = \emptyset$, $\alpha \in B$ and $(A \approx_\sigma B) \vdash \alpha$. Then there exists $X \subseteq A \approx_\sigma B$ such that $X \vdash \alpha$. Since $X \subseteq A \approx_\sigma B \subseteq A$ there $X' \subseteq X$ such that $X' \in A \perp\!\!\!\perp \alpha$. Due to observation 1 there exists $X'' \subseteq X'$ and $X'' \in A \perp\!\!\!\perp_P B$. By definition 10 there exists $\beta \in X''$ and $\beta \in \sigma(A \perp\!\!\!\perp_P B)$. Then it follows that $\beta \notin A \backslash \sigma(A \perp\!\!\!\perp_P B) = A \approx_\sigma B$, that contradicts $\beta \in X'' \subseteq A \approx_\sigma B$.

P-inclusion: It follows trivially from definition 11.

P-core-retainment: Let $\beta \in A$ and $\beta \notin A \approx_\sigma B$. Then $\beta \in \sigma(A \perp\!\!\!\perp_P B)$. It's follows by definition 10 that $\sigma(A \perp\!\!\!\perp_P B) \subseteq \cup(A \perp\!\!\!\perp_P B)$. Then there exists D such that $\beta \in D \in A \perp\!\!\!\perp_P B$. Let $C = D \setminus \{\beta\}$. Definition 8 yields $B \cap Cn(C) = \emptyset$ and $B \cap Cn(C \cup \{\beta\}) \neq \emptyset$.

P-uniformity: Suppose that every subset A' of A implies some element of B if and only if A' implies some element of C. By observation 2, $A \perp\!\!\!\perp_P B = A \perp\!\!\!\perp_P C$, from which it follows that $\sigma(A \perp\!\!\!\perp_P B) = \sigma(A \perp\!\!\!\perp_P C)$. Hence by definition 11 $A \approx_\sigma B = A \approx_\sigma C$.

Postulates to Construction: Let $\div$ be an operator for A that satisfies *P-success, P-inclusion, P-core-retainment,* and *P-uniformity*. We will prove that $\div$ is a kernel package contraction.

Let σ such that for all $B : \sigma(A \perp\!\!\!\perp_P B) = A \setminus (A \div B)$. We must prove that σ is a package incision function for A, proving that: (a) σ is a function, (b) σ satisfies conditions (1) and (2) from definition 10, and (c) that $A \div B = A \setminus \sigma(A \perp\!\!\!\perp_P B)$.

(a) σ *is a function*: Let B, $C \subseteq A$ such that $A \perp\!\!\!\perp_P B = A \perp\!\!\!\perp_P C$. By observation 2 it follows that every subset A' of A implies some element of B if and only if A' implies some element of C. Then by P-uniformity $A \div B = A \div C$. Hence by definition of σ, $\sigma(A \perp\!\!\!\perp_P B) = \sigma(A \perp\!\!\!\perp_P C)$.

(b) *Condition (1) hold:* Let $\beta \in \sigma(A \perp\!\!\!\perp_P B)$. By P-core-retainment there exists $C \subseteq A$ such that $B \cap Cn(C) = \emptyset$ and $B \cap Cn(C \cup \{\beta\}) \neq \emptyset$. By compactness it follows that there exists $X \subseteq C \cup \{\beta\}$ such that $B \cap Cn(X) \neq \emptyset$ and for all $Y \subset X$, $B \cap Cn(Y) = \emptyset$. Then by definition 8, $X \in A \perp\!\!\!\perp_P B$.

Due to $B \cap Cn(C) = \emptyset$, $B \cap Cn(X) \neq \emptyset$ and $X \subseteq C \cup \{\beta\}$ it follows that $\beta \in X$, and hence $\beta \in \cup(A \perp\!\!\!\perp_P B)$.

(b) *Condition (2) hold:* Assume that $\emptyset \neq X$ and $X \in (A \perp\!\!\!\perp_P B)$. It follows that $B \cap Cn(\emptyset) = \emptyset$ By P-success $B \cap Cn(A \div B) = \emptyset$. Since $B \cap Cn(X) \neq \emptyset$, $X \not\subseteq A \div B$ from which it follows that there exists $\beta \in X$ y and $\beta \notin A \div B$. $X \subseteq A$ yields $\beta \in A \setminus (A \div B)$, then $\beta \in \sigma(A \perp\!\!\!\perp_P B)$. Hence $\beta \in X \cap \sigma(A \perp\!\!\!\perp_P B)$.

(c) $A \div B = A \setminus \sigma(A \perp\!\!\!\perp_P B)$*:* It follows trivially from P-inclusion $A \div B \subseteq A$ and $\sigma(A \perp\!\!\!\perp_P B) = A \setminus A \div B$. This concludes the proof. ■

Proof of Theorem 7.

Construction to Postulates: Let $\approx_\sigma$ be a Multiple Package Kernel Contraction operator for A.

C-success: Let $B \not\subseteq Cn(\emptyset)$ and $B \subseteq Cn(A)$. Suppose by *reductio* that $B \subseteq Cn(A \approx_\sigma B)$. It follows by compactness that there exists X such that $X \subseteq A \approx_\sigma B$, $B \subseteq Cn(X)$ and such that for all $Y \subset X$ it follows that $B \not\subseteq Cn(Y)$. Then $X \in A \perp\!\!\!\perp_C B$ fro which it follows by definition 10 that $X \cap \sigma(A \perp\!\!\!\perp_C B) \neq \emptyset$. Hence $X \not\subseteq A \approx_\sigma B$. Absurd.

C-inclusion: It follows trivially from definition 11.

C-core-retainment: Let $\beta \in A$ and $\beta \notin A \approx_\sigma B$. Then by definition of $\approx_\sigma$, $\beta \in \sigma(A \perp\!\!\!\perp_C B)$. It follows by definition 10 that $\beta \in \cup(A \perp\!\!\!\perp_C B)$. Then there exists D such that $\beta \in D \in A \perp\!\!\!\perp_C B$. Let $C = D \setminus \{\beta\}$. Then by definition 9, $B \not\subseteq Cn(C)$ and $B \subseteq Cn(C \cup \{\beta\})$.

C-uniformity: Suppose that for every subset A' of A $B \subseteq Cn(A')$ if and only if $C \subseteq Cn(A')$. By observation 2 $A \perp\!\!\!\perp_C B = A \perp\!\!\!\perp_C C$. Then $\sigma(A \perp\!\!\!\perp_C B) = \sigma(A \perp\!\!\!\perp_C C)$. Hence, by definition 11 $A \approx_\sigma B = A \approx_\sigma C$.

Postulates to Construction: Let $\div$ an operator for A that satisfies *C-success, C-inclusion, C-core-retainment,* and *C-uniformity.* We will prove that $\div$ is a kernel choice contraction. Let σ such that for all $B : \sigma(A \perp\!\!\!\perp_C B) = A \setminus (A \div B)$. We must prove that σ is a choice incision function for A, proving that: (a) σ is a function, (b) σ satisfies conditions (1) and (2) from definition 10, and (c) that $A \div B = A \setminus \sigma(A \perp\!\!\!\perp_C B)$.

(a) σ *is a function*: Is proved in the same way of package, using C-uniformity instead of P-uniformity.

(b) *Condition (1) hold:* Let $\beta \in \sigma(A \perp\!\!\!\perp_C B)$. By C-core-retainment there exists $C \subseteq A$ such that $B \not\subseteq Cn(C)$ and $B \subseteq Cn(C \cup \{\beta\})$. By compactness it follows that there exists $X \subseteq C \cup \{\beta\}$ such that

$B \subseteq Cn(X)$ and for all $Y \subset X$, $B \not\subseteq Cn(Y)$. Then by definition 9, $X \in A \perp\!\!\!\perp_C B$. Hence $\beta \in \cup(A \perp\!\!\!\perp_C B)$.

(b) *Condition (2) hold:* Let $X \neq \emptyset$ and $X \in A \perp\!\!\!\perp_C B$. Then $B \subseteq Cn(X)$ and is minimal. Due to $X \neq \emptyset$, $B \not\subseteq Cn(\emptyset)$ and $B \subseteq Cn(A)$. Assume $X \subseteq Cn(A \approx_\sigma B)$, then $B \subseteq Cn(A \approx_\sigma B)$, absurd by C-success. Then $X \not\subseteq Cn(A \approx_\sigma B)$ from which it follows that there exists $\beta \in X$ such that $\beta \notin A \approx_\sigma B$. Hence $\beta \in \sigma(A \perp\!\!\!\perp_C B)$.

(c) $A \div B = A \setminus \sigma(A \perp\!\!\!\perp_C B)$*:* It follows trivially from C-inclusion $A \div B \subseteq A$ and $\sigma(A \perp\!\!\!\perp_C B) = A \setminus A \div B$. This concludes the proof. ■

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